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VECTOR POTENTIALS WITH MIXED BOUNDARY CONDITIONS. APPLICATION TO THE STOKES PROBLEM WITH PRESSURE AND NAVIER-TYPE BOUNDARY CONDITIONS[*]

CHÉRIF AMROUCHE[†] AND IMANE BOUSSETOUAN[‡]

Abstract. In a three-dimensional bounded possibly multiply connected domain, we prove the existence, uniqueness and regularity of some vector potentials, associated with a divergence-free function and satisfying mixed boundary conditions. For such a construction, the fundamental tool is the characterization of the kernel which is related to the topology of the domain. We also give several estimates of vector fields via the operators div and **curl** when mixing tangential and normal components on the boundary. Furthermore, we establish some Inf-Sup conditions that are crucial in the L^p-theory proofs. Finally, we apply the obtained results to solve the Stokes problem with a pressure condition on some part of the boundary and Navier-type boundary condition on the remaining part, where weak and strong solutions are considered.

Key words. Vector potentials, mixed boundary conditions, L^p theory, Stokes equations, Navier-type boundary condition.

AMS subject classifications. 35J05, 35J20, 35J25, 76D03, 76D07

1. Introduction.

A relevant problem in fluid mechanics is the appropriate choice of the boundary conditions type. Various physical phenomena, like lubrication or air and blood flows, require suitable mixed boundary conditions to be prescribed on the boundary [16, 20]. Problems involving such conditions have been widely discussed in the literature, from theoretical and numerical point of views : let us mention here only few selected references [10, 11, 13, 17, 18, 22]. Nevertheless, at our knowledge the theory of elliptic problems with mixed boundary conditions has not been fully investigated in complex 3D geometries.

Unless stated otherwise, we assume that Ω is a $\mathcal{C}^{1,1}$ domain in $\mathbb{R}^3$, possibly multiply connected. The boundary of the flow domain is decomposed of an inner and outer wall as $\Gamma = \Gamma_D \cup \Gamma_N$. Furthermore, we suppose that Γ_D and Γ_N are not empty and for the sake of simplification, $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

We do not assume that Γ_D and Γ_N are connected and we denote by Γ_D^ℓ, $0 \leq \ell \leq L_D$, the connected components of Γ_D and similarly by Γ_N^ℓ, $0 \leq \ell \leq L_N$ the connected components of Γ_N. Also, $\partial\Sigma$ stands for the union of the boundaries Σ_j of an admissible set of cuts $1 \leq j \leq J$ such that each surface Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$. The boundary of each Σ_j is contained in Γ and the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$. The open set $\Omega^o = \Omega \backslash \cup_{j=1}^J \Sigma_j$ is a simply-connected domain. More details will be given in Section 2.

It is known that a divergence-free vector field is the **curl** of another vector field called vector potential when adequate boundary conditions are imposed at any given part of the boundary. Furthermore, an amount that reflects the topological structure of the domain needs to be added as it plays an important role in the uniqueness results and in the well-posedness of the corresponding problems. The theory of vector potentials is very useful in the Maxwell's theory, in other words in electromagnetism.

Vector potentials on arbitrary Lipschitz domains have been treated by Mitrea et al [29]. Then, in the seminal work of Amrouche et al [2], the authors gave a fairly

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complete picture of the theory of vector potentials in non-smooth domains, in the Hilbert settings. These results were extended to the L^p-theory in [5]. In [33], an important estimate has been established via div and **curl** when $1 < p < \infty$ if and only if the first Betti number I vanishes, *i.e* Ω is simply connected in the case of $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ or if and only if the second Betti number J vanishes, *i.e* Ω has only one connected component of the boundary in the case of $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ , given by

$$\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \leq C\Big(\|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}\Big). \tag{1.1}$$

In [5], the authors generalized the inequality (1.1) to the case where Ω has arbitrary Betti numbers and for vector fields with vanishing tangential components or vanishing normal components on the boundary.

The main objective of this paper consists on a contribution to this topic that is focused on extending the previous results when mixing boundary conditions on the normal and tangential components of the vector potential where Ω has arbitrary Betti numbers, in the Hilbert and non-Hilbert cases. The methods of proofs are mainly based on the characterization of the kernel which is related to the geometrical properties of the domain. Since the boundary of the domain is decomposed into two parts, the dimension of the kernel depends on where the union of the boundaries of the admissible set of cuts $\partial\Sigma$ lies. Throughout this paper, we will deal separately with the case where $\partial\Sigma$ is included in Γ_N and the case where it is included in Γ_D because their treatments are entirely different in character. Our goal is also to improve the regularity of the obtained vector potentials to the L^p-theory for any $1 < p < +\infty$. For the general case $p \neq 2$, the standard arguments will not allow us to get the existence of the vector potentials. To overcome this obstacle, by use of the classical Helmholtz decomposition, we prove some important Inf-Sup conditions of the type :

$$\inf_{\substack{\boldsymbol{\varphi} \in \widetilde{\mathbf{V}}_0^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \ \sup_{\substack{\boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^{p}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\left| \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \right|}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta, \tag{1.2}$$

where $\beta > 0$ and the space $\widetilde{\mathbf{V}}_0^p(\Omega)$ will be defined later. It turns out that these conditions are the key point when solving various elliptic problems as the following one: find $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{\xi} = \mathbf{curl}\, \boldsymbol{v} \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\xi} \cdot \boldsymbol{n} = 0, \quad (\mathbf{curl}\, \boldsymbol{\xi} - \boldsymbol{v}) \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N, \\ \langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_N^\ell} = 0, \quad 1 \leq \ell \leq L_N, \end{cases} \tag{1.3}$$

where $\partial\Sigma \subset \Gamma_N$ and $\boldsymbol{v} \in \mathbf{L}^p(\Omega)$.

As an application, we consider stationary motions of viscous incompressible fluid in Ω governed by the Stokes system

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \qquad \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 \qquad\qquad \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where $\boldsymbol{u}$ is the velocity field, π the pressure and $\boldsymbol{f}$ denotes the external force. Here and in what follows, the unit outer normal to the boundary is denoted by $\boldsymbol{n}$ and the unit tangent vector by $\boldsymbol{\tau}$. We respectively define the normal and the tangential velocities by $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_n \boldsymbol{n}$.

Stokes and Navier-Stokes systems are often studied with the no-slip Dirichlet condition. However, this idea, although successful for some kind of flows from a mathematical point of view, is not well justified from a physical point of view. In fact, it has previously been shown that the conventional no-slip boundary condition predicts a singularity at a moving contact line and that forces us to take into account some form of slip [19]. In the last decades, several mathematical papers have been conducted in relation to the non standard boundary conditions involving some friction (see [15, 21, 31]). The L^p-theory for the Stokes problem with various types of boundary conditions can be found for instance in [28].

The Navier boundary conditions were proposed by Navier [30], these conditions assume that the tangential component of the strain tensor is proportional to the tangential component of the fluid velocity on the boundary, referred to as "stress-free" or "slip" boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{and} \quad 2\mu[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]_\tau+\alpha\boldsymbol{u}_\tau=\boldsymbol{0}, \tag{1.5}$$

where μ is the fluid viscosity, $\mathbb{D}(\boldsymbol{u})=1/2(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ is the strain rate tensor associated to the velocity field and α is a friction coefficient, which measures the tendency of the fluid to slip on the boundary. These conditions appear in the study of climate modeling and oceanic dynamics [27]. They are particularly used in the large eddy simulation for turbulent flows. Since the first work [32] treating the Stokes problem with Dirichlet boundary condition on some part of the boundary and (1.5) with ($\alpha=0$) on the other part, where the authors proved an existence result of strong (local) solutions, the interest in this kind of conditions has been increasing over the years (see for instance [25, 26]). In [7], the author has established the existence and uniqueness of solutions to the Stokes problem involving Navier conditions in the L^2-settings. This work was completed by Amrouche et al in [3] where the L^p-theory of such problems was developed. Recently in [1], the authors discussed the behavior of the weak and strong solutions with respect to the friction coefficient α assumed to be a function.

Let us consider any point P on Γ and choose any neighborhood W of P in Γ, small enough to allow the existence of C^2 curves on W. The lengths s_1, s_2 along each family of curves are a possible set of coordinates in W. The unit tangent vectors to each family of curves are denoted by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, with this notation we have $\boldsymbol{v}=\boldsymbol{v}_{\boldsymbol{\tau}}+(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ and $\boldsymbol{v}_{\boldsymbol{\tau}}=\sum_{k=1}^2 v_k\boldsymbol{\tau}$, where $v_k=\boldsymbol{v}\cdot\boldsymbol{\tau}_k$. Then we can prove that

$$2\mu[\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}=-\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}-2\Lambda\,\boldsymbol{u},$$

where Λ is the operator $\Lambda\,\boldsymbol{u}=\sum_{j=1}^2\left(\frac{\partial\boldsymbol{n}}{\partial\boldsymbol{s}_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\right)\boldsymbol{\tau}_j$.

One can observe that in the case of flat boundary and when $\alpha=0$, the Navier boundary condition (1.5) with a right hand side equal to $\boldsymbol{h}$ which is a given tangential vector field, may be replaced by the condition

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}, \tag{1.6}$$

which is called Navier-type boundary condition. In [4], the authors have shown the existence and uniqueness of weak, strong and very weak solutions to the Stokes problem subjected to Navier-type boundary conditions. We assume that (1.6) is imposed on Γ_D. Unfortunately, one cannot prescribe only the value of the pressure on the boundary, since such a problem is known to be ill-posed. We consider that the pressure values are prescribed, together with the condition of non-tangential flow on the remaining part of the boundary Γ_N

$$\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} \quad \text{and} \quad \pi=\pi_0 \quad \text{on} \quad \Gamma_N. \tag{1.7}$$

These conditions are used in Poiseuille flows, blood vessels or pipelines [6]. Recently in [14], the authors have considered the Stokes problem with (1.7) on a part of the boundary with a numerical approach applied in hemodynamics modeling of the cerebral venous network. Numerical analysis of the discrete corresponding problem has been performed in [13]. Stokes and Navier-Stokes systems including both conditions (1.6) and (1.7) were firstly treated in [18] where the authors assume that the boundary is divided into three parts and Dirichlet boundary condition is imposed on the third part. They proved the existence and uniqueness of a variational solution and they showed that it is a solution of the original problem in the Hilbert setting. Better regularity properties have been successfully demonstrated by Bernard. Indeed, if the given pressure on a part of the boundary is more regular then the variational solution satisfies $\Delta \boldsymbol{u} \in \mathbf{L}^2(\Omega)$ and the corresponding boundary conditions [11], then a $\mathbf{W}^{m,r}(\Omega)$ regularity is obtained for any $m \in \mathbb{N}$, $m \geq 2$, $r \geq 2$ [12].

In this paper, we follow another strategy based on the fact that the pressure can independently be obtained of the velocity field and is solution of an elliptic problem with Dirichlet boundary condition on a part of the boundary and Neumann boundary condition on the remaining part. Indeed, by setting $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi$ in the Stokes problem, we get a system of equations which only includes the velocity field.

$$-\Delta \boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega,$$

with the boundary condition (1.6) on Γ_D and $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ_N. Note that variational formulations have solutions that can be given by vector potentials of the velocity field of the Stokes problem [9]. We use the obtained Inf-Sup condition (1.2) to prove the existence of the velocity field in $\mathbf{W}^{1,p}(\Omega)$.

Let us outline the structure of this paper. In Section 2, we introduce the mathematical framework, we illustrate the geometry of the domain and we review some preliminary results.

In Section 3, we establish some estimates for vector fields dealing with mixed normal and tangential boundary conditions for any $1 < p < \infty$. Then, we characterize the kernels when $\partial\Sigma$ is included in Γ_D and then in Γ_D. Furthermore, we obtain in both cases some Fridriech's inequalities for any function $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ with $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_N by virtue of Peetre-Tartar Theorem.

Section 4 is devoted to the existence and uniqueness of vector potentials with divergence-free and satisfying vanishing tangential components on a part of the boundary and vanishing normal components on the other part, in the L^2-theory. We also point out the case of less standard but useful vector potentials that have non vanishing divergence and where Dirichlet boundary condition is imposed on a part of the boundary. In order to extend these results to the L^p-theory, we prove two Inf-Sup conditions when $\partial\Sigma$ is included either in Γ_N or in Γ_D that are necessary in the solvability of some elliptic problems as the system (1.3) and also in the last section.

Finally in Section 5, we focus the attention on the existence and uniqueness of the solution of Stokes problem with Navier-type boundary condition (1.6) on a part of the boundary and a pressure condition (1.7) on the other part and we give some regularity assertions to that solution. We restrict ourselves to the case where $\partial\Sigma$ lies in Γ_D in this section, the other case can be solved in a similar way.

The proofs of the Stokes problem are of great help in the analysis of the Navier-Stokes equations when mixing different boundary conditions, which is the main purpose of our forthcoming paper.

2. Functional spaces and notations. In this section, we give some basic notations, we introduce the functional spaces that are used and we describe the geometry of the domain in which we are working.

