# Anti-Ramsey number of edge-disjoint rainbow spanning trees 

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November 19, 2019


#### Abstract

An edge-colored graph $G$ is called rainbow if every edge of $G$ receives a different color. The anti-Ramsey number of $t$ edge-disjoint rainbow spanning trees, denoted by $r(n, t)$, is defined as the maximum number of colors in an edge-coloring of $K_{n}$ containing no $t$ edge-disjoint rainbow spanning trees. Jahanbekam and West [J. Graph Theory, 2016] conjectured that for any fixed $t, r(n, t)=\binom{n-2}{2}+t$ whenever $n \geq 2 t+2 \geq 6$. In this paper, we prove this conjecture. We also determine $r(n, t)$ when $n=2 t+1$. Together with previous results, this gives the anti-Ramsey number of $t$ edge-disjoint rainbow spanning trees for all values of $n$ and $t$.


## 1 Introduction

An edge-colored graph $G$ is called rainbow if every edge of $G$ receives a different color. The general anti-Ramsey problem asks for the maximum number of colors $A R(n, \mathcal{G})$ in an edge-coloring of $K_{n}$ containing no rainbow copy of any graph in a class $\mathcal{G}$. For some earlier results when $\mathcal{G}$ consists of a single graph, see the survey [9]. In particular, Montellano-Baallesteros and Neumann-Lara [13] showed a conjecture of Erdős, Simonovits and Sós [8] by computing $A R\left(n, C_{k}\right)$. Jiang and West [12] determined the anti-Ramsey number of the family of trees with $m$ edges.

Anti-Ramsey problems have also been investigated for rainbow spanning subgraphs. In particular, Hass and Young [10] showed that the anti-Ramsey number for perfect matchings (when $n$ is even) is $\binom{n-3}{2}+2$ for $n \geq 14$. For spanning trees, Bialostocki and Voxman [2] showed that the maximum number of colors in an edge-coloring of $K_{n}(n \geq 4)$ with no rainbow spanning tree is $\binom{n-2}{2}+1$. Jahanbekam and West [11] extended the investigations to finding the anti-Ramsey number of $t$ edge-disjoint rainbow spanning subgraphs of certain types including matchings, cycles and trees. In particular, for rainbow spanning trees, let $r(n, t)$ be the maximum number of colors in an edge-coloring of $K_{n}$ not having $t$ edge-disjoint rainbow spanning trees. Akbari and Alipour [1] showed that $r(n, 2)=\binom{n-2}{2}+2$ for $n \geq 6$. Jahanbekam and West [11] showed that

$$
r(n, t)= \begin{cases}\binom{n-2}{2}+t & \text { for } n>2 t+\sqrt{6 t-\frac{23}{4}}+\frac{5}{2} \\ \binom{n}{2}-t & \text { for } n=2 t\end{cases}
$$

and they made the following conjecture:
Conjecture 1. [11] $r(n, t)=\binom{n-2}{2}+t$ whenever $n \geq 2 t+2 \geq 6$.

[^0]In this paper, we show that the above conjecture holds and we also determine the value of $r(n, t)$ when $n=2 t+1$. Together with previous results ([2],[1],[11]), this gives the anti-Ramsey number of $t$ edge-disjoint rainbow spanning trees for all values of $n$ and $t$.

Theorem 1. For all positive integers $t$,

$$
r(n, t)= \begin{cases}\binom{n-2}{2}+t & \text { for } n \geq 2 t+2 \\ \binom{n-1}{2} & \text { for } n=2 t+1 \\ \binom{n}{2}-t & \text { for } n=2 t\end{cases}
$$

Remark 1. Note that if $n<2 t$, then $K_{n}$ does not have enough edges for $t$ edge-disjoint spanning trees.

The main tools we use are two structure theorems that characterize the existence of $t$ colordisjoint rainbow spanning trees or the existence of a color-disjoint extension of $t$ edge-disjoint rainbow spanning forests into $t$ edge-disjoint rainbow spanning trees. When $t=1$, Broersma and Li [3] showed that determining the largest rainbow spanning forest of a graph can be solved by applying the Matroid Intersection Theorem. The following characterization was established by Schrijver [16] using matroid methods, and later given graph theoretical proofs by Suzuki [17] and also by Carraher and Hartke [4].

Theorem 2. ([16, 17, 4]) An edge-colored connected graph $G$ has a rainbow spanning tree if and only if for every $2 \leq k \leq n$ and every partition of $G$ with $k$ parts, at least $k-1$ different colors are represented in edges between partition classes.

The above results can be generalized to $t$ color-disjoint rainbow spanning trees using similar matroid methods by Schrijver [16]. For the sake of self-completeness, we reproduce the proof using matroid methods in Section 2. We also give a new graph theoretical proof of Theorem 3.

Theorem 3. [16] An edge-colored multigraph $G$ has $t$ pairwise color-disjoint rainbow spanning trees if and only if for every partition $P$ of $V(G)$ into $|P|$ parts, at least $t(|P|-1)$ distinct colors are represented in edges between partition classes.

Remark 2. Recall the famous Nash-Williams-Tutte Theorem ([15, 18]): A multigraph contains $t$ edge-disjoint spanning trees if and only if for every partition $P$ of its vertex set, it has at least $t(|P|-1)$ cross-edges. Theorem 3 implies the Nash-Williams-Tutte Theorem by assigning every edge of the multigraph a distinct color.

Theorem 3 can be also generalized to extend edge-disjoint rainbow spanning forests to edgedisjiont rainbow spanning trees. Let $G$ be an edge-colored multigraph. Let $F_{1}, \ldots, F_{t}$ be $t$ edgedisjoint rainbow spanning forests. We are interested in whether $F_{1}, \ldots, F_{t}$ can be extended to $t$ edge-disjoint rainbow spanning trees $T_{1}, \ldots, T_{t}$ in $G$, i.e., $E\left(F_{i}\right) \subset E\left(T_{i}\right)$ for each $i$. We say the extension is color-disjoint if all edges in $\cup_{i}\left(E\left(T_{i}\right) \backslash E\left(F_{i}\right)\right)$ have distinct colors and these colors are different from the colors appearing in the edges of $\cup_{i} E\left(F_{i}\right)$. Using similar matroid methods or graph theoretical arguments, we can also obtain a criterion that characterizes the existence of a color-disjoint extension of rainbow spanning forests into rainbow spanning trees.

Theorem 4. A family of $t$ edge-disjoint rainbow spanning forests $F_{1}, \ldots, F_{t}$ has a color-disjoint extension in $G$ if and only if for every partition $P$ of $G$ into $|P|$ parts,

$$
\begin{equation*}
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right| \geq t(|P|-1) . \tag{1}
\end{equation*}
$$

Here $G^{\prime}$ is the spanning subgraph of $G$ by removing all edges with colors appearing in some $F_{i}$, and $c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)$ be the set of colors appearing in the edges of $G^{\prime}$ crossing the partition $P$.

It would be interesting to find a similar criterion for the existence of $t$ edge-disjoint rainbow trees in a general graph since applications of Theorem 3 and Theorem 4 usually require large number of colors in the host graph.

