Rigorous derivation from the water waves equations of some full dispersion shallow water models

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Abstract

In order to improve the frequency dispersion effects of irrotational shallow water models in coastal oceanography, several full dispersion versions of classical models were formally derived in the literature. The idea, coming from G. Whitham in [21], was to modify them so that their dispersion relation is the same as the water waves equations. In this paper we construct new shallow water approximations of the velocity potential then deducing ones on the vertically averaged horizontal component of the velocity. We make use of them to derive rigorously from the water waves equations two new Hamiltonian full dispersion models. This provides for the first time non-trivial precision results characterizing the order of approximation of the full dispersion models. They are non-trivial in the sense that they are better than the ones for the corresponding classical models.

1 Introduction

1.1 Motivations

In this work, we consider full dispersion models for the propagation of surface waves in coastal oceanography. It is a class of irrotational shallow water models which have the particularity of having the same dispersion relation as the water waves equations. The first nonlinear full dispersion model appearing in the literature was introduced formally by Whitham in [21, 22]. It is a modification of the Korteweg–de Vries (KdV) equations called the Whitham equations, see [15] for a rigorous comparison between those two equations. The goal here was to describe wave breaking phenomena [12] and Stokes waves of extreme amplitude [11]. Later on, the same kind of formal modifications has been made on other standard shallow water models such as the Boussinesq or Green-Naghdi systems, thus creating a whole class of full dispersion models. The motivation was to widen the range of validity of the shallow water models, see section 5.3 of [16], and to study the propagation of waves above obstacles, a situation where there is creation of high harmonics which are then freely released, see [1, 4, 16].

The models obtained by modifying the Boussinesq system, generally called Whitham-Boussinesq systems, have been the subject of active research, see [2, 7] for comparative studies, [5, 6, 9, 14] for the well-posedness theory, [8, 18] for some works on solitary waves solutions, and [13] for a study on modulational instability (this list is not exhaustive, see also [19, 20]). In the case of the modified Green-Nagdhi sytems, see [10] for a fully justified two-layer one.

However at the best of the author's knowledge no direct derivation of these models from the water waves equations has been done. In this paper we provide asymptotic approximations of the Dirichlet-to-Neumann operator. Then we use them to derive two different Hamiltonian full dispersion systems (see (1.8) and (1.9)) and justify them in the sense of consistency (see definition 1.10) of the water waves equations with these two models. Subsequently we deduce from them an improved precision result with respect to the one already known for the different full dispersion models appearing in the literature.

1.2 Consistency problem

Throughout this paper d will be the dimension of the horizontal variable (denoted $X \in \mathbb{R}^d$).

The starting point of this study is the adimensional water waves problem, that is ($d = 1, 2$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\mathcal{G}^\mu \psi = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{\mu}\frac{(\mathcal{G}^\mu\psi + \epsilon\mu\nabla\zeta\cdot\nabla\psi)^2}{2(1+\epsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \tag{1.1}$$

Here

- ∇ is the horizontal gradient, i.e.

$$\nabla := \begin{cases} \partial_x, & \text{when} \quad d = 1, \\ (\partial_x, \partial_y)^T, & \text{when} \quad d = 2. \end{cases}$$

- The free surface elevation is the graph of ζ, which is a function of time t and horizontal space $X \in \mathbb{R}^d$.

- $\psi(t, X)$ is the trace at the surface of the velocity potential.

- $\mathcal{G}^\mu$ is the Dirichlet-to-Neumann operator defined later in definition 1.4.

Moreover every variables and functions in (1.1) are compared with physical characteristic parameters of same dimension. Among those are the characteristic water depth H_0, the characteristic wave amplitude a_{surf} and the characteristic wavelength L_x. From these comparisons appear two adimensional parameters of main importance:

- $\mu := \frac{H_0^2}{L_x^2}$: the shallow water parameter,

- $\epsilon := \frac{a_{\text{surf}}}{H_0}$: the nonlinearity parameter.

We refer to [16] for details on the derivation of these equations.

Before giving the main definitions of this section, here are two assumptions maintained throughout this paper.

Hypotheses 1.1.
- *A fundamental hypothesis in this study will be the lower boundedness by a positive constant of the water depth (non-cavitation assumption):*

$$\exists h_{\min} > 0, \forall X \in \mathbb{R}^d, \quad h := 1 + \epsilon\zeta(t, X) \geq h_{\min}. \tag{1.2}$$

- *We suppose that the bottom of the sea is flat. The water domain is then defined by* $\Omega_t := \{(X, z) \in \mathbb{R}^{d+1} : -1 < z < \epsilon\zeta(X)\}$.

In what follows we need some notations on the functional setting of this paper.

Notations 1.2.
- *For any* $s \geq 0$ *we will denote* $H^s(\mathbb{R}^d)$ *the Sobolev space of order* s *in* $L^2(\mathbb{R}^d)$.

- *For any* $s \geq 1$ *we will denote* $\dot{H}^s(\mathbb{R}^d) := \{f \in L^2_{\text{loc}}(\mathbb{R}^d), \quad \nabla f \in H^{s-1}(\mathbb{R}^d)\}$ *the Beppo-Levi space of order* s.

- *The* $L^2(\mathbb{R}^d)$ *norm will be written* $|\cdot|_2$. *The* $L^2(\mathcal{S})$ *norm, where* $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ *(see definition 1.3), will be denoted* $||\cdot||_2$.

- *Denoting* $\Lambda^s := (1 - \Delta)^{s/2}$, *where* Δ *is the Laplace operator in* $\mathbb{R}^d$, *the* $H^s(\mathbb{R}^d)$ *norm will be* $|\cdot|_{H^s} := |\Lambda^s \cdot|_2$.

It is easier to work in a time independant water domain. For that reason, by the mean of a diffeomorphism defined in the next definition, we will straighten our problem.

Definition 1.3. *Let* $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ $(t_0 > d/2)$ *such that* (1.2) *is satisfied. We define the time-dependant trivial diffeomorphism mapping the flat strip* $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ *onto the water domain* Ω_t

$$\begin{array}{ccccc} \Sigma_t & : & \mathcal{S} := \mathbb{R}^d \times (-1, 0) & \to & \Omega_t := \{(X, z) \in \mathbb{R}^{d+1} : -1 < z < \epsilon\zeta(X)\} \\ & & (X, z) & \mapsto & (X, z + \epsilon\zeta(z + 1)). \end{array} \tag{1.3}$$

We can now define the Dirichlet-to-Neumann operator $\mathcal{G}$ in the flat strip $\mathcal{S}$, see the chapters 2 and 3 in [16].

Definition 1.4. *Let* $s \geq 0, t_0 > d/2$, $\psi \in \dot{H}^{s+3/2}(\mathbb{R}^d)$ *and* $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ *be such that* (1.2) *is satisfied. Using the trivial diffeomorphism* (1.3) *we introduce the potential velocity* ϕ *in the flat strip* $\mathcal{S}$ *through the following variable coefficients elliptic equation*

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \phi = 0 \quad \text{in} \quad \mathcal{S}, \\ \phi|_{z=0} = \psi, \quad \partial_z \phi|_{z=-1} = 0, \end{cases} \tag{1.4}$$

where ∇^μ *is the* $(d + 1)$*-gradient operator defined by* $\nabla^\mu = (\sqrt{\mu}\nabla^T, \partial_z)^T$,
and $P(\Sigma_t) = \begin{pmatrix} (1 + \epsilon\zeta)I_d & -\sqrt{\mu}\epsilon(z + 1)\nabla\zeta \\ -\sqrt{\mu}\epsilon(z + 1)\nabla\zeta^T & \frac{1+\mu\epsilon^2(z+1)^2|\nabla\zeta|^2}{1+\epsilon\zeta} \end{pmatrix}$.

Let's denote by h the water depth, $h = 1 + \epsilon\zeta$. We define the vertically averaged horizontal velocity $\overline{V}[\epsilon\zeta]\psi$ (denoted $\overline{V}$ when no confusion is possible) by the formula

$$\overline{V} = \frac{1}{h}\int_{-1}^{0}[h\nabla\phi - \epsilon(z+1)\nabla\zeta\partial_z\phi]\,\mathrm{dz}\,. \tag{1.5}$$

The Dirichlet-to-Neumann operator $\mathcal{G}^\mu[\epsilon\zeta]$ (denoted $\mathcal{G}^\mu$ when no confusion is possible) is then defined as

$$\begin{array}{cccc} \mathcal{G}^\mu & : & \dot{H}^{s+3/2}(\mathbb{R}^d) & \to & H^{s+1/2}(\mathbb{R}^d) \\ & & \psi & \mapsto & -\mu\nabla\cdot(h\overline{V}). \end{array} \tag{1.6}$$

Before stating the first result of this paper, we recall the definition of a Fourier multiplier.

Definition 1.5. *Let $u : \mathbb{R}^d \to \mathbb{R}^d$ be a tempered distribution, let $\widehat{u}$ be its Fourier transform. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a smooth function with polynomial decay. Then the Fourier multiplier associated with $F(\xi)$ is denoted $\mathrm{F}(D)$ (denoted F when no confusion is possible) and defined by the formula:*

$$\widehat{\mathrm{F}(D)u}(\xi) = F(\xi)\widehat{u}(\xi).$$

We need also other notations.

Notations 1.6. *All the results of this paper will use the following notations, where $C(\circ)$ means a constant depending on $\circ$.*
Let $t_0 > d/2$, $s \geq 0$, and $\mu_{\max} > 0$. Given sufficiently regular ζ and ψ satisfying hypothesis (1.2) we will write

- $M_0 := C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0}})$.
- $M(s) := C(M_0, |\zeta|_{H^{\max(t_0+1,s)}})$.
- $M := C(M_0, |\zeta|_{H^{t_0+2}})$.
- $N(s) := C(M(s), |\nabla\psi|_{H^s})$.

Remark 1.7. *In this paper, the notation t_0 is for a real number larger than $d/2$. However, it is not to be taken too large, we can consider $d/2 < t_0 \leq 2$. So that, when $s \geq 3$, $M(s)$ is in fact $M(s) := C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^s})$.*

The first result of this paper provides asymptotic expansions of the vertically averaged horizontal velocity (which implies ones of the Dirichlet-to-Neumann operator) at order $O(\mu\epsilon)$ or $O(\mu^2\epsilon)$ with estimations of error. It also provides an approximation of the velocity potential at the surface expressed in terms of the vertically averaged horizontal velocity at order $O(\mu^2\epsilon)$.

Proposition 1.8. *Let $s \geq 0$, and $\zeta \in H^{s+4}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$, and $\overline{V}$ be as in (1.5). Let also $\mathrm{F}_1 := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$, $\mathrm{F}_2 = \frac{3}{\mu|D|}(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|})$, and*

$\mathrm{F}_3 = \mathrm{F}_2 \circ \mathrm{F}_1{}^{-1}$ *be three Fourier multipliers.*
The following estimates hold:

$$\begin{cases} |\overline{V} - \mathrm{F}_1 \nabla\psi|_{H^s} \leq \mu\epsilon M(s+3)|\nabla\psi|_{H^{s+2}}, \\ |\overline{V} - \mathrm{F}_1\nabla\psi - \frac{\mu\epsilon}{3}[h\nabla\zeta\Delta\psi + \nabla(\zeta(1+h)\Delta\psi)]|_{H^s} \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}, \\ |\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\Delta\psi)|_{H^s} \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}, \\ |\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V})|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}}. \end{cases} \tag{1.7}$$

Remark 1.9.
- *From (1.6) we straightforwardly deduce corresponding estimates for the Dirichlet-to-Neumann operator which we omit to write down since we do not use them in our analysis.*
- *In fact we obtain estimates on the straightened velocity potential inside the fluid which would allow us to reconstruct the velocity field at precision $O(\mu^2\epsilon)$. Let $t_0 > d/2$, and ϕ be the solution of (1.4). Let also $\mathrm{F}_0 := \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$ be a Fourier multiplier depending on the transversal variable $z \in [-1,0]$. Then one has*

$$\begin{cases} ||\Lambda^s\nabla^\mu(\phi - \mathrm{F}_0\psi - \mu\epsilon\zeta(1+h)(\frac{z^2}{2}+z)\Delta\psi)||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}, \\ ||\Lambda^s\nabla^\mu(\phi - (\psi + h^2(\mathrm{F}_0-1)\psi)||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}. \end{cases}$$

To state the second result of this paper we need to define the notion of consistency of the water waves equations (1.1) with respect to a given model in the shallow water asymptotic regime at a certain order in μ and ϵ.