We follow the convention that C is a constant that may vary from expression to expression. We denote by X' the dual space of the space X and by $\langle\cdot,\cdot\rangle_{X,X'}$ the duality product between X and X'. Vector fields are designated by bold letters and their corresponding spaces by bold capital characters.

We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e* the differences of the traces for any $1 \leq j \leq J$. For any function $q \in W^{1,p}(\Omega^o)$, ∇q is the gradient of q in the sense of distributions in $\mathcal{D}'(\Omega^o)$ which belongs to $\mathbf{L}^p(\Omega^o)$ and it can be extended to $\mathbf{L}^p(\Omega)$. Therefore, to distinguish this extension from the gradient of q, we denote it by $\widetilde{\mathbf{grad}}\, q$.

Let us introduce for any $1 < p < \infty$ the following functional framework

$$\mathbf{H}^p(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \mathbf{L}^p(\Omega)\}$$
$$\mathbf{H}^p(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \mathbf{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in L^p(\Omega)\}$$

and we denote by $\mathbf{X}^p(\Omega)$ the space

$$\mathbf{X}^p(\Omega) = \mathbf{H}^p(\mathbf{curl},\Omega) \cap \mathbf{H}^p(\mathrm{div},\Omega)$$

provided with the norm

$$\|\boldsymbol{v}\|_{\mathbf{X}^p(\Omega)} = \Big(\|\boldsymbol{v}\|^p_{\mathbf{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^p_{\mathbf{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

We define also the following subspaces

$$\mathbf{X}^p_0(\Omega) = \{\boldsymbol{v} \in \mathbf{X}^p(\Omega),\quad \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D,\quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma_N\},$$
$$\widetilde{\mathbf{X}}^p_0(\Omega) = \{\boldsymbol{v} \in \mathbf{X}^p(\Omega),\quad \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N,\quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma_D\}$$

and the kernels

$$\mathbf{K}^p_0(\Omega) = \{\boldsymbol{v} \in \mathbf{X}^p_0(\Omega),\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega\},$$
$$\widetilde{\mathbf{K}}^p_0(\Omega) = \Big\{\boldsymbol{v} \in \widetilde{\mathbf{X}}^p_0(\Omega),\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega\Big\}.$$

Let us shed some light on the geometry of the domain here, we emphasize that Ω contains simply-connected obstacles denoted by $\Omega^0_D,\dots,\Omega^{L_D}_D$ and $\Omega^0_N,\dots,\Omega^{L_N}_N$, the non simply-connected ones are denoted by $\Omega^1_\Sigma,\dots,\Omega^J_\Sigma$. It is important to identify the components of each part of the boundary in the following cases as we will be confronted with in the whole paper:

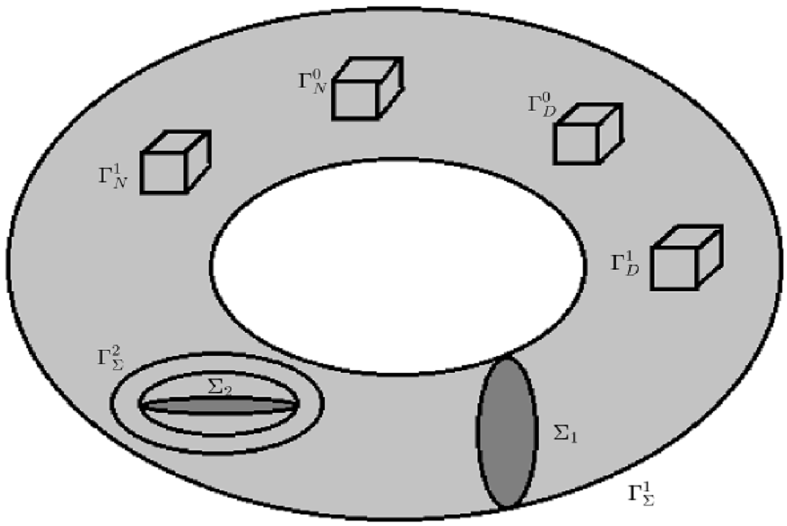

FIG. 1. *Lipschitz flow domain*

Case 1. When $\partial\Sigma \subset \Gamma_N$:

$$\Gamma_D = \bigcup_{\ell=0}^{L_D} \Gamma_D^\ell \quad \text{and} \quad \Gamma_N = \left(\bigcup_{\ell=0}^{L_N} \Gamma_N^\ell\right) \cup \left(\bigcup_{j=1}^{J} \Gamma_\Sigma^j\right),$$

where Γ_D^ℓ is the boundary of Ω_D^ℓ, Γ_N^ℓ is the boundary of Ω_N^ℓ and Γ_Σ^j is the boundary of Ω_Σ^j.

Case 2. When $\partial\Sigma \subset \Gamma_D$:

$$\Gamma_N = \bigcup_{\ell=0}^{L_N} \Gamma_N^\ell, \quad \Gamma_D = \left(\bigcup_{\ell=0}^{L_D} \Gamma_D^\ell\right) \cup \left(\bigcup_{j=1}^{J} \Gamma_\Sigma^j\right).$$

As shown in figure 1, if $\partial\Sigma \subset \Gamma_N$, this means that $\Gamma_N = \Gamma_\Sigma^1 \cup \Gamma_\Sigma^2 \cup \Gamma_N^0 \cup \Gamma_N^1$ and $\Gamma_D = \Gamma_D^0 \cup \Gamma_D^1$. In the other side, if $\partial\Sigma \subset \Gamma_D$, we interchange the notation in figure 1 such that $\Gamma_D = \Gamma_\Sigma^1 \cup \Gamma_\Sigma^2 \cup \Gamma_D^0 \cup \Gamma_D^1$ and $\Gamma_N = \Gamma_N^0 \cup \Gamma_N^1$.

Remark 2.1. We underline that in the case where $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ form an edge, we lose the H^2 regularity in some singularity points and for this reason, we avoid to work in this case and we consider only the simplified one $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$.

It is worth recalling the obtained results in [5] where Γ_i, $1 \le i \le I$ represent the connected components of the boundary Γ and Σ_j, $1 \le j \le J$, are the connected open surfaces called "cuts". The authors have established the following Friedriech's inequality concerning tangential vector fields $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$, $1 < p < \infty$ with $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ

$$(2.1) \qquad \|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \le C \left(\|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{i=1}^{I} \left| \int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n} \right| \right).$$

Similarly for normal vector fields, we have for any $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ

$$\|\nabla \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} \leq C\left(\|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{j=1}^{J}\left|\int_{\Sigma_j}\boldsymbol{u}\cdot\boldsymbol{n}\right|\right). \tag{2.2}$$

The above estimates are proved by use of some integral representations, Calderón Zygmund inequalities and the traces properties [5]. Note that as soon as $\boldsymbol{u}$ belongs to $\mathbf{H}^p(\mathbf{curl},\Omega)$, the tangential boundary component $\boldsymbol{u}\times\boldsymbol{n}$ is defined in $\mathbf{W}^{-1/p,p}(\Gamma)$ and in the case where $\boldsymbol{u}$ belongs to $\mathbf{H}^p(\operatorname{div},\Omega)$, the normal boundary component $\boldsymbol{u}\cdot\boldsymbol{n}$ is also defined in $W^{-1/p,p}(\Gamma)$. Moreover, we have the Green's formulas

$$\forall \boldsymbol{\varphi} \in \mathbf{W}^{1,p'}(\Omega),\ <\boldsymbol{u}\times\boldsymbol{n},\boldsymbol{\varphi}>_\Gamma = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{\varphi}\,dx, \tag{2.3}$$

where $<\cdot,\cdot>_\Gamma$ denotes the duality product between $\mathbf{W}^{-1/p,p}(\Gamma)$ and $\mathbf{W}^{1/p,p'}(\Gamma)$ and

$$\forall \varphi \in W^{1,p'}(\Omega),\ <\boldsymbol{u}\cdot\boldsymbol{n},\varphi>_\Gamma = \int_\Omega \boldsymbol{u}\cdot\nabla\varphi\,dx + \int_\Omega (\operatorname{div}\boldsymbol{u})\varphi\,dx, \tag{2.4}$$

where $<\cdot,\cdot>_\Gamma$ denotes the duality product between $W^{-1/p,p}(\Gamma)$ and $W^{1/p,p'}(\Gamma)$. In the case where the boundary conditions $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ or $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ are replaced by inhomogeneous ones, the authors have showed in [5] the following estimates

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{u}\cdot\boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)}\Big)$$

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{u}\times\boldsymbol{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma)}\Big).$$

3. Harmonic vector fields and Fridriech's inequalities.

An important tool to study, in the next section, the existence and the uniqueness of vector potentials, is the characterization of some kernels of harmonic vector fields. We establish also some Friedriech's inequalities which are essential to solve some elliptic problems. We give finally a new Stokes formula in a general pseudo-Lipschitz domain.

We assume that for any point x on the boundary $\partial\Omega$ there exists a system of orthogonal co-ordinates y_j, a hypercube U containing x ($U = \Pi_{i=1}^d]-a_i,a_i[$) and a function Φ of class $\mathcal{C}^{1,1}$ such that

$$\Omega\cap U = \{(y',y_d)\in U|\quad y_d < \Phi(y')\},$$
$$\partial\Omega\cap U = \{(y',y_d)\in U|\quad y_d = \Phi(y')\}.$$

The next lemma concerns the estimate of vector fields in the Hilbert case when tangential and normal boundary conditions are both applied *ie.* $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_N. In what follows, we assume that Ω is also connected.

LEMMA 3.1. *Assume that $\boldsymbol{u}\in\mathbf{H}^1(\Omega)$ with $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_N, then the following estimate is satisfied*

$$\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \leq C\left(\|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\right), \tag{3.1}$$

where C is a constant depending only on Ω.

Proof. To prove the estimate (3.1), we recall Theorem 3.1.1.2 in [24] which involves the curvature tensor of the boundary denoted by β and defined as

$$\beta(\boldsymbol{\zeta}, \boldsymbol{\kappa}) = \sum_{i,j=1}^{d-1} \frac{\partial^2 \Phi}{\partial y_i y_j}(0) \boldsymbol{\zeta}_i \boldsymbol{\kappa}_j,$$

and $\operatorname{Tr}\beta$ denotes the trace of this operator. We have the following relation

$$\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} = \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|^2_{L^2(\Omega)} - \int_{\Gamma_D} (\operatorname{Tr}\beta)(\boldsymbol{u}\cdot\boldsymbol{n})^2\, ds$$
$$- \int_{\Gamma_N} \beta(\boldsymbol{u}\times\boldsymbol{n}, \boldsymbol{u}\times\boldsymbol{n})\, ds.$$

For the boundary terms, we have

$$\left|\int_{\Gamma_N} \beta(\boldsymbol{u}\times\boldsymbol{n}, \boldsymbol{u}\times\boldsymbol{n})\, ds\right| \leq C \int_{\Gamma_N} |\boldsymbol{u}|^2\, ds \leq \frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C'\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.$$

With a similar inequality for the term on Γ_D, we get

$$\left|\int_{\Gamma_D} (\operatorname{Tr}\beta)(\boldsymbol{u}\cdot\boldsymbol{n})^2\, ds\right| \leq \frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + C''\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.$$

We deduce that (3.1) holds. ∎

THEOREM 3.2. *Let $\boldsymbol{u} \in \mathbf{X}^p(\Omega)$ such that $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_N, then $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and satisfies*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\Omega)\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)}, \tag{3.2}$$

where $C(\Omega)$ is a constant depending on p and Ω. The same result holds for the space $\widetilde{\mathbf{X}}^p(\Omega)$.

Proof. Let θ be a function defined in $C_0^\infty(\mathbb{R}^d)$, $0 \leq \theta \leq 1$ and satisfying $\theta = 1$ at the neighborhood of Γ_D and $\theta = 0$ at the neighborhood of Γ_N, we set $\eta = 1 - \theta$. As soon as $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_N, we deduce that $\theta\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ and $\eta\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ. Then, from Theorem 3.2 of [5] for $\theta\boldsymbol{u}$ we deduce that

$$\|\theta\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_1(\Omega)\|\theta\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)} \leq C_2(\Omega)\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)},$$

where $C_1(\Omega)$ and $C_2(\Omega)$ depend only on Ω and p. By using Theorem 3.4 of [5] for $\eta\boldsymbol{u}$, we get

$$\|\eta\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C_3(\Omega)\|\eta\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)} \leq C_4(\Omega)\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)}$$

where $C_3(\Omega)$ and $C_4(\Omega)$ depend only on Ω and p. Since $\boldsymbol{u} = \theta\boldsymbol{u} + \eta\boldsymbol{u}$, then by combining the obtained estimates, we obtain

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq \|\theta\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\eta\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C(\Omega)\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)},$$

where $C(\Omega) = C_2(\Omega) + C_4(\Omega)$. ∎

In order to avoid extra difficulties, we start by checking some results for the Laplace operator

$$\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_N. \tag{3.3}$$

We know that for a given $f \in L^2(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$. It is clear that the solution u belongs to $H^2(\Omega)$ because of the assumptions on Γ_D and Γ_N. We will give in the following corollary a brief proof to get this regularity.