Organization: The rest of the paper is organized as follows. In Section 2, we present the proofs of Theorem 3 and Theorem 4. In Section 3, we show Theorem 1.

## 2 Proof of Theorem 3

We first reproduce the proof of Theorem 3 using matroid methods. A matroid is defined as $M=$ $(E, \mathcal{I})$ where $E$ is the ground set and $\mathcal{I} \subseteq 2^{E}$ is a set containing subsets of $E$ (called indepedent sets) that satisfy (i) if $A \subseteq B \subseteq E$, and $B \in \mathcal{I}$, then $A \in \mathcal{I}$; (ii) if $A \in \mathcal{I}, B \in \mathcal{I}$ and $|A|>|B|$, then $\exists a \in A \backslash B$ such that $B \cup\{a\} \in \mathcal{I}$. Given a matroid $M=(E, \mathcal{I})$, the rank function $r_{M}: 2^{E} \rightarrow \mathbb{N}$ is defined as $r_{M}(S)=\max \{|I|: I \subseteq S, I \in \mathcal{I}\}$. Thus $r_{M}(E)$ is the size of the maximum independent set of $M$. Two matroids of interests here are the graphic matroid and the partition matroid. Given an edge-colored graph $G$, the graphic matroid of $G$ is the matroid $M=(E, \mathcal{I})$ where $E=E(G)$ and $\mathcal{I}$ is the set of forests in $G$. The partition matroid of $G$, is the matroid $M^{\prime}=\left(E^{\prime}, \mathcal{I}^{\prime}\right)$ where $E^{\prime}=E(G)$ and $\mathcal{I}$ is the set of rainbow subgraphs of $G$. Given $k$ matroids $\left\{M_{i}=\left(E_{i}, \mathcal{I}_{i}\right)\right\}_{i \in[k]}$, one can define the union of the $k$ matroids, $M_{1} \vee \cdots \vee M_{k}=(E, \mathcal{I})$, by $E=\bigcup_{i=1}^{k} E_{i}$ and $\mathcal{I}=$ $\left\{I_{1} \cup \cdots \cup I_{k}: I_{i} \in \mathcal{I}_{i}\right.$ for all $\left.i \in[k]\right\}$. It is well known in matroid theory $[6,14]$ that $M_{1} \vee \cdots \vee M_{k}$ is a matroid with rank function

$$
r(S)=\min _{T \subseteq S}\left(|S \backslash T|+\sum_{i=1}^{k} r_{M_{i}}\left(T \cap E_{i}\right)\right) .
$$

Given two matroids $M_{1}=\left(E, \mathcal{I}_{1}\right)$ and $M_{2}=\left(E, \mathcal{I}_{2}\right)$ on the same ground set with rank functions $r_{1}$ and $r_{2}$ respectively, consider the family of independent sets common to both matroids, i.e., $\mathcal{I}_{1} \cap \mathcal{I}_{2}$. The well-known Matroid Intersection Theorem [7] asserts that

$$
\max _{I \in \mathcal{I}_{1} \cap \mathcal{I}_{2}}|I|=\min _{U \subseteq E}\left(r_{1}(U)+r_{2}(E \backslash U)\right) .
$$

### 2.1 Proof of Theorem 3 using Matroid methods

Again we remark that the proof essentially follows the same approaches as Schrijver [16] and we only reproduce it here for the sake of completeness.

Proof of Theorem 3. The forward direction is clear. Thus it remains to show that if for every partition $P$ of $V(G)$ into $|P|$ parts, at least $t(|P|-1)$ distinct colors are represented in edges between partition classes, then there exist $t$ edge-disjoint rainbow spanning trees in $G$.

Given an edge-colored graph $G$, let $M=(E, \mathcal{I})$ be the graphic matroid of $G$ and $M^{\prime}=\left(E, \mathcal{I}^{\prime}\right)$ be the partition matroid of $G$. Moreover, let $M^{t}=M \vee M \vee \cdots \vee M=\left(E, \mathcal{I}^{t}\right)$, where we take $t$ copies of $M$. By the matriod union theorem, we obtain that

$$
r_{M^{t}}(S)=\min _{T \subseteq S}\left(|S \backslash T|+t \cdot r_{M}(T)\right)
$$

By the Matroid Intersection Theorem,

$$
\begin{aligned}
\max _{I \in \mathcal{I}^{t} \cap \mathcal{I}^{\prime}}|I| & =\min _{U \subseteq E}\left(r_{M^{t}}(U)+r_{M^{\prime}}(E \backslash U)\right) \\
& =\min _{U \subseteq E}\left(\min _{T \subseteq U}\left(|U \backslash T|+t \cdot r_{M}(T)\right)+r_{M^{\prime}}(E \backslash U)\right)
\end{aligned}
$$

Let $T, U \subseteq E$ be arbitrarily chosen such that $T \subseteq U$. Observe that $t \cdot r_{M}(T)=t(n-q(T))$, where $q(T)$ is the number of components of $G[T]$. Now we claim that

$$
|U \backslash T|+r_{M^{\prime}}(E \backslash U) \geq r_{M^{\prime}}(E \backslash T) \geq t(q(T)-1)
$$

Indeed, for any color $c$ appearing in some edge $e \in E \backslash T$, if $e \in E \backslash U$, then the color $c$ is counted in $r_{M^{\prime}}(E \backslash U)$; if $e \in U$, then that color is counted in $|U \backslash T|$. In particular, at least $t(q(T)-1)$ distinct colors are represented in edges between connected components of $T$, thus in $E \backslash T$. It follows that

$$
|U \backslash T|+t \cdot r_{M}(T)+r_{M^{\prime}}(E \backslash U) \geq t(q(T)-1)+t(n-q(T)) \geq t(n-1)
$$

which implies that $\max _{I \in \mathcal{I}^{t} \cap \mathcal{I}^{\prime}}|I| \geq t(n-1)$. By definition, we then have $t$ edge-disjoint rainbow spanning trees.

### 2.2 Proof of Theorem 3 using graph theoretical arguments

In this subsection, we give a new graph theoretical proof of Theorem 3. Given a graph $G$, we use $V(G), E(G)$ to denote its vertex set and edge set respectively. We use $\|G\|$ to denote the number of edges in $G$. Given a set of edges $E$, we use $c(E)$ to denote the set of colors that appear in $E$. For clarity, we abuse the notation to use $c(e)$ to denote the color of an edge $e$. We say a color $c$ has multiplicity $k$ in $G$ if the number of edges with color $c$ in $G$ is $k$. The color multiplicity of an edge in $G$ is the multiplicity of the color of the edge in $G$.

