# Surface dimension, tiles, and synchronising automata * 

Vladimir Yu. Protasov ${ }^{\dagger}$


#### Abstract

We study the surface regularity of compact sets $G \subset \mathbb{R}^{n}$ which is equal to the supremum of numbers $s \geq 0$ such that the measure of the set $G_{\varepsilon} \backslash G$ does not exceed $C \varepsilon^{s}, \varepsilon>0$, where $G_{\varepsilon}$ denotes the $\varepsilon$-neighbourhood of $G$. The surface dimension is by definition the difference between $n$ and the surface regularity. Those values provide a natural characterisation of regularity for sets of positive measure. We show that for self-affine attractors and tiles those characteristics are explicitly computable and find them for some popular tiles. This, in particular, gives a refined regularity scale for the multivariate Haar wavelets. The classification of attractors of the highest possible regularity is addressed. The relation between the surface regularity and the Hölder regularity of multivariate refinable functions and wavelets is found. Finally, the surface regularity is applied to the theory of synchronising automata, where it corresponds to the concept of parameter of synchronisation.


Keywords: Regularity, dimension, surface area, Minkowski content, self-affine attractors, tiles, spectral radius, multivariate Haar wavelets, finite deterministic automata, reset word, synchronisation

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## 1. Introduction

The well-known Minkowski -- Steiner formula defines the area, i.e., the ( $n-1$ )-dimensional volume, of the surface of a compact set $G \subset \mathbb{R}^{n}$ as the lower limit for the ratio $\frac{\left|G_{\varepsilon}\right|-|G|}{\varepsilon}$ as $\varepsilon \rightarrow+0$, where $G_{\varepsilon}$ is the $\varepsilon$-neighbourhood of $G$ and $|X|$ denotes the Lebesgue measure of the set $X$ (see, for instance, [14]). For sets with sufficiently regular surfaces, this limit is finite. This is the case, for example, if $G$ is convex. If this limit is infinite, then a natural characterisation of regularity of the surface is the supremum of $s \geq 0$ such that $\left|G_{\varepsilon}\right|-|G|$ does not exceed $C \varepsilon^{s}$ for all $\varepsilon>0$. This is, in a sense, analogous to the Hölder exponent of

[^0]a function while the surface area plays the role of Lipschitz constant. In this paper we show that for self-affine tiles and attractors, this characteristic is computable and gives a natural scale of regularity for those sets. It is related to the Hölder regularity of Haar wavelets in $\mathbb{R}^{n}$. Moreover, this characteristic can be applied in the study of synchronising automata, where it corresponds to their "rate of synchronisation".

We use the following notation: $\mathcal{B}(x, r)$ is the Euclidean ball of radius $r>0$ centered at a point $x \in \mathbb{R}^{n} ; A+B=\{a+b \mid a \in A, b \in B\}$ is the Minkowski sum of sets $A$ and $B$; $G_{\varepsilon}=G+\mathcal{B}(0, \varepsilon)$ is the $\varepsilon$-neighbourhood of a set $G$.

Definition 1 The surface regularity of a compact set $G \subset \mathbb{R}^{n}$ is

$$
s(G)=\sup \left\{s \geq 0|\quad| G_{\varepsilon}\left|-|G| \leq C \varepsilon^{s} \quad \forall \varepsilon>0\right\}\right.
$$

The surface dimension of $G$ is $\boldsymbol{d}=n-\boldsymbol{s}(G)$.
The surface dimension $\boldsymbol{d}$ of compact sets in $\mathbb{R}^{n}$ can take all values from 0 to $n$. The case of integer $\boldsymbol{d}$ corresponds to the upper Minkowski content, see [2, 3, 23, 34]. However, for the sets $G$ of positive measure, we always have $\boldsymbol{s}(G) \leq 1$ (Corollary 1 in the next section), and therefore, $\boldsymbol{d}(G) \in[n-1, n]$. The case $\boldsymbol{s}=1$ (i.e., $\boldsymbol{d}=n-1$ ) characterises sets with "regular" surfaces. For example, if $G$ is a union of finitely many convex sets or sets with piecewise-smooth boundary, then $s=1$. One can say that $s(G)$ measures the regularity of the boundary of $G$. On the other hand, it has no relation to the dimension of the topological boundary. This can be shown by simple examples. Consider the following "quasi-Cantor" set $G \subset \mathbb{R}$ : take a unit segment, remove the open interval of length $2^{-2^{1}}$ from the middle; in each of the two remaining segments remove the interval of length $2^{-2^{2}}$ from the middle, etc. In $k$ th iteration we have $2^{k}$ equal segments and from each of them we remove an interval of length $2^{-2^{k}}$ from the middle. The limit compact set $G$ has a positive measure. It is easily shown that $\boldsymbol{s}=1$ and hence $\boldsymbol{d}=0$. On the other hand, the boundary of $G$ coincides with $G$ and hence the Hausdorff dimension of its boundary is one and so it is not equal to $\boldsymbol{d}$.

An advantage of the surface regilarity and of the surface dimension is that they are both metric invariants of compact sets, i.e., these charactersistics are invariant with respect to bi-Lipschitz maps (Lipschitz maps with Lipschitz inverse). Hence, they provide characteristics of compact sets invariant under $C^{1}$-diffeomorphisms. In contrast to the topological or Hausdorff dimension it can distinct sets of positive measure, whose dimension is the same as the dimension of the entire space. That is why our main interest is in the sets of positive Lebesgue measure. In what follows we assume that $|G|>0$.

For sets of positive measure, a characteristic similar to the surface regularity is provided by the $L_{1}$-Hölder regularity of the indicator function. However, this charactersistic, in contrast to the surface regularity, is not bi-Lipschitz invariant. We show that the surface regularity does not exceed the Hölder regularity and can be strictly smaller even for tiles (Theorem 1). Then in Theorem 2 we establish a condition for a compact set that ensures that its surface and Hölder regularities coincide. In Section 3 we apply that result to self-affine attractors and tiles, which play an important role in construction of Haar and other wavelet systems in $\mathbb{R}^{n}$. In Section 5 we obtain formulas for the $L_{p}$-Hölder regularity of attractors
and tiles. This, in particular, makes it possible to compute the $L_{p}$-exponents of multivariate Haar wavelets and to range them by their regularity. In case of isotropic dilation matrix, those formulas can compute the surface regularities and surface dimensions. The computation of all those characteristics are reduced to finding the Perron eigenvalue of a special matrix. In Section 6 we compute surface dimensions of some popular self-affine tiles. Then we address the problem of characterising the self-affine attractors and tiles with the highest surface regularity. We make a conjecture that the only self-affine attractor with the surface regularity $s=1$ is a parallelepiped. So, the parallelepiped is the only regular attractor. In Section 7 this conjecture is proved for dimension $n=1$. Finally, we apply the surface regularity in the study of finite deterministic automata and establish a relation between the surface regularity and the rate of synchronisation (Section 8).

The following notation will be used: $|X|$ is the Lebesgue measure of a set $X$ or the cardinality of a finite set $X$, depending on the context; $L_{p}$ is the standard functional space with the norm $\|f\|_{p}=\left(\int|f|^{p} d t\right)^{1 / p}$. We use the standard notation $A^{*}=\bar{A}^{T}$ for the adjoint matrix to a matrix $A$; the spectral radius of $A$, i.e., the biggest modulus of eigenvalues, is denoted my $\rho(A)$.

## 2. The surface dimension and the Hölder regularity

The definition of the surface regularity is similar to the Hölder regularity of the characteristic function $\chi_{G}(x)$ in the space $L_{1}\left(\mathbb{R}^{n}\right)$. As usual, $\chi_{G}(x)=1$ if $x \in G$ and $\chi_{G}(x)=0$ otherwise. The $L_{p}$ Hölder regularity of a function $f \in L_{p}\left(\mathbb{R}^{n}\right)$ is defined as

$$
\boldsymbol{\alpha}_{p}(f)=\sup \left\{\alpha \geq 0 \mid\|f(\cdot+h)-f(\cdot)\|_{p} \leq C\|h\|^{\alpha} \forall h \in \mathbb{R}^{n}\right\}
$$

For a characteristic function of a compact set $G$, we denote shortly $\boldsymbol{\alpha}_{p}\left(\chi_{G}\right)=\boldsymbol{\alpha}_{p}(G)$ and call this value the Hölder $L_{p}$-regularity of $G$. The measure of the difference $G_{\varepsilon} \backslash G$ is the $L_{1}$-norm of the function $\chi_{G_{\varepsilon} \backslash G}$. Hence it is quite expected that $\boldsymbol{s}(G)$ can be related to $\boldsymbol{\alpha}_{1}(G)$. In what follows we omit the index 1 meaning that always $p=1$ if the the converse is not stated. Moreover, often we omit the set $G$ from the notation. Thus $\boldsymbol{\alpha}_{1}(G)=\boldsymbol{\alpha}$.

The following proposition shows that for every compact set, the Hölder regularity majorates the surface regularity.

Proposition 1 For every compact set in $\mathbb{R}^{n}$ of positive measure, we have $\boldsymbol{s} \leq \boldsymbol{\alpha}$.
Proof. Let $h \in \mathbb{R}^{n}$ be an arbitrary vector of length $\|h\|<\varepsilon$. Since $G+h \subset G+\mathcal{B}(0, \varepsilon)$, we see that the measure of the set $(G+h) \backslash G$ does not exceed the measure of $G_{\varepsilon} \backslash G$. Similarly, the measure of $(G-h) \backslash G$ does not exceed the same measure of $G_{\varepsilon} \backslash G$. Therefore, $\left\|\chi_{G}(\cdot+h)-\chi_{G}\right\|_{1} \leq 2\left|G_{\varepsilon} \backslash G\right|$. Computing logarithms of both parts and dividing by log $\frac{1}{\varepsilon}$, we conclude the proof.

Since the Hölder exponent never exceeds one, we obtain

Corollary 1 For a compact set of positive measure, $s \leq 1$ and respectively $n-1 \leq \boldsymbol{d} \leq n$.
In the sequel we always consider sets of positive measure, i.e., assume that $|G|>0$. There are examples when $\boldsymbol{s} \neq \boldsymbol{\alpha}$. Moreover, even for tiles on $\mathbb{R}$, it can happen that $\boldsymbol{s}<\boldsymbol{\alpha}$. A compact set in $\mathbb{R}$ is called a tile if its integer shifts cover $\mathbb{R}$ with intersections of zero measure.

Theorem 1 There is a tile in $\mathbb{R}$ for which $\boldsymbol{\alpha}=1$ and $\boldsymbol{s}=\frac{1}{3}$.
Proof. First we construct a compact set $G \subset \mathbb{R}$ with this property and then make a tile from it. Consider a sequence $x_{1}, x_{2}, \ldots$, where $x_{k}=\sum_{m=1}^{k} \frac{1}{m^{2}}$. Define the set $G$ as a union of segments $\left[x_{k}, x_{k}+2^{-k-2}\right], k \in \mathbb{N}$, plus the limit point $x_{\infty}=\sum_{m=1}^{\infty} \frac{1}{m^{2}}=\frac{\pi^{2}}{6}$.

Let us first compute $\boldsymbol{s}(G)$. Take arbitrary small $\varepsilon>0$ and denote by $N$ the minimal natural number such that for all $k \geq N$ the distance between points $x_{k}$ and $x_{k+1}$ is less than $2 \varepsilon$. Thus, $N$ is the smallest natural solution of inequality $\frac{1}{k^{2}}-\frac{1}{(k+1)^{2}}+2^{-k-2}<2 \varepsilon$. It is shown easily that $N \sim \varepsilon^{-1 / 3}$ as $\varepsilon \rightarrow 0$. The enlarged set $G_{\varepsilon}=G+[-\varepsilon, \varepsilon]$ contains $N-1$ segments:

$$
\left[x_{k}-\varepsilon, x_{k}+2^{-k-2}+\varepsilon\right], \quad k=1, \ldots, N-1
$$

and one big segment $\left[x_{N}-\varepsilon, x_{\infty}+\varepsilon\right]$ formed by all other segments for $k \geq N$. The total length of those $N$ segments is $2 \varepsilon+\sum_{k=N}^{\infty} \frac{1}{k^{2}}$ (the big segment) plus $\sum_{k=1}^{N-1}\left(2^{-k-2}+2 \varepsilon\right)$ (the remaining $N-1$ segments). Thus,

$$
\left|G_{\varepsilon}\right|=2 \varepsilon+\sum_{k=N}^{\infty} \frac{1}{k^{2}}+\sum_{k=1}^{N-1}\left(2^{-k-2}+2 \varepsilon\right)=2 N \varepsilon+\frac{1}{4}-2^{-N-2}+\sum_{k=N}^{\infty} \frac{1}{k^{2}} .
$$

On the other hand, $|G|=\frac{1}{4}$. Hence

$$
\left|G_{\varepsilon}\right|-|G|=2 N \varepsilon-2^{-N-2}+\sum_{k=N}^{\infty} \frac{1}{k^{2}} .
$$

Since $N \asymp \varepsilon^{-1 / 3}$ and $\sum_{k=N}^{\infty} \frac{1}{k^{2}} \asymp \frac{1}{N}$, we see that the value $\left|G_{\varepsilon}\right|-|G|$ is asymptotically equivalent to $2 \varepsilon^{2 / 3}+C \varepsilon^{1 / 3}$, where $C$ is a constant. We see that $\left|G_{\varepsilon}\right|-|G| \asymp C \varepsilon^{1 / 3}$ as $\varepsilon \rightarrow 0$, and therefore, $s=\frac{1}{3}$.

