Anti-Ramsey number of edge-disjoint rainbow spanning trees

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Abstract

An edge-colored graph G is called *rainbow* if every edge of G receives a different color. The *anti-Ramsey* number of t edge-disjoint rainbow spanning trees, denoted by r(n, t), is defined as the maximum number of colors in an edge-coloring of K_n containing no t edge-disjoint rainbow spanning trees. Jahanbekam and West [J. Graph Theory, 2016] conjectured that for any fixed $t, r(n, t) = \binom{n-2}{2} + t$ whenever $n \ge 2t + 2 \ge 6$. In this paper, we prove this conjecture. We also determine r(n, t) when n = 2t + 1. Together with previous results, this gives the anti-Ramsey number of t edge-disjoint rainbow spanning trees for all values of n and t.

1 Introduction

An edge-colored graph G is called *rainbow* if every edge of G receives a different color. The general *anti-Ramsey problem* asks for the maximum number of colors $AR(n, \mathcal{G})$ in an edge-coloring of K_n containing no rainbow copy of any graph in a class \mathcal{G} . For some earlier results when \mathcal{G} consists of a single graph, see the survey [9]. In particular, Montellano-Baallesteros and Neumann-Lara [13] showed a conjecture of Erdős, Simonovits and Sós [8] by computing $AR(n, C_k)$. Jiang and West [12] determined the anti-Ramsey number of the family of trees with m edges.

Anti-Ramsey problems have also been investigated for rainbow spanning subgraphs. In particular, Hass and Young [10] showed that the anti-Ramsey number for perfect matchings (when n is even) is $\binom{n-3}{2} + 2$ for $n \ge 14$. For spanning trees, Bialostocki and Voxman [2] showed that the maximum number of colors in an edge-coloring of K_n ($n \ge 4$) with no rainbow spanning tree is $\binom{n-2}{2} + 1$. Jahanbekam and West [11] extended the investigations to finding the anti-Ramsey number of t edge-disjoint rainbow spanning subgraphs of certain types including matchings, cycles and trees. In particular, for rainbow spanning trees, let r(n,t) be the maximum number of colors in an edge-coloring of K_n not having t edge-disjoint rainbow spanning trees. Akbari and Alipour [1] showed that $r(n,2) = \binom{n-2}{2} + 2$ for $n \ge 6$. Jahanbekam and West [11] showed that

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n > 2t + \sqrt{6t - \frac{23}{4}} + \frac{5}{2} \\ \binom{n}{2} - t & \text{for } n = 2t, \end{cases}$$

and they made the following conjecture:

Conjecture 1. [11] $r(n,t) = \binom{n-2}{2} + t$ whenever $n \ge 2t + 2 \ge 6$.

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In this paper, we show that the above conjecture holds and we also determine the value of r(n, t) when n = 2t + 1. Together with previous results ([2],[1],[11]), this gives the anti-Ramsey number of t edge-disjoint rainbow spanning trees for all values of n and t.

Theorem 1. For all positive integers t,

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n \ge 2t+2\\ \binom{n-1}{2} & \text{for } n = 2t+1\\ \binom{n}{2} - t & \text{for } n = 2t, \end{cases}$$

Remark 1. Note that if n < 2t, then K_n does not have enough edges for t edge-disjoint spanning trees.

The main tools we use are two structure theorems that characterize the existence of t colordisjoint rainbow spanning trees or the existence of a *color-disjoint* extension of t edge-disjoint rainbow spanning forests into t edge-disjoint rainbow spanning trees. When t = 1, Broersma and Li [3] showed that determining the largest rainbow spanning forest of a graph can be solved by applying the Matroid Intersection Theorem. The following characterization was established by Schrijver [16] using matroid methods, and later given graph theoretical proofs by Suzuki [17] and also by Carraher and Hartke [4].

Theorem 2. ([16, 17, 4]) An edge-colored connected graph G has a rainbow spanning tree if and only if for every $2 \le k \le n$ and every partition of G with k parts, at least k-1 different colors are represented in edges between partition classes.

The above results can be generalized to t color-disjoint rainbow spanning trees using similar matroid methods by Schrijver [16]. For the sake of self-completeness, we reproduce the proof using matroid methods in Section 2. We also give a new graph theoretical proof of Theorem 3.

Theorem 3. [16] An edge-colored multigraph G has t pairwise color-disjoint rainbow spanning trees if and only if for every partition P of V(G) into |P| parts, at least t(|P| - 1) distinct colors are represented in edges between partition classes.

Remark 2. Recall the famous Nash-Williams-Tutte Theorem ([15, 18]): A multigraph contains t edge-disjoint spanning trees if and only if for every partition P of its vertex set, it has at least t(|P| - 1) cross-edges. Theorem 3 implies the Nash-Williams-Tutte Theorem by assigning every edge of the multigraph a distinct color.

Theorem 3 can be also generalized to extend edge-disjoint rainbow spanning forests to edgedisjoint rainbow spanning trees. Let G be an edge-colored multigraph. Let F_1, \ldots, F_t be t edgedisjoint rainbow spanning forests. We are interested in whether F_1, \ldots, F_t can be extended to t edge-disjoint rainbow spanning trees T_1, \ldots, T_t in G, i.e., $E(F_i) \subset E(T_i)$ for each i. We say the extension is color-disjoint if all edges in $\cup_i (E(T_i) \setminus E(F_i))$ have distinct colors and these colors are different from the colors appearing in the edges of $\cup_i E(F_i)$. Using similar matroid methods or graph theoretical arguments, we can also obtain a criterion that characterizes the existence of a color-disjoint extension of rainbow spanning forests into rainbow spanning trees.