For any partition $P$ of the vertex set $V(G)$ and a subgraph $H$ of $G$, let $|P|$ denote the number of parts in the partition $P$ and let $\operatorname{cr}(P, H)$ denote the set of crossing edges in $H$ whose end vertices belong to different parts in the partition $P$. When $H=G$, we also write $\operatorname{cr}(P, G)$ as $\operatorname{cr}(P)$. Given two partitions $P_{1}: V=\cup_{i} V_{i}$ and $P_{2}: V=\cup_{j} V_{j}^{\prime}$, let the intersection $P_{1} \cap P_{2}$ denote the partition given by $V=\bigcup_{i, j} V_{i} \cap V_{j}^{\prime}$. Given a spanning disconnected subgraph $H$, there is a natural partition $P_{H}$ associated to $H$, which partitions $V$ into its connected components. Without loss of generality, we abuse our notation $\operatorname{cr}(H)$ to denote the crossing edges of $G$ corresponding to this partition $P_{H}$. Recall we want to show that an edge-colored multigraph $G$ has $t$ color-disjoint rainbow spanning trees if and only if for any partition $P$ of $V(G)$ (with $|P| \geq 2$ ),

$$
\begin{equation*}
|c(c r(P))| \geq t(|P|-1) \tag{2}
\end{equation*}
$$

Proof of Theorem 3. One direction is easy. Suppose that $G$ contains $t$ pairwise color-disjoint rainbow spanning trees $T_{1}, T_{2}, \ldots, T_{t}$. Then all edges in these trees have distinct colors. For any partition $P$ of the vertex set $V$, each tree contributes at least $|P|-1$ crossing edges, thus $t$ trees contribute at least $t(|P|-1)$ crossing edges and the colors of these edges are all distinct.

Now we prove the other direction. Assume that $G$ satisfies inequality (2). We would like to prove $G$ contains $t$ pairwise color-disjoint rainbow spanning trees. We will prove by contradiction. Assume that $G$ does not contain $t$ pairwise color-disjoint rainbow spanning trees. Let $\mathcal{F}$ be the collection of all families of $t$ color-disjoint rainbow spanning forests $\left\{F_{1}, \cdots, F_{t}\right\}$. Consider the following deterministic process:

Initially, set $C^{\prime}:=\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}\right)\right)$
while $C^{\prime} \neq \emptyset$ do
for each color $x$ in $C^{\prime}$, do
for $j$ from 1 to $t$, do
if color $x$ appears in $F_{j}$, then
delete the edge in color $x$ from $F_{j}$
endif
endfor
endfor
set $C^{\prime}:=\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}\right)\right)-C^{\prime}$

## endwhile

For $i \geq 0, F_{j}^{(i)}$ denote the rainbow spanning forest $F_{j}$ after $i$ iterations of the while loop. In particular, $F_{j}^{(0)}=F_{j}$ for all $j \in[t]$ and $F_{j}^{(\infty)}$ is the resulting rainbow spanning forest of $F_{j}$ after the process. Similarly, let $C_{i}$ denote the set $C^{\prime}$ after the $i$-th iteration of the while loop. Note that $C_{i}$ is the set of new colors crossing components of $F_{j} \mathrm{~s}$ after some edges are deleted in the $i$-th iteration.

Observe that since the procedure is deterministic, $\left\{F_{j}^{(i)}: j \in[t], i>0\right\}$ is unique for a fixed family $\left\{F_{1}, \cdots, F_{t}\right\}$. We define a preorder on $\mathcal{F}$. We say a family $\left\{F_{j}\right\}_{j=1}^{t}$ is less than or equal to another family $\left\{F_{j}^{\prime}\right\}_{j=1}^{t}$ if there is a positive integer $l$ such that

1. For $1 \leq i<l, \sum_{j=1}^{t}\left\|F_{j}^{(i)}\right\|=\sum_{j=1}^{t}\left\|F_{j}^{(i)}\right\|$.
2. $\sum_{j=1}^{t}\left\|F_{j}^{(l)}\right\|<\sum_{j=1}^{t}\left\|F_{j}^{(l)}\right\|$.

Since $G$ is finite, so is $\mathcal{F}$. There exists a maximal element $\left\{F_{1}, F_{2}, \cdots, F_{t}\right\} \in \mathcal{F}$. Run the deterministic process on $\left\{F_{1}, F_{2}, \cdots, F_{t}\right\}$.

The goal is to construct a common partition $P$ by refining $\operatorname{cr}\left(F_{j}\right)$ so that $|c(\operatorname{cr}(P))|<t(|P|-1)$. In particular, we will show that all forests in $\left\{F_{j}^{(\infty)}: j \in[t]\right\}$ admit the same partition $P$.

Claim (a): $\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i)}\right)\right) \subseteq\left(\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}\right)\right)\right) \cup\left(\bigcup_{j=1}^{t} c\left(F_{j}^{(i)}\right)\right)$.
Assume for the sake of contradiction that there is a color $x \in \bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i)}\right)\right) \backslash \bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}\right)\right)$ and there is no edge in color $x$ in all forests $F_{1}^{(i)}, \ldots, F_{t}^{(i)}$. Let $e$ be the edge such that $c(e)=x$ and $e \in \operatorname{cr}\left(F_{s}^{(i)}\right)$ for some $s \in[t]$. Observe that since $c(e) \notin \bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}\right)\right)$, it follows that $F_{s}^{(i-1)}+e$ contains a rainbow cycle, which passes through $e$ and another edge $e^{\prime} \in F_{s}^{(i-1)}$ joining two distinct components of $F_{s}^{(i)}$. Now let us consider a new family of rainbow spanning forests $\left\{F_{1}^{\prime}, \cdots, F_{t}^{\prime}\right\}$ where $F_{j}^{\prime}=F_{j}$ for $j \neq s$ and $F_{s}^{\prime}=F_{s}-e^{\prime}+e$. The color-disjoint property is maintained since the color of edge $e$ is not in any $F_{j}$. Observe that since $c(e) \notin \bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}\right)\right), F_{s}^{\prime(i)}$ will have one
fewer component than $F_{s}^{(i)}$. Thus we have

$$
\begin{gathered}
\sum_{j=1}^{t}\left\|F_{j}^{(k)}\right\|=\sum_{j=1}^{t}\left\|F_{j}^{\prime(k)}\right\| \text { for } k<i \\
\sum_{j=1}^{t}\left\|F_{j}^{(i)}\right\|>\sum_{j=1}^{t}\left\|F_{j}^{(i)}\right\|
\end{gathered}
$$

which contradicts our maximality assumption of $\left\{F_{i}: i \in[t]\right\}$. That finishes the proof of Claim (a).
Claim (a) implies that for each $x \in C_{i}$, there is an edge $e$ of color $x$ in exactly one of the forests in $\left\{F_{j}^{(i)}: j \in[t]\right\}$. Thus removing that edge in the next iteration will increase the sum of number of partitions exactly by 1 . Thus we have that

$$
\sum_{j=1}^{t}\left|P_{F_{j}^{(i+1)}}\right|=\sum_{j=1}^{t}\left|P_{F_{j}^{(i)}}\right|+\left|C_{i}\right| .
$$

It then follows that

$$
\begin{aligned}
\sum_{j=1}^{t}\left|P_{F_{j}^{(\infty)}}\right| & =\sum_{j=1}^{t}\left|P_{F_{j}}\right|+\sum_{i}\left|C_{i}\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}}\right|+\left|\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(\infty)}\right)\right)\right|
\end{aligned}
$$

Finally set the partition $P=\bigcap_{j=1}^{t} P_{F_{j}^{(\infty)}}$. We claim $P_{F_{j}^{(\infty)}}=P$ for all $j$. This is because all edges in $\operatorname{cr}\left(P_{F_{j}^{(\infty)}}\right) \cap \bigcup_{k=1}^{t} E\left(F_{k}^{(\infty)}\right)$ have been already removed. We then have

$$
\begin{aligned}
t|P| & =\sum_{j=1}^{t}\left|P_{F_{j}^{(\infty)}}\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}}\right|+\left|\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(\infty)}\right)\right)\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}}\right|+|c(\operatorname{cr}(P))| \\
& \geq t+1+|c(\operatorname{cr}(P))|
\end{aligned}
$$

We obtain

$$
|c(\operatorname{cr}(P))| \leq t(|P|-1)-1 .
$$

Contradiction.