Definition 1.10. *(Consistency)*
Let $\mu_{\max} > 0$. Let $\mathcal{A} \subset \{(\epsilon,\mu),\ \ 0 \leq \epsilon \leq 1,\ \ 0 \leq \mu \leq \mu_{\max}\}$ be the shallow water asymptotic regime. We denote by (A) and (A') two asymptotic models of the following form:

$$(A)\quad \begin{cases} \partial_t\zeta + \mathcal{N}^1_{(A)}(\zeta,\psi) = 0, \\ \partial_t\psi + \mathcal{N}^2_{(A)}(\zeta,\psi) = 0, \end{cases} \quad , \quad (A')\quad \begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t((Id + \mu T_{(A')}[h])\overline{V}) + \mathcal{N}^3_{(A')}(\zeta,\overline{V}) = 0, \end{cases}$$

where $\mathcal{N}^1_{(A)}$, $\mathcal{N}^2_{(A)}$ and $\mathcal{N}^3_{(A')}$ are nonlinear operators that depend respectively on the asymptotic model (A), (A) and (A'). And $T_{(A')}$ is an operator nonlinear in h and linear in $\overline{V}$ which depends on the asymptotic model (A').
We say that the water waves equations are consistent at order $O(\mu^k\epsilon^l)$ with respectively (A) or (A') in the regime $\mathcal{A}$ if there exists $n \in \mathbb{N}$ and $T > 0$ such that for all $s \geq 0$ and $p = (\epsilon,\mu) \in \mathcal{A}$, and for every solution $(\zeta,\psi) \in C([0,\frac{T}{\epsilon}]; H^{s+n}\times\dot{H}^{s+n+1})$ to the water waves equations (1.1) one has respectively

$$\begin{cases} \partial_t\zeta + \mathcal{N}^1_{(A)}(\zeta,\psi) = \mu^k\epsilon^l R_1, \\ \partial_t\psi + \mathcal{N}^2_{(A)}(\zeta,\psi) = \mu^k\epsilon^l R_2, \end{cases} \quad \text{or} \quad \begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t((Id + \mu T_{(A')}[h])\overline{V}) + \mathcal{N}^3_{(A')}(\zeta,\overline{V}) = \mu^k\epsilon^l R_3, \end{cases}$$

with respectively $|R_1|_{H^s}, |R_2|_{H^s} \leq N(s+n)$ or $|R_3|_{H^s} \leq N(s+n+1)$ on $[0,\frac{T}{\epsilon}]$.

For sufficiently regular initial data satisfying hypotheses 1.2, the existence and uniqueness of a solution of the water waves equations with existence time of order $1/\epsilon$ independent of μ and with the regularity we want is given by the theorem 4.16 in [16].

We now state our consistency results.

Proposition 1.11. *Let* F_1 *and* F_2 *be the Fourier multipliers defined in proposition 1.8. The water waves equations* (1.1) *are consistent at order* $O(\mu^2\epsilon)$ *in the shallow water regime* $\mathcal{A}$ *(see definition 1.10) with*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\nabla\psi) + \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2(\mathrm{F}_2[\Delta\psi])\Delta\psi = 0, \end{cases} \tag{1.8}$$

with $n = 4$.

Proposition 1.12. *Let* F_3 *be the Fourier multiplier defined in proposition 1.8. Let* $T[h]V := -\frac{1}{6h}(\nabla(h^3\mathrm{F}_3[\nabla\cdot V]) + \nabla(\mathrm{F}_3[h^3\nabla\cdot V]))$*. The water waves equations are consistent at order* $O(\mu^2\epsilon)$ *in the shallow water regime* $\mathcal{A}$ *with*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t((Id + \mu T[h])\overline{V}) + \nabla\zeta \quad + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])) \\ \qquad\qquad - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases} \tag{1.9}$$

with $n = 6$.

Setting $\mathrm{F}_3 = I_d$ in (1.8) we recover the Green-Naghdi system introduced in [3, 21]. Setting $\mathrm{F}_3 = I_d$ in (1.9) we recover the classical Green-Nagdhi system which has been proved to be consistent with precision $O(\mu^2)$ in [16, 17]. We refer to (1.8) and (1.9) as full dispersion Green-Nagdhi systems.

Remark 1.13. *The two full dispersion Green-Nagdhi systems* (1.8) *and* (1.9) *enjoy a canonical Hamiltonian formulation (see* (3.3) *and* (4.3)*).*

From these two previous propositions we are able to give results on the consistency with respect to water waves equations (1.1) of most of the full dispersion systems appearing in the literature. We give the examples of two systems that kept the author attention for there mathematical properties. The first one is a single layer, two dimensional generalisation with no surface tension of the model introduced in [10] to study high-frequency Kevin-Helmholtz instabilities. That is

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t(\overline{V} - \frac{\mu}{3h}\nabla(\sqrt{\mathrm{F}_3}h^3\sqrt{\mathrm{F}_3}[\nabla\cdot\overline{V}])) + \nabla\zeta \quad + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{3}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(\sqrt{\mathrm{F}_3}h^3\sqrt{\mathrm{F}_3}[\nabla\cdot\overline{V}])) \\ \qquad\qquad - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases} \tag{1.10}$$

Proposition 1.14. *The water waves equations are consistent at order* $O(\mu^2\epsilon)$ *in the shallow water regime* $\mathcal{A}$ *with the system* (1.10).

For the same reason as system (1.9), we refer to (1.10) as a full dispersion Green-Naghdi system.

The second one is a Whitham-Boussinesq system studied in [9]. They proved a local well-posedness result in dimension 2 and a global well-posedness result for small data in dimension 1. This system is

$$\begin{cases} \partial_t \zeta + \mathrm{F}_1 \Delta \psi + \epsilon \mathrm{F}_1 \nabla \cdot (\zeta \mathrm{F}_1 \nabla \psi) = 0, \\ \partial_t \nabla \psi + \nabla \zeta + \frac{\epsilon}{2} \nabla (\mathrm{F}_1 |\nabla \psi|)^2 = 0. \end{cases} \tag{1.11}$$

With the definition we gave of consistency (see definition 1.10) we have the following proposition.

Proposition 1.15. *The water waves equations (1.1) are consistent at order $O(\mu\epsilon)$ in the shallow water regime $\mathcal{A}$ with the system*

$$\begin{cases} \partial_t \zeta + \mathrm{F}_1 \Delta \psi + \epsilon \mathrm{F}_1 \nabla \cdot (\zeta \mathrm{F}_1 \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2} (\mathrm{F}_1 |\nabla \psi|)^2 = 0. \end{cases}$$

But one can easily adapt definition 1.10 to say that the water waves equations (1.1) are consistent at order $O(\mu\epsilon)$ with system (1.11).

Remark 1.16. *At the best of the author's knowledge, in the case of the full dispersion Green-Naghdi systems, before this work it was only known that the water waves equations are consistent in the shallow water regime with respect to system (1.10) at order $O(\mu^2)$ at worse (proposition 5.7 in [10]), that is the same precision as the one of the classical Green-Naghdi models, see chapter 5 in [16]. The use of proposition 1.8 allow us to improve the precision order by a factor ϵ, as conjectured in [10]. So that in a situation in which $\epsilon \sim \mu$ (long wave regime) we gain a power of μ, i.e. the full dispersion Green-Naghdi systems are precised at order $O(\mu^3)$ in the long wave regime, in the sense of consistency. Moreover even if μ is not so small, the latter systems stay good approximations of the water waves equations as long as ϵ is small enough, making them more robust than the corresponding classical Green-Naghdi models.*

The case of the Whitham-Boussinesq systems is a bit more subtle. Indeed, using the same argument as in the proof of proposition 5.7 in [10], one would obtain, in the shallow water regime, a precision order of $O(\mu^2 + \mu\epsilon)$ for these systems, that is the same as the one of the Boussinesq models, see chapter 5 in [16]. In this paper we prove that the precision order of the Whitham-Boussinesq systems is in fact $O(\mu\epsilon)$. So that the improvement can only be seen in a regime in which $\epsilon \ll \mu$. It still makes them more robust than the Boussinesq models for the same reason as the full dispersion Green-Naghdi systems.

1.3 outline

In section 2 we prove proposition 1.8. We begin in subsection 2.1 by constructing the shallow water expansions appearing in the proposition using a formal reasoning, see lemma 2.2. Then we use, in subsection 2.2, the fact that we have explicit candidates as approximations of the velocity potential to prove the estimates of proposition 1.8.

In section 3 we focus on system (1.8). First, in subsection 3.1, we derive formally (1.8) from Hamilton's equations associated with an approximated Hamiltonian based on proposition 1.8. In subsection 3.2 we prove rigorously the consistency of the water waves equations with system (1.8), i.e. we prove proposition 1.11.

In section 4 we focus on system (1.9) and do the same process as for system (1.8). In subsection 4.1 we get a second approximated Hamiltonian of the water waves system (1.1) and derive the Hamilton equations associated with, giving (1.9). In subsection 4.2 we prove proposition 1.12.

In section 5 we prove the consistency of the water waves equations with the systems (1.10) and (1.11). In subsection 5.1 we make use of proposition 1.11 to prove proposition 1.14. And in subsection 5.2 we use 1.12 to prove proposition 1.15.

2 Shallow water approximation of the vertically averaged horizontal component of the velocity

2.1 Formal construction

Here, we construct formally two different approximations of the velocity potential ϕ (see definition 1.4) at order $O(\mu^2\epsilon)$ (see just below notation 2.1) then deducing ones on the vertically averaged horizontal velocity $\overline{V}$ (see definition 1.4) with the same order of precision in terms of the trace at the surface of the velocity potential ψ (see also definition 1.4). And we also construct an approximation of this last quantity in terms of $\overline{V}$. For these purposes, we use a method similar to the one developed in chapter 5 of [16]. Everything can be proved in a functional framework and rigorous results will be provided in subsection 2.2.

Before starting the reasoning, for the sake of clarity we introduce a notation.

Notation 2.1. *Let $k \in \mathbb{N}$ and $l \in \mathbb{N}$. In all this paper, a function R is said to be of order $O(\mu^k\epsilon^l)$ if divided by $\mu^k\epsilon^l$ this function is uniformly bounded with respect to $(\mu,\epsilon) \in \mathcal{A}$ (defined in definition 1.4) in some Sobolev norm.*

Let us also recall (see again definition 1.4) that by definition the velocity potential ϕ satisfies an elliptic problem in the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$.
This problem is written in term of the velocity potential at the surface ψ:

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\phi = 0, \\ \phi_{|z=0} = \psi \quad , \quad \partial_z\phi_{|z=-1} = 0, \end{cases} \tag{2.1}$$

where ∇^μ stands for the $(d+1)$-gradient operator defined by $\nabla^\mu = (\sqrt{\mu}\nabla^T, \partial_z)^T$,
and $P(\Sigma_t) = \begin{pmatrix} (1+\epsilon\zeta)I_d & -\sqrt{\mu}\epsilon(z+1)\nabla\zeta \\ -\sqrt{\mu}\epsilon(z+1)\nabla\zeta^T & \frac{1+\mu\epsilon^2(z+1)^2|\nabla\zeta|^2}{1+\epsilon\zeta} \end{pmatrix}$.

Now we begin the constructions.
Step 1: The first step is to find approximations of ϕ, which satisfy the elliptic problem (2.1) up

to terms of order $O(\mu^2\epsilon)$. The functional meaning will be precised in the next subsection but can already be anticipated, we will work with Sobolev and Beppo-Levi spaces (see notations 1.2).

Lemma 2.2. *Let* $\mathrm{F}_0 = \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$ *be a Fourier multiplier depending on the transversal variable* $z \in [-1,0]$. *Let* ϕ *be the solution of the Laplace problem in the flat strip* (2.1). *We have the formal expansions*

$$\begin{cases} \phi_0 := \mathrm{F}_0\psi = \phi + O(\mu\epsilon), \\ \phi_{\mathrm{app}} := \mathrm{F}_0\psi - \mu\epsilon\zeta(1+h)(\frac{z^2}{2}+z)\Delta\psi = \phi + O(\mu^2\epsilon), \\ \widetilde{\phi}_{\mathrm{app}} := \psi + h^2(\mathrm{F}_0 - 1)\psi = \phi + O(\mu^2\epsilon). \end{cases}$$

Proof. The idea is to do a multi-scale expansion for the solution of the elliptic problem (2.1) by approximately solving (2.1) and using the technical lemma A.3.

Let us remark that multiplying the elliptic equation of (2.1) by the depth $h = 1+\epsilon\zeta$ allow us to decompose it into two parts:

$$h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\phi = (\partial_z^2\phi + \mu\Delta\phi) + \mu\epsilon A(\nabla,\partial_z)[\phi], \tag{2.2}$$

where $A(\nabla,\partial_z)$ is an operator defined as follow

$$\begin{aligned} A(\nabla,\partial_z)[\phi] = \nabla\cdot(\zeta\nabla\phi) + \zeta\nabla\cdot((1+\epsilon\zeta)\nabla\phi) + \epsilon|\nabla\zeta|^2\partial_z((z+1)^2\partial_z\phi) \\ - (1+\epsilon\zeta)(z+1)\nabla\cdot(\nabla\zeta\partial_z\phi) - (1+\epsilon\zeta)\nabla\zeta\cdot\partial_z((z+1)\nabla\phi). \end{aligned} \tag{2.3}$$

In the elliptic problem (2.1) let's only consider the part which is not of order $O(\mu\epsilon)$ and denote ϕ_0 its solution, i.e. ϕ_0 is the solution of the problem

$$\begin{cases} \partial_z^2\phi_0 + \mu\Delta\phi_0 = 0, \\ \phi_{0|_{z=0}} = \psi \quad , \quad \partial_z\phi_{0|_{z=-1}} = 0. \end{cases} \tag{2.4}$$

We get the expression of ϕ_0 by a Fourier analysis:

$$\phi_0 = \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}\psi. \tag{2.5}$$

Thus ϕ_0 is defined as a bounded Fourier multiplier applied to the trace at the surface of the velocity potential. And by lemma A.3, $\phi = \phi_0 + O(\mu\epsilon)$.
Now we seek ϕ_1 so that $\phi = \phi_0 + \mu\epsilon\phi_1 + O(\mu^2\epsilon)$. (2.2) and the lemma A.3 tell us that we just have to ask ϕ_1 to be the solution of the following problem:

$$\begin{cases} \partial_z^2\phi_1 = -A(\nabla,\partial_z)[\phi_0], \\ \phi_{1|_{z=0}} = 0 \quad , \quad \partial_z\phi_{1|_{z=-1}} = 0. \end{cases} \tag{2.6}$$

To solve (2.6), we integrate two times with respect to the transversal variable z, and we simplify the result thanks to the next formal property.