Now let us show that $\boldsymbol{\alpha}(G)=1$. If $k \leq \log _{2} \frac{1}{\varepsilon}$, i.e., $2^{-k}>\varepsilon$, then the $k$ th segment $\left[x_{k}, x_{k}+2^{-k}\right]$ intersects its copy shifted by $\varepsilon$. Therefore, the length of the symmetric difference of this segment with its copy is equal to $2 \varepsilon$. The number of those segments does not exceed $\log _{2} \frac{1}{\varepsilon}$. Hence, the total length of those symmetric differences is at most $2 \varepsilon \log _{2} \frac{1}{\varepsilon}$.

The total length of the remaining segments of the set $G$ is $\sum_{k>\log _{2} \frac{1}{\varepsilon}} 2^{-k}$, which is less then $2^{1-\log _{2} \frac{1}{\varepsilon}}=2 \varepsilon$. Therefore, the symmetric difference of this set with its shift to $\varepsilon$ has the length less that $4 \varepsilon$.

Summing over these two sets we have $\left\|\chi_{G}(\cdot)-\chi_{G}(\cdot+\varepsilon)\right\|_{1}<2 \varepsilon\left(2+\log _{2} \frac{1}{\varepsilon}\right)$. Therefore, $\boldsymbol{\alpha} \geq 1$. Since $\boldsymbol{\alpha}$ cannot be bigger than one, we conclude that $\boldsymbol{\alpha}=1$.

Thus, a compact set $G$ with $\boldsymbol{\alpha}=1$ and $\boldsymbol{s}=\frac{1}{3}$ is constructed. But this is not a tile. To make a tile form $G$ we take the unit segment $[0,1]$, unify it with the set $G$ and subtract the
set $G-1$ from it. Since $G \subset[0,2)$, we see that the obtained set $[0,1] \cup G \backslash(G-1)$ is a tile with the same parameters $\boldsymbol{s}$ and $\boldsymbol{\alpha}$.

Remark 1 The fact that $\boldsymbol{s} \leq \boldsymbol{\alpha}$ means that the characteristic $\boldsymbol{s}(G)$ provides a more refined analysis of a set $G$ than $\boldsymbol{\alpha}(G)$ and can distinct sets with the identical exponent $\boldsymbol{\alpha}$. For example, the set $G$ constructed in the proof of Theorem 1 has the maximal Hölder regularity $\boldsymbol{\alpha}=1$ as the segment $[0,1]$, while its surface regularity is lower: $s=\frac{1}{3}$ for $G$ instead of $s=1$ for the segment $[0,1]$. So, the Hölder regularity cannot distinct the set $G$ from a segment, but the surface regularity can.

A question arises what conditions of the set $G$ would guarantee that $\boldsymbol{s}(G)=\boldsymbol{\alpha}(G)$ ? Theorem 2 below establishes sufficient conditions. To formulate them we need one more notation. Let a compact set $G \subset \mathbb{R}^{n}$ be fixed. For a point $x \in \mathbb{R}^{n}$ and a number $r>0$, we denote

$$
\nu(x, r)=\frac{|\mathcal{B}(x, r) \cap G|}{|\mathcal{B}(x, r)|} .
$$

Thus, the number $\nu(x, r)$ shows which part of the volume of the ball $\mathcal{B}(x, r)$ is covered by $G$.
Theorem 2 If there are constants $c_{1}, c_{2}>0$ such that for every sufficiently small $\varepsilon>0$, the total measure of points $x$ of the set $G_{\varepsilon} \backslash G$ for which $\nu(x, 2 \varepsilon) \geq c_{1}$ is at least $c_{2}\left|G_{\varepsilon} \backslash G\right|$, then $s=\boldsymbol{\alpha}$.
If $x \in G_{\varepsilon}$, then the intersection of the ball $\mathcal{B}(x, 2 \varepsilon)$ with the set $G$ is at least nonempty. The assumption of Theorem 2 require that this intersection has not very small volume. If this condition is fulfilled for some significant part of points $x$ of the set $G_{\varepsilon}$, then $\boldsymbol{s}(G)=\boldsymbol{\alpha}(G)$.

In the proof we use the following technical
Lemma 1 For an arbitrary compact set $G \subset \mathbb{R}^{n}$ and for every $r>0$, we have

$$
\left|G_{2 r}\right|-|G| \leq 2^{n}\left(\left|G_{r}\right|-|G|\right) .
$$

Proof. Assume without loss of generality that $|G|=1$. Since $G_{r}$ is a Minkowski sum of $G$ and of a ball of radius $r$, we can apply the Brunn-Minkowski inequality and conclude that the function $f(r)=\sqrt[n]{\left|G_{r}\right|}$ is concave. Hence $f(r) \geq \frac{1}{2}(f(0)+f(2 r))$. Let $f(2 r)=1+a$. Since $f(0)=1$, we see that $f(r) \geq 1+\frac{a}{2}$. Therefore $2^{n}\left(\left|G_{r}\right|-|G|\right) \geq 2^{n}\left(\left(\frac{a}{2}+1\right)^{n}-1\right)=(a+2)^{n}$ $-2^{n}$. Since $\left|G_{2 r}\right|=(a+1)^{n}-1$, it remains to establish the inequality

$$
(a+2)^{n}-2^{n} \geq(a+1)^{n}-1
$$

Opening the brackets we have $\sum_{k=1}^{n}\binom{n}{k} 2^{n-k} a^{k} \geq \sum_{k=1}^{n}\binom{n}{k} a^{k}$, which is obvious.
Proof of Theorem 2. Denote $\chi(x)=\chi_{G}(x)$ and, for arbitrary $\varepsilon>0$, consider the following integral:

$$
I_{\varepsilon}=\int_{\substack{\left(x, y \in \mathbb{R}^{n} \times \mathbb{R}^{n} \\\| \| y-y \| \leq 2 \varepsilon\right.}}|\chi(x)-\chi(y)| d x d y
$$

This integral is computed over all pairs of points $(x, y)$ of the space $\mathbb{R}^{n}$ such that $\|x-y\| \leq 2 \varepsilon$. The function under the integral takes only two values: zero and one. It is equal to zero if both $x$ and $y$ belong to $G$ or both do not. Otherwise it is equal to one. In particular, this function is zero, whenever both $x, y$ are far from $G$. Therefore, it has a compact support, and hence it is integrable over $\mathbb{R}^{n} \times \mathbb{R}^{n}$.

We are going to prove that $s \geq \boldsymbol{\alpha}$ by computing the integral $I_{\varepsilon}$ in two ways. The first way is to integrate over the variable $x$ :

$$
\begin{gathered}
I_{\varepsilon}=\int_{h \in \mathcal{B}_{2 \varepsilon}} \int_{x \in \mathbb{R}^{n}}|\chi(x)-\chi(x+h)| d x d h=\int_{h \in \mathcal{B}_{2 \varepsilon}}\|\chi(\cdot)-\chi(\cdot+h)\|_{1} d h \leq \\
\left|\mathcal{B}_{2 \varepsilon}\right| \max _{h \in \mathcal{B}_{2 \varepsilon}}\|\chi(\cdot)-\chi(\cdot+h)\|_{1}
\end{gathered}
$$

Thus,

$$
\begin{equation*}
\max _{\|h\| \leq 2 \varepsilon}\|\chi(\cdot)-\chi(\cdot+h)\|_{1} \geq\left|\mathcal{B}_{2 \varepsilon}\right|^{-1} I_{\varepsilon} \tag{1}
\end{equation*}
$$

Now we compute the integral $I_{\varepsilon}$ differently. Note that $\chi(x)-\chi(x+h) \neq 0$ if and only if precisely one of the points $x$ and $x+h$ is out of $G$. On the other hand, $\|h\| \leq 2 \varepsilon$, hence that point belongs to $G_{2 \varepsilon} \backslash G$. Because of the symmetry we can always assume that $x \in G_{2 \varepsilon} \backslash G$. Thus,

$$
\begin{gathered}
I_{\varepsilon}=\int_{G_{2 \varepsilon} \backslash G}\left(\int_{\mathcal{B}_{2 \varepsilon}}|\chi(x)-\chi(x+h)| d h\right) d x=\int_{G_{2 \varepsilon} \backslash G}\left|\mathcal{B}_{2 \varepsilon}\right| \nu(x, 2 \varepsilon) d x \geq \\
\left|\mathcal{B}_{2 \varepsilon}\right| \int_{G_{\varepsilon} \backslash G} \nu(x, 2 \varepsilon) d x \geq\left|\mathcal{B}_{2 \varepsilon}\right| \cdot c_{1} \cdot\left|\left\{x \in G_{\varepsilon} \backslash G \mid \nu(x, 2 \varepsilon) \geq c_{1}\right\}\right| \geq c_{1} \cdot\left|\mathcal{B}_{2 \varepsilon}\right| \cdot c_{2} \cdot\left|G_{\varepsilon} \backslash G\right| .
\end{gathered}
$$

Now invoking Lemma 1 we get $\left|G_{\varepsilon} \backslash G\right| \geq 2^{-n}\left|G_{2 \varepsilon} \backslash G\right|$, consequently

$$
I_{\varepsilon} \geq c_{1} c_{2} 2^{-n}\left|\mathcal{B}_{2 \varepsilon}\right| \cdot\left|G_{2 \varepsilon} \backslash G\right| .
$$

Combining this with (1) we obtain

$$
\max _{\|h\| \leq 2 \varepsilon}\|\chi(\cdot)-\chi(\cdot+h)\|_{1} \geq c_{1} c_{2} 2^{-n}\left|G_{2 \varepsilon} \backslash G\right|
$$

for every sufficiently small $\varepsilon>0$. Taking now logarithm of both parts and a limit as $\varepsilon \rightarrow 0$ we complete the proof.

Equality $\boldsymbol{s}(G)=\boldsymbol{\alpha}(G)$ enables us to compute the surface dimension at least for some special classes of sets, because $\boldsymbol{\alpha}(G)$ can be expressed by the Hölder $L_{2}$-regularity as well as by the Sobolev regularity of the characteristic function. One of such classes of sets is the class of self-affine attractors. It plays an important role in many practical areas such as subdivision algorithms and wavelets. Moreover, for those sets the exponent of regularity $\boldsymbol{\alpha}(G)$ can be efficiently computed. This is done in Section 5 . Then in Section 8 we find a relation of this value to synchronising automata theory.

## 3. The surface dimension of self-affine attractors

Self-affine attractors are compact sets in $\mathbb{R}^{n}$ defined by an integer matrix and by a system of digits (integer points) associated to that matrix.

Let us have an integer $n \times n$ matrix $M$ which is supposed to be expanding, i.e., all its eigenvalues are strictly bigger than one by modulus. This matrix splits the integer lattice $\mathbb{Z}^{n}$ into $m=|\operatorname{det} M|$ quotient classes defined by the equivalence $x \sim y \Leftrightarrow y-x \in M \mathbb{Z}^{n}$. Choosing one representative $d_{i} \in \mathbb{Z}^{n}$ from each equivalence class, we obtain a set of digits $D=$ $\left\{d_{i}: i=0, \ldots, m-1\right\}$. We always assume that $0 \in D$ and naturally denote $d_{0}=0$.
For every integer point $d \in \mathbb{Z}^{n}$, we denote by $M_{d}$, the affine operator $M_{d} x=M x-d, x \in \mathbb{R}^{n}$. We use the notation $0 . a_{1} a_{2} \ldots=\sum_{i=1}^{\infty} M^{-i} a_{i}, a_{i} \in D$.