Theorem 4. A family of t edge-disjoint rainbow spanning forests F_1, \ldots, F_t has a color-disjoint extension in G if and only if for every partition P of G into |P| parts,

$$|c(cr(P,G'))| + \sum_{i=1}^{t} |cr(P,F_i)| \ge t(|P|-1).$$
(1)

Here G' is the spanning subgraph of G by removing all edges with colors appearing in some F_i , and c(cr(P,G')) be the set of colors appearing in the edges of G' crossing the partition P.

It would be interesting to find a similar criterion for the existence of t edge-disjoint rainbow trees in a general graph since applications of Theorem 3 and Theorem 4 usually require large number of colors in the host graph.

Organization: The rest of the paper is organized as follows. In Section 2, we present the proofs of Theorem 3 and Theorem 4. In Section 3, we show Theorem 1.

2 Proof of Theorem 3

We first reproduce the proof of Theorem 3 using matroid methods. A matroid is defined as $M = (E,\mathcal{I})$ where E is the ground set and $\mathcal{I} \subseteq 2^E$ is a set containing subsets of E (called indepedent sets) that satisfy (i) if $A \subseteq B \subseteq E$, and $B \in \mathcal{I}$, then $A \in \mathcal{I}$; (ii) if $A \in \mathcal{I}$, $B \in \mathcal{I}$ and |A| > |B|, then $\exists a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$. Given a matroid $M = (E,\mathcal{I})$, the rank function $r_M : 2^E \to \mathbb{N}$ is defined as $r_M(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}$. Thus $r_M(E)$ is the size of the maximum independent set of M. Two matroids of interests here are the graphic matroid and the partition matroid. Given an edge-colored graph G, the graphic matroid of G is the matroid $M = (E,\mathcal{I})$ where E = E(G) and \mathcal{I} is the set of forests in G. The partition matroid of G, is the matroid $M' = (E',\mathcal{I}')$ where E' = E(G) and \mathcal{I} is the set of rainbow subgraphs of G. Given k matroids $\{M_i = (E_i,\mathcal{I}_i)\}_{i \in [k]}\}$, one can define the union of the k matroids, $M_1 \vee \cdots \vee M_k = (E,\mathcal{I})$, by $E = \bigcup_{i=1}^k E_i$ and $\mathcal{I} = \{I_1 \cup \cdots \cup I_k : I_i \in \mathcal{I}_i$ for all $i \in [k]\}$. It is well known in matroid theory [6, 14] that $M_1 \vee \cdots \vee M_k$ is a matroid with rank function

$$r(S) = \min_{T \subseteq S} \left(|S \setminus T| + \sum_{i=1}^{k} r_{M_i}(T \cap E_i) \right).$$

Given two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ on the same ground set with rank functions r_1 and r_2 respectively, consider the family of independent sets common to both matroids, i.e., $\mathcal{I}_1 \cap \mathcal{I}_2$. The well-known Matroid Intersection Theorem [7] asserts that

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq E} \left(r_1(U) + r_2(E \setminus U) \right).$$

2.1 Proof of Theorem 3 using Matroid methods

Again we remark that the proof essentially follows the same approaches as Schrijver [16] and we only reproduce it here for the sake of completeness.

Proof of Theorem 3. The forward direction is clear. Thus it remains to show that if for every partition P of V(G) into |P| parts, at least t(|P| - 1) distinct colors are represented in edges between partition classes, then there exist t edge-disjoint rainbow spanning trees in G.

Given an edge-colored graph G, let $M = (E, \mathcal{I})$ be the graphic matroid of G and $M' = (E, \mathcal{I}')$ be the partition matroid of G. Moreover, let $M^t = M \vee M \vee \cdots \vee M = (E, \mathcal{I}^t)$, where we take tcopies of M. By the matriod union theorem, we obtain that

$$r_{M^t}(S) = \min_{T \subseteq S} \left(|S \setminus T| + t \cdot r_M(T) \right).$$

By the Matroid Intersection Theorem,

$$\max_{I \in \mathcal{I}^t \cap \mathcal{I}'} |I| = \min_{U \subseteq E} \left(r_{M^t}(U) + r_{M'}(E \setminus U) \right)$$
$$= \min_{U \subseteq E} \left(\min_{T \subseteq U} \left(|U \setminus T| + t \cdot r_M(T) \right) + r_{M'}(E \setminus U) \right)$$

Let $T, U \subseteq E$ be arbitrarily chosen such that $T \subseteq U$. Observe that $t \cdot r_M(T) = t(n - q(T))$, where q(T) is the number of components of G[T]. Now we claim that

$$|U \setminus T| + r_{M'}(E \setminus U) \ge r_{M'}(E \setminus T) \ge t(q(T) - 1).$$

Indeed, for any color c appearing in some edge $e \in E \setminus T$, if $e \in E \setminus U$, then the color c is counted in $r_{M'}(E \setminus U)$; if $e \in U$, then that color is counted in $|U \setminus T|$. In particular, at least t(q(T) - 1) distinct colors are represented in edges between connected components of T, thus in $E \setminus T$. It follows that

$$|U \setminus T| + t \cdot r_M(T) + r_{M'}(E \setminus U) \ge t(q(T) - 1) + t(n - q(T)) \ge t(n - 1),$$

which implies that $\max_{I \in \mathcal{I}^t \cap \mathcal{I}'} |I| \ge t(n-1)$. By definition, we then have t edge-disjoint rainbow spanning trees.

2.2 Proof of Theorem 3 using graph theoretical arguments

In this subsection, we give a new graph theoretical proof of Theorem 3. Given a graph G, we use V(G), E(G) to denote its vertex set and edge set respectively. We use ||G|| to denote the number of edges in G. Given a set of edges E, we use c(E) to denote the set of colors that appear in E. For clarity, we abuse the notation to use c(e) to denote the color of an edge e. We say a color c has *multiplicity* k in G if the number of edges with color c in G is k. The *color multiplicity* of an edge in G is the multiplicity of the color of the edge in G.