Let F_0 be the Fourier multiplier $\frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$. Then for any $z \in [-1,0]$ we have

$$\begin{cases} \frac{1-\mathrm{F}_0}{\mu|D|^2} = -\frac{z^2}{2} - z + O(\mu) \quad , \quad 1-(z+1)^2\mathrm{F}_0 = -z^2-2z+O(\mu), \\ \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} = 1+O(\mu) \quad , \quad \frac{z+1}{\sqrt{\mu}|D|}\frac{\sinh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)} = (z+1)^2 + O(\mu). \end{cases} \tag{2.7}$$

See proposition A.4 for a rigorous proof of these expansions.
This allows us after computations to obtain a simple approximation of ϕ_1:

$$\phi_1 = -\zeta(1+h)(\frac{z^2}{2}+z)\Delta\psi + O(\mu).$$

Thus we obtain the following expression of a first approximation ϕ_{app} of the velocity potential ϕ at order $O(\mu^2\epsilon)$:

$$\phi_{app} = \mathrm{F}_0\psi - \mu\epsilon\zeta(1+h)(\frac{z^2}{2}+z)\Delta\psi = \phi + O(\mu^2\epsilon). \tag{2.8}$$

Moreover using the first approximation of (2.7) we get a second approximation of ϕ:

$$\widetilde{\phi}_{app} = \psi + h^2(\mathrm{F}_0-1)\psi = \phi + O(\mu^2\epsilon). \tag{2.9}$$

$\square$

<u>Step 2</u>: The second step is to use both approximations ϕ_{app} and $\widetilde{\phi}_{\mathrm{app}}$ to get approximations of the vertically averaged horizontal velocity $\overline{V}$ at order $O(\mu^2\epsilon)$.

Proposition 2.3. *Let* $\mathrm{F}_1 := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ *and* $\mathrm{F}_2 = \frac{3}{\mu|D|^2}(1-\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|})$ *be two Fourier multipliers. Let* $\overline{V}$ *be the vertically averaged horizontal velocity. We have the formal expansions:*

$$\begin{cases} \overline{V}_{\mathrm{app}} := \mathrm{F}_1\nabla\psi + \frac{\mu\epsilon}{3}[h\nabla\zeta\Delta\psi + \nabla(\zeta(1+h)\Delta\psi)] = \overline{V} + O(\mu^2\epsilon), \\ \widetilde{\overline{V}}_{app} := \nabla\psi + \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\Delta\psi) = \overline{V} + O(\mu^2\epsilon). \end{cases}$$

Proof. For that purpose, we use the expression of $\overline{V}$ in term of the velocity potential ψ (see definition 1.4), that is:

$$\overline{V} = \frac{1}{h}\int_{-1}^{0}[h\nabla\phi - \epsilon(z+1)\nabla\zeta\partial_z\phi]\,\mathrm{d}z\,. \tag{2.10}$$

Replacing ϕ by ϕ_{app} in it, we get an approximation $\overline{V}_{\mathrm{app},0}$ of $\overline{V}$:

$$\begin{aligned} \overline{V}_{\mathrm{app},0} =& \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\nabla\psi - \epsilon\frac{\nabla\zeta}{h}(\psi - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\psi) \\ &+ \frac{\mu\epsilon}{3}[\frac{\nabla\zeta}{h}\epsilon\zeta(1+h)\Delta\psi + \nabla(\zeta(1+h)\Delta\psi)] = \overline{V} + O(\mu^2\epsilon). \end{aligned}$$

Moreover remark that formally we also have the following expansion

$$\psi - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\psi = -\frac{\mu}{3}\Delta\psi + O(\mu^2).$$

Using it we get a new approximation of $\overline{V}$, denoted $\overline{V}_{\mathrm{app}}$:

$$\overline{V}_{\mathrm{app}} = \mathrm{F}_1 \nabla\psi + \frac{\mu\epsilon}{3}[h\nabla\zeta\Delta\psi + \nabla(\zeta(1+h)\Delta\psi)] = \overline{V} + O(\mu^2\epsilon), \tag{2.11}$$

where $\mathrm{F}_1 = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$.

Replacing ϕ by $\widetilde{\phi}_{app}$ in (2.10) we get another useful approximation of $\overline{V}$ denoted $\widetilde{\overline{V}}_{\mathrm{app}}$:

$$\widetilde{\overline{V}}_{\mathrm{app}} = \nabla\psi + \frac{1}{h}\nabla(h^3(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1)\psi). \tag{2.12}$$

Let F_2 be the Fourier multiplier such that

$$\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1 = -\frac{\mu}{3}|D|^2\mathrm{F}_2, \tag{2.13}$$

then we write

$$\widetilde{\overline{V}}_{app} = \nabla\psi + \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\Delta\psi) = \overline{V} + O(\mu^2\epsilon). \tag{2.14}$$

□

Step 3: We construct approximations of $\nabla\psi$ in terms of $\overline{V}$ at order $O(\mu^2\epsilon)$.

Proposition 2.4. *Let $\mathrm{F}_3 = \mathrm{F}_2 \circ {\mathrm{F}_1}^{-1}$ be a Fourier multiplier where F_1 and F_2 are defined in proposition 2.3. Then we have the formal expansion:*

$$\nabla\psi = \overline{V} - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V}) + O(\mu^2\epsilon).$$

Proof. Using (2.11) in (2.14) we obtain

$$\nabla\psi = \overline{V} - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\mathrm{F}_1^{-1}\nabla\cdot\overline{V}) + O(\mu^2\epsilon).$$

Let F_3 be the Fourier multiplier defined by $\mathrm{F}_3 := \mathrm{F}_2\mathrm{F}_1^{-1}$. Then

$$\nabla\psi = \overline{V} - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V}) + O(\mu^2\epsilon). \tag{2.15}$$

□

Remark 2.5. *From (2.11), (2.14) and (2.15) we obtain formally the approximations displayed in proposition 1.8.*
Again the rigorous proof is given in the following subsection.

Remark 2.6. *The simplifications given by (2.7) allowed us to write simple approximations. Omitting this step of simplification in Step 1, we introduced the beginning of an iterative process which allows to construct approximations of order $O(\mu^k\epsilon)$ for any $k \in \mathbb{N}^*$.*

2.2 Rigorous expansions

In this subsection we prove rigorously the estimations of proposition 1.8.
For convenience we rewrite proposition 1.8 using (2.11), (2.14) and (2.15).

Proposition 2.7. *Let $s \geq 0$, and $\zeta \in H^{s+4}(\mathbb{R}^d)$ be such that 1.2 is satisfied. Let $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$, and $\overline{V}$ be as in (1.5). Let $\overline{V}_{\text{app}}$ be as in (2.11) and $\widetilde{\overline{V}}_{\text{app}}$ be as in (2.14). Let also $\mathrm{F}_1 := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$, $\mathrm{F}_2 := \frac{3}{\mu|D|}(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|})$, and $\mathrm{F}_3 = \mathrm{F}_2\mathrm{F}_1^{-1}$ be three Fourier multipliers. The following estimates hold:*

$$\begin{cases} |\overline{V} - \mathrm{F}_1\nabla\psi|_{H^s} \leq \mu\epsilon M(s+3)|\nabla\psi|_{H^{s+2}}, \\ |\overline{V} - \overline{V}_{\text{app}}|_{H^s} \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}, \\ |\overline{V} - \widetilde{\overline{V}}_{\text{app}}|_{H^s} \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}, \\ |\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V})|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}}. \end{cases} \tag{2.16}$$

Every steps in the previous subsection can be justified rigorously. But we won't follow the same path. We will mainly use the fact that we have explicit candidates for each approximations. It will give us sharper estimations.
<u>Step 1</u>: The first step is to write and prove a rigorous version of lemma 2.2.

Proposition 2.8. *Let $t_0 > d/2$, and $s \geq 0$. Let also ϕ_{app} and $\widetilde{\phi}_{\text{app}}$ be defined in lemma 2.2. We have the following estimates:*

$$\begin{cases} ||\Lambda^s\nabla^\mu(\phi - \widetilde{\phi}_{\text{app}})||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}, \\ ||\Lambda^s\nabla^\mu(\phi - \phi_{\text{app}})||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}. \end{cases}$$

Proof. We begin by computing the straightened Laplacian of $\widetilde{\phi}_{\text{app}}$, $h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\widetilde{\phi}_{\text{app}}$ (see definition 1.4 for the expression of $P(\Sigma_t)$). Let F_0 be the Fourier multiplier $\frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$ for any $z \in [-1, 0]$. Recalling $\widetilde{\phi}_{\text{app}} = \psi + h^2(\mathrm{F}_0 - 1)\psi = \mathrm{F}_0\psi + (h^2-1)(\mathrm{F}_0-1)\psi$ and $\partial_z^2\phi_0 + \mu\Delta\phi_0 = 0$ (see (2.4)), we get

$$\begin{aligned} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\widetilde{\phi}_{\text{app}} =& \mu(h^2-1)(\mathrm{F}_0-1)\Delta\psi + \mu\Delta((h^2-1)(\mathrm{F}_0-1)\psi) \\ &+ \mu\epsilon A(\nabla, \partial_z)(\mathrm{F}_0-1)\psi + \mu\epsilon A(\nabla, \partial_z)(h^2-1)(\mathrm{F}_0-1)\psi \end{aligned}$$

(see (2.3) for the definition of operator A). We estimate it thanks to product estimates A.1 and the following estimations on F_0 (where $a \lesssim b$ means there exists a constant $C > 0$ independent of μ such that $a \leq Cb$)

$$||\Lambda^s(\mathrm{F}_0 - 1)\psi||_2 \lesssim \mu|\nabla\psi|_{H^{s+1}}, \quad ||\Lambda^s\partial_z\mathrm{F}_0\psi||_2 \lesssim \mu|\nabla\psi|_{H^{s+1}}, \quad ||\Lambda^s\partial_z^2\mathrm{F}_0\psi||_2 \lesssim \mu|\nabla\psi|_{H^{s+1}},$$

stemming from the existence of $C > 0$ such that for any $z \in [-1, 0]$, $\xi \in \mathbb{R}^d$

$$|F_0(z,\xi) - 1| + |\partial_z F_0(z,\xi)| + |\partial_z^2 F_0(z,\xi)| \leq C|\xi|^2.$$

We get

$$||\Lambda^s h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \widetilde{\phi}_{\rm app}||_2 \leq \mu^2 \epsilon M(s+2)|\nabla\psi|_{H^{s+3}}. \tag{2.17}$$

Now let's define the function $\widetilde{u} := \phi - \widetilde{\phi}_{\rm app}$. It solves the following elliptic problem

$$\begin{cases} h\nabla^\mu \cdot P(\Sigma)\nabla^\mu \tilde{u} = -\mu^2\epsilon R, \\ \tilde{u}|_{z=0} = 0, \quad \partial_z \tilde{u}|_{z=-1} = 0, \end{cases}$$

where $R = \frac{1}{\mu^2\epsilon} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \widetilde{\phi}_{\rm app}$. So (2.17) gives a control on the remainder and we get from lemma A.3 one of the wanted estimations:

$$||\Lambda^s \nabla^\mu(\phi - \widetilde{\phi}_{\rm app})||_2 = ||\Lambda^s\nabla^\mu \widetilde{u}||_2 \leq \mu^2\epsilon M(s+1)||\Lambda^s R||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}. \tag{2.18}$$

Proceeding similarly as for (2.18) we get the estimates on ϕ_{app} defined by (2.8)

$$||\Lambda^s\nabla^\mu(\phi - \phi_{\rm app})||_2 \leq \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}. \tag{2.19}$$

□

<u>Step 2</u>: From the estimates on $\widetilde{\phi}_{\rm app}$ and $\phi_{\rm app}$ we get the ones on the error made when approximating $\overline{V}$ by $\widetilde{\overline{V}}_{\rm app}$ or $\overline{V}_{\rm app}$ using

$$\begin{cases} \overline{V} = \frac{1}{h}\int_{-1}^0 [h\nabla\phi - (z\nabla h + \epsilon\nabla\zeta)\partial_z\phi]\,\mathrm{d}z, \\ \widetilde{\overline{V}}_{\rm app} = \frac{1}{h}\int_{-1}^0 [h\nabla\widetilde{\phi}_{\rm app} - (z\nabla h + \epsilon\nabla\zeta)\partial_z\widetilde{\phi}_{\rm app}]\,\mathrm{d}z, \\ \overline{V}_{\rm app} = \frac{1}{h}\int_{-1}^0 [h\nabla\phi_{\rm app} - (z\nabla h + \epsilon\nabla\zeta)\partial_z\phi_{\rm app}]\,\mathrm{d}z\,. \end{cases} \tag{2.20}$$

Indeed for any u sufficiently regular and vanishing at $z = 0$ we have, thanks to Jensen inequality and Poincaré inequality (page 40 of [16]):

$$\begin{aligned} |\int_{-1}^0 u\,\mathrm{d}z\,|_{H^s}^2 = \int_{\mathbb{R}^d} |\Lambda^s \int_{-1}^0 u\,\mathrm{d}z\,|^2\,\mathrm{dX} \leq \int_{\mathbb{R}^d} (\int_{-1}^0 |\Lambda^s u|\,\mathrm{d}z)^2\,\mathrm{dX} \leq \int_{-1}^0\int_{\mathbb{R}^d} |\Lambda^s u|^2\,\mathrm{dX}\,\mathrm{d}z \\ \leq ||\Lambda^s\partial_z u||_2^2 \leq ||\Lambda^s\nabla^\mu u||_2^2. \end{aligned}$$