Corollary 1. The edge-colored complete graph $K_{n}$ has $t$ color-disjoint rainbow spanning trees if the number of edges colored with any fixed color is at most $n /(2 t)$.

Proof. Suppose $K_{n}$ does not have $t$ color-disjoint rainbow spanning trees, then there exists a partition $P$ of $V\left(K_{n}\right)$ into $r$ parts $(2 \leq r \leq n)$ such that the number of distinct colors in the crossing edges of $P$ is at most $t(r-1)-1$. Let $m$ be the number of edges crossing the partition $P$. It follows that

$$
m \leq(t(r-1)-1) \cdot \frac{n}{2 t} \leq \frac{n}{2}(r-1)-\frac{n}{2 t}
$$

On the other hand,

$$
m \geq\binom{ n}{2}-\binom{n-(r-1)}{2}
$$

Hence we have

$$
\binom{n}{2}-\binom{n-(r-1)}{2} \leq \frac{n}{2}(r-1)-\frac{n}{2 t}
$$

which implies

$$
(n-r)(r-1) \leq-\frac{n}{t}
$$

which contradicts that $2 \leq r \leq n$.
Remark: This result is tight since the total number of colors used in $K_{n}$ could be as small as $\binom{n}{2} /(n /(2 t))=t(n-1)$, but any $t$ color-disjoint rainbow spanning trees need $t(n-1)$ colors. On the contrast, a result by Carraher, Hartke and Horn [5] implies there are $\Omega(n / \log n)$ edge-disjoint rainbow spanning trees.

### 2.3 Proof of Theorem 4

Recall we want to show that any $t$ edge-disjoint rainbow spanning forests $F_{1}, \ldots, F_{t}$ have a colordisjoint extension to edge-disjoint rainbow spanning trees in $G$ if and only if

$$
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| \geq t(|P|-1)
$$

where $G^{\prime}$ is the spanning subgraph of $G$ by removing all edges with colors appearing in some $F_{j}$.
Proof. Again, the forward direction is trivial. We only need to show that condition (1) implies there exists a color-disjoint extension to edge-disjoint rainbow spanning trees. The proof is similar to the proof of Theorem 3. Consider a set of edge-maximal forests $F_{1}^{(0)}, \ldots, F_{t}^{(0)}$ which is a color-disjoint extension of $F_{1}, \ldots, F_{t}$. From $\left\{F_{j}^{(0)}\right\}$ we delete all edges (in $\left\{F_{j}^{(0)}\right\}$ ) of some color $c$ appearing in $\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(0)}, G^{\prime}\right)\right)$ to get a new set $\left\{F_{j}^{(1)}\right\}$. Repeat this process until we reach a stable set $\left\{F_{j}^{(\infty)}\right\}$. Since we only delete edges in $G^{\prime}$, we have $E\left(F_{j}\right) \subseteq E\left(F_{j}^{(\infty)}\right)$ for each $1 \leq j \leq t$. The edges and colors in $\cup_{j=1}^{t} E\left(F_{j}\right)$ will not affect the process. A similar claim still holds:

$$
\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i)}, G^{\prime}\right)\right) \subseteq\left(\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}, G^{\prime}\right)\right)\right) \cup\left(\bigcup_{j=1}^{t} c\left(E\left(F_{j}^{(i)}\right) \cap E\left(G^{\prime}\right)\right)\right)
$$

In particular, let $C_{i}=\left(\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i)}, G^{\prime}\right)\right)\right) \backslash\left(\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(i-1)}, G^{\prime}\right)\right)\right)$. Then we have

$$
\sum_{j=1}^{t}\left|P_{F_{j}^{(i+1)}}\right|=\sum_{j=1}^{t}\left|P_{F_{j}^{(i)}}\right|+\left|C_{i}\right|
$$

It then follows that

$$
\begin{aligned}
\sum_{j=1}^{t}\left|P_{F_{j}^{(\infty)}}\right| & =\sum_{j=1}^{t}\left|P_{F_{j}^{(0)}}\right|+\sum_{i}\left|C_{i}\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}^{(0)}}\right|+\left|\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(\infty)}, G^{\prime}\right)\right)\right|
\end{aligned}
$$

Finally set the partition $P=\bigcap_{j=1}^{t} P_{F_{j}^{(\infty)} \backslash E\left(F_{j}\right)}$. Clearly all edges in $\operatorname{cr}\left(P, G^{\prime}\right)$ are removed. All possible edges remaining in $G$ that cross the partition $P$ are exactly the edges in $\bigcup_{j=1}^{t} \operatorname{cr}\left(P, F_{j}\right)$.

We have

$$
\begin{aligned}
t|P| & =\sum_{j=1}^{t}\left|P_{F_{j}^{(\infty)}}\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}^{(0)}}\right|+\left|\bigcup_{j=1}^{t} c\left(\operatorname{cr}\left(F_{j}^{(\infty)}, G^{\prime}\right)\right)\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| \\
& =\sum_{j=1}^{t}\left|P_{F_{j}^{(0)}}\right|+\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| \\
& \geq t+1+\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| .
\end{aligned}
$$

We obtain

$$
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|+\sum_{j=1}^{t}\left|\operatorname{cr}\left(P, F_{j}\right)\right| \leq t(|P|-1)-1
$$

Contradiction.

## 3 Proof of Theorem 1

Recall that $r(n, t)$ is the maximum number of colors in an edge-coloring of the complete graph $K_{n}$ not having $t$ edge-disjoint rainbow spanning trees.

Lower Bound: Jahanbekam and West (See Lemma 5.1 in [11]) showed the following lower bound for $r(n, t)$.

Proposition 1. [11] For positive integers $n$ and $t$ such that $t \leq 2 n-3$, there is an edge-coloring of $K_{n}$ using $\binom{n-2}{2}+t$ colors that does not have $t$ edge-disjoint rainbow spanning trees. When $n=2 t+1$, the construction improves to $\binom{n-1}{2}$ colors. When $n=2 t$, it improves to $\binom{n}{2}-t$.