For any partition P of the vertex set V(G) and a subgraph H of G, let |P| denote the number of parts in the partition P and let $\operatorname{cr}(P, H)$ denote the set of crossing edges in H whose end vertices belong to different parts in the partition P. When H = G, we also write $\operatorname{cr}(P, G)$ as $\operatorname{cr}(P)$. Given two partitions $P_1: V = \bigcup_i V_i$ and $P_2: V = \bigcup_j V'_j$, let the intersection $P_1 \cap P_2$ denote the partition given by $V = \bigcup_{i \neq j} V_i \cap V'_j$. Given a spanning disconnected subgraph H, there is a natural partition

 P_H associated to H, which partitions V into its connected components. Without loss of generality, we abuse our notation cr(H) to denote the crossing edges of G corresponding to this partition P_H . Recall we want to show that an edge-colored multigraph G has t color-disjoint rainbow spanning trees if and only if for any partition P of V(G) (with $|P| \ge 2$),

$$|c(cr(P))| \ge t(|P| - 1).$$
(2)

Proof of Theorem 3. One direction is easy. Suppose that G contains t pairwise color-disjoint rainbow spanning trees T_1, T_2, \ldots, T_t . Then all edges in these trees have distinct colors. For any partition P of the vertex set V, each tree contributes at least |P| - 1 crossing edges, thus t trees contribute at least t(|P| - 1) crossing edges and the colors of these edges are all distinct.

Now we prove the other direction. Assume that G satisfies inequality (2). We would like to prove G contains t pairwise color-disjoint rainbow spanning trees. We will prove by contradiction. Assume that G does not contain t pairwise color-disjoint rainbow spanning trees. Let \mathcal{F} be the collection of all families of t color-disjoint rainbow spanning forests $\{F_1, \dots, F_t\}$. Consider the following deterministic process: Initially, set $C' := \bigcup_{j=1}^{t} c(cr(F_j))$ while $C' \neq \emptyset$ do for each color x in C', do for j from 1 to t, do if color x appears in F_j , then delete the edge in color x from F_j endif endifor

endfor

set
$$C' := \bigcup_{j=1}^{t} c(\operatorname{cr}(F_j)) - C'$$

endwhile

For $i \ge 0$, $F_j^{(i)}$ denote the rainbow spanning forest F_j after *i* iterations of the while loop. In particular, $F_j^{(0)} = F_j$ for all $j \in [t]$ and $F_j^{(\infty)}$ is the resulting rainbow spanning forest of F_j after the process. Similarly, let C_i denote the set C' after the *i*-th iteration of the while loop. Note that C_i is the set of new colors crossing components of F_j s after some edges are deleted in the *i*-th iteration.

Observe that since the procedure is deterministic, $\{F_j^{(i)} : j \in [t], i > 0\}$ is unique for a fixed family $\{F_1, \dots, F_t\}$. We define a *preorder* on \mathcal{F} . We say a family $\{F_j\}_{j=1}^t$ is less than or equal to another family $\{F'_j\}_{j=1}^t$ if there is a positive integer l such that

1. For
$$1 \le i < l$$
, $\sum_{j=1}^{t} \|F_j^{(i)}\| = \sum_{j=1}^{t} \|F'_j^{(i)}\|$.
2. $\sum_{j=1}^{t} \|F_j^{(l)}\| < \sum_{j=1}^{t} \|F'_j^{(l)}\|$.

Since G is finite, so is \mathcal{F} . There exists a maximal element $\{F_1, F_2, \dots, F_t\} \in \mathcal{F}$. Run the deterministic process on $\{F_1, F_2, \dots, F_t\}$.

The goal is to construct a common partition P by refining $\operatorname{cr}(F_j)$ so that $|c(\operatorname{cr}(P))| < t(|P|-1)$. In particular, we will show that all forests in $\{F_j^{(\infty)} : j \in [t]\}$ admit the same partition P.

Claim (a):
$$\bigcup_{j=1}^{t} c\left(\operatorname{cr}(F_{j}^{(i)})\right) \subseteq \left(\bigcup_{j=1}^{t} c\left(\operatorname{cr}(F_{j}^{(i-1)})\right)\right) \cup \left(\bigcup_{j=1}^{t} c(F_{j}^{(i)})\right).$$

Assume for the sake of contradiction that there is a color $x \in \bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i)})) \setminus \bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i-1)}))$ and there is no edge in color x in all forests $F_{1}^{(i)}, \ldots, F_{t}^{(i)}$. Let e be the edge such that c(e) = x and $e \in \operatorname{cr}(F_{s}^{(i)})$ for some $s \in [t]$. Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i-1)}))$, it follows that $F_{s}^{(i-1)} + e$ contains a rainbow cycle, which passes through e and another edge $e' \in F_{s}^{(i-1)}$ joining two distinct components of $F_{s}^{(i)}$. Now let us consider a new family of rainbow spanning forests $\{F_{1}', \cdots, F_{t}'\}$ where $F_{j}' = F_{j}$ for $j \neq s$ and $F_{s}' = F_{s} - e' + e$. The color-disjoint property is maintained since the color of edge e is not in any F_{j} . Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i-1)}))$, $F_{s}'^{(i)}$ will have one fewer component than $F_s^{(i)}$. Thus we have

$$\sum_{j=1}^{t} \|F_{j}^{(k)}\| = \sum_{j=1}^{t} \|F_{j}^{\prime(k)}\| \text{ for } k < i.$$
$$\sum_{j=1}^{t} \|F_{j}^{\prime(i)}\| > \sum_{j=1}^{t} \|F_{j}^{(i)}\|.$$

which contradicts our maximality assumption of $\{F_i : i \in [t]\}$. That finishes the proof of Claim (a).