Applying this last inequality to $\overline{V} - \widetilde{\overline{V}}_{\rm app}$ and $\overline{V} - \overline{V}_{\rm app}$ gives the desired estimations of proposition 2.7

$$\begin{cases} |\overline{V} - \widetilde{\overline{V}}_{\rm app}|_{H^s} \leq M(s+1)||\Lambda^{s+1}\nabla^\mu(\phi - \widetilde{\phi}_{\rm app})||_2 \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}, \\ |\overline{V} - \overline{V}_{\rm app}|_{H^s} \leq M(s+1)||\Lambda^{s+1}\nabla^\mu(\phi - \phi_{\rm app})||_2 \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}. \end{cases} \tag{2.21}$$

<u>Step 3</u>: We now prove the error estimates of proposition 2.7 on the approximation of $\nabla\psi$ by $\overline{V}$, i.e.

$$|\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V})|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}}.$$

The first thing we need is an estimation on $\phi - \phi_0$ (see (2.5)).
The straightened laplacian of ϕ_0 is $\mu\epsilon A(\nabla, \partial_z)\mathrm{F}_0\psi$. So by the same reasoning as above (see (2.18)) we get

$$||\Lambda^s\nabla^\mu(\phi - \phi_0)||_2 \leq M(s+1)||\Lambda^s\mu\epsilon A(\nabla,\partial_z)\mathrm{F}_0\psi||_2 \leq \mu\epsilon M(s+2)|\nabla\psi|_{H^{s+1}},$$

(Because $\forall z \in [-1,0]$, $\forall \xi \in \mathbb{R}^d$, $\mathrm{F}_0(z, \sqrt{\mu}|\xi|) \leq 1$).
And if we define $\overline{V}_0 := \frac{1}{h}\int_{-1}^0 [h\nabla\phi_0 - \epsilon(z+1)\nabla\zeta\partial_z\phi_0]\,\mathrm{dz}$ then using again Poincaré inequality we have

$$|\overline{V} - \overline{V}_0|_{H^s} \leq M(s+1)||\Lambda^{s+1}\nabla^\mu(\phi-\phi_0)||_2 \leq \mu\epsilon M(s+3)|\nabla\psi|_{H^{s+2}}. \tag{2.22}$$

Then remarking the following equality

$$\mathrm{F}_1\nabla\psi = \frac{1}{h}\int_{-1}^0 h\nabla\phi_0\,\mathrm{dz} = \overline{V}_0 + \frac{1}{h}\int_{-1}^0 \epsilon(z+1)\nabla\zeta\partial_z\phi_0\,\mathrm{dz},$$

from the previous estimation on $\overline{V} - \overline{V}_0$ (2.22), direct computations, product estimates A.1 and quotient estimates A.2 we get

$$\begin{aligned}|\overline{V} - \mathrm{F}_1\nabla\psi|_{H^s} \leq & |\overline{V} - \overline{V}_0|_{H^s} + |\frac{1}{h}\int_{-1}^0 \epsilon(z+1)\nabla\zeta\partial_z\phi_0\,\mathrm{dz}\,|_{H^s} \\ \leq & \mu\epsilon M(s+3)|\nabla\psi|_{H^{s+2}} + |\frac{\epsilon\nabla\zeta}{h}\int_{-1}^0 (z+1)\partial_z\mathrm{F}_0\psi|_{H^s} \\ \leq & \mu\epsilon M(s+3)|\nabla\psi|_{H^{s+2}}.\end{aligned}$$

This proves one of the inequality of proposition 2.7.
Now using the error estimates on $\widetilde{\overline{V}}_{\mathrm{app}}$ (2.21) and the upper bound $F_2(\sqrt{\mu}|\xi|) \leq \frac{1}{1+\frac{\mu|\xi|^2}{3}}$ we get

$$\begin{aligned}&|\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\nabla\cdot\mathrm{F}_1^{-1}\overline{V})|_{H^s} \\ \leq & |\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\nabla\cdot\nabla\psi)|_{H^s} + |\frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\nabla\cdot(\mathrm{F}_1^{-1}\overline{V} - \nabla\psi))|_{H^s} \\ \leq & \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}} + \mu M_0|\mathrm{F}_1^{-1}\overline{V} - \nabla\psi|_{H^s}.\end{aligned}$$

By using the upper bound $F_1^{-1}(\sqrt{\mu}|\xi|) \leq 1 + \sqrt{\mu}|\xi|$ we get

$$|\mathrm{F}_1^{-1}\overline{V} - \nabla\psi|_{H^s} \leq |\overline{V} - \mathrm{F}_1\nabla\psi|_{H^{s+1}}.$$

And at the end we proved

$$|\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_2\nabla\cdot F_1^{-1}\overline{V})|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}} \tag{2.23}$$

This conclude the proof of proposition 2.7.

3 Derivation and consistency of a first full dispersion Green-Naghdi system

3.1 Formal Derivation

Let H be the Hamiltonian of the Zakharov/Craig-Sulem's formulation of the water waves problem (1.1):

$$H = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2\mu}\int_{\mathbb{R}^d} \psi \mathcal{G}^\mu \psi \,\mathrm{dX}, \tag{3.1}$$

where ζ is the surface elevation, ψ is the velocity potential at the surface and $\mathcal{G}^\mu$ is the Dirichlet-to-Neumann operator. Let us recall that the expression of the Hamilton equations derived from an Hamiltonian, here $H(\zeta, \psi)$, is

$$\begin{cases} \partial_t \zeta = \delta_\psi H, \\ \partial_t \psi = -\delta_\zeta H, \end{cases}$$

where δ_ζ and δ_ψ are functional derivatives.
Using the definition of $\mathcal{G}^\mu$ in term of $\overline{V}$ (1.6) and a formal integration by parts, we get

$$H = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 \,\mathrm{dX} + \frac{1}{2}\int_{\mathbb{R}^d} h\overline{V}\cdot\nabla\psi \,\mathrm{dX}\,. \tag{3.2}$$

Replacing $\overline{V}$ by one of its approximation $\widetilde{\overline{V}}_{\mathrm{app}}$ (2.14), we obtain an approximation of the Hamiltonian at order $O(\mu^2\epsilon)$, denoted H_{app}

$$H_{\mathrm{app}} = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 \,\mathrm{dX} + \frac{1}{2}\int_{\mathbb{R}^d} h\nabla\psi\cdot\nabla\psi \,\mathrm{dX} + \frac{\mu}{6}\int_{\mathbb{R}^d} \nabla(h^3\mathrm{F}_2[\nabla\cdot\nabla\psi])\cdot\nabla\psi \,\mathrm{dX}\,. \tag{3.3}$$

Now let's differentiate this approximated Hamiltonian in the sense of functional derivatives with respect to ψ and ζ, we get

$$\begin{cases} \delta_\psi H_{\mathrm{app}} = -\nabla\cdot(h\nabla\psi) - \frac{\mu}{6}(\Delta(h^3\mathrm{F}_2[\Delta\psi]) + \Delta(\mathrm{F}_2[h^3\Delta\psi]), \\ \delta_\zeta H_{\mathrm{app}} = \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi. \end{cases} \tag{3.4}$$

We can now write down the Hamilton equations on the approximated Hamiltonian H_{app}

$$\begin{cases} \partial_t\zeta = -\nabla\cdot(h\nabla\psi) - \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big), \\ \partial_t\psi = -\zeta - \frac{\epsilon}{2}|\nabla\psi|^2 + \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi. \end{cases} \tag{3.5}$$

Remark 3.1. • *This system is the full dispersion equivalent of a Green-Naghdi system with variables (ζ, ψ) (set $\mathrm{F}_2 = I_d$ to get the latter), see [3] and [21]. This last one is never studied because of its ill-posedness at the linear level. But for (3.5) the ill-posedness is not clear. Indeed, by construction when linearizing this system around the rest state, we obtain the same system as the linearized water waves equations, that is*

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi = 0, \\ \partial_t\psi + \zeta = 0, \end{cases}$$

which is well-posed in Sobolev spaces.

- *The system* (3.5) *is Hamiltonian by construction. Hence smooth solutions preserve energy* H_{app} *in addition to mass* $\int_{\mathbb{R}^d}\zeta$ *and momentum* $\int_{\mathbb{R}^d}\zeta\nabla\psi$.

3.2 Consistency with respect to the water waves system

We now prove proposition 1.11, on the consistency at order $O(\mu^2\epsilon)$ with respect to the water waves system (see definition 1.10) of the first full dispersion system of Green-Naghdi type derived in the previous subsection (3.5). I recall the proposition here.

Proposition 3.2. *Let* F_1 *and* F_2 *be the Fourier multipliers defined in proposition 1.8. The water waves equations are consistent at order* $O(\mu^2\epsilon)$ *in the shallow water regime* $\mathcal{A}$ *(see definition 1.10) with the following full dispersion Hamiltonian Green-Naghdi system*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\nabla\psi) + \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi = 0, \end{cases} \tag{3.6}$$

with $n = 4$.

Proof. Let ζ and ψ be the solutions of the water waves system (1.1). Using the notations of definition 1.10 we have in our case

$$\begin{cases} \mathcal{N}^1_{(A)}(\zeta,\psi) := \nabla\cdot(h\nabla\psi) + \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big), \\ \mathcal{N}^2_{(A)}(\zeta,\psi) := \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi. \end{cases}$$

So we need to prove

$$\begin{cases} |\partial_t\zeta + \nabla\cdot(h\nabla\psi) + \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big)|_{H^s} \leq \mu^2\epsilon N(s+4), \\ |\partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi|_{H^s} \leq \mu^2\epsilon N(s+4). \end{cases} \tag{3.7}$$

Step 1: Let's prove the first estimate of (3.7).
Using the definition of the Dirichlet-to-Neumann operator $\mathcal{G}^\mu$ in term of the vertically averaged horizontal velocity $\overline{V}$, we know that the water waves solutions (ζ,ψ) satisfy

$$\partial_t\zeta + \nabla\cdot(h\overline{V}) = 0.$$

Moreover we found an approximation of $\overline{V}$ of order $O(\mu^2\epsilon)$. I recall it here

$$\widetilde{\overline{V}}_{\mathrm{app}} = \nabla\psi + \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\Delta\psi)$$

where F_2 is a Fourier multiplier defined as $\mathrm{F}_2 = \frac{3}{\mu|D|^2}\big(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\big)$. For which we got the following estimations (see proposition 2.7)

$$|\overline{V} - \widetilde{\overline{V}}_{\mathrm{app}}|_{H^s} \leq \mu^2\epsilon M(s+3)|\nabla\psi|_{H^{s+4}}.$$

So we have

$$
\begin{aligned}
&|\partial_t\zeta + \nabla\cdot(h\widetilde{\widetilde{V}}_{\mathrm{app}})|_{H^s} \leq |\partial_t\zeta + \nabla\cdot(h\overline{V})|_{H^s} + |\nabla\cdot(h(\overline{V} - \widetilde{\widetilde{V}}_{\mathrm{app}}))|_{H^s} \\
\leq& M(s+1)|\overline{V} - \widetilde{\widetilde{V}}_{\mathrm{app}}|_{H^{s+1}} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+5}}.
\end{aligned}
$$

Hence

$$
|\partial_t\zeta + \nabla\cdot(h\nabla\psi) + \frac{\mu}{3}\Delta(h^3\mathrm{F}_2[\Delta\psi])|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+5}}.
$$

To prove the first estimation in (3.7) it only remains to prove that there exists $k, l \in \mathbb{N}$ with $k, l \leq 4$ such that

$$
|\Delta(h^3\mathrm{F}_2[\Delta\psi]) - (\frac{1}{2}\Delta(h^3\mathrm{F}_2[\Delta\psi]) + \frac{1}{2}\Delta(\mathrm{F}_2[h^3\Delta\psi]))|_{H^s} \leq \mu\epsilon M(s+k)|\nabla\psi|_{H^{s+l}}.
$$

Seeing that

$$
\begin{aligned}
&\Delta(h^3\mathrm{F}_2[\Delta\psi]) - (\frac{1}{2}\Delta(h^3\mathrm{F}_2[\Delta\psi]) + \frac{1}{2}\Delta(\mathrm{F}_2[h^3\Delta\psi])) \\
=&\frac{1}{2}\Delta((h^3-1)(\mathrm{F}_2-1)[\Delta\psi]) - \frac{1}{2}\Delta((\mathrm{F}_2-1)[(h^3-1)\Delta\psi]),
\end{aligned}
$$

we only need to use the estimates on F_2 in proposition A.4, the product estimates A.1 and the fact that $|h^3-1|_{H^{s+4}} \leq \epsilon M(s+4)$ to get

$$
\begin{aligned}
&|\Delta((h-1)^3(\mathrm{F}_2-1)[\Delta\psi]) - \Delta((\mathrm{F}_2-1)[(h^3-1)\Delta\psi])|_{H^s} \\
\leq&|\Delta((h-1)^3(\mathrm{F}_2-1)[\Delta\psi])|_{H^s} + |\Delta((\mathrm{F}_2-1)[(h^3-1)\Delta\psi])|_{H^s} \leq \mu\epsilon M(s+4)|\nabla\psi|_{H^{s+5}}.
\end{aligned}
$$

