OBLIQUE INJECTION OF INCOMPRESSIBLE IDEAL FLUID FROM A SLOT INTO A FREE STREAM^{*}

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ABSTRACT. This paper deals with a two-phase fluid free boundary problem in a slotfilm cooling. We give two well-posedness results on the existence and uniqueness of the incompressible inviscid two-phase fluid with a jump relation on free interface. The problem formulates the oblique injection of an incompressible ideal fluid from a slot into a free stream. From the mathematical point of view, this work is motivated by the pioneer work [13] by A. Friedman, in which some well-posedness results are obtained in some special case. Furthermore, A. Friedman proposed an open problem in [14] on the existence and uniqueness of the injection flow problem for more general case. The main results in this paper solve the open problem and establish the well-posedness results on the physical problem.

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Key words: Existence and uniqueness, free boundary, two-phase fluid, contact discontinuity.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Introduction.** This paper is concerned with a two-phase free boundary problem produced when a secondary fluid (or injected or coolant) is injected obliquely at an angle from a slot into a cross flow fluid (see Figure 1). One important physical situation in which this problem arises in fuel injectors, smokestacks, the cooling of gas-turbine blades, and dilution holes in gas turbine combustors. Please see the review of this physical problem [17]. Many numerical simulations on this problem were investigated in [11, 20, 21] and the references therein.

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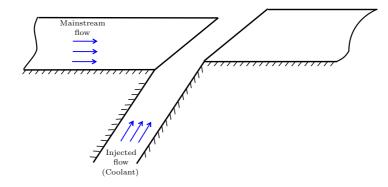


FIGURE 1. Injection of flow from a slot into a cross flow

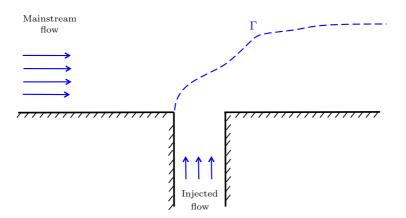


FIGURE 2. Horizontal blade surfaces

Mathematically, the motivation to investigate this free boundary problem follows from the work [13] by A. Friedman. He first considered the two-dimensional model and the simple situation of horizontal blade surface and the secondary fluid injected perpendicularly into a free stream in two dimensions (as in Figure 2). Here, for simplicity, we neglect the separation at the trailing edge B of the slot, such separation can be minimized in practice by slightly around the trailing edge. Also, we have assumed that the interface between the main stream flow and the secondary flow separates at the leading edge A, since the viscosity effects are ignored. Some existence and uniqueness of the solution to the two-phase fluid were established for simple special case in [13]. And furthermore, A. Friedman proposed an open problem in Page 69 in his survey [14], that

Problem (1). Extend the results of Theorem 9.1, 9.2 to more general geometries,

such as in Figure 9.3.

Please see Figure 3 for the Figure 9.3 in [14].

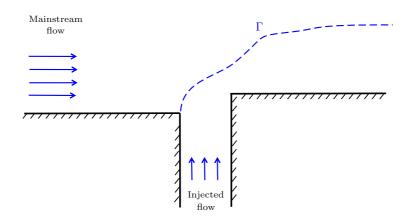


FIGURE 3. Figure 9.3 in [14]

The main purpose in this paper is to establish the existence and uniqueness of the free boundary problem on an incompressible inviscid fluid obliquely into a free stream (as in Figure 4) and solve the open problem proposed by A. Friedman.

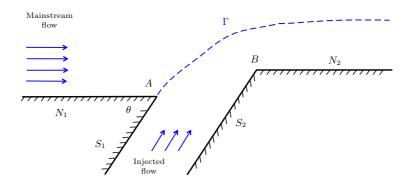


FIGURE 4. Oblique injection of flow into a free stream

In general, there is a discontinuity in the magnitude of velocity across the interface due to the Bernoulli's law. Therefore, the standard method of conformal mapping from the complex potential plane to the conjugate velocity plane will not be fruitful because the interface is mapping into unknown curve in the conjugate velocity plane. For the special case of $\theta = \frac{\pi}{2}$, the free boundary problem was reduced to a nonlinear singular integral differential equation in [22]. Along the variational arguments introduced in [3, 4, 5, 6], A. Friedman established the well-posedness results for the some special case (b = 0 and $\theta = \frac{\pi}{2}$) in Figure 2.

1.2. Notations and the free boundary problem. Before we state the main results in this paper, we will give the following notations of the geometry of the blade surface.

Denote

$$N_1 = \{(x,0) \mid x \le 0\}, \quad N_2 = \{(x,b) \mid x \ge a\}.$$

Here, a > 0, and we consider the general case that the blade surfaces are not horizontal, namely, b > 0. Let

$$S_1 = \{(x, y) \mid x = y \cot \theta, y \le 0\}$$
 and $S_2 = \{(x, y) \mid x = (y - b) \cot \theta + a, y \le b\},\$

where $\theta \in \left(0, \frac{\pi}{2}\right]$. Furthermore, we assume $a \sin \theta - b \cos \theta > 0$, which excludes the possibility of the intersection of N_1 and S_2 , θ is the inclination and the critical case $\theta = \frac{\pi}{2}$ means the normal injection. The leading edge A = (0,0) and the trailing edge B = (a,b).

Both of the mainstream flow and the secondary flow are assumed to be steady, incompressible, inviscid and irrotational. Denote by $(u_+, v_+), p_+, \rho_+$ the velocity, pressure and the constant density of the mainstream flow in Ω^+ , and $(u_-, v_-), p_-, \rho_-$ as the velocity, pressure and the constant density of the secondary flow in Ω^- . They are separated by a streamline, denoted as Γ . The pressure across the interface Γ has to be continuous, i.e., $p_+ = p_-$ on Γ . We assume that the mainstream flow is horizontal and possesses a uniform speed U_0 in upstream, without loss of generality, $U_0 = \frac{1}{\sqrt{\rho_+}}$. The secondary flow with mass flux Q_0 emerges from a slot, where the magnitude of Q_0 is unrestricted for the moment.

Denote by Ω the fluid field of the two-phase fluid, composed of the following domains

$$T_1 = \{ x \le 0, y \ge 0 \}, T_2 = \{ y \cot \theta \le x \le 0, y \le 0 \},$$

and

$$T_3 = \{0 \le x \le \min\{(y-b)\cot\theta + a, a\}\}$$
 and $T_4 = \{x \ge a, y \ge b\},\$

namely, $\Omega = \operatorname{int}(T_1 \cup T_2 \cup T_3 \cup T_4).$

Define a stream function ψ of the two-phase fluid as

$$\frac{\partial \psi}{\partial x} = -\sqrt{\rho_+}v_+$$
 and $\frac{\partial \psi}{\partial y} = \sqrt{\rho_+}u_+$ in the main fluid field Ω^+ ,

and

$$\frac{\partial \psi}{\partial x} = -\sqrt{\rho_-}v_-$$
 and $\frac{\partial \psi}{\partial y} = \sqrt{\rho_-}u_-$ in the secondary fluid field Ω^- .

On the solid boundaries, we impose that

$$\psi = 0 \text{ on } N_1 \cup S_1, \text{ and } \psi = -\frac{Q_0}{\sqrt{\rho_-}} \text{ on } N_2 \cup S_2.$$
 (1.1)

On the interface $\Gamma = \Omega \cap \{\psi = 0\}$, the Bernoulli's equation gives that

$$\rho_{-}(u_{-}^{2}+v_{-}^{2}) - \rho_{+}(u_{+}^{2}+v_{+}^{2}) = \text{constant}, \quad \text{on} \quad \Gamma,$$
(1.2)

the jump constant is denoted as λ . It is easy to see that $\lambda \in (-1, +\infty)$. The two-phase fluid we seek in this paper is the vortex sheet solution and the jump condition (1.2) is in fact the Rankine-Hugoniot jump condition to the vortex sheet. From the mathematical point of view, to attack the well-posedness of the problem on the injection of ideal fluid from a slot into a free stream in 1983, A. Friedman in [13] (see also the Chapter 9 in [14]) introduced the injection flow problems in two different situations. The injection flow problem 1. For any given $Q_0 > 0$, does there exist a unique injection flow (ψ, Γ) , such that the mainstream flow possesses uniform speed in upstream, and the interface Γ connects at A and extends to infinity?

The injection flow problem 2. For any given $\lambda \in (-1, +\infty)$, does there exist a unique injection flow (ψ, Γ) , such that the mainstream flow possesses uniform speed in upstream, and the interface Γ connects at A and extends to infinity?

Here, it is worth to mention that once the stream function ψ is solved,

$$(u_+, v_+) = \frac{1}{\sqrt{\rho_+}} \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$
 in the main fluid field Ω^+ ,

and

$$(u_{-}, v_{-}) = \frac{1}{\sqrt{\rho_{-}}} \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$
 in the secondary fluid field Ω^{-}

will be solved by the stream function.

Meanwhile, the existence and uniqueness of the injection flow problem 1 and 2 in some geometric special situation were established in Theorem 3.2 and Theorem 1.1 in [13], respectively. He assumed that the blade surface is horizontal and the injection is vertical, namely, b = 0 and $\theta = \frac{\pi}{2}$ (see Figure 2). Moreover, he proposed an open problem in [14] to extend the results in [13] to more general case as in Figure 3. This is the main motivation to investigate the oblique injection flow problem in this paper.

Next, we will define the solution to the injection flow problem 1 and problem 2, respectively.

Definition 1.1. (A solution to the injection flow problem 1).

For any given $Q_0 > 0$, a vector (ψ, Γ) is called a solution to the injection flow problem 1, provided that

(1) $\Delta \psi = 0$ in $\Omega \setminus \Gamma$, $\psi \in C^0(\overline{\Omega})$ and $\nabla \psi \in L^\infty(\Omega \setminus B_\varepsilon(B))$ for any $\varepsilon > 0$.

(2) ψ satisfies the Dirichlet boundary conditions (1.1).

(3) The free boundary $\Gamma : y = k(x)$ is C^1 -smooth strictly increasing function in $(0, +\infty)$, and k(x) > b for any $x \ge a$. Furthermore,

$$k(0) = 0, (1.3)$$

and there exists a $h \in (b, +\infty)$, such that

$$\lim_{x \to +\infty} k(x) = h \text{ and } \lim_{x \to +\infty} k'(x) = 0.$$

(4) ψ satisfies the Rankine-Hugoniot jump condition on Γ , namely,

$$\left(\frac{\partial\psi^{-}}{\partial\nu}\right)^{2} - \left(\frac{\partial\psi^{+}}{\partial\nu}\right)^{2} = \lambda \text{ on } \Gamma, \qquad (1.4)$$

where $\lambda = \frac{1}{\rho_-} \left(\frac{Q_0}{h-b}\right)^2 - 1$, $\psi^+ = \max\{\psi, 0\}$, $\psi^- = -\min\{\psi, 0\}$ and ν is the normal vector to Γ .

(5) Γ is continuously differentiable at A and

$$k'(0+0) = \begin{cases} \tan \theta, & \text{if } \lambda > 0, \text{ (see Figure 4)} \\ 0, & \text{if } \lambda < 0, \text{ (see Figure 5)} \\ \tan \frac{\theta}{2}, & \text{if } \lambda = 0, \text{ (see Figure 6).} \end{cases}$$
(1.5)

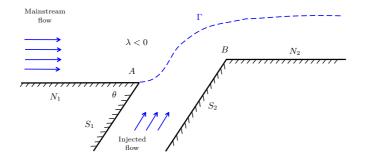


FIGURE 5. The case $\lambda < 0$.

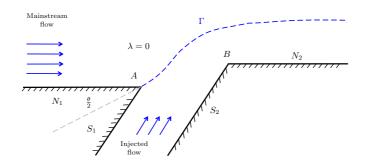


FIGURE 6. The case $\lambda = 0$

(6) ψ possesses the following asymptotic behaviors in far field

$$\psi(x,y) \to \begin{cases} \frac{Q_0(y-h)}{\sqrt{\rho_-}(h-b)}, & \text{if } b < y < h, \text{ as } x \to +\infty, \\ y-h, & \text{if } h < y < C, \text{ as } x \to +\infty, \text{ for any } C > 0, \\ \psi(x,y) \to y, \text{ if } 0 < y < C, \text{ as } x \to -\infty, \text{ for any } C > 0, \end{cases}$$

and

$$\left|\psi(x,y) - \frac{Q_0(y\cos\theta - x\sin\theta)}{\sqrt{\rho_-}(a\sin\theta - b\cos\theta)}\right| \to 0 \quad \text{uniformly in any compact subset of } S,$$

as $y \to -\infty$, where $S = \{(x, y) \mid y \cot \theta < x < (y - b) \cot \theta + a, -\infty < y < +\infty\}$. Furthermore,

$$\nabla \psi(x,y) \to (0,1), \text{ if } x^2 + y^2 \to +\infty, \operatorname{dist}((x,y),\Gamma) \to +\infty \text{ and with } \psi(x,y) > 0,$$

and

$$\left|\nabla\psi(x,y) - \frac{Q_0}{\sqrt{\rho_-}(a\sin\theta - b\cos\theta)} \left(-\sin\theta,\cos\theta\right)\right| \to 0 \text{ uniformly in any compact subset of } S$$

as $y \to -\infty$.

(7) The following estimates hold,

$$-h \le \psi^+(x,y) - y \le 0 \text{ in } \Omega \cap \{y > 0\}.$$
 (1.6)

Definition 1.2. (A solution to the injection flow problem 2). For some given appropriate $\lambda \in (-1, +\infty)$, (ψ, Γ) is called a solution to the injection flow problem 2, provided that the conditions (1) - (7) in Definition 1.1 hold.

Remark 1.1. k(0) = 0 is nothing but the *continuous fit condition* of the interface Γ , which gives that the interface Γ initiates at the leading edge A. Since the viscous effects are ignored here, and the boundary layer is not considered, the continuous fit condition seems to be reasonable. Moreover, the condition (1.5) is so-called *smooth fit condition* for $\lambda \neq 0$ (please see Figure 4 and Figure 6).

Remark 1.2. The conditions $\lim_{x\to+\infty} k(x) = h < +\infty$ and $\lim_{x\to+\infty} k'(x) = 0$ in (1.3) imply that the interface Γ is flat and does not oscillate in downstream.

Remark 1.3. To attack the injection flow problem 1, we first regard the constant λ as an undetermined parameter, and then the parameter λ will be determined uniquely by the continuous fit condition. It means that there exists a unique λ such that the interface connects at the leading edge point A. On another hand, the asymptotic behavior in downstream gives the relation $h = b + \frac{Q_0}{\sqrt{\rho_-(\lambda+1)}}$. Once the constant λ is fixed by the continuous fit condition, the asymptotic width h can be determined by the formula.

1.3. Main results. For the special case b = 0 and $\theta = \frac{\pi}{2}$, the existence and uniqueness were established in [13], and we will give the existence and uniqueness results on the injection flow problem in two situations in general case as follows.

Theorem 1.1. For any $Q_0 > 0$, there exist a unique $\lambda > -1$ and a unique solution (ψ, Γ) to the injection flow problem 1. Furthermore, the interface Γ is analytic, $u_{\pm} > 0$ in $\Omega^{\pm} \cup \Gamma$, and $v_{\pm} > 0$ in $\Omega^{\pm} \cup \Gamma$.

Remark 1.4. In [5], some well-posedness results on two fluids of steady incompressible inviscid flows issuing from two nozzles were established (see Figure 7). However, it is assumed that the two nozzles are symmetric with respect to x-axis and the upper boundary of the nozzle I coincides with the lower boundary of the nozzle II. Along the proof of Theorem 1.1 in this paper, we can extend the existence and uniqueness of the two fluids in [5] to the general case as Figure 8 (the nozzle are asymmetric and the nozzle walls do not coincide) without any additional difficulties.

Remark 1.5. In the previous work [13], A. Friedman showed that the free boundary Γ is only C^1 -smooth for b = 0 and $\theta = \frac{\pi}{2}$, and then the Rankine-Hugoniot (1.4) holds in weak

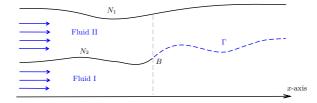


FIGURE 7. Two-phase fluid

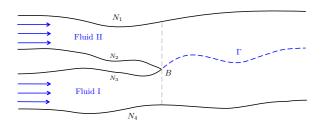


FIGURE 8. Two-phase fluid

sense. However, we would like to emphasize that here we showed that the free boundary is analytic and then the Rankine-Hugoniot (1.4) holds in classical sense.

On another hand, to obtain the well-posedness results on the injection flow problem 2, we will investigate the relationship of the constant λ and the flux of injection flow $Q_0 > 0$. In fact, we show that λ is strictly monotone increasing and continuous with respect to $Q_0 > 0$, denoted as $\lambda = \lambda(Q_0)$.

Theorem 1.2. For any $Q_0 > 0$, the solution $(\psi, \Gamma, \lambda(Q_0))$ established in Theorem 1.1 satisfies that

(1) $\lambda(Q_0)$ is strictly monotone increasing and continuous with respect to $Q_0 > 0$.

(2) There exists a $\underline{\lambda} \in (-1,0)$, such that $\lambda(Q_0) \to \underline{\lambda}$ as $Q_0 \to 0$.

(3) There exists a $\kappa \in (0, +\infty)$, such that $\frac{\lambda(Q_0)}{Q_0^2} \to \kappa$ as $Q \to +\infty$.

The second statement in Theorem 1.2 implies that the lower bound is $\underline{\lambda}$, so we can establish the well-posedness result to the injection flow problem 2.

Theorem 1.3. There exists a $\underline{\lambda} > -1$ ($\underline{\lambda}$ is given in Theorem 1.2), such that for any $\lambda > \underline{\lambda}$, there exist a unique $Q_0 > 0$ and a unique solution (ψ, Γ) to the injection flow problem 2. Furthermore, $u_{\pm} > 0$ in $\Omega^{\pm} \cup \Gamma$, and $v_{\pm} > 0$ in $\Omega^{\pm} \cup \Gamma$.