Claim (a) implies that for each $x \in C_i$, there is an edge e of color x in exactly one of the forests in $\{F_j^{(i)} : j \in [t]\}$. Thus removing that edge in the next iteration will increase the sum of number of partitions exactly by 1. Thus we have that

$$\sum_{j=1}^{t} |P_{F_{j}^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_{j}^{(i)}}| + |C_{i}|$$

It then follows that

$$\sum_{j=1}^{t} |P_{F_j^{(\infty)}}| = \sum_{j=1}^{t} |P_{F_j}| + \sum_{i} |C_i|$$
$$= \sum_{j=1}^{t} |P_{F_j}| + |\bigcup_{j=1}^{t} c(\operatorname{cr}(F_j^{(\infty)}))|$$

Finally set the partition $P = \bigcap_{j=1}^{t} P_{F_{j}^{(\infty)}}$. We claim $P_{F_{j}^{(\infty)}} = P$ for all j. This is because all edges in $cr(P_{F_{j}^{(\infty)}}) \cap \bigcup_{k=1}^{t} E(F_{k}^{(\infty)})$ have been already removed. We then have

$$t|P| = \sum_{j=1}^{t} |P_{F_{j}^{(\infty)}}|$$

= $\sum_{j=1}^{t} |P_{F_{j}}| + |\bigcup_{j=1}^{t} c(cr(F_{j}^{(\infty)}))|$
= $\sum_{j=1}^{t} |P_{F_{j}}| + |c(cr(P))|$
 $\geq t + 1 + |c(cr(P))|.$

We obtain

$$|c(cr(P))| \le t(|P| - 1) - 1.$$

Contradiction.

Corollary 1. The edge-colored complete graph K_n has t color-disjoint rainbow spanning trees if the number of edges colored with any fixed color is at most n/(2t).

Proof. Suppose K_n does not have t color-disjoint rainbow spanning trees, then there exists a partition P of $V(K_n)$ into r parts $(2 \le r \le n)$ such that the number of distinct colors in the crossing edges of P is at most t(r-1)-1. Let m be the number of edges crossing the partition P. It follows that

$$m \le (t(r-1)-1) \cdot \frac{n}{2t} \le \frac{n}{2}(r-1) - \frac{n}{2t}$$

On the other hand,

$$m \ge \binom{n}{2} - \binom{n - (r - 1)}{2}.$$

Hence we have

$$\binom{n}{2} - \binom{n - (r - 1)}{2} \le \frac{n}{2}(r - 1) - \frac{n}{2t}.$$

which implies

$$(n-r)(r-1) \le -\frac{n}{t}.$$

which contradicts that $2 \leq r \leq n$.

Remark: This result is tight since the total number of colors used in K_n could be as small as $\binom{n}{2}/(n/(2t)) = t(n-1)$, but any t color-disjoint rainbow spanning trees need t(n-1) colors. On the contrast, a result by Carraher, Hartke and Horn [5] implies there are $\Omega(n/\log n)$ edge-disjoint rainbow spanning trees.

2.3 Proof of Theorem 4

Recall we want to show that any t edge-disjoint rainbow spanning forests F_1, \ldots, F_t have a colordisjoint extension to edge-disjoint rainbow spanning trees in G if and only if

$$|c(cr(P,G'))| + \sum_{j=1}^{t} |cr(P,F_j)| \ge t(|P|-1).$$

where G' is the spanning subgraph of G by removing all edges with colors appearing in some F_j .

Proof. Again, the forward direction is trivial. We only need to show that condition (1) implies there exists a color-disjoint extension to edge-disjoint rainbow spanning trees. The proof is similar to the proof of Theorem 3. Consider a set of edge-maximal forests $F_1^{(0)}, \ldots, F_t^{(0)}$ which is a color-disjoint extension of F_1, \ldots, F_t . From $\{F_j^{(0)}\}$ we delete all edges (in $\{F_j^{(0)}\}$) of some color c appearing in $\bigcup_{j=1}^t c(\operatorname{cr}(F_j^{(0)}, G'))$ to get a new set $\{F_j^{(1)}\}$. Repeat this process until we reach a stable set $\{F_j^{(\infty)}\}$. Since we only delete edges in G', we have $E(F_j) \subseteq E(F_j^{(\infty)})$ for each $1 \leq j \leq t$. The edges and colors in $\cup_{j=1}^t E(F_j)$ will not affect the process. A similar claim still holds:

$$\bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i)}, G')) \subseteq \left(\bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(i-1)}, G'))\right) \cup \left(\bigcup_{j=1}^{t} c\left(E(F_{j}^{(i)}) \cap E(G')\right)\right)$$

In particular, let $C_i = \left(\bigcup_{j=1}^t c(\operatorname{cr}(F_j^{(i)}, G'))\right) \setminus \left(\bigcup_{j=1}^t c(\operatorname{cr}(F_j^{(i-1)}, G'))\right)$. Then we have

$$\sum_{j=1}^{t} |P_{F_{j}^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_{j}^{(i)}}| + |C_{i}|.$$

It then follows that

$$\begin{split} \sum_{j=1}^{t} |P_{F_{j}^{(\infty)}}| &= \sum_{j=1}^{t} |P_{F_{j}^{(0)}}| + \sum_{i} |C_{i}| \\ &= \sum_{j=1}^{t} |P_{F_{j}^{(0)}}| + |\bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(\infty)}, G'))| \end{split}$$

Finally set the partition $P = \bigcap_{j=1}^{t} P_{F_j^{(\infty)} \setminus E(F_j)}$. Clearly all edges in cr(P, G') are removed. All

possible edges remaining in G that cross the partition P are exactly the edges in $\bigcup_{j=1}^{t} \operatorname{cr}(P, F_j)$.