Definition 2 A self-affine attractor generated by an expanding matrix $M$ and by a digit set $D$ is the set

$$
\begin{equation*}
G=G(M, D)=\left\{0 . a_{1} a_{2} \ldots=\sum_{k=1}^{\infty} M^{-k} a_{k} \quad: \quad a_{k} \in D\right\} \tag{2}
\end{equation*}
$$

For any integer expanding matrix $M$ and for any digit set $D$, the self-affine attractor is a compact set with a nonempty interior [15, 16]. Moreover, its Lebesgue measure is always a positive integer. It is seen easily that $G=\bigcup_{d \in D} M_{d}^{-1} G$. Moreover, each set $M_{d}^{-1} G$ has measure $m^{-1}|G|$, hence the sum of measures of the sets $M_{d}^{-1} G, d \in D$, is exactly $m m^{-1}|G|=|G|$. Consequently, all those sets have intersections of zero measure. Thus $G$ is a disjunct (up to nill sets) sum of equal sets that are affinely similar to $G$. This justifies the terminology "self-affine". In what follows we say shortly attractor and always mean self-affine attractors from Definition 2.

Thus, an attractor is the set of points form $\mathbb{R}^{n}$ with zero integer part in their $M$-adic expansion. In this sense attractors play role of the unit segment in $\mathbb{R}$, but for the space $\mathbb{R}^{n}$ equipped with the $M$-adic system with digits from the set $D$.

The affine similarity implies that the characteristic function $\varphi=\chi_{G}(x)$ of an attractor satisfies the following functional equation with a contraction of the argument:

$$
\begin{equation*}
\varphi(x)=\sum_{d \in D} \varphi(M x-d) \quad \text { a.e. } \quad x \in \mathbb{R}^{n} \tag{3}
\end{equation*}
$$

This is a special case of a refinement equation (see Section 4). Therefore, the theory of refinement equations, which is well developed in the literature, can be applied in the study of attractors. Integer shifts of an attractor cover the space $\mathbb{R}^{n}$ with an integer number of layers (namely, with $|G|$ layers). This means that $\sum_{k \in \mathbb{Z}^{n}} \varphi(x+k) \equiv|G|$ a.e. If $|G|=1$, then $G$ called a self-affine tile.

Definition 3 A self-affine tile is an attractor of measure one.

Every integer expanding matrix and every set of digits generate an attractor, but this attractor is not always a tile. There is a criterion to determine whether the attractor generated by a matrix $M$ and by a digit set $D$ is a tile. It gives an answer in terms of eigenvalues of a certain integer matrix.

In case $n=1$, for $M=2$, there are only two digit sets $D=\{0,1\}$ and $D=\{0,-1\}$ for which the generated attractor is a tile. Already for $M=3$, the situation is more interesting: both digit sets $D=\{0,1,2\}$ and $D=\{0,1,5\}$ generate tiles and the second tile is not a segment. For $n=2,3$, every expanding matrix $M$ has at least one digit set $D$ generating a tile. However, for $n=4$, there are examples of matrices for which this is not true. Nevertheless, such a system of digits exists under quite general assumptions. For example, it exists whenever $|\operatorname{dim} M|>n$ [26]. This condition is indeed general taking into account that the matrix $M$ is expanding.

Definition $4 A$ tiling $\mathcal{G}$ generated by an integer expanding matrix $M$ and by a set of digits $D$ is a collection of sets $\mathcal{G}=\{k+G\}_{k \in \mathbb{Z}^{s}}$ such that
a) the union of the sets in $\mathcal{G}$ covers $\mathbb{R}^{s}$ and $|(\ell+G) \cap(k+G)|=0$, $\ell \neq k$;
b) $G=\cup_{d \in D} M_{d}^{-1} G$.

See [1, 5, 26] for the general discussion and more references. The characteristic function of a tile possesses orthonormal integer shifts. This property makes tiles very useful in the construction of the Multiresilutional analysis and wavelets systems on $\mathbb{R}^{n}$. In particular, multivariate Haar systems are obtained directly from tiles [5, 16, 25].

A matrix is called isotropic if it has equal by modulus eigenvalues and no nontrivial Jordan blocks. An isotropic matrix is similar to a multiple of an orthogonal matrix. Attractors and tiles generated by isotropic dilation matrices $M$ are very popular in applications and well studied in the literature, see [6, 17, 39] and references therein. The following theorem shows that at least for isotropic dilation matrices, the surface regularity of attractors is equal to the Hölder regularity.

Theorem 3 For every attractor with an isotropic dilation matrix, we have $\boldsymbol{s}=\boldsymbol{\alpha}$.
Proof. We need to show that an attractor generated by an isotropic dilation matrix satisfies assumptions of Theorem 2. Take a small $\varepsilon$ and a point $x \in G_{\varepsilon} \backslash G$. By definition, there is a point $y \in G$ such that $\|x-y\| \leq \varepsilon$. Denote $r=\rho(M)$. Let $k$ be the smallest number such that the diameter of the set $M^{-k} G$ is less that $\varepsilon$. Since $M$ is isotropic, we have $k \leq \frac{\log \varepsilon}{\log r}+C_{0}$, where $C_{0}$ does not depend on $\varepsilon$. The $k$ th iteration of the partition $G=\cap_{i=0}^{m-1} M^{-1}\left(G+d_{i}\right)$ covers the set $G$ with $m^{k}$ parts equal to $M^{-k} G$ each. Denote by $G_{k}$ a part that contains the point $y$. Since the diameter of $G_{k}$ is less than $\varepsilon$ and $\|x-y\| \leq \varepsilon$, we see that $G_{k} \subset \mathcal{B}(x, 2 \varepsilon)$. On the other hand, $G_{k} \subset G$. Hence, $G_{k}$ is contained in the intersection of the ball $\mathcal{B}(x, 2 \varepsilon)$ with $G$. Since $M$ is isotropic, we see that

$$
\left|G_{k}\right|=m^{-k}|G|=r^{-n k}|G| \geq \varepsilon^{n} r^{-n C_{0}}|G|
$$

(we used the inequality $k \leq \frac{\log r}{\log \varepsilon}+C_{0}$ ). Thus, the intersection of the ball $\mathcal{B}(x, 2 \varepsilon)$ with the set $G$ has the measure at least $C \varepsilon^{d}$, where $C$ is a constant not depending on $\varepsilon$. On the other
hand, $|\mathcal{B}(x, 2 \varepsilon)|=\varepsilon^{n}|\mathcal{B}|$. Hence, the ratio of measures of the intersection to the measure of the ball is at least $\frac{C}{|\mathcal{B}|}$. This is true for all points $x \in\left(G_{\varepsilon} \backslash G\right)$. Therefore, the assumptions of Theorem 2 are satisfied with the constants $c_{1}=\frac{C}{|\mathcal{B}|}, c_{2}=1$.

We believe that the assumption of isotropic dilation matrix $M$ can be omitted in Theorem 3 and that actually $\boldsymbol{s}=\boldsymbol{\alpha}$ for an arbitrary attractor.

Conjecture 1 Theorem 3 holds for arbitrary dilation matrix.
In the next section we show that the Hölder regularity of attractors can be efficiently computed. It can be expressed with the Perron eigenvalue of a special matrix. In case of an isotropic matrix $M$ this will give the values of the surface regularity and surface dimension of the attractors.

## 4. The $L_{p}$-regularity of multivariate refinable functions

In this section we consider the $L_{p}$-regularity of solutions of general refinement equations. We provide a method that allows, at least theoretically, to find the $L_{p}$-Hölder exponent of wavelets and of the limit functions of subdivision schemes on $\mathbb{R}^{n}$, see [9, 25]. Then, in Section 5, we apply the obtained results to the special refinement equations (3) for characteristic functions of attractors. As we will see, in that case the Hölder regularity in $L_{1}$ can be found within polynomial time. This, in particular, gives formulas the for $L_{p}$-Hölder regularity of Haar functions. This also makes it possible to compute the surface dimension of tiles provided $M$ is isotropic.

### 4.1 Refinement equation

Refinement equation is a functional equation of the type

$$
\begin{equation*}
\varphi(x)=\sum_{k \in \mathbb{Z}^{n}} c_{k} \varphi(M x-k), \quad x \in \mathbb{R}^{n} \tag{4}
\end{equation*}
$$

with a compactly supported set of coefficients $c_{k} \in \mathbb{C}$ (i.e., $c_{k}=0$ for all but finitely many $k$ ) and with a general integer expanding dilation matrix $M$. The set of coefficients $\boldsymbol{c}=\left\{c_{k}, k \in\right.$ $\left.\mathbb{Z}^{n}\right\}$ is called a mask of the equation. The theory of refinement equations is well developed in the literature due to their crucial role in the construction of wavelets [5, 12, 39], in the study of subdivision schemes for approximating functions and for curves and surfaces design [6, 10, 18], in some problems of combinatorics, number theory, and probability (see [6, 31] and references therein). The characteristic function of an attractor $\varphi=\chi_{G}$ is a solution of refinement equation with $c_{k}=1$ whenever $k \in D$ and $c_{k}=0$ otherwise (3).

A compactly supported function $\varphi \in L_{1}\left(\mathbb{R}^{n}\right)$ satisfying equation (4) is called a refinable function. It is well known that if such a solution exists, then it is unique up to normalization.

Moreover, if $\int_{\mathbb{R}^{n}} \varphi(x) d x \neq 0$, then $\sum_{k \in \mathbb{Z}^{n}} c_{k}=m$. We will focus on this case as in the most of literature. Under this assumption, the refinement equation always possesses a unique up to multiplication by a constant solution $\varphi$ in the space of tempered distributions. This solution is compactly supported [29]. For the special case (3) we have $c_{j}=1$ if $j \in D$, otherwise $c_{j}=$ 0 , and the (unique!) solution $\varphi$ is a characteristic function of the attractor $G=G(M, D)$. Furthermore, as in the most of literature we assume that the refinement equations satisfy the sum rules:

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}^{n}} c_{M k-d}=1, \quad d \in D \tag{5}
\end{equation*}
$$

The equation for attractors (3) always satisfies it because the set of coefficient $\left\{c_{M k-d},, k \in\right.$ $\mathbb{Z}\}$ consists of zeros except for one coefficient being equal to one.

### 4.2 The basic tile

There are several methods to analyse regularity of solutions of refinement equations. Some of them such as the matrix method can find the precise values of the Hölder exponent. The main idea is to pass from the refinement equation on $\mathbb{R}^{n}$ to an equation on a vector function defined on some basic tile. So, the matrix method requires an auxiliary tile $Q=$ $Q(M, \Delta)$ generated by the same matrix $M$ and by some set of digits $\Delta=\left\{\delta_{0}, \ldots, \delta_{m-1}\right\}$. An arbitrary tile generated by the matrix $M$ can play the role of a basic tile.

### 4.3 Invariant subsets of $\mathbb{Z}^{n}$

The first step to realize the matrix method is to choose a special finite subset of $\mathbb{Z}^{n}$. Let us have a refinement equation (4) with a mask $\boldsymbol{c}=\left\{c_{k}, k \in \mathbb{Z}^{n}\right\}$. Consider a map $\eta: 2^{\mathbb{Z}^{n}} \rightarrow 2^{\mathbb{Z}^{n}}$ that to every set of integers $X \subset \mathbb{Z}^{n}$ associates the set $M^{-1}(X+\operatorname{supp} \boldsymbol{c}-\Delta)$ $\cap \mathbb{Z}^{n}$. Let us recall that $\operatorname{supp} \boldsymbol{c}=\left\{k \in \mathbb{Z}^{n}, c_{k} \neq 0\right\}$. Since $\boldsymbol{c}$ is compactly supported, $|\operatorname{supp} \boldsymbol{c}|<\infty$.
Definition 5 Let a digit set $\Delta$ and a compactly supported mask $\boldsymbol{c}=\left\{c_{k}\right\}_{k \in \mathbb{Z}}$ be given. $A$ finite set $S \subset \mathbb{Z}^{n}$ is called invariant if $\eta S \subset S$.

Example 1 Consider the real line with the dilation $M=2$ and digits $\Delta=\{0,1\}$. Then for $\operatorname{supp} \boldsymbol{c}=\{0, N\}$, the set $S=\{0, \ldots, N-1\}$ is invariant. Indeed, $S+\operatorname{supp} \boldsymbol{c}=$ $\{0, \ldots, 2 N-1\}$, hence $S+\operatorname{supp} \boldsymbol{c}-\Delta=\{-1,0, \ldots, 2 N-1\}$. All integers in the set $\frac{1}{2}\{-1,0, \ldots, 2 N-1\}$ are $\{0, \ldots, N-1\}$, hence $\eta S=S$. Every segment of integers that contains $S$ is also an invariant set. The same holds for every mask with support that contains numbers $0, N$ and some integers (may be all) between them.

For the support supp $\boldsymbol{c}=\{0,7\}$ and $\Delta=\{0,1\}$, but with the dilation $M=-2$, the set $S=\{0, \ldots, N-1\}$ is nor longer invariant, but the set $S=\{-4,-2,-1,0,1\}$ is.