Remark 1.6. Similar to the Theorem 1.1, when the constant λ is imposed, the flux of the injection flux Q_0 can be regarded as a parameter to solve the injection flow problem 2. And the unique solvability of the flux Q_0 can be determined by the continuous fit condition. In particular, for $\lambda = 0$, the stream function is harmonic in the whole fluid field Ω , and the flux $Q_0 > 0$ is uniquely determined by the following formula

$$\left(a\sin\theta - b\cos\theta\right)^{\frac{\pi}{\theta}} = \left(\frac{\pi}{\theta} - 1\right) \left(\frac{Q_0}{\sqrt{\rho_-}}\right)^{\frac{\pi}{\theta}} + \frac{b\pi}{\theta} \left(\frac{Q_0}{\sqrt{\rho_-}}\right)^{\frac{\pi}{\theta}-1},$$

due to the conformal mapping in [18].

As we mentioned before, A. Friedman established the well-posedness results for the simple case of horizontal blade surfaces (b = 0), and proposed an open problem on the general case as shown in Figure 3. However, from the mathematical point of view, the extension to the present problem is not straightforward, and involves some additional difficulties. For the special case b = 0 (see Figure 2), consider the critical case $Q_0 = 0$, the injection flow vanishes and the mainstream flow is nothing but a trivial uniform flow. The free boundary is the segment connecting the leading edge A and the trailing edge B. However, for the general case $(b \neq 0)$, there does not exist a trivial flow for the critical case $Q_0 = 0$. This is the one of main differences and the difficulties here. This fact prevents us to establish the lower bound of λ while $Q_0 \to 0$. To overcome this difficulty, we will investigate the limiting flow $(Q_0 \rightarrow 0)$, and show that the free boundary initiates smoothly at A and terminated at the wall S_2 . Moreover, we will show that the intersection of the free boundary Γ and S_2 must below the trailing edge B. Another difference is that the domain is a star-sharped one with respect to B for the special case b = 0, we can take a rescaling transform to obtain the uniqueness. Furthermore, the property can not hold for the general case, and we have to develop a new method to obtain the uniqueness.

The basic idea in this paper is to seek a two-phase fluid with a smooth interface connecting at the leading edge A. A truncated injection flow problem is presented in Section 2, and furthermore, we give a result on existence and uniqueness in truncated fluid field. Section 3 studies some useful properties of the minimizer and free boundary in the truncated domain. In particular, we will establish the relationship between the jump constant λ and the injected flux Q_0 , which builds a bridge between the injection flow problem 1 and 2. Section 4 is devoted to the solution of the injection flow problem using some uniform estimates of the solution in truncated domain. The analysis reveals the existence and uniqueness of the two-phase fluid with C^1 -smooth interface, the fact firstly proved in [13] for special case. Our results solve the open problem on the well-posedness of an ideal fluid injected obliquely from a slot into a stream.

2. The truncated injection flow problem

To solve the injection flow problem, we first study the truncated injection flow problem with finite height in this section. To simplify notation, denote

$$Q = \frac{Q_0}{\sqrt{\rho_-}}$$

For any L > b, denote

$$N_L = \{(x, L) \mid -\infty < x < +\infty\}$$
 and $\Omega_L = \Omega \cap \{y < L\}$. (See Figure 9)

The definition of the truncated injection flow problem will be given in the following.

The truncated injection flow problem 1 corresponding to the injection flow problem 1 is as follows: For any given Q > 0, does there exist a unique λ_L and a unique injection flow

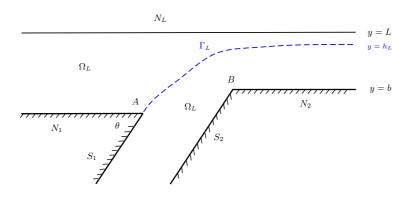


FIGURE 9. Truncated flow field

 (ψ_L, Γ_L) in the truncated domain Ω_L , such that the mainstream flow possesses uniform speed in upstream, and the interface Γ_L connects at A and extends to infinity?

Next, we will give the definition of the solution to the truncated injection flow problem 1.

Definition 2.1. (A solution to the truncated injection flow problem 1).

For any L > b, a vector (ψ_L, Γ_L) is called a solution to the truncated injection flow problem 1, provided that

(1) $\Delta \psi_L = 0$ in $\Omega_L \setminus \Gamma_L$, $\psi_L \in C^0(\overline{\Omega}_L)$ and $\nabla \psi_L \in L^\infty(\Omega_L \setminus B_\varepsilon(B))$ for any $\varepsilon > 0$.

(2) $\psi_L = 0$ satisfies the Dirichlet boundary conditions (1.1) and $\psi_L = L$ on N_L .

(3) The free boundary $\Gamma_L : y = k_L(x)$, and $k_L(x)$ is a C^1 -smooth strictly increasing function in $(0, +\infty)$, and $k_L(x) > b$ for any $x \ge a$. Furthermore,

$$k_L(0) = 0, (2.1)$$

and there exists a $h_L \in (b, L)$, such that

 $\lim_{x \to +\infty} k_L(x) = h_L \text{ and } \lim_{x \to +\infty} k'_L(x) = 0.$

(4) ψ_L satisfies the Rankine-Hugoniot jump condition on Γ_L , namely,

$$\left(\frac{\partial \psi_L^-}{\partial \nu}\right)^2 - \left(\frac{\partial \psi_L^+}{\partial \nu}\right)^2 = \lambda_L \quad \text{on} \quad \Gamma_L, \tag{2.2}$$

where $\lambda_L = \frac{Q^2}{(h_L - b)^2} - \frac{L^2}{(L - h_L)^2}$. (5) Γ_L is continuously differentiable at A and

$$k'_{L}(0+0) = \begin{cases} \tan \theta, & \text{if } \lambda_{L} > 0, \\ 0, & \text{if } \lambda_{L} < 0, \\ \tan \frac{\theta}{2}, & \text{if } \lambda_{L} = 0. \end{cases}$$
(2.3)

(6) ψ_L has the following asymptotic behaviors

$$\psi_L(x,y) \to \begin{cases} \frac{Q(y-h_L)}{h_L-b}, & \text{if } b < y < h_L, \text{ as } x \to +\infty, \\ \frac{L(y-h_L)}{L-h_L}, & \text{if } h_L < y < L, \text{ as } x \to +\infty, \end{cases}$$
$$\psi_L(x,y) \to y, \quad \text{if } 0 < y < L, \text{ as } x \to -\infty, \end{cases}$$

and

$$\left|\psi_L(x,y) - \frac{Q(y\cos\theta - x\sin\theta)}{a\sin\theta - b\cos\theta}\right| \to 0 \text{ uniformly in any compact subset of } S,$$

as $y \to -\infty$, where $S = \{(x, y) \mid y \cot \theta < x < (y - b) \cot \theta + a, -\infty < y < +\infty\}$. (7) $\frac{L(y - h_L)}{L - h_L} \le \psi_L^+(x, y) \le y$ in $\Omega_L \cap \{y > 0\}$.

Remark 2.1. It should be noted that $f(t) = \frac{Q^2}{(t-b)^2} - \frac{L^2}{(L-t)^2}$ is a strictly monotone decreasing function for $t \in (b, L)$. Therefore, the asymptotic height $h_L \in (b, L)$ of the free boundary can be determined uniquely by $\lambda = \frac{Q^2}{(h_L - b)^2} - \frac{L^2}{(L - h_L)^2}$.

2.1. Variational approach. To solve the truncated injection flow problem 1, as the first step, we introduce a truncated variational problem for any given parameter $\lambda \in (-\infty, +\infty)$ and Q > 0. Secondly, we will verify that there exists a unique parameter $\lambda = \lambda_L$, such that the interface Γ_L connects at the leading edge A. Finally, taking $L \to +\infty$ yields the existence of solution to the injection flow problem 1.

For any $\mu > 1$, denote

$$\Omega_{L,\mu} = \Omega_L \cap \{(x,y) \mid x > -\mu, y > -\mu\} \text{ and } \sigma_{L,\mu} = \{(-\mu,y) \mid 0 \le y \le L\},\$$

and

$$D_{1,L,\mu} = \Omega_{L,\mu} \cap \{(x,y) \mid x < 0, y < 0\} \text{ and } D_{2,L,\mu} = \Omega_{L,\mu} \cap \{(x,y) \mid x < 0, y > 0\},$$

and

$$N_{1,\mu} = N_1 \cap \{x \ge -\mu\}, \quad S_{1,\mu} = S_1 \cap \{y \ge -\mu\} \text{ and } N_{L,\mu} = N_L \cap \{x \ge -\mu\}$$

and

$$S_{\mu} = \{-\mu \tan \theta < x < a - (\mu + b) \tan \theta, y = -\mu\} \text{ and } S_{2,\mu} = S_2 \cap \{y \ge -\mu\}.$$

Please see Figure 10.

As mentioned in Remark 2.1, for any given λ and L > b, we can obtain a unique asymptotic height $h_L \in (b, L)$ of the interface $\Gamma_{\lambda,L}$. Then, we can define $\lambda_{1,L}$, $\lambda_{2,L}$ and $\lambda_{0,L}$ as follows

$$\lambda_{1,L} = \frac{Q}{h_L - b}, \quad \lambda_{2,L} = \frac{L}{L - h_L} \text{ and } \lambda_{0,L} = \min\{\lambda_{1,L}, \lambda_{2,L}\}.$$

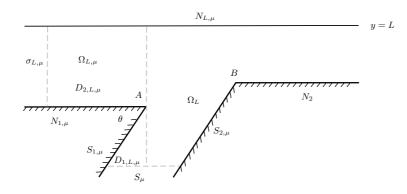


FIGURE 10. The nozzle walls and the ground

Obviously, $\lambda = \lambda_{1,L}^2 - \lambda_{2,L}^2$. Moreover, we give the following functional

$$J_{\lambda,L,\mu}(\psi) = \int_{\Omega_{L,\mu}} \left| \nabla \psi - (\lambda_{1,L} I_{\{\psi < 0\}} + \lambda_{2,L} I_{\{\psi > 0\}} + \lambda_{0,L} I_{\{\psi = 0\}}) I_{\{x > 0\}} e \right|^2 dx dy$$

where e = (0, 1) and I_D is the characteristic function of the set D. And the admissible set is defined as follows

$$\begin{split} K_{L,\mu} &= \{ \psi \mid \ \psi \in H^1_{loc}(\Omega_{L,\mu}), \ -Q \leq \psi \leq L \text{ a.e. in } \Omega_{L,\mu}, \ 0 \leq \psi \leq L \text{ a.e. in } D_{2,L,\mu}, \\ &-Q \leq \psi \leq 0 \text{ a.e. in } D_{1,L,\mu}, \ \psi = -Q \text{ on } S_\mu \cup S_{2,\mu} \cup N_2, \\ &\psi = L \text{ on } N_{L,\mu} \cup \sigma_{L,\mu}, \ \psi = \max\{L - (x+\mu)L, 0\} \text{ on } N_{1,\mu}, \\ &\psi = \min\{-Q + (y+\mu)Q, 0\} \text{ on } S_{1,\mu} \}. \end{split}$$

The truncated variational problem $(P_{\lambda,L,\mu})$: For any L > b, $\mu > 1$ and $\lambda \in (-\infty, +\infty)$, find a $\psi_{\lambda,L,\mu} \in K_{L,\mu}$ such that

$$J_{\lambda,L,\mu}(\psi_{\lambda,L,\mu}) = \min_{\psi \in K_{L,\mu}} J_{\lambda,L,\mu}(\psi).$$

Define the free boundary in the truncated domain as

$$\Gamma_{\lambda,L,\mu} = \Omega_{L,\mu} \cap \{x > 0\} \cap \{\psi_{\lambda,L,\mu} = 0\}.$$

For any L > b, $\mu > 1$ and $\lambda \in \mathbb{R}$, the existence and uniqueness of the minimizer to the truncated variational problem $(P_{\lambda,L,\mu})$ can be established along the proof of Theorem 2.1 and Lemma 2.2 in [13]. We state the results in the following.

Proposition 2.1. (Theorem 2.1 and Lemma 2.2 in [13]) For any L > b, $\mu > 1$ and $\lambda \in (-\infty, +\infty)$, there exists a unique minimizer $\psi_{\lambda,L,\mu}$ to the truncated variational problem

 $(P_{\lambda,L,\mu})$. Moreover,

$$\begin{split} \Gamma_{\lambda,L,\mu} &= \quad \Omega_{L,\mu} \cap \{x > 0\} \cap \{\psi_{\lambda,L,\mu} = 0\} \\ &= \quad \Omega_{L,\mu} \cap \{x > 0\} \cap \partial \{\psi_{\lambda,L,\mu} < 0\} \\ &= \quad \Omega_{L,\mu} \cap \{x > 0\} \cap \partial \{\psi_{\lambda,L,\mu} > 0\}, \end{split}$$

and $\psi_{\lambda,L,\mu}(x,y)$ is monotone increasing with respect to y and there exists a continuous function $k_{\lambda,L,\mu}(x)$ for x > 0, such that

$$\Gamma_{\lambda,L,\mu} = \{(x,y) \in \Omega_L \mid x > 0, y = k_{\lambda,L,\mu}(x)\}.$$

 $\psi_{\lambda,L,\mu}$ satisfies the free boundary condition in the weak sense, namely,

$$\lim_{\varepsilon \to 0^+, \delta \to 0^+} \left(\int_{\Omega_{L,\mu} \cap \{x > 0\} \cap \partial \{\psi_{\lambda,L,\mu} > \varepsilon\}} (|\nabla \psi_{\lambda,L,\mu}|^2 - \lambda_{2,L}^2) \eta \cdot \nu dS + \int_{\Omega_{L,\mu} \cap \{x > 0\} \cap \partial \{\psi_{\lambda,L,\mu} < -\delta\}} (|\nabla \psi_{\lambda,L,\mu}|^2 - \lambda_{1,L}^2) \eta \cdot \nu dS \right) = 0.$$

$$(2.4)$$

Furthermore, if $\lambda < 0$ and $|\lambda|$ is sufficiently large, then we have (1) $\psi_{\lambda,L,\mu}(x,y)$ is monotone decreasing with respect to x. (2) $k_{\lambda,L,\mu}(x)$ is monotone increasing with respect to x > 0. (3) $k_{\lambda,L,\mu}(0) = \lim_{x \to 0^+} k_{\lambda,L,\mu}(x)$ exists and $0 \le k_{\lambda,L,\mu}(x) \le L$.

2.2. The regularity of the free boundary $\Gamma_{\lambda,L,\mu}$. In Theorem 8.12 in [4], Alt, Caffarelli and Friedman proved that the free boundary $\Gamma_{\lambda,L,\mu}$ of the minimizer $\psi_{\lambda,L,\mu}$ is C^1 -smooth. Based on the significant work [7] by Caffarelli, we will obtain the higher regularity of the free boundary of the minimizer in this subsection. First, we give the definition of the weak solution of a free boundary problem as in [7].

Definition 2.2. Assume that G(t) is a continuous strictly monotone increasing function with respect to $t \in \mathbb{R}$, which satisfies that $G(t) \ge t$ and $t^{-C}G(t)$ is decreasing with respect to t > 0, for some large C > 0. Let E be a bounded open set in $\Omega_{L,\mu} \cap \{x > 0\}$. A continuous function ψ in E is called a *weak solution* of the free boundary problem, provided that ψ satisfies

- (1) $\Delta \psi = 0$ in $E^+(\psi) = E \cap \{\psi > 0\},\$
- (2) $\Delta \psi = 0$ in $E^{-}(\psi) = int(E \cap \{\psi \leq 0\}),$
- (3) (The weak free boundary condition) ψ satisfies the free boundary condition

 $\psi_{\nu}^+ = G(\psi_{\nu}^-)$ along $\mathcal{F}(\psi) = E \cap \partial \{\psi > 0\},$

in the following sense.

For any $X_0 \in \mathcal{F}(\psi)$, if $\mathcal{F}(\psi)$ has an one-side tangent ball at X_0 (i.e., there exists a ball $B_r(Y)$, such that $X_0 \in \partial B_r(Y)$ and $B_r(X)$ is contained either in $E^+(\psi)$ or in $E^-(\psi)$),

then

$$\psi(X) = \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- + o(|X - X_0|), \quad \beta \ge 0 \text{ and } \alpha = G(\beta),$$

where ν is the unit radial direction of $\partial B_r(Y)$ at X_0 pointing into $E^+(\psi)$, $\langle p, q \rangle^+ = \max\{p \cdot q, 0\}$ and $\langle p, q \rangle^- = \max\{-p \cdot q, 0\}$.

Next, we will obtain the analyticity of the free boundary $\Gamma_{\lambda,L,\mu}$ in the following, which implies that the Rankine-Hugoniot condition (2.2) on the free boundary holds in the classical sense. The main idea borrows from the works [1, 7, 19].

Theorem 2.2. The free boundary $\Gamma_{\lambda,L,\mu}$ is analytic.

Proof. Step 1. In this step, we will show that the minimizer $\psi_{\lambda,L,\mu}$ to the truncated variational problem $(P_{\lambda,L,\mu})$ is a weak solution in Definition 2.2.

Similar to Theorem 2.2 in [4], it is easy to verify that the minimizer $\psi_{\lambda,L,\mu}$ is harmonic in $E \setminus \{\psi_{\lambda,L,\mu} = 0\}$, where E is a bounded open set in $\Omega_{L,\mu} \cap \{x > 0\}$, which implies that $\psi_{\lambda,L,\mu}$ satisfies the conditions (1) and (2) in Definition 2.2. Next, it suffices to verify the condition (3) in Definition 2.2. Without loss of generality, we assume that $\lambda \leq 0$.

For any $X_0 \in \mathcal{F}(\psi_{\lambda,L,\mu})$, by means of Theorem 7.4 in [4], we have

$$\psi_{\lambda,L,\mu}(X) = \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- + o(|X - X_0|)$$

where $\alpha > 0$, $\beta > 0$ and $\lambda = \beta^2 - \alpha^2$.

Thus, $\alpha = G(\beta) = \sqrt{\beta^2 - \lambda}$. It is easy to see that $G(\beta)$ is strictly monotone increasing with respect to β and $\beta^{-1}G(\beta)$ is decreasing with respect to β .

Hence, we conclude that the minimizer $\psi_{\lambda,L,\mu}$ is a weak solution in Definition 2.2.

Step 2. Next, we will obtain the analyticity of the free boundary.

Since $\psi_{\lambda,L,\mu}$ is the weak solution in Definition 2.2, by using Theorem 1 in [7], we can conclude that the free boundary $\Omega_{L,\mu} \cap \partial \{\psi_{\lambda,L,\mu} > 0\}$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$.