So

$$
|\partial_t\zeta + \nabla\cdot(h\nabla\psi) + \frac{\mu}{6}\big(\Delta(\mathrm{F}_2[h^3\Delta\psi]) + \Delta(h^3\mathrm{F}_2[\Delta\psi])\big)|_{H^s} \leq \mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+5}}.
$$

Step 2: We now prove the second estimate of (3.7).
We know that the solutions of the water waves system (ζ, ψ) satisfy

$$
\partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}\frac{(\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\mu\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2\mu|\nabla\zeta|^2} = 0.
$$

Using quotient estimates A.2, product estimates A.1 and proposition A.7 we get

$$
\begin{aligned}
&|\frac{(\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2\mu|\nabla\zeta|^2} - (\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2|_{H^s} \\
=&|\frac{\mu\epsilon^2|\nabla\zeta|^2(\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2\mu|\nabla\zeta|^2}|_{H^s} \\
\leq&\mu\epsilon^2 M(s+1)|(\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2|_{H^s} \\
\leq&\mu\epsilon^2 M(s+1)|\frac{1}{\mu}\mathcal{G}^\mu\psi + \epsilon\nabla\zeta\cdot\nabla\psi|_{H^{s+2}}^2 \\
\leq&\mu\epsilon^2 M(s+1)(|\frac{1}{\mu}\mathcal{G}^\mu\psi|_{H^{s+2}} + \epsilon|\zeta|_{H^{s+3}}|\nabla\psi|_{H^{s+2}})^2 \\
\leq&\mu\epsilon^2 M(s+1)(M(s+4)|\nabla\psi|_{H^{s+3}})^2 \\
\leq&\mu\epsilon^2 C(M(s+4), |\nabla\psi|_{H^{s+3}}).
\end{aligned}
$$

So up to $O(\mu^2\epsilon)$ terms, we can replace the second equation of (1.1) by a simpler one, i.e. there exists $R_2 \in H^s$ such that $|R_2|_{H^s} \leq C(M(s+5), |\nabla\psi|_{H^{s+5}})$ and

$$\partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}(-\nabla\cdot(h\overline{V}) + \epsilon\nabla\zeta\cdot\nabla\psi)^2 = \mu^2\epsilon R_2. \tag{3.8}$$

Now we need a proposition proved in [16] (see proposition 3.37 and remark 3.40).

Proposition 3.3. *Let $t_0 > d/2$, $s \geq 0$, and $\zeta \in H^{\max(t_0+1,s+2)}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+2}(\mathbb{R}^d)$, and $\overline{V}$ be as in (1.5). Then we have the following error estimates*

$$\begin{cases} |\overline{V}| \leq M(s+2)|\nabla\psi|_{H^{s+2}}, \\ |\overline{V} - \nabla\psi|_{H^s} \leq \mu M(s+2)|\nabla\psi|_{H^{s+2}}. \end{cases}$$

Using proposition 3.3 and product estimates A.1 we have

$$\begin{aligned}
&|(-\nabla\cdot(h\overline{V}) + \epsilon\nabla\zeta\cdot\nabla\psi)^2 - (-\nabla\cdot(h\nabla\psi) + \epsilon\nabla\zeta\cdot\nabla\psi)^2|_{H^s} \\
=&|(-\nabla\cdot(h(\overline{V} - \nabla\psi))(-\nabla\cdot(h\overline{V}) + 2\epsilon\nabla\zeta\cdot\nabla\psi - \nabla\cdot(h\nabla\psi)|_{H^s} \\
\leq&|\nabla\cdot(h(\overline{V} - \nabla\psi)|_{H^{s+1}}| - \nabla\cdot(h\overline{V}) + 2\epsilon\nabla\zeta\cdot\nabla\psi - \nabla\cdot(h\nabla\psi)|_{H^{s+1}} \\
\leq&\mu C(M(s+4), |\nabla\psi|_{H^{s+4}}).
\end{aligned}$$

Hence up to $O(\mu^2\epsilon)$ terms, we can replace (3.8) by a simpler one, i.e. there exists $R_2 \in H^s$ such that $|R_2|_{H^s} \leq C(M(s+4), |\nabla\psi|_{H^{s+4}})$ and

$$\partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2(\Delta\psi)^2 = \mu^2\epsilon R_2. \tag{3.9}$$

Now it only remains to use the estimates on F_2 in proposition A.4 to get

$$|h^2\mathrm{F}_2[\Delta\psi]\Delta\psi - h^2(\Delta\psi)^2|_{H^s} \leq \mu C(M(s+2), |\nabla\psi|_{H^{s+3}})$$

So there exists $R_2 \in H^s$ such that $|R_2| \leq C(M(s+4), |\nabla\psi|_{H^{s+4}})$ and

$$\partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2\mathrm{F}_2[\Delta\psi]\Delta\psi = \mu^2\epsilon R_2.$$

Thus we proved the consistency of the water waves equations (1.1) at order $O(\mu^2\epsilon)$ in the shallow water regime with the system (3.5) with $n = 4$. □

4 Derivation and consistency of a second full dispersion Green-Naghdi system

4.1 Formal Derivation

In this subsection we explain formally how to obtain the second full dispersion Green-Naghdi system (1.9) using (2.15). But first let's symmetrize (2.15). It yields

$$h\nabla\psi = h\overline{V} - \frac{\mu}{6}(\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]) + \nabla(\mathrm{F}_3[h^3\nabla\cdot\overline{V}]) + O(\mu^2\epsilon). \tag{4.1}$$

We will define two operators $T[h]V = -\frac{1}{6h}(\nabla(h^3\mathrm{F}_3[\nabla\cdot V]) + \nabla(\mathrm{F}_3[h^3\nabla\cdot V]))$ and $I[h]V = h(V + \mu T[h]V)$, such that the previous approximation of $\nabla\psi$ can be written:

$$h\nabla\psi = I[h]\overline{V} + O(\mu^2\epsilon) \tag{4.2}$$

Remark 4.1. *The choice of the symmetrization is arbitrary. Here we use the same as for the first full dispersion Green-Naghdi system (3.5), for which the symmetrization naturaly comes up when asking the system to be Hamiltonian. See (5.1) for another full-dispersion Green-Naghdi system with a kind of symmetrization already appearing in the litterature [10].*

We suppose $I[h]$ invertible and do formally all the computations with this assumption.
Let H be the Hamiltonian of the Zakharov/Craig-Sulem formulation:

$$H = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2}\int_{\mathbb{R}^d} h\overline{V}\cdot\nabla\psi dX.$$

Using (4.2) an approximated Hamiltonian would be:

$$H_{\text{app}} = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2}\int_{\mathbb{R}^d} hI[h]^{-1}[h\nabla\psi]\cdot\nabla\psi dX. \tag{4.3}$$

Let's differentiate this Hamiltonian in the sense of functional derivative.
Through some computations we first get the derivative of H_{app} with respect to ψ

$$\delta_\psi H = -\nabla\cdot(hI[h]^{-1}[h\nabla\psi]).$$

To compute the derivative of H_{app} with respect to ζ we use the formula

$$I[h]\partial_\zeta(I[h]^{-1})I[h]V = -\partial_\zeta I[h]V. \tag{4.4}$$

From (4.4) and the fact that $I[h]^{-1}$ is a symmetric operator, we can compute the derivative in ζ of the second term of H_{app}. We get

$$\begin{aligned}
\delta_\zeta H =&\zeta + \frac{\epsilon}{2}I[h]^{-1}[h\nabla\psi]\cdot\nabla\psi - \frac{\epsilon}{2}I[h]^{-1}[h\nabla\psi]\cdot I[h]^{-1}[h\nabla\psi]\\
&+\frac{\mu\epsilon}{4}\nabla\cdot(I[h]^{-1}[h\nabla\psi])h^2\mathrm{F}_3[\nabla\cdot(I[h]^{-1}[h\nabla\psi])] + \frac{\mu\epsilon}{4}\nabla\cdot(\mathrm{F}_3[I[h]^{-1}[h\nabla\psi]])h^2\nabla\cdot(I[h]^{-1}[h\nabla\psi])\\
&+\frac{\epsilon}{2}I[h]^{-1}[h\nabla\psi]\cdot\nabla\psi.
\end{aligned}$$

Then we define $\overline{V}_{\text{app}} := (h(Id+\mu T[h]))^{-1}[h\nabla\psi] = I[h]^{-1}[h\nabla\psi]$. This new quantity approximates $\overline{V}$ at order $O(\mu^2\epsilon)$, i.e. $\overline{V}_{\text{app}} = \overline{V} + O(\mu^2\epsilon)$. The two functional derivatives $\delta_\psi H_{\text{app}}$ and $\delta_\zeta H_{\text{app}}$ allow us to obtain the equations of movement in ζ and $\overline{V}_{\text{app}}$.
First the conservation of mass:

$$\begin{aligned}
\partial_t\zeta = \delta_\psi H \iff \partial_t\zeta &= -\nabla\cdot(hI[h]^{-1}[h\nabla\psi])\\
&= -\nabla\cdot(h\overline{V}_{\text{app}}).
\end{aligned}$$

And next the conservation of momentum:

$$\begin{aligned}
&\partial_t\psi = -\delta_\zeta H\\
\iff &\partial_t\psi = -\zeta - \frac{\epsilon}{2}\nabla\psi\cdot I[h]^{-1}[h\nabla\psi] + \frac{\epsilon}{2}I[h]^{-1}[h\nabla\psi]\cdot I[h]^{-1}[h\nabla\psi]\\
&+\frac{\mu\epsilon}{4}\nabla\cdot(I[h]^{-1}[h\nabla\psi])h^2\mathrm{F}_3[\nabla\cdot(I[h]^{-1}[h\nabla\psi])]\\
&+\frac{\mu\epsilon}{4}\nabla\cdot(\mathrm{F}_3[I[h]^{-1}[h\nabla\psi]])h^2\nabla\cdot(I[h]^{-1}[h\nabla\psi]) - \frac{\epsilon}{2}I[h]^{-1}[h\nabla\psi]\cdot\nabla\psi.
\end{aligned}$$

Then applying ∇, we obtain the conservation of momentum

$$\partial_t((Id+\mu T[h])\overline{V}_{\text{app}}) = -\nabla\zeta - \frac{\epsilon}{2}\nabla(|\overline{V}_{\text{app}}|^2) + \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}_{\text{app}}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}_{\text{app}}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}_{\text{app}}]))$$
$$+\frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}_{\text{app}}]\nabla\cdot\overline{V}_{\text{app}}).$$

Thus we get the second full dispersion Green-Naghdi model (1.9) that we recall here

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t((Id+\mu T[h])\overline{V}) + \nabla\zeta & +\frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])) \\ & -\frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases}$$

4.2 Consistency with respect to the water waves equations

Taking the same notations and definitions as in subsection 3.2, we prove here proposition 1.12. For the sake of clarity we recall it here.

Proposition 4.2. *Let F_3 be the Fourier multiplier defined in proposition 1.8. Let $T[h]V := -\frac{1}{6h}(\nabla(h^3\mathrm{F}_3[\nabla\cdot V]) + \nabla(\mathrm{F}_3[h^3\nabla\cdot V]))$. The water waves equations are consistent at order $O(\mu^2\epsilon)$ in the shallow water regime $\mathcal{A}$ with*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t((Id+\mu T[h])\overline{V}) + \nabla\zeta & +\frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])) \\ & -\frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases} \tag{4.5}$$

with $n = 6$.

Remark 4.3. *The first equation of (4.5) is exact. There's nothing to prove for this one. All the work is on the second equation.*

Proof. Let ζ and ψ (so $\overline{V}$ through (1.5)) be the solutions of the water waves system (1.1). Using the notations of definition 1.10 we have in our case

$$\mathcal{N}^3_{(A')}(\zeta,\overline{V}) := \mu\partial_t(T[h]\overline{V}) + \nabla\zeta + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}]))$$
$$-\frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}). \tag{4.6}$$

So we need to prove

$$|\partial_t\overline{V} + \nabla\cdot(h\nabla\psi) + \mu\partial_t(T[h]\overline{V}) + \nabla\zeta + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}]))$$
$$-\frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V})|_{H^s} \leq \mu^2\epsilon N(s+7) \tag{4.7}$$

Step 1: Here we focus on the terms of the second equation of (4.5) which are not differentiated in time and prove that there exists $R_3 \in H^s(\mathbb{R}^d)$ such that $|R_3|_{H^s} \leq N(s+6)$ and

$$\partial_t \nabla\psi + \nabla\zeta + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])) \\ - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = \mu^2\epsilon R_3 \quad (4.8)$$

Taking (ζ,ψ) solutions of the water waves system (1.1) we proved in subsection 3.2 (see (3.9)) that there exists a remainder $R_2 \in H^s(\mathbb{R}^d)$ such that $|R_2|_{H^s} \leq N(s+4)$ and

$$\partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\mu\epsilon}{2}h^2(\Delta\psi)^2 = \mu^2\epsilon R_2. \quad (4.9)$$

Using proposition A.4 for F_3, also taking the gradient of equation (4.9), using proposition 3.3, product estimate A.1 and the boundedness of F_3 we get the existence of $R_3 \in H^s(\mathbb{R}^d)$ such that $|R_3|_{H^s} \leq N(s+5)$ and

$$\partial_t\nabla\psi + \nabla\zeta + \frac{\epsilon}{2}\nabla(|\nabla\psi|^2) - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = \mu^2\epsilon R_3.$$

In order to go further we need a symmetrized version of (2.23).

