SIAM J. CONTROL OPTIM.
Vol. 59, No. 2, pp. 825–856

SOCIAL OPTIMA IN MEAN FIELD LINEAR-QUADRATIC-GAUSSIAN CONTROL WITH VOLATILITY UNCERTAINTY*

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Abstract. This paper examines mean field linear-quadratic-Gaussian social optimum control with volatility-uncertain common noise. The diffusion terms in the dynamics of agents contain an unknown volatility process driven by a common noise. We apply a robust optimization approach in which all agents view volatility uncertainty as an adversarial player. Based on the principle of person-by-person optimality and a *two-step duality* technique for stochastic variational analysis, we construct an auxiliary optimal control problem for a representative agent. Through solving this problem combined with a consistent mean field approximation, we design a set of decentralized strategies, which are further shown to be asymptotically social optimal by perturbation analysis.

Key words. mean field game, social control, common noise, uncertainty, forward-backward stochastic differential equation

AMS subject classifications. 91A13, 91A23, 93E03, 93E20

DOI. 10.1137/19M1306737

1. Introduction.

1.1. Large population system and mean field game.

The large population (LP) systems have found wide applications across a broad spectrum, including economics, biology, engineering, and social science [9, 14, 20, 55]. The most salient feature of the LP system is the interactive *weakly* coupling structure across a large number of agents: each individual influence on the entire system is negligible, but their overall population impact is substantial and cannot be ignored. Recently, dynamic decisions of an LP system have become more important with the recent rapid growth of practical decision systems exhibiting large-scaled interactions. Subsequently, the mean field game (MFG) has drawn intensive research attention because it provides an effective theoretical scheme to analyze asymptotic behavior of *controlled* LP systems with competitive agents. In particular, it results in an MFG strategy which enables us to take advantage of mean field interaction to transform the analysis of the (high-dimensional) LP game to an optimization problem for one single representative agent (low-dimensional) with response to aggregation effects of other individuals.

Let us recall that MFG theory was initiated by the parallel works of Lasry and Lions [34] and Huang, Caines, and Malhamé [28] with a general aim to analyze the weakly coupled interactions of an LP system of rational agents with conflicting objectives. Specifically, [28] designed an ϵ-Nash equilibrium for a decentralized strategy

*Received by the editors December 13, 2019; accepted for publication (in revised form) November 24, 2020; published electronically March 1, 2021.
https://doi.org/10.1137/19M1306737
Funding: This work was funded in part by RGC Early Career Scheme (ECS) grants 502412P and 500613P, by the National Natural Science Foundation of China under grant 61773241, by NSF grant DMS-1812921, and in part by the PolyU-SDU Joint Research Centre on Financial Mathematics.
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with discount costs based on a Nash certainty equivalence (NCE) approach. NCE is also called the consistency condition (CC). Independently, [34] introduced a model of MFGs and studied the well-posedness of a limiting coupled partial differential equation systems, which was inspired by the McKean–Vlasov (MV) equation from physical particle systems. To date, MFGs have been extensively discussed and a quite rich literature has been accumulated centering it. Thus, instead of a comprehensive review, we are inclined to give a compact review of MFG works more relevant to the present work, along with the following classifications toward a clear comparison.

1.2. A compact literature review of MFG. First, depending on the state and cost setup, MFG studies can be classified into linear-quadratic-Gaussian (LQG) or more general nonlinear MV types. The LQG type presents its weak coupling through the so-called state average and is commonly adopted in MFG because of its analytical tractability and good modeling approximation to various nonlinearity. In this regard, the approximate Nash equilibrium of MFG can be represented, in its closed-loop sense, through a Riccati equation with a forcing-term equation to identify the limiting state average. Some relevant LQG MFG works include [28, 35, 6, 50, 10, 39] and references therein; our present work, to be specified soon, is also based on such an LQG setup. On the other hand, a nonlinear MV type of MFG formalizes weak coupling through so-called empirical distribution and is also of great importance because of its modeling generality. In this regard, the approximate Nash equilibrium can be resolved by coupling a (backward) Hamilton–Jacobi–Bellman equation with a (forward) Kolmogorov equation intended to identify the marginal distributions of optimal state. A large body of work is also devoted to this type; see [30, 16, 34, 33].

Second, depending on decision diversity and hierarchy, MFG can be classified as a homogeneous, heterogeneous, or mixed game. In a homogeneous game, all agents are *symmetric* (*exchangeable*) [33, 19] and *minor peers*. By symmetric, we mean all agents are endowed with identical coefficients not depending upon agent index and hence are statistically identical in decision behaviors; by minor peers, we mean a given single agent has no global influence on the population. In a heterogeneous game, all agents remain minor but may exhibit diversity in coefficient datum; thus, they are no longer symmetric in decision, and as a consequence, the resulting equilibrium should be parameterized by diversity cardinality (*discrete* or *continuum*). A mixed game is more distinctive especially in its decision hierarchical structure, rather than merely in system datum: it involves some *major* agents by imposing a dominant influence upon all minor agents, thus having some global effects which do not diminish even when the population size tends to infinity. This dominance may be modeled by the appearance of major states in dynamics or costs of all agents, with order $O(1)$ as agent number N tends to infinity. A mixed game is a realistic setup for modeling a monopoly in economic dynamics. Hence, it has attracted considerable attention. For instance, [27, 11] investigated LQG mixed MFGs with a major agent and various symmetric minor agents and provided ϵ-Nash equilibrium strategies. Wang and Zhang [51] studied a mixed game in a discrete-time case. Buckdahn, Li, and Peng [13] discussed a nonlinear mixed game using the probabilistic approach.

Third, a related but distinct concept to the mixed game is the MFG with common noise where common noise can be interpreted as some passive effect from an *uncontrolled* major player [26]. Such modeling can accommodate considerable situations by noting that common noise might represent an external factor affecting simultaneously all the agents participating in a game. This is well framed in reality, for instance, the physical environment for all particles or a financial policy for all market participants.

Consequently, mixed games with common noise are well motivated by a variety of applications in finance and economics, especially by the study of *system risk* [17, 22]. As a trade-off, the introduction of common noise also affixes more technical difficulties in its analysis. In particular, the limiting state average in the LQG type, or limiting empirical measure in the nonlinear type, becomes a *random* process driven by the tail filtration of all agents. Thus, the associated analysis ingredient becomes more complicated. Some relevant works include [17, 1, 21], [42], and in principle, MFG with common noise can be analyzed using the pure probabilistic approach with stochastic maximum principle and dynamic programming.

Last, existing MFG studies are tremendously rich and the current review only serves as a brief survey. For more comprehensive details, interested readers may refer to [15] for a useful overview of MFG. For an introduction to both theory and applications of MFG, see especially the Paris-Princeton Lectures [22] and the surveys [20], [14]. We also draw attention to the recent monographs concerning MFG, such as [9, 18].

1.3. Social optimum control by mean field analysis. Apart from noncooperative MFG, social optimum control by mean field analysis has also drawn increasing attention recently. The social optimum problem refers to an LP system in which all players cooperate to optimize some common social cost—the sum of individual costs. Social optima are linked to a type of team decision [23] but with highly complex interactions. All agents in a team decision access different information sets, thus social optima are *decentralized* and differ from classical vector optimization with a centralized designer. When player number $N \longrightarrow +\infty$, some mean field team-optimization problem is inspired to study the asymptotic behavior of an LP system with two approaches along this line: the direct method [34, 31] and a fixed-point method. We list a few relevant works for the second one. The work [29] considered social optima in mean field LQG control and provided an asymptotic team-optimal solution. Wang and Zhang [52] investigated a mean field social optimal problem in which a Markov jump parameter appears as a common source of randomness for all agents. The study of [32] designed socially optimal strategies by analyzing forward-backward stochastic differential equations (FBSDEs). For further literature, see [2] for team-optimal control with finite population and partial information, [44] for dynamic collective choice by finding a social optimum, [45, 46] for stochastic dynamic teams and their mean field limit, [41] for social optima in economic models subject to idiosyncratic shocks, and [47] for reinforcement learning algorithms for mean field teams.

1.4. Volatility uncertainty with common noise. Motivated by the aforementioned studies, the present study explores a class of robust cooperative mean field social optimum problems. Specifically, we focus on team optimization in an LQG setup with symmetric minor agents, driven by common noise but with uncertainty in its volatility term. More details of the motivation behind our problem are presented as follows.

In [25], the authors investigate mean field models with a *global* uncertainty term, which means that all players share a common unknown deterministic disturbance. They adopted the "soft constraint" approach [7] by removing the bound of disturbance while the effort is simultaneously penalized in cost function. The studies [8, 39] consider the case where each agent is paired with its *local* disturbance, and provide an ϵ-Nash equilibrium by tackling a Hamilton–Jacobi–Isaacs equation combined with fixed-point analysis. Another study relevant to our work is in [53], [54], which present robust analysis of mean-field social control with uncertain drift only. Because of the absence of volatility uncertainty therein, a closed-loop strategy with a consistency condition is still admissible in terms of a standard Riccati equation. In addition, asymptotic social optimality could still be verified in [53] directly based on a stationary condition of the strategy specified by Riccati equations obtained.

Unlike [53], [54], this paper is devoted to volatility uncertainty of social optimum control in a mean field LQG setup with common noise. Notice that various studies of mathematical finance (e.g., pricing and hedging [4, 40]) have remarkably focused on markets with uncertain volatility. In [12], uncertain volatility models are introduced to evaluate a scenario where the volatility coefficient of the pricing model cannot be determined exactly. Therefore, a practical motivation here is that, in many decision problems, a large number of coupled decision markers share a common noise but with uncertain volatility on it. For instance, volatility of trading prices in a financial market is often unknown and the *implied volatility* has thus been inspired and well studied. Subsequently, when some cooperative investors concern their team optima, it becomes necessary to study the social optimization with volatility uncertainty. Another example is system risk minimization in an interbanking system: all branches (of team formation) are subject to some uncertainty in common system noise, thus robust volatility analysis arises when seeking optima in joint operations. So, it is worthwhile to study the cooperative mean field model with volatility uncertainty [12, 37]. Moreover, for *linear* dynamics (e.g., wealth process in the Merton model), their volatilities are often inexact by allowing some modeling errors; thus, when some *quadratic hedging* is considered, the LQG setup is suggested and we adopt it here.

1.5. The analysis outline and comparison. Now, we outline our analysis components to be applied, along with the necessary literature comparison to other works. Recently, there have arisen various works (see, e.g., [3, 45, 46]) for mean field teams discussing the decentralized control and related asymptotic team optimality in the context of LP exchangeable agents. No uncertainty is formulated in their modelings and thus their analysis can be conducted in a *positive definite* setting. As a comparison, our setting here is mainly *indefinite* because of the uncertainty and soft-constraint introduced. Such indefiniteness brings difficulty when analyzing the related convexity that is crucial for the solvability of our problem. The works [8, 39, 5] consider some min-max problems in MFG or team settings to address the possible robustness. In these works, each agent is paired with a *local* drift uncertainty, and the ϵ-Nash/team decisions can be designed by solving the saddle-point conditions in an auxiliary robust control problem. In contrast, our present work differs essentially from them because our mean field social optimization is imposed with *global* uncertain volatility through the common noise. To handle the global influence of volatility uncertainty, we first solve a high-dimensional indefinite state-weights optimal control problem with respect to volatility uncertainty, and then construct an auxiliary control problem via a two-step duality procedure. Notably, even when the control for uncertainty is centralized, we can still reach a set of decentralized strategies for all agents. The consistency system is obtained through embedding representation of a nonstandard mean field type FBSDE. The above analysis techniques on volatility differ substantially from those of [8, 39, 5], where uncertainty is only imposed on the drift term.

In addition, at first glance, this present work seems somewhat similar to our previous works [53], [54], and [25]. However, various subtle and essential differences exist between them, in both setup and analysis. We highlight some key differences below for a more clear comparison.

(i) Our present study examines the uncertainty of team optimization; thus, a variational analysis should be conducted to test the response of related componentwise Fréchet differentials for a given agent. Such an analysis is not required in [25] when studying the uncertainty of MFGs when all agents are competitive.

(ii) In team optimization, a key step is to verify (uniform) convexity of the social cost functional, which is high-dimensional. For team optimization (e.g., [29]) with standard assumption (SA), such convexity follows directly because the SA weights are all positive (nonnegative) definite. However, it becomes more challenging in the present study because some weights are intrinsically indefinite due to the soft constraint and min-max setup here. Even though negative weight is also addressed in [25], (uniform) convexity therein is more tractable: only low-dimensional optimization needs to be treated in a *competitive* game context. More precisely, in [25], we need only to consider perturbation for a given single agent to verify the approximate Nash equilibrium by fixing other agents' strategies. However, the present study must consider team perturbation for all agents instead of a single one only; thus, the convexity involved is high-dimensional and indefinite, which becomes more technical to check.

(iii) Uncertainty (disturbance) in [25] is postulated to be *deterministic* on the *drift* term only. Thus, the related CC system by the fixed-point argument reduces to a forward-backward ordinary differential equation (FBODE), for which the well-posdness is more tractable. For instance, *the compatibility method* in [38] still works in [25] for such an FBODE but fails here to the more complicated FBSDE of the consistency condition due to volatility uncertainty.

(iv) Unlike [53], [54] for team optimization with drift uncertainty only, volatility uncertainty imposed here brings more technical difficulties. For example, more subtle estimates for a fully coupled consistency FBSDE system, especially for its (backward) adjoint solution in a common noise component, should be invoked. Instead of the Riccati equation approach, we mainly adopt the FBSDE analysis for the solvability of the related optimization problems. Moreover, for the auxiliary problem construction for social optimality, the related variational analysis becomes rather involved (see section 5). Furthermore, it differs fairly from that of [53], [54] mainly because of common noise and volatility uncertainty. More crucially, a *two-step duality* procedure (see section 4.2) should be applied and a new type of auxiliary problem is constructed, whereas in [53], [54], only single-step duality is required. In addition, different from [53], [54], the consistency system here requires a new *embedding representation* type. For the related asymptotic social optimality, the verification in [53], [54] can still proceed via standard Riccati decoupling and relevant estimation. Nevertheless, the verification of asymptotic team optimization in the current work becomes more complicated. In particular, Riccati decoupling is not well workable here, thus we adopt some Fréchet derivative and quadratic functional representation methods (see section 7.1).

To conclude, the main contributions of this paper can be summarized as follows:

(1) The volatility uncertainty of team optimization on common noise is introduced and formulated in a soft-constraint setting. Two sequential optimization problems are also formulated.

(2) An auxiliary control problem is constructed via a two-step duality procedure, and the consistency system is obtained through embedding representation of a nonstandard mean field type FBSDE. The related uniform convexity (concavity) is also established in the high-dimensional case.

(3) We obtain global solvability of related FBSDEs in some nontrivial and nonstandard case.

(4) The decentralized optimal team strategy is derived in an open-loop sense, and its asymptotic social optimality is verified in a robust social sense.

The rest of this paper is organized as follows. Section 2 formulates the volatility uncertainty with soft constraint; section 3 discusses the control problem with volatility uncertainty; section 4 investigates team optimization in person-by-person optimality; based on this, section 5 designs the decentralized strategies through a CC system; section 6 analyzes the well-posedness of FBSDEs, which arises from the consistency system; section 7 presents asymptotic robust social optimality of the decentralized strategy; and section 8 concludes the paper.