We have

$$\begin{split} t|P| &= \sum_{j=1}^{t} |P_{F_{j}^{(\infty)}}| + \sum_{j=1}^{t} |\operatorname{cr}(P, F_{j})| \\ &= \sum_{j=1}^{t} |P_{F_{j}^{(0)}}| + |\bigcup_{j=1}^{t} c(\operatorname{cr}(F_{j}^{(\infty)}, G'))| + \sum_{j=1}^{t} |\operatorname{cr}(P, F_{j})| \\ &= \sum_{j=1}^{t} |P_{F_{j}^{(0)}}| + |c(\operatorname{cr}(P, G'))| + \sum_{j=1}^{t} |\operatorname{cr}(P, F_{j})| \\ &\geq t + 1 + |c(\operatorname{cr}(P, G'))| + \sum_{j=1}^{t} |\operatorname{cr}(P, F_{j})|. \end{split}$$

We obtain

$$c(cr(P,G'))| + \sum_{j=1}^{t} |cr(P,F_j)| \le t(|P|-1) - 1.$$

Contradiction.

3 Proof of Theorem 1

Recall that r(n, t) is the maximum number of colors in an edge-coloring of the complete graph K_n not having t edge-disjoint rainbow spanning trees.

Lower Bound: Jahanbekam and West (See Lemma 5.1 in [11]) showed the following lower bound for r(n, t).

Proposition 1. [11] For positive integers n and t such that $t \le 2n-3$, there is an edge-coloring of K_n using $\binom{n-2}{2}+t$ colors that does not have t edge-disjoint rainbow spanning trees. When n = 2t+1, the construction improves to $\binom{n-1}{2}$ colors. When n = 2t, it improves to $\binom{n}{2} - t$.

This matches the upper bounds in Theorem 1. Hence we will skip the proof of lower bounds in the subsequent theorems. Moreover, we only consider the case $t \ge 2$ since the case t = 1 was already resolved in Bialostocki and Voxman [2]. In Section 3.1, we prove a technical lemma that will be used in the proof of Theorem 1. In Section 3.2, 3.3,3.4, we show Theorem 1 when n is in different range of values with respect to t.

3.1 Technical lemma

Lemma 1. Let G be an edge-colored graph with s colors c_1, \dots, c_s and |V(G)| = n = 2t + 2 where $t \ge 3$. For color c_i , let m_i be the number of edges of color c_i . Suppose $\sum_{i=1}^{s} (m_i - 1) = 3t$ and $m_i \ge 2$ for all $i \in [s]$. Then we can construct t edge-disjoint rainbow forests F_1, \dots, F_t in G such that if we define $G_0 = G - \bigcup_{i=1}^{t} E(F_i)$, then

$$|E(G_0)| \le 2t + 1.$$
(3)

and

$$\Delta(G_0) \le t + 1. \tag{4}$$

Proof. We consider two cases:

Case 1: $m_1 \ge 2t + 2$. Note that

$$\sum_{i=2}^{s} (m_i - 1) = 3t - (m_1 - 1) \le t - 1.$$

Thus, $s \leq t$. Let $d_i(v)$ be the number of edges in color c_i and incident to v in the current graph G. We construct the edge-disjoint rainbow forests F_1, F_2, \ldots, F_t in two rounds: In the first round, we greedily extract edges only in color c_1 . For $i = 1, \ldots, t$, at step i, pick a vertex v with maximum $d_1(v)$ (pick arbitrarily if tie). Pick an edge in color c_1 incident to v, assign it to F_i , and delete it from G.

We claim that after the first round $d_1(v) \leq t+1$ for any vertex v.

Suppose not, if $d_1(v) \ge t + 2$. Since n - 1 - (t + 2) < t, it follows that there exists another vertex u with $d_1(u) \ge d_1(v) - 1 \ge t + 1$.

This implies

$$m_1 \ge t + d_1(v) + d_1(u) - 1 \ge 3t + 2.$$

However,

$$m_1 - 1 \le \sum_{i=1}^{s} (m_i - 1) = 3t.$$

which gives us the contradiction.

In the second round, we greedily extract edges not in color c_1 . For $i = 1, \ldots, t$, at step i, among all vertices v with at least one neighboring edge not in color c_1 , pick a vertex v with maximum vertex degree d(v) (pick arbitrarily if tie). Pick an edge incident to v and not in color c_1 , assign it to F_i , and delete it from G.

If we succeed with selecting t edges not in color c_1 in the second round, we claim $d(v) \le t + 1$ for any vertex v. Suppose not, if $d(v) \ge t + 2$. Then there is another vertex u with $d(u) \ge d(v) - 1 \ge t + 1$. It implies

$$\sum_{i=1}^{s} m_i \ge 2t + d(u) + d(v) - 1 \ge 4t + 2.$$

However, since $s \leq t$, we have

$$\sum_{i=1}^{s} m_i \le 3t + s \le 4t.$$

Contradiction. Therefore it follows that $d(v) \le t + 1$. Moreover, $|E(G_0)| \le 4t - 2t \le 2t$.

If the process stops at step i = l < t, then all remaining edges in G_0 must be in color 1. Thus, by the previous claim, $\Delta(G_0) \le t + 1$. Moreover,

$$|E(G_0)| \le m_1 - t \le (3t+1) - t = 2t + 1.$$

In both cases above, F_1, \dots, F_t are edge-disjoint rainbow forests that satisfies inequality (3) and (4).

Case 2: $m_1 \le 2t + 1$.

Claim: There exists t edge-disjoint rainbow forests F_1, F_2, \dots, F_t such that $\Delta(G_0) \leq t + 1$. For $j = 1, 2, \dots, t$, we will construct a rainbow forest F_j by selecting a rainbow set of edges such that after deleting these edges from G, $\Delta(G_0) \leq 2t + 1 - j$. Notice that when j = t, we will have $\Delta(G_0) \leq t + 1$. Our procedure is as follows:

For step j, without loss of generality, let v_1, v_2, \dots, v_l be the vertices with degree 2t + 2 - jand let c_1, c_2, \dots, c_m be the set of colors of edges incident to v_1, v_2, \dots, v_l in G. If there is no such vertex, simply pick an edge incident to the max-degree vertex and assign it to F_j . Otherwise, we will construct an auxiliary bipartite graph $H = A \cup B$ where $A = \{v_1, \dots, v_l\}$ and $B = \{c_1, c_2, \dots, c_m\}$ and $v_x c_y \in E(H)$ if and only if there is an edge of color c_y incident to v_x . We claim that there exists a matching of A in H. Suppose not, then by Hall's theorem, there exists a set of vertices $A' = \{u_1, u_2, \dots, u_k\} \subseteq A$ such that |N(A')| < |A'| = k where $k \geq 2$. Without loss of generality, suppose $N(A) = \{c'_1, c'_2, \dots, c'_q\}$ where $q \leq k - 1$. Let m'_i be the number of edges of color c'_i remaining in G.

