arXiv:1706.08941v2 [math.NA] 10 Jul 2017

HYBRID LOCALIZED SPECTRAL DECOMPOSITION FOR MULTISCALE PROBLEMS

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Abstract. We consider a finite element method for elliptic equation with heterogeneous and possibly high-contrast coefficients based on primal hybrid formulation. A space decomposition as in FETI and BDCC allows a sequential computations of the unknowns through elliptic problems and satisfies equilibrium constraints. One of the resulting problems is non-local but with exponentially decaying solutions, enabling a practical scheme where the basis functions have an extended, but still local, support. We obtain quasi-optimal a priori error estimates for low-contrast problems assuming minimal regularity of the solutions.

To also consider the high-contrast case, we propose a variant of our method, enriching the space solution via local eigenvalue problems and obtaining optimal a priori error estimate that mitigates the effect of having coefficients with different magnitudes and again assuming no regularity of the solution. The technique developed is dimensional independent and easy to extend to other problems such as elasticity.

1. Introduction

Consider the problem of finding the weak solution $u : \Omega \to \mathbb{R}$ of

$$
\begin{aligned}
-\operatorname{div} \mathcal{A}\, \boldsymbol{\nabla}\, u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 for simplicity, and is an open bounded domain with polyhedral boundary $\partial\Omega$, the symmetric tensor $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}_{\text{sym}}$ is uniformly positive definite and bounded, and f is part of the given data.

It is hard to approximate such problem in its full generality using numerical methods, in particular because of the low regularity of the solution and its multiscale behavior. Most convergent proofs either assume extra regularity or special properties of the coefficients [1, 2, 4, 5, 19, 20, 31, 36, 51, 58, 59, 65–67]. Some methods work even considering that the solution has low regularity [3, 22, 35, 50, 60] but are based on ideas that differ considerably from what we advocate here and do not cover in depth the high-contrast case.

Date: July 10, 2017.

As in many multiscale methods previously considered, our starting point is the decomposition of the solution space into *fine* and *coarse* spaces that are adapted to the problem of interest. The exact definition of some basis functions requires solving global problems, but, based on decaying properties, only local computations are required, although these are not restricted to a single element. It is interesting to notice that, although the formulation is based on hybridization, the final numerical solution is defined by a sequence of elliptic problems.

The idea of using exponential decay to localize global problems was already considered by the interesting approach developed under the name of *Localized Orthogonal Decomposition* (LOD) [44, 46, 56] which are related to ideas of *Variational Multiscale Methods* [37, 38]. In their case, convergence follows from a special orthogonality property.

Another difficulty that hinders the development of efficient methods is the presence of high-contrast coefficients [12, 22, 35, 60]. In general when LOD or VMS methods are considered, high-contrast coefficients might slow down the exponential decay of the solutions, making the method not so practical. Here in this paper, in the presence of rough coefficients, spectral techniques are employed to overcome such hurdle, and by solving local eigenvalue problems we define a space where the exponential decay of solutions is insensitive to high-contrast coefficients.

We now further detail the problem under consideration. For almost all $\boldsymbol{x} \in \Omega$ let the positive constants $a_{\min}$ and $a_{\max}$ be such that

$$a_{\min}|\boldsymbol{v}|^2 \le a_-(\boldsymbol{x})|\boldsymbol{v}|^2 \le \mathcal{A}(\boldsymbol{x})\,\boldsymbol{v}\cdot\boldsymbol{v} \le a_+(\boldsymbol{x})|\boldsymbol{v}|^2 \le a_{\max}|\boldsymbol{v}|^2 \quad \text{for all } \boldsymbol{v} \in \mathbb{R}^d, \tag{2}$$

where $a_-(\boldsymbol{x})$ and $a_+(\boldsymbol{x})$ are the smallest and largest eigenvalues of $\mathcal{A}(\boldsymbol{x})$. Let $\rho \in L^\infty(\Omega)$ be chosen by the user and such that $\rho(\boldsymbol{x}) \in [\rho_{\min}, \rho_{\max}]$ almost everywhere for some positive constants $\rho_{\min}$ and $\rho_{\max}$. Consider g such that

$$f = \rho g,$$

and then the ρ-weighted $L^2(\Omega)$ norm $\|g\|_{L^2_\rho(\Omega)} := \|\rho^{1/2} g\|_{L^2(\Omega)} = \|f\|_{L^2_{1/\rho}(\Omega)}$ is finite. A reason to introduce such weight ρ is to balance u and f with respect to the tensor $\mathcal{A}$, establishing a priori error results without hidden constants that depend on $\mathcal{A}$. In principle any weight ρ can be chosen, and larger ρ implies in sharper a priori error estimates in the energy norm, at a larger size and cost of the finite element method. Natural choices are $\rho(\boldsymbol{x}) \in \{1, a_{\min}, a_-(\boldsymbol{x}), a_+(\boldsymbol{x}), a_{\max}\}$ or local choices depending also on the triangulation; the role of ρ is further addressed at the end of Section 4.

The remainder of the this paper is organized as follows. Section 2 describes a suitable primal hybrid formulation for the problem (1), which is followed in Section 3 by its a discrete formulation. A discrete space decomposition is introduced to transform the discrete saddle-point problem into a sequence of elliptic discrete problems. The analyze of the exponential decay of the multiscale basis function is considered in Section 3.1. To overcome the possible deterioration of the exponential decay for high-contrast coefficients, in Section 3.2 the *Localized Spectral Decomposition* (LSD) method is designed and fully analyzed. To allow an efficient pre-processing numerical scheme, Section 4 discusses how to reduce the right-hand side space dimension without losing a target accuracy, and also develops $L^2_\rho(\Omega)$ a priori error estimates. Section 5 gives a global overview of the LSD *algorithm* proposed. Appendix A provides some mathematical tools and Appendix B refers to a notation library for the paper.

2. Continuous Problem using Hybrid Formulation

We start by recasting the continuous problem in a weak formulation that depends on a simplicial regular mesh $\mathcal{T}_H$ and let $\mathcal{F}_H$ be the set of faces on $\mathcal{T}_H$. The extension to polyhedral regular meshes is straightforward. Without loss of generality we adopt above and in the remainder of the text the terminology of three-dimensional domains, denoting for instance the boundaries of the elements by faces. For a given element $\tau \in \mathcal{T}_H$ let $\partial\tau$ denote its boundary and $\boldsymbol{n}^\tau$ the unit size normal vector that points outward τ. We denote by $\boldsymbol{n}$ the outward normal vector on $\partial\Omega$. Consider now the following spaces:

$$
\begin{aligned}
H^1(\mathcal{T}_H) &= \{v \in L^2(\Omega) : v|_\tau \in H^1(\tau),\, \tau \in \mathcal{T}_H\}, \\
\Lambda(\mathcal{T}_H) &= \left\{ \prod_{\tau\in\mathcal{T}_H} \boldsymbol{\tau}\cdot\boldsymbol{n}^\tau|_{\partial\tau} : \boldsymbol{\tau} \in H(\mathrm{div};\Omega) \right\} \subsetneq \prod_{\tau\in\mathcal{T}_H} H^{-1/2}(\partial\tau).
\end{aligned} \tag{3}
$$

For w, $v \in H^1(\mathcal{T}_H)$ and $\mu \in \Lambda(\mathcal{T}_H)$ define

$$
(w,v)_{\mathcal{T}_H} = \sum_{\tau\in\mathcal{T}_H} \int_\tau wv\,d\boldsymbol{x}, \qquad (\mu,v)_{\partial\mathcal{T}_H} = \sum_{\tau\in\mathcal{T}_H} (\mu,v)_{\partial\tau},
$$

where $(\cdot,\cdot)_{\partial\tau}$ is the dual product involving $H^{-1/2}(\partial\tau)$ and $H^{1/2}(\partial\tau)$. Then

$$
(\mu,v)_{\partial\tau} = \int_\tau \operatorname{div}\boldsymbol{\sigma} v\,d\boldsymbol{x} + \int_\tau \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\, v\,d\boldsymbol{x}
$$

for all $\boldsymbol{\sigma} \in H(\mathrm{div};\tau)$ such that $\boldsymbol{\sigma}\cdot\boldsymbol{n}^\tau = \mu$. We also define the norms by

$$
\begin{aligned}
&\|\boldsymbol{\sigma}\|^2_{H_\mathcal{A}(\mathrm{div};\Omega)} = \|\mathcal{A}^{-1/2}\boldsymbol{\sigma}\|^2_{0,\Omega} + H^2\|\rho^{-1/2}\operatorname{div}\boldsymbol{\sigma}\|^2_{0,\Omega}, \\
&|v|^2_{H^1_\mathcal{A}(\mathcal{T}_H)} = \sum_{\tau\in\mathcal{T}_H} \|\mathcal{A}^{1/2}\,\boldsymbol{\nabla}\, v\|^2_{0,\tau}, \qquad \|\mu\|_{H^{-1/2}_\mathcal{A}(\mathcal{T}_H)} = \inf_{\substack{\boldsymbol{\sigma}\in H(\mathrm{div};\Omega)\\ \boldsymbol{\sigma}\,\boldsymbol{n}^\tau=\mu \text{ on } \partial\tau,\, \tau\in\mathcal{T}_H}} \|\boldsymbol{\sigma}\|_{H_\mathcal{A}(\mathrm{div};\Omega)}.
\end{aligned} \tag{4}
$$

We use analogous definitions on subsets of $\mathcal{T}_H$, in particular when the subset consists of a single element τ (and in this case we write τ instead of $\{\tau\}$). We note that since $a_{\min}$ and $\rho_{\min}$ are positive and $a_{\max}$ and $\rho_{\max}$ are bounded, then $\|\cdot\|_{H_{\mathcal{A}}(\operatorname{div};\Omega)}$ and $|\cdot|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}$ are equivalent to the usual norms $\|\cdot\|_{H(\operatorname{div};\Omega)}$ and $|\cdot|_{H^1(\mathcal{T}_H)}$.

In the primal hybrid formulation [57], find $u \in H^1(\mathcal{T}_H)$ and $\lambda \in \Lambda(\mathcal{T}_H)$ are such that

$$
\begin{aligned}
(\mathcal{A}\,\boldsymbol{\nabla}\, u, \boldsymbol{\nabla}\, v)_{\mathcal{T}_H} - (\lambda, v)_{\partial\mathcal{T}_H} &= (\rho g, v)_{\mathcal{T}_H} &&\text{for all } v \in H^1(\mathcal{T}_H),\\
(\mu, u)_{\partial\mathcal{T}_H} &= 0 &&\text{for all } \mu \in \Lambda(\mathcal{T}_H).
\end{aligned}
\tag{5}
$$

Following [57, Theorem 1], it is possible to show that the solution (u, λ) of (5) is such that $u \in H^1_0(\Omega)$ satisfies (1) in the weak sense and $\lambda = \mathcal{A}\,\boldsymbol{\nabla}\, u \cdot \boldsymbol{n}^\tau$ for all elements τ.

In the spirit of the Hybrid Multiscale Methods [1, 31, 32, 51] and FETI methods [15, 25, 27, 63], we consider the decomposition

$$
H^1(\mathcal{T}_H) = \mathbb{P}^0(\mathcal{T}_H) \oplus \widetilde{H}^1(\mathcal{T}_H),
$$

where $\mathbb{P}^0(\mathcal{T}_H)$ is the space of piecewise constants, and $\widetilde{H}^1(\mathcal{T}_H)$ is its $L^2_\rho(\tau)$ orthogonal complement, i.e., the space of functions with zero ρ-weighted average within each element $\tau \in \mathcal{T}_H$

$$
\begin{aligned}
\mathbb{P}^0(\mathcal{T}_H) &= \{v \in H^1(\mathcal{T}_H) :\ v|_\tau \text{ is constant}, \tau \in \mathcal{T}_H\},\\
\widetilde{H}^1(\mathcal{T}_H) &= \{\tilde{v} \in H^1(\mathcal{T}_H) :\ \int_\tau \rho\tilde{v}\, d\boldsymbol{x} = 0, \tau \in \mathcal{T}_H\}.
\end{aligned}
\tag{6}
$$

We then write $u = u^0 + \tilde{u}$, where $u^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\tilde{u} \in \widetilde{H}^1(\mathcal{T}_H)$, and find from (5) that

$$
\begin{aligned}
(\lambda, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v^0)_{\mathcal{T}_H} &&\text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H),\\
(\mu, u^0 + \tilde{u})_{\partial\mathcal{T}_H} &= 0 &&\text{for all } \mu \in \Lambda(\mathcal{T}_H),
\end{aligned}
\tag{7}
$$

and that

$$
(\mathcal{A}\,\boldsymbol{\nabla}\, \tilde{u}, \boldsymbol{\nabla}\, \tilde{v})_{\mathcal{T}_H} = (\lambda, \tilde{v})_{\partial\mathcal{T}_H} + (\rho g, \tilde{v})_{\mathcal{T}_H} \quad \text{for all } \tilde{v} \in \widetilde{H}^1(\mathcal{T}_H).
\tag{8}
$$

Let $T : \Lambda(\mathcal{T}_H) \to \widetilde{H}^1(\mathcal{T}_H)$ and $\tilde{T} : L^2(\Omega) \to \widetilde{H}^1(\mathcal{T}_H)$ be such that, for $\mu \in \Lambda(\mathcal{T}_H)$, $g \in L^2_\rho(\Omega)$ and $\tau \in \mathcal{T}_H$,

$$
\int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T\mu)\cdot\boldsymbol{\nabla}\,\tilde{v}\, d\boldsymbol{x} = (\mu, \tilde{v})_{\partial\tau}, \qquad \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(\tilde{T}g)\cdot\boldsymbol{\nabla}\,\tilde{v}\, d\boldsymbol{x} = (\rho g, \tilde{v})_\tau \quad \text{for all } \tilde{v} \in \widetilde{H}^1(\mathcal{T}_H).
\tag{9}
$$