COROLLARY 3.3. *For any $f \in L^2(\Omega)$, the solution $u \in H^1(\Omega)$ of the Problem (3.3) belongs to $H^2(\Omega)$ and satisfies the estimate*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{3.4}$$

Proof. We set $\boldsymbol{z} = \nabla u$, then $\boldsymbol{z} \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \boldsymbol{z} \in L^2(\Omega)$, $\mathbf{curl}\, \boldsymbol{z} \in \mathbf{L}^2(\Omega)$ with $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D, and $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ_N. We infer from Theorem 3.2 with $p = 2$ that $\boldsymbol{z} \in \mathbf{H}^1(\Omega)$. Since $\boldsymbol{z} = \nabla u \in \mathbf{H}^1(\Omega)$, therefore $u \in H^2(\Omega)$ and satisfies the estimate (3.4). ∎

In the case where the boundary conditions $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_N are replaced by inhomogeneous ones, the estimate (3.2) is generalized in the following corollary.

COROLLARY 3.4. *Let $\boldsymbol{u} \in \mathbf{X}^p(\Omega)$ such that $\boldsymbol{u} \times \boldsymbol{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ and $\boldsymbol{u} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma_N)$. Then $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and we have the following estimate*

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\Big(\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)} + \|\boldsymbol{u} \times \boldsymbol{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} + \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma_N)}\Big). \tag{3.5}$$

Proof. Arguing similar as in the proof of Theorem 3.2, the first property and the estimate (3.5) are easily deduced, thanks to Theorem 3.5 and Corollary 5.2 of [5]. ∎

More generally, we derive the following corollary in the same way.

COROLLARY 3.5. *Let $m \in \mathbb{N}^*$, Ω of class $\mathcal{C}^{m,1}$ and $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{u} \in W^{m-1,p}(\Omega)$ and $\mathbf{curl}\, \boldsymbol{u} \in \mathbf{W}^{m-1,p}(\Omega)$ such that $\boldsymbol{u} \times \boldsymbol{n} \in \mathbf{W}^{m-1/p,p}(\Gamma_D)$ and $\boldsymbol{u} \cdot \boldsymbol{n} \in W^{m-1/p,p}(\Gamma_N)$. Then $\boldsymbol{u} \in \mathbf{W}^{m,p}(\Omega)$ and we have the following estimate*

$$\begin{aligned}\|\boldsymbol{u}\|_{\mathbf{W}^{m,p}(\Omega)} \leq C\Big(&\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{W^{m-1,p}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\mathbf{W}^{m-1,p}(\Omega)} \\ &+ \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{W^{m-1/p,p}(\Gamma_N)} + \|\boldsymbol{u} \times \boldsymbol{n}\|_{\mathbf{W}^{m-1/p,p}(\Gamma_D)}\Big).\end{aligned}$$

The following lemma will serve as an argument in the forthcoming analysis.

LEMMA 3.6. *Assume that Ω is Lipschitz. Let $\partial\Sigma \subset \Gamma_N$, $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Omega)$ and $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ_N. Then, there exists a sequence $(\boldsymbol{\psi}_k)_k$ of functions in $\mathcal{D}(\widetilde{\Omega})$, where $\widetilde{\Omega} = \Omega \cup \Big(\cup_{\ell=0}^{L_D} \overline{\Omega}_D^\ell\Big)$ and $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^2(\operatorname{div}, \widetilde{\Omega})$ satisfying*

$$\boldsymbol{\psi}_k \to \widetilde{\boldsymbol{\psi}} \quad \text{in} \quad \mathbf{H}^2(\operatorname{div}, \widetilde{\Omega}), \quad \boldsymbol{\psi}_{k|\Omega} \to \boldsymbol{\psi} \quad \text{in} \quad \mathbf{H}^2(\operatorname{div}, \Omega).$$

Proof. For any $0 \leq \ell \leq L_D$, let us consider $\chi_\ell \in H^1(\Omega)$ solution of the problem

$$\begin{cases} \Delta\chi_\ell = c_\ell & \text{in} \quad \Omega_D^\ell, \\ \partial_{\boldsymbol{n}}\chi_\ell = \boldsymbol{\psi} \cdot \boldsymbol{n} & \text{on} \quad \Gamma_D^\ell. \end{cases}$$

where $c_\ell = \frac{1}{|\Omega|} \langle \boldsymbol{\psi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_D^\ell}$. We set $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$ in Ω and $\widetilde{\boldsymbol{\psi}} = \nabla\chi_\ell$ in Ω_ℓ. Let $\boldsymbol{\varphi} \in \mathcal{D}(\widetilde{\Omega})$, so we have

$$\left\langle \operatorname{div} \widetilde{\boldsymbol{\psi}}, \varphi \right\rangle = -\int_{\widetilde{\Omega}} \widetilde{\boldsymbol{\psi}} \cdot \nabla\varphi \, dx = -\int_{\Omega} \boldsymbol{\psi} \cdot \nabla\varphi \, dx - \sum_{\ell=0}^{L_D} \int_{\Omega_D^\ell} \nabla\chi_\ell \cdot \nabla\varphi \, dx$$

$$= \int_{\Omega} (\operatorname{div} \boldsymbol{\psi})\varphi \, dx - \sum_{\ell=0}^{L_D} \int_{\Gamma_D^\ell} (\boldsymbol{\psi} \cdot \boldsymbol{n})\varphi \, ds + \sum_{\ell=0}^{L_D} \int_{\Omega_\ell} \varphi\Delta\chi_\ell \, dx$$

$$+ \sum_{\ell=0}^{L_D} \int_{\Gamma_D^\ell} \varphi\partial_{\boldsymbol{n}}\chi_\ell \, ds = \int_{\Omega} (\operatorname{div} \boldsymbol{\psi})\varphi \, dx + \sum_{\ell=0}^{L_D} c_\ell \int_{\Omega_D^\ell} \varphi \, dx.$$

Thus, we obtain

$$\left| \left\langle \operatorname{div} \widetilde{\boldsymbol{\psi}}, \varphi \right\rangle \right| \leq \|\operatorname{div} \boldsymbol{\psi}\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + \sum_{\ell=0}^{L_D} |\Omega_D^\ell|^{\frac{1}{2}} |c_\ell| \|\varphi\|_{L^2(\Omega_\ell)}.$$

But

$$|c_\ell| \leq \frac{C(\Omega)}{|\Omega|} \left(\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \boldsymbol{\psi}\|_{L^2(\Omega)} \right),$$

which implies that $\widetilde{\boldsymbol{\psi}} \in \mathbf{H}^2_0(\operatorname{div}, \widetilde{\Omega})$. Therefore, there exists $\boldsymbol{\psi}_k \in \mathcal{D}(\widetilde{\Omega})$ such that

$$\boldsymbol{\psi}_k \to \widetilde{\boldsymbol{\psi}} \quad \text{in} \quad \mathbf{H}^2(\operatorname{div}, \widetilde{\Omega}) \quad \text{and} \quad \boldsymbol{\psi}_{k_{|\Omega}} \to \boldsymbol{\psi} \quad \text{in} \quad \mathbf{H}^2(\operatorname{div}, \Omega),$$

which is the required result. ∎

The next lemma is an extension of the Green's formula (2.4) in the case where $p = 2$ and is the equivalent version of Lemma 3.10 [2] when dealing with mixed boundary conditions. The proof below is more detailed and the dual space $\left[H^{1/2}(\Sigma_j)\right]'$ in [2] is everywhere replaced by the dual space $\left[H^{1/2}_{00}(\Sigma_j)\right]'$ which is more correct.

Lemma 3.7. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. If $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Omega)$, then the restriction of $\boldsymbol{\psi} \cdot \boldsymbol{n}$ to any Σ_j belongs to $\left[H^{1/2}_{00}(\Sigma_j)\right]'$ for any $1 \leq j \leq J$ and for any $\chi \in H^1(\Omega^o)$ with $\chi = 0$ on Γ, we have*

$$\sum_{j=1}^{J} \langle \boldsymbol{\psi} \cdot \boldsymbol{n}, [\chi]_j \rangle_{\Sigma_j} = \int_{\Omega^o} \boldsymbol{\psi} \cdot \nabla\chi \, dx + \int_{\Omega^o} \chi \operatorname{div} \boldsymbol{\psi} \, dx, \tag{3.6}$$

where

$$H^{1/2}_{00}(\Sigma_j) = \left\{ \mu \in H^{1/2}(\Sigma_j),\ \widetilde{\mu} \in H^{1/2}(\mathcal{M}_j) \right\}.$$

Moreover, if $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ_N then (3.6) holds for any $\chi \in H^1(\Omega^o)$ and $\chi = 0$ on Γ_D.

Proof. i) Let us consider the case where $\mu \in H^{1/2}_{00}(\Sigma_1)$, we extend the cut Σ_1 by Σ_1' which allows us to divide Ω into two parts Ω_1 and Ω_1' *i.e* $\Omega = \Omega_1 \cup \Sigma_1 \cup \Omega_1' \cup \Sigma_1'$. We set now $\Omega_1^o = \Omega_1 \cup \Omega_1' \cup \left(\cup_{j=2}^{J} \Omega_{\Sigma_j}\right)$. In other words, Ω_1^o is the open set $\Omega \backslash \left(\Sigma_1 \cup \Sigma_1'\right)$, to which we add the obstacles $\Omega_{\Sigma_2}, \ldots, \Omega_{\Sigma_J}$.

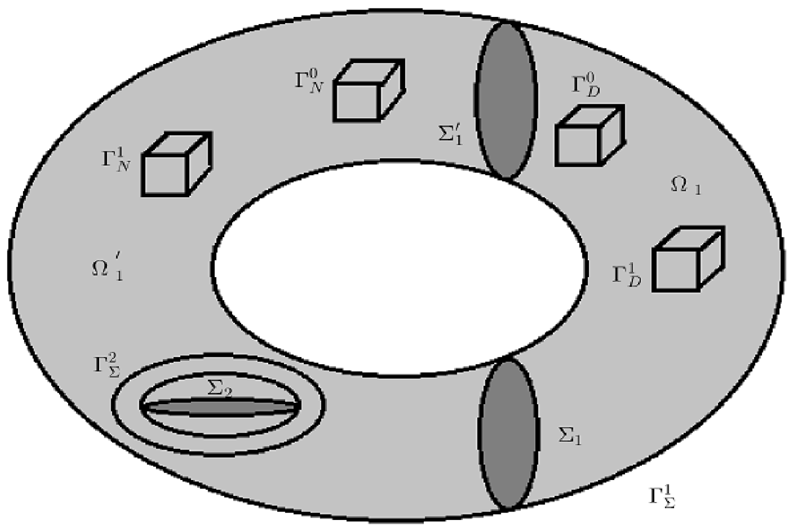

FIG. 2. $J = 2$

Now, we know that there exists $\varphi_1 \in H^1(\Omega_1)$ satisfying

$$\Delta\varphi_1 = 0 \text{ in } \Omega_1, \quad \varphi_1 = 0 \text{ on } \partial\Omega_1\backslash\Sigma_1, \quad \varphi_1 = \frac{\mu}{2} \text{ on } \Sigma_1,$$

and

$$\|\varphi_1\|_{H^1(\Omega_1)} \leq C\|\mu\|_{H^{1/2}_{00}(\Sigma_1)}.$$

In the same way, there exists $\varphi_1' \in H^1(\Omega_1'')$ where $\Omega_1'' = \Omega_1' \cup \left(\cup_{j=2}^J \Omega_{\Sigma_j}\right)$ satisfying

$$\Delta\varphi_1' = 0 \text{ in } \Omega_1'', \quad \varphi_1' = 0 \text{ on } \partial\Omega_1''\backslash\Sigma_1, \quad \varphi_1' = -\frac{\mu}{2} \text{ on } \Sigma_1,$$

and

$$\|\varphi_1'\|_{H^1(\Omega_1'')} \leq C\|\mu\|_{H^{1/2}_{00}(\Sigma_1)}.$$

Finally, we define the function φ as

$$\varphi = \begin{cases} \varphi_1 & \text{in} \quad \Omega_1 \\ \varphi_1' & \text{in} \quad \Omega_1'' \\ 0 & \text{on} \quad \Sigma_1'. \end{cases}$$

Furthermore, it satisfies

$$\varphi \in H^1(\Omega_1^o \cup \Sigma_1'), \quad [\varphi]_1 = \mu$$
$$\varphi = 0 \quad \text{on} \quad \Gamma_D \cup \Gamma_N, \quad [\varphi]_j = 0 \quad j = 2, \dots, J$$

and the estimate

$$\|\varphi\|_{H^1(\Omega_1^o \cup \Sigma_1')} \leq C\|\mu\|_{H^{1/2}_{00}(\Sigma_1)}.$$

We take now $\chi = \varphi|_{\Omega^o}$, then

$$\chi \in H^1(\Omega^o), \quad [\chi]_1 = \mu, \quad [\chi]_j = 0, \quad j = 2, \dots, J$$
$$\chi = 0 \quad \text{on} \quad \partial\Omega$$

$$\|\chi\|_{H^1(\Omega^o)} \le C\|\mu\|_{H^{1/2}_{00}(\Sigma_1)}.$$

We proceed similarly when $\mu \in H^{1/2}_{00}(\Sigma_j)$ with an adapted extension of the cut Σ_j for any $2 \le j \le J$.

ii) Now, let $\boldsymbol{\psi} \in \mathcal{D}(\overline{\Omega})$, then Green's formula gives for any $1 \le j \le J$

$$\langle \boldsymbol{\psi} \cdot \boldsymbol{n}, \mu \rangle_{\Sigma_j} = \int_{\Omega^o} \boldsymbol{\psi} \cdot \nabla \chi \, dx + \int_{\Omega^o} \chi \operatorname{div} \boldsymbol{\psi} \, dx. \tag{3.7}$$

Moreover, we have

$$|\langle \boldsymbol{\psi} \cdot \boldsymbol{n}, \mu \rangle_{\Sigma_j}| \le C\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\mathrm{div},\Omega)}\|\mu\|_{H^{1/2}_{00}(\Sigma_j)}.$$

As a consequence, $\boldsymbol{\psi} \cdot \boldsymbol{n} \in [H^{1/2}_{00}(\Sigma_j)]'$and

$$\|\boldsymbol{\psi} \cdot \boldsymbol{n}\|_{[H^{1/2}_{00}(\Sigma_j)]'} \le C\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\mathrm{div},\Omega)}.$$

Because of the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2(\mathrm{div}, \Omega)$, the last inequality holds for any function $\boldsymbol{\psi}$ in $\mathbf{H}^2(\mathrm{div}, \Omega)$. Finally, by using an adapted partition of unity and the Green's formula (3.7), we establish the relation (3.6).