For the real line with the dilation $M=3$, digits $\Delta=\{0,1,2\}$ and supp $\boldsymbol{c}=\{0,1,5\}$, the set $S=\{0,1,2\}$ is invariant.

For a given mask $\boldsymbol{c}$, we consider the following set:

$$
\begin{equation*}
\Gamma=\Gamma(M, \operatorname{supp} \boldsymbol{c})=\left\{x \in \mathbb{R}^{n} \mid x=\sum_{j=1}^{\infty} M^{-j} \gamma_{j}, \quad \gamma_{j} \in \operatorname{supp} \boldsymbol{c}\right\} \tag{6}
\end{equation*}
$$

If supp $\boldsymbol{c}$ is a digit set for $M$, then $\Gamma$ is an attractor. The set $\Gamma$ will be referred to as support set of the refinement equation. In general, this set may not coincide with $\operatorname{supp} \varphi$. However, we always have $\operatorname{supp} \varphi \subset \Gamma$. [5, Proposition 2.2]. Among all invariant integer sets $S$ we spot two ones:

Proposition 2 For every tile $Q$ and a mask $\boldsymbol{c}$, each of the following sets is invariant:
a) $S_{0}=\left\{a \in \mathbb{Z}^{n}| |(a+Q) \cap \Gamma \mid>0\right\}$;
b) $\bar{S}_{0}=\left\{a \in \mathbb{Z}^{n} \mid(a+Q) \cap \Gamma \neq \emptyset\right\}$.

Proof. We establish a), the proof of b) is the same, replacing positivity of the measure by the nonemptiness.

Take arbitrary $s \in S_{0}$ and show that if there exist $\delta \in \Delta$ and $\gamma \in \operatorname{supp} \boldsymbol{c}$ such that the point $x=M^{-1}(s+\gamma-\delta)$ is integer, then $x \in S_{0}$. This will imply that $S_{0}$ is invariant. We have $M x-s+\delta=\gamma$. Since the sets $s+Q$ and $\Gamma$ possess an intersection of a positive measure, so do the shifted sets $s+Q+(M x-s+\delta)$ and $\gamma+\Gamma$ because they are shifted by the same vector. Thus, the sets $Q+\delta+M x$ and $\gamma+\Gamma$ have an intersection of positive measure. Hence, so do $M^{-1}(\delta+Q)+x$ and $M^{-1}(\gamma+\Gamma)$. However, $M^{-1}(\delta+Q) \subset G$ and $M^{-1}(\gamma+\Gamma) \subset \Gamma$. Hence, the bigger sets $x+Q$ and $\Gamma$ also have an intersection of positive measure, therefore, $x \in S_{0}$.

For a given finite set $K \subset \mathbb{Z}^{n}$ we denote by $S_{K}$ the smallest invariant set of integers containing $K$. This set is merely an intersection of all invariant sets containing $K$.

Proposition 3 For the set $K=\{0\}$, we have $S_{K}=S_{0}$, where the set $S_{0}$ is defined in Proposition 2.

Proof. By Proposition 2, $S_{0}$ is an invariant set. Clearly, both $Q$ and $\Gamma$ contain a neighbourhood of zero since $0 \in D$ and $0 \in \operatorname{supp} \boldsymbol{c}$. Therefore, $S_{0}$ contains zero and hence contains the minimal invariant set $S_{K}$. Thus, $S_{K} \subset S_{0}$. If this inclusion is strict, then the set $\Gamma^{\prime}=Q+S_{K}$ does not cover $\Gamma$. On the other hand, $M^{-1}\left(\gamma+\Gamma^{\prime}\right) \subset \Gamma^{\prime}$ for all $\gamma \in \operatorname{supp} \boldsymbol{c}$. Indeed, for each $q \in S_{K}$, the set $M^{-1}(q+\gamma+Q)$ is a parallel shift of some $M^{-1}(\delta+Q)$ to an integer vector $x$. Hence $q+\gamma=\delta+M x$ and $x \in \eta S_{K}$. Consequently, $x \in S_{K}$ and $M^{-1}(q+\gamma+Q)=x+M^{-1}(d+Q) \subset x+\Gamma \subset \Gamma^{\prime}$. Thus, $M^{-1}\left(\gamma+\Gamma^{\prime}\right) \subset \Gamma^{\prime}$. Therefore, the fractal corresponding to the family of contractions $M^{-1}(\gamma+\cdot), \gamma \in \operatorname{supp} \boldsymbol{c}$ is contained in $\Gamma^{\prime}$. On the other hand, this fractal is $\Gamma$, and so $\Gamma \subset \Gamma^{\prime}$, which is a contradiction.

Proposition 3 makes it possible to obtain the set $S_{0}$ algorithmically within finite time, without computing the sets $Q$ and $\Gamma$. This was done in [10, we just slightly modify that construction.

## An algorithm to compute $S_{K}$ for a given set $K$.

Initialisation. We have a finite set $K \in \mathbb{Z}^{n}$. Denote $K_{0}=K$ and $S_{0}=K$. The set of digits $\Delta$ and a mask $\boldsymbol{c}$ are given.
Main loop. After the $(j-1)$ st iteration we have a finite set of integers $S_{j-1}$ and its subset $K_{j-1}$. Set $S_{j}=S_{j-1}, K_{j}=\emptyset$. For each points $k \in K_{j-1}, c \in \operatorname{supp} \boldsymbol{c}$ and $\delta \in \Delta$, we check whether or not the point $x=M^{-1}(k+c-\delta)$ is integer and does not belong to $S_{j-1}$. If so, we set $K_{j}=K_{j} \cup\{x\}$, otherwise we set $K_{j}=K_{j}$. After all triples $(k, c, \delta)$ are exhausted, we set $S_{j}=S_{j-1} \cup K_{j}$. If $K_{j}=\emptyset$, then STOP, the algorithm terminates and $S_{K}=S_{j}$. Otherwise go to the next iteration.

Proposition 4 For every finite set $K \subset \mathbb{Z}^{n}$, the algorithm terminates within finite time and the final set $S_{j}$ is equal to $S_{K}$.

Proof. By the construction, $S_{j}=\cup_{s=0}^{j} \eta^{s} K$. Consider the operator $\xi$ that associates to each finite set $K \subset \mathbb{Z}^{n}$ the set $\xi K=M^{-1}(K+\operatorname{supp} \boldsymbol{c}-\Delta)$. Clearly, $\eta K \subset \xi K$. Therefore, $S_{j}$ is contained in the set $\cup_{s=0}^{\infty} \xi^{s} K$, whose closure is a fractal set of the finitely many contractions $K \mapsto M^{-1}(K+c-\delta)$, where $c \in \operatorname{supp} \boldsymbol{c}, \delta \in \Delta$. Since this fractal set is compact, it contains only a finite number of integers. All the sets $S_{j}$ produced by the algorithm are contained in this finite set. Hence, for some $j$ we necessarily have $S_{j}=S_{j-1}$. Since $S_{j}=\cup_{s=0}^{j} \eta^{s} K=\cup_{s=0}^{\infty} \eta^{s} K$, we see that $\eta S_{j} \subset S_{j}$, so $S_{j}$ is invariant. On the other hand, each invariant set that contains $K$ must also contain $\eta^{s} K$ for all $s$, hence it contains $S_{j}$. Thus $S_{j}=S_{K}$.

### 4.4 Spectral factorisation of the dilation matrix

Now we need to use spectral properties of the dilation matrix $M$. All eigenvalues of $M$ are bigger than one by modulus. Let $r_{1}<\cdots<r_{q}$ be all possible absolute values of eigenvalues of $M$ and let exactly $n_{i}$ of them (counting multiplicities) be equal by modulus to $r_{i}, i=1, \ldots, q(M)$. Let $J_{i} \subset \mathbb{R}^{n}$ be the linear span of root subspaces of $M$ corresponding to all eigenvalues of modulus $r_{i}$. Thus, $\operatorname{dim}\left(J_{i}\right)=n_{i}$ and the operator $\left.M\right|_{J_{i}}$ has all its eigenvalues equal to $r_{i}$ in the absolute value. The space $\mathbb{R}^{n}$ is a direct sum of $J_{1}, \ldots, J_{q}$ :

$$
\mathbb{R}^{n}=\bigoplus_{i=1}^{q} J_{i}
$$

There exists an invertible transformation $B: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that $M$ has the following block diagonal structure

$$
B^{-1} M B=\left(\begin{array}{cccc}
\left.M\right|_{J_{1}} & 0 & \cdots & 0  \tag{7}\\
0 & \left.M\right|_{J_{2}} & & \vdots \\
\vdots & & \ddots & 0 \\
0 & \cdots & 0 & \left.M\right|_{J_{q}}
\end{array}\right)
$$

The subspaces $J_{k}$ will be referred to as spectral subspaces and (7) is a spectral factorisation. In particular, if the matrix $M$ is isotropic, then $q(M)=1$, in which case $J_{1}$ coincides with $\mathbb{R}^{n}$. The converse is not true: if all eigenvalues of $M$ have equal moduli and the $M$ has nontrivial Jordan blocks, then it is not isotropic, although $q=1$.

### 4.5 Wide simplex and admissible sets

The invariant set $S_{0}$ defined in Proposition 2 possesses the following key property: shifts of the tile $Q$ over vectors from $S_{0}$ cover the set $\Gamma$. The set $\bar{S}_{0}$ has more: shifts of $Q$ over vectors from $\bar{S}_{0}$ cover a neighbourhood of $\Gamma$. We need to define a property which is between those two. First we introduce two more notation.

A wide simplex is a simplex in $\mathbb{R}^{n}$ with one of vertices at the origin such that its interior intersects all spectral subspaces $J_{k}, k=1, \ldots, q$, of the matrix $M$.

The existence of wide simplices is easily shown (see also [9]). Moreover, a homothety about the origin respects wide simplices. Hence, every ball centered at the origin contains a wide simplex. Actually, even every half-ball contains a wide simplex.

Definition 6 Let $\Gamma$ be a support set of refinement equation defined by (6) and let $Q=$ $Q(M, \Delta)$ be a tile. A finite subset $S \subset \mathbb{Z}^{n}$ is called admissible if the set $S+Q$ contains the sum of $\Gamma$ with some wide simplex.

Since every ball centered at the origin contains a wide simplex, we see that $S$ is admissible whenever $S+Q$ contains a neighbourhood of $\Gamma$. In particular, the set $\bar{S}_{0}$ from Proposition 2 is always admissible. The set $S_{0}$ may be not, however, in most cases it is admissible as well. Actually we can always use the set $S=\bar{S}_{0}$. However, in some cases it is too large and can be replaced by a smaller admissible set. This is very important from the computational point of view, see Remark 3. This is the only reason for introducing the notions of wide simplices and of admissible sets.

### 4.6 The vector-function $v(x)$ and the transition matrices $T_{\delta}, \delta \in \Delta$

Now we are realising the main idea of the matrix approach to multivariate refinement equations. For an arbitrary refinement equation (4), we first choose a tile $Q=Q(M, \Delta)$ (basic tile) and then we pass from the function $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ to the vector-valued function $v$ : $Q \rightarrow \mathbb{R}^{N}$. To define this function, we take an arbitrary admissible invariant set $S \subset \mathbb{Z}^{n}$ (Definition 6) and denote $|S|=N$. Then $v(x)$ is defined as follows:

$$
\begin{equation*}
v: Q \rightarrow \mathbb{R}^{N}, \quad v(x)=v_{\varphi}(x)=(\varphi(x+k))_{k \in S}, \quad x \in Q \tag{8}
\end{equation*}
$$

For convenience, we enumerate the components of the vector $v$ by elements of the set $S$. Consider the $m$ following $N \times N$ transition matrices $T_{\delta}, \delta \in \Delta$, defined by the equality

$$
\begin{equation*}
\left(T_{\delta}\right)_{a b}=c_{M a-b+\delta}, \quad a, b \in S, \quad \delta \in \Delta \tag{9}
\end{equation*}
$$

Rows and columns of the transition matrices are enumerated by elements of the set $S$. We denote $\mathcal{T}=\left\{T_{\delta}: d \in \Delta\right\}$. The refinement equation on the function $\varphi(x)$ is equivalent to the following equation on the vector-valued function $v(x)$ :

$$
\begin{equation*}
v(x)=T_{\delta} v(M x-\delta), \quad x \in M^{-1}(Q+\delta), \quad \delta \in \Delta . \tag{10}
\end{equation*}
$$

Functional equations of this type are often called equation of self-similarity [33].
Remark 2 If in the definition of $v(x)$ we used an arbitrary invariant set $S$, then (rather surprisingly!) the $L_{p}$-regularity of $v$ might not be equal to the $L_{p}$-regularity of $\varphi$. The reason is that the function $v$ is defined on $Q$, while $\varphi$ is defined on the entire $\mathbb{R}^{n}$. That is why we had to use an admissible invariant set $S$. This guarantees that the union of translations $\bigcup_{k \in S}(k+Q)$ contains not only the support of $\varphi$ but a bigger set: the sum of this support with a wide simplex. Let us note that for measuring the Hölder regularity in $C\left(\mathbb{R}^{n}\right)$, this enlargement is not needed and any invariant set $S$ suffices [9]. The reason is that a continuous refinable function vanishes on the boundary of its support, which may not be true for an $L_{p}$ refinable function.