Proposition 4.4. *Let $s \geq 0$, and $\zeta \in H^{s+5}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$, and $\overline{V}$ be as in (1.5). Let also F_3 be the Fourier multiplier defined in proposition 1.8. The following estimate hold:*

$$|\overline{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^s} \leq \mu^2\epsilon M(s+5)|\nabla\psi|_{H^{s+5}}.$$

Proof. Using (2.23) and quotient estimates A.2 we get

$$\begin{aligned}
&|\overline{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^s} \\
\leq&|\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V})|_{H^s} + |\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3\nabla\cdot\overline{V}) - \frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}] + \mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^s} \\
\leq&\mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}} + \mu M_0|(h^3-1)(\mathrm{F}_3-1)[\nabla\cdot\overline{V}] - (\mathrm{F}_3-1)[(h^3-1)\nabla\cdot\overline{V}]|_{H^{s+1}} \\
\leq&\mu^2\epsilon M(s+4)|\nabla\psi|_{H^{s+4}} + \mu M_0|[\mathrm{F}_3-1; h^3-1]\nabla\cdot\overline{V}|_{H^{s+1}}.
\end{aligned}$$

Moreover using commutator estimates A.6 with $\mathrm{F}_3 - 1$ Fourier multiplier of order 2 and $\mathcal{N}^2(\mathrm{F}_3 - 1) \lesssim \mu$ (see definition A.5). Using also proposition A.4 for F_3 and proposition 3.3, we obtain

$$\begin{aligned}
&|[\mathrm{F}_3-1; h^3-1]\nabla\cdot\overline{V}|_{H^{s+1}} \\
\leq&|[(\mathrm{F}_3-1)\Lambda^{s+1}, h^3-1]\nabla\cdot\overline{V}|_2 + |[\Lambda^{s+1}, h^3-1](\mathrm{F}_3-1)[\nabla\cdot\overline{V}]|_2 \\
\leq&\mu\epsilon|\frac{h^3-1}{\epsilon}|_{H^{s+3}}|\nabla\cdot\overline{V}|_{H^{s+2}} + \mu\epsilon|\frac{h^3-1}{\epsilon}|_{H^{\max(t_0+1,s+1)}}|(\mathrm{F}_3-1)\nabla\cdot\overline{V}|_{H^s} \\
\leq&\mu\epsilon M(s+3)|\overline{V}|_{H^{s+3}} \leq \mu\epsilon M(s+5)|\nabla\psi|_{H^{s+5}}.
\end{aligned}$$

Here t_0 is a real number larger than $d/2$, see remark 1.7. ∎

Having in mind this symmetrized approximation of $\nabla\psi$ by $\overline{V}$ we estimate

$$\begin{aligned}
&|\frac{\epsilon}{2}\nabla(|\nabla\psi|^2)-(\frac{\epsilon}{2}\nabla(|\overline{V}|^2)-\frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])))|\\
\leq&|\frac{\epsilon}{2}\nabla(|\nabla\psi|^2)-\frac{\epsilon}{2}\nabla(|\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2)|_{H^s}\\
+&\frac{\mu^2\epsilon}{2}|\nabla(\frac{1}{6^2h^2}|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2)|_{H^s}\\
:=&I_1+I_2.
\end{aligned}$$

Remark that the second term of the last inequality I_2 is of order $O(\mu^2\epsilon)$. We will get an estimation of it a bit later.

For now let's use proposition 4.4, product estimates A.1, and quotient estimates A.2 to estimate the first term I_1:

$$\begin{aligned}
&|\frac{\epsilon}{2}\nabla(|\nabla\psi|^2)-\frac{\epsilon}{2}\nabla(|\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2)|_{H^s}\\
\leq&\frac{\epsilon}{2}||\nabla\psi|^2-|\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2|_{H^{s+1}}\\
\leq&\frac{\epsilon}{2}|\nabla\psi-(\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}]))|_{H^{s+1}}\\
\times&|\nabla\psi+\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}\\
\leq&\mu^2\epsilon^2M(s+6)|\nabla\psi|_{H^{s+6}}|\nabla\psi+\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}.
\end{aligned}$$

So to finish the second step of this proof we just need to estimate the two quantities allowing us to control I_1 and I_2:

$$\begin{cases}
|\nabla(\frac{1}{6^2h^2}|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2)|_{H^s},\\
|\nabla\psi+\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}.
\end{cases}$$

For the first one we can use quotient estimate A.2 and product estimates A.1 to show that

$$\begin{aligned}
|\nabla(\frac{1}{6^2h^2}|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2)|_{H^s}\leq&M(s+1)||\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2|_{H^{s+1}}\\
\leq&M(s+1)|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|^2_{H^{s+2}}.
\end{aligned}$$

And for the second one we can use proposition 3.3, quotient estimate A.2 and proposition 3.3 to get

$$\begin{aligned}
&|\nabla\psi+\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}\\
\leq&M(s+4)|\nabla\psi|_{H^{s+4}}+\mu M(s+2)|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}.
\end{aligned}$$

Hence for both quantities it only remains to estimate $|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}$. Let's do it.

Remark that the following inequality on F_3 (see proposition 1.8) holds:

$$|F_3(\xi)|=|\frac{3}{\mu|\xi|^2}(\frac{\sqrt{\mu}|\xi|}{\tanh(\sqrt{\mu}|\xi|)}-1)|\leq\frac{1}{1+\frac{\sqrt{\mu}|\xi|}{3}}. \tag{4.10}$$

Using product estimates A.1, (4.10) and proposition 3.3, we get

$$\begin{aligned}
&|\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+2}}\\
\leq&M(s+3)|\mathrm{F}_3[\nabla\cdot\overline{V}]|_{H^{s+3}}+|\mathrm{F}_3[h^3\nabla\cdot\overline{V}])|_{H^{s+3}}\\
\leq&M(s+3)|\overline{V}|_{H^{s+3}}\leq M(s+5)|\nabla\psi|_{H^{s+5}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&|\frac{\epsilon}{2}\nabla(|\nabla\psi|^2)-(\frac{\epsilon}{2}\nabla(|\overline{V}|^2)-\frac{\mu\epsilon}{6}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])))|_{H^s}\\
\leq&\mu^2\epsilon C(M(s+6),|\nabla\psi|_{H^{s+6}}).
\end{aligned}$$

And we proved (4.8).

To prove the consistency of the water waves equations (1.1) with respect to the second full dispersion Green-Naghdi system (4.5) at order $O(\mu^2\epsilon)$ it only remains to focus on the term differentiated in time and show that there exists $k\in\mathbb{N}$ with $k\leq 7$ such that

$$|\partial_t(\nabla\psi-(\overline{V}-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}])))|_{H^s}\leq\mu^2\epsilon N(s+k).$$

What we will prove is in fact

$$\begin{aligned}
&|\partial_t(\nabla\psi-(\overline{V}-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])))|_{H^s}\\
+&|\partial_t(\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}]))|_{H^s}\leq\mu^2\epsilon N(s+7).
\end{aligned}$$

<u>Step 2</u>: We estimate first

$$|\partial_t(\nabla\psi-(\overline{V}-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])))|_{H^s}. \tag{4.11}$$

In that end we denote $\widetilde{u}=\phi-\widetilde{\phi}_{\mathrm{app}}$, where ϕ is defined in definition 1.4 and $\widetilde{\phi}_{\mathrm{app}}$ in (2.9).
<u>Step 2.1</u>: Here we find a control on $\widetilde{u}$ and prove

$$||\Lambda^s\nabla^{\mu}\partial_t\widetilde{u}||_2\leq\mu^2\epsilon N(s+5). \tag{4.12}$$

By definition of ϕ and $\widetilde{\phi}_{\mathrm{app}}$ (see definition 1.4) we know that $\widetilde{u}$ solves an elliptic problem:

$$\begin{cases}h\nabla^{\mu}\cdot P(\Sigma_t)\nabla^{\mu}(\widetilde{u})=-\mu^2\epsilon R,\\ \widetilde{u}|_{z=0}=0,\quad\partial_z\widetilde{u}|_{z=-1}=0,\end{cases} \tag{4.13}$$

where

$$\begin{aligned}
R=\frac{1}{\mu^2\epsilon}\Big[&\mu(h^2-1)(\mathrm{F}_0-1)\Delta\psi+\mu\Delta((h^2-1)(\mathrm{F}_0-1)\psi)\\
&+\mu\epsilon A(\nabla,\partial_z)(\mathrm{F}_0-1)\psi+\mu\epsilon A(\nabla,\partial_z)(h^2-1)(\mathrm{F}_0-1)\psi\Big].
\end{aligned} \tag{4.14}$$

In add using proposition A.4 for F_0, and product estimates A.1, we have the following estimation of the remainder R:

$$||\Lambda^sR||_2\leq M(s+2)|\nabla\psi|_{H^{s+3}}. \tag{4.15}$$

Moreover we can differentiate in time the elliptic equation in (4.13) as follow

$$\begin{aligned}
&\partial_t(\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu(\widetilde{u})) = -\mu^2\epsilon\partial_t R \\
\iff &\nabla^\mu \cdot \partial_t(P(\Sigma_t))\nabla^\mu\widetilde{u} + \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\partial_t\widetilde{u} = -\mu^2\epsilon\partial_t R.
\end{aligned}$$

where here $R = (4.14)/h$ (I use the same notation for both remainders, thanks to quotient estimates A.2, the previous estimation holds).
It invites us to denote $v := \partial_t\widetilde{u}$ and decompose it into $v := v_1 + v_2$ where v_1 satisfy one elliptic problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu v_1 = -\mu^2\epsilon\partial_t R, \\ v_1|_{z=0} = 0, \quad \partial_z v_1|_{z=-1} = 0, \end{cases} \tag{4.16}$$

and v_2 satisfy, for $g := \partial_t P(\Sigma_t)\nabla^\mu\widetilde{u}$ (see definition 1.4 for the expression of $P(\Sigma_t)$), a second elliptic problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu v_2 = -\nabla^\mu \cdot g, \\ v_2|_{z=0} = 0, \quad v_2|_{z=-1} = -e_z \cdot g|_{z=-1}. \end{cases} \tag{4.17}$$

Thanks to the lemma A.3 we have a control on v_1 given by

$$||\Lambda^s\nabla^\mu v_1||_2 \leq \mu^2\epsilon M(s+1)||\Lambda^s\partial_t R||_2. \tag{4.18}$$

And having an explicit form of R we can easily find an estimation of $\partial_t R$ using quotient estimates A.2 and product estimates A.1:

$$||\Lambda^s\partial_t R||_2 \leq C(M(s+2), |\partial_t\zeta|_{H^{s+3}})|(\nabla\psi, \partial_t\nabla\psi)|_{H^{s+3}}. \tag{4.19}$$

Then using the water waves equations (1.1) we obtain estimates on the partial derivatives in time of ζ and $\nabla\psi$.

Lemma 4.5. *Let $s \geq 0$, and $\zeta \in H^{s+4}(\mathbb{R}^d)$. Let $\psi \in \dot{H}^{s+4}(\mathbb{R}^d)$.*
The two estimations hold:

$$\begin{cases} |\partial_t\zeta|_{H^{s+2}} \leq N(s+4), \\ |\partial_t\nabla\psi|_{H^{s+2}} \leq N(s+4). \end{cases}$$

Proof. For both estimations the tools are the same. We use the water waves equations (1.1), product estimates A.1, quotient estimates A.2, and proposition A.7.
Let's first prove the inequality on $\partial_t\zeta$. Denoting $s' = s+2$ and $\mathfrak{P}$ the Fourier multiplier defined as $\mathfrak{P} := \frac{|D|}{(1+\sqrt{\mu}|D|)^{1/2}}$ (I recall that $|D|$ means $|\xi|$ in Fourier space) we have

$$|\partial_t\zeta|_{H^{s'}} = |\frac{1}{\mu}\mathcal{G}^\mu[\epsilon\zeta]\psi|_{H^{s'}} \leq N(s'+2).$$

On the other hand for $\partial_t \nabla\psi$ we get

$$
\begin{aligned}
|\partial_t\psi|_{H^{s'}} \leq & |\zeta|_{H^{s'}} + ||\nabla\psi|^2|_{H^{s'}} + |\frac{(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu\psi + \epsilon\sqrt{\mu}\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2\mu|\nabla\zeta|^2}|_{H^{s'}} \\
\leq & |\zeta|_{H^{s'}} + |\nabla\psi|^2_{H^{s'}} + C(\mu_{\max}, \frac{1}{h_{\min}}, ||\nabla\zeta|^2|_{H^{s'}})|(\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu\psi + \epsilon\sqrt{\mu}\nabla\zeta\cdot\nabla\psi)^2|_{H^{s'}} \\
\leq & |\zeta|_{H^{s'}} + |\nabla\psi|^2_{H^{s'}} + C(\mu_{\max}, \frac{1}{h_{\min}}, |\zeta|_{H^{s'+1}}, |\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu\psi|_{H^{s'}}, |\nabla\zeta\cdot\nabla\psi|_{H^{s'}}) \\
\leq & N(s'+1).
\end{aligned}
$$

Thus

$$
|\partial_t\nabla\psi|_{H^{s'}} \leq |\partial_t\psi|_{H^{s'+1}} \leq N(s'+2).
$$

$\square$

Using the previous lemma, mixed with (4.18) and (4.19) we get the control on v_1:

$$
||\Lambda^s\nabla^\mu v_1||_2 \leq \mu^2\epsilon N(s+5).
$$

To get the control on v_2 we use a classical result for solutions of elliptic problems such as (4.17). It is the lemma A.8. Using also the fact that $g := \partial_t P(\Sigma_t)\nabla^\mu\widetilde{u}$ (see definition 1.4 for an expression of $P(\Sigma_t)$) it gives:

$$
\begin{aligned}
||\Lambda^s\nabla^\mu v_2||_2 \leq & M(s+1)||\Lambda^s g||_2 \\
||\Lambda^s h\nabla^\mu\cdot P(\Sigma_t)\nabla^\mu\widetilde{\phi}_{\text{app}}||_2 \leq & \mu^2\epsilon M(s+2)|\nabla\psi|_{H^{s+3}}.
\end{aligned}
$$

But $\widetilde{u}$ solves an elliptic problem for which we can use lemma A.3. Using also lemma 4.5 and (4.15) we get

$$
||\Lambda^s\nabla^\mu v_2||_2 \leq C(M(s+1), |\partial_t\zeta|_{H^{s+3}})\mu^2\epsilon M(s+1)||\Lambda^s R||_2 \leq N(s+5) \tag{4.20}
$$

At the end, joining together the control on v_1 (4.18) and the one on v_2 (4.20) we get (4.12).
Step 2.2: We can now give the control on (4.11). To do that we will first use (4.12) to prove the following lemma.