It follows from the above definition that $\tilde{T}g = 0$ if $g \in \mathbb{P}^0(\mathcal{T}_H)$, and that, for all $\mu \in \Lambda(\mathcal{T}_H)$,

$$
(\mu, \tilde{T}g)_{\partial\mathcal{T}_H} = \sum_{\tau\in\mathcal{T}_H} \int_\tau \mathcal{A}\,\boldsymbol{\nabla}(T\mu)\cdot\boldsymbol{\nabla}(\tilde{T}g)\, d\boldsymbol{x} = (\rho g, T\mu)_{\mathcal{T}_H}.
\tag{10}
$$

Note from (8) that $\tilde{u} = T\lambda + \tilde{T}g$, and substituting in (7), we have that $u^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\lambda \in \Lambda(\mathcal{T}_H)$ solve

$$\begin{aligned} (\mu, T\lambda)_{\partial\mathcal{T}_H} + (\mu, u^0)_{\partial\mathcal{T}_H} &= -(\mu, \tilde{T}g)_{\partial\mathcal{T}_H} \quad \text{for all } \mu \in \Lambda(\mathcal{T}_H), \\ (\lambda, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v^0)_{\mathcal{T}_H} \quad \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H). \end{aligned} \tag{11}$$

We use these unknowns u^0 and λ to reconstruct the u and the flux $\boldsymbol{\sigma}$ as follows:

$$u = u^0 + \tilde{u} = u^0 + T\lambda + \tilde{T}g, \qquad \boldsymbol{\sigma} = \mathcal{A}\,\boldsymbol{\nabla}(T\lambda + \tilde{T}g). \tag{12}$$

Remark 1. *With the above definitions, it is possible to rewrite the energy norm as below:*

$$|T\lambda|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \sum_{\tau\in\mathcal{T}_H} (\lambda, T\lambda)_{\partial\tau} = \|\mathcal{A}^{-1/2}\boldsymbol{\sigma}_\lambda\|^2_{L^2(\Omega)}, \quad \textit{where } \boldsymbol{\sigma}_\lambda = \mathcal{A}\,\boldsymbol{\nabla}\, T\lambda.$$

3. Hybrid Localized Finite Elements

Consider $\mathcal{F}_h$ be a partition of the faces of elements in $\mathcal{T}_H$, refining them in the sense that every (coarse) face of the elements in $\mathcal{T}_H$ can be written as a union of faces of $\mathcal{F}_h$. Let $\Lambda_h \subset \Lambda(\mathcal{T}_H)$ be the space of piecewise constants on $\mathcal{F}_h$, i.e.,

$$\Lambda_h = \{\mu_h \in \Lambda(\mathcal{T}_H) : \mu_h|_{F_h} \text{ is constant on each face } F_h \in \mathcal{F}_h\}.$$

To simplify the presentation we do not discretize $H^1(\tau)$ and $H(\mathrm{div};\tau)$ for $\tau \in \mathcal{T}_H$. We remark that the method develop here extends easily when we discretize $H(\mathrm{div};\tau)$ by simplices or cubical elements with lowest order Raviart–Thomas spaces [68], or discretize $H^1(\tau)$ fine enough to resolve the heterogeneities of $\mathcal{A}(x)$ and to satisfy inf-sup conditions with respect to the space Λ_h.

We pose then the problem of finding $u_h^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{aligned} (\mu_h, T\lambda_h)_{\partial\mathcal{T}_H} + (\mu_h, u_h^0)_{\partial\mathcal{T}_H} &= -(\mu_h, \tilde{T}g)_{\partial\mathcal{T}_H} \quad \text{for all } \mu_h \in \Lambda_h, \\ (\lambda_h, v^0)_{\partial\mathcal{T}_H} &= -(\rho g, v^0)_{\mathcal{T}_H} \quad \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H). \end{aligned} \tag{13}$$

Since (13) is finite dimensional, it is well-posed if and only if is injective. Assuming that $g = 0$, we easily gather that $\lambda_h = 0$ and $u_h^0 = 0$; see Lemma 8. We define our approximation as in (12), by

$$u_h = u_h^0 + T\lambda_h + \tilde{T}g, \qquad \boldsymbol{\sigma}_h = \mathcal{A}\,\boldsymbol{\nabla}(T\lambda_h + \tilde{T}g). \tag{14}$$

Simple substitutions yield that u_h, λ_h solve (5) if Λ is replaced by Λ_h, i.e.,

$$\begin{aligned} (\mathcal{A}\,\boldsymbol{\nabla}\, u_h, \boldsymbol{\nabla}\, v)_{\mathcal{T}_H} - (\lambda_h, v)_{\partial\mathcal{T}_H} &= (\rho g, v)_{\mathcal{T}_H} \quad \text{for all } v \in H^1(\mathcal{T}_H), \\ (\mu_h, u_h)_{\partial\mathcal{T}_H} &= 0 \quad \text{for all } \mu_h \in \Lambda_h. \end{aligned} \tag{15}$$

We assume that Λ_h is chosen fine enough so that

$$|u-u_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \big(\lambda-\lambda_h, T(\lambda-\lambda_h)\big)_{\mathcal{T}_H} \le \mathscr{H}^2 \|g\|^2_{L^2_\rho(\Omega)}, \tag{16}$$

where $\mathscr{H}$ represents a "target precision" the method should achieve. For instance, one could choose $\mathscr{H} = H$ or h^s, for $0 < s \le 1$. It must be mentioned that λ_h is never computed, the main goal of this paper is to develop an efficient approximation of order $\mathscr{H}$ for λ_h using $O(H^{-d})$ degrees of freedom.

Above, and in what follows, c denotes an arbitrary constant that does not depend on H, $\mathscr{H}$, h, $\mathcal{A}$, or ρ, it depends only on the shape regularity of the elements of $\mathcal{T}_H$.

Taking a further step, we decompose Λ_h into a space of "constants" plus "zero-average" functionals over the border of the elements of $\mathcal{T}_H$. For each $\tau_i \in \mathcal{T}_H$, let $\lambda_i^0 \in \Lambda_h$ such that

$$(\lambda_i^0, v)_{\partial\mathcal{T}_H} = \int_{\partial\tau_i} [\![v]\!]\, d\boldsymbol{x} \quad \text{for all } v \in H^1(\mathcal{T}_H), \tag{17}$$

where $[\![\cdot]\!]$ denotes the jump operator, defined as follows. For each face F belonging to the boundaries of two different elements τ_i, τ_j, fix $\boldsymbol{n}_F$ as the constant unitary normal vector pointing either inward or outward. If $\boldsymbol{n}_F$ is oriented from τ_i to τ_j, let $[\![v]\!] = v_i|_F - v_j|_F$, where $v_i = v|_{\tau_i}$, if not $[\![v]\!] = v_j|_F - v_i|_F$. As usual, if F belongs to $\partial\Omega$, then $\boldsymbol{n}_F = \boldsymbol{n}$ points outward and $[\![v]\!] = v$.

Remark 2. *It is also possible to define λ_i^0 explicitly. Let F be a face of an element τ. If F does not belong to $\partial\tau_i$ then $\lambda_i^0|_F = 0$. If it does, $\lambda_i^0|_{F\cap\partial\tau_i} = -\lambda_i^0|_{F\cap\partial\tau_j} = 1$ or -1 depending whether $\boldsymbol{n}_F$ points outward or inward of τ_i, respectively. Note that $\lambda_i^0 \in \Lambda_h$.*

Let N be the number of elements of $\mathcal{T}_H$ and

$$\begin{gathered}\Lambda^0 = \operatorname{span}\{\lambda_i^0 : i = 1,\dots,N\},\\ \widetilde{\Lambda}_h = \mathbb{P}^0(\mathcal{T}_H)^\perp = \{\mu_h \in \Lambda_h : (\mu_h, v^0)_{\partial\mathcal{T}_H} = 0 \text{ for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H)\}.\end{gathered} \tag{18}$$

We can now decompose $\Lambda_h = \Lambda^0 \oplus \tilde{\Lambda}_h$ as follows [9]. Given $\mu_h \in \Lambda_h$, let $\mu^0 \in \Lambda^0$ and $\tilde{\mu}_h \in \widetilde{\Lambda}_h$ such that

$$(\mu^0, v^0)_{\partial\mathcal{T}_H} = (\mu_h, v^0)_{\partial\mathcal{T}_H}, \qquad (\tilde{\mu}_h, v)_{\partial\mathcal{T}_H} = (\mu_h, v)_{\partial\mathcal{T}_H} - (\mu^0, v)_{\partial\mathcal{T}_H},$$

for all $v^0 \in \mathbb{P}^0(\mathcal{T}_H)$ and $v \in H^1(\mathcal{T}_H)$. Note that $\tilde{\mu}_h \in \widetilde{\Lambda}_h$ since $(\tilde{\mu}_h, v^0)_{\partial\mathcal{T}_H} = (\mu_h, v^0)_{\partial\mathcal{T}_H} - (\mu^0, v^0)_{\partial\mathcal{T}_H} = 0$, and $\mu_h = \mu^0 + \tilde{\mu}_h$.

We also decompose $\widetilde{\Lambda}_h = \widetilde{\Lambda}_h^0 \oplus \widetilde{\Lambda}_h^f$. Basically, the elements of $\widetilde{\Lambda}_h^0$ are constants on each element face of $\mathcal{T}_h$ but still with zero average over the element boundaries, and the elements of $\widetilde{\Lambda}_h^f$ have zero average on each face:

$$\begin{aligned} \widetilde{\Lambda}_h^0 &= \{\mu_h \in \widetilde{\Lambda}_h : \mu_h|_F \text{ is constant for each face } F \subset \partial\tau,\, \tau \in \mathcal{T}_H\}, \\ \widetilde{\Lambda}_h^f &= \{\mu_h \in \widetilde{\Lambda}_h : \int_F \mu_h \, ds = 0 \text{ for each face } F \subset \partial\tau,\, \tau \in \mathcal{T}_H\}. \end{aligned} \tag{19}$$

Considering again (13), from the decomposition for Λ_h, we gather that $\lambda_h = \lambda^0 + \tilde{\lambda}_h^0 + \tilde{\lambda}_h^f$. Thus, $u_h^0 \in \mathbb{P}^0(\mathcal{T}_H)$, $\lambda^0 \in \Lambda^0$, $\tilde{\lambda}_h^0 \in \widetilde{\Lambda}_h^0$ and $\tilde{\lambda}_h \in \tilde{\Lambda}_h^f$ solve

$$\begin{aligned} &(\lambda^0, v^0)_{\partial\mathcal{T}_H} && = -(\rho g, v^0)_{\mathcal{T}_H} && \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H), \\ &(\tilde{\mu}_h^f, T\lambda^0 + T\tilde{\lambda}_h^0 + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} && = -(\tilde{\mu}_h^f, \tilde{T}g)_{\partial\mathcal{T}_H} && \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^f, \\ &(\tilde{\mu}_h^0, T\lambda^0 + T\tilde{\lambda}_h^0 + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} && = -(\tilde{\mu}_h^0, \tilde{T}g)_{\partial\mathcal{T}_H} && \text{for all } \tilde{\mu}_h^0 \in \widetilde{\Lambda}_h^0, \\ &(\mu^0, T\lambda^0 + T\tilde{\lambda}_h^0 + T\tilde{\lambda}_h^f)_{\partial\mathcal{T}_H} + (\mu^0, u_h^0)_{\partial\mathcal{T}_H} && = -(\mu^0, \tilde{T}g)_{\partial\mathcal{T}_H} && \text{for all } \mu^0 \in \Lambda^0. \end{aligned} \tag{20}$$

It is possible to compute the unknowns step-by-step as we detail below. After that we discuss the well-posedness of each problem. The first equation of (20) determines λ^0. To deal with the second equation, we define the operator $P : H^1(\mathcal{T}_H) \to \tilde{\Lambda}_h^f$ such that, for $w \in H^1(\mathcal{T}_H)$,

$$(\tilde{\mu}_h^f, TPw)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, w)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^f, \tag{21}$$

i.e., $\big(\tilde{\mu}_h^f, (I - TP)w\big)_{\partial\mathcal{T}_H} = 0$. Note that PT is an orthogonal projection from Λ_h to $\tilde{\Lambda}_h^f$ since

$$(\tilde{\mu}_h^f, TPT\lambda_h)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, T\lambda_h)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^f.$$

The second equation of (20) becomes

$$\tilde{\lambda}_h^f = -P(T\lambda^0 + T\tilde{\lambda}_h^0 + \tilde{T}g). \tag{22}$$

Solving (21) efficiently is crucial for the good performance of the method, since it is the only large dimensional system of (20), in the sense that its size grows with order of h^{-d}. This issue is treated in Section 3.1 by taking into account the exponential decay of $PT(\lambda^0 + \tilde{\lambda}_h^0)$. It is also required to compute or to approximate $\tilde{T}g$ and $P\tilde{T}g$ efficiently. These issues are treated in Sections 3.1 and 4.