### 4.7 Special subspaces of $\mathbb{R}^{N}$

In the regularity analysis of refinable functions we deal with several linear affine subspaces in $\mathbb{R}^{N}$. First we define

$$
V=\left\{w=\left(w_{1}, \ldots, w_{N}\right) \in \mathbb{R}^{N} \quad: \quad \sum_{j=1}^{N} w_{j}=1\right\}
$$

It is well known that every compactly supported refinable function such that $\int_{\mathbb{R}^{n}} \varphi(x) d s=1$ possesses the partition of unity property:

$$
\sum_{k \in \mathbb{Z}^{n}} \varphi(x+k) \equiv 1
$$

Hence, after a multiplication of $\varphi$ by a constant it may be assumed that $v(x) \in V$ for almost all $x \in G$. We denote the linear part of the affine subspace $V$ by

$$
W=\left\{w=\left(w_{1}, \ldots, w_{N}\right) \in \mathbb{R}^{N} \quad: \quad \sum_{j=1}^{N} w_{j}=0\right\}
$$

Finally, define the space of differences of the vector-function $v=v_{\varphi}$ :

$$
\begin{equation*}
U=\operatorname{span}\{v(y)-v(x) \quad: \quad y, x \in Q\} \tag{11}
\end{equation*}
$$

Since $v(x) \in V$ for almost all $x \in Q$, we have $U \subset W$. The sum rules (5) imply that the column sums of each matrix $T_{\delta}$ are equal to one. Therefore, $T_{\delta} V \subset V$ and $T_{\delta} W \subset W$ for all $\delta \in \Delta$. Thus, $V$ is a common affine invariant subspace of the family $\mathcal{T}$ and $W$ is its common linear invariant subspace.

For $i=1, \ldots, q$, define the subspaces $U_{1}, \ldots U_{q}$ of the space $\mathbb{R}^{N}$ as follows:

$$
\begin{equation*}
U_{i}=\operatorname{span}\left\{v(y)-v(x) \quad: \quad x, y \in Q, y-x \in J_{i}\right\}, \quad i=1, \ldots, q(M) \tag{12}
\end{equation*}
$$

Note that $U_{i}$ are nonempty, due to the interior of $Q$ being nonempty. It is seen easily that the spaces $\left\{U_{i}\right\}_{i=1}^{q}$ span the whole space $U$, but their sum may not be direct. The subspaces $\left\{U_{i}\right\}_{i=1}^{q}$, unlike the subspaces $\left\{J_{i}\right\}_{i=1}^{q}$, may have nontrivial intersections. For example, they can all coincide with $U$. It turns out that all $U_{i}$ are common invariant subspaces for the matrices $T_{\delta}$.

Lemma 2 If $J$ is an invariant subspace for the matrix $M$, then $L=\operatorname{span}\{v(y)-v(x)$ : $y-x \in J\}$ is a common invariant subspace for all $T_{\delta}, \delta \in \Delta$.

Proof. If $u \in L$, then $u$ is a linear combination of several vectors of the form $v(y)-v(x)$ with $y-x \in J$. For every $\delta \in \Delta$ we define $x^{\prime}=M^{-1}(x+\delta), y^{\prime}=M^{-1}(y+\delta)$ and have

$$
v\left(y^{\prime}\right)-v\left(x^{\prime}\right)=T_{\delta}\left(v\left(M y^{\prime}-\delta\right)-v\left(M x^{\prime}-\delta\right)\right)=T_{\delta}(v(y)-v(x)) .
$$

Hence, $T_{\delta}(v(y)-v(x)) \in L$ for each pair $(x, y)$, and, therefore, $T_{\delta} u \in L$ for all $u \in L$.

### 4.8 The formula of regularity for refinable functions in $L_{p}$

This formula expresses the Hölder exponent of the refinable function with the $L_{p}$-spectral radius of matrices $T_{\delta}$ restricted to the subspaces $U_{i}$. For a given set of linear operators $\mathcal{A}=\left\{A_{0}, \ldots, A_{m-1}\right\}$ acting in $\mathbb{R}^{d}$ and for given $p \in[1,+\infty)$, the $L_{p}$-spectral radius ( $p$ radius) is defined by the formula:

$$
\rho_{p}=\rho_{p}(\mathcal{A})=\lim _{k \rightarrow \infty}\left(m^{-k} \sum_{A_{\ell_{i}} \in \mathcal{A}, i=1, \ldots, k}\left\|A_{\ell_{1}} \cdots A_{\ell_{k}}\right\|^{p}\right)^{1 / p k}
$$

The limit always exists and does not depend on the operator norm (see 30]. for more on properties of the $p$-radius). Clearly, for one operator, the value $\rho_{p}$ becomes the usual spectral radius, i.e., the largest by modulus eigenvalue. Already for two operators, the computation of the $p$-radius is a hard problem. For example, it is still not clear if the 1 -radius can be efficiently computed. On the other hand, for even integer $p$, the $p$-radius can be expressed by means of a usual spectral radius of some large matrix. For example, the 2-radius is equal
to the square root of the spectral radius of the following operator $\mathbf{A}$ acting on the space $\mathcal{M}_{d}$ of symmetric $d \times d$-matrices:

$$
\begin{equation*}
\mathbf{A}(X)=\frac{1}{m} \sum_{i=0}^{m-1} A_{i}^{*} X A_{i}, \quad X \in \mathcal{M}_{d} \tag{13}
\end{equation*}
$$

This operator acts on the $\frac{d(d+1)}{2}$-dimensional space $\mathcal{M}_{d}$ and obeys an invariant cone of positive semidefinite matrices. Hence, by the Krein-Rutman theorem [24], its largest by modulus eigenvalue $\lambda_{\max }$ (which can also be called Perron eigenvalue) is positive. The fact is $\rho_{2}\left(A_{0}, \ldots, A_{m-1}\right)=\sqrt{\lambda_{\max }}$ [28, 30].

The formula for $L_{p}$-regularity of univariate refinable functions was well-known [26, 30]. However, it offered a surprising resistance in extending to multivariate functions. For general dilation matrices $M$, this extension was done only in 2019 9]. The main idea is to find the Hölder exponent separately on the spectral subspaces $J_{i}$. The Hölder exponent of $\varphi$ along a subspace $J \subset \mathbb{R}^{n}$ is defined by

$$
\alpha_{p, J}(\varphi)=\sup \left\{\alpha \geq 0 \quad: \quad\|\varphi(\cdot+h)-\varphi(\cdot)\|_{p} \leq C\|h\|^{\alpha}, \quad h \in J\right\}
$$

The following theorem was proved in [9]. Let us remember that $\mathcal{T}=\left\{T_{\delta}, d \in \Delta\right\}$.
Theorem 4 Let $1 \leq p<\infty$. For a refinable function $\varphi \in L_{p}\left(\mathbb{R}^{n}\right)$, we have

$$
\begin{equation*}
\boldsymbol{\alpha}_{p, J_{i}}(\varphi)=\log _{1 / r_{i}} \rho_{p}\left(\left.\mathcal{T}\right|_{U_{i}}\right), \quad i=1, \ldots, q \tag{14}
\end{equation*}
$$

and, consequently,

$$
\begin{equation*}
\boldsymbol{\alpha}_{p}(\varphi)=\min _{i=1, \ldots, q} \log _{1 / r_{i}} \rho_{p}\left(\left.\mathcal{T}\right|_{U_{i}}\right) \tag{15}
\end{equation*}
$$

For isotropic matrices, when all $r_{i}$ are equal to $r=\rho(M)$, formula (15) looks as simple as for the univariate refinable functions: $\boldsymbol{\alpha}_{p}(\varphi)=\log _{1 / r} \rho_{p}\left(\left.\mathcal{T}\right|_{U}\right)$.

Remark 3 Theorem 4 expresses the $L_{p}$-Hölder regularity of a refinable function to $p$-radii of the transition matrices $T_{\delta}$ restricted to special common invariant subspaces. As we have mentioned, for even integer $p$, the $p$-radius can be computed as a Perron eigenvalue of some high-dimensional matrix, for other $p$ only approximate computational methods are known. At any rate, the complexity of computation depends significantly of the size of matrices $T_{\delta}$, which is $N=|S|$. That is why it is important to reduce the cardinality of the admissible set $S$. The set $S=\bar{S}_{0}$ is sometimes too large and it is possible to find a smaller admissible set using the notion of wide simplices.

## 5. Computing the surface regularity and surface dimension of attractors and tiles

The characteristic function of an attractor satisfies functional equation (3), which is the refinement equation with the coefficients $c_{k}=1$ if $k \in D$ and $c_{k}=0$ otherwise. Therefore, Theorem 14 can be applied directly for computing the Hölder regularity and (if the dilation matrix $M$ is isotropic) the surface regularity and the surface dimension of attractors. The specific mask containing only zeros and ones makes the computation easier. Moreover, it will allow us to come up with simpler formulas of regularity that do not involve the subspaces $U_{i}$, which are a priori, unknown (Theorem 6). This means that the same formulas can be applied to find the Hölder exponents and the surface regularity of multivariate Haar wavelets generated by arbitrary dilation matrices.

First of all, we observe that the refinement equation for attractors (3) admits the transition matrices $T_{\delta}$, which will be simple in the following sense:

Definition 7 A matrix is called simple if each its column contains precisely one entry equal to one and all others are zeros.

Proposition 5 Suppose $G(M, D) \subset \mathbb{R}^{n}$ is an attractor; then for every basic tile $Q(M, \Delta)$ and for every admissible invariant set $S$, the matrices $T_{\delta}, \delta \in \Delta$, are all simple.

Proof. We have $c_{k}=1$ if and only if $k \in D$, otherwise $c_{k}=0$. Therefore (formula (9)), $\left(T_{\delta}\right)_{a b}=c_{M a-b+\delta}=1$ if and only if $M a-b+\delta \in D$. If $b$ is fixed, then the set $b-\delta+D$ has precisely one common point with the lattice $M \mathbb{Z}^{n}$, since $D$ is a digit set. Therefore, one component of the $b$ th column of $T_{\delta}$ is one and the others are zeros.

All simple matrices form a miltiplicative matrix semigroup. Let us remember that $V$ is an affine hyperspace of $\mathbb{R}^{N}$ that consists of points with the sum of components being one and $W$ is its linear part. A simple matrix is column-stochastic, hence it respects both $V$ and $W$. As usual, $\mathcal{A}^{k}$ denotes the set of all products of matrices from $\mathcal{A}$ of length $k \geq \mathbb{N}$ (repetitions permitted). Clearly, $\left|\mathcal{A}^{k}\right|=m^{k}$. We write $\mathcal{A}_{0}^{k}$ for the set of matrices from $\mathcal{A}^{k}$ that have at most one positive row. If all matrices from $\mathcal{A}$ are simple, then so are all matrices from $\mathcal{A}^{k}$ and each matrix from $\mathcal{A}_{0}^{k}$ has one row of ones and all other elements are zeros.

Proposition 6 Let $\mathcal{A}=\left\{A_{0}, \ldots, A_{m-1}\right\}$ be a set of simple matrices. Then for every common invariant subspace $U \subset W$ of the matrices from $\mathcal{A}$, we have $\rho_{p}\left(\left.\mathcal{A}\right|_{U}\right)=\left[\rho_{1}\left(\left.\mathcal{A}\right|_{U}\right)\right]^{1 / p}$. In case $U=W$, the following formula holds:

$$
\begin{equation*}
\rho_{1}\left(\left.\mathcal{A}\right|_{U}\right)=\lim _{k \rightarrow \infty}\left[1-\frac{\left|\mathcal{A}_{0}^{k}\right|}{\left|\mathcal{A}^{k}\right|}\right]^{1 / k} \tag{16}
\end{equation*}
$$

Proof. We begin with the case $U=W$. The norm of every simple matrix restricted to $W$ is either zero (if this matrix has precisely one non-zero row) or between 1 and $\sqrt{\frac{n}{2}}$ otherwise.