Note that $k \neq 2$ since otherwise we will have one color with at least $2 \cdot (2t+2-j) - 1 \ge 2t+3$ edges, which contradicts our assumption in this case.

Notice that for every $i \in [k]$, u_i has at least (2t + 2 - j) edges incident to it. Moreover, at least j - 1 edges are already deleted from G in previous steps. Therefore, we have

$$\frac{k(2t+2-j)}{2} \le \sum_{i=1}^{q} m'_i \le \left(\sum_{i=1}^{q} (m'_i - 1)\right) + (k-1) \le 3t - (j-1) + (k-1)$$

It follows that

$$k \le 2 + \frac{2t}{2t - j} \le 4$$

Similarly, using another way of counting the edges incident to some u_i $(i \in [k])$, we have

$$k(2t+2-j) - \binom{k}{2} \le 3t - (j-1) + (k-1)$$

which implies that

$$t(2k-3) \le \frac{k(k-3)}{2} + j(k-1) \le \frac{k(k-3)}{2} + t(k-1).$$

It follows that $t \leq \frac{k(k-3)}{2(k-2)}$. Since $k \leq 4$ and k > 2, we obtain that $t \leq 1$, which contradicts our assumption that $t \geq 2$. Thus by contradiction, there exists a matching of A in H. This implies that there exists a rainbow set of edges E_j that cover all vertices with degree 2t+2-jin step j. We can then find a maximally acyclic subset F_j of E_j such that F_j is a rainbow forest and every vertex of degree 2t + 2 - j is adjacent to some edge in F_j . Delete edges of F_j from G and we have $\Delta(G_0) \leq 2t + 1 - j$. As a result, after t steps, we obtain t edge-disjoint rainbow forests F_1, \dots, F_t and $\Delta(G_0) \leq t + 1$. This finishes the proof of the claim.

Now let $\{F_1, F_2, \dots, F_t\}$ be an edge-maximal set of t edge-disjoint rainbow forests that satisfies $\Delta(G_0) \leq t+1$. We claim that $|E(G_0)| \leq 2t+1$. Suppose not, i.e., $|E(G_0)| \geq 2t+2$. It follows that $\sum_{i=1}^{t} |E(F_i)| \leq 6t - (2t+2) < 4t$, i.e. there exists a $j \in [t]$ such that F_j has at most 3 edges. Since F_j is edge maximal, none of the edges in G_0 can be added to F_j . We have three cases:

- Case 2a: $|E(F_j)| = 1$. It then follows that all edges in G_0 have the same color (call it c'_1) as the single edge in F_j . Thus we have a color with multiplicity at least 2t + 3, which contradicts that $m_1 < 2t + 2$.
- Case 2b: $|E(F_j)| = 2$. Similarly, we have that at least 2t + 1 edges in G_0 that share the same colors (call them c'_1, c'_2) as edges in F_j . It follows that $m_1 + m_2 \ge 2t + 3$. Similar to Case 1, in this case, we have that $s \le t + 1$ and $|E(G)| = 3t + s \le 4t + 1$. Since $|E(G_0)| \ge 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \le (4t + 1) (2t + 2) = 2t 1$. Hence there

exists some F_k such that $|E(F_k)| \leq 1$ and we are done by Case 2a.

Case 2c: $|E(F_j)| = 3$. Similarly, we have that at least 2t - 1 edges in G_0 share the same colors (call them c'_1, c'_2, c'_3) as edges in F_j . It follows that $m_1 + m_2 + m_3 \ge 2t + 2$. By inequality (5), we have that $s \le t + 4$ and $|E(G)| \le 4t + 4$. Since $|E(G_0)| \ge 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \le 2t + 2$. Since $t \ge 3$ by our assumption, there exists a $k \in [t]$ such that $|E(F_k)| \le 2$ and we are done by Case 2b and Case 2c.

Therefore, by contradiction, we have that $|E(G_0)| \leq 2t + 1$ and we are done.

3.2 Proof of Theorem 1 where n = 2t + 2

Proposition 2. For any $n = 2t + 2 \ge 6$, we have $r(n,t) = \binom{n-2}{2} + t = 2t^2$.

Proof. Note that the lower bound is shown by Jahanbekam and West in Proposition 1. For the upper bound, we will assume that $t \ge 3$ since the case when t = 2 is implied by the result of Akbari and Alipour [1]. We will show that any coloring of K_{2t+2} with $2t^2 + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this edge-colored graph G. Let m_i be the multiplicity of the color c_i in G. Without loss of generality, say the first s colors have multiplicity at least 2, i.e.

$$m_1 \ge m_2 \ge \cdots \ge m_s \ge 2$$

Let G_1 be the spanning subgraph of G consisting of all edges with color multiplicity greater than 1 in G. Let G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s} (m_i - 1) = \binom{n}{2} - (2t^2 + 1) = 3t.$$
(5)

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 3t + s \le 6t.$$

By Lemma 1, it follows that we can construct t edge-disjoint rainbow spanning forests F_1, \ldots, F_t in G such that if we define $G_0 = E(G_1) - \bigcup_{i=1}^t E(F_i)$, then

$$|E(G_0)| \le 2t + 1$$

and

$$\Delta(G_0) \le t+1.$$

Now we show that F_1, \ldots, F_t have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P. We will verify

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| \ge t(|P| - 1).$$
(6)

We will first verify the case when $3 \leq |P| \leq n$. Note that

$$|c(\operatorname{cr}(P), G_2)| + \sum_{i=1}^t |\operatorname{cr}(P, F_i)| - t(|P| - 1) \ge \binom{n}{2} - (2t+1) - \binom{n-|P|+1}{2} - t(|P|-1).$$

We want to show that the right hand side of the above inequality is nonnegative. Note that the function on the right hand side is concave downward with respect to |P|. Thus it is sufficient to verify it at |P| = 3 and |P| = n.