Denote $\psi = \psi_{\lambda,L,\mu}$ for simplicity. Since

$$\Gamma_{\lambda,L,\mu} = \Omega_{L,\mu} \cap \{x > 0\} \cap \partial\{\psi > 0\} = \Omega_{L,\mu} \cap \{x > 0\} \cap \{\psi = 0\},\$$

is $C^{1,\alpha}$ -smooth, which implies that the \mathcal{L}^2 -measure of the free boundary $\Gamma_{\lambda,L,\mu}$ is zero. Therefore, we can use a $C^{1,\alpha}$ transformation to flatten the free boundary. Then reflect ψ^+ to the full neighborhood of the free boundary, applying the Schauder estimates for elliptic equation in divergence form in Section 9 in [1], we can obtain the $C^{1,\alpha}$ regularity of ψ^+ up to the free boundary. Similarly, we can obtain the $C^{1,\alpha}$ regularity of ψ^- up to the free boundary. Moreover, it follows from (2.4) that

$$|\nabla\psi^{-}|^{2} - |\nabla\psi^{+}|^{2} = \lambda \quad \text{on the free boundary.}$$
(2.5)

If $\lambda = 0$, it follows from Theorem 2.2 in [4] that ψ is harmonic in $\Omega_{L,\mu} \cap \{x > 0\}$. By means of the monotonicity of $\psi(x, y)$ with respect to y, the strong maximum principle gives that $\partial_y \psi > 0$ in $\Omega_{L,\mu} \cap \{x > 0\}$. Hence, the implicit function theorem gives that the level set $\Omega_{L,\mu} \cap \{x > 0\} \cap \{\psi = 0\}$ is analytic. If $\lambda \neq 0$, without loss of generality, we assume that 0 is a free boundary point of ψ , $|\nabla \psi^+(0)| \neq 0$ and the inner normal to $\Gamma_{\lambda,L,\mu}$ at 0 is in the direction of the positive *y*-axis. Extend $\tilde{\psi}$ as a $C^{1,\alpha}$ function into a full neighborhood of $0 \in \Gamma_{\lambda,L,\mu}$, such that $\tilde{\psi} = \psi$ in $\{\psi > 0\}$. In view of $|\nabla \psi^+(0)| \neq 0$, one has

$$\dot{\psi}_y(0) > 0.$$
 (2.6)

Define a mapping as follows,

$$S = TX = (s,t) \triangleq (x,\psi(x,y)), \quad X = (x,y).$$

By virtue of (2.6), it is easy to check that

$$\det\left(\frac{\partial S}{\partial X}\right) = \tilde{\psi}_y(x, y) > 0 \text{ in a small neighborhood of } 0.$$

And thus the mapping T is a local diffeomorphism near 0.

Denote the inverse transform as

$$\begin{cases} x = s \\ y = \phi(s, t). \end{cases}$$
(2.7)

Therefore, the free boundary $\Gamma_{\lambda,L,\mu}$ is transformed into t = 0, and we have

$$\left(\frac{\partial X}{\partial S}\right) = \left(\frac{\partial S}{\partial X}\right)^{-1} = \left(\begin{array}{cc} 1 & 0\\ -\frac{\tilde{\psi}_x}{\tilde{\psi}_y} & \frac{1}{\tilde{\psi}_y} \end{array}\right).$$

Consequently, one has

and

$$\phi_s = \frac{\partial y}{\partial s} = -\frac{\psi_x}{\tilde{\psi}_y}, \quad \phi_t = \frac{\partial y}{\partial t} = \frac{1}{\tilde{\psi}_y},$$
$$\frac{\partial t}{\partial x} = \tilde{\psi}_x = -\frac{\phi_s}{\phi_t}, \quad \frac{\partial t}{\partial y} = \tilde{\psi}_y = \frac{1}{\phi_t}.$$
(2.8)

It follows from (2.8) that

$$\mathcal{Q}\phi = \partial_s \left(\frac{\phi_s}{\phi_t}\right) + \partial_t \left(-\frac{1+\phi_s^2}{2\phi_t^2}\right) = 0$$
 in the neighborhood of 0.

Denote $A_1(\phi_s, \phi_t) = \frac{\phi_s}{\phi_t}$ and $A_2(\phi_s, \phi_t) = -\frac{1+\phi_s^2}{2\phi_t^2}$. It is easy to check that

$$\mathcal{A} = \frac{\partial(A_1, A_2)}{\partial(\phi_s, \phi_t)} = \frac{1}{\phi_t^3} \begin{pmatrix} \phi_t^2 & -\phi_s \phi_t \\ & \\ -\phi_s \phi_t & 1 + \phi_s^2 \end{pmatrix}.$$

Direct straightforward computations give that the matrix \mathcal{A} has two eigenvalues

$$\lambda_1 = \frac{1 + \phi_s^2 + \phi_t^2 + \sqrt{((1 + \phi_t)^2 + \phi_s^2)((1 - \phi_t)^2 + \phi_s^2)}}{2\phi_t^3},$$

and

$$\lambda_2 = \frac{2}{\left(1 + \phi_s^2 + \phi_t^2 + \sqrt{((1 + \phi_t)^2 + \phi_s^2)((1 - \phi_t)^2 + \phi_s^2)}\right)\phi_t}.$$

In view of (2.6), one has

$$\lambda_1 > 0$$
 and $\lambda_2 > 0$ in a small neighborhood of 0.

Thus, $\mathcal{Q}\phi = 0$ is a quasilinear elliptic equation in a neighborhood E of 0. Furthermore, ϕ satisfies the Neumann type boundary condition as follows,

$$Q\phi = 0 \qquad \text{in } D,$$

$$\left(\frac{\phi_t}{\sqrt{1+\phi_s^2}} = g(s) \qquad \text{on } \bar{D} \cap \{t=0\},$$

$$\text{nd } g(s,0) = \frac{1}{\sqrt{|\nabla\psi^-(s,0)|^2 - \lambda}}.$$
(2.9)

where $D = E \cap \{t > 0\}$ and $g(s, 0) = \frac{1}{\sqrt{|\nabla \psi^-(s, 0)|^2 - \lambda}}$. Noting that $\tilde{\psi}$ is in $C^{1,\alpha}$ near 0, we have that the coefficients of \mathcal{Q} and g(s) are $C^{0,\alpha}$.

Noting that ψ is in $C^{2,\alpha}$ near 0, we have that the coefficients of \mathcal{Q} and g(s) are $C^{2,\alpha}$. By using the elliptic regularity in Section 9 in [1], we obtain that $\phi(S)$ is $C^{2,\alpha}$ near 0. Furthermore, the free boundary $\Gamma_{\lambda,L,\mu}$ can be described by $y = \phi(s,0) = \phi(x,0)$, and thus the free boundary $\Gamma_{\lambda,L,\mu}$ is $C^{2,\alpha}$ near 0.

Applying the Schauder estimates for elliptic equations in [1], we can obtain the $C^{2,\alpha}$ regularity of ψ^+ and ψ^- up to the free boundary $\Gamma_{\lambda,L,\mu}$. By using the above arguments, we can conclude that the free boundary $\Gamma_{\lambda,L,\mu}$ is $C^{3,\alpha}$.

Along the bootstrap arguments, the C^{∞} regularity of the free boundary $\Gamma_{\lambda,L,\mu}$ can be established. Finally, with the aid of the results in Section 6.7 in [19], we can obtain that $\phi(s,t)$ is analytic in $t \geq 0$. Hence, we obtain the analyticity of the free boundary $\Gamma_{\lambda,L,\mu}$.

It follows from Lemma 2.4 and Lemma 2.5 in [13] that we can obtain the existence of $\lambda_{L,\mu}$, such that the continuous fit condition of $\Gamma_{\lambda_{L,\mu},L,\mu}$ at A holds for any L > b and $\mu > 1$. Similar to the arguments on the compressible subsonic flows in infinitely long nozzle in [8, 9, 10, 23, 24, 25], we can obtain the asymptotic behavior of the flows in downstream and in upstream. We omit the details here.

Proposition 2.3. For any Q > 0, L > b and $\mu > 1$, there exists a $\lambda_{L,\mu} \in \mathbb{R}$ with $|\lambda_{L,\mu}| \leq C$ (the constant C is independent of μ), such that $k_{\lambda_{L,\mu},L,\mu}(0) = 0$ and $\psi_{\lambda_{L,\mu},L,\mu}$ is Lipschitz continuous in $\overline{\Omega}_{L,\mu} \setminus B_{\varepsilon}(B)$ for any small $\varepsilon > 0$. Furthermore,

$$k_{\lambda_{L,\mu},L,\mu}(x) \to h_{\lambda_{L,\mu}} \in (b,L) \text{ and } \psi_{\lambda_{L,\mu},L,\mu}(x,y) \to \psi_0(y)$$

in any compact subset of (b, L) as $x \to +\infty$, where $h_{\lambda_L,\mu}$ is determined uniquely by

$$\lambda_{L,\mu} = \frac{Q^2}{(h_{\lambda_{L,\mu}} - b)^2} - \frac{L^2}{(L - h_{\lambda_{L,\mu}})^2} \quad and \quad \psi_0(y) = \begin{cases} \frac{Q(y - h_{\lambda_{L,\mu}})}{h_{\lambda_{L,\mu}} - b}, & \text{if } b < y < h_{\lambda_{L,\mu}}, \\ \frac{L(y - h_{\lambda_{L,\mu}})}{L - h_{\lambda_{L,\mu}}}, & \text{if } h_{\lambda_{L,\mu}} < y < L, \end{cases}$$

2.3. The existence of the truncated injection flow problem 1. In this subsection, we will investigate the existence of the truncated injection flow problem 1. Moreover, the positivity of horizontal velocity and vertical velocity will be obtained.

Theorem 2.4. For any Q > 0 and L > b, there exist a λ_L and a solution $(\psi_{\lambda_L}, \Gamma_{\lambda_L})$ to the truncated injection flow problem 1.

Proof. Since $\psi_{\lambda_{L,\mu},L,\mu}$ is Lipschitz continuous in $\Omega_{L,\mu}$ and $|\lambda_{L,\mu}| \leq C$ (the constant C > 0 is independent of μ), we can take a sequence $\{\mu_n\}$ with $\mu_n \to +\infty$, then there exist a λ_L and a $\psi_{\lambda_L,L} \in H^1_{loc}(\Omega_L)$, such that

$$\lambda_{L,\mu_n} \to \lambda_L,$$

and

 $\psi_{\lambda_{L,\mu_n},L,\mu_n} \to \psi_{\lambda_L,L}$ in $H^1_{loc}(\Omega_L)$ and uniformly in any compact subset of Ω_L ,

as $\mu_n \to +\infty$.

Next, we divide six steps to verify that $(\psi_{\lambda_L}, \Gamma_{\lambda_L})$ satisfies the conditions in Definition 2.1.

Step 1. For any Q > 0 and L > b, denote

$$D_{1,L} = \Omega_L \cap \{(x,y) \mid x < 0, y < 0\}$$
 and $D_{2,L} = \Omega_L \cap \{(x,y) \mid x < 0, y > 0\}.$

By virtue of Lemma 6.2 in [4], we can show that $\psi_{\lambda_L,L}$ is a local minimizer to the variational problem $(P_{\lambda_L,L})$, namely,

$$P_{\lambda_L,L}$$
: $J_D(\psi_{\lambda_L,L}) = \min J_D(\psi)$ for any $\psi \in K_L$ and $\psi = \psi_{\lambda_L,L}$ on ∂D ,

where

$$J_D(\psi) = \int_D \left| \nabla \psi - (\lambda_{1,L} I_{\{\psi < 0\}} + \lambda_{2,L} I_{\{\psi > 0\}} + \lambda_{0,L} I_{\{\psi = 0\}}) I_{\{x > 0\}} e \right|^2 dx dy$$

and

$$K_L = \{ \psi \mid \psi \in H^1_{loc}(\Omega_L), -Q \le \psi \le L \text{ a.e. in } \Omega_L, \quad 0 \le \psi \le L \text{ a.e. in } D_{2,L}, \\ -Q \le \psi \le 0 \text{ a.e. in } D_{1,L}, \ \psi = -Q \text{ on } S_2 \cup N_2, \\ \psi = L \text{ on } N_L, \ \psi = 0 \text{ on } N_1 \cup S_1 \}.$$

for any bounded domain $D \subset \Omega_L$. Therefore, the conditions (1) and (2) in Definition 2.1 have be verified.

Step 2. We can conclude that $\psi_{\lambda_L,L}(x, y)$ is monotone increasing with respect to y and decreasing with respect to x, which follows from the monotonicity of $\psi_{\lambda_{L,\mu},L,\mu}(x, y)$. Furthermore, the free boundary

$$\Gamma_{\lambda_L,L} = \Omega_L \cap \{x > 0\} \cap \{\psi_{\lambda_L,L} = 0\}$$

of the minimizer $\psi_{\lambda_L,L}$ is given by a continuous function $y = k_{\lambda_L,L}(x)$ for any x > 0. In particular, $k_{\lambda_L,L}(0) = 0$. In view of Proposition 2.3, one has

$$k_{\lambda_L,L}(x) \to h_L \text{ and } \psi_{\lambda_L,L}(x,y) \to \psi_0(y)$$
 (2.10)

in any compact subset of (b, L) as $x \to +\infty$, where h_L is determined uniquely by

$$\lambda_L = \frac{Q^2}{(h_L - b)^2} - \frac{L^2}{(L - h_L)^2} \text{ and } \psi_0(x, y) = \begin{cases} \frac{Q(y - h_L)}{h_L - b}, & \text{if } b < y < h_L, \\ \frac{L(y - h_L)}{L - h_L}, & \text{if } h_L < y < L. \end{cases}$$

Furthermore, by virtue of the analyticity of the free boundary $\Gamma_{\lambda_L,L}$ and (2.10), one has

 $k'_{\lambda_L,L}(x) \to 0 \text{ as } x \to +\infty.$

Next, we will show that $k_{\lambda_L,L}(x)$ is strictly monotone increasing with respect to x > 0. If not, there exist $x_1, x_2 \in (0, +\infty)$ with $x_1 < x_2$, such that $k_{\lambda_L,L}(x_1) = k_{\lambda_L,L}(x_2)$. The monotonicity of $\psi_{\lambda_L,L}(x, y)$ with respect to x and y gives that there exists a small r > 0, such that

$$\psi_{\lambda_L,L} > 0$$
 in $B_r(X_0) \cap \{y > y_0\}$ and $\psi_{\lambda_L,L} < 0$ in $B_r(X_0) \cap \{y < y_0\}$,

where $X_0 = (x_0, y_0)$ with $x_0 = \frac{x_1 + x_2}{2}$ and $y_0 = k_{\lambda_L, L}(x_1)$. Denote $I_0 = \{(x, y_0) \mid -r < x - x_0 < r\}$, $B^+ = B_{2r}(X_0) \cap \{y > y_0\}$ and $B^- = B_{2r}(X_0) \cap \{y < y_0\}$. Set $\psi_1 = \psi_{\lambda_L, L}$ in B^- and $\psi_2 = \psi_{\lambda_L, L}$ in B^+ . Since $\psi_{\lambda_L, L}(x, y)$ is decreasing with respect to x, the strong maximum principle gives that

$$\partial_x \psi_1 < 0$$
 in B^- and $\partial_x \psi_2 < 0$ in B^+

In view of that $\partial_x \psi_1 = \partial_x \psi_2 = 0$ on I_0 , it follows from Hopf's lemma that

$$\frac{\partial}{\partial y}\partial_x\psi_1 > 0 \text{ and } \frac{\partial}{\partial y}\partial_x\psi_2 < 0 \text{ on } I_0.$$
 (2.11)

Noting that $\psi_1 = \psi_2 = 0$ on I_0 , thanks hopf's lemma, one has

 $\partial_u \psi_1 > 0$ and $\partial_u \psi_2 > 0$ on I_0 ,

which together with (2.11) give that

$$\frac{\partial(\partial_y \psi_1)^2}{\partial x} > 0 \quad \text{and} \quad \frac{\partial(\partial_y \psi_2)^2}{\partial x} < 0 \quad \text{on } I_0.$$

Thus one has

$$\frac{\partial}{\partial x} \left(|\nabla \psi_{\lambda_L,L}^-|^2 - |\nabla \psi_{\lambda_L,L}^+|^2 \right) = \frac{\partial}{\partial x} \left((\partial_y \psi_1)^2 - (\partial_y \psi_2)^2 \right) > 0 \quad \text{on } I_0,$$

which contradicts to the fact $|\nabla \psi_{\lambda_L,L}^-|^2 - |\nabla \psi_{\lambda_L,L}^+|^2 = \lambda_L$ on I_0 . **Step 2** It follows from (2.4) that

$$\lim_{\varepsilon \to 0^+, \delta \to 0^+} \left(\int_{\Omega_L \cap \{x > 0\} \cap \partial \{\psi_{\lambda_L, L} > \varepsilon\}} (|\nabla \psi_{\lambda_L, L}|^2 - \lambda_{2, L}^2) \eta \cdot \nu dS + \int_{\Omega_L \cap \{x > 0\} \cap \partial \{\psi_{\lambda_L, L} < -\delta\}} (|\nabla \psi_{\lambda_L, L}|^2 - \lambda_{1, L}^2) \eta \cdot \nu dS \right) = 0.$$

Since the free boundary $\Gamma_{\lambda_L,L}$ is analytic, $\psi_{\lambda_L,L}$ is C^2 up to the free boundary $\Gamma_{\lambda_L,L}$. Thus, the condition (4) in Definition 2.1 holds.

Step 4. In this step, we will show that the free boundary $\Gamma_{\lambda_L,L}$ is continuous differentiable at A, and (2.3) holds.

For any r > 0, define a blow-up sequence $\{\psi_n\}$, such that $\psi_n(\tilde{X}) = \frac{\psi_{\lambda_L,L}(r_nX)}{r_n}$ in $B_r(0)$ with $\tilde{X} = (\tilde{x}, \tilde{y})$ and $r_n \to 0$. We next consider the following three cases.