Replacing the norms of all matrix products in the definition of $L_{1}$-spectral radius by those numbers we see that the quantity (16) is equal to $\rho_{1}\left(\left.\mathcal{A}\right|_{W}\right)$. Now consider the case of general $U \subset W$. Denote by $\mathcal{H}$ the set of all simple $N \times N$ matrices. This set is finite and for every $k$, all product from $\mathcal{A}^{k}$ belong to this set. Hence, the number $\left\|\left.\Pi\right|_{U}\right\|$ for arbitrary $\Pi \in \mathcal{A}^{k}$ and $k \in \mathbb{N}$, can take a finite number of values. Consequently, $\left\|\left.\Pi\right|_{U}\right\|^{p} \asymp\left\|\left.\Pi\right|_{U}\right\|$ and this equivalence is defined by two absolute constants. Therefore, the values $m^{-k} \sum_{\Pi \in \mathcal{A}^{k}}\left\|\left.\Pi\right|_{U}\right\|^{p}$ and $m^{-k} \sum_{\Pi \in \mathcal{A}^{k}}\left\|\left.\Pi\right|_{U}\right\|^{p}$ are equivalent by the same two constants. Hence, $\rho_{p}\left(\left.\mathcal{A}\right|_{U}\right)=$ $\left[\rho_{1}\left(\left.\mathcal{A}\right|_{U}\right)\right]^{1 / p}$, which concludes the proof.

Applying Theorem 4 and Proposition 6 we obtain
Theorem 5 For every attractor $G$, we have

$$
\boldsymbol{\alpha}=2 \min _{1, \ldots, q} \log _{\frac{1}{r_{i}}} \rho_{2}\left(\left.\mathcal{T}\right|_{U_{i}}\right)
$$

If the matrix $M$ is isotropic, then

$$
s=\boldsymbol{\alpha}=2 \log _{\frac{1}{r}} \rho_{2}\left(\left.\mathcal{T}\right|_{U}\right)
$$

This theorem allows us to find the Hölder regularity of any attractor as the largest eigenvalue of the operator A defined by (16) for $m$ operators $A_{\delta}=\left.T_{\delta}\right|_{U}, \delta \in \Delta$. If in addition the dilation matrix is isotropic, then the surface regularity is equal to the same value. In case $G$ is a tile, everything can be rewritten in a simpler terms. Moreover, in this case we can use the fact that all matrices $T_{\delta}$ are simple and the $L_{1}$-spectral radius $\rho_{1}\left(\left.\mathcal{T}\right|_{W}\right)$ obtains a combinatorial form (16).

Theorem 6 For every time $G$ generated by an isotropic matrix $M$, we have

$$
s=\boldsymbol{\alpha}=2 \log _{\frac{1}{r}} \rho_{2}\left(\left.\mathcal{T}\right|_{W}\right)
$$

This value is equal to the right hand side of equality (16).
The main advantage of this theorem is that the expression for the Hölder exponent and for the surface regularity of a tile does not depend on the subspace $U$ (which may be different for different tiles) and can be expressed by the 1-radius of the matrices $T_{\delta}$ restricted to the standard subspace $W=\left\{w \in \mathbb{R}^{N} \mid \sum_{i} w_{i}=0\right\}$. The proof requires one auxiliary result. As usual we denote by $\mathcal{T}^{k}$ the set of all products of length $k$ of matrices from $\mathcal{T}$.

Lemma 3 Let $G(M, D)$ be an attractor and $S \subset \mathbb{Z}^{d}$ be an admissible set for $G$; then there is $p \in N$ such that for every $i \in S \backslash \bar{S}_{0}$, all matrices from $\mathcal{T}^{p}$ have zero ith row.

Proof. Consider the transition operator

$$
\boldsymbol{T} f(x)=\sum_{k \in D} f(M x-k)
$$

Clearly, for the function $\varphi=\chi_{G}$, we have $\boldsymbol{T} \varphi=\varphi$. If $f$ is the indicator function of some compact set $K \subset \mathbb{R}^{n}$, then the Hausdorff distance between the support of the function $\boldsymbol{T}^{p} f$ and $G$ tends to zero as $p \rightarrow \infty$. Indeed, since $M^{-1}$ has spectral radius smaller than one, there exists a norm in $\mathbb{R}^{n}$ such that in the corresponding operator norm $q=\left\|M^{-1}\right\|<1$. In this norm, the distance between the support of $\boldsymbol{T}^{p} f$ and $G$ is at most $q^{j}$ times the distance between the support of $f$ and $G$. Since $i \notin \bar{S}_{0}$, it follows that $i+Q$ does not intersect $G$. Therefore, for all sufficiently large $p$, the support of $\boldsymbol{T}^{p} f$ does not intersect the set $i+Q$. Let now $K=j+Q$ for some $j \in S$ and $f=\chi_{K}$. Then for every sequence $\ell_{1}, \ldots, \ell_{k}$, the element in the $i$ th row and $j$ th column of the matrix $\Pi=T_{\ell_{1}} \cdots T_{\ell_{p}}$ is equal to $\boldsymbol{T}^{p} f\left(i+0 . \ell_{1} \ldots \ell_{k}\right)$. However, $i+0 . d_{1} \ldots d_{k} \in i+Q$, hence $\boldsymbol{T}^{p} f\left(i+0 . \ell_{1} \ldots \ell_{k}\right)$ and so $\Pi_{i j}=0$. Thus, for all sufficiently large $i$, the $i$ th row of every product $\Pi \in \mathcal{T}^{p}$ is zero.

Proof of Theorem 6. Since $G$ is a tile, for almost all $x$, the vector $v(x)$ has only one non-zero component, which is equal to 1 . Namely, $v_{k}=1$ if $x \in k+G$. Denote $\mathcal{I} \in S \cap \bar{S}_{0}$ and $L=\operatorname{span}\left\{e_{i}-e_{j} \mid i, j \in \mathcal{I}\right\}$. Then $U=L$. Furthermore, by Lemma 3, there is $p$ such that for all $i \notin \mathcal{I}$, the $i$ th row of every matrix from $\mathcal{T}^{p}$ is zero. Therefore, $\rho_{1}\left(\left.\mathcal{T}\right|_{W}\right)=\rho_{1}^{1 / p}\left(\left.\mathcal{T}^{p}\right|_{W}\right)=\rho_{1}^{1 / p}\left(\left.\mathcal{T}^{p}\right|_{L}\right)=\rho_{1}\left(\left.\mathcal{T}\right|_{L}\right)$ and hence $\rho_{1}\left(\left.\mathcal{T}\right|_{U}\right)=\rho_{1}\left(\left.\mathcal{T}\right|_{W}\right)$. On the other hand, since $M$ is isotropic, then all $r_{i}$ are equal to $r$, therefore formula (15) reads $\boldsymbol{\alpha}=\log _{\frac{1}{r}} \rho_{1}\left(\left.\mathcal{T}\right|_{U}\right)$. This completes the proof.

Figure 1: The one-dimensional tile with $M=3, D=\{0,1,5\}: s=0.1977$.. and $\boldsymbol{d}=0.8022$..

## 6. Examples

Example 2 The univariate tile $G(M, D)$ with $M=3$ and $D=\{0,1,5\}$. Its characteristic function $\varphi=\chi_{G}$ satisfies the refinement equation

$$
\varphi(x)=\varphi(3 x)+\varphi(3 x-1)+\varphi(3 x-5), \quad x \in \mathbb{R},
$$

Since every point of $G$ has the form $x=\sum_{k=1}^{\infty} \ell_{k} 3^{-k}$ with $\ell_{k} \in D$, we have $x \leq 5 \sum_{k=1}^{\infty} 3^{-k} \leq \frac{5}{2}$. Therefore $G \subset[0,2.5]$. This tile is shown in fig. 1 .
Taking the basic tile $Q=[0,1]$ generated by the digit set $\Delta=\{0,1,2\}$, we have $\cup_{k=0,1,2}(k+$ $Q)=[0,3]$. Therefore, $S=\{0,1,2\}$ is an admissible invariant set. Hence, $N=3$ and there are three transition matrices:

$$
T_{0}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), \quad T_{1}=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right) \quad \text { and } \quad T_{2}=\left(\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) .
$$

The subspace $W$ is two-dimensional. Choosing the basis $e_{1}=(1,-1,0)^{T}, e_{2}=(0,1,-1)^{T}$ of $W$, we obtain the matrices $A_{\delta}=\left.T_{\delta}\right|_{W}, \delta=0,1,2$ :

$$
A_{0}=\left(\begin{array}{cc}
1 & 0 \\
1 & -1
\end{array}\right), \quad A_{1}=\left(\begin{array}{ll}
0 & 1 \\
0 & 1
\end{array}\right) \quad \text { and } \quad A_{2}=\left(\begin{array}{rr}
-1 & 0 \\
0 & 0
\end{array}\right)
$$

To compute $\rho\left(\left.\mathcal{T}\right|_{W}\right)=\rho_{2}\left(A_{0}, A_{1}, A_{2}\right)$ we build the matrix of the operator $\mathbf{A}$ defined by (13). The space of symmetric $2 \times 2$ matrices has dimension 3 , and

$$
\mathbf{A}=\frac{1}{3}\left(\begin{array}{ccc}
2 & 1 & 2 \\
1 & 2 & 2 \\
0 & -1 & -1
\end{array}\right) .
$$

For this matrix, $\lambda_{\max }=\frac{1+\sqrt{2}}{3}$, and therefore $\rho_{2}\left(\left.\mathcal{T}\right|_{W}\right)=\sqrt{\frac{1+\sqrt{2}}{3}}$. Now by Theorem 6 . we have $s(G)=\boldsymbol{\alpha}(G)=-\log _{3} \frac{1+\sqrt{2}}{3}=0.1977 \ldots$. Hence, the surface dimension of $G$ is $\boldsymbol{d}=1-s=0.8022 \ldots$.


Figure 2: The Square: $s=1$ and $\boldsymbol{d}=1$


Figure 3: The Dragon: $s=0.4763$.. and $\boldsymbol{d}=1.5236$..

Example 3 Two-digit tiles on the plane. There are only three attractors in $\mathbb{R}^{2}$ up to affine similarity which are generated by two digits, i.e., when $m=2$, see [40. In all the three cases the digit set can be $D=\{(0,0) ;(1,0)\}$. In this case all those three attractors are tiles.

The first one is a unit square (fig. 2), it is generated by the matrix $M=\left(\begin{array}{c}0-2 \\ 1\end{array} 0\right.$. . Of course, for this tile $\boldsymbol{s}=1$ and $\boldsymbol{d}=1$.

The second type is more interesting. This is the Dragon (fig. 3) generated by the matrix

$$
M=\left(\begin{array}{rr}
1 & 1 \\
-1 & 1
\end{array}\right)
$$

(rotation on $45^{\circ}$ with the expanding by $\sqrt{2}$ ). Denote this tile by $G$.
To compute $s$ we choose the set $\bar{S}_{0}$ (Proposition 2 ), in which case the basis tile $Q$ coincides with $G$. We have $\bar{S}_{0}=(0,0),( \pm 1,0),(0, \pm 1),( \pm 1, \pm 1)$ (seven points), therefore $N=7$ and the matrices $T_{0}$ and $T_{1}$ are $7 \times 7$. Computing $\rho_{2}\left(\left.\mathcal{T}\right|_{W}\right)$ we obtain $\boldsymbol{\alpha}=0.4763 \ldots$ Therefore $s=0.4763$.. and $\boldsymbol{d}=1.5236 \ldots$


Figure 4: The Bear: $s=0.7892$.. and $\boldsymbol{d}=1.2107$..

The third two-digit tile is generated by the matrix

$$
M=\left(\begin{array}{rr}
1 & -2 \\
1 & 0
\end{array}\right)
$$

As it is seen in fig. 4, if is natural to call it Bear. It has the same set $\bar{S}_{0}$ and hence its transition matrices $T_{0}, T_{1}$ are again $7 \times 7$. Computing $\rho_{2}\left(\left.\mathcal{T}\right|_{W}\right)$ we obtain $\boldsymbol{\alpha}=0.7892 \ldots$ Therefore $\boldsymbol{s}=0.7892$.. and $\boldsymbol{d}=1.2107 \ldots$ Thus, the Bear has a bigger surface regularity than the Dragon.

Example 4 Plane tiles with $M=2 I$. In this case $m=|\operatorname{det} M|=4$, hence there will be four digits. Different choices of these digits define different attractors. For example, for $D=\{(0,0) ;(1,0) ;(1,0) ;(1,1)\}$, we obtain a unit square. In this case, of course, $\boldsymbol{\alpha}=1$ and $\boldsymbol{d}=1$. Changing one digit: $(1,1)$ to $(-1,-1)$, we obtain the tile depicted in fig. 5

For this tile $\boldsymbol{s}=0.4150 \ldots$ and respectively $\boldsymbol{d}=1.5849 \ldots$. So, its regularity is close to the Dragon. It looks similar to the Sierpinski carpet ans can be called quasi Sierpinski tile. In contrast to the Sierpinski carpet, which has measure zero, it has measure 1 as a tile. This tile was considered in [39].