This matches the upper bounds in Theorem 1. Hence we will skip the proof of lower bounds in the subsequent theorems. Moreover, we only consider the case $t \geq 2$ since the case $t=1$ was already resolved in Bialostocki and Voxman [2]. In Section 3.1, we prove a technical lemma that will be used in the proof of Theorem 1. In Section 3.2, 3.3,3.4, we show Theorem 1 when $n$ is in different range of values with respect to $t$.

### 3.1 Technical lemma

Lemma 1. Let $G$ be an edge-colored graph with $s$ colors $c_{1}, \cdots, c_{s}$ and $|V(G)|=n=2 t+2$ where $t \geq 3$. For color $c_{i}$, let $m_{i}$ be the number of edges of color $c_{i}$. Suppose $\sum_{i=1}^{s}\left(m_{i}-1\right)=3 t$ and $m_{i} \geq 2$ for all $i \in[s]$. Then we can construct $t$ edge-disjoint rainbow forests $F_{1}, \ldots, F_{t}$ in $G$ such that if we define $G_{0}=G-\bigcup_{i=1}^{t} E\left(F_{i}\right)$, then

$$
\begin{equation*}
\left|E\left(G_{0}\right)\right| \leq 2 t+1 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta\left(G_{0}\right) \leq t+1 \tag{4}
\end{equation*}
$$

Proof. We consider two cases:
Case 1: $m_{1} \geq 2 t+2$. Note that

$$
\sum_{i=2}^{s}\left(m_{i}-1\right)=3 t-\left(m_{1}-1\right) \leq t-1 .
$$

Thus, $s \leq t$. Let $d_{i}(v)$ be the number of edges in color $c_{i}$ and incident to $v$ in the current graph $G$. We construct the edge-disjoint rainbow forests $F_{1}, F_{2}, \ldots, F_{t}$ in two rounds: In the first round, we greedily extract edges only in color $c_{1}$. For $i=1, \ldots, t$, at step $i$, pick a vertex $v$ with maximum $d_{1}(v)$ (pick arbitrarily if tie). Pick an edge in color $c_{1}$ incident to $v$, assign it to $F_{i}$, and delete it from $G$.
We claim that after the first round $d_{1}(v) \leq t+1$ for any vertex $v$.
Suppose not, if $d_{1}(v) \geq t+2$. Since $n-1-(t+2)<t$, it follows that there exists another vertex $u$ with $d_{1}(u) \geq d_{1}(v)-1 \geq t+1$.
This implies

$$
m_{1} \geq t+d_{1}(v)+d_{1}(u)-1 \geq 3 t+2
$$

However,

$$
m_{1}-1 \leq \sum_{i=1}^{s}\left(m_{i}-1\right)=3 t
$$

which gives us the contradiction.
In the second round, we greedily extract edges not in color $c_{1}$. For $i=1, \ldots, t$, at step $i$, among all vertices $v$ with at least one neighboring edge not in color $c_{1}$, pick a vertex $v$ with maximum vertex degree $d(v)$ (pick arbitrarily if tie). Pick an edge incident to $v$ and not in color $c_{1}$, assign it to $F_{i}$, and delete it from $G$.
If we succeed with selecting $t$ edges not in color $c_{1}$ in the second round, we claim $d(v) \leq t+1$ for any vertex $v$. Suppose not, if $d(v) \geq t+2$. Then there is another vertex $u$ with $d(u) \geq$ $d(v)-1 \geq t+1$. It implies

$$
\sum_{i=1}^{s} m_{i} \geq 2 t+d(u)+d(v)-1 \geq 4 t+2
$$

However, since $s \leq t$, we have

$$
\sum_{i=1}^{s} m_{i} \leq 3 t+s \leq 4 t
$$

Contradiction. Therefore it follows that $d(v) \leq t+1$. Moreover, $\left|E\left(G_{0}\right)\right| \leq 4 t-2 t \leq 2 t$.
If the process stops at step $i=l<t$, then all remaining edges in $G_{0}$ must be in color 1. Thus, by the previous claim, $\Delta\left(G_{0}\right) \leq t+1$. Moreover,

$$
\left|E\left(G_{0}\right)\right| \leq m_{1}-t \leq(3 t+1)-t=2 t+1 .
$$

In both cases above, $F_{1}, \cdots F_{t}$ are edge-disjoint rainbow forests that satisfies inequality (3) and (4).

Case 2: $m_{1} \leq 2 t+1$.
Claim: There exists $t$ edge-disjoint rainbow forests $F_{1}, F_{2}, \cdots, F_{t}$ such that $\Delta\left(G_{0}\right) \leq t+1$.
For $j=1,2, \ldots, t$, we will construct a rainbow forest $F_{j}$ by selecting a rainbow set of edges such that after deleting these edges from $G, \Delta\left(G_{0}\right) \leq 2 t+1-j$. Notice that when $j=t$, we will have $\Delta\left(G_{0}\right) \leq t+1$. Our procedure is as follows:

For step $j$, without loss of generality, let $v_{1}, v_{2}, \cdots, v_{l}$ be the vertices with degree $2 t+2-j$ and let $c_{1}, c_{2}, \cdots, c_{m}$ be the set of colors of edges incident to $v_{1}, v_{2}, \cdots, v_{l}$ in $G$. If there is no such vertex, simply pick an edge incident to the max-degree vertex and assign it to $F_{j}$. Otherwise, we will construct an auxiliary bipartite graph $H=A \cup B$ where $A=\left\{v_{1}, \cdots, v_{l}\right\}$ and $\mathrm{B}=\left\{c_{1}, c_{2}, \ldots, c_{m}\right\}$ and $v_{x} c_{y} \in E(H)$ if and only if there is an edge of color $c_{y}$ incident to $v_{x}$. We claim that there exists a matching of $A$ in $H$. Suppose not, then by Hall's theorem, there exists a set of vertices $A^{\prime}=\left\{u_{1}, u_{2}, \cdots u_{k}\right\} \subseteq A$ such that $\left|N\left(A^{\prime}\right)\right|<\left|A^{\prime}\right|=k$ where $k \geq 2$. Without loss of generality, suppose $N(A)=\left\{c_{1}^{\prime}, c_{2}^{\prime}, \cdots, c_{q}^{\prime}\right\}$ where $q \leq k-1$. Let $m_{i}^{\prime}$ be the number of edges of color $c_{i}^{\prime}$ remaining in $G$.
Note that $k \neq 2$ since otherwise we will have one color with at least $2 \cdot(2 t+2-j)-1 \geq 2 t+3$ edges, which contradicts our assumption in this case.