Finally, we assume that $\boldsymbol{\psi} \in \mathbf{H}^2(\mathrm{div}, \Omega)$ and $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ_N, then it is easily checked that the Green's formula (3.6) is valid by means of Lemma 3.6. ◻

In order to ensure the uniqueness of the first vector potential, we are interested here in the characterization of the kernel $\mathbf{K}^2_0(\Omega)$ in the case where $\partial\Sigma$ is included in Γ_N.

Proposition 3.8. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. Then the dimension of the space $\mathbf{K}^2_0(\Omega)$ is equal to $L_D \times J$ and it is spanned by the functions $\widetilde{\mathbf{grad}}\, q^\ell_j$, for $1 \le j \le J$ and $1 \le \ell \le L_D$, where each q^ℓ_j is the unique solution in $\mathbf{H}^1(\Omega^o)$ of the problem*

$$\begin{cases} -\Delta q^\ell_j = 0 \quad \text{in} \quad \Omega^o, \\ \dfrac{\partial q^\ell_j}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_N, \\ q^\ell_j|_{\Gamma^0_D} = 0, \quad q^\ell_j|_{\Gamma^m_D} = \text{const}, \quad 1 \le m \le L_D, \\ \left[q^\ell_j\right]_k = \text{const} \quad \text{and} \quad \left[\dfrac{\partial q^\ell_j}{\partial \boldsymbol{n}}\right]_k = 0, \quad 1 \le k \le J, \\ \left\langle \dfrac{\partial q^\ell_j}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \le k \le J, \\ \left\langle \dfrac{\partial q^\ell_j}{\partial \boldsymbol{n}}, \quad 1 \right\rangle_{\Gamma^0_D} = -1 \quad \text{and} \quad \left\langle \dfrac{\partial q^\ell_j}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma^m_D} = \delta_{\ell m}, \quad 1 \le m \le L_D. \end{cases} \tag{3.8}$$

Proof. **Step 1.** We define the space $\Theta^1(\Omega^o)$ as

$$\Theta^1(\Omega^o) = \left\{ \begin{array}{l} r \in H^1(\Omega^o);\ [r]_j = \text{const},\ 1 \le j \le J, \\ r|_{\Gamma^0_D} = 0 \quad r|_{\Gamma^m_D} = \text{const},\ 1 \le m \le L_D \end{array} \right\}.$$

We look for $q_j^\ell \in \Theta^1(\Omega^o)$ such that

$$\forall r \in \Theta^1(\Omega^o), \quad \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r \, dx = [r]_j + r|_{\Gamma_D^\ell}. \tag{3.9}$$

Since $\Theta^1(\Omega^o)$ is a closed subspace of $H^1(\Omega^o)$, using Lax-Milgram lemma, Problem (3.9) has a unique solution.

(i) Now let $q_j^\ell \in \Theta^1(\Omega^o)$ be solution of (3.9), by taking $r \in \mathcal{D}(\Omega)$, we get

$$\left\langle \operatorname{div}(\widetilde{\mathbf{grad}}\, q_j^\ell), r \right\rangle = -\int_\Omega \widetilde{\mathbf{grad}}\, q_j^\ell \cdot \nabla r \, dx = -\int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = 0,$$

which implies that $\operatorname{div}(\widetilde{\mathbf{grad}}\, q_j^\ell) = 0$ in Ω and then $\Delta q_j^\ell = 0$ in Ω^o.

(ii) We choose $r \in H_0^1(\Omega)$ and from Green's formula, we obtain

$$\int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r \, dx = \sum_{k=1}^{J} \int_{\Sigma_k} \left[\frac{\partial q_j^\ell}{\partial \boldsymbol{n}} \right]_k r = [r]_j = 0,$$

which means that $\left[\frac{\partial q_j^\ell}{\partial \boldsymbol{n}} \right]_k = 0$ for any $1 \le k \le J$. Furthermore, using (3.9) with $r \in H^1(\Omega)$ such that $r = 0$ on Γ_D, and by applying again Green's formula, we deduce that

$$0 = \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = \left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, r \right\rangle_{\Gamma_N}.$$

Therefore $\frac{\partial q_j^\ell}{\partial \boldsymbol{n}} = 0$ on Γ_N.

(iii) From Lemma 3.7, we have for any $r \in H^1(\Omega^o)$ such that $r = 0$ on Γ_D

$$\sum_{k=1}^{J} \left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, [r]_k \right\rangle_{\Sigma_k} = \int_{\Omega^o} \nabla q_j^\ell \cdot \nabla r = [r]_j \, .$$

In particular, if we choose $r \in \Theta^1(\Omega^o)$ with $r = 0$ on Γ_D, we get

$$\sum_{k=1}^{J} [r]_k \left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, 1 \right\rangle_{\Sigma_k} = [r]_j \, ,$$

from which we easily derive the relations $\left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}$.

(iv) In the same way, if $r \in H^1(\Omega)$ with $r|_{\Gamma_D^m} = \text{const}$, $1 \le m \le L_D$ and $r|_{\Gamma_D^0} = 0$, we have

$$\sum_{m=1}^{L_D} r|_{\Gamma_D^m} \left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, 1 \right\rangle_{\Gamma_D^m} = r|_{\Gamma_D^\ell},$$

from which we deduce the relations $\left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, 1 \right\rangle_{\Gamma_D^m} = \delta_{\ell m}$ for any $1 \le \ell \le L_D$ and then $\left\langle \nabla q_j^\ell \cdot \boldsymbol{n}, 1 \right\rangle_{\Gamma_D^0} = -1$.

Step 2. Conversely, it is easy to check that every solution of Problem (3.8) also solves (3.9).

Step 3. Since $q_j^\ell \in H^1(\Omega^o)$ and $[q_j^\ell]_k = \text{const}$, for any $1 \le k \le J$, we deduce from Lemma 3.11 of [2] that $\mathbf{curl}\, \widetilde{\mathbf{grad}}\, q_j^\ell = \mathbf{0}$ in Ω and then $\widetilde{\mathbf{grad}}\, q_j^\ell \in \mathbf{K}_0^2(\Omega)$. From the last properties in (3.8), it is readily checked that the functions $\widetilde{\mathbf{grad}}\, q_j^\ell$ are linearly independent for $1 \le j \le J$ and $1 \le \ell \le L_D$.

It remains to show that they span $\mathbf{K}_0^2(\Omega)$. Let $\boldsymbol{w} \in \mathbf{K}_0^2(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{\ell=1}^{L_D}\sum_{j=1}^{J}\left(\frac{1}{L_D}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j} + \frac{1}{J}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^\ell}\right)\widetilde{\mathbf{grad}}\, q_j^\ell. \tag{3.10}$$

Since $\boldsymbol{w} \in \boldsymbol{K}_0^2(\Omega)$, then

$$\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D} = \left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma} = \int_\Omega \operatorname{div}\boldsymbol{w}\, dx = 0. \tag{3.11}$$

Therefore using (3.10), we infer that for any $1 \le m \le L_D$

$$\begin{aligned}\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m} &= \left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m} - \sum_{j=1}^{J}\left(\frac{1}{L_D}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j} + \frac{1}{J}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m}\right)\\ &= -\frac{1}{L_D}\sum_{j=1}^{J}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j}\end{aligned}$$

Clearly from this relation, we get after summing

$$0 = \sum_{m=1}^{L_D}\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m} = -\sum_{j=1}^{J}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j},$$

which implies that $\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m} = 0$ for any $1 \le m \le L_D$. In the same way for any $1 \le j \le J$, we deduce from (3.11) and (3.10) that

$$\begin{aligned}\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} &= \left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} - \sum_{\ell=1}^{L_D}\left(\frac{1}{L_D}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} + \frac{1}{J}\left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^\ell}\right)\\ &= \left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} - \left\langle \boldsymbol{w}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} = 0.\end{aligned}$$

From the above properties, it is obvious that $\boldsymbol{u}$ belongs to $\mathbf{K}_0^2(\Omega)$. Furthermore, it satisfies

$$\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^m} = 0, \quad \forall 0 \le m \le L_D \quad \text{and} \quad \left\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_k} = 0, \quad \forall 1 \le k \le J. \tag{3.12}$$

Since Ω^o is simply connected and $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ in Ω^o then $\boldsymbol{u} = \nabla q$, where $q \in H^1(\Omega^o)$. Furthermore, $\operatorname{div}\boldsymbol{u} = 0$ in Ω then $\Delta q = 0$ in Ω^o. Because $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_N, we get $\frac{\partial q}{\partial \boldsymbol{n}} = 0$ on Γ_N. As $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω then $\left[\frac{\partial q}{\partial \boldsymbol{n}}\right]_j = 0$ for any $1 \le j \le J$. As $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ also in Ω, Lemma 3.11 of [2] implies that $[q]_j = \text{const}$ for any $1 \le j \le J$. Therefore for any $r \in \Theta^1(\Omega^o)$, we have by (3.6) and (3.12) that

$$\int_{\Omega^o} \nabla q\cdot\nabla r\, dx = \sum_{j=1}^{J}[r]_j\left\langle \frac{\partial q}{\partial \boldsymbol{n}},\ 1\right\rangle_{\Sigma_j} - \sum_{m=1}^{L_D} r|_{\Gamma_D^m}\left\langle \frac{\partial q}{\partial \boldsymbol{n}},\ 1\right\rangle_{\Gamma_D^m} = 0.$$

This implies that q is solution of (3.9) with a second hand side equal to zero, which means that $q = 0$ and then $\boldsymbol{u}$ is zero and this ends the proof. ∎

Let us state an immediate consequence of Proposition 3.8.

COROLLARY 3.9. *Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial\Sigma \subset \Gamma_N$. On the space $\mathbf{X}^p_0(\Omega)$, the semi-norm*

$$\boldsymbol{u} \mapsto \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_D} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Gamma_D^\ell} + \sum_{j=1}^{J} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Sigma_j} \tag{3.13}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$).

Proof. The proof consists in applying Peetre-Tartar theorem (cf. Ref. [23]), with the following correspondence: $\mathbf{E}_1 = \mathbf{X}^p_0(\Omega)$ equiped with the graph norm, $\mathbf{E}_2 = L^p(\Omega) \times \mathbf{L}^p(\Omega)$, $\mathbf{E}_3 = \mathbf{L}^p(\Omega)$, $A\boldsymbol{u} = (\operatorname{div} \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{u})$ and $B = Id$, the identity operator of $\mathbf{E}_1$ into $\mathbf{E}_3$. Then $\|\boldsymbol{u}\|_{\mathbf{E}_1} \simeq \|A\boldsymbol{u}\|_{\mathbf{E}_2} + \|\boldsymbol{u}\|_{\mathbf{E}_3}$ since $\mathbf{X}^p_0(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. Note that the imbedding of $\mathbf{X}^p_0(\Omega)$ into $\mathbf{L}^p(\Omega)$ is compact and the canonical imbedding Id of $\mathbf{E}_1$ into $\mathbf{E}_3$ is also compact. Let $M : \mathbf{X}^p_0(\Omega) \mapsto \mathbf{K}^p_0(\Omega)$ be the following continuous linear mapping

$$M\boldsymbol{u} = \sum_{\ell=1}^{L_D}\sum_{j=1}^{J} \left(\frac{1}{L_D} \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} + \frac{1}{J} \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} \right) \widetilde{\mathbf{grad}}\, q_j^\ell.$$

We set

$$\|M\boldsymbol{u}\|_{\mathbf{K}^p_0(\Omega)} = \sum_{\ell=1}^{L_D} \left|\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell}\right| + \sum_{j=1}^{J} \left|\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}\right|.$$

Let us check that if $\boldsymbol{u} \in KerA = \mathbf{K}^p_0(\Omega)$, then $M\boldsymbol{u} = \mathbf{0}$ if and only if $\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} = 0$ for any $1 \le \ell \le L_D$ and $\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$, for any $1 \le j \le J$, which means that $\boldsymbol{u} = \mathbf{0}$. So by Peetre-Tartar theorem we deduce that

$$\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)} \le C \left(\|A\boldsymbol{u}\|_{\mathbf{E}_2} + \|M\boldsymbol{u}\|_{\mathbf{K}^p_0(\Omega)} \right)$$

and then estimate (3.13). ∎

We introduce the following space

$$\Theta^1(\Omega) = \left\{ r \in H^1(\Omega), \quad r|_{\Gamma_D^0} = 0 \quad \text{and} \quad r|_{\Gamma_D^m} = \text{const}, \quad 1 \le m \le L_D \right\}.$$

In the case where $\partial\Sigma$ is included in Γ_D, the characterization of the kernel $\mathbf{K}^2_0(\Omega)$ is considered in the following proposition.