2. Problem formulation. We denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, $\mathbb{R}^{n\times k}$ the set of all $n \times k$ matrices, and $\otimes$ the Kronecker product. We use $|\cdot|$ to denote the norm of a Euclidean space, or the Frobenius norm of matrices. For a vector or matrix M, M^T denotes its transpose; for two vectors x, y, $\langle x, y\rangle = x^T y$. For symmetric matrix Q and a vector z, $|z|_Q^2 = z^T Qz$, and $Q > 0$ ($Q \geq 0$) means that Q is positive (nonnegative) definite. Consider a finite time horizon $[0, T]$ for $T > 0$; for a given filtration $\mathbb{G} \triangleq \{\mathcal{G}_t\}_{0\leq t\leq T}$, denote $L^2_{\mathbb{G}}(0, T; \mathbb{R}^\ell)$ ($L^2_{\mathbb{G}}(\Omega; C([0, T]; \mathbb{R}^\ell))$) the space of all $\mathbb{R}^\ell$-valued $\mathcal{G}_t$-progressively measurable (continuous) processes $s(\cdot)$ satisfying $\|s\|^2_{L_2} := \mathbb{E}\int_0^T |s(t)|^2 dt < \infty$ ($\|s\|^2_{\max} := \mathbb{E}\sup_{0\leq t\leq T} |s(t)|^2 < \infty$). For convenience of presentation, we may use c (or $c_1, c_2, \ldots$) to denote a generic constant which does not depend on the population size N of the LP system and may vary from place to place.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a sequence of independent one-dimensional Brownian motions $\{W_i(t), i = 0, 1, \ldots, N\}$ are defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration of $\{W_i(t), i = 0, 1, \ldots, N\}$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider a linear stochastic LP system with N agents (or particles), in which the ith agent $\mathcal{A}_i$ evolves by

$$\begin{aligned} dx_i(t) &= [Ax_i(t) + Bu_i(t) + f(t)]dt + [Du_i(t) + \sigma(t)]dW_i(t) \\ &\quad +[C_0x_i(t) + D_0u_i(t) + \sigma_0(t)]dW_0(t), \quad x_i(0) = x_0, \quad i = 1, \ldots, N, \end{aligned} \tag{2.1}$$

where $x_i(\cdot)$ and $u_i(\cdot)$ are state and input of agent $\mathcal{A}_i$, valued in $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively, and $x_0 \in \mathbb{R}^n$ is a constant vector; coefficients A, B, D, C_0, D_0 are constant matrices of suitable sizes; $W_i(\cdot)$ is a Brownian motion representing the idiosyncratic noise for agent $\mathcal{A}_i$; and $W_0(\cdot)$ is a Brownian motion representing a common noise shared by all agents (a similar setup can be found in [14, 26]). For $i = 0, 1, \ldots, N$, let $\mathbb{F}^i = \{\mathcal{F}^i_t\}_{0\leq t\leq T}$ be the natural filtration of $W_i(\cdot)$ augmented by all the $\mathbb{P}$-null sets. Then, $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T} = \{\sigma(\bigcup_{i=0}^N \mathcal{F}^i_t)\}_{0\leq t\leq T}$ is called the *centralized information.* σ_0 is unknown volatility but note that it might not be only $\mathbb{F}^0 = \{\mathcal{F}^0_t\}_{0\leq t\leq T}$-adapted with $\mathcal{F}^0$ the information generated by common noise $W_0(\cdot)$.

Remark 2.1. The individual diffusion part of (2.1) driven by W_i does not include a term like Cx_i as in standard LQ control literature, mainly due to two concerns. First, introduction of Cx_i will bring considerable technical difference in relevant analysis and we plan to address it in future work; second, the current setup is still rather general, especially including risky investments as its special case (i.e., $\sigma = 0$). For simplicity, we assume that all the agents have the same initial state. It is not hard to extend our results to the case that initial states of agents are independent and identically distributed random variables.

When $D, D_0 \neq 0$, the control process enters diffusion terms (driven by $W_i(\cdot)$, $W_0(\cdot)$) of (2.1), and in this case (2.1) is said to be *diffusion-controlled*. The study of diffusion-controlled systems has attracted extensive attention, mainly because of their modeling power and application potential in operational research and mathematical finance, etc. The readers may refer to [58, 9, 48] for relevant studies of LQ diffusion-controlled systems and related applications in mean-variance and portfolio selection problems. By comparison, the *drift-controlled* (i.e., $D = D_0 = 0, B \neq 0$) system is more classical in the LQ literature and has been broadly adopted in most MFG or team studies (e.g., [28, 29, 50]). Besides modeling, the diffusion-controlled system also differs from the drift-controlled one in relevant analysis, for example, in the study of related Riccati equations and Hamiltonian systems.

Given state dynamics (2.1), the cost functional of $\mathcal{A}_i$ is given by

$$\begin{aligned} J_i(u) = \frac{1}{2}\mathbb{E}\int_0^T &\Big\{\big|x_i(t) - \Gamma x^{(N)}(t) - \eta(t)\big|_Q^2 + |u_i(t)|_R^2\Big\}dt \\ &+\frac{1}{2}\mathbb{E}|x_i(T) - \Gamma_0 x^{(N)}(T) - \eta_0|_G^2, \end{aligned} \tag{2.2}$$

where $x^{(N)} = \frac{1}{N}\sum_{j=1}^N x_j$ is the weakly coupled state average, and $u = \{u_1, \ldots, u_N\} \in \mathbb{R}^{r\times N}$ is the team strategy. The admissible strategy set of $\mathcal{A}_i$ is in the *distributed* sense:

$$\begin{aligned} \mathcal{U}_i^r = \Big\{u_i(\cdot) \in\ & L^2_{\mathbb{H}^i}(0,T;\mathbb{R}^r) : \\ & \mathbb{H}^i = \{\mathcal{H}_t^i\}_{0\le t\le T}, \mathcal{H}_t^i \triangleq \sigma\big\{\mathcal{F}_t^0 \cup \mathcal{F}_t^i \cup \sigma(x_i(s), 0 \le s \le t)\big\}\Big\}. \end{aligned}$$

Here, $\{\mathcal{H}_t^i\}$ denotes the decentralized (or distributed) information for the individual agent $\mathcal{A}_i$. Note that x_i is not $\{\mathcal{F}_t^i\}$-adapted because of the state average coupling $x^{(N)}$; thus, the inclusions of $\sigma(x_i(s))$ and $\mathcal{F}_t^i$ are both necessary in the above formulation. For comparison, the *centralized strategy* set is

$$\mathcal{U}_c^r = \Big\{u_i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^r)\Big\}.$$

Denote the social cost under volatility with soft constraint by

$$J_{\rm soc}^{(N)}(u,\sigma_0) = \sum_{i=1}^N \left(J_i(u) - \frac{1}{2}\mathbb{E}\int_0^T |\sigma_0(t)|_{R_0}^2 dt\right)$$

with R_0 being the attenuation parameter of soft constraint (see [7]). The main goal of the current paper is to seek a set of distributed strategies to minimize the social cost under soft constraint for system (2.1)–(2.2), i.e.,

$$\text{(P) minimize}_{u_i\in\mathcal{U}_i^r} J_{\rm soc}^{\rm wo}(u) \quad \text{with} \quad J_{\rm soc}^{\rm wo}(u) \triangleq \sup_{\sigma_0\in\mathcal{U}_c^n} J_{\rm soc}^{(N)}(u,\sigma_0)$$

over $\{u = (u_1, \ldots, u_i, \ldots, u_N), u_i \in \mathcal{U}_i^r, i = 1, \ldots, N\}$, where $J_{\rm soc}^{\rm wo}(u)$ is the social cost under the worst-case volatility.

To simplify the analysis, we introduce the following hypothesis.

(H1) The state and cost functional coefficients satisfy

$$\begin{cases} A, C_0, \Gamma, \Gamma_0 \in \mathbb{R}^{n\times n}, & B, D, D_0 \in \mathbb{R}^{n\times r}, \\ Q \ge 0, R > 0, R_0 > 0, G \ge 0, & f, \sigma, \sigma_0, \eta, \eta_0 \in L^2_{\mathbb{F}}(0,T,\mathbb{R}^n). \end{cases}$$

Under (H1), by [58], for any x_0 and $u_i \in \mathcal{U}_c^r$, (2.1) admits a unique strong solution,

$$\mathbf{x}^T(\cdot) = (x_1^T(\cdot), \ldots, x_i^T(\cdot), \ldots, x_N^T(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^{nN})),$$

where the following estimates hold true: for some c_1 independent of N,

$$\mathbb{E}\sup_{0\le t\le T}|\mathbf{x}(t)|^2 \le c_1\mathbb{E}\Bigg[N|x_0|^2 + N\bigg(\int_0^T |f(s)|ds\bigg)^2 + N\int_0^T \big(|\sigma_0(s)|^2 + |\sigma(s)|^2\big)ds + \sum_{i=1}^N \int_0^T |u_i(s)|^2 ds\Bigg].$$

3. The control problem with respect to volatility uncertainty. From now on, the time variable t might be suppressed when no confusion occurs. Let $u_i \in \mathcal{U}_c^r, i = 1,\dots,N$, be fixed. The optimal control problem with volatility uncertainty can be formulated as

$$\text{(P1) maximize}_{\sigma_0\in\mathcal{U}_c^n} J_{\rm soc}^{(N)}(u,\sigma_0)$$

which is equivalent to

(P1′) Minimize $\check{J}_{\rm soc}^{(N)}$ over $\sigma_0 \in \mathcal{U}_c^n$, where

$$\check{J}_{\rm soc}^{(N)}(\sigma_0) = \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^T \Big\{ -\big|x_i - \Gamma x^{(N)} - \eta\big|_Q^2 + |\sigma_0|_{R_0}^2\Big\}dt - \frac{1}{2}\mathbb{E}|x_i(T) - \Gamma_0 x^{(N)}(T) - \eta_0|_G^2.$$

Hereafter, the following notation will be used to enable more compact representation. Let $\mathbf{u} = (u_1^T,\dots,u_N^T)^T$, $\mathbf{1} = (1,\dots,1)^T$, $\sigma_i = (0,\dots,0,\sigma^T,0,\dots,0)^T$, $\mathbf{A} = \mathrm{Diag}(A,\dots,A)$, $\mathbf{B} = \mathrm{Diag}(B,\dots,B)$, $\mathbf{D}_i = \mathrm{Diag}(0,\dots,0,D,0,\dots,0)$, $\mathbf{C}_0 = \mathrm{Diag}(C_0,\dots,C_0)$, $\mathbf{D}_0 = \mathrm{Diag}(D_0,\dots,D_0)$, and $\mathbf{x}_0 = (x_0^T,\dots,x_0^T)$.

Remark 3.1. Hereafter, whenever necessary, we may exchange the usage of notation $u = (u_1,\dots, u_N) \in \mathbb{R}^{r\times N}$ and $\mathbf{u} = (u_1^T,\dots,u_N^T)^T \in \mathbb{R}^{rN}$ by noting they both represent the team decision profile among all agents but only differ in formations.

With the above notation, we can rewrite dynamics of all agents in a more compact form:

$$\begin{aligned} d\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}\mathbf{u}(t)dt + \mathbf{1}\otimes f(t)dt + \sum_{i=1}^N [\mathbf{D}_i\mathbf{u}(t) + \sigma_i(t)]dW_i(t) \\ &\quad + [\mathbf{C}_0\mathbf{x}(t) + \mathbf{D}_0\mathbf{u}(t) + \mathbf{1}\otimes\sigma_0(t)]dW_0(t),\ \mathbf{x}(0) = \mathbf{x}_0. \end{aligned}$$

Recall $\otimes$ denotes the Kronecker product. Also, we introduce the following notations:

$$\text{(3.1)}\qquad \begin{cases} \Xi_1 := \Gamma^T Q + Q\Gamma - \Gamma^T Q\Gamma, & \Xi_2 := Q\eta - \Gamma^T Q\eta, \\ \Xi_1^G := \Gamma_0^T G + G\Gamma_0 - \Gamma_0^T G\Gamma_0, & \Xi_2^G := G\eta_0 - \Gamma_0^T G\eta_0. \end{cases}$$

By rearranging the integrand of $\check{J}_{\rm soc}^{(N)}$, we have

$$\text{(3.2)}\qquad \check{J}_{\rm soc}^{(N)} = \frac{1}{2}\mathbb{E}\int_0^T \Big(-|\mathbf{x}|_{\hat{\mathbf{Q}}}^2 + 2\hat{\eta}^T\mathbf{x} + N|\sigma_0|_{R_0}^2\Big)dt - \frac{1}{2}\mathbb{E}\big(|\mathbf{x}(T)|_{\hat{\mathbf{G}}}^2 - 2\hat{\eta}_0^T\mathbf{x}(T)\big),$$

where $\hat{\eta} = \mathbf{1}\otimes\Xi_2$, $\hat{\eta}_0 = \mathbf{1}\otimes\Xi_2^G$, and $\hat{\mathbf{Q}} = (\hat{Q}_{ij})$, $\hat{\mathbf{G}} = (\hat{G}_{ij})$ are given respectively by

$$\text{(3.3)}\qquad \hat{Q}_{ii} = Q - \Xi_1/N,\ \hat{Q}_{ij} = -\Xi_1/N,\ \hat{G}_{ii} = G - \Xi_1^G/N,\ \hat{G}_{ij} = -\Xi_1^G/N,\ 1\le i\ne j\le N.$$

Denote

$$\mathbf{\Gamma}_i = \Big[-\frac{\Gamma}{N}, \dots, -\frac{\Gamma}{N}, I - \frac{\Gamma}{N}, -\frac{\Gamma}{N}, \dots, -\frac{\Gamma}{N} \Big],$$

where $I - \frac{\Gamma}{N}$ is the ith element. Note $\hat{\mathbf{Q}} = \sum_{i=1}^N \mathbf{\Gamma}_i^T Q \mathbf{\Gamma}_i$. Then

$$\lambda_{min}(Q) \sum_{i=1}^N \mathbf{\Gamma}_i^T \mathbf{\Gamma}_i \le \hat{\mathbf{Q}} \le \lambda_{max}(Q) \sum_{i=1}^N \mathbf{\Gamma}_i^T \mathbf{\Gamma}_i.$$

For further analysis, we assume

(H2) $\breve{J}_{\rm soc}^{(N)}(\sigma_0)$ of (P1$'$) is convex in σ_0;

(H2$'$) $\breve{J}_{\rm soc}^{(N)}(\sigma_0)$ of (P1$'$) is uniformly convex in σ_0.

We have the following equivalent conditions that ensure (H2).

PROPOSITION 3.2. *The following statements are equivalent:*

(i) $\breve{J}_{\rm soc}^{(N)}(\sigma_0)$ *is convex in* σ_0.

(ii) *For any* $\sigma_0 \in \mathcal{U}_c^n$,

$$\mathbb{E}\int_0^T \Big(-\mathbf{z}^T \hat{\mathbf{Q}} \mathbf{z} + N\sigma_0^T R_0 \sigma_0 \Big) dt - \mathbb{E}|\mathbf{z}(T)|^2_{\hat{\mathbf{G}}} \ge 0,$$

where $\mathbf{z} \in \mathbb{R}^{nN}$ *satisfies*

$$\begin{cases} d\mathbf{z} = \mathbf{A}\mathbf{z}dt + (\mathbf{C}_0\mathbf{z} + \mathbf{1} \otimes \sigma_0)dW_0, \\ \mathbf{z}(0) = 0. \end{cases}$$

(iii) $\bar{J}_i'(\sigma_0)$ *is convex in* σ_0, *where*

$$\bar{J}_i'(\sigma_0) \triangleq \mathbb{E}\int_0^T \Big\{ -\big|(I-\Gamma)z_i\big|^2_Q + |\sigma_0|^2_{R_0} \Big\} dt - \mathbb{E}|(I-\Gamma_0)z_i(T)|^2_G$$

subject to

$$dz_i(t) = Az_i(t)dt + [C_0 z_i(t) + \sigma_0(t)]dW_0(t), \ z_i(0) = 0. \tag{3.4}$$

Proof. (i) $\Leftrightarrow$ (ii) is given in [25]. From (3.4), we have $z_1 = z_2 = \cdots = z_N = z^{(N)}$. Thus,

$$\begin{aligned} &\mathbb{E}\int_0^T \Big(-|\mathbf{z}|^2_{\hat{\mathbf{Q}}} + N|\sigma_0|^2_{R_0} \Big) dt - \mathbb{E}|\mathbf{z}(T)|^2_{\hat{\mathbf{G}}} \\ &\quad = \sum_{i=1}^N \mathbb{E}\int_0^T \Big(-\big|z_i - \Gamma z_i\big|^2_Q + |\sigma_0|^2_{R_0} \Big) dt - \sum_{i=1}^N \mathbb{E}|(I-\Gamma_0)z_i(T)|^2_G \\ &\quad = N\Big[\mathbb{E}\int_0^T \Big(-\big|(I-\Gamma)z_i\big|^2_Q + |\sigma_0|^2_{R_0} \Big) dt - \mathbb{E}|(I-\Gamma_0)z_i(T)|^2_G \Big], \end{aligned} \tag{3.5}$$

which implies that (ii) is equivalent to (iii). ∎

Denote $\hat{\mathbf{1}} = \mathbf{1} \otimes I$. By [49], if the Riccati equation

$$\begin{aligned} &\dot{\mathbf{P}} + \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{C}_0^T \mathbf{P}\mathbf{C}_0 - \hat{\mathbf{Q}} - \big(\hat{\mathbf{1}}^T \mathbf{P}\mathbf{C}_0\big)^T \big[NR_0 + \hat{\mathbf{1}}^T \mathbf{P}\hat{\mathbf{1}}\big]^{-1} \hat{\mathbf{1}}^T \mathbf{P}\mathbf{C}_0 = 0, \\ &\mathbf{P}(T) = -\hat{\mathbf{G}}, \end{aligned} \tag{3.6}$$

admits a solution such that $NR_0 + \hat{\mathbf{1}}^T\mathbf{P}\hat{\mathbf{1}} > 0$, then $\breve{J}^{(N)}_{\text{soc}}(\sigma_0)$ is uniformly convex, which further gives that (H2$'$) holds. The above condition (3.6) is of high-dimension $nN\times nN$ which is not feasible to verify. Alternatively, we give the following necessary and sufficient condition with low-dimensionality.