Notice that for every $i \in[k], u_{i}$ has at least $(2 t+2-j)$ edges incident to it. Moreover, at least $j-1$ edges are already deleted from $G$ in previous steps. Therefore, we have

$$
\frac{k(2 t+2-j)}{2} \leq \sum_{i=1}^{q} m_{i}^{\prime} \leq\left(\sum_{i=1}^{q}\left(m_{i}^{\prime}-1\right)\right)+(k-1) \leq 3 t-(j-1)+(k-1) .
$$

It follows that

$$
k \leq 2+\frac{2 t}{2 t-j} \leq 4
$$

Similarly, using another way of counting the edges incident to some $u_{i}(i \in[k])$, we have

$$
k(2 t+2-j)-\binom{k}{2} \leq 3 t-(j-1)+(k-1)
$$

which implies that

$$
t(2 k-3) \leq \frac{k(k-3)}{2}+j(k-1) \leq \frac{k(k-3)}{2}+t(k-1) .
$$

It follows that $t \leq \frac{k(k-3)}{2(k-2)}$. Since $k \leq 4$ and $k>2$, we obtain that $t \leq 1$, which contradicts our assumption that $t \geq 2$. Thus by contradiction, there exists a matching of $A$ in $H$. This implies that there exists a rainbow set of edges $E_{j}$ that cover all vertices with degree $2 t+2-j$ in step $j$. We can then find a maximally acyclic subset $F_{j}$ of $E_{j}$ such that $F_{j}$ is a rainbow
forest and every vertex of degree $2 t+2-j$ is adjacent to some edge in $F_{j}$. Delete edges of $F_{j}$ from $G$ and we have $\Delta\left(G_{0}\right) \leq 2 t+1-j$. As a result, after $t$ steps, we obtain $t$ edge-disjoint rainbow forests $F_{1}, \cdots, F_{t}$ and $\Delta\left(G_{0}\right) \leq t+1$. This finishes the proof of the claim.

Now let $\left\{F_{1}, F_{2}, \cdots, F_{t}\right\}$ be an edge-maximal set of $t$ edge-disjoint rainbow forests that satisfies $\Delta\left(G_{0}\right) \leq t+1$. We claim that $\left|E\left(G_{0}\right)\right| \leq 2 t+1$. Suppose not, i.e., $\left|E\left(G_{0}\right)\right| \geq 2 t+2$. It follows that $\sum_{i=1}^{t}\left|E\left(F_{i}\right)\right| \leq 6 t-(2 t+2)<4 t$, i.e. there exists a $j \in[t]$ such that $F_{j}$ has at most 3 edges. Since $F_{j}$ is edge maximal, none of the edges in $G_{0}$ can be added to $F_{j}$. We have three cases:

Case 2a: $\left|E\left(F_{j}\right)\right|=1$. It then follows that all edges in $G_{0}$ have the same color (call it $c_{1}^{\prime}$ ) as the single edge in $F_{j}$. Thus we have a color with multiplicity at least $2 t+3$, which contradicts that $m_{1}<2 t+2$.
Case 2b: $\left|E\left(F_{j}\right)\right|=2$. Similarly, we have that at least $2 t+1$ edges in $G_{0}$ that share the same colors (call them $c_{1}^{\prime}, c_{2}^{\prime}$ ) as edges in $F_{j}$. It follows that $m_{1}+m_{2} \geq 2 t+3$. Similar to Case 1, in this case, we have that $s \leq t+1$ and $|E(G)|=3 t+s \leq 4 t+1$. Since $\left|E\left(G_{0}\right)\right| \geq 2 t+2$, that implies that $\sum_{i=1}^{t}\left|E\left(F_{i}\right)\right| \leq(4 t+1)-(2 t+2)=2 t-1$. Hence there exists some $F_{k}$ such that $\left|E\left(F_{k}\right)\right| \leq 1$ and we are done by Case 2 a.
Case 2c: $\left|E\left(F_{j}\right)\right|=3$. Similarly, we have that at least $2 t-1$ edges in $G_{0}$ share the same colors (call them $c_{1}^{\prime}, c_{2}^{\prime}, c_{3}^{\prime}$ ) as edges in $F_{j}$. It follows that $m_{1}+m_{2}+m_{3} \geq 2 t+2$. By inequality (5), we have that $s \leq t+4$ and $|E(G)| \leq 4 t+4$. Since $\left|E\left(G_{0}\right)\right| \geq 2 t+2$, that implies that $\sum_{i=1}^{t}\left|E\left(F_{i}\right)\right| \leq 2 t+2$. Since $t \geq 3$ by our assumption, there exists a $k \in[t]$ such that $\left|E\left(F_{k}\right)\right| \leq 2$ and we are done by Case $2 b$ and Case $2 c$.

Therefore, by contradiction, we have that $\left|E\left(G_{0}\right)\right| \leq 2 t+1$ and we are done.

### 3.2 Proof of Theorem 1 where $n=2 t+2$

Proposition 2. For any $n=2 t+2 \geq 6$, we have $r(n, t)=\binom{n-2}{2}+t=2 t^{2}$.
Proof. Note that the lower bound is shown by Jahanbekam and West in Proposition 1. For the upper bound, we will assume that $t \geq 3$ since the case when $t=2$ is implied by the result of Akbari and Alipour [1]. We will show that any coloring of $K_{2 t+2}$ with $2 t^{2}+1$ distinct colors contains $t$ edge-disjoint rainbow spanning trees. Call this edge-colored graph $G$. Let $m_{i}$ be the multiplicity of the color $c_{i}$ in $G$. Without loss of generality, say the first $s$ colors have multiplicity at least 2, i.e.

$$
m_{1} \geq m_{2} \geq \cdots \geq m_{s} \geq 2
$$

Let $G_{1}$ be the spanning subgraph of $G$ consisting of all edges with color multiplicity greater than 1 in $G$. Let $G_{2}$ be the spanning subgraph consisting of the remaining edges. We have

$$
\begin{equation*}
\sum_{i=1}^{s}\left(m_{i}-1\right)=\binom{n}{2}-\left(2 t^{2}+1\right)=3 t \tag{5}
\end{equation*}
$$

In particular, we have

$$
\left|E\left(G_{1}\right)\right|=\sum_{i=1}^{s} m_{i}=3 t+s \leq 6 t
$$

By Lemma 1, it follows that we can construct $t$ edge-disjoint rainbow spanning forests $F_{1}, \ldots, F_{t}$ in $G$ such that if we define $G_{0}=E\left(G_{1}\right)-\bigcup_{i=1}^{t} E\left(F_{i}\right)$, then

$$
\left|E\left(G_{0}\right)\right| \leq 2 t+1
$$

and

$$
\Delta\left(G_{0}\right) \leq t+1 .
$$

Now we show that $F_{1}, \ldots, F_{t}$ have a color-disjoint extension to $t$ edge-disjoint rainbow spanning trees. Consider any partition $P$. We will verify

$$
\begin{equation*}
\left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right| \geq t(|P|-1) \tag{6}
\end{equation*}
$$

We will first verify the case when $3 \leq|P| \leq n$. Note that

$$
\left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right|-t(|P|-1) \geq\binom{ n}{2}-(2 t+1)-\binom{n-|P|+1}{2}-t(|P|-1) .
$$

We want to show that the right hand side of the above inequality is nonnegative. Note that the function on the right hand side is concave downward with respect to $|P|$. Thus it is sufficient to verify it at $|P|=3$ and $|P|=n$.