Now, we can write the third equation of (20) as

$$(\tilde{\mu}_h^0, T\tilde{\lambda}_h^0)_{\partial\mathcal{T}_H} = -(\tilde{\mu}_h^0, \tilde{T}g)_{\partial\mathcal{T}_H} - \big(\tilde{\mu}_h^0, T\lambda^0 - TP(T\lambda^0 + T\tilde{\lambda}_h^0 + \tilde{T}g)\big)_{\partial\mathcal{T}_H} \tag{23}$$

and then

$$(\tilde{\mu}_h^0, T\tilde{\lambda}_h^0 - TPT\tilde{\lambda}_h^0)_{\partial\mathcal{T}_H} = -(\tilde{\mu}_h^0, \tilde{T}g - TP\tilde{T}g)_{\partial\mathcal{T}_H} - (\tilde{\mu}_h^0, T\lambda^0 - TPT\lambda^0)_{\partial\mathcal{T}_H}.$$

Since $PT\tilde{\mu}_h^0 \in \tilde{\Lambda}_h^f$,

$$\big(\tilde{\mu}_h^0 - PT\tilde{\mu}_h^0, (I-TP)T\tilde{\lambda}_h^0\big)_{\partial\mathcal{T}_H} \\ = -(\tilde{\mu}_h^0 - PT\tilde{\mu}_h^0, (I-TP)\tilde{T}g)_{\partial\mathcal{T}_H} - (\tilde{\mu}_h^0 - PT\tilde{\mu}_h^0, (I-TP)T\lambda^0)_{\partial\mathcal{T}_H}.$$

Thus, $\tilde{\lambda}_h^0$ is computed from

$$\big((I-PT)\tilde{\mu}_h^0, T(I-PT)\tilde{\lambda}_h^0\big)_{\partial\mathcal{T}_H} = -\big((I-PT)\tilde{\mu}_h^0, (I-TP)\tilde{T}g\big)_{\partial\mathcal{T}_H} \\ - \big((I-PT)\tilde{\mu}_h^0, T(I-PT)\lambda^0\big)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^0 \in \widetilde{\Lambda}_h^0, \tag{24}$$

and $\tilde{\lambda}_h^f$ is recovered from (22).

Finally, the fourth equation of (20) yields u_h^0, and the post-processing (14) recovers the main variables:

$$u_h = u_h^0 + T(\lambda^0 + \tilde{\lambda}_h^0 + \tilde{\lambda}_h^f) + \tilde{T}g, \qquad \boldsymbol{\sigma}_h = \mathcal{A}\,\boldsymbol{\nabla}[T(\lambda^0 + \tilde{\lambda}_h^0 + \tilde{\lambda}_h^f) + \tilde{T}g]. \tag{25}$$

To show the existence and uniqueness of solutions for (20), we proceed by parts. The existence of solution for the first equation follows from Lemma 8. Solving the second equation is equivalent to (21), and such system is well-posed due to the coercivity of $(\cdot, T\cdot)_{\partial\mathcal{T}_H}$ on $\tilde{\Lambda}_h^f$; see [1,31] and [26,42,63]. The same arguments hold for the third equation of (20), rewritten in (23). Another way to see this is to consider (24) with zero right hand side. From the coercivity of $(\cdot, T\cdot)_{\partial\mathcal{T}_H}$ on $\widetilde{\Lambda}$ we have $(I-PT)\tilde{\lambda}_h^0 = 0$. But since $\widetilde{\Lambda}_h^0 \cap \tilde{\Lambda}_h^f = \{0\}$, then $\tilde{\lambda}_h^0 = 0$. Finally, the fourth equation of (20) is again finite dimension, and if $(\mu^0, u_h^0)_{\partial\mathcal{T}_H} = 0$ for all $\mu^0 \in \Lambda^0$, then, from Lemma 8, $u_h^0 = 0$.

3.1. **Decaying Low-Contrast.** It is essential for the performing method that the static condensation is done efficiently. We prove next that the solutions decay exponentially fast, so instead of solving the problems in the whole domain, we actually solve it locally. We note that the idea of performing global static condensation goes back to the Variational Multiscale Finite Element Method–VMS [37, 38]. Recently variations of the VMS and denoted by Localized Orthogonal Decomposition Methods–LOD were introduced and analyzed in [44–46, 56].

For $K \in \mathcal{T}_H$, define $\mathcal{T}_0(K) = \emptyset$, $\mathcal{T}_1(K) = \{K\}$, and for $j = 1, 2, \ldots$ let

$$\mathcal{T}_{j+1}(K) = \{\tau \in \mathcal{T}_H : \overline{\tau} \cap \overline{\tau}_j \neq \emptyset \text{ for some } \tau_j \in \mathcal{T}_j(K)\}.$$

We now establish the following fundamental result for low-contrast. The technique used for the proof is extended in Lemma 1' for the high contrast case.

Lemma 1. *Let $v \in H^1(\mathcal{T}_H)$ where $\operatorname{supp} v \subset K \in \mathcal{T}_H$, and $\tilde{\mu}_h^f = Pv$. Then, for any integer $j \geq 1$,*

$$|T\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))} \leq d^2\alpha|T\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K))},$$

where $\alpha = \gamma\kappa\beta^2_{H/h}$, γ is a positive constant that depends only on the shape regularity of $\mathcal{T}_H$,

$$\beta_{H/h} = 1+\log(H/h), \quad \kappa = \max_{\tau\in\mathcal{T}_H}\kappa^\tau, \quad \kappa^\tau = \frac{a^\tau_{max}}{a^\tau_{min}}, \quad a^\tau_{max} = \sup_{\boldsymbol{x}\in\tau} a_+(\boldsymbol{x}), \quad a^\tau_{min} = \inf_{\boldsymbol{x}\in\tau} a_-(\boldsymbol{x}).$$

Proof. Choose $\tilde{\nu}_h^f \in \tilde{\Lambda}_h^f$ defined by $\tilde{\nu}_h^f|_F = 0$ if F is a face of an element of $\mathcal{T}_j(K)$ and $\tilde{\nu}_h^f|_F = \tilde{\mu}_h^f|_F$ otherwise. We obtain

$$\begin{aligned}
|T\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))} &= \sum_{\tau\in\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K)} (\tilde{\mu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau} \\
&= \sum_{\tau\in\mathcal{T}_H}(\tilde{\nu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau} - \sum_{\tau\in\mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K)}(\tilde{\mu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau} + \sum_{\tau\in\mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K)}(\tilde{\mu}_h^f - \tilde{\nu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau} \\
&= -|T\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K))} + \sum_{\tau\in\mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K)}(\tilde{\mu}_h^f - \tilde{\nu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau},
\end{aligned}$$

where we used that $\sum_{\tau\in\mathcal{T}_H}(\tilde{\nu}_h^f, T\tilde{\mu}_h^f)_{\partial\tau} = 0$ due to the definition of T and the local support of v.

Next, let F be a face of an element $\tau \in \mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K)$ and let χ_F be the characteristic function of F being identically equal to one on F and zero on $\partial\tau\setminus F$. See that $\chi_F(\tilde{\mu}_h^f - \tilde{\nu}_h^f)$ vanishes for faces F on $\tau \in \mathcal{T}_{j+1}(K)\setminus\mathcal{T}_j(K)$ that are not shared by an element in $\mathcal{T}_j(K)$ however it is not known a priori how many, possibly node. For the shared faces, $\chi_F(\tilde{\mu}_h^f - \tilde{\nu}_h^f) = \chi_F\tilde{\mu}_h^f$ and denote $\mu_F = \chi_F\tilde{\mu}_h^f$. Since it is possible that all d faces of τ share faces of elements of $\mathcal{T}_j$, hence, the following bound always holds:

$$|T(\tilde{\mu}_h^f - \tilde{\nu}_h^f)|^2_{H^1_{\mathcal{A}}(\tau)} \leq d\sum_{F\subset\partial\tau}|T\chi_F\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\tau)},$$

and it remains to estimate $|T\mu_F|_{H^1_{\mathcal{A}}(\tau)}$. Let us first define $T_{\mathcal{I}}$ by (9) with $\mathcal{A} = \mathcal{I}$, the identity tensor, that is, $T_{\mathcal{I}}$ is the classical harmonic extension with Neumann boundary condition. From Lemma 9 and a direct application of [68, Lemma 4.4] since both μ_F and $\tilde{\mu}_h^f$ have zero average on $\partial\tau$, we have

$$\begin{aligned}
|T\mu_F|^2_{H^1_{\mathcal{A}}(\tau)} &\leq \frac{1}{a^\tau_{\min}}|T_{\mathcal{I}}\mu_F|^2_{H^1(\tau)} \leq \frac{c_1}{a^\tau_{\min}}|\mu_F|^2_{H^{-1/2}(\partial\tau)} \leq \frac{c_2}{a^\tau_{\min}}\beta^2_{H/h}|\tilde{\mu}_h^f|^2_{H^{-1/2}(\partial\tau)} \\
&\leq \frac{\gamma}{a^\tau_{\min}}\beta^2_{H/h}|T_{\mathcal{I}}\tilde{\mu}_h^f|^2_{H^1(\tau)} \leq \gamma\frac{a^\tau_{\max}}{a^\tau_{\min}}\beta^2_{H/h}|T\tilde{\mu}_h^f|^2_{H^1_{\mathcal{A}}(\tau)},
\end{aligned}$$

and then the result follows. ∎

Remark 3. *Even though the proof of Lemma 1 concentrates the analysis to hybrid discretization and $u_h \in H^1(\mathcal{T}_H)$, the analysis also extends easily to mixed finite element discretizations or to finite dimensional approximations of $H^1(\mathcal{T}_H)$. We also could have used the flux approach for the proof, that is, let $\boldsymbol{\sigma}^F = \mathcal{A}\,\boldsymbol{\nabla}\, T\mu_F$, $\boldsymbol{\sigma}_{\mathcal{I}}^F = \boldsymbol{\nabla}\, T_1\mu_F$, $\boldsymbol{\sigma} = \mathcal{A}\,\boldsymbol{\nabla}\, T\tilde{\mu}_h^f$ and $\boldsymbol{\sigma}_{\mathcal{I}} = \boldsymbol{\nabla}\, T_{\mathcal{I}}\tilde{\mu}_h^f$, or the corresponding ones arising from the lower-order Raviart–Thomas case (associate to a triangulation $\mathcal{T}_h(\tau)$). We would have*

$$\begin{aligned}\|\mathcal{A}^{-1/2}\boldsymbol{\sigma}^F\|_{L^2(\tau)}^2 &\le \frac{1}{a_{min}^\tau}\|\boldsymbol{\sigma}_{\mathcal{I}}^F\|_{L^2(\tau)}^2 \le \frac{c_1}{a_{min}^\tau}|\mu_F|_{H^{-1/2}(\partial\tau)}^2 \le \frac{c_2}{a_{min}^\tau}\beta_{H/h}^2|\tilde{\mu}_h^f|_{H^{-1/2}(\partial\tau)}^2 \\ &\le \frac{\gamma}{a_{min}^\tau}\beta_{H/h}^2\|\boldsymbol{\sigma}_{\mathcal{I}}\|_{L^2(\tau)}^2 \le \gamma\frac{a_{max}^\tau}{a_{min}^\tau}\beta_{H/h}^2\|\mathcal{A}^{-1/2}\boldsymbol{\sigma}\|_{L^2(\tau)}^2.\end{aligned}$$

Remark 4. *We note that* [68, Lemma 4.4] *is based on $H^{-1/2}(\tau)$ norms and therefore it holds whether we use $H^1(\tau), H(\mathrm{div};\tau)$ or corresponding discretized versions inside τ. We point out that the h in $\log(H/h)$ is related to the space Λ_h, not to the interior. The α in this paper is estimated as the worst case scenario, that is, using Lemma 9 and* [68, Lemma 4.4]*. For particular cases of coefficients $\mathcal{A}$ and discretizations for $H^1(\tau)$ or $H(\mathrm{div};\tau)$, sharper estimated for α can be derived using weighted and generalized Poincaré inequalities techniques and partitions of unity that conform with $\mathcal{A}$ in order to avoid large energies on the interior extensions* [7, 8, 18, 29, 42, 52–56, 60].

Remark 5. *The result of Lemma 1 also holds if* $\operatorname{supp} v \subset \mathcal{T}_i(K)$ *for some positive integer* $i < j$.

Theorem 2. *Let $v \in H^1(\mathcal{T}_H)$ such that* $\operatorname{supp} v \subset K$, *and $\tilde{\mu}_h^f = Pv$. Then, for any integer $j \ge 1$,*

$$|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))}^2 \le e^{-\frac{j}{1+d^2\alpha}}|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H)}^2. \tag{26}$$

Proof. If $\tilde{\mu}_h^f = Pv$ where $\operatorname{supp} v \subset K$, then using Lemma 1 we have

$$|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))}^2 \le d^2\alpha|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_j(K))}^2 - d^2\alpha|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))}^2$$

and then

$$|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_{j+1}(K))}^2 \le \frac{d^2\alpha}{1+d^2\alpha}|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_j(K))}^2 \le e^{-\frac{1}{1+d^2\alpha}}|T\tilde{\mu}_h^f|_{H_{\mathcal{A}}^1(\mathcal{T}_H\setminus\mathcal{T}_j(K))}^2,$$

and the theorem follows. □

Note that if the coefficient $\mathcal{A}$ is nearly constant and isotropic inside each element τ, the exponential decay will depend only mildly on $\beta_{H/h}$. However, the decay of Pv deteriorates as the contrast κ gets larger. In Section 3.2 we modify the method to consider high contrast and eliminate the α dependence.

We now localize Pv since it decays exponentially when v has local support. We consider two families of localizations. The first family $P^{K,j}$ is based on elements $K \in \mathcal{T}_H$ and utilized to localize $\tilde{T}g$, while the second family $P^{F,j}$ is based on faces $F \in \mathcal{F}_H$ with the purpose to localize $T\lambda_h$.