Proposition 3.3. *The following statements are equivalent:*
(i) $\breve{J}^{(N)}_{\text{soc}}(\sigma_0)$ *is uniformly convex in* σ_0.
(ii) $\bar{J}'_i(\sigma_0)$ *is uniformly convex in* σ_0.
(iii) *The equation*

$$\dot{K}+KA+A^TK+C_0^TKC_0-C_0^TK(K+R_0)^{-1}KC_0+\Xi_1-Q=0,\ K=\Xi_1^G-G \tag{3.7}$$

admits a solution such that $K+R_0>0$.

Proof. (i) By (3.5) and [36], we have (i)$\Leftrightarrow$(ii). (ii)$\Leftrightarrow$(iii) is implied from [49]. $\square$

By examining the variation of $\breve{J}^{(N)}_{\text{soc}}$, we obtain the following result.

Theorem 3.4. *Suppose that* $R_0>0$*; then for any fixed admissible strategy set* $u=(u_1,\dots,u_N)\in\prod_{i=1}^N\mathcal{U}_i^r$*, problem* (P1$'$) *has a minimizer* $\sigma_0^*(u)$ *if and only if* (H2) *holds and the following forward-backward equation system admits a solution* $(x_i,p_i,\{\beta_i^j\}_{j=0}^N)$*:*

$$\begin{cases} dx_i=(Ax_i+Bu_i+f)dt+(Du_i+\sigma)dW_i+\left(C_0x_i+D_0u_i-\dfrac{R_0^{-1}}{N}\displaystyle\sum_{j=1}^N\beta_j^0\right)dW_0,\\ dp_i=-\left[A^Tp_i+C_0^T\beta_i^0-Qx_i+\Xi_1x^{(N)}+\Xi_2\right]dt+\beta_i^0dW_0+\displaystyle\sum_{j=1}^N\beta_i^jdW_j,\\ x_i(0)=x_0,\quad p_i(T)=(-G)x_i(T)+\Xi_1^Gx^{(N)}(T)+\Xi_2^G,\quad i=1,\dots,N. \end{cases} \tag{3.8}$$

In this case, the minimizer $\sigma_0^*(u)=-\frac{R_0^{-1}}{N}\sum_{j=1}^N\beta_j^0$.

Proof. The "if" part follows directly by the standard completion of square technique for (P1$'$) and stationary condition reasoning for quadratic functional.

For the "only if" part, suppose σ_0^* is a minimizer to problem (P1$'$). x_i^* is the optimal state of agent i under the volatility σ_0^*. $x_*^{(N)}=\frac{1}{N}\sum_{j=1}^Nx_j^*$. For $i=1,2,\dots,N$, denote $\delta x_i=x_i-x_i^*$ the increment of x_i along with the variation $\delta\sigma_0=\sigma_0-\sigma_0^*$. Similarly, $\delta x^{(N)}=\frac{1}{N}\sum_{j=1}^N\delta x_j$ and $\delta\breve{J}^{(N)}_{\text{soc}}(\sigma_0^*,\delta\sigma_0)=\breve{J}^{(N)}_{\text{soc}}(\sigma_0)-\breve{J}^{(N)}_{\text{soc}}(\sigma_0^*)+o(||\delta\sigma_0||_{L^2})$, the Fréchet differential of $\breve{J}^{(N)}_{\text{soc}}$ on σ_0^* along with direction $\delta\sigma_0$. By (2.1),

$$d(\delta x_i)=A(\delta x_i)dt+[C_0(\delta x_i)+\delta\sigma_0]dW_0,\quad \delta x_i(0)=0,\quad i=1,2,\dots,N. \tag{3.9}$$

By the standard variational principle, we have the following stationary condition on the Fréchet differential:

$$\begin{aligned} 0&=\delta\breve{J}^{(N)}_{\text{soc}}(\sigma_0^*,\delta\sigma_0)\\ &=\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{\langle -Q\big[x_i^*-(\Gamma x_*^{(N)}+\eta)\big],\delta x_i-\Gamma\delta x^{(N)}\rangle+\langle R_0\sigma_0^*,\delta\sigma_0\rangle\Big\}dt\\ &\quad+\sum_{i=1}^N\mathbb{E}\big\{\langle -G\big[x_i^*(T)-(\Gamma_0x_*^{(N)}(T)+\eta_0)\big],\delta x_i(T)-\Gamma\delta x^{(N)}(T)\rangle\big\}. \end{aligned} \tag{3.10}$$

Introduce the adjoint equation

(3.11)
$$\begin{aligned} dp_i =& - \left[A^T p_i + C_0^T \beta_i^0 + \Gamma^T Q\big[(I-\Gamma)x_*^{(N)} - \eta\big] - Q\big[x_i^* - (\Gamma x_*^{(N)} + \eta)\big]dt \right. \\ &+ \beta_i^0 dW_0 + \beta_i^i dW_i + \sum_{j\neq i} \beta_i^j dW_j, \quad p_i(T) = (-G)x_i^*(T) + \Xi_1^G x_*^{(N)}(T) + \Xi_2^G. \end{aligned}$$

Then by Itô's formula,

$$\begin{aligned} &\mathbb{E}[\langle (-G)x_i(T) + \Xi_1^G x^{(N)}(T) + \Xi_2^G, x_i(T)\rangle] \\ &= \mathbb{E}\int_0^T \big[\langle -\big[A^T p_i + C_0^T \beta_i^0 + \Gamma^T Q\big[(I-\Gamma)x_*^{(N)} - \eta\big] \\ &\quad - Q\big[x_i^* - (\Gamma x_*^{(N)} + \eta)\big], \delta x_i\rangle + \langle p_i, A\delta x_i\rangle + \langle \beta_i^0, C_0 \delta x_i + \delta\sigma_0\rangle\big] dt. \end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned} &\sum_{i=1}^N \mathbb{E}\int_0^T \big\langle -Q\big(x_i^* - (\Gamma x_*^{(N)} + \eta)\big), \Gamma \delta x^{(N)}\big\rangle dt \\ &\quad = \sum_{j=1}^N \mathbb{E}\int_0^T \big\langle -\Gamma^T Q\big[(I-\Gamma)x_*^{(N)} - \eta\big], \delta x_j\big\rangle dt. \end{aligned}$$

It follows by (3.10)–(3.12) that

$$\begin{aligned} 0 =& \mathbb{E}\sum_{i=1}^N \int_0^T \Big[\big\langle -Q\big(x_i^* - (\Gamma x_*^{(N)} + \eta)\big), \delta x_i - \Gamma\delta x^{(N)}\big\rangle + \langle R_0 \sigma_0^*, \delta\sigma_0\rangle\Big] dt \\ &+ \sum_{i=1}^N \mathbb{E}\int_0^T \big[\langle -\big[A^T p_i + C_0^T \beta_i^0 + \Gamma^T Q\big[(I-\Gamma)x_*^{(N)} - \eta\big] \\ &- Q\big[x_i^* - (\Gamma x_*^{(N)} + \eta)\big], \delta x_i\rangle + \langle p_i, A\delta x_i\rangle + \langle \beta_i^0, C_0\delta x_i + \delta\sigma_0\rangle\big] dt \\ =& \mathbb{E}\int_0^T \Big\langle N R_0 \sigma_0^* + \sum_{i=1}^N \beta_i^0, \delta\sigma_0\Big\rangle dt, \end{aligned}$$

which leads to

$$\sigma_0^* = -\frac{R_0^{-1}}{N}\sum_{i=1}^N \beta_i^0.$$

Thus, the Hamiltonian system (3.8) admits a solution $(x_i^*, p_i, \{\beta_i^j\}_{j=0}^N)$. ☐

Let $p^{(N)} = \frac{1}{N}\sum_{i=1}^N p_i$ and $\beta_0^{(N)} = \frac{1}{N}\sum_{i=1}^N \beta_i^0$. It follows from (3.8) that

$$\begin{cases} dx^{(N)} = (Ax^{(N)} + Bu^{(N)} + f)dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^N (Du_i + \sigma)dW_i \\ \qquad\qquad + (C_0 x^{(N)} + D_0 u^{(N)} - R_0^{-1}\beta_0^{(N)})dW_0, \qquad x^{(N)}(0) = x, \\ dp^{(N)} = -\Big[A^T p^{(N)} + C_0^T \beta_0^{(N)} - (Q - \Xi_1)x^{(N)} + \Xi_2\Big]dt + \beta_0^{(N)} dW_0 \\ \qquad\qquad + \dfrac{1}{N}\displaystyle\sum_{i=1}^N\sum_{j=1}^N \beta_i^j dW_j, \qquad p^{(N)}(T) = (\Xi_1^G - G)x^{(N)}(T) + \Xi_2^G. \end{cases}$$

Letting $N \to \infty$, we obtain an approximation as follows:

$$\begin{cases} d\hat{x} =(A\hat{x} + B\hat{u} + f)dt + (C_0\hat{x} + D_0\hat{u} - R_0^{-1}\hat{\beta}_0)dW_0, \\ d\hat{p} = -\left[A^T\hat{p} + C_0^T\hat{\beta}_0 - (Q - \Xi_1)\hat{x} + \Xi_2\right]dt + \hat{\beta}_0 dW_0, \\ \hat{x}(0) = x_0, \quad \hat{p}(T) = (\Xi_1^G - G)\hat{x}(T) + \Xi_2^G. \end{cases} \tag{3.13}$$

4. The control problem of agent i: Person-by-person optimality.

4.1. Some variational analysis.

When the volatility $\sigma_0^* = -\frac{R_0^{-1}}{N}\sum_{j=1}^N \beta_j^0$ is applied, we turn to study the outer minimization problem for team agents.

(P2): Minimize $J_{\text{soc}}^{\text{wo}}(u)$ over $\{u = (u_1, \dots, u_N) | u_i \in \mathcal{U}_c^r\}$, where

$$\begin{aligned} J_{\text{soc}}^{\text{wo}}(u) &\triangleq J_{\text{soc}}^{(N)}(u, \sigma_0^*(u)) \\ &= \frac{1}{2}\sum_{i=1}^N \mathbb{E}\int_0^T \left\{\left|x_i - \Gamma x^{(N)} - \eta\right|_Q^2 + |u_i|_R^2 - |\sigma_0^*(u)|_{R_0}^2\right\}dt \\ &\quad + \frac{1}{2}\mathbb{E}|x_i(T) - \Gamma_0 x^{(N)}(T) - \eta_0|_G^2 \end{aligned} \tag{4.1}$$

subject to

$$\begin{cases} dx_i = (Ax_i + Bu_i + f)dt + (Du_i + \sigma)dW_i + \left(C_0 x_i + D_0 u_i - \dfrac{R_0^{-1}}{N}\displaystyle\sum_{k=1}^N \beta_k^0\right)dW_0, \\ dp_i = -\,(A^T p_i + C_0^T\beta_i^0 - Qx_i + \Xi_1 x^{(N)} + \Xi_2)dt + \beta_i^0 dW_0 + \displaystyle\sum_{k=1}^N \beta_i^k dW_k, \\ x_i(0) = x_0, \quad p_i(T) = (-G)x_i(T) + \Xi_1^G x^{(N)}(T) + \Xi_2^G. \end{cases} \tag{4.2}$$

For further analysis, we introduce the following assumption.

(H3) $J_{\text{soc}}^{\text{wo}}(u)$ of (P2) is convex in u.

Suppose $\bar{u} = (\bar{u}_1, \dots, \bar{u}_i, \dots, \bar{u}_N)$ and $\bar{x} = (\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_N)$ are respectively the centralized optimal control and states of (P2) and we make the following person-by-person optimality variation around its optimal point. We now perturb the control of $\mathcal{A}_i$ to be u_i and keep $(\bar{u}_1, \dots, \bar{u}_{i-1}, \bar{u}_{i+1}, \dots, \bar{u}_N)$, the strategies of all other agents fixed. Let $\delta u_i = u_i - \bar{u}_i$ and $\delta u_i \in \mathcal{U}_c^r$. Denote $\delta x_j = x_j - \bar{x}_j$, $\delta p_j = p_j - \bar{p}_j$, and $\delta\beta_j^k = \beta_j^k - \bar{\beta}_j^k, j, k = 1, \dots, N$, the corresponding (forward, adjoint) state variation. By (3.8) and (4.2), we have

$$\begin{cases} d(\delta x_i) = (A\delta x_i + B\delta u_i)dt + (D\delta u_i)dW_i \\ \qquad\qquad + \left(C_0\delta x_i + D_0\delta u_i - \dfrac{R_0^{-1}}{N}\displaystyle\sum_{k=1}^N \delta\beta_k^0\right)dW_0, \ \delta x_i(0) = 0, \\ d(\delta p_i) = -\left(A^T\delta p_i + C_0^T\delta\beta_i^0 - Q\delta x_i + \Xi_1\delta x^{(N)}\right)dt + \delta\beta_i^0 dW_0 \\ \qquad\qquad + \delta\beta_i^i dW_i + \displaystyle\sum_{k\neq i}\delta\beta_i^k dW_k, \ \delta p_i(T) = (-G)\delta x_i(T) + \Xi_1^G\delta x^{(N)}(T), \end{cases} \tag{4.3}$$

and for $j \neq i$,

$$
\begin{cases} d(\delta x_j) = A\delta x_j dt + \left(C_0 \delta x_j - \dfrac{R_0^{-1}}{N} \displaystyle\sum_{l=1}^{N} \delta\beta_l^0 \right) dW_0,\ \delta x_j(0) = 0, \\ d(\delta p_j) = -\left(A^T \delta p_j + C_0^T \delta\beta_j^0 - Q\delta x_j + \Xi_1 \delta x^{(N)} \right) dt + \delta\beta_j^0 dW_0 + \delta\beta_j^j dW_j \\ \qquad + \displaystyle\sum_{l\neq j} \delta\beta_j^l dW_l,\ \delta p_j(T) = (-G)\delta x_j(T) + \Xi_1^G \delta x^{(N)}(T). \end{cases} \tag{4.4}
$$

This implies that for any $j, j' \neq i$, $\delta x_j = \delta x_{j'}$, which further gives

$$
\delta p_j = \delta p_{j'}, \qquad \delta\beta_j^0 = \delta\beta_{j'}^0, \qquad \text{for} \quad j, j' \neq i. \tag{4.5}
$$