When $|P|=3$, we have

$$
\binom{n}{2}-(2 t+1)-\binom{n-2}{2}-2 t=0
$$

When $|P|=n$, we have

$$
\binom{n}{2}-(2 t+1)-t(n-1)=0
$$

It remains to verify the inequality (6) for $|P|=2$. By Theorem 4, we have $\left|E\left(G_{0}\right)\right| \leq 2 t+1$. If each part of $P$ contains at least 2 vertices, then we have

$$
\begin{aligned}
& \left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right|-t(|P|-1) \\
& \quad \geq\binom{ n}{2}-\left|E\left(G_{0}\right)\right|-\left(\binom{n-2}{2}+1\right)-t \\
& \quad \geq\binom{ n}{2}-(2 t+1)-\left(\binom{n-2}{2}+1\right)-t \\
& \quad=t-1 \geq 0
\end{aligned}
$$

Otherwise, $P$ is of the form $V(G)=\{v\} \cup B$ for some $v \in V(G)$ and $B=V(G) \backslash\{v\}$. By Lemma 1 , we have $d_{G_{0}} \leq t+1$. Thus,

$$
\left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right|-t(|P|-1) \geq(n-1)-d_{G_{0}}(v)-t \geq 2 t+1-(t+1)-t=0
$$

Therefore, by Theorem $4, F_{1}, \ldots, F_{t}$ have a color-disjoint extension to $t$ edge-disjoint rainbow spanning trees.

### 3.3 Proof of Theorem 1 where $n \geq 2 t+3$

Proposition 3. For any $n \geq 2 t+2 \geq 6$, we have $r(n, t)=\binom{n-2}{2}+t$.
Proof. Again, the lower bound is due to Proposition 1. For the upper bound, we will show that every edge-coloring of $K_{n}$ with exactly $\binom{n-2}{2}+t+1$ distinct colors has $t$ edge-disjoint spanning trees. Call this edge-colored graph $G$.

Given a vertex $v$, we define $D(v)$ to be the set of colors $C$ such that every edge with colors in $C$ is incident to $v$. Given a vertex $v$ and a set of colors $C$, define $\Gamma(v, C)$ as the set of edges incident to $v$ with colors in $C$. For ease of notation, we let $\Gamma(v)=\Gamma(v, D(v))$.

For fixed $t$, we will prove the theorem by induction on $n$. The base case is when $n=2 t+2$, which is proven in Proposition 2. Let's now consider the theorem when $n \geq 2 t+3$.

Case 1: there exists a vertex $v \in V(G)$ with $|\Gamma(v)| \geq t$ and $|D(v)| \leq n-3$.
In this case, we set $G^{\prime}=G-\{v\}$. Note that $G^{\prime}$ is an edge-colored complete graph with at least $\binom{n-2}{2}+t+1-(n-3)=\binom{n-3}{2}+t+1$ distinct colors. Moreover $\left|G^{\prime}\right| \geq 2 t+2$. Hence by induction, there exists $t$ edge-disjoint rainbow spanning trees in $G^{\prime}$. Note that by our definition of $D(v)$, none of the colors in $D(v)$ appear in $E\left(G^{\prime}\right)$. Moreover, since $|\Gamma(v)| \geq t$, we can extend the $t$ edge-disjoint rainbow spanning trees in $G^{\prime}$ to $G$ by adding one edge in $\Gamma(v)$ to each of the rainbow spanning trees in $G^{\prime}$.

Case 2: Suppose we are not in Case 1. We first claim that there exists two vertices $v_{1}, v_{2} \in V(G)$ such that $\left|\Gamma\left(v_{1}\right)\right| \leq t-1$ and $\left|\Gamma\left(v_{2}\right)\right| \leq t-1$.
Otherwise, there are at least $n-1$ vertices $u$ with $|\Gamma(u)| \geq t$. Since we are not in Case 1 , it follows that all these vertices $u$ also satisfy $|D(u)| \geq n-2$. Hence by counting the number of distinct colors in $G$, we have that

$$
\frac{(n-1)(n-2)}{2} \leq\binom{ n-2}{2}+t+1 .
$$

which implies that $n \leq t+3$, giving us the contradiction.

Now suppose $\left|\Gamma\left(v_{1}\right)\right| \leq t-1$ and $\left|\Gamma\left(v_{2}\right)\right| \leq t-1$. Let $D=D\left(v_{1}\right) \cup D\left(v_{2}\right)$. Add new colors to $D$ until $\left|\Gamma\left(v_{1}, D\right)\right| \geq t,\left|\Gamma\left(v_{2}, D\right)\right| \geq t+1$ and $|D| \geq t+1$. Call the resulting color set $S$. Note that

$$
t+1 \leq|S| \leq 2 t+1 \leq n-2 .
$$

Now let $G^{\prime}=G-\left\{v_{1}, v_{2}\right\}$ and delete all edges of colors in $S$ from $G^{\prime}$.
We claim that $G^{\prime}$ has t color-disjoint rainbow spanning trees.
By Theorem 3, it is sufficient to verify the condition that for any partition $P$ of $V\left(G^{\prime}\right)$,

$$
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right| \geq t(|P|-1)
$$

Observe

$$
\begin{aligned}
& \left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|-t(|P|-1) \\
& \geq \left\lvert\, c\left(E\left(G^{\prime}\right) \left\lvert\,-\binom{n-1-|P|}{2}-t(|P|-1)\right.\right.\right. \\
& \geq\binom{ n-2}{2}+t+1-|S|-\binom{n-1-|P|}{2}-t(|P|-1) \\
& \geq\binom{ n-2}{2}+t+1-(n-2)-\binom{n-1-|P|}{2}-t(|P|-1) .
\end{aligned}
$$

Note the expression above is concave downward as a function of $|P|$. It is sufficient to check the value at 2 and $n-2$. When $|P|=2$, we have

$$
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|-t(|P|-1) \geq\binom{ n-2}{2}+t+1-(n-2)-\binom{n-3}{2}-t=0
$$

When $|P|=n-2$, we have

$$
\begin{aligned}
\left|c\left(\operatorname{cr}\left(P, G^{\prime}\right)\right)\right|-t(|P|-1) & \geq\binom{ n-2}{2}+t+1-(n-2)-t(n-3) \\
& =\frac{(n-4)(n-2 t-3)}{2} \\
& \geq 0 .
\end{aligned}
$$