Case 1. $\lambda_L < 0$. Taking a sequence $\{X_n\}$ with $X_n \in \Gamma_{\lambda_L,L}$ and $X_n \to 0$, set $r_n = |X_n|$. By virtue of the non-degeneracy Theorem 3.1 in [4], one has

$$\frac{1}{r_n} \oint_{\partial B_{\frac{r_n}{2}}(X_n)} \psi^+_{\lambda_L,L}(X) dS_X \ge c |\lambda_L|^{\frac{1}{2}}, \tag{2.12}$$

where c > 0 is a constant independent of n. Denote $\tilde{Y}_n = \frac{X_n}{r_n}$, it follows from (2.12) that

$$\int_{\partial B_{\frac{1}{2}}(\tilde{Y}_n)} \psi_n^+(\tilde{X}) dS_{\tilde{X}} \ge c |\lambda_L|^{\frac{1}{2}} \text{ and } |\tilde{Y}_n| = 1.$$
(2.13)

It follows from the similar arguments in Pages 444-445 in [4] that there exist a subsequence $\{\psi_n\}$ and a blow-up limit $\psi_0 \in H^1_{loc}(\mathbb{R}^2)$, such that

 $\psi_n(\tilde{X}) \to \psi_0(\tilde{X})$ uniformly in bounded sets, and $\nabla \psi_n \to \nabla \psi_0$ a.e. in \mathbb{R}^2 . (2.14) Furthermore,

$$\partial\{\psi_n > 0\} \to \partial\{\psi_0 > 0\}$$
 locally in the Hausdorff metric. (2.15)

In particular, 0 is the free boundary point of ψ_0 .

In view of (2.13) and (2.14), there exist two subsequences $\{\tilde{Y}_n\}$ and $\{\psi_n\}$, such that $\tilde{Y}_n \to \tilde{Y}_0$ and

$$\int_{\partial B_{\frac{1}{2}}(\tilde{Y}_0)} \psi_0^+(\tilde{X}) dS_{\tilde{X}} \ge c |\lambda_L|^{\frac{1}{2}} \text{ and } |\tilde{Y}_0| = 1.$$
(2.16)

By using the monotonicity formula lemma 5.1 in [4], one has

$$\frac{1}{r^4} \int_{B_r(\tilde{Y}_0)} |\nabla \psi_0^+(\tilde{X})|^2 d\tilde{X} \cdot \int_{B_r(\tilde{Y}_0)} |\nabla \psi_0^-(\tilde{X})|^2 d\tilde{X} = \gamma \ge 0,$$

for any r > 0, which together with Lemma 6.6 in [4] gives that

 ψ_0 is either a 2-plane solution, or a 1-plane solution, or identically zero. (2.17)

Since $\theta > 0$, one has

$$\psi_{\lambda_L,L} \equiv 0 \quad \text{in } S \text{ and } \frac{\mathcal{L}^2(B_r(A) \cap S)}{\mathcal{L}^2(B_r(A))} = \frac{\theta}{2\pi} > 0, \tag{2.18}$$

for any r > 0, where $S = \{(x, y) \mid x \le y \cot \theta, y \le 0\}$. By virtue of (2.17) and (2.18), one has

 ψ_0 is either a 1-plane solution, or identically zero,

which together with (2.16) gives that

$$\psi_0 \ge 0 \quad \text{and} \quad \psi_0 \not\equiv 0.$$
 (2.19)

Since $\psi_{\lambda_L,L}(X) \leq 0$ in $\{(x,y) \mid y \leq 0\}$ and $\psi_{\lambda_L,L}(X) > 0$ in $\{(x,y) \mid x < 0, y > 0\}$, by virtue of (2.14) and (2.19), we have

$$\psi_0(\tilde{x}, \tilde{y}) \equiv 0 \quad \text{in } \{(\tilde{x}, \tilde{y}) \mid \tilde{y} \le 0\}, \tag{2.20}$$

and

$$\psi_0(\tilde{x}, \tilde{y}) > 0 \text{ in } \{(\tilde{x}, \tilde{y}) \mid \tilde{x} < 0, \tilde{y} > 0\}.$$
 (2.21)

Thus, it follows from (2.21) that

$$\psi_0(\tilde{x}, \tilde{y}) = \max\{\alpha \tilde{y}, 0\} \quad \text{and} \quad \alpha > 0. \tag{2.22}$$

By virtue of the proof of Lemma 6.2 in [4], we can conclude that ψ_0 is a local minimizer for the variational problem

$$J_R(\psi_0) = \min J_R(\psi) \text{ for any } \psi - \psi_0 \in H_0^1(B_R) \text{ and } R > 0,$$

where $B_R = B_R(0)$ and the functional

$$J_{R}(\psi) = \int_{B_{R}} \left| \nabla \psi - (\lambda_{1,L} I_{\{\psi \le 0\}} + \lambda_{2,L} I_{\{\psi > 0\}}) I_{\{\tilde{x} > 0\}} e \right|^{2} d\tilde{x} d\tilde{y}$$

with $\lambda_{1,L} = \frac{Q}{h_L - b}$ and $\lambda_{2,L} = \frac{L}{L - h_L}$. Next, we claim that ψ_0 is a local minimizer for the variational problem, namely,

$$J_R^0(\psi_0) = \min J_R^0(\psi)$$
 for any $\psi - \psi_0 \in H_0^1(B_R)$ and $R > 0$,

where the functional

$$J_R^0(\psi) = \int_{B_R} |\nabla \psi|^2 + (\lambda_{2,L}^2 - \lambda_{1,L}^2) I_{\{\psi>0\}} I_{\{\tilde{x}>0\}} d\tilde{x} d\tilde{y}.$$

In fact, for any $\psi - \psi_0 \in H_0^1(B_R)$, one has

$$\begin{split} &\int_{B_R} |\nabla \psi_0|^2 - \lambda_L^2 I_{\{\psi_0 > 0\}} I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y} - \int_{B_R} |\nabla \psi|^2 - \lambda_L^2 I_{\{\psi > 0\}} I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y} \\ &= \int_{B_R} |\nabla \psi_0|^2 + (\lambda_{1,L}^2 I_{\{\psi_0 \le 0\}} + \lambda_{2,L}^2 I_{\{\psi_0 > 0\}}) I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y} \\ &- \int_{B_R} |\nabla \psi|^2 + (\lambda_{1,L}^2 I_{\{\psi \le 0\}} + \lambda_{2,L}^2 I_{\{\psi > 0\}}) I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y} \\ &= \int_{B_R} |\nabla \psi_0 - (\lambda_{1,L} I_{\{\psi_0 \le 0\}} + \lambda_{2,L} I_{\{\psi_0 > 0\}}) I_{\{\tilde{x} > 0\}} e|^2 d\tilde{x} d\tilde{y} \\ &- \int_{B_R} |\nabla \psi - (\lambda_{1,L} I_{\{\psi \le 0\}} + \lambda_{2,L} I_{\{\psi > 0\}}) I_{\{\tilde{x} > 0\}} e|^2 d\tilde{x} d\tilde{y} \\ &\leq 0, \end{split}$$

where we have used the fact

$$\int_{B_R} \nabla \psi_0 \cdot eI_{\{\psi_0 \le 0\}} I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y} = \int_{B_R} \nabla \psi \cdot eI_{\{\psi \le 0\}} I_{\{\tilde{x} > 0\}} d\tilde{x} d\tilde{y},$$

and

$$\int_{B_R} \nabla \psi_0 \cdot eI_{\{\psi_0>0\}} I_{\{\tilde{x}>0\}} d\tilde{x} d\tilde{y} = \int_{B_R} \nabla \psi \cdot eI_{\{\psi>0\}} I_{\{\tilde{x}>0\}} d\tilde{x} d\tilde{y}.$$

For the minimal functional $J_R^0(\psi_0)$, it follows from Theorem 2.5 in [2] that

$$|\nabla \psi_0|^2 = \lambda_{2,L}^2 - \lambda_{1,L}^2 = -\lambda_L$$
 on the free boundary of ψ_0 ,

which implies that

$$\alpha = \sqrt{-\lambda_L}.$$

Hence, one has

$$\psi_n(\tilde{x}, \tilde{y}) \to \sqrt{-\lambda_L} \tilde{y}^+$$
 as $r_n \to 0$.

By using the similar arguments in Lemma 11.2 in Chapter 3 in [12], we can conclude that $k'_{\lambda_{L},L}(0+0) = 0$.

Case 2. $\lambda_L > 0$. Similar to Case 1, we can conclude that $k'_{\lambda_L,L}(0+0) = \tan \theta$.

Case 3. $\lambda_L = 0$. It is easy to check that $\psi_{\lambda_L,L}$ is a harmonic function across the free boundary in Ω_L . By using a conformal mapping $\tilde{\psi}(z) = \psi_{\lambda_L,L} \left(z^{\frac{\pi}{2\pi-\theta}} \right)$ with z = x + iy, such that $\tilde{\psi}(z)$ becomes a harmonic function in $B_r(0) \cap \{\text{Im}z > 0\}$, $\tilde{\psi} = 0$ on $B_r(0) \cap \partial \{\text{Im}z > 0\}$, where z = x + iy. Furthermore, $N_1 \cup S_1$ is mapped into the real axis and the free boundary $\Gamma_{\lambda_L,L}$ is mapped into a continuous arc γ initiating at 0. Then the harmonic function $\tilde{\psi}(z)$ has harmonic continuation across Imz = 0 in $B_r(0)$. It follows that the level set $\{\tilde{\psi} = 0\}$ consists arcs forming equal angles at A. Since $\tilde{\psi}$ vanishes only on γ , which implies that the continuous arc γ must intersect $\partial \{\text{Im}z > 0\}$ orthogonally at A, namely, $k'_{\lambda_L,L}(0+0) = \tan \frac{\theta}{2}$.

Hence, the condition (5) in Definition 2.1 is obtained.

Step 5. In this step, we will verify that $\psi_{\lambda_L,L}$ satisfies the condition (6) in Definition 2.1. The asymptotic behavior of $\psi_{\lambda_L,L}$ in downstream has been obtained in Step 2. Next, we consider the asymptotic behavior of $\psi_{\lambda_L,L}$ in upstream. Define a blow-up sequence $\psi_n(x,y) = \psi_{\lambda_L,L}(x-n,y)$ for $x < \frac{n}{2}$. By using the elliptic regularity in [16], there exists a subsequence $\{\psi_n\}$, such that

$$\psi_n(x,y) \to \overline{\psi}_0(x,y)$$
 in $C^{2,\alpha}(D)$,

for any compact subset D of $E = \{-\infty < x < +\infty\} \times \{0 < y < L\}$, and

$$\Delta \bar{\psi}_0 = 0 \text{ in } E,$$

$$\bar{\psi}_0(x,0) = 0 \text{ and } \bar{\psi}_0(x,L) = L \text{ for } -\infty < x < +\infty,$$

$$0 \le \bar{\psi}_0(x,y) \le L \text{ in } E.$$

Then the above boundary value problem has a unique solution

$$\bar{\psi}_0(x,y) = y$$
 in E ,

which implies that

$$\psi_{\lambda_L,L}(x,y) \to y \text{ for } 0 < y < L, \text{ as } x \to -\infty.$$
 (2.23)

Denote $\tilde{x} = x \sin \theta - y \cos \theta$ and $\tilde{y} = y \sin \theta + x \cos \theta$, let

$$\psi(\tilde{x}, \tilde{y}) = \psi_{\lambda_L, L}(\tilde{x}\sin\theta + \tilde{y}\cos\theta, \tilde{y}\sin\theta - \tilde{x}\cos\theta)$$

and $\tilde{\psi}_n(\tilde{x}, \tilde{y}) = \tilde{\psi}(\tilde{x}, \tilde{y} - n)$. By virtue of the elliptic regularity, there exists a subsequence $\{\tilde{\psi}_n\}$, such that

$$\tilde{\psi}_n(\tilde{x}, \tilde{y}) \to \tilde{\psi}_0(\tilde{x}, \tilde{y}) \text{ in } C^{2,\alpha}(D),$$

for any compact subset D of $\tilde{E} = \{0 < \tilde{x} < a \sin \theta - b \cos \theta\} \times \{-\infty < \tilde{y} < +\infty\}$, and $\tilde{\psi}_0$ satisfies that

$$\begin{cases} \Delta \tilde{\psi}_0 = 0 \text{ in } \tilde{E}, \\ \tilde{\psi}_0(0, \tilde{y}) = 0 \text{ and } \tilde{\psi}_0(a \sin \theta - b \cos \theta, \tilde{y}) = -Q \text{ for } -\infty < \tilde{y} < +\infty \\ -Q \le \tilde{\psi}_0(\tilde{x}, \tilde{y}) \le 0 \text{ in } \tilde{E}. \end{cases}$$

Then one has

$$\tilde{\psi}_0(\tilde{x}, \tilde{y}) = -\frac{Q}{a\sin\theta - b\cos\theta}\tilde{x}$$
 in \tilde{E} ,

which implies that

$$\left| \psi_{\lambda_L,L}(x,y) - \frac{Q(y\cos\theta - x\sin\theta)}{a\sin\theta - b\cos\theta} \right| \to 0 \text{ uniformly in any compact subset of } S, \quad (2.24)$$

as $y \to -\infty$, where $S = \{(x,y) \mid y\cot\theta < x < (y-b)\cot\theta + a, -\infty < y < +\infty\}.$

Step 6. Finally, we will verify the condition (7) in Definition 2.1 and complete the proof. For any $\varepsilon > 0$, by virtue of (2.10) and (2.23), there exists a large $\mu_0 > 0$, such that

$$\psi_{\lambda_L,L} - y \le \varepsilon \text{ in } \Omega_L^+ \cap \{x \le -\mu_0\} \text{ and } \psi_{\lambda_L,L} \le y \text{ in } \Omega_L^+ \cap \{x \ge \mu_0\}$$

where $\Omega_L^+ = \Omega_L \cap \{\psi_{\lambda_L,L} > 0\}$. This together with the maximum principle gives that

$$\psi_{\lambda_L,L} - y \le \varepsilon \text{ in } \Omega_L^+ \cap \{-\mu_0 \le x \le \mu_0\}$$

Therefore, we have

$$\psi_{\lambda_L,L} - y \le \varepsilon \quad \text{in} \quad \Omega_L^+. \tag{2.25}$$

Taking $\varepsilon \to 0$ in (2.25), one has

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$$\psi^+_{\lambda_L,L}(x,y) \le y \text{ in } \Omega_L \cap \{y > 0\}.$$

Similarly, we can show that

$$\psi_{\lambda_L,L}^+(x,y) \ge \frac{L(y-h_L)}{L-h_L}$$
 in $\Omega_L \cap \{y>0\}.$

Finally, we will obtain the positivity of horizontal velocity and vertical velocity in the following.

Lemma 2.5. The horizontal velocity and vertical velocity are positive in Ω_L , namely,

$$u > 0 \text{ and } v > 0 \text{ in } \Omega_L^- \cup \Gamma_{\lambda_L,L}, \quad u > 0 \text{ and } v > 0 \text{ in } \Omega_L^+ \cup \Gamma_{\lambda_L,L},$$

where $\Omega_L^- = \Omega_L \cap \{\psi_{\lambda_L,L} < 0\}$ and $\Omega_L^+ = \Omega_L \cap \{\psi_{\lambda_L,L} > 0\}.$

Proof. Denote $\omega_1(x,y) = \partial_y \psi_{\lambda_L,L}(x,y)$ in Ω_L^- and $\omega_2(x,y) = \partial_y \psi_{\lambda_L,L}(x,y)$ in Ω_L^+ , it is easy to check that

$$\Delta \omega_1 = 0$$
 in Ω_L^- , and $\Delta \omega_2 = 0$ in Ω_L^+

Since $\psi_{\lambda_L,L}(x, y)$ is monotone increasing with respect to y, which together with the strong maximum principle gives that

$$\omega_1(x,y) > 0$$
 in Ω_L^- , and $\omega_2(x,y) > 0$ in Ω_L^+ .

Next, we claim that

$$\omega_1(x,y) > 0$$
 and $\omega_2(x,y) > 0$ on $\Gamma_{\lambda_L,L}$

Suppose not, without loss of generality, we assume that there exists an $x_0 \in (0, +\infty)$, such that $\omega_1(X_0) = 0$ with $X_0 = (x_0, k_{\lambda_L, L}(x_0))$. We consider the following two cases.

Case 1. $\lambda_L = 0$. Then we have that $\psi_{\lambda_L,L}$ is harmonic in Ω_L , the strong maximum principle gives that $\omega_1(X_0) > 0$, which contradicts to our assumption.

Case 2. $\lambda_L \neq 0$. Since the free boundary $\Gamma_{\lambda_L,L}$ is analytic at $X_0, \omega_1(X_0) = 0$ implies that the normal vector of $\Gamma_{\lambda_L,L}$ is parallel to (1,0), which implies that $\omega_2(X_0)$ is also zero. Thus, it follows from (2.5) that $|\partial_x \psi_{\lambda_L,L}^-(X_0)|^2 - |\partial_x \psi_{\lambda_L,L}^+(X_0)|^2 = \lambda_L$.

Without loss of generality, we assume that the outer normal vector of $\partial \{\psi_{\lambda_L,L} > 0\}$ at X_0 is $\nu = (1,0)$. Thanks to Hopf's lemma, one has

$$\partial_x \psi_{\lambda_L,L}^+ = \frac{\partial \psi_{\lambda_L,L}^+}{\partial \nu} < 0 \quad \text{and} \quad \partial_x \psi_{\lambda_L,L}^- = \frac{\partial \psi_{\lambda_L,L}^-}{\partial \nu} > 0 \quad \text{on} \quad X_0, \tag{2.26}$$

and

$$\partial_{xy}^2 \psi_{\lambda_L,L}^+ = \partial_x \omega_1 = \frac{\partial \omega_1}{\partial \nu} < 0 \quad \text{and} \quad \partial_{xy}^2 \psi_{\lambda_L,L}^- = \partial_x \omega_2 = \frac{\partial \omega_2}{\partial \nu} < 0 \quad \text{on} \quad X_0.$$
(2.27)

Since $|\nabla \psi_{\lambda_L,L}^-|^2 - |\nabla \psi_{\lambda_L,L}^+|^2 = \lambda_L$ on the free boundary $\Gamma_{\lambda_L,L}$, one has

$$0 = \frac{\partial (|\nabla \psi_{\lambda_L,L}^-|^2 - |\nabla \psi_{\lambda_L,L}^+|^2)}{\partial s} = 2(\partial_{xy}\psi_{\lambda_L,L}^- \partial_x \psi_{\lambda_L,L}^- - \partial_{xy}\psi_{\lambda_L,L}^+ \partial_x \psi_{\lambda_L,L}^+) \quad \text{at} \quad X_0, \quad (2.28)$$

where s = (0, 1) is the tangential direction of $\Gamma_{\lambda_L, L}$ at X_0 . On the other hand, it follows from (2.26) and (2.27) that

$$\partial_{xy}^2 \psi_{\lambda_L,L}^- \partial_x \psi_{\lambda_L,L}^- - \partial_{xy}^2 \psi_{\lambda_L,L}^+ \partial_x \psi_{\lambda_L,L}^+ < 0 \text{ at } X_0,$$

which contradicts to (2.28).