Let $\mathbb{E}_{\mathcal{F}^0}[\cdot] \triangleq \mathbb{E}[\cdot|\mathcal{F}_t^0]$ (suppressing t). Note that W_j is independent of W_0. It follows from (4.3) that

$$
\begin{cases} d(\mathbb{E}_{\mathcal{F}^0}[\delta x_i]) = \left(A\mathbb{E}_{\mathcal{F}^0}[\delta x_i] + B\mathbb{E}_{\mathcal{F}^0}[\delta u_i] \right) dt \\ \qquad + \left(C_0 \mathbb{E}_{\mathcal{F}^0}[\delta x_i] + D_0 \mathbb{E}_{\mathcal{F}^0}[\delta u_i] - \dfrac{R_0^{-1}}{N} \displaystyle\sum_{k=1}^{N} \mathbb{E}_{\mathcal{F}^0}[\delta\beta_k^0] \right) dW_0, \\ d(\mathbb{E}_{\mathcal{F}^0}[\delta p_i]) = -\left(A^T \mathbb{E}_{\mathcal{F}^0}[\delta p_i] + C_0^T \mathbb{E}_{\mathcal{F}^0}[\delta\beta_i^0] - Q\mathbb{E}_{\mathcal{F}^0}[\delta x_i] + \Xi_1 \mathbb{E}_{\mathcal{F}^0}[\delta x^{(N)}] \right) dt \\ \qquad + \mathbb{E}_{\mathcal{F}^0}[\delta\beta_i^0] dW_0, \\ \mathbb{E}_{\mathcal{F}_0^0}[\delta x_i(0)] = 0, \quad \mathbb{E}_{\mathcal{F}_T^0}[\delta p_i(T)] = (-G)\mathbb{E}_{\mathcal{F}_T^0}(\delta x_i(T)) + \Xi_1^G \mathbb{E}_{\mathcal{F}_T^0}(\delta x^{(N)}(T)). \end{cases} \tag{4.6}
$$

It follows from (4.4) that for $j \neq i$

$$
\begin{cases} d(\mathbb{E}_{\mathcal{F}^0}[\delta x_j]) = A\mathbb{E}_{\mathcal{F}^0}[\delta x_j] dt + \left(C_0 \mathbb{E}_{\mathcal{F}^0}[\delta x_j] - \dfrac{R_0^{-1}}{N} \displaystyle\sum_{k=1}^{N} \mathbb{E}_{\mathcal{F}^0}[\delta\beta_k^0] \right) dW_0, \\ d(\mathbb{E}_{\mathcal{F}^0}[\delta p_j]) = -\left(A^T \mathbb{E}_{\mathcal{F}^0}[\delta p_j] + C_0^T \mathbb{E}_{\mathcal{F}^0}[\delta\beta_j^0] - Q\mathbb{E}_{\mathcal{F}^0}[\delta x_j] \right. \\ \qquad \left. + \Xi_1 \mathbb{E}_{\mathcal{F}^0}[\delta x^{(N)}] \right) dt + \mathbb{E}_{\mathcal{F}^0}[\delta\beta_j^0] dW_0, \\ \mathbb{E}_{\mathcal{F}_0^0}[\delta x_j(0)] = 0, \mathbb{E}_{\mathcal{F}_T^0}[\delta p_j(T)] = (-G)\mathbb{E}_{\mathcal{F}_T^0}(\delta x_j(T)) + \Xi_1^G \mathbb{E}_{\mathcal{F}_T^0}(\delta x^{(N)}(T)). \end{cases} \tag{4.7}
$$

Denote $\delta J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}, \delta u_i)$ the Fréchet differential of $J_{\mathrm{soc}}^{\mathrm{wo}}$ at $\bar{u}$ along with direction δu_i:

$$
J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}+\delta u_i) - J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}) = \delta J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}, \delta u_i) + o(\|\delta u_i\|_{L^2}) = \langle \mathcal{D}_{u_i} J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}), \delta u_i \rangle + o(\|\delta u_i\|_{L^2}), \tag{4.8}
$$

where $\mathcal{D}_{u_i} J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u})$ is the Fréchet derivative of $J_{\mathrm{soc}}^{\mathrm{wo}}$ at $\bar{u}$ with componentwise variation $(0, \ldots, \delta u_i^T, \ldots, 0)$. Then, from (4.5), we can obtain that for $j \neq i$,

$$
\begin{aligned} &\delta J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u}, \delta u_i) \\ &\quad = \mathbb{E} \int_0^T \left[\left\langle Q(\bar{x}_i - \Gamma \bar{x}^{(N)} - \eta), \delta x_i \right\rangle - \left\langle \Gamma^T Q(\bar{x}_i - \Gamma \bar{x}^{(N)} - \eta), \delta x^{(N)} \right\rangle \right. \\ &\qquad + \left\langle Q\left(\left(I - \frac{N-1}{N}\Gamma \right) \bar{x}^{(N)} - \frac{\bar{x}_i}{N} - \frac{N-1}{N}\eta \right), N\delta x_j \right\rangle \end{aligned}
$$

$$
\begin{aligned}
&-\left\langle \frac{1}{N}\sum_{j\neq i}\Gamma^T Q(\bar{x}_j-\Gamma\bar{x}^{(N)}-\eta),\delta x_i\right\rangle-\big\langle R_0^{-1}\bar{\beta}_0^{(N)},\delta\beta_i^0\big\rangle-\big\langle R_0^{-1}\bar{\beta}_0^{(N)},(N-1)\delta\beta_j^0\big\rangle\\
&-\left\langle \Gamma^T Q\left(\bar{x}^{(N)}-\frac{\bar{x}_i}{N}-\frac{N-1}{N}\Gamma\bar{x}^{(N)}-\frac{N-1}{N}\eta\right),(N-1)\delta x_j\right\rangle+\big\langle R\bar{u}_i,\delta u_i\big\rangle\Bigg]dt\\
&+\mathbb{E}\big[\big\langle G(\bar{x}_i(T)-\Gamma_0\bar{x}^{(N)}(T)-\eta_0),\delta x_i(T)\big\rangle-\big\langle \Gamma_0^T G(\bar{x}_i(T)-\Gamma_0\bar{x}^{(N)}(T)-\eta_0),\\
&\quad\delta x^{(N)}(T)\big\rangle+\sum_{j\neq i}\big\langle G(\bar{x}_j(T)-\Gamma_0\bar{x}^{(N)}(T)-\eta_0),\delta x_j(T)\big\rangle\\
&-\sum_{j\neq i}\big\langle \Gamma_0^T G(\bar{x}_j(T)-\Gamma_0\bar{x}^{(N)}(T)-\eta_0),\delta x^{(N)}(T)\big\rangle\big].
\end{aligned}
$$

When $N\to+\infty$, from (3.13), we further have

$$
\begin{aligned}
&\lim_{N\to+\infty}\delta J_{\mathrm{soc}}^{\mathrm{wo}}(\bar{u},\delta u_i):=\delta\hat{J}_i(\bar{u},\delta u_i)=\langle\mathcal{D}_{u_i}\hat{J}_i(\bar{u}),\delta u_i\rangle\\
&\quad=\mathbb{E}\int_0^T\Big[\big\langle Q\bar{x}_i,\delta x_i\big\rangle-\big\langle Q(\Gamma\hat{x}+\eta)+\Gamma^T Q((I-\Gamma)\hat{x}-\eta),\delta x_i\big\rangle+\big\langle R\bar{u}_i,\delta u_i\big\rangle\\
&\qquad-\big\langle R_0^{-1}\hat{\beta}_0,\delta\beta_i^0\big\rangle-\big\langle R_0^{-1}\hat{\beta}_0,\delta\beta^*\big\rangle\\
&\qquad+\big\langle Q((I-\Gamma)\hat{x}-\eta)-\Gamma^T Q((I-\Gamma)\hat{x}-\eta),\delta x^*\big\rangle\Big]dt\\
&\qquad+\mathbb{E}\big[\big\langle G\bar{x}_i(T),\delta x_i(T)\big\rangle-\big\langle G(\Gamma_0\hat{x}(T)+\eta_0),\delta x_i(T)\big\rangle\\
&\qquad+\big\langle G((I-\Gamma_0)\hat{x}(T)-\eta_0),\delta x^*(T)\big\rangle-\big\langle\Gamma_0^T G((I-\Gamma_0)\hat{x}(T)-\eta_0),\delta x_i(T)\big\rangle\\
&\qquad-\big\langle\Gamma_0^T G((I-\Gamma_0)\hat{x}(T)-\eta_0),\delta x^*(T)\big\rangle\big],
\end{aligned}
\tag{4.9}
$$

where $\delta\hat{J}_i(\bar{u},\delta u_i)$ is the Fréchet differential of some auxiliary cost functional $\hat{J}_i$, to be constructed later (see (P3) in section 5), $\mathcal{D}_{u_i}\hat{J}_i(\bar{u})$ is the related Fréchet derivative, and state average limits $(\hat{x},\hat{\beta}_0)$ are to be determined by the CC system in section 5; moreover, by (4.7), the quantities

$$
\delta x^*:=N\mathbb{E}_{\mathcal{F}^0}[\delta x_j],\qquad \delta p^*:=N\mathbb{E}_{\mathcal{F}^0}[\delta p_j],\qquad \delta\beta^*:=N\mathbb{E}_{\mathcal{F}^0}[\delta\beta_j^0],\qquad \text{for } j\neq i,
\tag{4.10}
$$

do not depend on N and satisfy the following equations:

$$
\begin{cases}
d(\delta x^*)=A(\delta x^*)dt+[C_0(\delta x^*)-R_0^{-1}(\delta\beta^*+\mathbb{E}_{\mathcal{F}^0}[\delta\beta_i^0])]dW_0,\ \delta x^*(0)=0,\\
d(\delta p^*)=-\big[A^T(\delta p^*)+C_0^T(\delta\beta^*)-Q(\delta x^*)+\Xi_1(\mathbb{E}_{\mathcal{F}^0}[\delta x_i]+\delta x^*)\big]dt\\
\qquad\qquad+(\delta\beta^*)dW_0,\ \delta p^*(T)=\Xi_1^G\mathbb{E}_{\mathcal{F}_T^0}(\delta x_i(T))-(G-\Xi_1^G)\delta x^*(T).
\end{cases}
\tag{4.11}
$$

Remark 4.1. When studying the asymptotic behavior of (4.9) with $N\longrightarrow+\infty$, the following remainder term needs to be considered:

$$
\begin{aligned}
\epsilon_1^{(N)}:=\mathbb{E}\int_0^T&\Big[-\langle\Xi_1(\bar{x}^{(N)}-\hat{x}),\delta x_i\rangle-\langle R_0^{-1}(\bar{\beta}_0^{(N)}-\hat{\beta}_0),\delta\beta_i^0+(N-1)\delta\beta_j^0\rangle\\
&+\langle(Q-\Xi_1)(\bar{x}^{(N)}-\hat{x}),N\delta x_j\rangle\Big]dt-\mathbb{E}[\langle\Xi_1^G(\bar{x}^{(N)}(T)-\hat{x}(T)),\delta x_i(T)\rangle]\\
&+\mathbb{E}[\langle(G-\Xi_1^G)(\bar{x}^{(N)}(T)-\hat{x}(T)),N\delta x_j(T)\rangle].
\end{aligned}
$$

Because $\|\delta u_i\|_{L^2}<\infty$, $\epsilon_1^{(N)}$ should be an infinitesimal term with same order to $\|\bar{x}^{(N)}-\hat{x}\|_{\max}+\|\bar{\beta}_0^{(N)}-\hat{\beta}_0\|_{L_2}$ $(N\to\infty)$. Actually, from (3.8) and (3.13) we may obtain

$\|\bar{x}^{(N)} - \hat{x}\|^2_{\max} + \|\bar{\beta}_0^{(N)} - \hat{\beta}_0\|^2_{L_2} = O(\frac{1}{N})$ (the rigorous proof will be given in section 7). Thus, $\epsilon_1^{(N)} = O(\frac{1}{\sqrt{N}})\|\delta u_i\|_{L^2}\|\bar{u}\|_{L^2}$.

4.2. Duality derivation. A key point in analyzing the social optimization problem is to formulate some auxiliary control problem for a given agent, based on $\delta\hat{J}_i = \lim_{N\to+\infty}\delta J^{\mathrm{wo}}_{\mathrm{soc}}$ of (4.9); thus the decentralized strategy can be derived via some MFG procedure. Such an auxiliary problem can be derived via some variational analysis (see [53] for related variational analysis but with only the drift-controlled term). Due to volatility uncertainty, all states of agents are coupled via some high-dimensional FBSDE system. Therefore, related variational analysis becomes fairly different from that of [53] and depends on a *two-step duality* procedure, as discussed below.

Step 1 (duality independent of $(\delta x^*, \delta p^*)$). The first step removes the dependence of $\delta\hat{J}_i(\bar{u}, \delta u_i)$ on $(\delta x^*, \delta p^*)$, the variational process common to all agents. To this end, introduce the adjunct FBSDE:

$$\begin{cases} dy = f_0 dt + z dW_0(t), \quad y(T) = \left(G - \Xi_1^G\right)\hat{x}(T) - \Xi_2^G, \\ dh = f_1 dt + f_2 dW_0(t), \quad h(0) = 0, \end{cases} \tag{4.12}$$

where the drivers $(f_0; f_1, f_2)$ are to be determined. Note $h(0) = 0$, and

$$\delta p^*(T) - \Xi_1^G \mathbb{E}_{\mathcal{F}_T^0}(\delta x_i)(T) - (G - \Xi_1^G)\delta x^*(T) = 0.$$

By Itô's formula,

$$\begin{aligned} 0 = \mathbb{E}\int_0^T & \Big\{ \big\langle h, -\big(A^T\delta p^* + C_0^T(\delta\beta^*) + \Xi_1\mathbb{E}_{\mathcal{F}^0}(\delta x_i) - (Q - \Xi_1)(\delta x^*)\big) \\ & - \Xi_1^G\big(A\mathbb{E}_{\mathcal{F}^0}(\delta x_i) + B\mathbb{E}_{\mathcal{F}^0}(\delta u_i) - (G - \Xi_1^G)A(\delta x^*)\big)\big\rangle \\ & + \big\langle \delta p^* - \Xi_1^G\mathbb{E}_{\mathcal{F}^0}(\delta x_i) - (G - \Xi_1^G)\delta x^*(t), f_1\big\rangle \\ & + \big\langle f_2, \delta\beta^* - \Xi_1^G\left(C_0\mathbb{E}_{\mathcal{F}^0}(\delta x_i) + D_0\mathbb{E}_{\mathcal{F}^0}(\delta u_i)\right) \\ & -(G - \Xi_1^G)\left[C_0(\delta x^*) - R_0^{-1}\left(\delta\beta^* + \mathbb{E}_{\mathcal{F}^0}(\delta\beta_i^0)\right)\right]\big\rangle \Big\} dt. \end{aligned} \tag{4.13}$$

Using Itô formula to $\langle \delta x^*, y\rangle$, we have

$$\begin{aligned} & \mathbb{E}\left\langle (G - \Xi_1^G)\hat{x}(T) - \Xi_2^G, \delta x^*(T)\right\rangle \\ & = \mathbb{E}\left[\int_0^T \langle \delta x^*, f_0\rangle + \langle A^T y, \delta x^*\rangle + \langle z, C_0\delta x^* - R_0^{-1}(\delta\beta^* + \mathbb{E}_{\mathcal{F}^0}(\delta\beta_i^0))\rangle\right] dt. \end{aligned}$$

It follows from (4.12) and (4.13) that

$$\begin{aligned} & \mathbb{E}\left\langle (G - \Xi_1^G)\hat{x}(T) - \Xi_2^G, \delta x^*(T)\right\rangle \\ & = \mathbb{E}\int_0^T \Big[\big\langle \delta x^*, f_0 + A^T y + C_0^T z + (Q - \Xi_1)^T h - A^T(G - \Xi_1^G)^T h \\ & \quad - (G - \Xi_1^G)^T f_1 - C_0^T(G - \Xi_1^G) f_2\big\rangle + \langle \delta p^*, -Ah + f_1\rangle \\ & \quad + \langle \delta\beta^*, -R_0^{-1}z - C_0 h + f_2 + R_0^{-1}(G - \Xi_1^G) f_2\rangle \\ & \quad + \langle \mathbb{E}_{\mathcal{F}^0}(\delta u_i), -B^T\Xi_1^G h - D_0^T\Xi_1^G f_2\rangle + \langle \mathbb{E}_{\mathcal{F}^0}(\delta\beta_i^0), -R_0^{-1}z + R_0^{-1}(G - \Xi_1^G) f_2\rangle \\ & \quad + \langle \mathbb{E}_{\mathcal{F}^0}(\delta x_i), -\Xi_1^T h - A^T(\Xi_1^G)^T h - (\Xi_1^G)^T f_1 - C_0^T\Xi_1^G f_2\rangle\Big] dt. \end{aligned} \tag{4.14}$$