PROPOSITION 3.10. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. Then the dimension of the space $\mathbf{K}^2_0(\Omega)$ is equal to L_D and it is spanned by the functions ∇q_ℓ, $1 \le \ell \le L_D$ where each q_ℓ is the unique solution in $H^1(\Omega)$, of the problem*

$$\begin{cases} -\Delta q_\ell = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial q_\ell}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_N, \\ q_\ell|_{\Gamma_D^0} = 0 \quad \text{and} \quad q_\ell|_{\Gamma_D^m} = \text{const}, \quad 1 \le m \le L_D, \\ \left\langle \frac{\partial q_\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_D^0} = -1 \quad \text{and} \quad \left\langle \frac{\partial q_\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_D^m} = \delta_{\ell m}, \quad 1 \le m \le L_D. \end{cases} \tag{3.14}$$

Proof. It is obvious that the problem: find $q_\ell \in \Theta^1(\Omega)$ such that

$$\forall r \in \Theta^1(\Omega), \quad \int_\Omega \nabla q_\ell \cdot \nabla r \, dx = r|_{\Gamma_D^\ell} \tag{3.15}$$

has a unique solution and each solution q_ℓ of (3.14) also solves (3.15). Conversely, using (3.15) with $r \in \mathcal{D}(\Omega)$, we obtain $\Delta q_\ell = 0$ in Ω. By using Green's formula in (3.15) with $r \in H^1(\Omega)$ and $r = 0$ on Γ_D, thus $\frac{\partial q_\ell}{\partial \boldsymbol{n}} = 0$ on Γ_N. By taking $r \in \Theta^1(\Omega)$, we have

$$\sum_{m=1}^{L_D} r|_{\Gamma_D^m} \langle \nabla q_\ell \cdot \boldsymbol{n}, 1\rangle_{\Gamma_D^m} = r|_{\Gamma_D^\ell}$$

and then we derive the last equalities in (3.14). The functions ∇q_ℓ are linearly independent and belong to $\mathbf{K}_0^2(\Omega)$. To prove that they span $\mathbf{K}_0^2(\Omega)$, we take a function $\boldsymbol{w} \in \mathbf{K}_0^2(\Omega)$ and we consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{\ell=1}^{L_D} \langle \boldsymbol{w} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} \nabla q_\ell$$

which remains in $\mathbf{K}_0^2(\Omega)$ and satisfies $< \boldsymbol{u} \cdot \boldsymbol{n}, 1 >_{\Gamma_D^m} = 0$ for any $1 \le m \le L_D$ and also for $m = 0$ since $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Note that $\boldsymbol{w} = \nabla q$ with $q \in H^1(\Omega^o)$. But if we take another admissible set of cuts denoted by Σ'_j, $1 \le j \le J$, we will obtain that $\boldsymbol{w} = \nabla q'$ with $q' \in H^1(\Omega'^o)$. But, for any fixed $1 \le j \le J$, the function $q' \in H^1(W_j)$ where W_j is a neighborhood of Σ_j. Since $\nabla q = \nabla q'$ in $W_j \backslash \Sigma_j$, we deduce that there exist two constants c_j^+ and c_j^- such that $q' = q + c_j^+$ in $W_j^+ \backslash \Sigma_j$ and $q' = q + c_j^-$ in $W_j^- \backslash \Sigma_j$, where W_j^+ (resp W_j^-) is a part of W_j located on one side of Σ_j (resp on the other side). This means that $[q]_j = \text{const}$. Since $\Delta \boldsymbol{w} = 0$ in Ω, we have that $\boldsymbol{w} \in \mathcal{C}^\infty(\Omega)$. Furthermore, q is constant on any connected component Γ_D^ℓ and $\partial\Sigma \subset \Gamma_D$, we infer that $c_j^+ = c_j^-$ *i.e* $[q]_j = 0$ and then $q \in H^1(\Omega)$. Since $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_N we have

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{u} \, dx = \int_\Omega \boldsymbol{u} \cdot \nabla q \, dx = \sum_{m=1}^{L_D} q|_{\Gamma_D^m} < \boldsymbol{u} \cdot \boldsymbol{n}, 1 >_{\Gamma_D^m} = 0.$$

thus $\boldsymbol{u}$ is zero and this ends the proof. ◻

As previously, Proposition 3.10 has a corollary about equivalent norms.

COROLLARY 3.11. *Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial\Sigma \subset \Gamma_D$. On the space $\mathbf{X}_0^p(\Omega)$, the semi-norm*

$$\boldsymbol{u} \mapsto \|\operatorname{div} \boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_D} | < \boldsymbol{u} \cdot \boldsymbol{n}, 1 >_{\Gamma_D^\ell} | \tag{3.16}$$

is a norm equivalent to the norm $\| \cdot \|_{\mathbf{X}^p(\Omega)}$ (resp. $\| \cdot \|_{\mathbf{W}^{1,p}(\Omega)}$).

Proof. By applying again the Peetre-Tartar theorem with the same correspondences of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ as in Corollary 3.9. Let $M : \mathbf{X}_0^p(\Omega) \mapsto \mathbf{K}_0^p(\Omega)$ be the following mapping

$$M\boldsymbol{u} = \sum_{\ell=1}^{L_D} < \boldsymbol{u} \cdot \boldsymbol{n}, 1 >_{\Gamma_D^\ell} \nabla q_\ell.$$

We set

$$\|M\boldsymbol{u}\|_{\mathbf{K}_0^p(\Omega)} = \sum_{\ell=1}^{L_D} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Gamma_D^\ell} |.$$

It is clear that if $\boldsymbol{u} \in KerA = \mathbf{K}_0^p(\Omega)$, then $M\boldsymbol{u} = \mathbf{0}$ if and only if $< \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Gamma_D^\ell} = 0$ for any $1 \le \ell \le L_D$ which means that $\boldsymbol{u} = \mathbf{0}$. So by Peetre-Tartar theorem, we deduce that

$$\|\boldsymbol{u}\|_{\mathbf{X}^p(\Omega)} \le C\left(\|A\boldsymbol{u}\|_{\mathbf{E}_2} + \|M\boldsymbol{u}\|_{\mathbf{K}_0^p(\Omega)}\right)$$

and this finishes the proof. □

Remark 3.12. Assume that Ω is Lipschitz (resp. $\mathcal{C}^{1,1}$) and $\partial\Sigma \subset \Gamma_N$, then on the space $\widetilde{\mathbf{X}}_0^p(\Omega)$, the following semi-norm

$$\boldsymbol{u} \mapsto \|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_N} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Gamma_N^\ell} | \tag{3.17}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$). Similarly when $\partial\Sigma \subset \Gamma_D$, the following semi-norm

$$u \mapsto \|\operatorname{div}\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_N} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Gamma_N^\ell} + \sum_{j=1}^{J} | < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Sigma_j}, \tag{3.18}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbf{X}^p(\Omega)}$ (resp. $\|\cdot\|_{\mathbf{W}^{1,p}(\Omega)}$) on $\widetilde{\mathbf{X}}_0^p(\Omega)$.

The following propositions concern the characterization of the kernel $\widetilde{\mathbf{K}}_0^2(\Omega)$ where Γ_N and Γ_D are swapped. The proofs are exactly the same as in Proposition 3.8 and Proposition 3.10 respectively.

PROPOSITION 3.13. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. Then the dimension of the space $\widetilde{\mathbf{K}}_0^2(\Omega)$ is equal to $L_N \times J$ and it is spanned by the functions $\widetilde{\mathbf{grad}}\, s_j^\ell$, $1 \le j \le J$ and $1 \le \ell \le L_N$ where each s_j^ℓ is the unique solution in $\mathbf{H}^1(\Omega^o)$ of the problem*

$$\begin{cases} -\Delta s_j^\ell = 0 \quad \text{in} \quad \Omega^o, \\ \frac{\partial s_j^\ell}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_D, \\ s_j^\ell|_{\Gamma_N^0} = 0 \quad \text{and} \quad s_j^\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \le m \le L_N, \\ \left[s_j^\ell\right]_k = \text{const} \quad \text{and} \quad \left[\frac{\partial s_j^\ell}{\partial \boldsymbol{n}}\right]_k = 0, \quad 1 \le k \le J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \le k \le J, \\ \left\langle \frac{\partial s_j^\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_N^0} = -1 \quad \text{and} \quad \left\langle \frac{\partial s_j^\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \le m \le L_N. \end{cases} \tag{3.19}$$

PROPOSITION 3.14. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. Then the dimension of the space $\widetilde{\mathbf{K}}_0^2(\Omega)$ is equal to L_N and it is spanned by the functions ∇s_ℓ, $1 \le \ell \le L_N$ where each s_ℓ is the unique solution in $H^1(\Omega)$, of the problem*

$$\begin{cases} -\Delta s_\ell = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial s_\ell}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_D, \\ s_\ell|_{\Gamma_N^0} = 0 \quad \text{and} \quad s_\ell|_{\Gamma_N^m} = \text{const}, \quad 1 \le m \le L_N, \\ \left\langle \frac{\partial s_\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_N^0} = -1 \quad \text{and} \quad \left\langle \frac{\partial s_\ell}{\partial \boldsymbol{n}}, 1 \right\rangle_{\Gamma_N^m} = \delta_{\ell m}, \quad 1 \le m \le L_N. \end{cases} \tag{3.20}$$

Remark 3.15. Observe that if Ω is of class $\mathcal{C}^{1,1}$, then for any $1 < p < \infty$, we have

$$\mathbf{K}_0^p(\Omega) \hookrightarrow \bigcap_{q \geq 1} \mathbf{W}^{1,q}(\Omega).$$

We prove this result for any $1 < p < 3$. Let $\mathbf{u} \in \mathbf{K}_0^p(\Omega)$, we know that $\mathbf{u} \in \mathbf{W}^{1,1}(\Omega) \hookrightarrow \mathbf{L}^{3/2}(\Omega)$. Then, $\mathbf{u} \in \mathbf{K}_0^{3/2}(\Omega)$. By using Theorem 3.2, we infer that $\mathbf{u} \in \mathbf{K}_0^{3/2}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$. Now, we assume that $p \geq 3$ and due to Theorem 3.2 again, we have $\mathbf{u} \in \mathbf{W}^{1,3}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega)$ for any $q < \infty$. Thanks to Theorem 3.2, $\boldsymbol{u} \in \mathbf{W}^{1,q}(\Omega)$ and then the kernel $\mathbf{K}_0^p(\Omega)$ does not depend on p.

Now, we state in the following lemma another preliminary result (which was proven in a different form by Mitrea, Lemma 4.1 p.144 [29]), that is necessary in the next section.

Lemma 3.16. *Let $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$ with $\boldsymbol{\varphi} \times \boldsymbol{n} \in \mathbf{L}^2(\Gamma)$. Then*

$$\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n}) = \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{n} \quad \text{in} \quad H^{-1/2}(\Gamma). \tag{3.21}$$

In particular if $\boldsymbol{\varphi} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D, we have $\mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on Γ_D.

Proof. For any $\chi \in H^1(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$, we have from Green's formula

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \nabla \chi \, dx = \langle \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{n}, \chi \rangle_{H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}.$$

Let us introduce the following Hilbert space:

$$E(\Omega) = \left\{ \chi \in H^1(\Omega);\ \chi_{|\Gamma} \in H^1(\Gamma) \right\}.$$

For any $\chi \in E(\Omega)$ and $\boldsymbol{\varphi} \in \mathbf{H}^2(\mathbf{curl}, \Omega)$, we have the following relation

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \nabla \chi \, dx = -\int_\Gamma (\boldsymbol{\varphi} \times \boldsymbol{n}) \cdot \nabla_\tau \chi, \tag{3.22}$$

that we prove by using the fact that (see [8])

$$\mathcal{D}(\overline{\Omega}) \quad \text{is dense in} \quad E(\Omega).$$

That implies that

$$\langle \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} = \langle \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{n}, \chi \rangle_{H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)}$$

and

$$| \langle \operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n}), \chi \rangle_{H^{-1}(\Gamma) \times H^1(\Gamma)} | \leq C(\Omega) \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)} \|\chi\|_{H^1(\Omega)}.$$

Now, let $\mu \in H^1(\Gamma)$. We know that there exists $\chi \in H^1(\Omega)$ (in fact $\chi \in H^{3/2}(\Omega)$) such that $\chi = \mu$ on Γ with the estimate $\|\chi\|_{H^1(\Omega)} \leq C(\Omega)\|\mu\|_{H^{1/2}\Gamma}$. As $H^1(\Gamma)$ is dense in $H^{1/2}(\Gamma)$, we deduce that $\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n}) \in H^{-1/2}(\Gamma)$ and

$$\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n}) = \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{n} \quad \text{with} \quad \|\operatorname{div}_\Gamma(\boldsymbol{\varphi} \times \boldsymbol{n})\|_{H^{-1/2}(\Gamma)} \leq C(\Omega)\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{L}^2(\Omega)}.$$

$\square$

4. Vector potentials. This section presents the first main results of this paper related to the existence and uniqueness of vector potentials satisfying mixed boundary conditions in the Hilbert case and then in the L^p-theory, when $\partial\Sigma$ is included in Γ_N or in Γ_D.