For each fixed element K and positive integer j, let $\tilde{\Lambda}_h^{f,K,j} \subset \tilde{\Lambda}_h^f$ be the set of functions of $\tilde{\Lambda}_h^f$ which vanish on faces of elements in $\mathcal{T}_H \backslash \mathcal{T}_j(K)$. We introduce the operator $P^{K,j}$: $H^1(\mathcal{T}_H) \to \tilde{\Lambda}_h^{f,K,j}$ such that, for $v \in H^1(\mathcal{T}_H)$,

$$(\tilde{\mu}_h^f, TP^{K,j}v)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, v_K)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^{f,K,j}.$$

where v_K is equal to v on K. For $v \in H^1(\mathcal{T}_H)$ we define then $\tilde{P}^jTv \in \tilde{\Lambda}_h^f$ by

$$\tilde{P}^j v = \sum_{K \in \mathcal{T}_H} P^{K,j} v_K. \tag{27}$$

We next introduce a new localization, this time based on faces. For a fixed $F \in \mathcal{F}_H$ shared by elements τ_1^F and τ_2^F or shared by only one element τ^F, define $\mathcal{T}_0(F) = \emptyset$, $\mathcal{T}_1(F) = \{\tau_1^F, \tau_2^F\}$ or $\mathcal{T}_1(F) = \{\tau^F\}$, and for $j = 1, 2, \ldots$ let

$$\mathcal{T}_{j+1}(F) = \{\tau \in \mathcal{T}_H : \bar{\tau} \cap \bar{\tau}_j \neq \emptyset \text{ for some } \tau_j \in \mathcal{T}_j(F)\}.$$

Let $\tilde{\Lambda}_h^{f,F,j} \subset \tilde{\Lambda}_h^f$ be the set of functions of $\tilde{\Lambda}_h^f$ vanishing on faces of elements in $\mathcal{T}_H \backslash \mathcal{F}_j(F)$. Let us decompose $\lambda_h \in \Lambda_h$ into $\lambda_h = \sum_{F \in \mathcal{F}_H} \lambda_h^F$ where $\lambda_h^F = \lambda_h$ on the face F and zero everywhere else on $\mathcal{F}_H$. The operator $P^{F,j} : \Lambda_h \to \tilde{\Lambda}_h^{f,F,j}$ is defined as

$$(\tilde{\mu}_h^f, TP^{F,j}T\lambda_h)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, T\lambda_h^F)_{\partial\tau_1^F} + (\tilde{\mu}_h^f, T\lambda_h^F)_{\partial\tau_2^F} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^{f,F,j}.$$

We define then $P^jT\lambda_h \in \tilde{\Lambda}_h^f$ by

$$P^jT\lambda_h = \sum_{F \in \mathcal{F}_H} P^{F,j}T\lambda_h. \tag{28}$$

Lemma 3. *Let $\tilde{\lambda}_h^f \in \tilde{\Lambda}_h^f$ and P^j be defined as above. Then $P^jT\tilde{\lambda}_h^f = \tilde{\lambda}_h^f$.*

Proof. Since

$$(\tilde{\mu}_h^f, T\tilde{\lambda}_h^{F,f})_{\partial\tau_1^F} + (\tilde{\mu}_h^f, T\tilde{\lambda}_h^{F,f})_{\partial\tau_2^F} = (\tilde{\mu}_h^f, T\tilde{\lambda}_h^{F,f})_{\partial\mathcal{T}_H}$$

and $\tilde{\lambda}_h^{F,f} \in \tilde{\Lambda}_h^{f,F,j}$, then the results follows from the existence and uniqueness of $P^{F,j}T\tilde{\lambda}_h^f$ and (28) $\square$

The reason to introduce $P^jT\lambda_h$ is because we could not prove that $\tilde{P}^jT\tilde{\lambda}_h^f = \tilde{\lambda}_h^f$ and this property is fundamental on Theorems 5 and 5'.

Lemma 4. *Consider $v \in H^1(\mathcal{T}_H)$, and the operators P defined by (21) and $\tilde{P}^j$ by (28). Then*

$$|T(P-\tilde{P}^j)v|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le cj^{2d}\alpha^2 e^{-\frac{j-2}{1+\alpha}}|v|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}. \tag{29}$$

Proof. For $K \in \mathcal{T}_H$, let $\tilde{\mu}_h^{f,K} = Pv_K$ and $\tilde{\mu}_h^{f,K,j} = P^{j,K}v_K$, and $\tilde{\psi}_h^f = \sum_{K\in\mathcal{T}_H}(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})$. For each $K \in \mathcal{T}_H$, let $\tilde{\psi}_h^{f,K} \in \tilde{\Lambda}_h^f$ be defined by $\tilde{\psi}_h^{f,K}|_F = 0$ if F is a face of an element of $\mathcal{T}_j(K)$ and $\tilde{\psi}_h^{f,K}|_F = \tilde{\psi}_h^f|_F$, otherwise. We obtain

$$|T\tilde{\psi}_h^f|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \sum_{K\in\mathcal{T}_H}\sum_{\tau\in\mathcal{T}_H}(\tilde{\psi}_h^f - \tilde{\psi}_h^{f,K}, T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j}))_{\partial\tau} + (\tilde{\psi}_h^{f,K}, T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j}))_{\partial\tau}. \tag{30}$$

See that the second term of (30) vanishes since

$$\sum_{\tau\in\mathcal{T}_H}(\tilde{\psi}^{f,K}, T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j}))_{\partial\tau} = \sum_{\tau\in\mathcal{T}_H}(\tilde{\psi}^{f,K}, T\tilde{\mu}_h^{f,K})_{\partial\tau} = 0.$$

For the first term of (30), as in Lemma 1, we use a direct application of [68, Lemma 4.4] yielding

$$\begin{aligned}
\sum_{\tau\in\mathcal{T}_H}\big(\tilde{\psi}_h^f - \tilde{\psi}_h^{f,K}, T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})\big)_{\partial\tau} &\le \sum_{\tau\in\mathcal{T}_{j+1}(K)} |T(\tilde{\psi}_h^f - \tilde{\psi}_h^{f,K})|_{H^1_{\mathcal{A}}(\tau)}|T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})|_{H^1_{\mathcal{A}}(\tau)} \\
&\le d\alpha^{1/2}|T\tilde{\psi}_h^f|_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K))}|T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})|_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K))}.
\end{aligned}$$

Let $\nu_h^{f,K,j} \in \tilde{\Lambda}_h^{f,K,j}$ be equal to zero on all faces of elements of $\mathcal{T}_H\backslash\mathcal{T}_j(K)$ and equal to $\tilde{\mu}_h^{f,K}$ otherwise. Using Galerkin best approximation property, [68, Lemma 4.4] and Theorem 2 we obtain

$$\begin{aligned}
|T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})|^2_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K))} \le |T(\tilde{\mu}_h^{f,K} - \tilde{\mu}_h^{f,K,j})|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &\le |T(\tilde{\mu}_h^{f,K} - \nu_h^{f,K,j})|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \\
\le d^2\alpha|T\tilde{\mu}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H\backslash\mathcal{T}_{j-1}(K))} &\le d^2\alpha e^{-\frac{j-2}{1+d^2\alpha}}|T\tilde{\mu}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}.
\end{aligned}$$

We gather the above results to obtain

$$\begin{aligned}
|T\tilde{\psi}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &\le d^2\alpha e^{-\frac{j-2}{2(1+d^2\alpha)}}\sum_{K\in\mathcal{T}_H}|T\tilde{\psi}_h^{f,K}|_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K))}|T\tilde{\mu}_h^{f,K}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \\
&\le d^2\alpha e^{-\frac{j-2}{2(1+d^2\alpha)}}cj^d|T\tilde{\psi}_h^{f,K}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\Big(\sum_{K\in\mathcal{T}_H}|T\tilde{\mu}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\Big)^{1/2}.
\end{aligned}$$

We finally gather that

$$|T\tilde{\mu}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = (\tilde{\mu}_h^{f,K}, TPv_K)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^{f,K}, v_K)_{\partial\mathcal{T}_H} = \int_K \mathcal{A}\,\boldsymbol{\nabla}(T\tilde{\mu}_h^{f,K})\cdot\boldsymbol{\nabla}(v_K)\,d\boldsymbol{x}$$

and from Cauchy–Schwarz, $|T\tilde{\mu}_h^{f,K}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le |v_K|_{H^1_{\mathcal{A}}(K)}$, we have

$$\sum_{K \in \mathcal{T}_H} |T\tilde{\mu}_h^{f,K}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le |v|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}.$$

□

Remark 6. *For $\lambda_h \in \Lambda_h$, we have*

$$|T(P - P^j)T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le cj^{2d}\alpha^2 e^{-\frac{j-2}{1+\alpha}} |T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}. \tag{31}$$

By replacing $\mathcal{T}_j(K)$ by $\mathcal{T}_j(F)$ and v_K by λ_h^F, the same proofs for Lemma 1, Theorem 2 and Lemma 4 hold true.

Recall from (22) and (25) that $u_h = u_h^0 + T\lambda_h + \tilde{T}g$, where

$$\lambda_h = (I - PT)\lambda^0 + (I - PT)\tilde{\lambda}_h^0 - P\tilde{T}g.$$

Motivated by the above decaying results and (20)–(24), we define the solution of the localized algorithm by

$$u_h^j = u_h^{0,j} + T\lambda_h^j + \tilde{T}g, \quad \text{where } \lambda_h^j = (I - P^jT)\lambda^0 + (I - P^jT)\tilde{\lambda}_h^{0,j} - \tilde{P}^j\tilde{T}g, \tag{32}$$

and λ^0 solves the first equation of (20). Also, similarly to (24), $\tilde{\lambda}_h^{0,j} \in \widetilde{\Lambda}_h^0$ solves

$$\begin{aligned} \big((I - P^jT)\tilde{\mu}_h^0, T(I - P^jT)\tilde{\lambda}_h^{0,j}\big)_{\partial\mathcal{T}_H} &= -\big((I - P^jT)\tilde{\mu}_h^0, (I - T\tilde{P}^j)\tilde{T}g\big)_{\partial\mathcal{T}_H} \\ &\quad - \big((I - P^jT)\tilde{\mu}_h^0, T(I - P^jT)\lambda^0\big)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^0 \in \widetilde{\Lambda}_h^0, \end{aligned} \tag{33}$$

and similarly to the fourth equation of (20), we obtain $u_h^{0,j}$ by

$$(\mu^0, u_h^{0,j})_{\partial\mathcal{T}_H} = -(\mu^0, T\lambda^0 + T\tilde{\lambda}_h^{0,j} + T\tilde{\lambda}_h^{f,j})_{\partial\mathcal{T}_H} - (\mu^0, \tilde{T}g)_{\partial\mathcal{T}_H} \quad \text{for all } \mu^0 \in \Lambda^0.$$

and as in (22) we have defined

$$\tilde{\lambda}_h^{f,j} = -P^j(T\lambda^0 + T\tilde{\lambda}_h^{0,j}) - \tilde{P}^j\tilde{T}g. \tag{34}$$

A fundamental difference between our discretization and some multiscale mixed finite elements methods such as the ones in [2, 44] is that here we avoid solving a saddle point problem by computing $\lambda_h^{0,j} = \lambda^0$ using the first equation of (20), while there the equation (33) is extended to the whole space $\Lambda^0 \oplus \widetilde{\Lambda}_h^0$ and solved together with an equation for $u_h^{0,j}$.

Theorem 5. *Let u_h and u_h^j be defined by (25) and (32). Then there exists a constant c such that*

$$|u_h - u_h^j|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le cj^{2d}d^4\alpha^2 e^{-\frac{j-2}{1+d^2\alpha}} \big(|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + |\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\big).$$

Proof. It follows immediately from the definitions of u_h and u_h^j that

$$|u_h - u_h^j|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = |T(\lambda_h - \lambda_h^j)|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \big((\lambda_h - \lambda_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H}.$$

Defining $\nu_h^j = (I - P^jT)\lambda^0 + (I - P^jT)\tilde{\lambda}_h^0 - \tilde{P}^j\tilde{T}g$, it follows that

$$\big((\lambda_h - \lambda_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H} = \big((\lambda_h - \nu_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H} + \big((\nu_h^j - \lambda_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H}. \tag{35}$$

Since $\tilde{\lambda}_h^0 - \tilde{\lambda}_h^{0,j} \in \tilde{\Lambda}_h^0$ and $\nu_h^j - \lambda_h^j = (I - P^jT)(\tilde{\lambda}_h^0 - \tilde{\lambda}_h^{0,j}) \in \tilde{\Lambda}_h$ then by using (33) and (20), respectively, the second term of the right-hand side of (35) vanishes. Indeed, from (32), (33), (34),

$$\big((I - P^jT)\tilde{\mu}_h^0, T\lambda_h^j\big)_{\partial\mathcal{T}_H} = -\big((I - P^jT)\tilde{\mu}_h^0, \tilde{T}g\big)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^0 \in \widetilde{\Lambda}_h^0,$$

and from (20) we have

$$\big((I - P^jT)\tilde{\mu}_h^0, T\lambda_h\big)_{\partial\mathcal{T}_H} = -\big((I - P^jT)\tilde{\mu}_h^0, \tilde{T}g\big)_{\partial\mathcal{T}_H}$$

since $\tilde{\mu}_h^0 \in \tilde{\Lambda}_h^0$ and $P^jT\tilde{\mu}_h^0 \in \tilde{\Lambda}_h^f$. Thus, choosing $\tilde{\mu}_h^0 = \tilde{\lambda}_h^0 - \tilde{\lambda}_h^{0,j}$,

$$\big((\nu_h^j - \lambda_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H} = \big((I - P^jT)(\tilde{\lambda}_h^0 - \tilde{\lambda}_h^{0,j}), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H} = 0.$$