Set

$$\mathbb{I}^G := \big[I + R_0^{-1}(G - \Xi_1^G)\big]^{-1} \tag{4.15}$$

(note that $\mathbb{I}^G = I$ if $G = 0$). Comparing the coefficients, we obtain

$$\begin{cases} f_1 = Ah, \quad f_2 = \mathbb{I}^G(R_0^{-1}z + C_0h + R_0^{-1}\hat{\beta}_0) \\ f_0 = -\,(A^Ty + C_0^Tz + (Q-\Xi_1)h - A^T(G-\Xi_1^G)h - (G-\Xi_1^G)f_1 \\ \qquad - C_0^T(G-\Xi_1^G)f_2 + (Q-\Xi_1)\hat{x} - \Xi_2). \end{cases} \tag{4.16}$$

Then, we have

$$\begin{cases} dy = -\Big[A^Ty + C_0^Tz + (Q-\Xi_1)h - A^T(G-\Xi_1^G)h - (G-\Xi_1^G)Ah \\ \qquad - C_0^T(G-\Xi_1^G)\mathbb{I}^G(R_0^{-1}z + C_0h + R_0^{-1}\hat{\beta}_0) + (Q-\Xi_1)\hat{x} - \Xi_2\Big]dt + zdW_0(t), \\ dh = Ahdt + \mathbb{I}^G(R_0^{-1}z + C_0h + R_0^{-1}\hat{\beta}_0)dW_0(t), \\ y(T) = (G-\Xi_1^G)\hat{x}(T) - \Xi_2^G, \quad h(0) = 0. \end{cases} \tag{4.17}$$

Let $\xi_1 = (Q-\Xi_1)\hat{x} - \Xi_2$ and $\xi_2 = -R_0^{-1}\hat{\beta}_0$. From (4.14), we obtain

$$\begin{aligned} &\mathbb{E}\left\langle (G-\Xi_1^G)\hat{x}(T) - \Xi_2^G, \delta x^*(T)\right\rangle + \mathbb{E}\int_0^T [\langle \delta x^*, \xi_1\rangle + \langle \delta\beta^*, \xi_2\rangle]dt \\ &\quad = \mathbb{E}\int_0^T \Big[\left\langle \mathbb{E}_{\mathcal{F}^0}(\delta\beta_i^0), -R_0^{-1}z + R_0^{-1}(G-\Xi_1^G)f_2\right\rangle \\ &\qquad + \left\langle \mathbb{E}_{\mathcal{F}^0}(\delta u_i), -B^T\Xi_1^Gh - D_0^T\Xi_1^Gf_2\right\rangle \\ &\qquad + \left\langle \mathbb{E}_{\mathcal{F}^0}(\delta x_i), -\Xi_1h - A^T\Xi_1^Gh - \Xi_1f_1 - C_0^T\Xi_1^Gf_2\right\rangle\Big]dt. \end{aligned}$$

Then a direct computation from (4.9) shows that

$$\begin{aligned} &\delta\hat{J}_i(\bar{u}, \delta u_i)) = \lim_{N\to+\infty} \delta J_{\rm soc}^{\rm wo}(\bar{u}, \delta u_i) \\ &\quad = \mathbb{E}\int_0^T \big[\langle Q\bar{x}_i, \delta x_i\rangle + \langle R\bar{u}_i, \delta u_i\rangle - \langle \Xi_1\hat{x} + \Xi_2, \delta x_i\rangle - \langle R_0^{-1}\hat{\beta}_0, \delta\beta_i^0\rangle\big]dt \\ &\qquad + \mathbb{E}[\langle G\hat{x}_i(T), \delta x_i(T)\rangle - \langle \Xi_1^G\hat{x}(T) + \Xi_2^G, \delta x_i(T)\rangle] \\ &\qquad + \mathbb{E}\int_0^T \Big[\langle -R_0^{-1}z + R_0^{-1}(G-\Xi_1^G)\mathbb{I}^G(R_0^{-1}z + C_0h + R_0^{-1}\hat{\beta}_0), \delta\beta_i^0\rangle \\ &\qquad + \langle \delta x_i, -\Xi_1h - A^T\Xi_1^Gh - \Xi_1^GAh - C_0^T\Xi_1^Gf_2\rangle - \langle \delta u_i, B^T\Xi_1^Gh + D_0^T\Xi_1^Gf_2\rangle\Big]dt. \end{aligned} \tag{4.18}$$

Let $\xi_3 = -R_0^{-1}(z + \hat{\beta}_0) + R_0^{-1}(G-\Xi_1^G)f_2$. Then we will consider the following term in Step 2:

$$\mathbb{E}\int_0^T \big[-\langle R_0^{-1}\hat{\beta}_0, \delta\beta_i^0\rangle + \langle -R_0^{-1}z + R_0^{-1}(G-\Xi_1^G)f_2, \delta\beta_i^0\rangle\big]dt = \mathbb{E}\int_0^T \langle \xi_3, \delta\beta_i^0\rangle dt.$$

Step 2 (duality independent of $(\delta\beta_i^0)$). The second step removes the dependence of $\delta\hat{J}_i(\bar{u},\delta u_i)$ on *backward* variational process $\delta\beta_i^0$. Thus, the derived auxiliary problem will end up with a *forward* LQ control on $(\delta u_i,\delta x_i)$ only. To this end, first introduce the adjoint process

$$d\Phi = g_1 dt + g_2 dW_0(t),\ \ \Phi(0)=0,$$

with g_1, g_2 to be determined. Note

$$\begin{cases} d\delta p_i = -\Big(A^T\delta p_i + C_0^T\delta\beta_i^0 - Q\delta x_i + \Xi_1\delta x^{(N)}\Big)dt + \delta\beta_i^0 dW_0 + \displaystyle\sum_{k=1}^N \delta\beta_i^k dW_k(t),\\ \delta p_i(T) = (-G)\delta x_i(T) + \Xi_1^G\delta x^{(N)}(T). \end{cases}$$

Then, by Itô's formula, we obtain

$$\begin{aligned} 0 = \mathbb{E}\int_0^T &\big[\langle\delta p_i, -A\Phi+g_1\rangle + \langle\delta\beta_i^0, -C_0\Phi+g_2\rangle \\ &+ \langle\delta x_i, Q^T\Phi + A^TG\Phi + Gg_1 + C_0^TGg_2\rangle - \langle\delta u_i, B^TG\Phi + D_0^TGg_2\rangle\big]dt, \end{aligned} \tag{4.19}$$

which implies $g_1 = A\Phi$, and $g_2 = C_0\Phi - \xi_3 = C_0\Phi + R_0^{-1}(z+\hat{\beta}_0) - R_0^{-1}(G-\Xi_1^G)f_2$. Then we have

$$d\Phi = A\Phi dt + \big[C_0\Phi + R_0^{-1}(z+\hat{\beta}_0) - R_0^{-1}(G-\Xi_1^G)f_2\big]dW_0(t), \Phi(0)=0, \tag{4.20}$$

which with (4.16) gives $\Phi = h$. From (4.19),

$$\mathbb{E}\int_0^T\langle\delta\beta_i^0,\xi_3\rangle dt = \mathbb{E}\int_0^T\big[\langle\delta x_i, Q^T\Phi + A^TG\Phi + Gg_1 + C_0^TGg_2\rangle + \langle\delta u_i, B^TG\Phi + D_0^TGg_2\rangle\big]dt.$$

From this with (4.18), the variational functional becomes

$$\begin{aligned} \delta\hat{J}_i(\bar{u},\delta u_i) = \mathbb{E}\int_0^T &\Big[\langle Q\bar{x}_i,\delta x_i\rangle + \langle R\bar{u}_i,\delta u_i\rangle \\ &-\langle\Xi_1\hat{x}+\Xi_2,\delta x_i\rangle + \big\langle Q^Th + A^TGh + GAh + C_0^TGg_2,\delta x_i\big\rangle \\ &-\langle\Xi_1 h + A^T\Xi_1^Gh + \Xi_1^GAh + C_0^T\Xi_1^Gf_2,\delta x_i\rangle \\ &+\langle B^TGh + D_0^TGf_2,\delta u_i\rangle - \langle B^T\Xi_1^Gh + D_0^T\Xi_1^Gf_2,\delta u_i\rangle\Big]dt \\ &+\mathbb{E}[\langle G\bar{x}_i(T),\delta x_i(T)\rangle - \langle\Xi_1^G\hat{x}(T) + \Xi_2^G,\delta x_i(T)\rangle], \end{aligned} \tag{4.21}$$

where $g_2 = f_2 = \mathbb{I}^G(R_0^{-1}(z+\hat{\beta}_0) + C_0h)$.

Remark 4.2. Relation $\Phi = h$ is not a coincidence; instead it is implied by some structural similarity. In fact, the dynamics coefficients $(A^T, C_0^T, Q, \Xi_1,\cdots)$ in (4.3) and (4.11) are exactly matching via the following correspondences: $\delta p_i \longleftrightarrow \delta p^*$, $\delta\beta_i^0 \longleftrightarrow \delta\beta^*$, $\delta x_i \longleftrightarrow \delta x^*$, and $\delta x^{(N)} \longleftrightarrow \mathbb{E}_{\mathcal{F}^0}[\delta x_i] + \delta x^*$. Meanwhile, their terminal conditions also follow similar matching. Also, even with $h=\Phi$, two-step duality is still needed because Steps 1 and 2 deal with different variations and thus cannot be covered by each other.

5. Decentralized robust team strategy design. By (3.13) and limiting social variational functional (4.21), we construct the following auxiliary control problem, for a representative agent, still indexed by $\mathcal{A}_i$.

(P3): Minimize $\hat{J}_i(u_i)$ over $u_i \in \mathcal{U}_i$, with state dynamics and cost functional:

$$
\begin{aligned}
&dx_i = (Ax_i + Bu_i + f)dt + (Du_i + \sigma)dW_i + (C_0x_i + D_0u_i - R_0^{-1}\hat{\beta}_0)dW_0, \ x_i(0) = x_0,\\
&\hat{J}_i(u_i) = \frac{1}{2}\mathbb{E}\Bigg\{ \int_0^T |x_i|_Q^2 + |u_i|_R^2 - 2\langle \Xi_1\hat{x} + \Xi_2, x_i\rangle + 2\langle (Q-\Xi_1)h, x_i\rangle\\
&\qquad + 2\langle A^TGh + GAh + C_0^TGg_2 - A^T\Xi_1^G h - \Xi_1^G Ah - C_0^T\Xi_1^G g_2, x_i\rangle\\
&\qquad + 2\langle B^T(G-\Xi_1^G)h + D_0^T(G - \Xi_1^G g_2), u_i\rangle dt\\
&\qquad + \langle Gx_i(T), x_i(T)\rangle - 2\langle \Xi_1^G\hat{x}(T) + \Xi_2^G, x_i(T)\rangle \Bigg\}.
\end{aligned}
\tag{5.1}
$$

Here the triple $(\hat{x}, \hat{\beta}_0, h)$ satisfies the following limiting (off-line) system parameterized by undetermined process $\hat{u}$:

$$
\begin{cases}
d\hat{x} = (A\hat{x} + B\hat{u} + f)dt + (C_0\hat{x} + D_0\hat{u} - R_0^{-1}\hat{\beta}_0)dW_0,\\
d\hat{p} = -\left(A^T\hat{p} + C_0^T\hat{\beta}_0 - Q\hat{x} + \Xi_1\hat{x} + \Xi_2\right)dt + \hat{\beta}_0 dW_0,\\
dy = -\left(A^Ty + C_0^Tz + (Q-\Xi_1)h + (Q-\Xi_1)\hat{x} - \Xi_2\right)dt + \left(A^T(G-\Xi_1^G)h\right.\\
\qquad \left. + (G-\Xi_1^G)Ah + C_0^T(G-\Xi_1^G)f_2\right)dt + zdW_0,\\
dh = Ahdt + \left(I + R_0^{-1}(G-\Xi_1^G)\right)^{-1}\left(C_0h + R_0^{-1}(z+\hat{\beta}_0)\right)dW_0,\\
\hat{x}(0) = x_0, \ \hat{p}(T) = \left(\Xi_1^G - G\right)\hat{x}(T) + \Xi_2^G, \ y(T) = (G-\Xi_1^G)\hat{x}(T) - \Xi_2^G, \ h(0) = 0.
\end{cases}
\tag{5.2}
$$

Remark 5.1. FBSDE (5.2) can be decomposed into subsystems $(\hat{x}, \hat{p}, \hat{\beta}_0)$ and (h, y, z), which are decoupled for each other. Thus, solvability of (5.2) reduces to that of (h, y, z) and $(\hat{x}, \hat{p}, \hat{\beta}_0)$ separately. Section 6 will discuss the global solvability of subsystem (h, y, z), and a similar analysis can be applied to $(\hat{x}, \hat{p}, \hat{\beta}_0)$ considering these two subsystems have similar coupling structures. Moreover, parameter process $\hat{u}$ will be further determined by some CC system through the MFG argument.

Let $\hat{u}(t) \in \mathcal{F}_t^0$ be fixed. We study the decentralized open-loop strategy and related CC system. We have the following result by the maximum principle.