## 7. Attractors of the highest regularity

The highest possible surface regularity of any set is one. It is attained, for example, for sets bounded by surfaces of finite area: for convex sets, for sets with piecewise smooth boundaries, or for finite unions of such sets. We conjecture that for self-affine attractors, this situation is impossible apart from the case of parallelepipeds.

Conjecture 2 If an attractor satisfies $\boldsymbol{s}=1$, then it is a parallelepiped.


Figure 5: The quasi-Sierpinski tile. $s=0.4150 \ldots, \boldsymbol{d}=1.5849 \ldots$

For two-digit attractors on the plane this is true, since there are only three types of such attractors [40]. In Example 3 we analysed all of them: for the square we have $s=1$, for Dragon and for Bear, $s<1$. In every dimension $n$, for each $m \geq 2$, there are finitely many, up to an affine similarity, pairs $(M, D)$, $|\operatorname{det} M|=|D|=m$, for which the corresponding attractors are a parallelepipeds. Such pairs are all classified in [41]. For them, of course, $s=1$. Conjecture 2 claims that for all other attractors $s<1$. This means, in particular, that an attractor which is not a parallelepiped cannot be presented as a finite union of regular sets (either convex or with a piecewise-smooth boundary).

We can prove only the univariate version of Conjecture 2;
Theorem 7 If an attractor $G \subset \mathbb{R}$ is such that $\boldsymbol{s}=1$, then $G$ is a segment.
We prove more: if $\boldsymbol{\alpha}(G)=1$, then $G$ is a segment. The proof uses some facts from the theory of univariate refinement equations, from approximation theory, and from combinatorics. Since $M$ is a number, we assume in the proof that $M>0$, the case of negative $M$ is considered in the same way. Thus, $M=m$. We begin with proving several auxiliary results.

Lemma 4 Let a set $P \subset \mathbb{R}$ consist of finitely many disjoint segments with integer ends. Suppose several translates of $P$ form a disjoint (up to sets of measure zero) partition of some segment. Then all the segments of the set $P$ have the same length and all distances between them are multiples of that length.

In the proof it will be convenient to use words "left" and "right" for the standard orientation on the real line.

Proof. Without loss of generality it can be assumed that translates of $P$ to positive numbers cover a segment without overlaps. Denote the most left segment of $P$ by $\alpha$ and the next segment by $\beta$. The distance between $\alpha$ and $\beta$ must be filled with several translates of $\alpha$, hence this distance is a multiple of $|\alpha|$. The first translate of $P$ maps the segment $\beta$ to the segment $\beta+|\alpha|$. The gap between those two segments is of length $|\alpha|-|\beta|$, it must be filled
with several translates of the segment $\alpha$. Therefore, $|\alpha|-|\beta|=k|\alpha|$ for some non-negative integer $k$. Hence $k=0$ and $|\beta|=|\alpha|$. Then by the same argument we show that the next segment of $P$ has length $|\alpha|$ and that the distance from $\beta$ to that segment is a multiple of $|\alpha|$, etc.

A compactly-supported function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy the Strang-Fix condition of order $\ell \in \mathbb{Z} \cup\{0\}$ if linear combinations of its integer translates generate all algebraic polynomials of order $\leq \ell$. This condition is important in approximation theory, see [35].

Lemma 5 A characteristic function of a compact set cannot satisfy the Strang-Fix condition of order bigger than zero.

Proof. We need to show that integer translates of $f$ cannot generate a linear function. Assume the contrary: $\sum_{k \in \mathbb{Z}} a_{k} f(x-k) \equiv x$ for some coefficients $\left\{a_{k}\right\}_{k \in \mathbb{Z}}$. Let $\operatorname{supp} f \subset$ $[-N, N]$. Then for every $x \in[0,1]$ we have: $\sum_{k=-N}^{N} a_{k} f(x-k)=x$. On the other hand, $f$ takes only the values 0 and 1 , hence the function $\sum_{k=-N}^{N} a_{k} f(x-k)$ takes finitely many values. This is a contradiction since the function $x$ takes infinitely many values on $[0,1]$.

The following result is crucial in the proof of Theorem 7.
Proposition 7 If $\boldsymbol{\alpha}(G)=1$, then $G$ is a union of segments with integer ends.
Proof. By [29, Theorem 7.1.1], if a refinable function $\varphi$ is such that $\boldsymbol{\alpha}_{1}(\varphi)=1$ and it does not satisfy the Strang-Fix condition of order 1 , then it is $L_{1}$-Lipschitz, i.e there is a constant $C>0$ for which

$$
\begin{equation*}
\|\varphi(x+h)-\varphi(h)\|_{1} \leq C h, \quad h \geq 0 \tag{17}
\end{equation*}
$$

By Lemma 5, this is true for the function $\varphi=\chi_{G}$. We take the system of digits $\Delta=$ $\{0, \ldots, M-1\}$ with the corresponding tile $Q=[0,1]$. Then, up to an integer translate, it can be assumed that all digits from $D$ are nonnegative and as always $d_{0}=0$. Let $N$ be the number bigger by one than the largest digit form $D$. Since, $D \subset[0, N-1]$, we have $G=\operatorname{supp} \varphi \subset[0, N-1]$. Since $\cup_{k=0, \ldots, N-1}(Q+k)=[0, N]$, we see that the set $S=\{0, \ldots, N-1\}$ is admissible. We consider the corresponding vector-function $v(x)=$ $(\varphi(x), \ldots, \varphi(x+N-1))^{T} \in \mathbb{R}^{N}$, and the transition $N \times N$ matrices $T_{0}, \ldots, T_{M-1}$ defined in formula (9) for the sequence $c_{k}=1$ of $k \in D$ and $c_{k}=0$ otherwise. For every $h \in(0,1)$, denote $\varphi_{h}(x)=\varphi(x+h)-\varphi(x)$ and $v_{h}(x)=\left(\varphi_{h}(x), \ldots, \varphi_{h}(x+N-1)\right)^{T}$. Applying the self-similarity equation (10) we obtain

$$
\begin{equation*}
v_{M^{-1} h}(x)=T_{\delta} v_{h}(M x-\delta), \quad \delta \in \Delta . \tag{18}
\end{equation*}
$$

Denote by $\mathcal{R}_{k}$ the set of all $M$-adic rational numbers of order $k$ on the interval $[0,1)$, thus $R_{k}=\left\{0 . \delta_{1} \ldots \delta_{k} \mid \delta_{i} \in \Delta, i=1, \ldots, k\right\}$. For every $q=0 . \delta_{1} \ldots \delta_{k} \in \mathcal{R}_{k}$, let $\Pi_{q}=$ $T_{\delta_{1}} \cdots T_{\delta_{k}}$. Iterating $k$ times equation (18) we obtain

$$
\begin{equation*}
v_{M^{-k} h}(x)=\Pi_{q} v_{h}\left(M^{k}(x-q)\right), \quad x \in\left[q, q+M^{k}\right), q \in \mathcal{R}_{k} \tag{19}
\end{equation*}
$$

Denote $A_{\delta}=\left.T_{\delta}\right|_{U}$. It was proved in [29, Theorem 5.2.2] that a refinable function $\varphi$ is $L_{1^{-}}$ Lipschitz if and only if for every $k$, we have $\sum_{q \in \mathcal{R}_{k}}\left\|\left.\Pi_{q}\right|_{U}\right\| \leq C_{0}$, where $C_{0}$ is some constant. On the other hand, all the products $\Pi_{q}, q \in \mathcal{R}_{k}$, are simple matrices. Denote by $C_{1}$ the smallest positive norm of all simple matrices restricted to $U$. Thus, for every $q \in \mathcal{R}_{k}$, either $\left.\Pi_{q}\right|_{U}=0$ or $\left\|\left.\Pi_{q}\right|_{U}\right\| \geq C_{1}$. Hence, for every $k \in \mathbb{N}$, the set $\left\{\left.\Pi_{q}\right|_{U} \mid q \in \mathcal{R}_{k}\right\}$ contains at most $r=\left[\frac{C_{0}}{C_{1}}\right]$ nonzero operators. Apply this fact to equation 19) taking into account that the vector $v_{h}\left(M^{k}(x-q)\right)$ belongs to $U$. We see that on all but $r$ intervals $\left[q, q+M^{-k}\right)$, the function $v_{M^{-k} h}(x)$ is an identical zero. This holds for every $h \in(0,1)$ and the set of segments on which this function is zero is the same for all $h$. Therefore on all but $r$ those segments, the function $v(x)$ is an identical constant. This is true for all $k \in \mathbb{N}$. Hence, the function $v(x)$ is piecewise-constant with at most $r$ points of discontinuity. Consequently, the function $\varphi$ is piecewise-constant with at most $r+N+1$ points of discontinuity (in the $N+1$ integer points of the segment $[0, N]$ the function $\varphi$ may also be discontinuous). By [27, Theorem 1] for every piecewise-constant refinable function with finitely many points of discontinuity, all those points are integer. Hence, $G$ is a union of segments with integer ends.

Proof of Theorem 7. By Proposition 7, the set $G$ consists of several segments with integer ends. Without loss of generality it can be assumed that the number $M$ is positive and exceeds the diameter of the set $G$. Otherwise we iterate the refinement equation for $\varphi=\chi_{G}$ several times, say $k$ times, and obtain a refinement equation with the factor $M^{k}$, then we just replace $M$ by $M^{k}$. Denote by $\alpha$ the most left segment of $G$. We have $G=\cup_{d \in D} M^{-1}(G+d)$, hence $M G=\cup_{d \in D}(G+d)$. The most left segment of $G$ is $M \alpha$, its length $M|\alpha|$ exceeds the diameter of $G$ (since $M>\operatorname{diam}(G))$. Moreover, the distances from this segment to other segments of $M G$ being multiples of the number $M$ also exceed the diameter of $G$. Therefore, $M \alpha=\cup_{d \in D^{\prime}}(G+d)$, where $D^{\prime}$ is some subset of $D$. Thus, several translates of the set $G$ form the segment $M \alpha$. Applying now Lemma 4 to the set $P=G$, we obtain that all segments of the set $G$ have the same length $|\alpha|$ and all distances between them are multiples of this number. Therefore, all distances between the segments of the set $M G$ are multiples of $M|\alpha|$. Hence, each segment of the set $M G$ is $M \alpha+k M|\alpha|$ with some $k \in \mathbb{N}$. Consequently, for that segment we have $M \alpha+k M|\alpha|=\cup_{d \in D^{\prime}+k M|\alpha|}(G+d)$. Therefore, $D^{\prime}+k M|\alpha| \subset D$. However, the sets $D^{\prime}$ and $D^{\prime}+k M|\alpha|$ are equal modulo $M$. This is impossible, since all elements of the digit set $D$ are different modulo $M$.

## 8. Application to synchronising automata

The theory of synchronising automata originated in 1960s has found numerous applications in engineering and computer science. It is actively developing in the modern literature, see [11, 22, 37, 38] and references therein.

Suppose some system can be at $N$ different states. There are $m$ actions that change the states of the system. The $k$ th action changes the states according to a prescribed mapping $f_{k}$ defined on the set of states. If we enumerate the states by numbers from 1 to $N$, then the
$k$ th action changes the $j$ th state to the state $f_{k}(j), j=1, \ldots, N$. The set of states and of actions is called deterministic finite automaton. It can be defined by the directed graph $G(V, E)$ with coloured edges. The vertices correspond to the states and edges of each color are associated to actions. There is an edge of $k$ th colour from the vertex $j$ to the vertex $i$ if $f_{k}(j)=i$. Thus, $G$ has $N$ vertices and $m$ outgoing edges from every vertex - one edge of each colour. The edges of the $k$ th colour generate an adjacency matrix $B_{k}$. We have $\left(B_{k}\right)_{i j}=1$ if $f_{k}(j)=i$ and $\left(B_{k}\right)_{i j}=0$ if $f_{k}(j) \neq i$. Thus, the matrix $B_{k}$ is simple in the sense of Definition 77 every column possesses exactly one element equal to one and all others being zeros. We see that a deterministic finite automaton is completely defined by the family of simple $N \times N$ matrices $\mathcal{B}=\left\{B_{0}, \ldots, B_{m-1}\right\}$. Conversely, every family of simple $N \times N$ matrices defines an automaton. Here it will be more convenient to enumarate the actions (colours) not from 1 to $m$ as in the most of literature on automata but from 0 to $m-1$.