We define the following Banach space:

$$\widetilde{\mathbf{V}}_0^p(\Omega) = \left\{ \ \boldsymbol{v} \in \widetilde{\mathbf{X}}_0^p(\Omega),\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0,\ 1 \le \ell \le L_N \ \right\}.$$

4.1. The Hilbert case ($p = 2$). The following theorem is an extension of Theorem 3.12 of [2] when $\Gamma_D \neq \emptyset$.

THEOREM 4.1. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_N$. A function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\begin{aligned} &\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D, \\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0, \quad 0 \le \ell \le L_N, \end{aligned} \tag{4.1}$$

if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{X}^2(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \operatorname{\mathbf{curl}} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega, \\ &\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N, \\ &\langle \boldsymbol{\psi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} = 0, \quad 0 \le \ell \le L_D, \\ &\langle \boldsymbol{\psi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J. \end{aligned} \tag{4.2}$$

The function $\boldsymbol{\psi}$ is unique and satisfies the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{X}^2(\Omega)} \le C\|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}. \tag{4.3}$$

Proof. **Step 1. Uniqueness.** Clearly, the uniqueness of the function $\boldsymbol{\psi}$ will follow from the characterization of the kernel $\mathbf{K}_0^2(\Omega)$ given in Proposition 3.8. Suppose that $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$ where $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ satisfy (4.2), thus $\boldsymbol{\psi}$ belongs to $\mathbf{K}_0^2(\Omega)$ and from the last properties in (4.2), we deduce that $\boldsymbol{\psi} = \mathbf{0}$.

Step 2. Necessary conditions. Let us prove that (4.2) implies (4.1). It is obvious that if $\boldsymbol{u} = \operatorname{\mathbf{curl}} \boldsymbol{\psi}$ then $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Since $\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0}$ on Γ_D then due to Lemma 3.16, we have $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_D. For $0 \le \ell \le L_D$, let μ_ℓ be a function of $\mathcal{C}^\infty(\overline{\Omega})$ which is equal to 1 in the neighborhood of Γ_N^ℓ and vanishes in the neighborhood of Γ_N^m where $0 \le m \le L_N$ and $\ell \neq m$ and in the neighborhood of Γ_D. Proceeding as in the proof of Lemma 3.5 [2], we have

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = \langle \operatorname{\mathbf{curl}}(\mu_\ell\boldsymbol{\psi})\cdot\boldsymbol{n}, 1\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \int_\Omega \operatorname{div} \operatorname{\mathbf{curl}}(\mu_\ell\boldsymbol{\psi})\, dx = 0.$$

Step 3. Existence. We know that there exists (see Lemma 3.5 in [2]) $\boldsymbol{\psi}_0 \in \mathbf{H}^1(\Omega)$ such that $\boldsymbol{u} = \operatorname{\mathbf{curl}} \boldsymbol{\psi}_0$ and $\operatorname{div} \boldsymbol{\psi}_0 = 0$ in Ω. Let $\chi \in H^1(\Omega)$ such that

$$\Delta\chi = 0 \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = \boldsymbol{\psi}_0\cdot\boldsymbol{n} \quad \text{on} \quad \Gamma_N.$$

Setting now $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla\chi$, then $\operatorname{\mathbf{curl}} \boldsymbol{\psi}_1 = \boldsymbol{u}$ and $\operatorname{div} \boldsymbol{\psi}_1 = 0$ in Ω with $\boldsymbol{\psi}_1\cdot\boldsymbol{n} = 0$ on Γ_N. We define the bilinear form $a(.,.)$ as

$$a(\boldsymbol{\xi}, \boldsymbol{\varphi}) = \int_\Omega \operatorname{\mathbf{curl}} \boldsymbol{\xi}\cdot\operatorname{\mathbf{curl}} \boldsymbol{\varphi}\, dx.$$

From (3.17), the bilinear form a is coercive on $\widetilde{\mathbf{V}}_0^2(\Omega)$ and the following problem:

$$\text{(4.4)}\qquad \begin{aligned}&\text{Find}\quad \boldsymbol{\xi}\in\widetilde{\mathbf{V}}_0^2(\Omega)\quad\text{such that for any}\quad \boldsymbol{\varphi}\in\widetilde{\mathbf{V}}_0^2(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx=\int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx-\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx\end{aligned}$$

admits a unique solution. Next, we want to extend (4.4) to any test function in $\widetilde{\mathbf{X}}_0^2(\Omega)$:

$$\text{(4.5)}\qquad \begin{aligned}&\text{Find}\quad \boldsymbol{\xi}\in\widetilde{\mathbf{V}}_0^2(\Omega)\quad\text{such that for any}\quad \boldsymbol{\varphi}\in\widetilde{\mathbf{X}}_0^2(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx=\int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx-\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx.\end{aligned}$$

Indeed, it is easy to check that any solution of (4.5) also solves (4.4). On the other side, let $\boldsymbol{\xi}\in\widetilde{\mathbf{V}}_0^2(\Omega)$ solution of (4.4) and $\boldsymbol{\varphi}\in\widetilde{\mathbf{X}}_0^2(\Omega)$. Then, there exists a unique $\theta\in H^1(\Omega)$ satisfying

$$\text{(4.6)}\qquad \Delta\theta=\operatorname{div}\boldsymbol{\varphi}\quad\text{in}\quad\Omega,\quad \theta=0\quad\text{on}\quad\Gamma_N\quad\text{and}\quad \frac{\partial\theta}{\partial\boldsymbol{n}}=0\quad\text{on}\quad\Gamma_D.$$

We set

$$\text{(4.7)}\qquad \widetilde{\boldsymbol{\varphi}}=\boldsymbol{\varphi}-\nabla\theta-\sum_{\ell=1}^{L_N}\langle(\boldsymbol{\varphi}-\nabla\theta)\cdot\boldsymbol{n},1\rangle_{\Gamma_N^\ell}\,\nabla s_\ell.$$

Therefore $\widetilde{\boldsymbol{\varphi}}\in\widetilde{\mathbf{V}}_0^2(\Omega)$, and we observe then that

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx=\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,dx&=\int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,dx-\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\widetilde{\boldsymbol{\varphi}}\,dx\\ &=\int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx-\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx,\end{aligned}$$

where we observe that

$$\text{(4.8)}\qquad \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\nabla\theta\,dx=\langle\boldsymbol{u}\cdot\boldsymbol{n},\theta\rangle_\Gamma=0$$

since $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ_D and $\theta=0$ on Γ_N and thanks to (4.1)

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\nabla s_\ell\,dx=\langle\boldsymbol{u}\cdot\boldsymbol{n},s_\ell\rangle_{\Gamma_N}=\sum_{\ell=1}^{L_N}s_\ell\,\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_N^\ell}=0.$$

From (4.5), we deduce that $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}=\mathbf{0}$ in Ω and $(\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{\psi}_0)\times\boldsymbol{n}=\mathbf{0}$ on Γ_D. It follows that the function

$$\text{(4.9)}\qquad \boldsymbol{\psi}=\widetilde{\boldsymbol{\psi}}-\sum_{\ell=1}^{L_D}\sum_{j=1}^{J}\left(\frac{1}{L_D}\left\langle\widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n},1\right\rangle_{\Sigma_j}+\frac{1}{J}\left\langle\widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n},1\right\rangle_{\Gamma_D^\ell}\right)\widetilde{\mathbf{grad}}\,q_j^\ell,$$

with $\widetilde{\boldsymbol{\psi}}=\boldsymbol{\psi}_1-\mathbf{curl}\,\boldsymbol{\xi}$ satisfies the properties (4.2) of Theorem 4.1. Finally, it is easy to get the estimate (4.3). ∎

Remark 4.2. If Ω is of class $\mathcal{C}^{1,1}$, the vector potential $\boldsymbol{\psi}$ belongs to $\mathbf{H}^1(\Omega)$. Indeed, $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{\xi}\in\mathbf{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{z}=0$, $\mathbf{curl}\,\boldsymbol{z}=\mathbf{0}$ in Ω and $\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{\psi}_0\times\boldsymbol{n}$ on Γ_D. Since $\boldsymbol{\xi}\times\boldsymbol{n}=\mathbf{0}$ on Γ_N we have $\boldsymbol{z}\cdot\boldsymbol{n}=0$ on Γ_N which implies that $\mathbf{curl}\,\boldsymbol{\xi}$ belongs to $\mathbf{H}^1(\Omega)$.

We consider the following space

$$\widetilde{\mathbf{W}}^p_\Sigma(\Omega) = \Big\{\boldsymbol{v} \in \widetilde{\mathbf{X}}^p_0(\Omega),\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J$$
$$\text{and} \quad \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma^\ell_N} = 0, \quad 0 \le \ell \le L_N\Big\}.$$

The following theorem, which is an extension of Theorem 3.17 of [2] when $\Gamma_N \neq \emptyset$, consists on the existence and the uniqueness of a vector potential when $\partial\Sigma$ is included in Γ_D.

THEOREM 4.3. *Assume that Ω is Lipschitz and $\partial\Sigma \subset \Gamma_D$. A function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\begin{aligned} &\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D,\\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma^\ell_N} = 0, \quad 0 \le \ell \le L_N,\\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{aligned} \tag{4.10}$$

if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{X}^2(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in} \quad \Omega,\\ &\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N,\\ &\langle \boldsymbol{\psi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma^\ell_D} = 0, \quad 0 \le \ell \le L_D. \end{aligned} \tag{4.11}$$

This function $\boldsymbol{\psi}$ is unique and it satisfies

$$\|\boldsymbol{\psi}\|_{\mathbf{X}^2(\Omega)} \le C\|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}.$$

Proof. **Step 1. Uniqueness.** The uniqueness of the vector potential $\boldsymbol{\psi}$ is a consequence of the characterization of the kernel $\mathbf{K}^2_0(\Omega)$ given in Proposition 3.10.

Step 2. Necessary conditions. As in Step 2 of the proof of Theorem 4.1, if $\boldsymbol{\psi}$ satisfies (4.11), we check that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ satisfies (4.10). Clearly, the fluxes over Γ^ℓ_N are equal to zero and by Lemma 3.16, $\mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n} = 0$ on Γ_D. Hence $\mathbf{curl}\,\boldsymbol{\psi}$ satisfies the assumptions of Lemma 3.7 where Γ_N is replaced by Γ_D and then $\mathbf{curl}\,\boldsymbol{\psi}\cdot\boldsymbol{n} \in [H^{\frac{1}{2}}(\Sigma_j)]'$ for any $1 \le j \le J$. Moreover, we have

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega), \quad \forall \mu \in L^2(\Sigma_j), \quad \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n},\, \mu\rangle_{\Sigma_j} = \langle \mathbf{grad}\,\mu \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_{\Sigma_j}.$$

By choosing $\mu = 1$, we get

$$\langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J,$$

and by the density of $\mathcal{D}(\Sigma_j)$ in $[H^{\frac{1}{2}}(\Sigma_j)]'$, this last relation holds for $\boldsymbol{\varphi} = \boldsymbol{\psi}$, which proves the last equality of (4.10).