When |P| = 3, we have

$$\binom{n}{2} - (2t+1) - \binom{n-2}{2} - 2t = 0.$$

When |P| = n, we have

$$\binom{n}{2} - (2t+1) - t(n-1) = 0.$$

It remains to verify the inequality (6) for |P| = 2. By Theorem 4, we have $|E(G_0)| \le 2t + 1$. If each part of P contains at least 2 vertices, then we have

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1)$$

$$\geq \binom{n}{2} - |E(G_0)| - \binom{n-2}{2} + 1 - t$$

$$\geq \binom{n}{2} - (2t+1) - \binom{n-2}{2} + 1 - t$$

$$= t - 1 \ge 0.$$

Otherwise, P is of the form $V(G) = \{v\} \cup B$ for some $v \in V(G)$ and $B = V(G) \setminus \{v\}$. By Lemma 1, we have $d_{G_0} \leq t + 1$. Thus,

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P| - 1) \ge (n - 1) - d_{G_0}(v) - t \ge 2t + 1 - (t + 1) - t = 0.$$

Therefore, by Theorem 4, F_1, \ldots, F_t have a color-disjoint extension to t edge-disjoint rainbow spanning trees.

3.3 Proof of Theorem 1 where $n \ge 2t+3$

Proposition 3. For any $n \ge 2t + 2 \ge 6$, we have $r(n,t) = \binom{n-2}{2} + t$.

Proof. Again, the lower bound is due to Proposition 1. For the upper bound, we will show that every edge-coloring of K_n with exactly $\binom{n-2}{2} + t + 1$ distinct colors has t edge-disjoint spanning trees. Call this edge-colored graph G.

Given a vertex v, we define D(v) to be the set of colors C such that every edge with colors in C is incident to v. Given a vertex v and a set of colors C, define $\Gamma(v, C)$ as the set of edges incident to v with colors in C. For ease of notation, we let $\Gamma(v) = \Gamma(v, D(v))$.

For fixed t, we will prove the theorem by induction on n. The base case is when n = 2t + 2, which is proven in Proposition 2. Let's now consider the theorem when $n \ge 2t + 3$.

Case 1: there exists a vertex $v \in V(G)$ with $|\Gamma(v)| \ge t$ and $|D(v)| \le n-3$.

In this case, we set $G' = G - \{v\}$. Note that G' is an edge-colored complete graph with at least $\binom{n-2}{2} + t + 1 - (n-3) = \binom{n-3}{2} + t + 1$ distinct colors. Moreover $|G'| \ge 2t + 2$. Hence by induction, there exists t edge-disjoint rainbow spanning trees in G'. Note that by our definition of D(v), none of the colors in D(v) appear in E(G'). Moreover, since $|\Gamma(v)| \ge t$, we can extend the t edge-disjoint rainbow spanning trees in G' to G by adding one edge in $\Gamma(v)$ to each of the rainbow spanning trees in G'.

Case 2: Suppose we are not in Case 1. We first claim that there exists two vertices $v_1, v_2 \in V(G)$ such that $|\Gamma(v_1)| \leq t - 1$ and $|\Gamma(v_2)| \leq t - 1$.

Otherwise, there are at least n-1 vertices u with $|\Gamma(u)| \ge t$. Since we are not in Case 1, it follows that all these vertices u also satisfy $|D(u)| \ge n-2$. Hence by counting the number of distinct colors in G, we have that

$$\frac{(n-1)(n-2)}{2} \le \binom{n-2}{2} + t + 1.$$

which implies that $n \leq t+3$, giving us the contradiction.

Now suppose $|\Gamma(v_1)| \leq t-1$ and $|\Gamma(v_2)| \leq t-1$. Let $D = D(v_1) \cup D(v_2)$. Add new colors to D until $|\Gamma(v_1, D)| \geq t$, $|\Gamma(v_2, D)| \geq t+1$ and $|D| \geq t+1$. Call the resulting color set S. Note that

$$t+1 \le |S| \le 2t+1 \le n-2.$$

Now let $G' = G - \{v_1, v_2\}$ and delete all edges of colors in S from G'.

We claim that G' has t color-disjoint rainbow spanning trees.

By Theorem 3, it is sufficient to verify the condition that for any partition P of V(G'),

$$|c(cr(P,G'))| \ge t(|P|-1).$$

Observe

$$\begin{aligned} |c(\operatorname{cr}(P,G'))| &- t(|P|-1) \\ &\geq |c(E(G'))| - \binom{n-1-|P|}{2} - t(|P|-1) \\ &\geq \binom{n-2}{2} + t + 1 - |S| - \binom{n-1-|P|}{2} - t(|P|-1) \\ &\geq \binom{n-2}{2} + t + 1 - (n-2) - \binom{n-1-|P|}{2} - t(|P|-1). \end{aligned}$$

Note the expression above is concave downward as a function of |P|. It is sufficient to check the value at 2 and n-2. When |P|=2, we have

$$|c(cr(P,G'))| - t(|P| - 1) \ge {\binom{n-2}{2}} + t + 1 - (n-2) - {\binom{n-3}{2}} - t = 0.$$

When |P| = n - 2, we have

$$\begin{aligned} |c(\operatorname{cr}(P,G'))| - t(|P|-1) &\geq \binom{n-2}{2} + t + 1 - (n-2) - t(n-3) \\ &= \frac{(n-4)(n-2t-3)}{2} \\ &\geq 0. \end{aligned}$$

Here we use the assumption $n \geq 2t + 3$ in the last step. Now it remains to extend the t color-disjoint spanning trees we found to G by using only the colors in S. Let e_1, \dots, e_k be the edges in G incident to v_1 with colors in S. Let e'_1, \dots, e'_l be the edges in $G \setminus \{v_1\}$ incident to v_2 with colors in S. With our selection of S, it follows that $k, l \geq t$. Now construct an auxiliary bipartite graph H with partite sets $A = \{e_1, \dots, e_k\}$ and $B = \{e'_1, \dots, e'_l\}$ such that $e_i e'_i \in E(H)$ if and only if e_i, e'_j have different colors in G.