Similarly, we can show that

$$v > 0$$
 in $\Omega_L^- \cup \Gamma_{\lambda_L,L}$ and $v > 0$ in $\Omega_L^+ \cup \Gamma_{\lambda_L,L}$.

2.4. The uniqueness of the truncated injection flow problem 1. In this subsection, we will obtain the uniqueness of the truncated injection flow problem 1 for any given Q > 0 and L > b.

Lemma 2.6. For any Q > 0 and L > b, there exist a unique λ_L and a unique solution $(\psi_{\lambda_L,L}, \Gamma_{\lambda_L,L})$ to the truncated injection flow problem 1.

Proof. Suppose that there exist two different solutions $\psi = \psi_{\lambda_L,L}$ and $\tilde{\psi} = \tilde{\psi}_{\tilde{\lambda}_L,L}$. We divide two steps to complete the proof.

Step 1. First, we show that

$$\lambda_L = \tilde{\lambda}_L.$$

Suppose not, without loss of generality, we assume that $\lambda_L < \tilde{\lambda}_L$. Noting that

$$\lambda_L = \frac{Q^2}{(h_L - b)^2} - \frac{L^2}{(L - h_L)^2}$$
 and $\tilde{\lambda}_L = \frac{Q^2}{(\tilde{h}_L - b)^2} - \frac{L^2}{(L - \tilde{h}_L)^2}$

this together with Remark 2.1 implies that

$$\lim_{x \to +\infty} k_{\lambda_L,L}(x) = h_L > \tilde{h}_L = \lim_{x \to +\infty} \tilde{k}_{\tilde{\lambda}_L,L}(x).$$
(2.29)

Then,

$$k_{\lambda_L,L}(x) > \tilde{k}_{\tilde{\lambda}_L,L}(x)$$
 for sufficiently large $x > 0.$ (2.30)

Define a function $\psi_{\varepsilon}(x,y) = \psi(x,y-\varepsilon)$ for any $\varepsilon \ge 0$, and $\Gamma_{\lambda_L,L}^{\varepsilon} : y = k_{\lambda_L,L}(x) + \varepsilon$ as the free boundary of ψ_{ε} . Take $\varepsilon_0 \ge 0$ to be the smallest one such that

 $\psi_{\varepsilon_0}(X) \le \tilde{\psi}(X)$ in Ω_L and $\psi_{\varepsilon_0}(X_0) = \tilde{\psi}(X_0)$ for some $X_0 \in \bar{\Omega}_L$. (2.31)

We consider the following two cases for ε_0 .

Case 1. $\varepsilon_0 > 0$. The strong maximum principle gives that $X_0 \notin \Omega_L \cap \{\tilde{\psi} < 0\} \cup \{\psi_{\varepsilon_0} > 0\}$). Suppose not, without loss of generality, we assume that there exists $X_0 \in \Omega_L \cap \{\tilde{\psi} < 0\}$, such that $-Q < \psi_{\varepsilon_0}(X_0) = \tilde{\psi}(X_0) < 0$. The continuity of ψ_{ε_0} and $\tilde{\psi}$ implies that there exists a small r > 0, such that

$$-Q < \psi_{\varepsilon_0}(X) < 0$$
 and $-Q < \tilde{\psi}(X) < 0$ in $B_r(X_0)$.

Since ψ_{ε_0} and $\tilde{\psi}$ are harmonic in $B_r(X_0)$, the strong maximum principle gives that $\psi_{\varepsilon_0} \equiv \tilde{\psi}$ in $B_r(X_0)$, due to $\psi_{\varepsilon_0}(X_0) = \tilde{\psi}(X_0)$. Applying the strong maximum principle again, we can obtain a contradiction to the boundary value of $\tilde{\psi}$.

Since $\varepsilon_0 > 0$, it follows from (2.30) that $|X_0| < +\infty$. Therefore, choose X_0 to be a free boundary point of ψ_{ε_0} and $\tilde{\psi}$, and one has $\psi_{\varepsilon_0}(X_0) = \tilde{\psi}(X_0) = 0$. In view of (2.31), the strong maximum principle gives that

 $\psi_{\varepsilon_0} < \tilde{\psi} \quad \text{in} \quad \Omega_L \cap \{\tilde{\psi} < 0\} \quad \text{and} \quad \psi_{\varepsilon_0} < \tilde{\psi} \quad \text{in} \quad \Omega_L \cap \{\psi_{\varepsilon_0} > 0\}.$

Since the free boundaries $\Gamma_{\lambda_L,L}^{\varepsilon_0}$ and $\tilde{\Gamma}_{\tilde{\lambda}_L,L}$ are analytic at X_0 , thanks to Hopf's lemma, one has

$$|\nabla \psi_{\varepsilon_0}^-| = -\frac{\partial \psi_{\varepsilon_0}^-}{\partial \nu} > -\frac{\partial \tilde{\psi}^-}{\partial \nu} = |\nabla \tilde{\psi}^-| \text{ and } |\nabla \psi_{\varepsilon_0}^+| = \frac{\partial \psi_{\varepsilon_0}^+}{\partial \nu} < \frac{\partial \tilde{\psi}^+}{\partial \nu} = |\nabla \tilde{\psi}^+| \text{ at } X_0,$$

where ν is the inner normal vector to $\partial{\{\tilde{\psi} > 0\}}$ at X_0 . Those give that

$$\lambda_L = |\nabla \psi_{\varepsilon_0}^-|^2 - |\nabla \psi_{\varepsilon_0}^+|^2 > |\nabla \tilde{\psi}^-|^2 - |\nabla \tilde{\psi}^+|^2 = \tilde{\lambda}_L \text{ at } X_0,$$

which contradicts to our assumption.

Case 2. $\varepsilon_0 = 0$. We first claim that

$$\lambda_L \cdot \tilde{\lambda}_L > 0. \tag{2.32}$$

Suppose not, if $\tilde{\lambda}_L > 0 \ge \lambda_L$, by virtue of (2.3), one has

$$\tilde{k}_{\lambda_L,L}'(0+0) = \tan \theta > \tan \frac{\theta}{2} \ge k_{\tilde{\lambda}_L,L}'(0+0),$$

which implies that $k_{\lambda_L,L}(x) < \tilde{k}_{\tilde{\lambda}_L,L}(x)$ for small x > 0. This leads a contradiction to the fact $\psi \ge \tilde{\psi}$ in Ω . Similarly, we can obtain a contradiction if $\tilde{\lambda}_L = 0 > \lambda_L$.

In view of the claim (2.32), without loss of generality, we assume that $0 > \tilde{\lambda}_L > \lambda_L$ and take $X_0 = A$. Define two blow-up sequences $\{\psi_n\}$ and $\{\tilde{\psi}_n\}$ with $\psi_n(\tilde{X}) = \frac{\psi(r_n \tilde{X})}{r_n}$ and $\tilde{\psi}_n(\tilde{X}) = \frac{\tilde{\psi}(r_n \tilde{X})}{r_n}$. Let ψ_0 and $\tilde{\psi}_0$ be the blow-up limits of ψ_n and $\tilde{\psi}_n$ as $r_n \to 0$, respectively.

It follows from the similar arguments in Step 3 in the proof of Theorem 2.4 that ψ_0 and $\tilde{\psi}_0$ satisfy that

$$\psi_0(\tilde{x}, \tilde{y}) = \max\{\alpha \tilde{y}, 0\}, \quad \alpha > 0 \text{ and } \alpha^2 = -\lambda_L,$$
(2.33)

and

$$\tilde{\psi}_0(\tilde{x}, \tilde{y}) = \max\{\tilde{\alpha}\tilde{y}, 0\}, \quad \tilde{\alpha} > 0 \quad \text{and} \quad \tilde{\alpha}^2 = -\tilde{\lambda}_L.$$
(2.34)

The fact (2.31) implies that

$$\tilde{\psi}_0 \ge \psi_0 \text{ in } B_1(0) \cap \{\tilde{\psi}_0 > 0\},\$$

which together with (2.33) and (2.34) implies that

$$\sqrt{-\tilde{\lambda}_L} = \frac{\partial \tilde{\psi}_0}{\partial \nu} \ge \frac{\partial \psi_0}{\partial \nu} = \sqrt{-\lambda_L} \text{ at } 0,$$

where $\nu = (0, 1)$ is inner normal vector. This contradicts to our assumption $0 > \tilde{\lambda}_L > \lambda_L$.

Step 2. In this step, we will show that $\psi = \tilde{\psi}$. It follows from the asymptotic behavior of ψ and $\tilde{\psi}$ that

$$\lim_{x \to +\infty} k_{\lambda_L, L}(x) = \lim_{x \to +\infty} \tilde{k}_{\lambda_L, L}(x).$$

Without loss of generality, we assume that there exists $x_0 > 0$, such that

$$k_{\lambda_L,L}(x_0) < \tilde{k}_{\lambda_L,L}(x_0). \tag{2.35}$$

Consider a function $\psi_{\varepsilon}(x,y) = \psi(x,y-\varepsilon)$ for $\varepsilon \ge 0$, and choosing the smallest $\varepsilon_0 \ge 0$ such that

$$\psi_{\varepsilon_0}(X) \leq \tilde{\psi}(X)$$
 in Ω , and $\psi_{\varepsilon_0}(X_0) = \tilde{\psi}(X_0)$ for some $X_0 \in \overline{\Omega}$.

It follows from (2.35) that $\varepsilon_0 > 0$, which implies that $|X_0| < +\infty$. By using the similar arguments in Step 1, we can let X_0 be the free boundary point of ψ_{ε_0} and $\tilde{\psi}$. Applying Hopf's lemma at X_0 , one has

$$\lambda_L = |\nabla \psi_{\varepsilon_0}^-|^2 - |\nabla \psi_{\varepsilon_0}^+|^2 > |\nabla \tilde{\psi}^-|^2 - |\nabla \tilde{\psi}^+|^2 = \lambda_L \text{ at } X_0$$

which is impossible.

Hence, we obtain the uniqueness of the solution to the truncated injection flow problem 1 for any L > b.

2.5. The relation between λ_L and Q. In Lemma 2.6, the uniqueness of λ_L is obtained for any Q > 0 and L > b. Then we can denote $\lambda_L = \lambda_L(Q)$ for any Q > 0 and L > b. We investigate the relation between $\lambda_L(Q)$ and Q for any fixed L > b, and show that $\lambda_L(Q)$ is strictly monotone increasing and continuous with respect to Q > 0 for any L > b.

Lemma 2.7. For any L > b, $\lambda_L(Q)$ is strictly monotone increasing and continuous with respect to Q.

Proof. For any $Q_1 > Q_2 > 0$, there exist a unique $\lambda_L(Q_1)$ and a unique $\lambda_L(Q_2)$, such that $(\psi_{\lambda_L(Q_1),L}, \Gamma_{\lambda_L(Q_1),L})$ and $(\psi_{\lambda_L(Q_2),L}, \Gamma_{\lambda_L(Q_1),L})$ are the solutions to the truncated injection flow problem 1. We next show that

$$\lambda_L(Q_1) > \lambda_L(Q_2)$$
 for any $Q_1 > Q_2 > 0$.

If not, suppose that there exist $Q_1 > Q_2 > 0$, such that $\lambda_L(Q_1) \leq \lambda_L(Q_2)$. Denote $\psi_1 = \psi_{\lambda_L(Q_1),L}$ and $\psi_2 = \psi_{\lambda_L(Q_2),L}$ for simplicity. We next consider the following two cases.

Case 1. $\lambda_L(Q_1) = \lambda_L(Q_2) = 0$. Then ψ_1 and ψ_2 are harmonic functions in Ω_L . For the harmonic function ψ_1 in Ω_L , along the conformal mapping, it follows from (7) in pp.292 in [18] that

$$\left(\frac{U_1}{U}\right)^{1-\frac{\pi}{\theta}}\frac{d_1}{d} - \left(\frac{U_2}{U}\right)^{1-\frac{\pi}{\theta}}\frac{d_2}{d} = 1$$

where d = L - b, $d_1 = L$, $d_2 = a \sin \theta - b \cos \theta$, $U = \frac{L + Q_1}{d}$, $U_1 = \frac{L}{d_1}$ and $U_2 = \frac{Q_1}{d_2}$. Then we have

$$d_{2}^{\frac{\pi}{\theta}}Q_{1}^{1-\frac{\pi}{\theta}} = L - (L-b)^{\frac{\pi}{\theta}}(L+Q_{1})^{1-\frac{\pi}{\theta}}.$$
(2.36)

Define $f(Q) = d_2^{\frac{\pi}{\theta}}Q^{1-\frac{\pi}{\theta}} - L + (L-b)^{\frac{\pi}{\theta}}(L+Q)^{1-\frac{\pi}{\theta}}$ for Q > 0, it is easy to check that f(Q) is strictly monotone decreasing with respect to Q > 0, which implies that $Q_1 > 0$ is uniquely determined by (2.36). Similarly, for the harmonic function ψ_2 in Ω_L , one has

$$d_2^{\frac{\pi}{\theta}}Q_2^{1-\frac{\pi}{\theta}} = L - (L-b)^{\frac{\pi}{\theta}}(L+Q_2)^{1-\frac{\pi}{\theta}},$$

which implies that $Q_1 = Q_2$. This leads a contradiction.

Case 2. $\lambda_L(Q_1) \neq 0$ or $\lambda_L(Q_2) \neq 0$.

Recalling Remark 2.1, there exists a unique $h_{i,L}$ (i = 1, 2), such that

$$\lambda_L(Q_1) = \frac{Q_1^2}{(h_{1,L} - b)^2} - \frac{L^2}{(L - h_{1,L})^2}$$
 and $\lambda_L(Q_2) = \frac{Q_2^2}{(h_{2,L} - b)^2} - \frac{L^2}{(L - h_{2,L})^2}$.

Moreover, $Q_1 > Q_2$ implies that

$$h_{1,L} > h_{2,L}$$

and furthermore,

$$k_{\lambda_L(Q_1),L}(x) > k_{\lambda_L(Q_2),L}(x)$$
 for sufficiently large $x > 0$.

Define a function $\psi_{1,\varepsilon}(x,y) = \psi_1(x,y-\varepsilon)$ for $\varepsilon \ge 0$, and let $\varepsilon_0 \ge 0$ be the smallest one such that

$$\psi_{1,\varepsilon_0}(X) \leq \psi_2(X)$$
 in Ω_L and $\psi_{1,\varepsilon_0}(X_0) = \psi_2(X_0)$ for some $X_0 \in \overline{\Omega}_L$.

We consider the following two subcases.

Subcase 2.1. $\varepsilon_0 > 0$. Similar to Case 1 in the proof of Lemma 2.6, we can conclude that $X_0 \notin \Omega_L \cap (\{\psi_2 < 0\} \cup \{\psi_{1,\varepsilon_0} > 0\})$ and $|X_0| < +\infty$. Therefore, choose X_0 be a free boundary point of ψ_{1,ε_0} and ψ_2 . Thanks to Hopf's lemma, one has

$$\lambda_L(Q_1) = |\nabla \psi_{1,\varepsilon_0}^-|^2 - |\nabla \psi_{1,\varepsilon_0}^+|^2 > |\nabla \psi_2^-|^2 - |\nabla \psi_2^+|^2 = \lambda_L(Q_2) \text{ at } X_0,$$

which contradicts to our assumption $\lambda_L(Q_1) \leq \lambda_L(Q_2)$.

Subcase 2.2. $\varepsilon_0 = 0$. Along the proof of the claim (2.32), we have that $\lambda_L(Q_1) \cdot \lambda_L(Q_2) > 0$. Without loss of generality, we assume that $0 > \lambda_L(Q_2) \ge \lambda_L(Q_1)$. Taking $X_0 = A$, the strong maximum principle gives that

$$\psi_1 < \psi_2 \text{ in } \Omega \cap \{\psi_1 > 0\}.$$

We next show that

$$k_{\lambda_L(Q_1),L}(x) > k_{\lambda_L(Q_2),L}(x)$$
 for any $x > 0.$ (2.37)

If not, there exists an $x_1 \in (0, +\infty)$, such that $k_{\lambda_L(Q_1),L}(x_1) = \tilde{k}_{\lambda_L(Q_2),L}(x_1)$. Taking $X_0 = (x_1, k_{\lambda_L(Q_1),L})$ as the free boundary point, we can obtain a contradiction by using the similar arguments in Subcase 2.1,

Since N_1 is $C^{2,\alpha}$ -smooth, by using Hopf's lemma, one has

$$\frac{\partial \psi_1}{\partial \nu} < \frac{\partial \psi_2}{\partial \nu} \quad \text{on } N_1 \cap \{x < 0\}, \ \nu \text{ is the inner normal vector of } N_1.$$
(2.38)

In view of (2.37) and (2.38), for small r > 0, there exists a small $\delta > 0$, such that

$$(1+\delta)\psi_1 \le \psi_2 \text{ on } \partial(B_r(0) \cap \{\psi_1 > 0\}).$$

The maximum principle gives that

$$(1+\delta)\psi_1 \le \psi_2 \text{ in } B_r(0) \cap \{\psi_1 > 0\}.$$
 (2.39)

Define two blow-up sequences $\{\psi_{1,n}\}$ and $\{\psi_{2,n}\}$ with $\psi_{1,n}(\tilde{X}) = \frac{\psi_1(r_n X)}{r_n}$ and $\psi_{2,n}(\tilde{X}) = \psi_1(r_n X)$

 $\frac{\psi_2(r_n \tilde{X})}{r_n}$. Since $\psi_{1,n}$ and $\tilde{\psi}_{1,n}$ are Lipschitz continuous, we can denote $\psi_{1,0}$ and $\psi_{2,0}$ as the blow-up limit of $\psi_{1,n}$ and $\psi_{2,n}$, respectively. Furthermore,

$$\psi_{1,0}(\tilde{x}, \tilde{y}) = \max\{\sqrt{-\lambda_L(Q_1)}\tilde{y}, 0\}$$
 and $\psi_{2,0}(\tilde{x}, \tilde{y}) = \max\{\sqrt{-\lambda_L(Q_2)}\tilde{y}, 0\},\$

and

$$\psi_{2,0} \ge (1+\delta)\psi_{1,0}$$
 in $\{\tilde{y} > 0\}$.