By Cauchy–Schwarz inequality, we have from (35) that

$$\begin{aligned}|u_h - u_h^j|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \big((\lambda_h - \nu_h^j), T(\lambda_h - \lambda_h^j)\big)_{\partial\mathcal{T}_H} &= (\lambda_h - \nu_h^j, u_h - u_h^j)_{\partial\mathcal{T}_H} \\ &\le |T(\lambda_h - \nu_h^j)|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}|u_h - u_h^j|_{H^1_{\mathcal{A}}(\mathcal{T}_H)},\end{aligned}$$

since $\lambda_h - \nu_h^j \in \widetilde{\Lambda}_h$. We now use Lemma 3 where $(P - P^j)T\tilde{\lambda}_h^f = 0$ and then Lemma 4 and Remark 6 to obtain

$$\begin{aligned}|u_h - u_h^j|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le |T(\lambda_h - \nu_h^j)|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &= |T(P - P^j)T\lambda_h + T(P - \tilde{P}^j)\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \\ &\le cj^{2d}d^4\alpha^2 e^{-\frac{j-2}{1+d^2\alpha}}\big(|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + |\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\big).\end{aligned}$$

□

Remark 7. *We note that* $|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le 2|u_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + 2|\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}$*, therefore*

$$|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + |\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le 4|u - u_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + 4|u|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + 3|\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}.$$

Defining global and local Poincaré constants to obtain

$$\|u\|_{L^2_\rho(\Omega)} \le C_{G,P}|u|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}, \qquad |\tilde{T}g|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le c_P H|u|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\|g\|_{L^2_\rho(\Omega)},$$

it follows from Theorem 5 that if j is taken such that

$$cj^{2d}d^4\alpha^2 e^{-\frac{j-2}{1+d^2\alpha}}(4\mathscr{H}^2 + 4C^2_{P,G} + 3c_p^2H^2) \le \mathscr{H}^2,$$

then, from (16),

$$|u-u_h^j|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le 2\mathscr{H}\|g\|_{L^2_\rho(\Omega)}. \tag{36}$$

Since in general c_pH is dominated by $C_{P,G}$, then $j=\mathcal{O}(d^2\kappa\log(C_{P,G}/\mathscr{H})\log^2(H/h))$, and we have that j is large not only in the high contrast case (α large), but also if $h\ll H$. We also have a lightly log *dependence of j with respect to the global Poincaré constant $C_{P,G}$. However, if we choose $\rho(\boldsymbol{x})=a_{min}$ then $C_{P,G}=\mathcal{O}(1)$.*

3.2. **Decaying High-Contrast.** The main bottle-neck in dealing with high-contrast coefficients is that α also becomes high, therefore j has to be large as well, as seen in Theorems 2 and 5. To deal with this situation, we use a subspace of $\tilde{\Lambda}_h^f$ to augment $\widetilde{\Lambda}_h^0$ by selecting eigenfunctions associated to a proper generalized eigenvalue problem associated to each face of the mesh $\mathcal{T}_H$. In order to define these generalized eigenvalue problems, we first introduce some theoretical tools for high-contrast coefficients.

Let $\tau\in\mathcal{T}_H$, F a face of $\partial\tau$, and let $F_\tau^c=\partial\tau\backslash F$. Define

$$\tilde{\Lambda}_h^\tau=\{\tilde{\mu}_h|_{\partial\tau}:\,\tilde{\mu}_h\in\widetilde{\Lambda}_h^f\},\qquad \tilde{\Lambda}_h^F=\{\tilde{\mu}_h|_F:\,\tilde{\mu}_h\in\tilde{\Lambda}_h^f\},\qquad \tilde{\Lambda}_h^{F_\tau^c}=\{\tilde{\mu}_h|_{F_\tau^c}:\,\tilde{\mu}_h\in\tilde{\Lambda}_h^f\}.$$

For any given $\tilde{\mu}_h^\tau\in\tilde{\Lambda}_h^\tau$, denote $\tilde{\mu}_h^\tau=\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\}\in\tilde{\Lambda}_h^F\times\tilde{\Lambda}_h^{F_\tau^c}$, and define

$$\begin{aligned} T_{FF}^\tau:\tilde{\Lambda}_h^F\to(\tilde{\Lambda}_h^F)',\qquad & T_{F^cF}^\tau:\tilde{\Lambda}_h^F\to(\Lambda_h^{F_\tau^c})',\\ T_{FF^c}^\tau:\tilde{\Lambda}_h^{F_\tau^c}\to(\tilde{\Lambda}_h^F)',\qquad & T_{F^cF^c}^\tau:\tilde{\Lambda}_h^{F_\tau^c}\to(\tilde{\Lambda}_h^{F_\tau^c})', \end{aligned}$$

by

$$\begin{aligned} (\tilde{\mu}_h,T\tilde{\mu}_h)_{\partial\tau}&=:\big(\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\},T\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\}\big)_{\partial\tau}\\ &=(\tilde{\mu}_h^F,T_{FF}^\tau\tilde{\mu}_h^F)_F+(\tilde{\mu}_h^F,T_{FF^c}^\tau\tilde{\mu}_h^{F_\tau^c})_F+(\tilde{\mu}_h^{F_\tau^c},T_{F^cF}^\tau\tilde{\mu}_h^F)_{F_\tau^c}+(\tilde{\mu}_h^{F_\tau^c},T_{F^cF^c}^\tau\tilde{\mu}_h^{F_\tau^c})_{F_\tau^c}. \end{aligned}$$

We remind that $T:\widetilde{\Lambda}_h^\tau\to\widetilde{H}^1(\tau)$, satisfies

$$(\mathcal{A}\,\boldsymbol{\nabla}(T\tilde{\mu}_h^\tau),\boldsymbol{\nabla}\,v)_\tau=(\tilde{\mu}_h^\tau,v)_{\partial\tau}\quad\text{for all } v\in\widetilde{H}^1(\tau). \tag{37}$$

Note that $\mathcal{A}\,\boldsymbol{\nabla}(T\tilde{\mu}_h^\tau)\cdot\boldsymbol{n}^\tau=\tilde{\mu}_h$ on $\partial\tau$.

It follows from the properties of T that both T_{FF}^τ and $T_{F^cF^c}^\tau$ are symmetric and positive definite matrices, and follow from Schur complement arguments that for any $\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\}\in\tilde{\Lambda}_h$

$$\begin{aligned} (\tilde{\mu}_h^F,\hat{T}_{FF}^\tau\tilde{\mu}_h^F)_F&=\min_{\tilde{\nu}_h^{F_\tau^c}\in\tilde{\Lambda}_h^{F_\tau^c}}(\{\tilde{\mu}_h^F,\tilde{\nu}_h^{F_\tau^c}\},T\{\tilde{\mu}_h^F,\tilde{\nu}_h^{F_\tau^c}\})_{\partial\tau}\\ &\le\big(\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\},T\{\tilde{\mu}_h^F,\tilde{\mu}_h^{F_\tau^c}\}\big)_{\partial\tau}=(\tilde{\mu}_h^\tau,T\tilde{\mu}_h^\tau)_{\partial\tau}, \end{aligned} \tag{38}$$

where

$$\hat{T}^\tau_{FF} = T^\tau_{FF} - T^\tau_{FF^c}(T^\tau_{F^cF^c})^{-1}T^\tau_{F^cF},$$

and the minimum is attained at $\tilde{\nu}_h^{F^c_\tau} = -(T^\tau_{F^cF^c})^{-1}T^\tau_{F^cF}\tilde{\mu}_h^F$. In what follows, to take into account high contrast coefficients, we consider the following generalized eigenvalue problem: Find eigenpairs $(\alpha_i^F, \tilde{\mu}_{h,i}^F) \in (\mathbb{R}, \tilde{\Lambda}_h^F)$, where $\alpha_1^F \le \alpha_2^F \le \alpha_3^F, \dots$, such that

(1) If the face F is shared by elements τ and τ' we solve

$$(\tilde{\nu}_h^F, (T^\tau_{FF} + T^{\tau'}_{FF})\tilde{\mu}_{h,i}^F)_F = \alpha_i^F(\tilde{\nu}_h^F, (\hat{T}^\tau_{FF} + \hat{T}^{\tau'}_{FF})\tilde{\mu}_{h,i}^F)_F \quad \text{for all } \tilde{\nu}_h^F \in \tilde{\Lambda}_h^F.$$

(2) If the face F is on the boundary $\partial\Omega$ we solve

$$(\tilde{\nu}_h^F, T^\tau_{FF}\tilde{\mu}_{h,i}^F)_F = \alpha_i^F(\tilde{\nu}_h^F, \hat{T}^\tau_{FF}\tilde{\mu}_{h,i}^F)_F, \quad \text{for all } \tilde{\nu}_h^F \in \tilde{\Lambda}_h^F.$$

Now we decompose $\tilde{\Lambda}_h^F := \tilde{\Lambda}_h^{F,\triangle} \oplus \tilde{\Lambda}_h^{F,\Pi}$ where, for a given $\alpha_{\text{stab}} \ge 1$,

$$\tilde{\Lambda}_h^{F,\triangle} := \operatorname{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F < \alpha_{\text{stab}}\}, \qquad \tilde{\Lambda}_h^{F,\Pi} := \operatorname{span}\{\tilde{\mu}_{h,i}^F : \alpha_i^F \ge \alpha_{\text{stab}}\}. \tag{39}$$

The eigenfunctions $\tilde{\mu}_{h,i}^F$ are chosen to be orthonormal with respect to $(\cdot, (\hat{T}^\tau_{FF} + \hat{T}^{\tau'}_{FF})\cdot)_F$ if F is an interior face, and with $(\cdot, \hat{T}^\tau_{FF}\cdot)_F$ if $F \subset \partial\Omega$. Note that α_{stab} is defined by the user to replace α in the proof of Lemma 1. From (38) with $\tilde{\mu}_h^{F^c_\tau} = 0$, we have that $\alpha_i^F \ge 1$, and if we take α_{stab} larger than α, introduced in Lemma 1, then $\tilde{\Lambda}_h^{F,\Pi}$ is empty.

Generalized eigenvalue problems of this type have appeared in the literature to make preconditioners robust with respect to coefficients [6,11,13,40,41,47,48,62]. In particular [49] shows, for a related problem, that the $\alpha_i^F - 1$ decays exponentially to zero since, when h goes to zero, the operators $\hat{T}_{FF}$ (related to $(H^{1/2}(F))'$) and T_{FF} (related to $(H^{1/2}_{00}(F))'$) differ only by a compact operator. In [28] is shown that the number eigenvalues that are very large is related to the number of connected sub-regions on $\bar{\tau} \cup \bar{\tau}'$ with large coefficients surrounded by regions with small coefficients. Generalized eigenvalue problems also have been used on overlapping domain decomposition solvers [14,23,28,61]. The design of robust discretizations with respect to coefficients using domain decomposition ideas have been studied in [33,34,36] assuming some regularity on the solution, and in [28] for a class of problems when the weighted Poincaré constant (see [18,52,53] is not large, otherwise the exponential decay of the multiscale functions deteriorates. See also [21,22] where a priori error estimates are obtained in terms of spectral norms.

In order to define our LSD–Localized Spectral Decomposition Method for high-contrast coefficients, let us introduce the non-localized version. Let us first define

$$\begin{aligned} \tilde{\Lambda}_h^{\Pi} &= \{\tilde{\mu}_h \in \widetilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\Pi} \text{ for all } F \in \partial\mathcal{T}_H\}, \\ \tilde{\Lambda}_h^{\triangle} &= \{\tilde{\mu}_h \in \widetilde{\Lambda}_h^f : \tilde{\mu}_h|_F \in \tilde{\Lambda}_h^{F,\triangle} \text{ for all } F \in \partial\mathcal{T}_H\}. \end{aligned} \tag{40}$$

We now follow the same procedure as in (20) except that now we replace $\widetilde{\Lambda}_h^0$ by

$$\widetilde{\Lambda}_h^{0,\Pi} := \widetilde{\Lambda}_h^0 \oplus \tilde{\Lambda}_h^{\Pi},$$

replace $\tilde{\Lambda}_h^f$ by $\tilde{\Lambda}_h^{\triangle}$ and replace P by $P^{\triangle} : H^1(\mathcal{T}_H) \to \tilde{\Lambda}_h^{\triangle}$ such that, for $w \in H^1(\mathcal{T}_H)$,

$$(\tilde{\mu}_h^{\triangle}, TP^{\triangle}w)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^{\triangle}, w)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^{\triangle} \in \tilde{\Lambda}_h^{\triangle}. \tag{41}$$

We obtain

$$\begin{aligned} u_h = u_h^0 + T\lambda_h + \tilde{T}g, \qquad \tilde{\lambda}_h^{\triangle} = -P^{\triangle}(T\lambda^0 + T\tilde{\lambda}_h^{0,\Pi} + \tilde{T}g),\\ \lambda_h = \lambda^0 + \tilde{\lambda}_h^{0,\Pi} + \tilde{\lambda}_h^{\triangle} = (I - P^{\triangle}T)(\lambda^0 + \tilde{\lambda}_h^{0,\Pi}) - P^{\triangle}\tilde{T}g. \end{aligned} \tag{42}$$

Note that u_h, u_h^0 and λ^0 in (42) are the same as in (25). Also, since the space $\widetilde{\Lambda}_h = \widetilde{\Lambda}_h^0 \oplus \tilde{\Lambda}_h^{\Pi} \oplus \tilde{\Lambda}_h^{\triangle}$ is a direct sum,

$$\tilde{\lambda}_h^{0,\Pi} = \tilde{\lambda}_h^0 + \tilde{\lambda}_h^{\Pi}, \qquad \tilde{\lambda}_h^{\Pi} + \tilde{\lambda}_h^{\triangle} = \tilde{\lambda}_h^f, \tag{43}$$

where $\tilde{\lambda}_h^0$ and $\tilde{\lambda}_h^f$ are the same as in (25) and $\tilde{\lambda}_h^{\Pi} \in \tilde{\Lambda}_h^{\Pi}$.