Lemma 4.6. *Let $s \geq 0$, and $\zeta \in H^{s+6}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+7}(\mathbb{R}^d)$, and $\overline{V}$ be as in (1.5). Let also F_1 and F_2 be the Fourier multipliers defined in proposition 1.8. The following estimates hold:*

$$
\begin{cases}
|\partial_t(\overline{V} - \mathrm{F}_1\nabla\psi)|_{H^s} \leq \mu\epsilon N(s+4), \\
|\partial_t(\overline{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_2[\Delta\psi]))|_{H^s} \leq \mu^2\epsilon N(s+6).
\end{cases} \tag{4.21}
$$

Proof. Let's first prove the second inequality. If we denote $\mu^2\epsilon R = \overline{V} - \widetilde{\overline{V}}_{\text{app}}$, having in mind $\overline{V}$ and $\widetilde{\overline{V}}_{\text{app}}$ written as in (2.20), and the fact that through computations $\widetilde{\overline{V}}_{\text{app}} = \nabla\psi +$

$\frac{1}{h}\nabla(h^3(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}-1)\psi))$ (see (2.12)), we have the following equality

$$|\partial_t(\overline{V}-\nabla\psi-\frac{1}{h}\nabla(h^3(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}-1)\psi))|_{H^s}=\mu^2\epsilon|\partial_t R|_{H^s}$$
$$=|\int_{-1}^{0}(\nabla\partial_t\widetilde{u}-\frac{1}{h}(z\nabla h+\epsilon\nabla\zeta)\partial_z\partial_t\widetilde{u})dz-\int_{-1}^{0}\partial_t(\frac{z\nabla h+\epsilon\nabla\zeta}{h})\partial_z\widetilde{u}dz|_{H^s}.$$

So Poincaré's inequality (page 40 of [16]) mixed up with product and quotient estimates A.1, A.2, leads us to

$$\mu^2\epsilon|\partial_t R|_{H^s}\leq||\Lambda^{s+1}\nabla^{\mu}\partial_t\widetilde{u}||_2\leq\mu^2\epsilon N(s+6).$$

Moreover we defined F_2 as

$$(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}-1)\psi=-\frac{\mu}{3}|D|^2\mathrm{F}_2\psi.$$

Hence we come up with the estimation

$$|\partial_t(\overline{V}-\nabla\psi-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_2[\Delta\psi]))|_{H^s}\leq\mu^2\epsilon N(s+6). \tag{4.22}$$

To prove the first inequality of (4.21) we just need to do the same reasoning from step 2.1 to this point but taking instead $\widetilde{u}=\phi-\phi_0$, where ϕ_0 is defined in (2.5). $\square$

Having in mind this proposition, we decompose (4.11) in two parts:

$$|\partial_t(\overline{V}-\nabla\psi-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\mathrm{F}_1^{-1}[\nabla\cdot\overline{V}]))|_{H^s}$$
$$\leq|\partial_t(\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot(\overline{V}-\mathrm{F}_1\nabla\psi)]))|_{H^s}+|\partial_t(\overline{V}-\nabla\psi-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_2[\Delta\psi]))|_{H^s}.$$

The bound on the second term is given by the second inequality of proposition 4.6.
The first term can be decomposed in three parts:

$$|\partial_t(\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot(\overline{V}-\mathrm{F}_1\nabla\psi)]))|_{H^s}$$
$$\leq|\frac{\mu\epsilon}{3h^2}\partial_t\zeta\nabla(h^3\mathrm{F}_3[\nabla\cdot(\overline{V}-\mathrm{F}_1\nabla\psi)])|_{H^s}+|\frac{\mu\epsilon}{3h}\nabla(3h^2\partial_t\zeta\mathrm{F}_3[\nabla\cdot(\overline{V}-\mathrm{F}_1\nabla\psi)])|_{H^s}$$
$$+|\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\partial_t(\overline{V}-\mathrm{F}_1\nabla\psi)])|_{H^s}.$$

Each of this terms are bounded using quotient estimates A.2, product estimates A.1, proposition 3.3 and the first inequality of proposition 4.6.
At the end we get what we wanted to prove in this step 2:

$$|\partial_t(\overline{V}-\nabla\psi-\frac{\mu}{3h}\nabla(h^3\mathrm{F}_2\mathrm{F}_1^{-1}[\nabla\cdot\overline{V}]))|_{H^s}\leq\mu^2\epsilon N(s+6).$$

Step 3: The last step is to bound

$$|\partial_t(\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}]))|_{H_s} \tag{4.23}$$

The main key is commutator estimates A.6. We decompose (4.23) into three parts:

$$\begin{aligned}
&|\partial_t(\frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])-\frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}]+\mathrm{F}_3[h^3\nabla\cdot\overline{V}]))|_{H_s}\\
\leq&\mu\epsilon|\frac{\partial_t\zeta}{h^2}[\nabla(h^3-1)(\mathrm{F}_3-1)[\nabla\cdot\overline{V}]-(\mathrm{F}_3-1)[h^3-1\nabla\cdot\overline{V}])]|_{H^s}\\
+&\mu\epsilon|\frac{1}{h}\nabla(\partial_t\zeta h^2(\mathrm{F}_3-1)[\nabla\cdot\overline{V}]-(\mathrm{F}_3-1)[\partial_t\zeta h^2\nabla\cdot\overline{V}])|_{H^s}\\
+&\mu\epsilon|\frac{1}{h}\nabla(\frac{h^3-1}{\epsilon}(\mathrm{F}_3-1)[\nabla\cdot\partial_t\overline{V}]-(\mathrm{F}_3-1)[\frac{h^3-1}{\epsilon}\nabla\cdot\partial_t\overline{V}])|_{H^s}\\
:=&T_1+T_2+T_3.
\end{aligned}$$

Using quotient estimates A.2, product estimates A.1 and lemma 4.5 we get

$$\begin{cases}
T_1\leq\mu\epsilon M(s+4)|[\mathrm{F}_3-1,h^3-1]\nabla\cdot\overline{V}|_{H^{s+1}},\\
T_2\leq\mu\epsilon M_0|[\mathrm{F}_3-1,\partial_t\zeta h^2]\nabla\cdot\overline{V}|_{H^{s+1}},\\
T_3\leq\mu\epsilon M_0|[\mathrm{F}_3-1,\frac{h^3-1}{\epsilon}]\nabla\cdot\partial_t\overline{V}|_{H^{s+1}}.
\end{cases}$$

But using the fact that for any $s\geq0$ the operator Λ^s and F_3-1 commute we have

$$\begin{cases}
|[\mathrm{F}_3-1,h^3-1]\nabla\cdot\overline{V}|_{H^{s+1}} & \leq|[(\mathrm{F}_3-1)\Lambda^{s+1},h^3-1]\nabla\cdot\overline{V}|_2\\
& +|[\Lambda^{s+1},h^3-1](\mathrm{F}_3-1)[\nabla\cdot\overline{V}]|_2,\\
|[\mathrm{F}_3-1,\partial_t\zeta h^2]\nabla\cdot\overline{V}|_{H^{s+1}} & \leq|[(\mathrm{F}_3-1)\Lambda^{s+1},\partial_t\zeta h^2]\nabla\cdot\overline{V}|_2\\
& +|[\Lambda^{s+1},\partial_t\zeta h^2](\mathrm{F}_3-1)[\nabla\cdot\overline{V}]|_2,\\
|[\mathrm{F}_3-1,\frac{h^3-1}{\epsilon}]\nabla\cdot\partial_t\overline{V}|_{H^{s+1}} & \leq|[(\mathrm{F}_3-1)\Lambda^{s+1},\frac{h^3-1}{\epsilon}]\nabla\cdot\partial_t\overline{V}|_2\\
& +|[\Lambda^{s+1},\frac{h^3-1}{\epsilon}](\mathrm{F}_3-1)[\nabla\cdot\partial_t\overline{V}]|_2.
\end{cases}$$

So using commutator estimates A.6 with F_3-1 of order 2 and $\mathcal{N}^2(\mathrm{F}_3-1)\lesssim\mu$ (see definition A.5 for the definition of $\mathcal{N}^2(\mathrm{F}_3-1)$), product estimates A.1, lemma 4.5, proposition 3.3 and the first inequality in proposition 4.6 we obtain

$$\begin{cases}
T_1\leq\mu\epsilon M(s+4)(\mu|h^3-1|_{H^{s+3}}|\nabla\cdot\overline{V}|_{H^{s+2}}+|h^3-1|_{H^{\max(t_0+1,s+1)}}|(\mathrm{F}_3-1)[\nabla\cdot\overline{V}]|_{H^s}),\\
T_2\leq\mu\epsilon M(s+4)(\mu|\partial_t\zeta h^2|_{H^{s+3}}|\nabla\cdot\overline{V}|_{H^{s+2}}+|\partial_t\zeta h^2|_{H^{\max(t_0+1,s+1)}}|(\mathrm{F}_3-1)[\nabla\cdot\overline{V}]|_{H^s}),\\
T_3\leq\mu\epsilon M_0(\mu|\frac{h^3-1}{\epsilon}|_{H^{s+3}}|\nabla\cdot\partial_t\overline{V}|_{H^{s+2}}+|\nabla\frac{h^3-1}{\epsilon}|_{H^{\max(t_0+1,s+1)}}|(\mathrm{F}_3-1)[\nabla\cdot\partial_t\overline{V}]|_{H^s}).
\end{cases}$$

Here t_0 is a real number larger than t_0, see remark 1.7.
Hence using product estimates A.1, lemma 4.5, proposition 3.3 and the first inequality in proposition 4.6 we end up with

$$\begin{cases}
T_1\leq\mu^2\epsilon M(s+5)|\nabla\psi|_{H^{s+5}}\leq N(s+5),\\
T_2\leq\mu^2\epsilon M(s+5)|\nabla\psi|_{H^{s+5}}\leq N(s+5),\\
T_3\leq\mu^2\epsilon N(s+7).
\end{cases}$$

It finishes the step 3 and the proof of proposition 4.2, i.e. we proved the consistency of the water waves at order $O(\mu^2\epsilon)$ in the shallow water regime with the full dispersion Green-Naghdi system (4.5) (with $n=6$). □

Remark 4.7. *The $n = 6$ regularity asked for deriving (4.5) appeared only in the last step of the proof when we wanted to pass from a non-symmetric system to a symmetric one. Only $n = 5$ is asked for the solutions of the water waves equations (1.1) to prove the consistency at order $O(\mu^2\epsilon)$ with respect to the system*

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t(\overline{V} - \frac{\mu}{3h}\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])) + \nabla\zeta \quad + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{3}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(h^3\mathrm{F}_3[\nabla\cdot\overline{V}])) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases} \tag{4.24}$$

However system (4.24) does not have a Hamiltonian formulation.

5 Consistency of other full dispersion models appearing in the literature

5.1 Full dispersion Green-Naghdi system

In (4.1) we chose a kind of symmetrization which were naturally induced by an analogy with the one appearing in the first full dispersion Green-Naghdi system (3.5) we derived in this paper. Another kind of symmetrization appears in the literature for a full dispersion Green-Naghdi system, see [10]. They introduced a two-layer full dispersion Green-Naghdi system with surface tension in order to be able to study high-frequency Kevin-Helmholtz instabilities. In dimension $d = 2$, without surface tension, their system for a one-layer fluid is

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t(\overline{V} - \frac{\mu}{3h}\nabla(\sqrt{\mathrm{F}_3}h^3\sqrt{\mathrm{F}_3}[\nabla\cdot\overline{V}])) + \nabla\zeta \quad + \frac{\epsilon}{2}\nabla(|\overline{V}|^2) - \frac{\mu\epsilon}{3}\nabla(\frac{1}{h}\overline{V}\cdot\nabla(\sqrt{\mathrm{F}_3}h^3\sqrt{\mathrm{F}_3}[\nabla\cdot\overline{V}])) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{\mu\epsilon}{2}\nabla(h^2\mathrm{F}_3[\nabla\cdot\overline{V}]\nabla\cdot\overline{V}) = 0. \end{cases} \tag{5.1}$$

Proposition 5.1. *Let F_3 be the Fourier multiplier defined in proposition 1.8. The water waves equations are consistent at order $O(\mu^2\epsilon)$ in the shallow water regime $\mathcal{A}$ with the system (5.1).*

Proof. I will only do a formal proof. The rigorous one would use the same tools as the proof of proposition 1.12 (see subsection 4.2).