This give that

$$\sqrt{-\tilde{\lambda}_L(Q_2)} = \frac{\partial \psi_{2,0}}{\partial \nu} \ge (1+\delta) \frac{\partial \psi_{1,0}}{\partial \nu} = (1+\delta) \sqrt{-\lambda_L(Q_1)} \text{ at } 0$$

where $\nu = (0,1)$ is the inner normal vector. This contradicts to our assumption $0 > \lambda_L(Q_2) \ge \lambda_L(Q_1)$.

Next, we will show that $\lambda_L(Q)$ is continuous with respect to Q. Since $\lambda_L(Q)$ is strictly monotone increasing with respect to Q, it suffices to show that

$$\lambda_L(Q+0) = \lambda_L(Q-0)$$
 for any $Q > 0$.

where $\lambda_L(Q+0) = \lim_{Q_n \to Q^+} \lambda_L(Q_n)$ and $\lambda_L(Q-0) = \lim_{Q_n \to Q^-} \lambda_L(Q_n)$.

Suppose not, then there exists a $Q_0 > 0$, such that $\lambda_L(Q_0 + 0) > \lambda_L(Q_0 - 0)$. For a sequence $\{Q_n\}$ with $Q_n \downarrow Q_0$, there exist a unique $\lambda_L(Q_n)$ and a unique solution $\psi_{\lambda_L(Q_n),L}$ to the truncated injection flow problem 1. Then there exists a subsequence $\{Q_n\}$, such that

$$\lambda_L(Q_n) \to \lambda_L(Q_0+0)$$

and

 $\psi_{\lambda_L(Q_n),L} \to \psi_{\lambda_L(Q_0+0),L}$ in $H^1_{loc}(\Omega_L)$ and uniformly in any compact subset of Ω_L .

It is easy to check that $(\psi_{\lambda_L(Q_0+0),L}, \Gamma_{\lambda_L(Q_0+0),L})$ is a solution to the truncated injection flow problem 1.

Similarly, there exists a solution $(\psi_{\lambda_L(Q_0-0),L}, \Gamma_{\lambda_L(Q_0-0),L})$ to the truncated injection flow problem 1.

For the given $Q_0 > 0$, the uniqueness of λ_L and $\psi_{\lambda_L,L}$ gives that $\lambda_L(Q_0+0) = \lambda_L(Q_0-0)$ and $\psi_{\lambda_L(Q_0+0),L} = \psi_{\lambda_L(Q_0-0),L}$, which leads a contradiction.

Next, we will obtain the upper bound and the lower bound of $\lambda_L(Q)$. It should be noted that the monotonicity of $\lambda_L(Q)$ implies that the lower bound of $\lambda_L(Q)$ follows from the limit $\lim_{Q\to 0} \lambda_L(Q)$, which means the injection flow vanishes. To see this, we have to investigate the one-phase flow above a blade surface with unit velocity in upstream. In the special case b = 0 (see Figure 2), the problem is so simple and the one-phase flow for Q = 0 is nothing but the uniform flow $(u, v) = \frac{1}{\sqrt{\rho_+}}(1, 0)$ with free boundary $\Gamma = \{0 < x < a, y = 0\}$. And it is clear that the limit $\lambda_L(Q) \to -1$ as $Q \to 0$. However, at the present situation (b > 0), the one-phase fluid problem is unclear and complicated, which is the one of main differences and difficulties here. Therefore, there is an important observation that for the limit case Q = 0, the free boundary initiates at the leading edge A and touches the boundary S_2 below the trailing edge B (see Figure 11). Based on this important observation, we will show that the limit $\lim_{Q\to 0} \lambda_L(Q)$ is not -1 but a constant $\lambda_L(0) > -1$. This is also a difference from the special case b = 0.

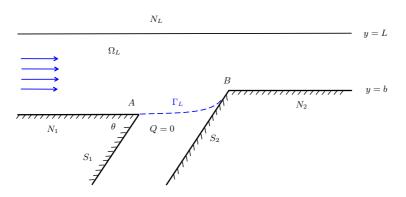


FIGURE 11. The critical case Q = 0

Lemma 2.8. For any L > b, there exist a $\lambda_L(0) \in (-1,0)$ and a $\kappa_L \in (0,+\infty)$, such that

$$\lambda_L(Q) \to \lambda_L(0) \quad as \quad Q \to 0,$$

and

$$\frac{\lambda_L(Q)}{Q^2} \to \kappa_L \quad as \quad Q \to +\infty.$$

Furthermore, if L is sufficiently large, κ_L is a uniform constant independent of L.

Proof. Step 1. The limit $Q \to 0$. For any sequence $\{Q_n\}$ with $Q_n > 0$ and $Q_n \to 0$, there exists a subsequence $\{Q_n\}$, such that

$$\lambda_L(Q_n) \to \lambda_L(0),$$

and

 $\psi_{\lambda_L(Q_n),L} \to \psi_{\lambda_L(0),L}$ in $H^1_{loc}(\Omega_L)$ and uniformly in any compact subset of Ω_L ,

as $Q_n \to 0$. Furthermore, $\psi_{\lambda_L(0),L}(x, y)$ is monotone increasing with respect to y and decreasing with respect to x. The monotonicity of $\psi_{\lambda_L(0),L}(x, y)$ implies that there exists a monotone increasing function $y = k_{\lambda_L(0),L}(x)$ for x > 0, such that

$$\Omega_L \cap \{x > 0\} \cap \{\psi_{\lambda_L(0),L} > 0\} = \Omega_L \cap \{x > 0\} \cap \{y > k_{\lambda_L(0),L}(x)\},\$$

and

$$k_{\lambda_L(0),L}(0) = 0.$$

We first claim that

$$k_{\lambda_L(0),L}(x) \equiv b \text{ for any } x \in (a, +\infty).$$
(2.40)

Suppose not, there exists an $x_0 \in (a, +\infty)$, such that $k_{\lambda_L(0),L}(x_0) = b$ and $k_{\lambda_L(0),L}(x) > b$ for any $x \in (x_0, +\infty)$. By using the asymptotic behavior of $\psi_{\lambda_L(0),L}$ in the downstream, one has

$$\lim_{x \to +\infty} k_{\lambda_L(0),L}(x) = h_0 \in (b,L) \text{ and } -\lambda_L(0) = \frac{L^2}{(L-h_0)^2}.$$
 (2.41)

It follows from the results in Section 9 in [3] and Section 11 in Chapter 3 in [12] that the continuous fit condition implies the smooth fit condition, namely, $N_1 \cup \Gamma_{\lambda_L(0),L}$ is C^1 smooth at A and $k'_{\lambda_L(0),L}(0+0) = 0$. Furthermore, $\nabla \psi_{\lambda_L(0),L}$ is uniformly continuous in a $\{\psi_{\lambda_L(0),L} > 0\}$ -neighborhood of A. Define $\omega(y) = \max\{y, 0\}$ for $y \in (-\infty, L)$, it is easy to check that $\omega(y) \ge \psi_{\lambda_L(0), L}(x, y)$ in Ω_L . In view of $\omega(0) = \psi_{\lambda_L(0), L}(0, 0) = 0$, one has

$$1 = \frac{\partial \omega}{\partial \nu} \ge \frac{\psi_{\lambda_L(0),L}}{\partial \nu} = \sqrt{-\lambda_L(0)} \text{ at } A,$$

where $\nu = (0, 1)$ is the inner normal vector. This contradicts to the fact that $\lambda_L(0) = L^2$

$$-\frac{1}{(L-h_0)^2} < -1 \text{ in } (2.41).$$

Next, we will show that

$$\lambda_L(0) > -1. \tag{2.42}$$

If not, we assume that $\lambda_L(0) \leq -1$. For any small r > 0, it follows from the proof of (2.39) that there exists a small $\delta > 0$, such that

$$\omega \ge (1+\delta)\psi_{\lambda_L(0),L} \quad \text{in} \quad B_r(A) \cap \{\psi_{\lambda_L(0),L} > 0\},\$$

which gives that

$$1 = \frac{\partial \omega}{\partial \nu} \ge (1+\delta) \frac{\psi_{\lambda_L(0),L}}{\partial \nu} = (1+\delta) \sqrt{-\lambda_L(0)} \text{ at } A.$$

This contradicts to our assumption $\lambda_L(0) \leq -1$.

Moreover, we will show that

$$\Gamma_{\lambda_L(0),L} \cap N_2 = \emptyset. \tag{2.43}$$

If not, it follows from (2.40) that $k_{\lambda_L(0),L}(a) = b$. Similarly, we have that $N_2 \cup \Gamma_{\lambda_L(0),L}$ is C^1 -smooth at B and $\nabla \psi_{\lambda_L(0),L}$ is uniformly continuous in a $\{\psi_{\lambda_L(0),L} > 0\}$ -neighborhood of B. Define $\omega_1(y) = \frac{L}{L-b} \max\{y-b,0\}$, it is easy to check that $\omega_1(y) \leq \psi_{\lambda_L(0),L}(x,y)$ in Ω_L , and thus

$$1 < \frac{L}{L-b} = \frac{\partial \omega_1}{\partial \nu} \le \frac{\psi_{\lambda_L(0),L}}{\partial \nu} = \sqrt{-\lambda_L(0)} \text{ at } B,$$

where $\nu = (0, 1)$ is the inner normal vector. This contradicts to (2.42).

Since $\lambda_L(Q)$ is strictly decreasing with respect to Q, we can obtain the uniqueness of $\lambda_L(0)$.

Step 2. The limit $Q \to +\infty$. We will show that there exists a positive constant κ_L , such that

$$rac{\lambda_L(Q)}{Q^2} o \kappa_L \quad ext{ as } \ Q o +\infty.$$

For any fixed L > b, set $\psi_Q = \frac{\psi_{\lambda_L(Q),L}}{Q}$ and $\lambda_Q = \frac{\lambda_L(Q)}{Q^2}$. Then ψ_Q solves the following free boundary value problem

$$\begin{split} \Delta \psi_Q &= 0 \text{ in } \Omega_L \cap \{\psi_Q < 0\}, \quad \Delta \psi_Q = 0 \text{ in } \Omega_L \cap \{\psi_Q > 0\}, \\ |\nabla \psi_Q^-|^2 - |\nabla \psi_Q^+|^2 &= \lambda_Q \quad \text{on } \Gamma_{\lambda_L(Q),L}, \\ \psi_Q &= 0 \text{ on } N_1 \cup S_1 \cup \Gamma_{\lambda_L(Q),L}, \ \psi_Q &= -1 \text{ on } N_2 \cup S_2, \quad \psi_Q = \frac{L}{Q} \text{ on } N_L. \end{split}$$

By virtue of non-degeneracy Theorem 3.1 in [4], we have that if $\lambda_L(Q) > 0$, then

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \psi_{\lambda_L(Q),L}^- dS \le c\sqrt{\lambda_L(Q)} \text{ implies } \psi_{\lambda_L(Q),L} \equiv 0 \text{ in } B_{\frac{r}{2}}(X_0), \tag{2.44}$$

and if $\lambda_L(Q) < 0$, then

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \psi^+_{\lambda_L(Q),L} dS \le c\sqrt{-\lambda_L(Q)} \text{ implies } \psi_{\lambda_L(Q),L} \equiv 0 \text{ in } B_{\frac{r}{2}}(X_0), \qquad (2.45)$$

for any disc $B_r(X_0) \subset \Omega_L$ with $B_{\frac{r}{2}}(X_0) \subset \Omega_L \cap \{(x,y) \mid x > 0, y > 0\}$. Here, c > 0is a constant independent of $\lambda_L(Q)$ and L. Therefore, there exists a constant $r_0 > 0$ independent of $\lambda_L(Q)$ and L, such that $B_{r_0}(X_0) \subset \Omega_L$ with $B_{\frac{r_0}{2}}(X_0) \subset \Omega_L \cap \{(x,y) \mid x > 0, y > 0\}$ and $B_{\frac{r_0}{2}}(X_0) \cap \Gamma_{\lambda_L(Q),L} \neq \emptyset$, and it follows from (2.44) and (2.45) that

$$\frac{Q}{r_0} \ge \frac{1}{r_0} \left| \int_{\partial B_{r_0}(X_0)} \psi_{\lambda_L(Q),L} dS \right| \ge c |\lambda_L(Q)|^{\frac{1}{2}},$$

for any Q > 0. This implies that

 $|\lambda_Q| \le C, \quad C > 0 \text{ is a constant independent of } Q \text{ and } L.$ (2.46)

For any sequence $\{Q_n\}$ with $Q_n \to +\infty$, there exists a subsequence $\{Q_n\}$, such that

$$\lambda_{Q_n} \to \kappa_L,$$

and

 $\psi_{Q_n} \to \bar{\psi}_{\kappa_L}$ in $H^1_{loc}(\Omega_L)$ and uniformly in any compact subset of Ω_L ,

as $Q_n \to +\infty$. The monotonicity of $\psi_{\lambda_L(Q),L}(x, y)$ with respect to x and y gives that $\bar{\psi}_{\kappa_L}(x, y)$ is monotone increasing with respect to y and decreasing with respect to x.

Since $0 \leq \psi_Q \leq \frac{L}{Q}$ in $\Omega_L \cap \{(x,y) \mid x \leq 0, y \geq 0\}$, we have that $\bar{\psi}_{\kappa_L} = 0$ in $\Omega_L \cap \{(x,y) \mid x \leq 0, y \geq 0\}$. Denote $E_L = \Omega_L \setminus (\{(x,y) \mid x \leq 0, y \geq 0\})$, then $\bar{\psi}_{\kappa_L}$ is a solution of the following free boundary value problem

$$\Delta \bar{\psi}_{\kappa_L} = 0 \text{ in } E_L \cap \{ \bar{\psi}_{\kappa_L} < 0 \},$$

$$\bar{\psi}_{\kappa_L} = 0, \quad \left| \frac{\partial \bar{\psi}_{\kappa_L}}{\partial \nu} \right|^2 = \kappa_L \quad \text{on } \Gamma_{\kappa_L},$$

$$\bar{\psi}_{\kappa_L} = 0 \text{ on } N_L^+ \cup I_L \cup S_1, \quad \bar{\psi}_{\kappa_L} = -1 \text{ on } N_2 \cup S_2,$$

$$(2.47)$$

where $N_L^+ = N_L \cap \{x \ge 0\}$, $I_L = \{(0, y) \mid 0 \le y \le L\}$ and $\Gamma_{\kappa_L} = E_L \cap \partial \{\bar{\psi}_{\kappa_L} < 0\}$ is the free boundary of $\bar{\psi}_{\kappa_L}$. Furthermore, the free boundary Γ_{κ_L} is C^1 -smooth at the initial point A, and which is given by

$$\Gamma_{\kappa_L} = \{(x, y) \mid x = g_{\kappa_L}(y), 0 < y < h_L\}, \ g_{\kappa_L}(y) \text{ is increasing with respect to } y,$$

where either $h_L < L$, $g_{\kappa_L}(h_L - 0) = +\infty$ or $h_L = L$, $g_{\kappa_L}(h_L - 0) \le +\infty$.

We first show that

$$\kappa_L > 0. \tag{2.48}$$

Suppose that $\kappa_L = 0$. By virtue of (4.6) in [5], for any free boundary point X_0 and $\varepsilon > 0$, we have

$$\frac{1}{r} \left| \oint_{\partial B_r(X_0)} \psi_{Q_n} dS \right| \le C |\lambda_{Q_n}|^{\frac{1}{2}}, \tag{2.49}$$

if $B_r(X_0) \subset \Omega_L \cap \{\varepsilon < y < L - \varepsilon\}$ and *n* is sufficiently large, where C > 0 is a constant depending only on ε . Taking $Q_n \to +\infty$ in (2.49) yields that

$$\frac{1}{r} \left| \oint_{\partial B_r(X_0)} \bar{\psi}_{\kappa_L} dS \right| \le C |\kappa_L|^{\frac{1}{2}} = 0$$

which together with $\bar{\psi}_{\kappa_L} \equiv 0$ in $\Omega_L \cap \{x < 0, y > 0\}$ implies that

$$\bar{\psi}_{\kappa_L} \equiv 0$$
 in $\{(x, y) \mid 0 < x < \varepsilon, \varepsilon < y < L - \varepsilon\}.$

By using the unique continuation, we can conclude that $\psi_{\kappa_L} \equiv 0$ in E_L , which contradicts to the fact $\bar{\psi}_{\kappa_L} = -1$ on N_2 .

Finally, we will investigate the relation between κ_L and h_{κ_L} , where $h_{\kappa_L} \in (b, L]$ is the asymptotic height of the free boundary Γ_{κ_L} . Consider the following two cases.

Case 1. $h_{\kappa_L} < L$ and $g_{\kappa_L}(h_{\kappa_L} - 0) = +\infty$. (See Figure 12)

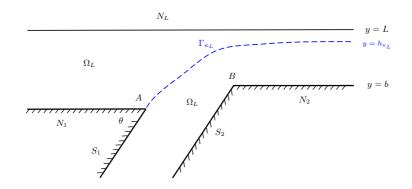


FIGURE 12. Case 1

Similar to Step 2 in the proof of Theorem 2.4, we can obtain that

$$\kappa_L = \frac{1}{(h_{\kappa_L} - b)^2}.$$
 (2.50)

Case 2. $h_{\kappa_L} = L$ and $g_{\kappa_L}(L-0) \in (0, +\infty)$. (See Figure 13).