The Lemma 1 is now replaced by the following.

Lemma 1'. *Let $v \in H^1(\mathcal{T}_H)$ such that* $\operatorname{supp} v \subset K$, *and $\tilde{\mu}_h^{\triangle} = P^{\triangle}v$. Then, for any integer $j \geq 1$,*

$$|T\tilde{\mu}_h^{\triangle}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H\backslash\mathcal{T}_{j+1}(K))} \leq d^2\alpha_{\text{stab}}|T\tilde{\mu}_h^{\triangle}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K)\backslash\mathcal{T}_{j-1}(K))}.$$

Proof. Following the steps of the proof of Lemma 1, it remains to estimate $|T\chi_F\tilde{\mu}_h^{\triangle}|_{H^1_{\mathcal{A}}(\tau)}$ for faces F of elements $\tau \in \mathcal{T}_{j+1}(K)\backslash\mathcal{T}_j(K)$ which are also shared by an element in $\mathcal{T}_j(K)$. Let $\tilde{\mu}_h^{\triangle} = \{\tilde{\mu}_h^{F,\triangle}, \tilde{\mu}_h^{F_\tau^c,\triangle}\} \in \tilde{\Lambda}_\tau^{F,\triangle} \times \tilde{\Lambda}_\tau^{F_\tau^c,\triangle}$ and consider first the case (1) that the face F is shared by two elements $\tau \in \mathcal{T}_{j+1}(K)\backslash\mathcal{T}_j(K)$ and $\tau \in \mathcal{T}_j(K)\backslash\mathcal{T}_{j-1}(K)$. Since $\tilde{\mu}_h^{F,\triangle} \in \tilde{\Lambda}_h^{F,\triangle}$ we obtain using (38) that

$$\begin{aligned} |T(\chi_F\tilde{\mu}_h^{\triangle})|^2_{H^1_{\mathcal{A}}(\tau)} = (\tilde{\mu}_h^{F,\triangle}, T_{FF}^{\tau}\tilde{\mu}_h^{F,\triangle})_F \leq d^2\alpha_{\text{stab}}(\tilde{\mu}_h^{F,\triangle}, (\hat{T}_{FF}^{\tau} + \hat{T}_{FF}^{\tau'})\tilde{\mu}_h^{F,\triangle})_F\\ \leq d^2\alpha_{\text{stab}}\big((\tilde{\mu}_h^{\triangle}, T\tilde{\mu}_h^{\triangle})_{\partial\tau} + (\tilde{\mu}_h^{\triangle}, T\tilde{\mu}_h^{\triangle})_{\partial\tau'}\big). \end{aligned}$$

Analogous arguments hold also for the case (2) where $F \subset \partial\Omega$. The result follows using the same arguments used in the proof of Lemma 1.

□

Note that because $\tau' \in \mathcal{T}_j(K)\backslash\mathcal{T}_{j-1}(K)$, the bound in Lemma 1' is with respect to $H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K)\backslash\mathcal{T}_{j-1}(K))$ rather than $H^1_{\mathcal{A}}(\mathcal{T}_{j+1}(K)\backslash\mathcal{T}_j(K))$, hence as we see below, the term $(j+1)$ in Theorem 2 is replaced by the integer part of $(j+1)/2$ in Theorem 2'. The reason for that is because on the proof of Theorem 2' we now need to apply recursively Lemma 1' every two layers of elements instead of every one layer as done on the proof of Theorem 2.

Following similar analysis as before, Theorem 2 holds with α replaced by α_{stab}.

Theorem 2'. *Let $v \in H^1(\mathcal{T}_H)$ such that* $\operatorname{supp} v \subset K$*, and $\tilde{\mu}_h^\triangle = P^\triangle v$. Then, for any integer $j \geq 1$,*

$$|T\tilde{\mu}_h^\triangle|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H\backslash\mathcal{T}_{j+1}(K))} \leq e^{-\frac{[(j+1)/2]}{1+d^2\alpha_{\mathrm{stab}}}}|T\tilde{\mu}_h^\triangle|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)},$$

where $[s]$ is the integer part of s.

Taking advantage of such exponential decay, we define $\tilde{P}^{\triangle,j}$ similarly to $\tilde{P}^j$. Fixing K, j, let $\tilde{\Lambda}_h^{\triangle,K,j} \subset \tilde{\Lambda}_h^\triangle$ be the set of functions of $\tilde{\Lambda}_h^\triangle$ vanishing on faces of elements in $\mathcal{T}_H\backslash\mathcal{T}_j(K)$, and $P^{\triangle,K,j}: H^1(\mathcal{T}_H) \to \tilde{\Lambda}_h^{\triangle,K,j}$ such that, for $v \in H^1(\mathcal{T}_H)$,

$$(\tilde{\mu}_h^{\triangle,K,j}, TP^{\triangle,K,j}v)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^{\triangle,K,j}, v_K)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^{\triangle,K,j} \in \tilde{\Lambda}_h^{\triangle,K,j},$$

where we again let v_K to be equal to v on K and zero otherwise. We define then $\tilde{P}^{\triangle,j}$ by

$$\tilde{P}^{\triangle,j}v = \sum_{K\in\mathcal{T}_H} P^{\triangle,K,j}v_K. \tag{44}$$

In a similar way to P^j we define

$$P^{\triangle,j}T\lambda_h = \sum_{F\in\mathcal{F}_H} P^{\triangle,F,j}T\lambda_h^F. \tag{45}$$

where $P^{\triangle,F,j}T\lambda_h \in \tilde{\Lambda}_h^{\triangle,K,j}$ is such that

$$(\tilde{\mu}_h^f, TP^{\triangle,F,j}T\lambda_h)_{\partial\mathcal{T}_H} = (\tilde{\mu}_h^f, T\lambda_h^F)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^f \in \tilde{\Lambda}_h^{\triangle,K,j}.$$

Then, Lemma 4 and Remark 6 hold now with

$$|T(P^\triangle - \tilde{P}^{\triangle,j})v|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq cj^{2d}d^4\alpha^2_{\mathrm{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\mathrm{stab}}}}|v|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \tag{46}$$

and

$$|T(P^\triangle - P^{\triangle,j})T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq cj^{2d}d^4\alpha^2_{\mathrm{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\mathrm{stab}}}}|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}. \tag{47}$$

The LSD method is defined by computing

$$u_h^{\mathrm{LSD},j} = u_h^{\mathrm{LSD},0,j} + T\lambda_h^{\mathrm{LSD},j} + \tilde{T}g, \qquad \lambda_h^{\mathrm{LSD},j} = \lambda^0 + \tilde{\lambda}_h^{0,\Pi,j} + \tilde{\lambda}_h^{\triangle,j}, \tag{48}$$

based on modifications of (33), (22), and the fourth equation of (20). Indeed, define $\tilde{\lambda}_h^{0,\Pi,j} \in \tilde{\Lambda}_h^{0,\Pi}$ from

$$\begin{aligned}\big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, T(I - P^{\triangle,j}T)\tilde{\lambda}_h^{0,\Pi,j}\big)_{\partial\mathcal{T}_H} &= -\big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, (I - T\tilde{P}^{\triangle,j})\tilde{T}g\big)_{\partial\mathcal{T}_H} \\ &\quad - \big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, T(I - P^{\triangle,j}T)\lambda^0\big)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^{0,\Pi} \in \tilde{\Lambda}_h^{0,\Pi},\end{aligned} \tag{49}$$

and compute $\tilde{\lambda}_h^{\triangle,j} \in \tilde{\Lambda}_h^{\triangle}$ and $u_h^{0,\mathrm{LSD},j} \in \mathbb{P}^0(\mathcal{T}_H)$ from

$$\tilde{\lambda}_h^{\triangle,j} = -P^{\triangle,j}(T\lambda^0 + T\tilde{\lambda}^{0,\Pi,j}) - \tilde{P}^{\triangle,j}\tilde{T}g, \tag{50}$$

$$(\mu^0, u_h^{0,\mathrm{LSD},j})_{\partial\mathcal{T}_H} = -(\mu^0, T\lambda^0 + T\tilde{\lambda}_h^{0,\Pi,j} + T\tilde{\lambda}_h^{\triangle,j})_{\partial\mathcal{T}_H} - (\mu^0, \tilde{T}g)_{\partial\mathcal{T}_H} \quad \text{for all } \mu^0 \in \Lambda^0. \tag{51}$$

A new version of Theorem 5 follows.

Theorem 5'. *Let u_h be defined by (42) or (25) and $u_h^{LSD,j}$ by (48). Then there exists a constant c such that*

$$|u_h - u_h^{LSD,j}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq cj^{2d}d^4\alpha^2_{\mathrm{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\mathrm{stab}}}}\big(|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + |\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\big).$$

Proof. Proceeding as in the proof of Theorem 5, however now replacing $\tilde{\Lambda}_h^f$ by $\tilde{\Lambda}_h^{\triangle}$ and $\tilde{\Lambda}_h^0$ by $\tilde{\Lambda}_h^{0,\Pi}$, it follows that

$$|u_h - u_h^{\mathrm{LSD},j}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = |T(\lambda_h - \lambda_h^{\mathrm{LSD},j})|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = \big((\lambda_h - \lambda_h^{\mathrm{LSD},j}), T(\lambda_h - \lambda_h^{\mathrm{LSD},j})\big)_{\partial\mathcal{T}_H}.$$

Defining $\nu_h^{\mathrm{LSD},j} = (I - P^{\triangle,j}T)\lambda^0 + (I - P^{\triangle,j}T)\tilde{\lambda}_h^{0,\Pi} - \tilde{P}^{\triangle,j}\tilde{T}g$, then

$$\begin{aligned}
&\big((\lambda_h - \lambda_h^{\mathrm{LSD},j}), T(\lambda_h - \lambda_h^{\mathrm{LSD},j})\big)_{\partial\mathcal{T}_H} \\
&\quad = \big((\lambda_h - \nu_h^{\mathrm{LSD},j}), T(\lambda_h - \lambda_h^{\mathrm{LSD},j})\big)_{\partial\mathcal{T}_H} + \big((\nu_h^{\mathrm{LSD},j} - \lambda_h^{\mathrm{LSD},j}), T(\lambda_h - \lambda_h^{\mathrm{LSD},j})\big)_{\partial\mathcal{T}_H}.
\end{aligned} \tag{52}$$

Since $\nu_h^{\mathrm{LSD},j} - \lambda_h^{\mathrm{LSD},j} = (I - P^{\triangle,j}T)(\tilde{\lambda}_h^{0,\Pi} - \tilde{\lambda}^{0,\Pi,j}) \in \tilde{\Lambda}_h$, then, as in the proof of Theorem 5, the second term of the right-hand side of (52) vanishes and

$$|u_h - u_h^{\mathrm{LSD},j}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq |T(\lambda_h - \nu_h^{\mathrm{LSD},j})|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}|u_h - u_h^{\mathrm{LSD},j}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}.$$

Using similar arguments as in Lemma 3 to have $(P^{\triangle} - P^{\triangle,j})T\tilde{\lambda}_h^{\triangle} = 0$, (46) and (47), we obtain

$$\begin{aligned}
|u_h - u_h^{\mathrm{LSD},j}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &\leq \big(|T(P^{\triangle} - P^{\triangle,j})T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + T(P^{\triangle} - \tilde{P}^{\triangle,j})\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\big) \\
&\leq cj^{2d}d^4\alpha^2_{\mathrm{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\mathrm{stab}}}}\big(|T\lambda_h|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + |\tilde{T}g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}\big).
\end{aligned} \tag{53}$$

$\square$

As in Remark 7, by choosing j such that

$$cj^{2d}d^4\alpha^2_{\mathrm{stab}}e^{-\frac{[(j-3)/2]}{1+d^2\alpha_{\mathrm{stab}}}}(4\mathscr{H}^2 + 4C^2_{P,G} + 3c_p^2H^2) \leq \mathscr{H}^2$$

then Theorem 5' and (16) implies

$$|u - u_h^{\mathrm{LSD},j}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \leq 2\mathscr{H}|g|_{L^2_\rho(\Omega)}. \tag{54}$$

Also note that $j = \mathcal{O}(4d^2\alpha_{\mathrm{stab}}\log(C_{P,G}/\mathscr{H}))$.