Step 3. Existence. As in Step 3 of the proof of Theorem 4.1, we set $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla\chi$ and we consider the same bilinear form a which is coercive on $\widetilde{\mathbf{W}}^2_\Sigma(\Omega)$ thanks to (3.18. Consequently, the following problem

$$\begin{aligned} &\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}^2_\Sigma(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}^2_\Sigma(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx, \end{aligned} \tag{4.12}$$

admits a unique solution. We will now extend (4.12) to any test function in $\widetilde{\mathbf{X}}^2_0(\Omega)$:

$$\text{(4.13)}\qquad \begin{aligned} &\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}^2_\Sigma(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}^2_0(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx. \end{aligned}$$

Indeed, it is easy to check that any solution of (4.13) also solves (4.12). On the other side, let $\boldsymbol{\xi} \in \widetilde{\mathbf{W}}^2_\Sigma(\Omega)$ solution of (4.12) and $\boldsymbol{\varphi} \in \widetilde{\mathbf{X}}^2_0(\Omega)$. Setting $\overline{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\theta$ with θ defined in (4.6), we verify easily that the following function

$$\text{(4.14)}\qquad \widetilde{\boldsymbol{\varphi}} = \overline{\boldsymbol{\varphi}} - \sum_{\ell=1}^{L_N}\sum_{j=1}^{J}\left(\frac{1}{L_N}\langle\overline{\boldsymbol{\varphi}}\cdot\boldsymbol{n},1\rangle_{\Sigma_j} + \frac{1}{J}\langle\overline{\boldsymbol{\varphi}}\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_N}\right)\widetilde{\mathbf{grad}}\,s^\ell_j$$

belongs to $\widetilde{\mathbf{W}}^2_\Sigma(\Omega)$ and as in the proof of Theorem 4.1 we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx.$$

It follows from this equality that the function

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \mathbf{curl}\,\boldsymbol{\xi} - \sum_{\ell=1}^{L_D}\langle(\boldsymbol{\psi}_1 - \mathbf{curl}\,\boldsymbol{\xi})\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_D}\,\nabla q_\ell$$

belongs to $\mathbf{X}^2(\Omega)$ and we can verify that $\boldsymbol{\psi}$ satisfies the properties (4.11) of Theorem 4.3. ∎

Remark 4.4. As previously, if Ω is of class $\mathcal{C}^{1,1}$ then the obtained vector potential belongs to $\mathbf{H}^1(\Omega)$.

4.2. Other potentials. In this subsection, we turn our attention to another kind of vector potentials. Indeed, we assume that $\operatorname{div}\boldsymbol{u} = 0$ in Ω and we look for the conditions to impose on $\boldsymbol{u}$ such that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$ in Ω and $\boldsymbol{\psi} = \mathbf{0}$ on a part of the boundary. As previously, we consider the case where $\partial\Sigma$ is included in Γ_N or in Γ_D. In the next, we require the following preliminaries.

We define the space

$$\mathbf{H}^2(\operatorname{div},\Delta;\Omega) = \left\{\boldsymbol{v} \in \mathbf{H}^2(\operatorname{div},\Omega);\ \Delta(\operatorname{div}\boldsymbol{v}) \in L^2(\Omega)\right\},$$

endowed with the scalar product

$$((\boldsymbol{u},\boldsymbol{v}))_{\mathbf{H}^2(\operatorname{div},\Delta;\Omega)} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx + \int_\Omega(\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{v})\,dx + \int_\Omega \Delta(\operatorname{div}\boldsymbol{u})\Delta(\operatorname{div}\boldsymbol{v})\,dx,$$

which is a Hilbert space.

Lemma 4.5. *Assume that Ω is Lipschitz. Then*

$$\mathcal{D}(\overline{\Omega}) \quad \textit{is dense in the space} \quad \mathbf{H}^2(\operatorname{div},\Delta;\Omega).$$

Proof. Let $\ell \in \left[\mathbf{H}^2(\operatorname{div},\Delta;\Omega)\right]'$ such that for any $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})$, $\langle\ell,\boldsymbol{v}\rangle = 0$. Since $\mathbf{H}^2(\operatorname{div},\Delta;\Omega)$ is a Hilbert space, we can associate to ℓ a function $\boldsymbol{f}$ in $\mathbf{H}^2(\operatorname{div},\Delta;\Omega)$ such that for any $\boldsymbol{v} \in \mathbf{H}^2(\operatorname{div},\Delta;\Omega)$, we have

$$\langle\ell,\boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx + \int_\Omega(\operatorname{div}\boldsymbol{f})(\operatorname{div}\boldsymbol{v})\,dx + \int_\Omega \Delta(\operatorname{div}\boldsymbol{f})\Delta(\operatorname{div}\boldsymbol{v})\,dx.$$

We set now $F = \operatorname{div} \boldsymbol{f}$, and $G = \Delta F$ and we denote by $\widetilde{\boldsymbol{f}}$, $\widetilde{F}$ and $\widetilde{G}$ the extensions of $\boldsymbol{f}$, F and G respectively to $\mathbb{R}^3$. Assume now that $\ell = \mathbf{0}$ in $\mathcal{D}(\overline{\Omega})$, then for any $\boldsymbol{\varphi} \in \mathcal{D}(\overline{\Omega})$, we have

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\varphi} + \int_{\mathbb{R}^3} \widetilde{F} \operatorname{div} \boldsymbol{\varphi} + \int_{\mathbb{R}^3} \widetilde{G} \Delta \operatorname{div} \boldsymbol{\varphi} = 0,$$

which means that

$$\widetilde{\boldsymbol{f}} = \nabla(\widetilde{F} + \Delta \widetilde{G}) \quad \text{in} \quad \mathbb{R}^3.$$

Since $\widetilde{\boldsymbol{f}} \in \mathbf{L}^2(\mathbb{R}^3)$ and $\nabla \widetilde{F} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ then $\nabla(\Delta \widetilde{G}) \in \mathbf{H}^{-1}(\mathbb{R}^3)$ and $\Delta \widetilde{G} \in L^2(\mathbb{R}^3)$. As $\widetilde{G} \in L^2(\mathbb{R}^3)$, we deduce that $\widetilde{G} \in H^2(\mathbb{R}^3)$ and thus $G \in H^2_0(\Omega)$. So there exists ψ_k in $\mathcal{D}(\Omega)$ such that $\psi_k \to G$ in $H^2(\Omega)$. Furthermore, since $\Delta \widetilde{G} = \widetilde{\Delta G}$, we have $\widetilde{F + \Delta G} \in H^1(\mathbb{R}^3)$. In other words,

$$F + \Delta G \in H^1_0(\Omega).$$

Then, for any $\boldsymbol{v}$ in $\mathbf{H}^2(\operatorname{div}, \Delta; \Omega)$ we have

$$\begin{aligned}
\langle \ell, \boldsymbol{v} \rangle &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_\Omega F \operatorname{div} \boldsymbol{v}\, dx + \lim_{k \to \infty} \int_\Omega \psi_k \Delta \operatorname{div} \boldsymbol{v}\, dx \\
&= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_\Omega F \operatorname{div} \boldsymbol{v}\, dx + \lim_{k \to \infty} \int_\Omega (\Delta \psi_k) \operatorname{div} \boldsymbol{v}\, dx \\
&= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int_\Omega (F + \Delta G) \operatorname{div} \boldsymbol{v}\, dx \\
&= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx - \int_\Omega \nabla(F + \Delta G) \cdot \boldsymbol{v}\, dx = 0.
\end{aligned}$$

This ends the proof. ∎

LEMMA 4.6. *Let $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div}, \Delta, \Omega)$.*

i) Then, $\partial_n(\operatorname{div} \boldsymbol{\psi}) \in \left[H^{3/2}_{00}(\Sigma_j)\right]'$ for any $1 \leq j \leq J$ and we have the following Green's formula for any $r \in H^2(\Omega^o)$ such that $r = \partial_n r = 0$ on Γ and $[\partial_n r]_k = 0$ for any $1 \leq k \leq J$:

$$\int_{\Omega^o} r \Delta(\operatorname{div} \boldsymbol{\psi})\, dx - \int_{\Omega^o} (\operatorname{div} \boldsymbol{\psi}) \Delta r\, dx = \sum_{k=1}^{J} \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), [r]_k \rangle_{\Sigma_k}. \tag{4.15}$$

ii) Moreover the following Green's formula holds for any $r \in H^2(\Omega^o)$ such that $\partial_n r = 0$ on Γ and $[\partial_n r]_k = 0$ for any $1 \leq k \leq J$:

$$\begin{aligned}
\int_{\Omega^o} r \Delta(\operatorname{div} \boldsymbol{\psi})\, dx - \int_{\Omega^o} (\operatorname{div} \boldsymbol{\psi}) \Delta r\, dx \quad &= \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), r \rangle_\Gamma + \\
&+ \sum_{k=1}^{J} \langle \partial_n(\operatorname{div} \boldsymbol{\psi}), [r]_k \rangle_{\Sigma_k}.
\end{aligned} \tag{4.16}$$

Proof. **i)** Let $\mu \in H^{3/2}_{00}(\Sigma_j)$, then there exists $\varphi \in H^2(\Omega^o)$ such that

$$[\varphi]_k = \mu \delta_{jk}, \quad [\partial_n \varphi]_k = 0 \quad \text{for all } k = 1, \ldots, J \quad \text{and} \quad \varphi = \partial_n \varphi = 0 \quad \text{on} \quad \Gamma.$$

Furthermore, it satisfies

$$\|\varphi\|_{H^2(\Omega^o)} \leq C\|\mu\|_{H^{3/2}_{00}(\Sigma_j)}.$$

Let $\boldsymbol{\psi} \in \mathcal{D}(\overline{\Omega})$. Then, the Green's formula gives

$$\int_{\Omega^o} \varphi\Delta(\operatorname{div}\boldsymbol{\psi})\,dx - \int_{\Omega^o} (\operatorname{div}\boldsymbol{\psi})\Delta\varphi\,dx = \langle\partial_n(\operatorname{div}\boldsymbol{\psi}), \mu\rangle_{\Sigma_j}. \tag{4.17}$$

Therefore

$$|\langle\partial_n(\operatorname{div}\boldsymbol{\psi}), \mu\rangle_{\Sigma_j}| \leq C\|\mu\|_{H^{3/2}_{00}(\Sigma_j)}\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\operatorname{div},\Delta;\Omega)},$$

which proves that $\partial_n(\operatorname{div}\boldsymbol{\psi})|_{\Sigma_j} \in \left[H^{3/2}_{00}(\Sigma_j)\right]'$ and

$$\|\partial_n(\operatorname{div}\boldsymbol{\psi})\|_{\left[H^{3/2}_{00}(\Sigma_j)\right]'} \leq C\|\boldsymbol{\psi}\|_{\mathbf{H}^2(\operatorname{div},\Delta;\Omega)}.$$

We deduce from the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2(\operatorname{div},\Delta;\Omega)$, that the last inequality holds for any $\boldsymbol{\psi}$ in $\mathbf{H}^2(\operatorname{div},\Delta;\Omega)$ and we get the formula (4.17). Finally, by an adequate partition of unity, we obtain the required formula (4.15).

ii) As a consequence, using the density of $\mathcal{D}(\overline{\Omega})$ in $\mathbf{H}^2(\operatorname{div},\Delta;\Omega)$, we deduce now the following Green's formula: for any $r \in H^2(\Omega^o)$ such that $\partial_n r = 0$ on Γ and $[\partial_n r]_k = 0$ for any $1 \leq k \leq J$ and for any $\boldsymbol{\psi} \in \mathbf{H}^2(\operatorname{div},\Delta;\Omega)$:

$$\begin{aligned}\int_{\Omega^o} r\Delta(\operatorname{div}\boldsymbol{\psi})\,dx - \int_{\Omega^o} (\operatorname{div}\boldsymbol{\psi})\Delta r\,dx = \quad & \langle\partial_n(\operatorname{div}\boldsymbol{\psi}), r\rangle_{\Gamma} + \\ & + \sum_{k=1}^{J} \langle\partial_n(\operatorname{div}\boldsymbol{\psi}), [r]_k\rangle_{\Sigma_k}.\end{aligned} \tag{4.18}$$

Observe that the regularity $\mathcal{C}^{1,1}$ of the domain Ω implies that

$$\partial_n(\operatorname{div}\boldsymbol{\psi}) \in H^{-3/2}(\Gamma).$$

This finishes the proof. ∎

Let us define the kernel

$$\mathbf{B}^p_0(\Omega) = \left\{ \begin{array}{l} \boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega);\ \operatorname{div}(\Delta\boldsymbol{w}) = 0,\ \mathbf{curl}\,\boldsymbol{w} = \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{w} = \mathbf{0} \text{ on } \Gamma_D,\ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ and } \partial_{\boldsymbol{n}}(\operatorname{div}\boldsymbol{w}) = 0 \text{ on } \Gamma_N \end{array} \right\},$$

and the space $\Theta^2(\Omega^o)$ by

$$\Theta^2(\Omega^o) = \left\{ \begin{array}{l} r \in H^2(\Omega^o); \quad r|_{\Gamma^0_D} = 0, \quad r|_{\Gamma^m_D} = \text{const},\ 1 \leq m \leq L_D \\ [r]_j = \text{const},\ [\partial_{\boldsymbol{n}} r]_j = 0, 1 \leq j \leq J, \quad \frac{\partial r}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma \end{array} \right\}.$$

Remark 4.7. Suppose that

$$r \in H^2(\Omega^o); \quad [r]_j = \text{const} \quad \text{and} \quad [\partial_{\boldsymbol{n}} r]_j = 0 \quad \text{for any } 1 \leq j \leq J.$$

Since for any $1 \leq j \leq J$, we have $[\nabla r \times \boldsymbol{n}]_j = 0$ then $[\nabla r]_j = 0$, which means that $\widetilde{\mathbf{grad}}\, r \in \mathbf{H}^1(\Omega)$.

The next proposition states the characterization of the kernel $\mathbf{B}_0^2(\Omega)$ when $\partial\Sigma$ is included in Γ_N.