THEOREM 5.2. *Suppose that* $Q \geq 0, G \geq 0$ *and* $R > 0$. *Then the following backward stochastic differential equation (BSDE) admits a (unique) solution:*

$$
\begin{aligned}
dk_i = &-\left[A^Tk_i + C_0^T\zeta_0 + Qx_i - \Xi_1\hat{x} + (Q-\Xi_1)h - \Xi_2 + \mathcal{K}(G, g_2) - \mathcal{K}(\Xi_1^G, g_2)\right]dt\\
&+ \zeta_0 dW_0 + \zeta_i dW_i, \qquad k_i(T) = Gx_i(T) - \Xi_1^G\hat{x}(T) - \Xi_2^G,
\end{aligned}
\tag{5.3}
$$

where $\mathcal{K}(G,g_2) = A^T Gh + GAh + C_0^T Gg_2$, $\mathcal{K}(\Xi_1^G,g_2) = A^T\Xi_1^G h + \Xi_1^G Ah + C_0^T\Xi_1^G g_2$ *and*

$$\check{u}_i = -R^{-1}(B^T k_i + D_0^T\zeta_0 + D^T\zeta_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2), \quad i=1,\dots,N. \tag{5.4}$$

Proof. Since $Q \geq 0, G \geq 0$, and $R > 0$, (P3) is uniformly convex, which implies the unique solvability of (P3). Assume that $\check{u}_i$ is the unique optimal control of problem (P3) and $\check{x}_i$ is the state equation under $\check{u}_i$. Then

$$\begin{aligned} 0 &= \delta\hat{J}_i(\check{u}_i,\delta u_i) \\ &= \mathbb{E}\int_0^T \left[\langle Q\check{x}_i,\delta x_i\rangle + \langle R\check{u}_i,\delta u_i\rangle - \langle \Xi_1\hat{x}+\Xi_2,\delta x_i\rangle + \langle (Q-\Xi_1)h,\delta x_i\rangle\right] \\ &\quad + \langle \mathcal{K}(G,g_2)-\mathcal{K}(\Xi_1^G,g_2),\delta x_i\rangle + \langle B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2,\delta u_i\rangle dt \\ &\quad + \langle G\check{x}_i(T),\delta x_i(T)\rangle - \langle \Xi_1^G\hat{x}(T)+\Xi_2^G,\delta x_i(T)\rangle. \end{aligned} \tag{5.5}$$

Given $\hat{x}$ and h, (5.3) is a standard linear BSDE and thus has a unique solution (k_i,ζ_0,ζ_i). Then

$$\begin{aligned} &\langle G\check{x}_i(T) - \Xi_1^G\hat{x}(T) - \Xi_2^G,\delta x_i(T)\rangle \\ &\quad = \mathbb{E}\int_0^T \Big\{\langle -(Qx_i - \Xi_1\hat{x} - (Q-\Xi_1)h - \Xi_2 - \mathcal{K}(G,g_2) + \mathcal{K}(\Xi_1^G,g_2)),\delta x_i\rangle \\ &\qquad + \langle k_i, B\delta u_i\rangle + \langle \zeta_0, D_0\delta u_i\rangle + \langle \zeta_i, D\delta u_i\rangle\Big\}dt. \end{aligned} \tag{5.6}$$

From this and (5.5), we have

$$0 = \mathbb{E}\int_0^T \langle R\check{u}_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2 + B^T k_i + D_0^T\zeta_0 + D^T\zeta_i,\delta u_i\rangle dt, \tag{5.7}$$

which implies the open-loop optimal strategy:

$$\begin{aligned} \check{u}_i &= -R^{-1}\left(B^T k_i + D_0^T\zeta_0 + D^T\zeta_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) \\ &= -R^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2). \end{aligned}$$

Note that here,

$$v_i := B^T k_i + D^T\zeta_i + D_0\zeta_0. \tag{5.8}$$

□

After the strategy (5.4) is applied, we obtain the following state equation:

$$\begin{aligned} dx_i &= \left[Ax_i - BR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) + f\right]dt \\ &\quad + \left[-DR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) + \sigma\right]dW_i \\ &\quad + \left[C_0x_i - DR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) - R_0^{-1}\hat{\beta}_0\right]dW_0. \end{aligned}$$

Consequently, the consistency argument implies the following CC system to $(\hat{x}, \hat{\beta}_0, h)$:
(5.9)

$$\begin{cases} dx_i = \big[Ax_i - BR^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) + f\big]\,dt \\ \qquad + \Big[C_0x_i - D_0R^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) - R_0^{-1}\hat{\beta}_0\Big]\,dW_0 \\ \qquad - \big[DR^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) - \sigma\big]\,dW_i, \\ dk_i = -\big[A^Tk_i + C_0^T\zeta_0 + Qx_i - \Xi_1\mathbb{E}_{\mathcal{F}^0}[x_i] - \Xi_2 + (Q-\Xi_1)h \\ \qquad + \mathcal{K}(G,g_2) - \mathcal{K}(\Xi_1^G, g_2)\big]dt + \zeta_0 dW_0 + \zeta_i dW_i, \\ d\hat{p} = -\big[A^T\hat{p} + C_0^T\hat{\beta}_0 - (Q-\Xi_1)\mathbb{E}_{\mathcal{F}^0}[x_i] + \Xi_2\big]dt + \hat{\beta}_0 dW_0, \\ dy = -\big[A^Ty + C_0^Tz + (Q-\Xi_1)h + (Q-\Xi_1)\mathbb{E}_{\mathcal{F}_T^0}[x_i] - \Xi_2\big]dt \\ \qquad + [A^T(G-\Xi_1^G)h + (G-\Xi_1^G)Ah + C_0^T(G-\Xi_1^G)g_2]dt + z\,dW_0, \\ dh = Ah\,dt + [I + R_0^{-1}(G-\Xi_1^G)]^{-1}[C_0h + R_0^{-1}(z+\hat{\beta}_0)]dW_0, \\ x_i(0) = x_0, k_i(T) = Gx_i(T) - \Xi_1^G\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] - \Xi_2^G, \hat{p}(T) = (\Xi_1^G - G)\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] + \Xi_2^G, \\ y(T) = (G-\Xi_1^G)\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] - \Xi_2^G,\ h(0) = 0. \end{cases}$$

Remark 5.3. CC system (5.9) differs from those in the classical MFG literature (e.g., [28, 35, 50, 39]) by noting the evolution dynamics of $\hat{x}$ is not explicitly specified here. Instead, it is characterized by some implicit representation $\hat{x} = \mathbb{E}_{\mathcal{F}^0}[x_i]$ which is embedded into an augmented mean field type FBSDE system of $(x_i, k_i, \hat{p}, y, h)$ driven by a generic Brownian motion W_i independent of common noise W_0. CC system (5.9) is symmetric for all agents and thus such representation is uniquely defined.

Such a difference in CC representation is mainly caused by the presence of adjoint process term $D^T\zeta_i$ in the decentralized strategy design (refer to (5.8)). Thus, an explicit representation of $\hat{x}$ becomes unavailable, and a similar CC representation was derived in [24].

6. Well-posedness of relevant FBSDEs. Our study in previous sections, especially the one related to decentralized strategy design and CC systems, involves various (fully coupled) FBSDEs or Riccati equations. Keeping this in mind, this section aims to discuss the existence and uniqueness of their (global) solvability. Note that because of the introduction of soft constraints, these equations are intrinsically *nonstandard* (i.e., control/state weights are indefinite), thus their global solvability becomes more technical. Moreover, due to the uncertainty on volatility, it is necessary to treat the adjoint states of FBSDE which closely connect to volatility, the diffusion term in BSDE formulation. As a result, the relevant analysis becomes more complex considering the adjoint states are of less regularity property.

We consider the solvability of FBSDE (5.2) from section 5. A similar analysis can be applied to CC system (5.9) for which the arguments become more lengthy. By partial coupling of Remark 5.1, it suffices to consider the following subsystem constructed by (h, y, z):

$$(6.1)\qquad \begin{cases} dh = Ah\,dt + \mathbb{I}^G\Big(C_0h + R_0^{-1}(z+\hat{\beta}_0)\Big)dW_0(t), \\ dy = -\big(A^Ty + C_0^Tz + (Q-\Xi_1)h + (Q-\Xi_1)\hat{x} - \Xi_2\big)\,dt \\ \qquad + \big(A^T(G-\Xi_1^G)h + (G-\Xi_1^G)Ah + C_0^T(G-\Xi_1^G)f_2\big)\,dt + z\,dW_0(t), \\ h(0) = 0, \quad y(T) = (G-\Xi_1^G)\hat{x}(T) - \Xi_2^G. \end{cases}$$

Equation (6.1) is a fully coupled FBSDE involving forward state h, backward state y, and adjoint state z. Moreover, it is *nonstandard* or *indefinite* because of the volatility uncertainty (thus, unlike *definite* case, some weights are singular or negative due to its minmax feature). It is known that (global) solvability of such an indefinite FBSDE is by no means unconditional: to ensure its well-posedness, it is always necessary to impose some additional compatibility conditions. Also, direction computation indicates the *monotonicity method*, which is well applied to nonstandard FBSDE, fails to work here.

Reduction decoupling method. Our method is the reduction decoupling method proposed in [56], which leads to the global solvability by imposing conditions on orthogonality of C_0. Let $\Psi_1(\cdot, s)$ be the solution of the following ODE:

$$\begin{cases} \dfrac{d}{dt}\Psi_1(t,s) = \begin{pmatrix} A & 0 \\ \widehat{A} & -A^T \end{pmatrix} \Psi_1(t,s), \quad t \in [s,T], \\ \Psi_1(s,s) = I, \end{cases}$$

where

$$\widehat{A} \triangleq (\Xi_1 - Q) + A^T(G - \Xi_1^G) + (G - \Xi_1^G)A + C_0^T(G - \Xi_1^G)\mathbb{I}^G C_0.$$

Denote $\Psi_1(t) = \Psi_1(t,0)$. Then we have

$$\Psi_1(t) = \exp\begin{pmatrix} At & 0 \\ \hat{A}t & -A^T t \end{pmatrix} = \begin{pmatrix} \exp(At) & 0 \\ \sum_{n=0}^{\infty} \frac{\Lambda_n t^n}{n!} & \exp(-A^T t) \end{pmatrix},$$

where

$$\Lambda_n \triangleq \widehat{A}A^{n-1} - A^T\widehat{A}A^{n-2} + \cdots + (-A^T)^{k-1}\widehat{A}A^{n-k} + \cdots + (-A^T)^{n-1}\widehat{A}.$$

If $A = A^T$ and $A\widehat{A} = \widehat{A}A$, then

$$\Lambda_n = \begin{cases} \widehat{A}A^{n-1}, & n = 2k-1, \\ 0, & n = 2k, \end{cases}$$

and

$$\sum_{n=1}^{\infty} \frac{\Lambda_n t^n}{n!} = \widehat{A}\sum_{k=1}^{\infty} \frac{A^{2k-2}t^{2k-1}}{(2k-1)!}.$$

Further, if A is invertible, then

$$\sum_{n=1}^{\infty} \frac{\Lambda_n t^n}{n!} = \widehat{A}A^{-1}\frac{e^{At} - e^{-At}}{2}.$$

We have

$$\begin{aligned} &(0, I)\begin{pmatrix} \exp(AT) & 0 \\ \sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!} & \exp(-A^T T) \end{pmatrix} \begin{pmatrix} \mathbb{I}^G C_0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!} & \exp(-A^T T) \end{pmatrix} \begin{pmatrix} \mathbb{I}^G C_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!}\mathbb{I}^G C_0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, if and only if $\sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!}\mathbb{I}^G C_0 = 0$, then

$$(0, I)\Psi_1(T)\begin{pmatrix} \mathbb{I}^G C_0 & 0 \\ 0 & 0 \end{pmatrix} = 0. \tag{6.2}$$

Note

$$(0, I)\Psi_1(T) \begin{pmatrix} 0 \\ -C_0^T + C_0^T(G - \Xi_1^G)\mathbb{I}^G R_0^{-1} \end{pmatrix} \\ = \exp(-A^T)(-C_0^T + C_0^T(G - \Xi_1^G)\mathbb{I}^G R_0^{-1}) = 0$$

implies that $C_0^T[I - (G - \Xi_1^G)\mathbb{I}^G R_0^{-1}] = 0$, i.e., $C_0^T \mathbb{I}^G = 0$. Since $C_0 \neq 0$ and $\mathbb{I}^G$ is invertible, we have

$$(0, I)\Psi_1(T) \begin{pmatrix} 0 \\ -C_0^T + C_0^T(G - \Xi_1^G)\mathbb{I}^G R_0^{-1} \end{pmatrix} \neq 0.$$

Note that

$$(0, I)\Psi_1(T) \begin{pmatrix} 0 \\ I \end{pmatrix} = (0, I) \begin{pmatrix} \exp(AT) & 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!} & \exp(-A^T T) \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \\ = \left(\sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!} \quad \exp(-A^T T) \right) \begin{pmatrix} 0 \\ I \end{pmatrix} = \exp(-A^T T).$$

We have $(0, I)\Psi_1(T)\binom{0}{I}$ is invertible, and

$$\begin{aligned} (0, I)\Psi_1(T, t)(R_0^{-1}\mathbb{I}^G, I)^T &= (0, I) \begin{pmatrix} \exp[A(T-t)] & 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda_n (T-t)^n}{n!} & \exp[-A^T(T-t)] \end{pmatrix} \begin{pmatrix} \mathbb{I}_G R_0^{-1} \\ I \end{pmatrix} \\ &= \left(\sum_{n=1}^{\infty} \frac{\Lambda_n (T-t)^n}{n!}, \ \exp(-A^T(T-t) \right) \begin{pmatrix} \mathbb{I}_G R_0^{-1} \\ I \end{pmatrix} \\ &= \sum_{n=1}^{\infty} \frac{\Lambda_n (T-t)^n}{n!} \mathbb{I}_G R_0^{-1} + \exp[-A^T(T-t)]. \end{aligned} \tag{6.3}$$

From the above analysis and Theorem 3.2 in [56], we have the following sufficient condition for solvability of FBSDE (6.1).

PROPOSITION 6.1. *Let* (H1) *hold. Then* (6.1) *is solvable if* $\sum_{n=1}^{\infty} \frac{\Lambda_n T^n}{n!} \mathbb{I}^G C_0 = 0$ *and* $(0, I)\Psi_1 \ (T, \cdot)(R_0^{-1}\mathbb{I}^G, I)^T$ *is full-rank.*

EXAMPLE 6.2. *Consider the system* (2.1)–(2.2) *with parameters*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \ C_0 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0 \\ 0 & 0.4 \end{pmatrix}, \ R_0 = \begin{pmatrix} 0.1 & 0 \\ 0 & 2 \end{pmatrix}, \\ \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \ G = 0.$$

We have $\widehat{A} = \begin{pmatrix} 0 & 0 \\ 0 & -0.1 \end{pmatrix}$ *and* $\Lambda_n = 0, n = 1, 2, \ldots$. *From* (6.3), *we have*

$$(0, I)\Psi_1(T, \cdot)(R_0^{-1}\mathbb{I}^G, I)^T = \exp[-A^T(T-t)]$$

is of row full-rank. By Proposition 6.1, *FBSDE* (6.1) *is solvable.*

7. Asymptotic optimality. Based on results of section 6, we may assume the off-line system (5.2) and consistency system (5.9) are well-posed (we do not specify which concrete conditions lead to it because our analysis below only requires the well-posedness of these FBSDE systems); thus the decentralized control set $\breve{u} = (\breve{u}_1, \dots, \breve{u}_N)$ is well-defined through (5.4). The main theorem of this section states the asymptotic robust social optimality of decentralized decision $\breve{u}$.

DEFINITION 7.1. *A set of control laws* $\breve{u} = (\breve{u}_1, \dots, \breve{u}_N)$ *has asymptotic robust social optimality if*

$$\left| \frac{1}{N} J^{\mathrm{wo}}_{\mathrm{soc}}(\breve{u}) - \frac{1}{N} \inf_{u_i \in \mathcal{U}_c} J^{\mathrm{wo}}_{\mathrm{soc}}(u) \right| = o(1). \tag{7.1}$$

THEOREM 7.2. *Assume that* (H1), (H2′), *and* (H3) *hold, and* (5.2) *and* (5.9) *admit a unique solution, respectively. Then the set of control laws* (5.4) *has asymptotic robust social optimality with*

$$\left| \frac{1}{N} J^{\mathrm{wo}}_{\mathrm{soc}}(\breve{u}) - \frac{1}{N} \inf_{u_i \in \mathcal{U}_c} J^{\mathrm{wo}}_{\mathrm{soc}}(u) \right| = O\left(\frac{1}{\sqrt{N}} \right).$$

7.1. A quadratic functional representation. To verify asymptotic social optimality (7.1), it is helpful to construct some quadratic representation of worse-case functional $J^{\mathrm{wo}}_{\mathrm{soc}}(u)$ for $u = (u_1, \dots, u_N) \in \mathbb{R}^{r \times N}$. First, recall the compact notation introduced in section 3, and denote $\mathbf{R} = \mathrm{Diag}(R, \dots, R)$, $\bar{\beta}^i = [(\beta^i_1)^T, \dots, (\beta^i_N)^T]^T$, $i = 0, 1, \dots, N$. Then we can rewrite state (4.2) and cost functional (4.1) as follows:

$$\begin{cases} d\mathbf{x} = (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{1} \otimes f)dt + \displaystyle\sum_{i=1}^{N} (\mathbf{D}_i \mathbf{u} + \sigma_i) dW_i \\ \qquad + (\mathbf{C}_0 \mathbf{x} + \mathbf{D}_0 \mathbf{u} - \dfrac{1}{N}(\mathbf{1}\mathbf{1}^T \otimes R_0^{-1})\bar{\beta}^0 dW_0, \\ d\mathbf{p} = -(\mathbf{A}^T \mathbf{p} - \hat{\mathbf{Q}}\mathbf{x} + \hat{\eta} + \mathbf{C}_0^T \bar{\beta}^0) dt + \displaystyle\sum_{i=1}^{N} \bar{\beta}^i dW_i + \bar{\beta}^0 dW_0, \\ \mathbf{x}(0) = \mathbf{x}_0, \ \mathbf{p}(T) = -\hat{\mathbf{G}}\mathbf{x}(T) + \hat{\eta}_0, \end{cases} \tag{7.2}$$

and

$$\begin{aligned} J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}) = {} & \frac{1}{2}\mathbb{E} \int_0^T \left(|\mathbf{x}|^2_{\hat{\mathbf{Q}}} - 2\hat{\eta}^T \mathbf{x} + |\mathbf{u}|^2_{\mathbf{R}} - \frac{1}{N} |\bar{\beta}^0|^2_{\mathbf{1}\mathbf{1}^T \otimes R_0^{-1}} \right) dt \\ & + \frac{1}{2}\mathbb{E}\big(|\mathbf{x}(T)|^2_{\hat{\mathbf{G}}} - 2\hat{\eta}_0^T \mathbf{x}(T) \big), \end{aligned} \tag{7.3}$$

where $\hat{\eta} = \mathbf{1} \otimes \Xi_2$, $\hat{\eta}_0 = \mathbf{1} \otimes \Xi_2^G$, and $\hat{\mathbf{Q}} = (\hat{Q}_{ij})$, $\hat{\mathbf{G}} = (\hat{G}_{ij})$ are given by (3.3). Recall by Remark 3.1 we may exchange the usage $\mathbf{u} = (u_1^T, \dots, u_N^T)^T$ with $u = (u_1, \dots, u_N)$.