Remark 8. *We remark that the post-processed stress* $\boldsymbol{\sigma}_h^{LSD,j} = \mathcal{A}\,\boldsymbol{\nabla}[T(\lambda^0+\tilde\lambda_h^{0,\Pi,j}+\tilde\lambda_h^{\triangle,j})+\tilde T g]$ *is in equilibrium, in the sense that*

$$\int_\tau \boldsymbol{\sigma}_h^{LSD,j}\cdot\boldsymbol{\nabla}\,v\,d\boldsymbol{x} = \int_\tau \rho g v\,d\boldsymbol{x} \quad \textit{for all } v\in H_0^1(\tau). \tag{55}$$

Indeed, given $v\in H_0^1(\tau)$, *let* v^0 *be constant and* $\tilde v\in H^1(\tau)$ *with zero* ρ*-average such that* $v = v^0+\tilde v$. *Hence*

$$\begin{aligned}\int_\tau gv\,d\boldsymbol{x} = \int_\tau \mathcal{A}\,\boldsymbol{\nabla}\,\tilde T g\cdot\boldsymbol{\nabla}\,\tilde v\,d\boldsymbol{x} + \int_\tau g v^0\,d\boldsymbol{x}& \\ = \int_\tau [\boldsymbol{\sigma}_h^{LSD,j} - \mathcal{A}\,\boldsymbol{\nabla}\,T(\lambda^0+\tilde\lambda_h^{0,\Pi,j}+\tilde\lambda_h^{\triangle,j})]\cdot\boldsymbol{\nabla}\,\tilde v\,d\boldsymbol{x} + \int_\tau g v^0\,d\boldsymbol{x}.&\end{aligned}$$

However, $\tilde v = -v^0$ *on* $\partial\tau$ *since* $v\in H_0^1(\tau)$, *and then*

$$\begin{aligned}-\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\,T(\lambda^0+\tilde\lambda_h^{0,\Pi,j}+\tilde\lambda_h^{\triangle,j})\cdot\boldsymbol{\nabla}\,\tilde v\,d\boldsymbol{x} + \int_\tau g v^0\,d\boldsymbol{x} = -(\lambda^0+\tilde\lambda_h^{0,\Pi,j}+\tilde\lambda_h^{\triangle,j},\tilde v)_{\partial\tau} + \int_\tau g v^0\,d\boldsymbol{x}& \\ = (\lambda^0+\tilde\lambda_h^{0,\Pi,j}+\tilde\lambda_h^{\triangle,j},v^0)_{\partial\tau} + \int_\tau g v^0\,d\boldsymbol{x} = 0,&\end{aligned}$$

where the final estimate follows from (7). *Thus,* (55) *holds.*

4. Finite Dimensional Right-Hand Side

We first note that the solution given by the four-steps of (20) is exact, in the sense that it solves (13) as well, and hence the solution error vanishes. To compute $\tilde\lambda_h^0$ and $\tilde\lambda_h^f$ is necessary to compute $P\tilde T g$, and that hampers efficiency, and it is the only part of algorithm that does not permit exact pre-processing. In order to allow pre-processing, we replace the space $L^2_\rho(\Omega)$, which contains g, by a finite dimensional one in such a way that the solution error is $O(\mathscr{H})$. If v_i is a basis function of this finite dimensional space, $P\tilde T v_i$ can be built in advance as a pre-processing computation. To guarantee order $\mathscr{H}$ convergence in the energy norm, it is enough to define the basis using elementwise generalized eigenvalues problems. For each $\tau\in\mathcal{T}_H$, find eigenpairs $(\sigma_i, v_i)\in(\mathbb{R}, H^1(\tau))$, $i\in\mathbb{N}$, such that

$$\int_\tau \mathcal{A}\,\boldsymbol{\nabla}\,v_j\cdot\boldsymbol{\nabla}\,w\,d\boldsymbol{x} = \sigma_j\int_\tau \rho v_i w\,d\boldsymbol{x} \quad \text{for all } w\in H^1(\tau) \tag{56}$$

Let us order the eigenvalues $0=\sigma_1<\sigma_2\le\sigma_3\le\dots$. First note that $\sigma_i\to\infty$ since $H^1(\tau)$ is compactly embedded in $L^2(\tau)$ and $\mathcal{A}$ and ρ are uniform positive definite and bounded. Define J_τ as the minimum integer such that $\sigma_{J_\tau+1}^{-1}\le c\mathscr{H}^2$, where $\mathscr{H}$ was introduced in (16), and define the space

$$\mathcal{F}_{J_\tau} = \operatorname{span}\{v_1,\dots,v_{J_\tau}\}.$$

Then we obtain

$$v \in \mathcal{F}_{J_\tau}^{\perp_{L^2_\rho}} \implies \int_\tau \rho v^2 \, d\boldsymbol{x} \le c\mathscr{H}^2 \int_\tau \mathcal{A} \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} v \, d\boldsymbol{x}.$$

Indeed, let $v \in \mathcal{F}_{J_\tau}^{\perp_{L^2_\rho}}$. Using the fact that the eigenfunctions $v_{\tau,i}$ define an orthogonal basis in both $H^1_{\mathcal{A}}(\tau)$ semi-norm and $L^2_\rho(\tau)$ norm, we can write $v = \sum_{i \ge J_\tau+1} \alpha_i v_i$, and then

$$\begin{aligned} \int_\tau \rho v^2 \, d\boldsymbol{x} &= \sum_{i \ge J_\tau+1} \alpha_i^2 \int_\tau \rho v_i^2 \, d\boldsymbol{x} = \sum_{i \ge J_\tau+1} \alpha_i^2 \sigma_i^{-2} \int_\tau \mathcal{A} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_i \, d\boldsymbol{x} \le \\ &\sigma_{J_\tau+1}^{-2} \sum_{i \ge J_\tau+1} \alpha_i^2 \int_\tau \mathcal{A} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_i \, d\boldsymbol{x} \le (c\mathscr{H})^2 \sum_{i \ge J_\tau+1} \alpha_i^2 \int_\tau \mathcal{A} \boldsymbol{\nabla} v_i \cdot \boldsymbol{\nabla} v_i \, d\boldsymbol{x} \\ &= (c\mathscr{H})^2 \int_\tau \mathcal{A} \boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} v \, d\boldsymbol{x}. \end{aligned} \tag{57}$$

Clearly, $\mathcal{F}_{J_\tau}$ is nonempty since it contains the constant function.

Let $\mathcal{F}_J \subset H^1(\mathcal{T}_H)$ be the space of functions whose restrictions to τ are in $\mathcal{F}_{J_\tau}$, and let $\Pi_J g \in \mathcal{F}_J$ be the $L^2_\rho(\Omega)$ orthogonal projection of g on $\mathcal{F}_J$. Note that the computation of $\tilde{T}g$ becomes now trivial since, on each element τ and $i > 1$, $\tilde{T} v_i = \sigma_i^{-1} v_i$. This follows from the second equation of (9), and (56).

Lemma 6. *Consider $u_J \in H^1_0(\Omega)$ weakly satisfying $-\operatorname{div} \mathcal{A} \boldsymbol{\nabla} u_J = \rho \Pi_J g$. Then*

$$|u - u_J|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le c\mathscr{H} \|g\|_{L^2_\rho(\Omega)} = c\mathscr{H} \|f\|_{L^2_{1/\rho}(\Omega)}.$$

Proof. Note that the error $e_J = u - u_J \in H^1_0(\Omega)$ weakly satisfies $-\operatorname{div} \mathcal{A} \boldsymbol{\nabla} e_J = \rho(I - \Pi_J)g$, and note that u_J does not necessary belong to $\mathcal{F}_J$. We have

$$\begin{aligned} |e_J|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} &= (\mathcal{A} \boldsymbol{\nabla} e_J, \boldsymbol{\nabla} e_J)_{\mathcal{T}_H} = (\rho(I - \Pi_J)g, e_J)_{\mathcal{T}_H} = (\rho(I - \Pi_J)g, e_J - \Pi_J e_J)_{\mathcal{T}_H} \\ &\le \|g\|_{L^2_\rho(\Omega)} \|e_J - \Pi_J e_J\|_{L^2_\rho(\Omega)} \le c\mathscr{H} \|g\|_{L^2_\rho(\Omega)} |e_J - \Pi_J e_J|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le c\mathscr{H} \|g\|_{L^2_\rho(\Omega)} |e_J|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}, \end{aligned}$$

where we have used that $\Pi_J e_J \in \mathcal{F}_J$, and Π_J is an orthogonal projection in both $L^2_\rho(\Omega)$ and $H^1_{\mathcal{A}}(\mathcal{T}_H)$ inner-products. □

Remark 9. *The above arguments can be extended to the discrete case u_h and $u_{h,J}$ solutions of (20) with ρg and $\rho \Pi_J g$ as the forcing term, respectively. Denote $\delta g = (I - \Pi_J)g$. The solution error $e_{h,J} = u_h - u_{h,J}$ can be decomposed as $e_{h,J} = e^0_{h,J} + T\delta\lambda_h + \tilde{T}\delta g$, where $\delta\lambda_h = (I - PT)\delta\tilde{\lambda}^0 - P\tilde{T}\delta g$. Note that $\delta\lambda^0$ vanishes. Using the definitions of T and $\tilde{T}$, and (13) we obtain*

$$|e_{h,J}|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = |T\delta\lambda_h + \tilde{T}\delta g|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)} = (e_{h,J}, \rho\delta g)_{L^2(\Omega)} \le c\mathscr{H} |e_{h,J}|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \|g\|_{L^2_\rho(\Omega)}. \tag{58}$$

Remark 10. *On Section 3 we have assumed ρg as the right-hand side. In case we use $\rho\Pi_J g$ as the right hand side, we obtain the same bound since $\|\Pi_J g\|_{L^2_\rho(\Omega)} \le \|g\|_{L^2_\rho(\Omega)}$. Thus, if $u^j_{h,J}$ is the localized solution of the method with $\rho\Pi_J g$ as the right-hand side , then*

$$\|u_h - u^j_{h,J}\|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} \le \|u_h - u_{h,J}\|_{H^1_{\mathcal{A}}(\mathcal{T}_H)} + \|u_{h,J} - u^j_{h,J}\|_{H^1_{\mathcal{A}}(\mathcal{T}_H)},$$

and a final estimate follows from (58) *and Theorem 5 or 5' with g replaced by $\Pi_J g$.*

We now discuss the role of the choice of ρ based on local arguments. Observe that as ρ increases, the eigenvalues σ_j of (56) decreases, then the size of the space $\mathcal{F}_J$ increases, the error $|e_J|^2_{H^1_{\mathcal{A}}(\mathcal{T}_H)}$ decreases, and also note that $\|f\|_{L^2_{1/\rho}(\Omega)}$ decreases. Also see that when H gets smaller, the dimension of the space $\mathcal{F}_J$ decreases. A physical interpretation for that follows. For the case which $\mathcal{A}$ is isotropic on an element τ, that is, $\mathcal{A}(\boldsymbol{x}) = a(\boldsymbol{x})I$, it has been shown [22] that an interesting choice is $\rho(\boldsymbol{x}) = a(\boldsymbol{x})$ for $\boldsymbol{x} \in \tau$, since J_τ can be estimated from above by the number of inclusions with large coefficients inside τ, plus the number channels with large coefficients crossing τ, that is, these are related to explicit modes that do not satisfy the local weighted Poincaré inequality. As H decreases, some of these modes disappear and eventually J_τ vanishes, that is, all functions satisfy the weighted Poincaré inequality. We note however that decreasing H too much might not be an option since the upscaled system can becomes prohibitively large. We believe that $\rho(\boldsymbol{x}) = a_+(\boldsymbol{x})$ is a judicious choice; it yields sharp estimates and the J_τ has a physical interpretation.

5. Algorithms

In this section we give a practical guideline on how both “versions” of the method could be implemented. Actually, the low-contrast case is a particular, simpler, instance of the high-contrast one, and can be chosen by picking $\alpha_{\text{stab}} > \alpha$. So we highlight the high-contrast case here, and rewrite some defining equations for the convenience of the reader.

(1) **Partition of Ω:** The starting point is the generation of a partition $\mathcal{T}_H$ of Ω into triangles (2D) or tetrahedra (3D)
(2) **The right-hand side:** Define ρ; see the discussion at the end of Section 4. If a pre-processing as described in Section 4 is desired, then compute $\Pi_J g$ as described in Lemma 6, and proceed with the computation replacing g by $\Pi_J g$. This is particularly useful in the case of multiple right-hand sides
(3) **Space definitions:** Define $\mathbb{P}^0(\mathcal{T}_H)$ as in (6), Λ^0 as in (18); define $\widetilde{\Lambda}^0_h$ and $\tilde{\Lambda}^f_h$ as in (19). Define $\tilde{\Lambda}^\Pi_h$ and $\tilde{\Lambda}^\triangle_h$ by (40), and set $\widetilde{\Lambda}^{0,\Pi}_h = \widetilde{\Lambda}^0_h \oplus \tilde{\Lambda}^\Pi_h$. In the low-contrast case, $\tilde{\Lambda}^\triangle_h = \tilde{\Lambda}^f_h$, $\widetilde{\Lambda}^{0,\Pi}_h = \widetilde{\Lambda}^0_h$

(4) **Find λ^0:** Find $\lambda^0 \in \Lambda^0$ from the first equation of (20),

$$(\lambda^0, v^0)_{\partial\mathcal{T}_H} = -(\rho g, v^0)_{\mathcal{T}_H} \quad \text{for all } v^0 \in \mathbb{P}^0(\mathcal{T}_H)$$

(5) **Define $\tilde{P}^{\triangle,j}$ and $P^{\triangle,j}$:** From (44) and (45), or compute $\tilde{P}^{\triangle,j} = \tilde{P}^j$ and $P^{\triangle,j} = P^j$ from (27) and (28) in the low-contrast case, respectively

(6) **Find $\tilde{\lambda}_h^{0,\Pi,j}$:** Find $\tilde{\lambda}_h^{0,\Pi,j} \in \widetilde{\Lambda}_h^{0,\Pi}$ from (49),

$$\begin{aligned}\big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, T(I - P^{\triangle,j}T)\tilde{\lambda}_h^{0,\Pi,j}\big)_{\partial\mathcal{T}_H} &= -\big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, (I - TP^{\triangle,j})\tilde{T}g\big)_{\partial\mathcal{T}_H} \\ &\quad - \big((I - P^{\triangle,j}T)\tilde{\mu}_h^{0,\Pi}, T(I - P^{\triangle,j}T)\lambda^0\big)_{\partial\mathcal{T}_H} \quad \text{for all } \tilde{\mu}_h^{0,\Pi} \in \tilde{\Lambda}_h^{0,\Pi}\end{aligned}$$