We claim that there is a matching of size t in H. Let M be the maximum matching in H. Without loss of generality, suppose $e_1e'_1, \dots, e_me'_m \in M$ where m < t. It follows that $\{e_j : m < j \leq k\} \cup \{e'_j : m < j \leq l\}$ all have the same color (otherwise we can extend the matching). Without loss of generality, they all have color x. Now observe that for every matched edge $e_ie'_i$, exactly one of the two end vertices must be in color x. Otherwise, we can extend the matching by pairing e_i with e'_t and e_t with e'_i . This implies that H has at most t colors, which contradicts that $|S| \geq t + 1$.

Hence there is a matching of size t in H. Since none of the edges in G' have colors in S, it follows that we can extend the t color-disjoint rainbow spanning trees in G' to t edge-disjoint rainbow spanning trees in G.

Hence in all of the three cases, we obtain that G has t edge-disjoint rainbow spanning trees.

3.4 Theorem **1** where n = 2t + 1

Proposition 4. For positive integers $t \ge 1$ and n = 2t + 1, we have $r(n, t) = \binom{n-1}{2} = 2t^2 - t$.

Proof. Again, the lower bound is due to Proposition 1. Now we prove that any edge-coloring of K_{2t+1} with $2t^2 - t + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this edge-colored graph G. The proof approach is similar to the case when n = 2t + 2. Let m_i be the multiplicity of the color c_i in G. Without loss of generality, say the first s colors have multiplicity greater than or equal to 2:

$$m_1 \ge m_2 \ge \cdots \ge m_s \ge 2.$$

Let G_1 be the spanning subgraph consisting of all edges whose color multiplicity is greater than 1 in G. Let G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s} (m_i - 1) = \binom{n}{2} - (2t^2 - t + 1) = 2t - 1.$$
(7)

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 2t - 1 + s \le 4t - 2.$$

Claim: we can construct t edge-disjoint rainbow forests F_1, \ldots, F_t in G_1 such that if we let $G_0 = G_1 \setminus \bigcup_{i=1}^{t} E(F_i)$, then $|E(G_0)| \leq t$. Again, for the proof of the claim, we consider two cases:

Case 1: $m_1 \ge t+2$. By equation (7), we have that $s \le (2t-1) - (t+1) + 1 = t-1$. We construct t edge-disjoint rainbow forests F_1, \dots, F_t as follows: First take t edges of color c_1 and add one edge to each of F_1, \dots, F_t . Next, pick one edge from each of the remaining s-1 colors and add each of them to a distinct F_i .

Clearly, we can obtain t edge-disjoint rainbow forests in this way. Furthermore,

$$|E(G_0)| \le 2t - 1 + s - (t + s - 1) = t.$$

which proves the claim.

Case 2: $m_1 < t+2$. Let F_1, \ldots, F_t be the edge-maximal family of rainbow spanning forests in G_1 . Let $G_0 = G_1 \setminus \bigcup_{i=1}^t E(F_i)$. Suppose that $|E(G_0)| > t$. Then $\sum_{i=1}^t |E(F_i)| \le 2t - 1 + s - (t+1) = t + s - 2.$

Since $s \leq 2t - 1$, it follows that there exists some j such that $|E(F_j)| \leq 2$.

- Case 2a: $|E(F_j)| = 1$. Since $\{F_1, \ldots, F_t\}$ is edge-maximal and $|E(G_0)| \ge t + 1$, it follows that all edges in G_0 share the same color (call it c'_1) as the single edge in F_j . Thus $m_1 \ge t + 2$, which contradicts that $m_1 < t + 2$ since we are in Case 2.
- Case 2b: $|E(F_j)| = 2$. Similarly, at least t edges in G_0 share the same colors (call them c'_1 , c'_2) as the two edges in F_j . It follows that $m_1 + m_2 \ge t + 2$. Hence $s \le t + 1$. Now since $|E(G_0)| \ge t + 1$, it follows that

$$\sum_{i=1}^{t} |E(F_i)| \le 2t - 1 + s - (t+1) = t + s - 2 \le 2t - 1,$$

Hence there exists some forest with only one edge, in which case we are done by Case 2a.

Hence by contradiction, we obtain that $|E(G_0)| \leq t$, which completes the proof of the claim.

Now we show that F_1, \ldots, F_t have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P. We will verify

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| \ge t(|P| - 1).$$

We have

$$|c(\operatorname{cr}(P), G_2)| + \sum_{i=1}^{t} |\operatorname{cr}(P, F_i)| - t(|P| - 1) \ge \binom{n}{2} - t - \binom{n - |P| + 1}{2} - t(|P| - 1).$$

Note that the function on right is concave downward on |P|. It is enough to verify it at |P| = 2 an |P| = n. When |P| = 2, we have

$$\binom{n}{2} - t - \binom{n-1}{2} - t = n - 1 - 2t \ge 0.$$

When |P| = n, we have

$$\binom{n}{2} - t - t(n-1) = 0.$$

By Theorem 4, F_1, \ldots, F_t have a color-disjoint extension to t edge-disjoint rainbow spanning trees.

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