Moreover, by the superposition property of linear system (7.2), a straightforward calculation implies that for any $(\mathbf{u}^1, \mathbf{u}^2; \mathbf{x}_0^1, \mathbf{x}_0^2; \hat{\eta}_0^1, \hat{\eta}_0^2)$,

$$\begin{aligned} & J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}^1 + \mathbf{u}^2; \mathbf{x}_0^1 + \mathbf{x}_0^2; \hat{\eta}_0^1 + \hat{\eta}_0^2) + J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}^1 - \mathbf{u}^2; \mathbf{x}_0^1 - \mathbf{x}_0^2; \hat{\eta}_0^1 - \hat{\eta}_0^2) \\ & \quad = 2\left(J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}^1; \mathbf{x}_0^1; \hat{\eta}_0^1) + J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}^2; \mathbf{x}_0^2; \hat{\eta}_0^2) \right). \end{aligned}$$

Thus, $J^{\mathrm{wo}}_{\mathrm{soc}}$ satisfies the parallelogram law and it is a quadratic functional with respect to control process $\mathbf{u}(\cdot)$ and initial-terminal condition pair $(\mathbf{x}_0, \hat{\eta}_0)$. Then, by the symmetric property of $J^{\mathrm{wo}}_{\mathrm{soc}}(u)$ to inputs $(\mathbf{u}(\cdot);\mathbf{x}_0;\hat{\eta}_0)$, the following quadratic representation holds true:

$$
\begin{aligned}
J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u}(\cdot);\mathbf{x}_0;\hat{\eta}_0) = {}& \langle \mathbf{M}_1(\mathbf{u}),\mathbf{u}\rangle + 2\langle \mathbf{M}_{12}(\mathbf{x}_0,\hat{\eta}_0),\mathbf{u}\rangle + \langle \mathbf{M}_2(\mathbf{x}_0,\hat{\eta}_0),(\mathbf{x}_0,\hat{\eta}_0)\rangle \\
&+ 2\langle \mathbf{M}_{13},\mathbf{u}\rangle + 2\langle \mathbf{M}_{23},(\mathbf{x}_0,\hat{\eta}_0)\rangle + \mathbf{M}_3
\end{aligned} \tag{7.4}
$$

for linear bounded self-adoint operators $\mathbf{M}_1 : \mathcal{U}_c^{\otimes N} \to \mathcal{U}_c^{\otimes N}$, $\mathbf{M}_2 : \mathbb{S}^{nN\times nN}$, $\mathbf{M}_3 \in \mathbb{R}$, and linear bounded operators $\mathbf{M}_{12}:\ \mathbb{R}^{nN}\times\mathbb{R}^{nN} \to \mathcal{U}_c^{\otimes N}$, $\mathbf{M}_{13} \in \mathcal{U}^{\otimes N}[0,T]$, $\mathbf{M}_{23} \in \mathbb{R}^{nN}$, where $\mathcal{U}_c^{\otimes N} = \underbrace{\mathcal{U}_c\times\cdots\times\mathcal{U}_c}_{N-\mathrm{fold}}$, where $\langle\cdot\rangle$ denotes the inner product in the sense of $dt\otimes d\mathbb{P}$. More precisely, we have the following representations. For operator $\mathbf{M}_1$,

$$
\begin{cases}
\mathbf{M}_1(\mathbf{u}) = \mathbf{R}\mathbf{u} + \mathbf{B}^T\mathbf{m}_1 + \displaystyle\sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_1^i + \mathbf{D}_0^T\mathbf{n}_1^0; \\
\langle \mathbf{M}_1(\mathbf{u}),\mathbf{u}\rangle = \mathbb{E}\displaystyle\int_0^T \langle \mathbf{R}\mathbf{u} + \mathbf{B}^T\mathbf{m}_1 + \sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_1^i + \mathbf{D}_0^T\mathbf{n}_1^0, \mathbf{u}\rangle ds,
\end{cases}
$$

with

$$
\begin{cases}
d\mathbf{m}_1 = -(\mathbf{A}^T\mathbf{m}_1 + \mathbf{C}_0^T\mathbf{n}_1^0 + \hat{\mathbf{Q}}\mathbf{x}_1 + \hat{\mathbf{Q}}\mathbf{y}_1)dt + \displaystyle\sum_{i=1}^N \mathbf{n}_1^i dW_i + \mathbf{n}_1^0 dW_0, \\
d\mathbf{y}_1 = \mathbf{A}\mathbf{y}_1 dt + (\mathbf{C}_0\mathbf{y}_1 + N\mathbf{R}\otimes\mathbf{q}_1^0 + \mathbf{R}\otimes\mathbf{n}_1^0)dW_0, \\
\mathbf{y}_1(0) = 0, \qquad \mathbf{m}_1(T) = \hat{\mathbf{G}}(\mathbf{y}_1(T) + \mathbf{x}_1(T)),
\end{cases} \tag{7.5}
$$

$$
\begin{cases}
d\mathbf{x}_1 = (\mathbf{A}\mathbf{x}_1 + \mathbf{B}\mathbf{u})dt + \displaystyle\sum_{i=1}^N \mathbf{D}_i\mathbf{u}dW_i + (\mathbf{C}_0\mathbf{x}_1 + \mathbf{D}_0\mathbf{u} - \frac{1}{N}(\mathbf{1}\mathbf{1}^T\otimes R_0^{-1})\mathbf{q}_1^0 dW_0, \\
d\mathbf{p}_1 = -(\mathbf{A}^T\mathbf{p}_1 - \hat{\mathbf{Q}}\mathbf{x}_1 + \mathbf{C}_0^T\mathbf{q}_1^0)dt + \displaystyle\sum_{i=1}^N \mathbf{q}_1^i dW_i + \mathbf{q}_1^0 dW_0, \\
\mathbf{x}_1(0) = 0, \quad \mathbf{p}_1(T) = -\hat{\mathbf{G}}\mathbf{x}_1(T).
\end{cases} \tag{7.6}
$$

For operator $\mathbf{M}_{12}$, we have

$$
\begin{cases}
\mathbf{M}_{12}(\mathbf{x},\hat{\eta}_0) = \mathbf{B}^T\mathbf{m}_2 + \mathbf{D}_0^T\mathbf{n}_2^0 + \displaystyle\sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_2^i; \\
\langle \mathbf{M}_{12}(\mathbf{x},\hat{\eta}_0),\mathbf{u}\rangle = \mathbb{E}\displaystyle\int_0^T \langle \mathbf{B}^T\mathbf{m}_2 + \mathbf{D}_0^T\mathbf{n}_2^0 + \sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_2^i, \mathbf{u}\rangle dt,
\end{cases}
$$

with

$$
\begin{cases}
d\mathbf{m}_2 = -(\mathbf{A}^T\mathbf{m}_2 + \mathbf{C}_0^T\mathbf{n}_2^0 + \hat{\mathbf{Q}}\mathbf{x}_2 + \hat{\mathbf{Q}}\mathbf{y}_2)dt + \mathbf{n}_2^0 dW_0 + \displaystyle\sum_{i=1}^N \mathbf{n}_2^i dW_i, \\
d\mathbf{y}_2 = \mathbf{A}\mathbf{y}_2 dt + (\mathbf{C}_0\mathbf{y}_2 + N\mathbf{R}\otimes\mathbf{q}_2^0 + \mathbf{R}\otimes\mathbf{n}_2^0)dW_0, \\
\mathbf{m}_2(T) = \hat{\mathbf{G}}(\mathbf{y}_2(T) + \mathbf{x}_2(T)), \quad \mathbf{y}_2(0) = 0,
\end{cases}
$$

and

$$\begin{cases} d\mathbf{x}_2 = \mathbf{A}\mathbf{x}_2 dt + (\mathbf{C}_0\mathbf{x}_2 - \mathbf{R}\otimes\mathbf{q}_2^0)dW_0, \\ d\mathbf{p}_2 = -(\mathbf{A}^T\mathbf{p}_2 - \hat{\mathbf{Q}}\mathbf{x}_2 + \mathbf{C}_0^T\mathbf{q}_2^0)dt + \sum_{i=1}^N \mathbf{q}_2^i dW_i + \mathbf{q}_2^0 dW_0, \\ \mathbf{x}_2(0) = \mathbf{x}_0, \quad \mathbf{p}_2(T) = -\hat{\mathbf{G}}\mathbf{x}_2(T) + \hat{\eta}_0. \end{cases}$$

For operator $\mathbf{M}_{13}$, we have

$$\begin{cases} \mathbf{M}_{13} = \mathbf{B}^T\mathbf{m}_{13} + \mathbf{D}_0^T\mathbf{n}_{13}^0 + \sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_{13}^i, \\ \langle \mathbf{M}_{13}, \mathbf{u}\rangle = \mathbb{E}\int_0^T \langle \mathbf{B}^T\mathbf{m}_{13} + \mathbf{D}_0^T\mathbf{n}_{13}^0 + \sum_{i=1}^N \mathbf{D}_i^T\mathbf{n}_{13}^i, \mathbf{u}\rangle dt, \end{cases}$$

where

$$\begin{cases} d\mathbf{m}_{13} = -(\mathbf{A}^T\mathbf{m}_{13} + \mathbf{C}_0^T\mathbf{n}_{13}^0 + \hat{\mathbf{Q}}\mathbf{x}_3 + \mathbf{Q}\mathbf{y}_{13} - 2\hat{\eta}_s)ds + \mathbf{n}_{13}^0 dW_0 + \sum_{i=1}^N \mathbf{n}_{13}^i dW_i, \\ d\mathbf{y}_{13} = \mathbf{A}\mathbf{y}_{13}dt + (\mathbf{C}_0\mathbf{y}_{13} + N\mathbf{R}\otimes\mathbf{q}_{13}^0 + \mathbf{R}\otimes\mathbf{n}_{13}^0)dW_0, \\ \mathbf{y}_{13}(0) = 0, \quad \mathbf{m}_{13}(T) = \hat{\mathbf{G}}(\mathbf{y}_{13}(T) + \mathbf{x}_3(T)) - \hat{\eta}_0, \end{cases}$$

and

$$\begin{cases} d\mathbf{x}_3 = (\mathbf{A}\mathbf{x}_3 + \mathbf{1}\otimes\mathbf{f})dt + \sum_{i=1}^N \sigma_i dW_i + (\mathbf{C}_0\mathbf{x}_3 - \mathbf{R}\otimes\mathbf{q}_3^0)dW_0, \\ d\mathbf{p}_3 = -(\mathbf{A}^T\mathbf{p}_3 - \hat{\mathbf{Q}}\mathbf{x}_3 + \hat{\eta} + \mathbf{C}_0^T\mathbf{q}_3^0)dt + \sum_{i=1}^N \mathbf{q}_3^i dW_i + \mathbf{q}_3^0 dW_0, \\ \mathbf{x}_3(0) = \mathbf{x}_0, \quad \mathbf{p}_3(T) = -\hat{\mathbf{G}}\mathbf{x}_3(T). \end{cases}$$

$\mathbf{M}_2, \mathbf{M}_{23}$, and $\mathbf{M}_3$ can be defined similarly. With the above presentations, the Fréchet differential of $J_{\mathrm{soc}}^{\mathrm{wo}}$ along the variation $\delta\mathbf{u}$ can be represented as

$$\delta J_{\mathrm{soc}}^{\mathrm{wo}}(\mathbf{u}, \delta\mathbf{u}) = 2\langle \mathbf{M}_1\mathbf{u} + \mathbf{M}_{12}(\mathbf{x}, \hat{\eta}_0) + \mathbf{M}_{13}, \delta\mathbf{u}\rangle.$$

7.2. Asymptotic optimality: Four-step procedure. Given the quadratic representation of $J_{\mathrm{soc}}^{\mathrm{wo}}(u) := J_{\mathrm{soc}}^{(N)}(u, \sigma_0^*(u))$ by (7.4), we can verify the asymptotic robust optimality stated in Theorem 7.2 through the following steps.

Step 1. We first analyze the asymptotic convergence of the realized state system. When each agent $\mathcal{A}_i$ applies the open-loop decentralized strategy $\breve{u}_i$ as

$$\breve{u}_i = -R^{-1}\left(v_i + B^T(G - \Xi_1^G)h + D_0^T(G - \Xi_1^G)g_2\right),$$

then the corresponding realized state $\check{x}_i$ is given by the following fully coupled FBSDE subsystem together with backward and adjoint states $(\check{p}_i, \check{\beta}_i^0, \{\check{\beta}_i^k\}_{k=1}^N)$:

$$
\begin{cases}
d\check{x}_i = \big[A\check{x}_i - BR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) + f\big]\,dt \\
\qquad + \big[-DR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) + \sigma\big]\,dW_i \\
\qquad + \big[C_0\check{x}_i - DR^{-1}\left(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\right) - \sigma_0^*(\check{u})\big]\,dW_0, \\
d\check{p}_i = -\,(A^T\check{p}_i + C_0^T\check{\beta}_i^0 - Q\check{x}_i + \Xi_1\check{x}^{(N)} + \Xi_2)dt + \check{\beta}_i^0 dW_0 + \displaystyle\sum_{k=1}^N \check{\beta}_i^k dW_k, \\
\check{x}_i(0) = x, \quad \check{p}_i(T) = (-G)\check{x}_i(T) + \Xi_1^G\check{x}^{(N)}(T) + \Xi_2^G,
\end{cases} \tag{7.7}
$$

where $\check{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \check{x}_i, \sigma_0^*(\check{u}) = \frac{R_0^{-1}}{N}\sum_{k=1}^N \check{\beta}_k^0$. Moreover, $v_i = B^Tk_i + D_0^T\zeta_0 + D^T\zeta_i$ is defined through the following CC system for a representative agent $\mathcal{A}_i$:

$$
\begin{cases}
dx_i = \big[Ax_i - BR^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) + f\big]\,dt \\
\qquad + \Big[C_0x_i - D_0R^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) - R_0^{-1}\hat{\beta}_0\Big]\,dW_0 \\
\qquad - \big[DR^{-1}(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) - \sigma\big]\,dW_i, \\
dk_i = -\,\big[A^Tk_i + C_0^T\zeta_0 + Qx_i - \Xi_1\mathbb{E}_{\mathcal{F}^0}[x_i] - \Xi_2 + (Q-\Xi_1)h \\
\qquad + \mathcal{K}(G,g_2) - \mathcal{K}(\Xi_1^G, g_2)\big]dt + \zeta_0 dW_0 + \zeta_i dW_i, \\
d\hat{p} = -\,\big[A^T\hat{p} + C_0^T\hat{\beta}_0 - (Q-\Xi_1)\mathbb{E}_{\mathcal{F}^0}[x_i] + \Xi_2\big]dt + \hat{\beta}_0 dW_0, \\
dy = -\,\big[A^Ty + C_0^Tz + (Q-\Xi_1)h + (Q-\Xi_1)\mathbb{E}_{\mathcal{F}^0}[x_i] - \Xi_2\big]dt \\
\qquad + [A^T(G-\Xi_1^G)h + (G-\Xi_1^G)Ah + C_0^T(G-\Xi_1^G)g_2]dt + zdW_0, \\
dh = Ahdt + [I + R_0^{-1}(G-\Xi_1^G)]^{-1}[C_0h + R_0^{-1}(z+\hat{\beta}_0)]dW_0
\end{cases} \tag{7.8}
$$

with the initial-terminal condition

$$
\begin{cases}
x_i(0) = x, \quad k_i(T) = Gx_i(T) - \Xi_1^G\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] - \Xi_2^G, \\
\hat{p}(T) = (\Xi_1^G - G)\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] + \Xi_2^G, \ \ y(T) = (G-\Xi_1^G)\mathbb{E}_{\mathcal{F}_T^0}[x_i(T)] - \Xi_2^G, \ \ h(0) = 0.
\end{cases} \tag{7.9}
$$

Note that all such N-subsystems $\left(\check{p}_j, \check{\beta}_j^0, \{\check{\beta}_j^k\}_{k=1}^N\right)_{j=1}^N$ of (7.7) are further coupled via the worst-volatility $\sigma_0^* = \sum_{k=1}^N \check{\beta}_k^0$ and they thus frame a fully coupled and highly dimensional FBSDE system in $L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^{nN})) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nN})) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{nN^2})$. Regarding system (7.7)–(7.8), we have the following prior estimate.