A finite sequence of actions (colours) is called a synchronising sequence or reset word if application of this sequence of actions sends the system to one and the same state, independently of the initial state. In terms of the matrices $\left\{B_{k}\right\}_{k=0}^{m-1}$ a reset word is a sequence of numbers $k_{1}, \ldots, k_{s}$ from $\{0, \ldots, m-1\}$ such that the corresponding product $\Pi=B_{k_{s}} \cdots B_{k_{1}}$ has a row of ones. Since $\Pi$ is a simple matrix it follows that all other entries of $\Pi$ are zeros.

In practice a reset word allows the user to make a reset the system i.e., sending it to the initial state even if its current state is unknown. There are lots of applications of this notion in computer science, electronics, robotics, etc. There are efficient polynomial time algorithms to decide the existence of a reset word and to find it [38]. On the other hand, finding the shortest possible reset word is an NP-complete problem [13, 38]. There is a famous Cerný conjecture (1964) claiming that if a reset word exists then the shortest reset word has length at most $(N-1)^{2}$. This lower bound is sharp [7]. The conjecture is still open and the best known upper bounds is cubic in $N$ [36].

Now come back for a moment to the self affine attractors. Let $G$ be an attractor. The transition matrices $T_{\delta}, \delta \in \Delta$, are all simple. Therefore the family $\mathcal{T}$ of these matrices generates a deterministic finite automaton. What is the sense of the the surface regularity $s(G)$ is terms of the automaton of the family $\mathcal{T}$ ? Can it be of interest to the automata theory? To answer this question we introduce a concept of parameter of synchronisation. In the following theorem we use the same subspace $W=\left\{x \in \mathbb{R}^{N} \mid \sum_{i=1}^{N} x_{i}=0\right\}$ and denote $A_{k}=\left.B_{k}\right|_{W}$, where $B_{k}$ is an adjacency matrix of an automaton.
Theorem 8 Let a deterministic finite automaton be given. For a natural $k$, let $P_{k}$ be the probability that a random word of length $k$ of the alphabet $\{0, \ldots, m-1\}$ is not a reset word. Then there exists a limit $p=\lim _{k \rightarrow \infty}\left[P_{k}\right]^{1 / k}$. This limit is equal to the spectral radius $\rho(\mathbf{A})$ of the operator

$$
\begin{equation*}
\mathbf{A}(X)=\frac{1}{m} \sum_{i=0}^{m-1} A_{i}^{*} X A_{i}, \quad X \in \mathcal{M}_{N-1} \tag{20}
\end{equation*}
$$

acting on the $N(N-1) / 2$-dimensional space $\mathcal{M}_{N-1}$ of symmetric matrices of size $N$.
The automaton has a reset word if and only if $\rho(\mathbf{A})<1$.
Proof. Denote by $C_{1}$ and $C_{2}$ respectively the smallest and the largest strictly positive norms of all simple matrices restricted to the subspace $W$. Since there are finitely many simple
matrices, it follows that $C>0$. If a word $\ell_{1} \ldots \ell_{k}$ is reset, then the product $\Pi=B_{\ell_{1}} \cdots B_{\ell_{k}}$ has a row of ones and therefore $\left.\Pi\right|_{W}=A_{\ell_{1}} \cdots A_{\ell_{k}}=0$. Otherwise, $C_{1} \leq\left\|A_{\ell_{1}} \cdots A_{\ell_{k}}\right\| \leq C_{2}$. For every $k$, the number of nonzero products among all products $A_{b_{1}} \cdots A_{b_{k}}$ is equal to $m^{k} P_{k}$. Therefore, the value

$$
S_{k}=m^{-k} \sum_{b_{1}, \ldots, b_{k}}\left\|A_{b_{1}} \cdots A_{b_{k}}\right\|^{2}
$$

is between $C_{1}^{2} P_{k}$ and $C_{2}^{2} P_{k}$. The power $1 / k$ of this value tends to $\rho_{2}^{2}$, where $\rho_{2}$ is the $L_{2^{-}}$ spectral radius of the family $A_{0}, \ldots, A_{m-1}$. Hence the limit $\lim _{k \rightarrow \infty}\left[P_{k}\right]^{1 / k}$ exists and is equal to $\rho_{2}^{2}$, which is in turn equal to the spectral radius of the operator 20 .

Since all norms $\left\|A_{b_{1}} \cdots A_{b_{k}}\right\|$ are bounded above by $C_{2}$, it follows that $\rho_{2} \leq 1$, and hence $p=\rho_{2}^{2} \leq 1$. It is well-known that for every family of operators $\left\{A_{0}, \ldots, A_{m-1}\right\}$, there exists a constant $C>0$ such that $S_{k} \geq C \rho_{2}^{2 k}$ for every $k \in \mathbb{N}$. Hence if $p=1$, then $S_{k} \geq C$ for all $k$. However, if there exists at least one zero product of those operators, then $S_{k} \rightarrow 0$ as $k \rightarrow \infty$. Therefore, if $p=1$, then there is no zero product, which means that there is no product of operators $B_{0}, \ldots, B_{m-1}$ with a row of ones, i.e., there is no reset word.

We call the number $p$ the parameter of synchronisation of the automaton. It has the following meaning. Assume we do not know a reset word and instead take a random sequence of actions of length $k$; then we obtain a reset word apart from the probability approximately $p^{k}$. Thus, the parameter $p$ shows the degree of random synchronisation of an automaton. As we see from Theorem 8, this parameter can be effectively computed merely by finding the largest eigenvalue of operator (20). The following theorem reveals a curious relation between the parameter of synchronisation and the surface regularity of a self-affine tile.

Theorem 9 Let a tile $G(M, D)$ be given and its dilation matrix $M$ be isotropic; then for the automaton defined by the transition matrices $T_{\delta}, \delta \in \Delta$, the parameter of synchronisation $p$ is equal to $r^{-s}$, where $r=\rho(M)$ and $\boldsymbol{s}$ is the surface regularity of $G$.

Proof. Applying Theorem 6 we obtain $s=2 \log _{1 / r} \rho_{2}$ and hence $\rho_{2}^{2}=r^{-s}$. By Theorem 8 . $p=\rho_{2}^{2}$, which completes the proof.

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## References

[1] S. Akiyama, N. Gjini On the connectedness of self-affine attractors Archiv der Mathematik, 82 (2004), no 2, 153 - 163.
[2] A. Akopyan, A. Hubard, and R.Karasev, Lower and upper bounds for the waists of different spaces, Topol. Methods Nonlinear Anal. 53 (2019), no. 2, 457 - 490,
[3] L. Ambrosio, A. Colesanti, and E.,Villa, Outer Minkowski content for some classes of closed sets Mathematische Annalen 342 (2008), no 4, 727 - 748.
[4] C.A. Cabrelli, C. Heil and U.M. Molter, Accuracy of lattice translates of several multidimensional refinable functions, J. Approx. Theory, 95 (1998), 5-52.
[5] C. A. Cabrelli, C. Heil and U.M. Molter, Self-similarity and multiwavelets in higher dimensions, Memoirs Amer. Math. Soc., 170 (2004), no. 807.
[6] A. S. Cavaretta, W. Dahmen, and C.A. Micchelli, Stationary subdivision, Memoirs Amer. Math. Soc. 93 (1991), no. 453.
[7] J. Černý, Poznámka $k$ homogénnym experimentom $s$ konečnými automatmi, Matematicko-fyzikálny časopis Slovenskej Akadémie Vied, 14 (1964), 208 - 216 (in Slovak).
[8] M. Charina, Vector multivariate subdivision schemes: Comparison of spectral methods for their regularity analysis, Appl. Comp. Harm. Anal., 32 (2012), 86 - 108.
[9] M. Charina, V.Yu. Protasov, Regularity of anisotropic refinable functions, Appl. Comp. Harm. Anal., 47 (2019), no. 3, 795 - 821.
[10] M. Charina, T. Mejstrik, Multiple multivariate subdivision schemes: matrix and operator approaches, J. Comput. Appl. Math. 349 (2019), 279 - 291.
[11] P.-Y. Chevalier, J.M. Hendrickx, R.M. Jungers, Reachability of consensus and synchronizing automata, Proceedings of 54th IEEE Conference on Decision and Control (CDC), 2015.
[12] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61, SIAM, Philadelphia, 1992.
[13] D. Eppstein, Reset sequences for monotonic automata, SIAM J. Comput., 19 (1990), No 3, $500-510$.
[14] H. Federer, Geometric Measure Theory, Springer-Verlag, NY (1969)
[15] K. Gröchenig and A. Haas, Self-similar lattice tilings, J. Fourier Anal. Appl., 2 (1994), 131 - 170.
[16] K. Gröchenig, W.R. Madych, Multiresolution analysis, Haar bases and self-similar tilings of $\mathbb{R}^{n}$, IEEE Trans. Inform. Theory, 38 (1992), 556 - 568.
[17] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, J. Approx. Theory, 124 (2003), $44-88$.
[18] R.-Q. Jia, Subdivision schemes in $L_{p}$ spaces, Advances Comp. Math., 3 (1995), 455 454.
[19] M. Kolountzakis, The study of translational tiling with Fourier analysis, Fourier analysis and convexity (2004), 131 -- 187, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004.
[20] M. Kolountzakis, M. Matolcsi, Tilings by translation, La Gaceta de la Real Sociedad Espanola, 13 (2010), 4
[21] M. Kolountzakis, Y. Wang, The structure of multiplicative tilings of the real line, J. Fourier Anal. Appl. 25 (2019), no. 3, 1248 -- 1265.
[22] Z. Kohavi, Switching and finite automata theory, McGraw-Hill, New York (1970).
[23] S.G Krantz, H.R. Parks, The geometry of domains in space, Birkhäuser Advanced Texts: Basler Lehrbücher, Boston, MA: Birkhäuser Boston
[24] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Transl. (1950), no 26, 128 pp
[25] A. Krivoshein, V. Yu. Protasov, and M. A. Skopina, Multivariate wavelets frames, Springer, 2016
[26] J. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^{n}$. II. Lattice tilings, J. Fourier Anal. Appl. 3 (1997), 83 - 102.
[27] W. Lawton, S. L. Lee, and Z. Shen, Characterization of compactly supported refinable splines, Adv. Comput. Math. 3 (1995), no. 1-2, 137 -145,
[28] K.-S. Lau, J. Wang, Characterization of $L_{p}$ solutions for two-scale dilations equations, SIAM J. Math. Anal. 26 (1995), 1025 - 1046.
[29] I. Novikov, V. Yu. Protasov, and M. A. Skopina, Wavelets theory, AMS, Translations Mathematical Monographs, 239 (2011).
[30] V. Yu. Protasov, The generalized spectral radius. A geometric approach, Izvestiya Math., 61 (1997), 995 - 1030.
[31] V. Yu. Protasov, Fractal curves and wavelets, Izvestiya Math., 70 (2006), 123 - 162.
[32] V. Yu. Protasov, Spectral decomposition of 2-block Toeplitz matrices and refinement equations, St.Petersburg Math. J., 18 (2007), 607 - 646.
[33] V. Yu. Protasov, Extremal $L_{p}$-norms and self-similar functions, Lin. Alg. Appl., 428 (2008), 2339 - 2357.
[34] M. Resman, Invariance of the normalized Minkowski content with respect to the ambient space, Chaos, Solitons \& Fractals, 57 (2012), 123-128.
[35] G. Strang, G.J.Fix, An analysis of the finite element method, Prentice-Hall Series in Automatic Computation. Englewood Cliffs.-NJ; Prentice-Hall, Inc. XIV (1973).
[36] M. Szykula, Improving the upper bound on the length of the shortest reset words, (2017) arXiv:1702.05455
[37] A.N. Trahtman, The Černý conjecture for aperiodic automata, Discr. Math. Theoret. Comput. Sci. 9 (2007), No 2, 3 - 10.
[38] M.V Volkov, Synchronizing automata and the Černý conjecture, Language and automata theory and applications, 11 - 27, Lecture Notes in Comput. Sci., 5196, Springer, Berlin (2008).
[39] P. Wojtaszczyk, Mathematical introduction to wavelets, London Math. Soc. student texts, 37, Cambridge Univ. Press (1997).
[40] T. Zaitseva, Haar wavelets and subdivision algorithms on the plane, Adv. Syst. Sci. Appl. 17 (2017), no 3, 49 - 57.
[41] T. Zaitseva, Simple tiles and attractors, Sb. Math. 211 (2020), no 9.


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    ${ }^{\dagger}$ University of L’Aquila (Italy), Moscow State University (Russia), e-mail: v-protassov@yandex.ru