Here we use the assumption $n \geq 2 t+3$ in the last step. Now it remains to extend the $t$ color-disjoint spanning trees we found to $G$ by using only the colors in $S$. Let $e_{1}, \cdots, e_{k}$ be the edges in $G$ incident to $v_{1}$ with colors in $S$. Let $e_{1}^{\prime}, \cdots e_{l}^{\prime}$ be the edges in $G \backslash\left\{v_{1}\right\}$ incident to $v_{2}$ with colors in $S$. With our selection of $S$, it follows that $k, l \geq t$. Now construct an auxiliary bipartite graph $H$ with partite sets $A=\left\{e_{1}, \cdots, e_{k}\right\}$ and $B=\left\{e_{1}^{\prime}, \cdots, e_{l}^{\prime}\right\}$ such that $e_{i} e_{j}^{\prime} \in E(H)$ if and only if $e_{i}, e_{j}^{\prime}$ have different colors in $G$.
We claim that there is a matching of size $t$ in $H$. Let $M$ be the maximum matching in $H$. Without loss of generality, suppose $e_{1} e_{1}^{\prime}, \cdots, e_{m} e_{m}^{\prime} \in M$ where $m<t$. It follows that $\left\{e_{j}: m<j \leq k\right\} \cup\left\{e_{j}^{\prime}: m<j \leq l\right\}$ all have the same color (otherwise we can extend the matching). Without loss of generality, they all have color $x$. Now observe that for every matched edge $e_{i} e_{i}^{\prime}$, exactly one of the two end vertices must be in color $x$. Otherwise, we can extend the matching by pairing $e_{i}$ with $e_{t}^{\prime}$ and $e_{t}$ with $e_{i}^{\prime}$. This implies that $H$ has at most $t$ colors, which contradicts that $|S| \geq t+1$.
Hence there is a matching of size $t$ in $H$. Since none of the edges in $G^{\prime}$ have colors in $S$, it follows that we can extend the $t$ color-disjoint rainbow spanning trees in $G^{\prime}$ to $t$ edge-disjoint rainbow spanning trees in $G$.

Hence in all of the three cases, we obtain that $G$ has $t$ edge-disjoint rainbow spanning trees.
3.4 Theorem 1 where $n=2 t+1$

Proposition 4. For positive integers $t \geq 1$ and $n=2 t+1$, we have $r(n, t)=\binom{n-1}{2}=2 t^{2}-t$.

Proof. Again, the lower bound is due to Proposition 1. Now we prove that any edge-coloring of $K_{2 t+1}$ with $2 t^{2}-t+1$ distinct colors contains $t$ edge-disjoint rainbow spanning trees. Call this edge-colored graph $G$. The proof approach is similar to the case when $n=2 t+2$. Let $m_{i}$ be the multiplicity of the color $c_{i}$ in $G$. Without loss of generality, say the first $s$ colors have multiplicity greater than or equal to 2 :

$$
m_{1} \geq m_{2} \geq \cdots \geq m_{s} \geq 2
$$

Let $G_{1}$ be the spanning subgraph consisting of all edges whose color multiplicity is greater than 1 in $G$. Let $G_{2}$ be the spanning subgraph consisting of the remaining edges. We have

$$
\begin{equation*}
\sum_{i=1}^{s}\left(m_{i}-1\right)=\binom{n}{2}-\left(2 t^{2}-t+1\right)=2 t-1 . \tag{7}
\end{equation*}
$$

In particular, we have

$$
\left|E\left(G_{1}\right)\right|=\sum_{i=1}^{s} m_{i}=2 t-1+s \leq 4 t-2
$$

Claim: we can construct $t$ edge-disjoint rainbow forests $F_{1}, \ldots, F_{t}$ in $G_{1}$ such that if we let $G_{0}=$ $G_{1} \backslash \bigcup_{i=1}^{t} E\left(F_{i}\right)$, then $\left|E\left(G_{0}\right)\right| \leq t$. Again, for the proof of the claim, we consider two cases:
Case 1: $m_{1} \geq t+2$. By equation (7), we have that $s \leq(2 t-1)-(t+1)+1=t-1$. We construct $t$ edge-disjoint rainbow forests $F_{1}, \cdots, F_{t}$ as follows: First take $t$ edges of color $c_{1}$ and add one edge to each of $F_{1}, \cdots F_{t}$. Next, pick one edge from each of the remaining $s-1$ colors and add each of them to a distinct $F_{i}$.
Clearly, we can obtain $t$ edge-disjoint rainbow forests in this way. Furthermore,

$$
\left|E\left(G_{0}\right)\right| \leq 2 t-1+s-(t+s-1)=t
$$

which proves the claim.
Case 2: $m_{1}<t+2$. Let $F_{1}, \ldots, F_{t}$ be the edge-maximal family of rainbow spanning forests in $G_{1}$.
Let $G_{0}=G_{1} \backslash \bigcup_{i=1}^{t} E\left(F_{i}\right)$. Suppose that $\left|E\left(G_{0}\right)\right|>t$. Then

$$
\sum_{i=1}^{t}\left|E\left(F_{i}\right)\right| \leq 2 t-1+s-(t+1)=t+s-2
$$

Since $s \leq 2 t-1$, it follows that there exists some $j$ such that $\left|E\left(F_{j}\right)\right| \leq 2$.
Case 2a: $\left|E\left(F_{j}\right)\right|=1$. Since $\left\{F_{1}, \ldots, F_{t}\right\}$ is edge-maximal and $\left|E\left(G_{0}\right)\right| \geq t+1$, it follows that all edges in $G_{0}$ share the same color (call it $c_{1}^{\prime}$ ) as the single edge in $F_{j}$. Thus $m_{1} \geq t+2$, which contradicts that $m_{1}<t+2$ since we are in Case 2.
Case 2b: $\left|E\left(F_{j}\right)\right|=2$. Similarly, at least $t$ edges in $G_{0}$ share the same colors (call them $c_{1}^{\prime}$, $c_{2}^{\prime}$ ) as the two edges in $F_{j}$. It follows that $m_{1}+m_{2} \geq t+2$. Hence $s \leq t+1$.
Now since $\left|E\left(G_{0}\right)\right| \geq t+1$, it follows that

$$
\sum_{i=1}^{t}\left|E\left(F_{i}\right)\right| \leq 2 t-1+s-(t+1)=t+s-2 \leq 2 t-1
$$

Hence there exists some forest with only one edge, in which case we are done by Case 2a.

Hence by contradiction, we obtain that $\left|E\left(G_{0}\right)\right| \leq t$, which completes the proof of the claim.

Now we show that $F_{1}, \ldots, F_{t}$ have a color-disjoint extension to $t$ edge-disjoint rainbow spanning trees. Consider any partition $P$. We will verify

$$
\left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right| \geq t(|P|-1)
$$

We have

$$
\left|c\left(\operatorname{cr}(P), G_{2}\right)\right|+\sum_{i=1}^{t}\left|\operatorname{cr}\left(P, F_{i}\right)\right|-t(|P|-1) \geq\binom{ n}{2}-t-\binom{n-|P|+1}{2}-t(|P|-1) .
$$

Note that the function on right is concave downward on $|P|$. It is enough to verify it at $|P|=2$ an $|P|=n$. When $|P|=2$, we have

$$
\binom{n}{2}-t-\binom{n-1}{2}-t=n-1-2 t \geq 0 .
$$

When $|P|=n$, we have

$$
\binom{n}{2}-t-t(n-1)=0 .
$$

By Theorem 4, $F_{1}, \ldots, F_{t}$ have a color-disjoint extension to $t$ edge-disjoint rainbow spanning trees.

Acknowledgement The authors thank an anonymous referee for the valuable comments, in particular, for pointing out that Schrijver's theorem implies Theorem 3.

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