Denote $X_0 = (g_{\kappa_L}(L-0), L)$. Similarly, we have that $N_L \cup \Gamma_{\kappa_L}$ is C^1 -smooth at X_0 and its tangent is in the direction of positive *x*-axis. Moreover, $\nabla \bar{\psi}_{\kappa_L}$ is uniformly continuous in a $\{\bar{\psi}_{\kappa_L} < 0\}$ -neighborhood of X_0 . Define $\omega(y) = \frac{1}{L-b} \max\{y-b,0\} - 1$, it is easy to check that

$$\bar{\psi}_{\kappa_L}(x,y) \ge \omega(y)$$
 in E_L ,

which implies that

$$\sqrt{\kappa_L} = \frac{\partial \psi_{\kappa_L}}{\partial \nu} \le \frac{\partial \omega}{\partial \nu} = \frac{1}{L-b}$$
 at X_0 ,

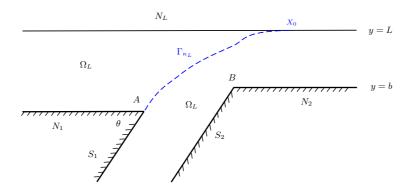


FIGURE 13. Case 2

where $\nu = (0, 1)$ is the outer normal vector. This implies that

$$\kappa_L \le \frac{1}{(L-b)^2}.\tag{2.51}$$

By using the similar arguments in the proof of Lemma 2.6, we can obtain the uniqueness of κ_L and $\bar{\psi}_{\kappa_L}$ to the free boundary problem (2.47). Hence, one has

$$\frac{\lambda_L(Q)}{Q^2} \to \kappa_L,$$

and

$$\frac{\psi_{\lambda_L(Q),L}}{Q} \to \bar{\psi}_{\kappa_L}$$
 uniformly in E_L ,

as $Q \to +\infty$.

Step 3. Finally, we will show that κ_L is a uniform constant for any large L, namely, there exists a L_0 , such that $\kappa_{L_1} = \kappa_{L_2}$ for any $L_2 > L_1 > L_0$. It follows from (2.46) that there exists a positive constant C_2 independent of L, such that

$$\kappa_L \le C_2. \tag{2.52}$$

By using the bounded gradient lemma 5.1 in Chapter 3 in [12], one has

$$|\nabla \bar{\psi}_{\kappa_L}| \le C \sqrt{\kappa_L} \quad \text{in} \quad D \subset \Omega_L, \tag{2.53}$$

where $D \cap \Gamma_{\kappa_L} \neq \emptyset$ and the constant C depends only on D. Denote $D = \Omega_L \cap B_{2a}(0)$, it is easy to check that $D \cap \Gamma_{\kappa_L} \neq \emptyset$. Then there exist two points $X_1 \in \overline{D} \cap S_2$ and $X_2 \in D \cap \Gamma_{\kappa_L}$, such that $X_t = tX_1 + (1-t)X_2 \in D$ for any $t \in (0,1)$. It follows from (2.53) that

$$1 = \bar{\psi}_{\kappa_L}(X_2) - \bar{\psi}_{\kappa_L}(X_1) \le |\nabla \bar{\psi}_{\kappa_L}(X_{t_0})| |X_1 - X_2| \le C\sqrt{\kappa_L},$$

for $t_0 \in (0, 1)$, where C is a constant independent of L. This implies that there exists a positive constant C_1 independent of L, such that

$$\kappa_L \ge C_1 > 0. \tag{2.54}$$

It follows from (2.50)-(2.54) that

$$h_{\kappa_L} \le b + \frac{1}{\sqrt{\kappa_L}} \le L_0, \tag{2.55}$$

where L_0 is a constant independent of L.

Suppose that there exist two solutions $(\bar{\psi}_{\kappa_{L_1}}, \kappa_{L_1})$ and $(\bar{\psi}_{\kappa_{L_2}}, \kappa_{L_2})$ to the free boundary problem (2.47), with $L_2 > L_1 > L_0$.

By virtue of (2.55), we have

the free boundary of $\bar{\psi}_{\kappa_{L_1}}$ lies below $\{y = L_1\},\$

and

the free boundary of
$$\bar{\psi}_{\kappa_{L_2}}$$
 lies below $\{y = L_1\}$.

Applying the similar arguments in the proof of Lemma 2.6, we can obtain that $\bar{\psi}_{\kappa_{L_1}} = \bar{\psi}_{\kappa_{L_2}}$ and $\kappa_{L_1} = \kappa_{L_2}$.

Remark 2.2. By virtue of Lemma 2.8, there exists a constant $\kappa \in (0, +\infty)$, such that

$$\frac{\lambda_L(Q)}{Q^2} \to \kappa \text{ and } \frac{\psi_{\lambda_L(Q),L}}{Q} \to \bar{\psi}_{\kappa} \text{ uniformly in } E_L,$$

as $Q \to +\infty$, for any $L > L_0$, where E_L is defined as in (2.47).

Next, we will give the uniform estimate of the asymptotic height h_L of the free boundary.

Lemma 2.9. For any Q > 0, there exists a positive constant C independent of L, such that

$$h_L \leq C$$
,

where h_L is the asymptotic height of the free boundary of $\psi_{\lambda_L(Q),L}$.

Proof. Suppose not, we assume that there exists a sequence $\{L_n\}$ with $L_n \to +\infty$, such that $h_{L_n} \to +\infty$. Note that

$$\lambda_{L_n}(Q) = \frac{Q^2}{(h_{L_n} - b)^2} - \frac{L_n^2}{(L_n - h_{L_n})^2}.$$
(2.56)

Denote $(\psi_{\lambda_{L_n}(Q),L_n}, \lambda_{L_n}(Q))$ as the corresponding solution to the truncated injection flow problem 1 for any Q > 0. By virtue of (2.46), there exists a subsequence $\{L_n\}$, such that

$$\lambda_{L_n}(Q) \to \lambda,$$

and

 $\psi_{\lambda_{L_n}(Q),L_n} \to \psi_{\lambda}$ in $H^1_{loc}(\Omega)$ and uniformly in any compact subset of Ω ,

as $L_n \to +\infty$. Moreover, $\psi_{\lambda}(x, y)$ is monotone increasing with respect to y and decreasing with respect to x, which implies that the free boundary of ψ_{λ} can be denoted as

$$\Gamma_{\lambda} = \Omega \cap \{x > 0\} \cap \{\psi_{\lambda} = 0\} : y = k_{\lambda}(x) \text{ for any } x > 0$$

Here, $k_{\lambda}(x)$ is continuous and strictly monotone increasing with respect to x, $k_{\lambda}(0) = 0$ and $k_{\lambda}(x) \to +\infty$ as $x \to +\infty$. It follows from (2.56) that

$$\lambda \le -1. \tag{2.57}$$

Furthermore, the free boundary Γ_{λ} is continuous differentiable at A, namely, $k'_{\lambda}(0+0) = 0$.

In view of the condition (7) in Definition 2.1, one has

$$\psi_{\lambda_{L_n}(Q),L_n}(x,y) \le \max\{y,0\}$$
 in Ω_{L_n}

The strong maximum principle gives that

$$\psi_{\lambda}(x,y) < y \text{ in } \Omega \cap \{\psi_{\lambda} > 0\}.$$

Therefore, for small r > 0, there exists a small $\delta > 0$, such that

$$\max\{y,0\} \ge (1+\delta)\psi_{\lambda} \text{ on } \partial(B_r(0) \cap \{\psi_{\lambda} > 0\}).$$

It follows from the maximum principle that

$$\max\{y,0\} \ge (1+\delta)\psi_{\lambda} \quad \text{in} \quad B_r(0) \cap \{\psi_{\lambda} > 0\}.$$

$$(2.58)$$

Define a blow-up sequence $\tilde{\psi}_n(\tilde{X}) = \frac{\psi_\lambda(r_n\tilde{X})}{r_n}$ with $r_n \to 0$, it follows from (2.58) that

$$\max\{\tilde{y}, 0\} \ge (1+\delta)\tilde{\psi}_n(\tilde{X}) \text{ in } B_1(0) \cap \{\psi_n > 0\}.$$
(2.59)

Denote $\tilde{\psi}_0$ as the blow-up limit of $\tilde{\psi}_n$, it follows from (2.59) and the similar arguments in the proof of Theorem 2.4 that

$$\tilde{\psi}_0(\tilde{X}) = \max\{\sqrt{-\lambda}\tilde{y}, 0\} \text{ and } \max\{\tilde{y}, 0\} \ge (1+\delta)\tilde{\psi}_0(\tilde{X}) \text{ in } B_1(0) \cap \{\psi_0 > 0\}.$$

This gives that

$$1 \ge (1+\delta)\frac{\partial \tilde{\psi}_0}{\partial \nu} = (1+\delta)\sqrt{-\lambda}$$
 at 0

where $\nu = (0, 1)$ is the inner normal vector. It leads a contradiction with (2.57).

3. The proof of The main results

Based on the results in previous sections, we will complete the proof of Theorem 1.1 -Theorem 1.3 in this section.

Theorem 3.1. For any Q > 0, there exist a unique $\lambda > -1$ and a unique solution $(\psi_{\lambda}, \Gamma_{\lambda})$ to the injection flow problem 1.

Proof. Step 1. It follows from (2.46) that there exists a positive constant C independent of Q and L, such that

$$|\lambda_L| \leq CQ^2$$
 for any $L > b$ and $Q > 0$.

By virtue of Lemma 2.9, one has

$$h_L \le C$$
 and $\lambda_L = \frac{Q^2}{(h_L - b)^2} - \frac{L^2}{(L - h_L)^2} = \frac{Q^2}{(h_L - b)^2} - \frac{1}{\left(1 - \frac{h_L}{L}\right)^2}.$

Then there exist a sequence $\{L_n\}$, a constant λ and a h > b, such that

$$\lambda_{L_n} \to \lambda, \quad h_{L_n} \to h,$$

and

$$\psi_{\lambda_{L_n},L_n} \to \psi_{\lambda}$$
 in $H^1_{loc}(\Omega)$ and uniformly in any compact subset of Ω ,

as $L_n \to +\infty$. Obviously, $\lambda = \frac{Q^2}{(h-b)^2} - 1$. By using the similar arguments in Lemma 6.2 in [4], we can show that ψ_{λ} is a local minimizer to the variational problem (P_{λ}) , namely,

$$P_{\lambda}: J_D(\psi_{\lambda}) = \min J_D(\psi) \text{ for any } \psi \in K \text{ and } \psi = \psi_{\lambda} \text{ on } \partial D,$$

where

$$J_D(\psi) = \int_D \left| \nabla \psi - (\lambda_1 I_{\{\psi < 0\}} + \lambda_2 I_{\{\psi > 0\}} + \lambda_0 I_{\{\psi = 0\}}) I_{\{x > 0\}} e \right|^2 dx dy$$

for any bounded domain $D \subset \Omega$, where $\lambda_1 = \frac{Q}{h-b}$ and $\lambda_2 = 1$.

Step 2. Since ψ_{λ} is a local minimizer, we can conclude that ψ_{λ} is a harmonic in $\Omega \setminus \Gamma$. Moreover, the free boundary $\Gamma_{\lambda} : y = k_{\lambda}(x)$ of ψ_{λ} satisfies the continuous fit condition $k_{\lambda}(0) = 0$ and the smooth fit condition (1.5), where $k_{\lambda}(x)$ is continuous and strictly monotone increasing with respect to x, and $k_{\lambda}(x) \to h$ as $x \to +\infty$. It follows from the condition (7) in Definition 2.1 that

$$y - h \le \psi_{\lambda}^+(x, y) \le y \quad \text{in } \Omega \cap \{y > 0\}.$$

$$(3.1)$$

Hence, the conditions (1)-(5) and (7) in Definition 1.1 hold.

Step 3. In this step, we will verify the condition (6) in Definition 1.1. Denote $\phi(x,y) = \psi_{\lambda}(x,y) - y$ and $\phi_n(x,y) = \phi(x-n,y)$, it follows from (3.1) that

$$\Delta \phi_n = 0 \quad \text{and} \quad -h \le \phi_n \le 0 \quad \text{in} \quad \{x < n, y > 0\}$$

By using the elliptic estimate, there exists a subsequence $\{\phi_n\}$, such that

$$\phi_n \to \phi_0 \text{ in } \{-\infty < x < +\infty, 0 < y < +\infty\},\$$

and ϕ_0 satisfies

 $\Delta \phi_0 = 0$ and $-h \le \phi_0 \le 0$ in $\{-\infty < x < +\infty, 0 < y < +\infty\}$, and $\phi_0(x, 0) = 0$. Then $\phi_0 \equiv 0$ in $\{-\infty < x < +\infty, 0 < y < +\infty\}$, and thus

$$\psi_{\lambda}(x,y) \to y$$
 uniformly in any compact subset of $(0, +\infty)$, as $x \to -\infty$, (3.2)

Along the similar arguments in the proof of (2.24), one has

$$\left|\psi_{\lambda}(x,y) - \frac{Q(y\cos\theta - x\sin\theta)}{a\sin\theta - b\cos\theta}\right| \to 0 \quad \text{uniformly in any compact subset of } S, \qquad (3.3)$$

as $y \to -\infty$, where $S = \{(x, y) \mid y \cot \theta < x < (y - b) \cot \theta + a, -\infty < y < +\infty\}.$

Next, we consider the asymptotic behavior of ψ_{λ} in the downstream. For any blow-up sequence $\psi_n(x,y) = \psi_{\lambda_L,L}(x+n,y)$ for $x > -\frac{n}{2}$, such that

 $\psi_n(x,y) \to \psi_0(x,y)$ uniformly in any compact subset of $(0,+\infty)$, as $x \to +\infty$,

and ψ_0 satisfies

$$\Delta \psi_0 = 0 \text{ in } \mathbb{R}^2_b \setminus \{y = h\},$$

$$\psi_0(x, b) = -Q \text{ and } \psi_0(x, h) = 0 \text{ for } -\infty < x < +\infty,$$

$$0 \le \psi_0(x, y) \le h \text{ in } \{-\infty < x < +\infty\} \times \{0 < y < h\}$$

where $\mathbb{R}_{b}^{2} = \{(x, y) \mid -\infty < x < +\infty, y > b\}.$

Then one has that $\{y = h\}$ is the free boundary of ψ_0 and $\psi_0 = \frac{Q(y-b)}{h-b} - Q$ in $\{-\infty < x < +\infty\} \times \{0 < y < h\}$. In view of the condition (3) in Definition 1.1, we can conclude that $\frac{\partial \psi_0(x, h+0)}{\partial y} = 1$, and thus $\psi_0(x, y) = y - h$ in $\{-\infty < x < +\infty\} \times \{y > h\}$. Therefore, the boundary value problem above possesses a unique solution

$$\psi_0(x,y) = \begin{cases} \frac{Q(h-y)}{h-b}, & \text{if } b < y < h, \\ y-h, & \text{if } h < y < +\infty. \end{cases}$$

Finally, we will verify the convergence of $\nabla \psi_{\lambda}$ in the far field. For any sequence $X_n = (x_n, y_n) \in \Omega \cap \{\psi_{\lambda} > 0\}$ with $\rho_n = |X_n| \to +\infty$, we next consider the following two cases.

Case 1. $y_n > \varepsilon |x_n|$ for $\varepsilon > 0$, or $x_n < 0$ and $\frac{y_n}{x_n} \to 0$. Define $\tilde{Y}_n = \frac{X_n}{\rho_n}$ and a blow-up sequence

$$\tilde{\psi}_{\rho_n}(\tilde{X}) = \frac{\psi_\lambda(\rho_n X)}{\rho_n}$$

Then one has

 $\tilde{Y}_n \to \tilde{Y}_0 = (\tilde{x}_0, \tilde{y}_0)$ and $\tilde{\psi}_{\rho_n} \to \tilde{\phi}$ uniformly in any compact subset of $\mathbb{R}^2 \cap \{\tilde{y} > 0\}$. By virtue of (3.1), one has

$$\tilde{y} - \frac{h}{\rho_n} \le \tilde{\psi}_{\rho_n}(\tilde{X}) \le \tilde{y} \text{ in } \mathbb{R}^2 \cap \{\psi_{\rho_n} > 0\},\$$

which implies that $\tilde{\phi}(\tilde{X}) = \max{\{\tilde{y}, 0\}}.$

If $y_n > \varepsilon |x_n|$ for $\varepsilon > 0$, it is easy to check that

$$|\tilde{Y}_0| = 1$$
 and $\tilde{y}_0 \ge \frac{\varepsilon}{\sqrt{1 + \varepsilon^2}}$.

By virtue of elliptic regularity, one has

$$\tilde{\psi}_{\rho_n} \to \tilde{\phi} \text{ in } C^{2,\alpha}(B_r(\tilde{Y})), \ \alpha \in (0,1),$$

for $0 < r < \frac{\varepsilon}{4\sqrt{1+\varepsilon^2}}$. Thus one has

$$\nabla \tilde{\psi}_{\rho_n}\left(\frac{X_n}{\rho_n}\right) \to \nabla \tilde{\phi}(\tilde{Y}_0) = (0,1).$$

If $x_n < 0$ and $\frac{y_n}{x_n} \to 0$, one has

$$\tilde{Y}_0 = (-1, 0).$$

Applying elliptic estimates, one has

$$\tilde{\psi}_{\rho_n} \to \tilde{\phi} \text{ in } C^{2,\alpha}(D \cup T), \ \alpha \in (0,1),$$

where $D = B_{2r}(\tilde{Y}) \cap \{\tilde{y} > 0\}$ and $T = \{(\tilde{x}, 0) \mid |\tilde{x} + 1| < r\}$ for r > 0. Consequently,

$$\nabla \tilde{\psi}_{\rho_n}\left(\frac{X_n}{\rho_n}\right) \to \nabla \tilde{\phi}(\tilde{Y}_0) = (0,1).$$

Case 2. $x_n > 0$ and $\frac{y_n}{x_n} \to 0$ and $y_n - k_\lambda(x_n) \to +\infty$ as $n \to +\infty$. Define a

blow-up sequence $\psi_{r_n}(\tilde{X}) = \frac{\psi_{\lambda}(Z_n + r_n \tilde{X})}{r_n}$ and ψ_0 is the blow-up limit of ψ_{r_n} , where $r_n = y_n - k_{\lambda}(x_n)$ and $Z_n = (x_n, k_{\lambda}(x_n))$. The inequality (3.1) gives that

$$\tilde{y} + \frac{k_{\lambda}(x_n) - h}{r_n} \le \psi_{r_n}^+(\tilde{X}) \le \tilde{y} + \frac{k_{\lambda}(x_n)}{r_n} \quad \text{in} \quad \{\tilde{y} > 0\},$$

which implies that

$$\psi_0(\tilde{X}) = \max\{\tilde{y}, 0\} \text{ in } \{\tilde{y} > 0\}.$$

Since $-Q \leq \psi_{\lambda} < 0$ in $\Omega \cap \{\psi_{\lambda} < 0\}$, which implies that $\psi_0^- = 0$. Therefore, ψ_0 is 1-plane solution, and $\psi_0(\tilde{X}) = \max\{\tilde{y}, 0\}$. The elliptic regularity gives that

$$\psi_{r_n} \to \psi_0$$
 in $C^{2,\alpha}(B_{\frac{1}{4}}(X_1)), \ \alpha \in (0,1), \ X_1 = (0,1).$

Thus one has

$$\nabla \psi_{\lambda}(X_n) = \nabla \psi_{r_n}(X_1) \to (0,1) \text{ as } n \to +\infty.$$

This gives that $\nabla \psi_{\lambda}(x,y) \to \nabla \psi(X_1) = (0,1)$ as $x^2 + y^2 \to +\infty$ with $\operatorname{dist}((x,y),\Gamma) \to +\infty$ and x > 0.