PROPOSITION 7.3. *Let* (H1), (H2′) *hold. Assume* (5.2) *and* (5.9) *admit a unique solution, respectively. Then*

$$
\mathbb{E}\sup_{0\le t\le T}\left(|\check{x}^{(N)} - \mathbb{E}_{\mathcal{F}^0}[x_i]|^2 + |\check{p}^{(N)} - \hat{p}|^2\right) + \mathbb{E}\int_0^T |R_0\sigma_0^*(\check{u}) - \hat{\beta}_0|^2 dt \le c_0\left(\frac{1}{N}\right) \tag{7.10}
$$

for some constant $c_0 > 0$ independent of N and i. Here, $\check{p}^{(N)} = \frac{1}{N}\sum \check{p}_i$.

Proof. Making the state aggregation of (7.7), we have

$$\begin{cases} d\check{x}^{(N)} = \Big[A\check{x}^{(N)} - BR^{-1}\Big(v^{(N)} + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\Big) + f\Big]\,dt \\ \qquad + \dfrac{1}{N}\displaystyle\sum_{i=1}^N \big[-DR^{-1}\big(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\big) + \sigma\big]\,dW_i \\ \qquad + \Big[C_0\check{x}^{(N)} - DR^{-1}\Big(v^{(N)} + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\Big) - \sigma_0^*(\check{u})\Big]\,dW_0, \\ d\check{p}^{(N)} = -\big[A^T\check{p}^{(N)} + C_0^T\check{\beta}_0^{(N)} + (\Xi_1 - Q)\check{x}^{(N)} + \Xi_2\big]dt + \check{\beta}_0^{(N)}dW_0 + \dfrac{1}{N}\displaystyle\sum_{i=1}^N\sum_{k=1}^N \check{\beta}_i^k dW_k, \\ \check{x}^{(N)}(0) = x_0, \quad \check{p}^{(N)}(T) = (\Xi_1^G - G)\check{x}^{(N)}(T) + \Xi_2^G, \end{cases} \tag{7.11}$$

where $v^{(N)} = \frac{1}{N}\sum_{i=1}^N v_i$. By (7.8), $\mathbb{E}_{\mathcal{F}^0}[x_i]$ satisfies

$$\begin{aligned} &d\mathbb{E}_{\mathcal{F}^0}[x_i] = \big[A\mathbb{E}_{\mathcal{F}^0}[x_i] - BR^{-1}(\hat{v} + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) + f\big]\,dt \\ &\quad + \big[C_0\mathbb{E}_{\mathcal{F}^0}[x_i] - D_0R^{-1}(\hat{v} + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2) - R_0^{-1}\hat{\beta}_0\big]dW_0, \\ &\mathbb{E}_{\mathcal{F}^0}[x_i](0) = x_0, \end{aligned} \tag{7.12}$$

where $\hat{v} = \mathbb{E}_{\mathcal{F}^0}[B^Tk_i + D_0^T\zeta_0 + D^T\zeta_i]$. Assume (5.9) admits a unique solution and thus its state component $(k_i, \zeta_0, \zeta_i, h)$ should have an upper bound in their L^2-norms. Thus, $\sup_{0\le t\le T}\mathbb{E}I_N^2(t) = O(\frac{1}{N})$ with

$$I_N := \frac{1}{N}\sum_{i=1}^N\int_0^T \big[-DR^{-1}\big(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\big) + \sigma\big]\,dW_i.$$

Moreover, well-posedness of (5.9) implies some compatibility condition holds true and the iterative scheme of coupled FBSDE works. Then, we can apply the standard continuity-dependence estimate between system (7.7) and system (7.8) to get the estimate (7.10). ∎

Step 2. Given Step 1, we have the estimate to the realized social cost $J_{\mathrm{soc}}^{\mathrm{wo}}(\check{u})$.

PROPOSITION 7.4. *There exists a constant c_1 independent of N such that*

$$J_{\mathrm{soc}}^{\mathrm{wo}}(\check{u}) \le Nc_1.$$

Proof. Consider the following intermediate state:

$$\begin{aligned} dx_i = &\big[Ax_i - BR^{-1}\big(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\big) + f\big]\,dt \\ &+ \big[-DR^{-1}\big(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\big) + \sigma\big]\,dW_i \\ &+ \big[C_0x_i - DR^{-1}\big(v_i + B^T(G-\Xi_1^G)h + D_0^T(G-\Xi_1^G)g_2\big) - R_0^{-1}\hat{\beta}_0\big]dW_0. \end{aligned}$$

By Proposition 7.3 and the standard FBSDE estimate, the following estimate holds:

$$\mathbb{E}\sup_{0\le t\le T}|\check{x}_i(t) - x_i(t)|^2 \le \frac{c_1}{N}.$$

Then,

$$\begin{aligned}J^{\mathrm{wo}}_{\mathrm{soc}}(\check u) =&\frac12\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{\big|(x_i-\Gamma\hat x-\eta)+(\check x_i-x_i)+\Gamma(\hat x-\Gamma\check x^{(N)})\big|^2_Q\\&\qquad\qquad+|\check u_i|^2_R-|(\sigma_0^*(\check u)-\hat\beta_0)+\hat\beta_0|^2_{R_0}\Big\}dt\\&+\frac12\mathbb{E}|(x_i(T)-\Gamma_0\hat x(T)-\eta_0)+(\check x_i(T)-x_i(T))+\Gamma_0(\check x^{(N)}(T)-(\check x_i(T))|^2_G\\\le& Nc_2\left(\|f\|_{L^2}+\|\sigma\|_{L^2}+\|\Xi_2\|_{L^2}+\|\Xi_2^G\|_{L^2}+O\left(\frac1N\right)\right)\le Nc.\end{aligned}$$ □

Step 3. This step aims to address the convexity of $J^{\mathrm{wo}}_{\mathrm{soc}}(u)$ of (P2). By its quadratic representation (7.4), it is equivalent to $\langle\mathbf{M}_1(\mathbf{u}),\mathbf{u}\rangle\ge0$. Here,

$$\mathbf{M}_1(\mathbf{u})=\mathbf{R}\mathbf{u}+\mathbf{B}^T\mathbf{m}_1+\sum_{i=1}^N\mathbf{D}_i^T\mathbf{n}_1^i+\mathbf{D}_0^T\mathbf{n}_1^0$$

with $(\mathbf{m}_1,\mathbf{n}_1^i,\mathbf{n}_1^0)$ given by (7.5). By examining its coupling structure of (7.5)–(7.6), it can be further reformulated via the following problem:

$$J^0_{\mathrm{soc}}(u)=\frac12\sum_{i=1}^N\mathbb{E}\int_0^T\Big\{\big|\grave x_i-\Gamma\grave x^{(N)}\big|^2_Q+|u_i|^2_R-|\grave\beta_0^{(N)}|^2_{R_0^{-1}}\Big\}dt,$$

where $\grave\beta_0^{(N)}=\frac1N\sum_{i=1}^N\grave\beta_i^0$, and for $i=1,\dots,N$,

$$\begin{cases}d\grave x_i=(A\grave x_i+Bu_i)dt+Du_idW_i+\left(C_0\grave x_i+D_0u_i-\dfrac{R_0^{-1}}{N}\displaystyle\sum_{k=1}^N\grave\beta_k^0\right)dW_0,\\ d\grave p_i=-(A^T\grave p_i+C_0^T\grave\beta_i^0-Q\grave x_i+\Xi_1\grave x^{(N)})dt+\grave\beta_i^0dW_0+\displaystyle\sum_{k=1}^N\grave\beta_i^kdW_k,\\ \grave x_i(0)=0,\quad\grave p_i(T)=(-G)\grave x_i(T)+\Xi_1^G\grave x^{(N)}(T).\end{cases}$$

Then $J^{\mathrm{wo}}_{\mathrm{soc}}(u)$ of (P2) is convex if and only if $J^0_{\mathrm{soc}}(u)\ge0$. Noticing the upper bound of realized cost functional $J^{\mathrm{wo}}_{\mathrm{soc}}(\check u)$ by Proposition 7.4, it suffices to consider the perturbation control $\tilde u$ satisfying $J^{\mathrm{wo}}_{\mathrm{soc}}(\tilde u)\le J^{\mathrm{wo}}_{\mathrm{soc}}(\check u)\le Nc_1$. This further implies that

$$\|\tilde u\|^2_{L^2}:=\sum_{i=1}^N\mathbb{E}\int_0^T|\tilde u_i(t)|^2dt\le Nc\tag{7.13}$$

by noting (P2) is convex. Also, it implies $\|\delta\tilde u\|^2_{L^2}:=\sum_{i=1}^N\mathbb{E}\int_0^T|\delta\tilde u_i(t)|^2dt\le Nc_1$ with $\delta u_i=\check u_i-\tilde u_i$.

Step 4. This step discusses the Fréchet differential of $J^{\mathrm{wo}}_{\mathrm{soc}}(u)$. Recalling the quadratic functional (7.4) and notation exchange between $\mathbf{u}$ and u, we have

$$\begin{aligned}J^{\mathrm{wo}}_{\mathrm{soc}}(\mathbf{u})=&\langle\mathbf{M}_1(\mathbf{u}),\mathbf{u}\rangle+2\langle\mathbf{M}_{12}(\mathbf{x}_0,\hat\eta_0),\mathbf{u}\rangle+\langle\mathbf{M}_2(\mathbf{x}_0,\hat\eta_0),(\mathbf{x}_0,\hat\eta_0)\rangle\\&+2\langle\mathbf{M}_{13},\mathbf{u}\rangle+2\langle\mathbf{M}_{23},(\mathbf{x}_0,\hat\eta_0)\rangle+\mathbf{M}_3\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathbf{M}_1(\check{\mathbf{u}}), \check{\mathbf{u}}\rangle + 2\langle \mathbf{M}_{12}(\mathbf{x}_0, \hat{\eta}_0), \check{\mathbf{u}}\rangle + \langle \mathbf{M}_2(\mathbf{x}_0, \hat{\eta}_0), (\mathbf{x}_0, \hat{\eta}_0)\rangle \\
&\quad + 2\langle \mathbf{M}_{13}, \check{\mathbf{u}}\rangle + 2\langle \mathbf{M}_{23}, (\mathbf{x}_0, \hat{\eta}_0)\rangle + \mathbf{M}_3 \qquad (= J^{\mathrm{wo}}_{\mathrm{soc}}(\check{\mathbf{u}})) \\
&\quad + \langle \mathbf{M}_1(\mathbf{u} - \check{\mathbf{u}}), \mathbf{u} - \check{\mathbf{u}}\rangle + 2\langle \mathbf{M}_{13}, \mathbf{u} - \check{\mathbf{u}}\rangle \qquad (= J^{0}_{\mathrm{soc}}(\mathbf{u} - \check{\mathbf{u}})) \\
&\quad + 2\langle \mathbf{M}_1(\mathbf{u}) + \mathbf{M}_{12}(\mathbf{x}, \hat{\eta}_0) + \mathbf{M}_{13}, \mathbf{u} - \check{\mathbf{u}}\rangle \qquad (= \langle \mathcal{D}_{\mathbf{u}} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{\mathbf{u}}), \mathbf{u} - \check{\mathbf{u}}\rangle) \\
&= J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}) + J^{0}_{\mathrm{soc}}(u - \check{u}) + \sum_{i=1}^{N} \langle \mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}), u_i - \check{u}_i\rangle,
\end{aligned}
$$

where $\mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u})$ given by (4.8) is the componentwise Fréchet derivative of $J^{\mathrm{wo}}_{\mathrm{soc}}$ at $\check{u}$ on the ith-component coordinate. Moreover, for u, by examining the person-by-person optimality procedures in section 4.1, and duality expression (4.21) for auxiliary cost $\hat{J}_i$, we have

$$
\|\mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}) - \mathcal{D}_{u_i} \hat{J}_i(\check{u})\|_{L^2} \le \frac{c}{\sqrt{N}} \|\check{u}\|_{L^2}
$$

for some constant c independent of N and $\check{u}$.

Proof of Theorem 7.2. Notice that

$$
\left| \frac{1}{N} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}) - \frac{1}{N} \inf_{u_i \in \mathcal{U}_c} J^{\mathrm{wo}}_{\mathrm{soc}}(u) \right| = O\left(\frac{1}{\sqrt{N}}\right)
$$

is equivalent to

$$
\inf_{u_i \in \mathcal{U}_c} J^{\mathrm{wo}}_{\mathrm{soc}}(u) \le J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}) \le \inf_{u_i \in \mathcal{U}_c} J^{\mathrm{wo}}_{\mathrm{soc}}(u) + O(\sqrt{N}).
$$

The first inequality is trivial. For the second inequality, we need only consider the perturbed control u satisfying $J^{\mathrm{wo}}_{\mathrm{soc}}(u) \le J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u})$ which is bounded in its L^2-norm by Step 2, namely $||u||^2_{L^2} \le cN$ with c independent of N. Now, by Steps 3 and 4, for all such perturbed u,

$$
\begin{aligned}
J^{\mathrm{wo}}_{\mathrm{soc}}(u) - J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}) &= J^{0}_{\mathrm{soc}}(u - \check{u}) + \sum_{i=1}^{N} \langle \mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}), u_i - \check{u}_i\rangle \\
&\ge \gamma ||\delta u||^2_{L^2} + \sum_{i=1}^{N} \langle \mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}), \delta u_i\rangle.
\end{aligned} \tag{7.14}
$$

Moreover, by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\sum_{i=1}^{N} \langle \mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u}), \delta u_i\rangle &\le \sqrt{\sum_{i=1}^{N} \|\mathcal{D}_{u_i} J^{\mathrm{wo}}_{\mathrm{soc}}(\check{u})\|^2_{L^2} \sum_{i=1}^{N} \|\delta u_i\|^2_{L_2}} \\
&\le c \sqrt{\sum_{i=1}^{N} O\left(\frac{1}{N}\right) \|\check{u}\|^2_{L^2} \sum_{i=1}^{N} \|\delta u_i\|^2_{L_2}} = O(\sqrt{N}),
\end{aligned} \tag{7.15}
$$

where the last inequality is due to (5.5) and Proposition 7.3 of Step 1. Also, note that $\mathcal{D}_{u_i} \hat{J}_i(\check{u}) = 0$ for $i = 1, \ldots, N$ due to the person-by-person optimality and Theorem 5.2. Thus, the asymptotic optimality (7.1) follows directly by (7.14) and (7.15). ∎

8. Concluding remarks. This paper investigated mean field LQG social control with volatility uncertain common noise. Based on a two-step duality, we construct an auxiliary optimal control problem. By solving this problem combined with consistent mean field approximations, we design a set of decentralized strategies and verify their asymptotically social optimality. An interesting work for further study is to consider the closed-loop team strategy, or the case when the state variable enters the term driven by W_i.

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