(7) **Find $\tilde{\lambda}_h^{\triangle,j}$:** Find $\tilde{\lambda}_h^{\triangle,j} \in \tilde{\Lambda}_h^{\triangle}$ from (50),

$$\tilde{\lambda}_h^{\triangle,j} = -P^{\triangle,j}(T\lambda^0 + T\tilde{\lambda}^{0,\Pi,j} + \tilde{T}g)$$

(8) **Find $u_h^{0,\mathrm{LSD},j}$:** Find $u_h^{0,\mathrm{LSD},j} \in \mathbb{P}^0(\mathcal{T}_H)$ from (51),

$$(\mu^0, u_h^{0,\mathrm{LSD},j})_{\partial\mathcal{T}_H} = -(\mu^0, T\lambda^0 + T\tilde{\lambda}_h^{0,\Pi,j} + T\tilde{\lambda}_h^{\triangle,j})_{\partial\mathcal{T}_H} - (\mu^0, \tilde{T}g)_{\partial\mathcal{T}_H} \quad \text{for all } \mu^0 \in \Lambda^0$$

(9) **Compute $u_h^{\mathrm{LSD},j}$ and $\boldsymbol{\sigma}_h^{\mathrm{LSD},j}$:** From (48) and Remark 8,

$$u_h^{\mathrm{LSD},j} = u_h^{\mathrm{LSD},0,j} + T\lambda_h^{\mathrm{LSD},j} + \tilde{T}g, \qquad \lambda_h^{\mathrm{LSD},j} = \lambda^0 + \tilde{\lambda}_h^{0,\Pi,j} + \tilde{\lambda}_h^{\triangle,j},$$
$$\boldsymbol{\sigma}_h^{\mathrm{LSD},j} = \mathcal{A}\,\boldsymbol{\nabla}[T(\lambda^0 + \tilde{\lambda}_h^{0,\Pi,j} + \tilde{\lambda}_h^{\triangle,j}) + \tilde{T}g]$$

Appendix A. Auxiliary results

Lemma 7. *Let λ_i^0 defined as in (17). Then $\{\lambda_i^0\}_{i=1}^N$ are linearly independent.*

Proof. Assume that there exist constants $\beta_1, \dots, \beta_N$ such that $\sum_{i=1}^N \beta_i\lambda_i^0 = 0$. Consider τ_i, τ_j two adjacent triangles sharing a face F and let $v \in H^1(\mathcal{T}_H)$ with support in τ_i such that $v|_{F'} = 0$ if $F' \neq F$ and $\int_F v = 1$. Then

$$0 = \sum_{i=1}^N \beta_i(\lambda_i^0, v)_{\partial\mathcal{T}_H} = \beta_i + \beta_j.$$

If $F \subset \partial\Omega \cap \partial\tau_i$ then using the same arguments we have $\beta_i = 0$. Then $\beta_i = 0$ for all $i = 1, \dots, N$ by using the connectivity of $\mathcal{T}_H$. □

Lemma 8. *Let $\mu \in \Lambda_h$. Then the problem of finding μ^0 such that $(\mu^0, v^0)_{\partial\mathcal{T}_H} = (\mu, v^0)_{\partial\mathcal{T}_H}$ for all $v^0 \in \mathbb{P}^0(\mathcal{T}_H)$ is well-posed.*

Proof. Since we are dealing with finite dimensional spaces, it is enough to prove that the $N \times N$ matrix with components $(\lambda_i^0, v_j^0)_{\partial\mathcal{T}_H}$ is non-singular, where v_i^0 is the characteristic function of the element τ_i. Note that it follows from the definition of λ_i^0 that

$$|(\lambda_i^0, v_j^0)_{\partial\mathcal{T}_H}| = \begin{cases} |\partial\tau_i| & \text{if } i = j, \\ |\partial\tau_i \cap \partial\tau_j| & \text{otherwise,} \end{cases}$$

and then the matrix is diagonally dominant. Consider now an element τ_i such that $\partial\tau_i \cap \partial\Omega \neq \emptyset$. Then,

$$|(\lambda_i^0, v_i^0)_{\partial\mathcal{T}_H}| = |\partial\tau_i| > \sum_{j \neq i} |\partial\tau_i \cap \partial\tau_j|.$$

We note that the matrix is irreducible since given any two distinct elements τ_i and τ_k, there exists a path of adjacent faces connecting τ_i to τ_k. Then the matrix is irreducibly diagonally dominant, and from [64, Theorem 1.11], it is non-singular. □

Lemma 9. *Let $\mu \in \Lambda_h$ and $\tau \in \mathcal{T}_H$, assume that* (2) *holds, and denote by $T_{\mathcal{I}}$ the harmonic extension defined by* (9) *replacing $\mathcal{A}$ by the identity operator. It follows then that*

$$\frac{1}{a_{\min}^\tau}|T_{\mathcal{I}}\mu|^2_{H^1(\tau)} \geq |T\mu|^2_{H^1_{\mathcal{A}}(\tau)} \geq \frac{1}{a_{\max}^\tau}|T_{\mathcal{I}}\mu|^2_{H^1(\tau)}.$$

Proof. First note that, for any non-vanishing λ,

$$\begin{aligned} \inf_{v \in \widetilde{H}^1(\tau)} \frac{1}{2}(\lambda \boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_\tau - (\mu, v)_{\partial\tau} &= \inf_{v \in \widetilde{H}^1(\tau)} \frac{1}{2}(\lambda \boldsymbol{\nabla} \lambda^{-1} v, \boldsymbol{\nabla} \lambda^{-1} v)_\tau - (\mu, \lambda^{-1} v)_{\partial\tau} \\ &= \frac{1}{\lambda} \inf_{v \in \widetilde{H}^1(\tau)} \frac{1}{2}(\boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_\tau - (\mu, v)_{\partial\tau}. \end{aligned}$$

Then

$$\begin{aligned} -\frac{1}{2}(\mu, T\mu)_{\partial\tau} = \inf_{v \in \widetilde{H}^1(\tau)} \frac{1}{2}(\mathcal{A} \boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_\tau - (\mu, v)_{\partial\tau} &\leq \inf_{v \in \widetilde{H}^1(\tau)} a_{\max}^\tau \frac{1}{2}(\boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_\tau - (\mu, v)_{\partial\tau} \\ &= -\frac{1}{2a_{\max}^\tau}(\mu, T_{\mathcal{I}}\mu)_{\partial\tau}. \end{aligned}$$

Similarly,

$$-\frac{1}{2}(\mu, T\mu)_{\partial\tau} \geq \inf_{v \in \widetilde{H}^1(\tau)} a_{\min}^\tau \frac{1}{2}(\boldsymbol{\nabla} v, \boldsymbol{\nabla} v)_\tau - (\mu, v)_{\partial\tau} = -\frac{1}{2a_{\min}^\tau}(\mu, T_{\mathcal{I}}\mu)_{\partial\tau}.$$

□

Appendix B. Notations

Inner products, dualities and norms:

- $(\cdot,\cdot)_{\mathcal{T}_H}$, $(\cdot,\cdot)_{\partial\mathcal{T}_H}$, $(\cdot,\cdot)_{\partial\tau}$: inner and duality products; page 3
- $\|\cdot\|_{L^2_\rho(\Omega)}$: ρ-weighted $L^2(\Omega)$ norm; page 2
- $\|\cdot\|_{H_{\mathcal{A}}(\operatorname{div};\Omega)}$, $\|\cdot\|_{H^1_{\mathcal{A}}(\mathcal{T}_H)}$, $\|\cdot\|_{H^{-1/2}_{\mathcal{A}}(\mathcal{T}_H)}$: equation (4)

Functional spaces:

- $H^1(\mathcal{T}_H)$ and $\Lambda(\mathcal{T}_H)$: piecewise H^1 functions, and trace of $H(\operatorname{div};\Omega)$ functions; equation (3)
- $\widetilde{H}^1(\tau)$: local space; page 15
- $\mathbb{P}^0(\mathcal{T}_H)$, $\widetilde{H}^1(\mathcal{T}_H)$: piecewise constants and zero average functions; page 4
- $\mathcal{F}_J$: eigenspace generating a finite dimension space for the right-hand side; page 20
- Λ_h: piecewise constants on the refined skeleton; page 5
- Λ^0: constants flux traces on element boundaries; equation (18)
- $\widetilde{\Lambda}_h = \mathbb{P}^0(\mathcal{T}_H)^\perp$: zero average flux traces on element boundaries; equation (18)
- $\widetilde{\Lambda}^0_h$: constants on faces with zero average on element boundaries; equation (19)
- $\widetilde{\Lambda}^f_h$: zero average flux traces on faces; equation (19)
- $\tilde{\Lambda}^{f,\tau,j}_h \subset \tilde{\Lambda}^f_h$: page 11
- $\tilde{\Lambda}^\tau_h$: local space; page 15
- $\tilde{\Lambda}^F_h$, $\tilde{\Lambda}^{F^c_\tau}_h$: restriction of $\tilde{\Lambda}^f_h$ to F, F^c_τ; page 15
- $\tilde{\Lambda}^{F,\triangle}_h$, $\tilde{\Lambda}^{F,\Pi}_h$, $\tilde{\Lambda}^\Pi_h$, $\tilde{\Lambda}^\triangle_h$: spectral spaces; equations (39), (40)
- $\widetilde{\Lambda}^{0,\Pi}_h := \widetilde{\Lambda}^0_h \oplus \tilde{\Lambda}^\Pi_h$: replaces $\widetilde{\Lambda}^0_h$ in the high-contrast case; page 17

Operators

- $P : H^1(\mathcal{T}_H) \to \tilde{\Lambda}^f_h$: non-local operator; equation (21)
- $\tilde{P}^j$, P^j, $P^{K,j}$, $P^{F,j}$, $P^\triangle$, $\tilde{P}^{\triangle,j}$, $P^{\triangle,j}$, $P^{\triangle,K,j}$, $P^{\triangle,F,j}$: pages 11, 17, 18
- Π_J: $L^2_\rho(\Omega)$ orthogonal projection on $\mathcal{F}_J$; page 21
- T, $\tilde{T}$, T^τ_{FF}, $T^\tau_{FF^c}$, $T^\tau_{F^cF}$, $T^\tau_{F^cF^c}$, $\widehat{T}^\tau_{FF}$: local operators; pages 4 and 15

Unknowns:

- λ: trace of the elementwise fluxes; equation (5)
- λ_h: traces of "surrogate" flux; equation (13)
- λ^0, $\tilde{\lambda}^0_h$, $\tilde{\lambda}^f_h$: decompose λ_h; page 7
- λ^j_h, $\tilde{\lambda}^{0,j}_h$, $\tilde{\lambda}^{f,j}_h$: solutions using the P^j operator; equations (32) and (34)
- $\tilde{\lambda}^{0,\Pi}_h$, $\tilde{\lambda}^\triangle_h$: components of the solution using the $P^\triangle$ operator; equations (41), (42)
- $\tilde{\lambda}^{0,\Pi,j}_h$, $\tilde{\lambda}^{\triangle,j}_h$: components of the solution using the $P^{\triangle,j}$ operator; equation (48)

- $\boldsymbol{\sigma}$: flux; equation (12)
- $\boldsymbol{\sigma}_h$: "surrogate" flux; equation (14)
- $\boldsymbol{\sigma}_h^{\mathrm{LSD},j}$: LSD flux; Remark 8
- u: solution of the original problem; equation (1)
- u^0, $\tilde{u}$: average and zero average components of u; equations (7) and (8)
- u_h: solution of the "surrogate" discrete problem; equation (14)
- u_h^0: average of u_h; equation (13)
- u_h^j, $u_h^{0,j}$: solutions using the P^j operator; equation (32)
- $u_h^{\mathrm{LSD},j}$, $u_h^{0,\mathrm{LSD},j}$: solutions using the $P^{\triangle,j}$ operator; equation (48)
- v_1, v_2, v_3, $\ldots$: eigenfunctions generating a finite dimension space for the right-hand side; equation (56)

Other notations:

- $a_{\min}$, $a_{\max}$, a_-, a_+, $a_{\min}^\tau$, $a_{\max}^\tau$: bounds for the eigenvalues of $\mathcal{A}$; equation (2) and page 9
- $\mathcal{A}$: symmetric coefficients tensor; equation (1)
- α_{stab}: controls the decay rate of the solutions to the non-local problems; equation (39)
- c_P, $C_{P,G}$: local and global weighted Poincaré inequality constants; page 14
- $\beta_{H/h} = 1 + \log(H/h)$: page 9
- d: dimension; page 1
- $\mathcal{F}_h$: partition of the faces of elements in $\mathcal{T}_H$; page 5
- $F_\tau^c = \partial\tau \backslash F$: $\partial\tau$ except the face F; page 15
- f: right-hand side of the original problem; equation (1)
- g: related to f by $f = \rho g$; page 2
- γ: constant depending on shape regularity of $\mathcal{T}_H$; page 9
- H, h: coarse and fine mesh characteristic lengths
- $\mathscr{H}$: the method's "target precision"; equation (16)
- $\kappa = \max_{\tau \in \mathcal{T}_H} \kappa^\tau$, $\kappa^\tau = a_{\max}^\tau / a_{\min}^\tau$: page 9
- $\boldsymbol{n}^\tau$: unit size normal vector pointing outward τ; page 3
- Ω, $\partial\Omega$: bi- or tri-dimensional domain and its boundary; page 1
- ρ, $\rho_{\min}$, $\rho_{\max}$: weight and its bounds; page 2
- τ, $\partial\tau$: typical element in $\mathcal{T}_H$ and its boundary
- $\mathcal{T}_H$: partition of Ω; page 3
- $\mathcal{T}_j(K)$: submesh of $\mathcal{T}_H$; page 8

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