Step 2. In this step, we will obtain the uniqueness of the injection flow problem 1. Suppose that there exists another different solution $(\tilde{\psi}_{\tilde{\lambda}}, \tilde{\lambda})$ to the injection flow problem 1. In view of (1.6), one has

$$\psi_{\lambda}(X) - y = o(|X|), \quad \psi_{\lambda}(X) > 0, \text{ as } |X| \to +\infty,$$
(3.4)

and

$$\tilde{\psi}_{\tilde{\lambda}}(X) - y = o(|X|), \quad \tilde{\psi}_{\tilde{\lambda}}(X) > 0, \text{ as } |X| \to +\infty.$$
(3.5)

Without loss of generality, we assume that $\lambda \leq \tilde{\lambda}$. It is easy to check that

$$\lim_{x \to +\infty} k_{\lambda}(x) = h = \frac{Q}{\sqrt{1+\lambda}} + b \ge \frac{Q}{\sqrt{1+\tilde{\lambda}}} + b = \tilde{h} = \lim_{x \to +\infty} \tilde{k}_{\tilde{\lambda}}(x).$$
(3.6)

Define $\psi_{\lambda,\varepsilon}(x,y) = \psi_{\lambda}(x,y-\varepsilon)$ for any $\varepsilon \ge 0$. Since the asymptotic heights of the free boundaries Γ_{λ} and $\tilde{\Gamma}_{\tilde{\lambda}}$ are finite, it follows from (3.6) that we can take $\varepsilon_0 \ge 0$ to be the smallest one, such that

the free boundary of $\psi_{\lambda,\varepsilon_0}$ lies above the free boundary of $\psi_{\tilde{\lambda}}$. (3.7)

Denote $\Omega^+ = \Omega \cap \{\psi_{\lambda,\varepsilon_0} > 0\} \cap \{\tilde{\psi}_{\tilde{\lambda}} > 0\}$ and $\omega(X) = \tilde{\psi}_{\tilde{\lambda}} - \psi_{\lambda,\varepsilon_0}$, it follows from (3.4), (3.5) and (3.7) that

$$\omega(X) \ge 0 \text{ on } \partial \Omega^+ \text{ and } \lim_{r \to +\infty} \frac{m(r)}{r} \to 0,$$

where r = |X| and $m(r) = \min_{|X|=r} \omega(X)$. Applying the Phragmèn-Lindelöf theorem in [15], one has

$$\omega(X) \ge 0$$
 in Ω^+ ,

which implies that

$$\tilde{\psi}_{\tilde{\lambda}} \ge \psi_{\lambda,\varepsilon_0} \quad \text{in } \Omega \cap \{\psi_{\lambda,\varepsilon_0} > 0\}.$$
(3.8)

By virtue of the asymptotic behavior of $\psi_{\lambda,\varepsilon_0}$ and $\tilde{\psi}_{\tilde{\lambda}}$, it follows from the similar arguments in the step 4 in the proof of Theorem 2.4 that

$$\tilde{\psi}_{\tilde{\lambda}} \ge \psi_{\lambda,\varepsilon_0} \quad \text{in } \Omega \cap \{\tilde{\psi}_{\tilde{\lambda}} < 0\}.$$
(3.9)

Next, we consider two cases in the following.

Case 1. $\varepsilon_0 > 0$. In view of (3.7), we can take a free boundary point X_0 with $|X_0| < +\infty$. Applying the strong maximum principle, one has

$$\tilde{\psi}_{\tilde{\lambda}} > \psi_{\lambda,\varepsilon_0} \text{ in } \Omega \cap \{\psi_{\lambda,\varepsilon_0} > 0\} \text{ and } \tilde{\psi}_{\tilde{\lambda}} > \psi_{\lambda,\varepsilon_0} \text{ in } \Omega \cap \{\tilde{\psi}_{\tilde{\lambda}} < 0\}.$$

Since the free boundary $\tilde{\Gamma}_{\tilde{\lambda}}$ and $\Gamma_{\lambda}^{\varepsilon_0}$ are analytic at X_0 , it follows from Hopf's lemma that

$$|\nabla \psi_{\lambda,\varepsilon_0}^-| = -\frac{\partial \psi_{\lambda,\varepsilon_0}^-}{\partial \nu} > -\frac{\partial \tilde{\psi}_{\tilde{\lambda}}^-}{\partial \nu} = |\nabla \tilde{\psi}_{\tilde{\lambda}}^-| \text{ and } |\nabla \psi_{\lambda,\varepsilon_0}^+| = \frac{\partial \psi_{\lambda,\varepsilon_0}^+}{\partial \nu} < \frac{\partial \tilde{\psi}_{\tilde{\lambda}}^+}{\partial \nu} = |\nabla \tilde{\psi}_{\tilde{\lambda}}^+| \text{ at } X_0,$$

where ν is the inner normal vector to $\partial \{\tilde{\psi}_{\tilde{\lambda}} > 0\}$ at X_0 . Those give that

$$\lambda = |\nabla \psi_{\lambda,\varepsilon_0}^-|^2 - |\nabla \psi_{\lambda,\varepsilon_0}^+|^2 > |\nabla \tilde{\psi}_{\tilde{\lambda}}^-|^2 - |\nabla \tilde{\psi}_{\tilde{\lambda}}^+|^2 = \tilde{\lambda} \quad \text{at} \quad X_0,$$

which contradicts to our assumption $\lambda \leq \lambda$.

Case 2. $\varepsilon_0 = 0$. Similar to (2.32), we can show that $\lambda \cdot \tilde{\lambda} > 0$. Without loss of generality, one assume that $0 > \tilde{\lambda} \ge \lambda$. Therefore, we can obtain a contradiction by using the similar arguments in Subcase 2.1 in the proof of Lemma 2.5.

By virtue of Theorem 3.1, we complete the proof of Theorem 1.1.

Due to the uniqueness of λ for any Q > 0, we can define a function $\lambda = \lambda(Q)$ for any Q > 0. We next consider the relation between $\lambda(Q)$ and Q > 0, and complete the proof of Theorem 1.2.

Proof of Theorem 1.2. (1). For any $Q_1 > Q_2 > 0$, there exist two solutions $(\psi_{\lambda(Q_1)}, \lambda(Q_1))$ and $(\psi_{\lambda(Q_2)}, \lambda(Q_2))$ to the injection flow problem 1. We next show that

$$\lambda(Q_1) > \lambda(Q_2) \quad \text{for any } Q_1 > Q_2 > 0.$$

If not, then there exist $Q_1 > Q_2 > 0$, such that $\lambda(Q_1) \leq \lambda(Q_2)$, and we consider the following two cases.

Case 1. $\lambda(Q_1) = \lambda(Q_2) = 0$. Since $\lambda(Q_1)$ and $\psi_{\lambda(Q_1)}$ are unique for any given $Q_1 > 0$, there exists a sequence $\lambda_{L_n}(Q_1)$ with $\lambda_{L_n}(Q_1) = 0$, such that

 $\psi_{\lambda_{L_n}(Q_1),L_n} \to \psi_{\lambda(Q_1)}$ in $H^1_{loc}(\Omega)$ and uniformly in any compact subset of Ω ,

as $L_n \to +\infty$. By virtue of (2.36), one has

$$(a\sin\theta - b\cos\theta)^{\frac{\pi}{\theta}}Q_1^{1-\frac{\pi}{\theta}} = L_n - (L_n - b)^{\frac{\pi}{\theta}}(L_n + Q_1)^{1-\frac{\pi}{\theta}}$$

Set $t_n = L_n + Q_1$, one has

$$(a\sin\theta - b\cos\theta)^{\frac{\pi}{\theta}}Q_1^{1-\frac{\pi}{\theta}} = t_n \left(1 - \left(1 - \frac{Q_1 + b}{t_n}\right)^{\frac{\pi}{\theta}}\right) - Q_1 \to \left(\frac{\pi}{\theta} - 1\right)Q_1 + \frac{b\pi}{\theta},$$

as $t_n \to +\infty$. Then one has

$$(a\sin\theta - b\cos\theta)^{\frac{\pi}{\theta}} = \left(\frac{\pi}{\theta} - 1\right)Q_1^{\frac{\pi}{\theta}} + \frac{b\pi}{\theta}Q_1^{\frac{\pi}{\theta} - 1}.$$
(3.10)

Similarly, we have

$$(a\sin\theta - b\cos\theta)^{\frac{\pi}{\theta}} = \left(\frac{\pi}{\theta} - 1\right)Q_2^{\frac{\pi}{\theta}} + \frac{b\pi}{\theta}Q_2^{\frac{\pi}{\theta} - 1},$$

which together with (3.10) implies that $Q_1 = Q_2$. This leads a contradiction.

Case 2. $\lambda(Q_1) \neq 0$ or $\lambda(Q_2) \neq 0$.

Since $Q_1 > Q_2$, one has

$$h_1 = \frac{Q_1}{\sqrt{\lambda(Q_1) + 1}} + b > \frac{Q_2}{\sqrt{\lambda(Q_2) + 1}} + b = h_2 \text{ and } k_{\lambda(Q_1)}(x) > k_{\lambda(Q_2)}(x)$$
(3.11)

for sufficiently large x > 0.

Define a function $\psi_{\lambda(Q_1),\varepsilon}(x,y) = \psi_{\lambda(Q_1)}(x,y-\varepsilon)$ for $\varepsilon \ge 0$. In view of (3.11), let $\varepsilon_0 \ge 0$ be the smallest one, such that

the free boundary of $\psi_{\lambda(Q_1),\varepsilon_0}$ lies above the free boundary of $\psi_{\lambda(Q_2)}$.

Similar to the proof of Lemma 3.1, by using the Phragmèn-Lindelöf theorem in [15] and the asymptotic behavior of $\psi_{\lambda(Q_1),\varepsilon_0}$ and $\psi_{\lambda(Q_2)}$, we have

$$\psi_{\lambda(Q_1),\varepsilon_0} \leq \psi_{\lambda(Q_2)} \text{ in } \Omega \cap \{\psi_{\lambda(Q_1),\varepsilon_0} > 0\} \text{ and } \psi_{\lambda(Q_1),\varepsilon_0} \leq \psi_{\lambda(Q_2)} \text{ in } \Omega \cap \{\psi_{\lambda(Q_2)} < 0\}.$$

Then we can obtain a contradiction by using the similar arguments in the proof of Lemma 3.1.

By virtue of the uniqueness of the solution $(\psi_{\lambda}, \lambda)$, it follows from the similar arguments in the proof of Lemma 2.7 that $\lambda(Q)$ is continuous for any Q > 0.

(2). Next, we will show that there exists a $\underline{\lambda} \in (-1,0)$, such that $\lambda(Q) \to \underline{\lambda}$ as $Q \to 0$. The monotonicity of $\lambda(Q)$ gives that there exists $\underline{\lambda} \ge -1$, such that $\lambda(Q) \to \underline{\lambda}$ as $Q \to 0$. It suffices to exclude the case $\lambda = -1$. For any sequence $\{Q_n\}$ with $Q_n > 0$ and $Q_n \to 0$, such that

 $\lambda(Q_n) \to \underline{\lambda}, \ \psi_{\lambda(Q_n)} \to \psi_{\underline{\lambda}} \text{ in } H^1_{loc}(\Omega) \text{ and uniformly in any compact subset of } \Omega,$ as $Q_n \to 0$. Moreover, $\partial_x \psi_{\underline{\lambda}} \leq 0$ and $\partial_y \psi_{\underline{\lambda}} \geq 0$ in $\Omega \cap \{\psi_{\underline{\lambda}} > 0\}$, which implies that

$$\Omega \cap \{x > 0\} \cap \{\psi_{\lambda} > 0\} = \Omega \cap \{x > 0\} \cap \{y > k_{\lambda}(x)\},\$$

where $k_{\underline{\lambda}}(x)$ is monotone increasing for x > 0, and $k_{\underline{\lambda}}(0) = 0$.

Suppose that $\underline{\lambda} = -1$. Define $\omega(y) = \max\{y, 0\}$, it follows from the condition (7) in Definition 1.1 that $\omega(y) \ge \psi_{\underline{\lambda}}(x, y)$ in Ω . The strong maximum principle gives that

$$\psi_{\lambda} < y \text{ in } \Omega \cap \{\psi_{\lambda} > 0\}.$$

For any small r > 0, it follows from the proof of (2.39) that there exists a small $\delta > 0$, such that

$$\omega \ge (1+\delta)\psi_{\underline{\lambda}}$$
 in $B_r(A) \cap \{\psi_{\underline{\lambda}} > 0\},\$

which gives that

$$1 \ge (1+\delta)\frac{\psi_{\underline{\lambda}}}{\partial\nu} = (1+\delta)\sqrt{-\underline{\lambda}} = 1+\delta \quad \text{at} \quad A,$$
(3.12)

where $\nu = (0, 1)$ is inner normal vector. This leads a contradiction.

Similar to the proof of (2.40), one has

$$k_{\lambda}(x) \equiv b \text{ for any } x \in (a, +\infty).$$
 (3.13)

In fact, if there exists an $x_0 \in (a, +\infty)$, such that $k_{\underline{\lambda}}(x_0) = b$ and $k_{\underline{\lambda}}(x) > b$ for any $x \in (x_0, +\infty)$. The asymptotic behavior of $\psi_{\underline{\lambda}}$ gives that

$$\underline{\lambda} = -1,$$

which contradicts to $\underline{\lambda} > -1$. Similar to the proof of (2.43), we can show that $\overline{\Gamma}_{\underline{\lambda}} \cap N_2 = \emptyset$.

(3). In this step, we will show that

$$\frac{\lambda(Q)}{Q^2} \to \kappa \in (0, +\infty) \text{ as } Q \to +\infty.$$

Set $\psi_Q = \frac{\psi_{\lambda(Q)}}{Q}$, $\lambda_Q = \frac{\lambda(Q)}{Q^2}$ and h_Q is the asymptotic height of the free boundary of ψ_Q . By virtue of Lemma 2.8, one has

$$\lambda_Q = \frac{1}{(h_Q - b)^2} - \frac{1}{Q^2} \le C, \text{ where } C > 0 \text{ is a constant independent of } Q.$$
(3.14)

It follows from the proof of (2.48) that there exists a c > 0 independent of Q, such that

$$\lambda_Q \ge c > 0. \tag{3.15}$$

In view of (3.14) and (3.15), there exist two positive constants C_1 and C_2 independent of Q, such that

$$C_1 \le h_Q - b \le C_2. \tag{3.16}$$

Therefore, for any sequence $\{Q_n\}$ with $Q_n \to +\infty$, such that

$$\lambda_{Q_n} \to \kappa, \quad h_{Q_n} \to h_{\kappa} \text{ and } \psi_{\lambda_{Q_n}} \to \bar{\psi}_{\kappa} \text{ uniformly in } \Omega, \text{ as } Q_n \to +\infty.$$

It is easy to check that $\bar{\psi}_{\kappa} = 0$ in $\Omega \cap \{(x, y) \mid x \leq 0, y \geq 0\}$. Similar to Lemma 2.8, $\bar{\psi}_{\kappa}$ is a solution of the following free boundary problem

$$\Delta \bar{\psi}_{\kappa} = 0 \text{ in } E \cap \{ \bar{\psi}_{\kappa} < 0 \},$$

$$\bar{\psi}_{\kappa} = 0, \quad \left| \frac{\partial \bar{\psi}_{\kappa}}{\partial \nu} \right|^{2} = \kappa \quad \text{on } \Gamma_{\kappa},$$

$$\bar{\psi}_{\kappa} = 0 \text{ on } S_{1}, \quad \bar{\psi}_{\kappa} = -1 \text{ on } N_{2} \cup S_{2},$$

(3.17)

where $E = \Omega \setminus (\{(x, y) \mid x \leq 0, y \geq 0\})$ and $\Gamma_{\kappa} = E \cap \{\bar{\psi}_{\kappa} < 0\}$ is the free boundary of $\bar{\psi}_{\kappa}$.

By using the similar arguments in the proof of Lemma 2.8, we can obtain the uniqueness of $(\bar{\psi}_{\kappa}, \kappa)$ to the free boundary problem (3.17).

Based on the proof of Theorem 1.2, we can obtain the existence and uniqueness of the solution to the injection flow problem 2.

Corollary 3.2. For any $\lambda \in (\underline{\lambda}, +\infty)$, there exist a unique $Q = Q(\lambda) > 0$ and a unique solution (ψ_Q, Γ_Q) to the injection flow problem 2. Furthermore, (1) $Q(\underline{\lambda} + 0) = \lim_{\lambda \to \underline{\lambda}^+} Q(\lambda) = 0$ and $\frac{Q^2(\lambda)}{\lambda} \to \frac{1}{\kappa}$ as $\lambda \to +\infty$. (2) Q(0) > 0 is uniquely determined by

$$(a\sin\theta - b\cos\theta)^{\frac{\pi}{\theta}} = \left(\frac{\pi}{\theta} - 1\right)Q^{\frac{\pi}{\theta}} - \frac{b\pi}{\theta}Q^{\frac{\pi}{\theta} - 1}.$$

Hence, Theorem 1.3 follows from Corollary 3.2 immediately.

Conflict of interest. The authors declare that they have no conflict of interest.

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