It is easy to see that

$$2\sqrt{\mathrm{F}_3}[h^3\sqrt{\mathrm{F}_3}[V]] = h^3\mathrm{F}_3[V] + \mathrm{F}_3[h^3V] + O(\mu). \tag{5.2}$$

It only remains to use proposition 1.12 together with product estimates A.1, quotient estimates A.2 and the estimates on F_3 of proposition A.4 to get the result. $\square$

The difference between (1.9) and (5.1) in the mathematical point of view is of importance. Indeed the operator

$$h(I_d - \frac{\mu}{3h}\nabla(\sqrt{\mathrm{F}_3}h^3\sqrt{\mathrm{F}_3}[\nabla\cdot\circ]))$$

is invertible because one can decompose it in the following way:

$$hI_d + \mu(\frac{h}{\sqrt{3}}\sqrt{\mathrm{F}_3}\nabla\cdot)^* h(\frac{h}{\sqrt{3}}\sqrt{\mathrm{F}_3}\nabla\cdot), \tag{5.3}$$

giving the positiveness of the operator and the coercivity of the bilinear form associated with. The Lax-Milgram theorem conclude [10].

However we don't have a similar decomposition as (5.3) for the operator $h(I_d - \frac{\mu}{6h}\nabla(h^3\mathrm{F}_3[\nabla\cdot \circ] + \mathrm{F}_3[h^3\nabla\cdot\circ]))$. To ensure the invertibility it seems that we need an additionnal hypothesis on the smallness of $\epsilon\zeta$.

5.2 Full dispersion Boussinesq systems

In the literature several Whitham-Boussinesq systems (or full dispersion Boussinesq systems) are introduced, see [2, 5, 6, 7, 8, 9, 13, 18, 20]. We pay a particular attention to the one studied in [9] for which they proved a local well-posedness result in dimension $d = 2$ and a global well-posedness result for small data in dimension $d = 1$. We recall it for $d = 2$

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi + \epsilon\mathrm{F}_1\nabla\cdot(\zeta\mathrm{F}_1\nabla\psi) = 0, \\ \partial_t\nabla\psi + \nabla\zeta + \frac{\epsilon}{2}\nabla(\mathrm{F}_1|\nabla\psi|)^2 = 0. \end{cases} \tag{5.4}$$

Proposition 5.2. *The water waves equations (1.1) are consistent at order $O(\mu\epsilon)$ in the shallow water regime $\mathcal{A}$ with the following system*

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi + \epsilon\mathrm{F}_1\nabla\cdot(\zeta\mathrm{F}_1\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}(\mathrm{F}_1|\nabla\psi|)^2 = 0. \end{cases} \tag{5.5}$$

Proof. Again I will only do a formal proof. The rigorous one would use the same tools as the one of proposition 1.11.

To do so let's use the fact that we proved the consistency of the water waves equations at order $O(\mu^2\epsilon)$ with system (1.8), we discard all the terms of order $O(\mu\epsilon)$ of the latter. We obtain a formal consistency of the water waves system at order $O(\mu\epsilon)$ with the system

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi + \epsilon\nabla\cdot(\zeta\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 = 0, \end{cases}$$

(We used the identity $\Delta\psi + \frac{\mu}{3}\Delta\mathrm{F}_2\Delta\psi = \mathrm{F}_1\Delta\psi$).
It only remains to have in mind product estimates A.1, and the estimates on F_1 of proposition A.4 to see that taking (ζ,ψ) solutions of the water waves system (1.1), one has

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi + \epsilon\mathrm{F}_1\nabla\cdot(\zeta\mathrm{F}_1\nabla\psi) = O(\mu\epsilon), \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}(\mathrm{F}_1|\nabla\psi|)^2 = O(\mu\epsilon). \end{cases}$$

This conclude the formal demonstration. ☐

Remark 5.3.

- *We could easily adapt definition 1.10 to match with system (5.4). And say that the water waves equations (1.1) are consistent with this Whitham-Boussinesq system at order $O(\mu\epsilon)$.*

- *Using the same tools we could prove the consistency at order $O(\mu\epsilon)$ of the water waves equations (1.1) with the other Whitham-Boussinesq systems of the literature.*

- *As the proof of proposition 5.2 makes clear, the water waves equations are consistent at order $O(\mu\epsilon)$ with every systems*

$$\begin{cases} \partial_t\zeta + \mathrm{F}_1\Delta\psi + \epsilon\mathrm{G}_1\nabla\cdot(\zeta\mathrm{G}_2\nabla\psi) = 0, \\ \partial_t\nabla\psi + \nabla\zeta + \frac{\epsilon}{2}\mathrm{G}_3\nabla(|\mathrm{G}_4\nabla\psi|^2) = 0. \end{cases}$$

  where the Fourier multipliers G_1, G_2, G_3 and G_4 are approximations of identity of the type $\mathrm{G}_i = 1 + O(\mu)$. However, the well-posedness properties of the system will depend on the characteristics of the Fourier multipliers and in particular the order of their symbol (definition A.5). We postpone the study of the well-posedness of such systems to a future work.

A Technical tools

Proposition A.1. *(Product estimates)*

1. *Let $t_0 > d/2$, $s \geq -t_0$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^d), g \in H^s(\mathbb{R}^d)$. Then $fg \in H^s(\mathbb{R}^d)$ and*

$$|fg|_{H^s} \lesssim |f|_{H^{\max(t_0,s)}}|g|_{H^s}$$

2. *Let $s_1, s_2 \in \mathbb{R}$ be such that $s_1 + s_2 \geq 0$. Then for all $s \leq s_j$ $(j = 1, 2)$ and $s < s_1 + s_2 - d/2$, and all $f \in H^{s_1}(\mathbb{R}^d), g \in H^{s_2}(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and*

$$|fg|_{H^s} \lesssim |f|_{H^{s_1}}|g|_{H^{s_2}}$$

Proof. See Appendix B.1 in [16]. □

Proposition A.2. *(Quotient estimates) Let $t_0 > d/2, s \geq -t_0$ and $c_0 > 0$. Also let $f \in H^s(\mathbb{R}^d)$ and $g \in H^s \cap H^{t_0}(\mathbb{R}^d)$ be such that for all $X \in \mathbb{R}^d$, one has $1 + g(X) \geq c_0$. Then $\frac{f}{1+g}$ belongs to $H^s(\mathbb{R}^d)$ and*

$$|\frac{f}{1+g}|_{H^s} \leq C(\frac{1}{c_0}, |g|_{H^{\max(t_0,s)}})|f|_{H^s}$$

Proof. See Appendix B.1 in [16]. □

Lemma A.3. *Let $P(\Sigma_t)$ be as in definition 1.4. Let $h \in L_z^2 H_X^s((-1,0)\times\mathbb{R}^d)$ and $u \in L_z^2 H_X^{s+1} \cap H_z^1 H_X^s((-1,0)\times\mathbb{R}^d)$ $(s \geq 0)$ solve the boundary value problem*

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu u = h, \\ u|_{z=0} = 0, \quad \partial_z u|_{z=-1} = 0. \end{cases}$$

Then one has

$$||\Lambda^s \nabla^\mu u||_2 \leq M(s+1)||\Lambda^s h||_2.$$

Proof. See lemma 3.43 in [16]. □

Proposition A.4. *Let $s \geq 0$, $z \in (-1,0)$ and ψ such that $\nabla\psi \in H^{s+1}(\mathbb{R}^d)$, then we have the following estimations*

$$\begin{cases} |(\frac{1-\mathrm{F}_0}{\mu|D|^2} + \frac{z^2}{2} + z)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(1-(z+1)^2\mathrm{F}_0 + z^2 + 2z)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \\ |(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(\frac{z+1}{\sqrt{\mu}|D|}\frac{\sinh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)} - (z+1)^2)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \\ |(\mathrm{F}_2 - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(\mathrm{F}_3 - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \end{cases}$$

An estimation of order $O(\mu^2)$ for $\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ will also be useful. If $\nabla\psi \in H^{s+3}$ then

$$|(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1 + \frac{1}{3}\mu|D|^2)\psi|_{H^s} \lesssim \mu^2|\nabla\psi|_{H^{s+3}}$$

Proof. All the proves are similar and the main key is the Taylor-Lagrange formula. All these estimations except the last one are on the same form where G is a smooth function on $(0,+\infty)$, continuous in 0.

$$|(G(\sqrt{\mu}|D|) - G(0))\psi|_{H^s} \leq \mu|\nabla\psi|_{H^{s+1}} \iff |(G(\sqrt{\mu}|\xi|) - G(0))\hat{\psi}|_{H^s} \leq \mu|\nabla\psi|_{H^{s+1}}$$

For the last one it would be

$$\begin{aligned} &|(G(\sqrt{\mu}|D|) - G(0) - \mu|D|^2 G''(0))\psi|_{H^s} \leq \mu^2|\nabla\psi|_{H^{s+3}} \\ \iff &|(G(\sqrt{\mu}|\xi|) - G(0) - \mu|\xi|^2 G''(0))\hat{\psi}|_2 \leq \mu^2|\nabla\psi|_{H^{s+3}} \end{aligned}$$

If we succeed in proving that the second derivative of G is bounded in $[0,+\infty)$, and that $G'(0) = 0$ then we can use the Taylor-Lagrange formula stating that for all $x \in [0,+\infty)$ there exists $\theta \in [0,1]$ such that

$$G(x) - G(0) = \frac{x^2}{2}G''(\theta x)$$

then the boundedness of G'' allow us to write

$$|G(x) - G(0)| \leq |G''|_\infty x^2$$

Replacing x by $\sqrt{\mu}|\xi|$ in the last inequality we obtain

$$|(G(\sqrt{\mu}|\xi|) - G(0))\hat{\psi}|_2 \leq \mu||\xi|^2\hat{\psi}|_2 \leq \mu|\nabla\psi|_{H^1}$$

Thus it is sufficient to prove the boundedness in $C^2([0,+\infty))$ of G and the fact that $G'(0)=0$ for the following functions:

$$\begin{cases} G_1(x) = \frac{1}{x^2}(1-\frac{\cosh((z+1)x)}{\cosh(x)}), \quad G_2(x) = 1-(z+1)^2\frac{\cosh((z+1)x)}{\cosh(x)} \\ G_3(x) = \frac{\tanh(x)}{x}, \quad G_4(x) = \frac{z+1}{x}\frac{\sinh((z+1)x)}{\cosh(x)} \\ G_5(x) = \frac{3}{x\tanh(x)} - \frac{3}{x^2}, \quad G_6(x) = \frac{3}{\mu}(1-\frac{\tanh(x)}{x}) \end{cases}$$

The end of the proof is let to the reader. □

Definition A.5. *We say that a Fourier multiplier* $\mathrm{F}(D)$ *is of order* s $(s\in\mathbb{R})$ *and write* $\mathrm{F}\in\mathcal{S}^s$ *if* $\xi\in\mathbb{R}^d\mapsto F(\xi)\in\mathbb{C}$ *is smooth and satisfies*

$$\forall\xi\in\mathbb{R}^d, \forall\beta\in\mathbb{N}^d, \quad \sup_{\xi\in\mathbb{R}^d}\langle\xi\rangle^{|\beta|-s}|\partial^\beta F(\xi)|<\infty.$$

We also introduce the seminorm

$$\mathcal{N}^s(F) = \sup_{\beta\in\mathbb{N}^d, |\beta\leq 2+d+\lceil\frac{d}{2}\rceil}\sup_{\xi\in\mathbb{R}^d}\langle\xi\rangle^{|\beta|-s}|\partial^\beta F(\xi)|.$$

Proposition A.6. *Let* $t_0>d/2$, $s\geq 0$ *and* $F\in\mathcal{S}^s$. *If* $f\in H^s\cap H^{t_0+1}$ *then, for all* $g\in H^{s-1}$,

$$|[F(D),f]g|_2 \leq \mathcal{N}^s(F)|f|_{H^{\max(t_0+1,s)}}|g|_{H^{s-1}}.$$

Proof. See Appendix B.2 in [16] for a proof of this proposition. □

Proposition A.7. *Let* $s\geq 2$. *Let* $\zeta\in H^{s+2}(\mathbb{R}^d)$ *be such that* (1.2) *is satisfied and* $\psi\in\dot{H}^{s+1}(\mathbb{R}^d)$. *Then one has*

$$\begin{cases} |\frac{1}{\mu}\mathcal{G}^\mu\psi|_{H^s} \leq M(s+2)|\nabla\psi|_{H^{s+1}}, \\ |\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu\psi|_{H^s} \leq \mu^{1/4}M(s+1)|\nabla\psi|_{H^s}. \end{cases}$$

Proof. This is a direct consequence of theorem 3.15 in [16]. □

Lemma A.8. *Let* $t_0>d/2$, $s\geq 0$. *Let* $P(\Sigma_t)$ *be as in definition 1.4. Let* $g(X,z)$ *be a function on* $\mathcal{S}:=\mathbb{R}^d\times(-1,0)$ *sufficiently regular such that its trace at* $z=-1$ *makes sense. Let* u *solve the boundary value problem*

$$\begin{cases} \nabla^\mu\cdot P(\Sigma_t)\nabla^\mu u = -\nabla^\mu\cdot g, \\ u|_{z=0}=0, \quad v_2|_{z=-1} = -e_z\cdot g|_{z=-1}. \end{cases}$$

Then one has

$$||\Lambda^s\nabla^\mu u||_2 \leq M(s+1)||\Lambda^s g||_2.$$

Proof. See lemma 2.38 in [16]. □

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