PROPOSITION 4.8. *If $\partial\Sigma \subset \Gamma_N$, the dimension of the space $\mathbf{B}_0^2(\Omega)$ is equal to $L_D \times J$ and it is spanned by the functions $\widetilde{\mathbf{grad}}\,\chi_j^\ell$, $1 \le j \le J$ and $1 \le \ell \le L_D$, where each χ_j^ℓ is the unique solution in $H^2(\Omega^o)$ of the problem*

$$(4.19)\quad \begin{cases} \Delta^2\chi_j^\ell = 0 \quad \text{in} \quad \Omega^o, \\ \frac{\partial\chi_j^\ell}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma, \quad \partial_{\boldsymbol{n}}(\Delta\,\chi_j^\ell) = 0 \quad \text{on} \quad \Gamma_N \\ \chi_j^\ell|_{\Gamma_D^0} = 0, \quad \chi_j^\ell|_{\Gamma_D^m} = \text{const}, \quad 1 \le m \le L_D, \\ \left[\chi_j^\ell\right]_k = \text{const}, \ \left[\partial_{\boldsymbol{n}}\chi_j^\ell\right]_k = \left[\Delta\chi_j^\ell\right]_k = \left[\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell)\right]_k = 0, \ 1 \le k \le J, \\ \left\langle\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell), 1\right\rangle_{\Sigma_k} = \delta_{jk}, \quad 1 \le k \le J, \\ \left\langle\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell), 1\right\rangle_{\Gamma_D^0} = -1 \quad \text{and} \quad \left\langle\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell), 1\right\rangle_{\Gamma_D^m} = \delta_{\ell m}, \quad 1 \le m \le L_D. \end{cases}$$

ii) Moreover if Ω is of class $\mathcal{C}^{2,1}$, then $\widetilde{\mathbf{grad}}\,\chi_j^\ell \in \mathbf{H}^2(\Omega)$.

Proof. **Step 1.** Note that $\Theta^2(\Omega^o)$ is a closed subspace of $H^2(\Omega^o)$. Then from Lax Milgram theorem the problem

$$(4.20)\quad \begin{aligned} &\text{Find} \quad \chi_j^\ell \in \Theta^2(\Omega^o) \quad \text{such that} \\ &\forall r \in \Theta^2(\Omega^o), \quad \int_{\Omega^o} \Delta\chi_j^\ell \Delta r\,dx = -\left[r\right]_j - r|_{\Gamma_D^\ell} \end{aligned}$$

admits a unique solution. Moreover, for any $r \in \mathcal{D}(\Omega)$, we have

$$\left\langle \operatorname{div}\Delta(\widetilde{\mathbf{grad}}\,\chi_j^\ell), r \right\rangle = -\int_\Omega \operatorname{div}(\widetilde{\mathbf{grad}}\,\chi_j^\ell)\Delta r\,dx = -\int_{\Omega^o} \Delta\chi_j^\ell \Delta r\,dx = 0,$$

in other words $\operatorname{div}\Delta(\widetilde{\mathbf{grad}}\,\chi_j^\ell) = 0$ in Ω and thus $\Delta^2\chi_j^\ell = 0$ in Ω^o.

Step 2. It remains to show the properties concerning the jumps of $\Delta\chi_j^\ell$ and $\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell)$ over Σ_j and those concerning the fluxes. Taking $r \in H_0^2(\Omega)$, then

$$0 = \int_{\Omega^o} \Delta\chi_j^\ell \Delta r\,dx = -\sum_{k=1}^{J} \left\langle \left[\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell)\right]_{\Sigma_k}, r \right\rangle_k + \sum_{k=1}^{J} \left\langle \left[\Delta\chi_j^\ell\right]_k, \partial_{\boldsymbol{n}} r \right\rangle_{\Sigma_k}.$$

Consequently

$$\left[\partial_{\boldsymbol{n}}(\Delta\chi_j^\ell)\right]_k = \left[\Delta\chi_j^\ell\right]_k = 0, \quad 1 \le k \le J.$$

Taking now $r \in H^2(\Omega)$ with $r = 0$ on Γ_D and $\partial_{\boldsymbol{n}} r = 0$ on Γ and Green's formula leads to

$$0 = \int_{\Omega^o} \Delta\chi_j^\ell \Delta r\,dx = -\left\langle \partial_{\boldsymbol{n}}\Delta\chi_j^\ell, r \right\rangle_{\Gamma_N},$$

i.e $\partial_{\boldsymbol{n}}\Delta\chi_j^\ell = 0$ on Γ_N.

Choosing now $r \in H^2(\Omega) \cap \Theta^2(\Omega^o)$, we deduce that

$$(4.21)\quad \sum_{m=1}^{L_D} r_{|\Gamma_D^m} \left\langle \partial_{\boldsymbol{n}}\Delta\chi_j^\ell, 1 \right\rangle_{\Gamma_D^m} = r_{|\Gamma_D^\ell}$$

and then

$$\left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Gamma_D^m} = \delta_{m\ell}, \quad 1 \leq m \leq L_D. \tag{4.22}$$

Since $\operatorname{div} \Delta(\widetilde{\mathbf{grad}}\, \chi_j^{\ell}) = 0$ in Ω then $\left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Gamma_D^0} = -1$. Observe now that for any $r \in \Theta^2(\Omega^o)$, we have from Lemma 4.6

$$\sum_{m=1}^{L_D} r_{|_{\Gamma_D^m}} \left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Gamma_D^m} + \sum_{k=1}^{J} [r]_k \left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Sigma_k} = r_{|_{\Gamma_D^{\ell}}} + [r]_j.$$

Then due to (4.22), we have

$$\sum_{k=1}^{J} [r]_k \left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Sigma_k} = [r]_j.$$

We finally infer that $\left\langle \partial_{\boldsymbol{n}} \Delta \chi_j^{\ell}, 1 \right\rangle_{\Sigma_k} = \delta_{jk}$.

Step 3. It is obvious that any solution of (4.19) also solves (4.20).

Step 4. It is readily checked that the functions $\widetilde{\mathbf{grad}}\, \chi_j^{\ell}$ are linearly independent for any $1 \leq j \leq J$ and $1 \leq \ell \leq L_D$. To prove that they span $\mathbf{B}_0^2(\Omega)$, we consider $\boldsymbol{w} \in \mathbf{B}_0^2(\Omega)$ and the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{\ell=1}^{L_D} \sum_{j=1}^{J} \left(\frac{1}{L_D} \left\langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{w}), 1 \right\rangle_{\Sigma_j} + \frac{1}{J} \left\langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{w}), 1 \right\rangle_{\Gamma_D^{\ell}} \right) \widetilde{\mathbf{grad}}\, \chi_j^{\ell}$$

remains in $\mathbf{B}_0^2(\Omega)$ and satisfies $\langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{u}), 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$ and $\langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{u}), 1 \rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$.

As $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ in Ω^o, there exists a function $q \in H^2(\Omega^o)$ such that $\boldsymbol{u} = \nabla q$ in Ω^o, with $\Delta^2 q = 0$ in Ω^o since $\Delta(\operatorname{div} \boldsymbol{u}) = 0$ in Ω. Since $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ with $\boldsymbol{u} = \mathbf{0}$ on Γ_D and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_N, we deduce that $\partial_{\boldsymbol{n}} q = 0$ on Γ and $q = \text{const}$ on Γ_D^{ℓ} for any $0 \leq \ell \leq L_D$. Moreover, we can take the constant equal to zero on Γ_D^0. Now, we choose the extension of ∇q denoted $\widetilde{\mathbf{grad}}\, q$ such that $\widetilde{\mathbf{grad}}\, q = \boldsymbol{u}$ in Ω. As $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ in Ω then $\mathbf{curl}\, \widetilde{\mathbf{grad}}\, q = 0$ and thus the jump of q is zero almost everywhere across each cut Σ_j (see Lemma 3.11 [2]), which means that $q \in H^1(\Omega)$ and $\boldsymbol{u} = \widetilde{\mathbf{grad}}\, q = \nabla q$ in Ω. As $\boldsymbol{u}$ belongs to $\mathbf{H}^1(\Omega)$, we infer that $q \in H^2(\Omega)$ and that $\Delta^2 q = 0$ in Ω due to the fact that $\Delta(\operatorname{div} \boldsymbol{u}) = 0$ in Ω. Since $\partial_{\boldsymbol{n}} q = 0$ on Γ, $\partial_{\boldsymbol{n}}(\Delta q) = 0$ on Γ_N and $q = \text{const}$ on Γ_D^{ℓ} for any $0 \leq \ell \leq L_D$, we have by using the Green formula

$$\begin{aligned} 0 &= \int_\Omega |\Delta q|^2 \, dx - \langle \Delta q, \partial_{\boldsymbol{n}} q \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} + \langle \partial_{\boldsymbol{n}} \Delta q, q \rangle_{H^{-3/2}(\Gamma) \times H^{3/2}(\Gamma)} \\ &= \int_\Omega |\Delta q|^2 \, dx + \sum_{m=1}^{L_D} q|_{\Gamma_D^m} \langle \partial_{\boldsymbol{n}} \Delta q, 1 \rangle_{\Gamma_D^m}. \end{aligned}$$

As $\langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{u}), 1 \rangle_{\Gamma_D^m} = \langle \partial_{\boldsymbol{n}} \Delta q, 1 \rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$, we deduce that $\Delta q = 0$ in Ω which means that q is constant because $\partial_{\boldsymbol{n}} q = 0$ on Γ and consequently $\boldsymbol{u}$ is equal to zero.

To finish the proof, the point ii) is an immediate consequence of Corollary 3.5. ∎

THEOREM 4.9. *Assume that $\partial\Sigma \subset \Gamma_N$, a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\begin{aligned}&\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D,\\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0, \quad 0 \le \ell \le L_N,\end{aligned} \tag{4.23}$$

if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that

$$\begin{aligned}&\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}(\Delta\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega,\\ &\boldsymbol{\psi} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}}(\operatorname{div}\boldsymbol{\psi}) = 0 \quad \text{on} \quad \Gamma_N,\\ &\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Gamma_D^\ell} = 0, \quad 0 \le \ell \le L_D,\\ &\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J.\end{aligned} \tag{4.24}$$

This function $\boldsymbol{\psi}$ is unique.

Remark 4.10. Since $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ and $\operatorname{div}(\Delta\boldsymbol{\psi}) = 0$ in Ω, then by using Lemma 4.6, the quantities $\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Gamma_D^\ell}$ and $\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Sigma_j}$ make sense.

Proof. The uniqueness is deduced from the characterization of the kernel $\mathbf{B}_0^2(\Omega)$ and the necessary conditions are proved in the same way as in the proof of Theorem 4.1.

Let us consider a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfying (4.23) to which we associate the vector potential $\boldsymbol{\psi}$ defined in Theorem 4.1 that we will denote hereinafter by $\widetilde{\boldsymbol{\psi}}$. We consider now the following problem

$$\begin{cases}\Delta^2\lambda = 0 \quad \text{in} \quad \Omega,\\ \lambda = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \partial_n(\Delta\lambda) = 0 \quad \text{on} \quad \Gamma_N,\\ \frac{\partial\lambda}{\partial\boldsymbol{n}} = \widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n} \quad \text{on} \quad \Gamma.\end{cases}$$

This problem admits a solution in $H^2(\Omega)$ since $\widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n} \in H^{\frac{1}{2}}(\Gamma)$ and the following function

$$\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} - \nabla\lambda + \sum_{\ell=1}^{L_D}\sum_{j=1}^{J}\left(\frac{1}{L_D}\,\langle \partial_{\boldsymbol{n}}(\Delta\,\chi), 1\rangle_{\Sigma_j} + \frac{1}{J}\,\langle \partial_{\boldsymbol{n}}(\Delta\,\chi), 1\rangle_{\Gamma_D^\ell}\right)\widetilde{\mathbf{grad}}\,\chi_j^\ell$$

satisfies the properties (4.24) of Theorem 4.9. ∎

We define the space $\Theta^2(\Omega)$ by

$$\Theta^2(\Omega) = \left\{ r \in H^2(\Omega);\, r|_{\Gamma_D^0} = 0,\ r|_{\Gamma_D^m} = \text{const},\ 1 \le m \le L_D,\ \frac{\partial r}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma \right\}.$$

Let us consider in the next proposition the dimension of the kernel $\mathbf{B}_0^2(\Omega)$ in the case where $\partial\Sigma$ is included in Γ_D.

PROPOSITION 4.11. *If $\partial\Sigma \subset \Gamma_D$, the dimension of the space $\mathbf{B}_0^2(\Omega)$ is equal to L_D and it is spanned by the function $\nabla\chi_\ell$, $1 \le \ell \le L_D$ where each χ_ℓ is the unique solution in $H^2(\Omega)$, of the problem*

$$\begin{cases}\Delta^2\chi_\ell = 0 \quad \text{in} \quad \Omega,\\ \frac{\partial\chi_\ell}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma, \quad \partial_{\boldsymbol{n}}(\Delta\chi_\ell) = 0 \quad \textit{on} \quad \Gamma_N,\\ \chi_\ell|_{\Gamma_D^0} = 0 \quad \text{and} \quad \chi_\ell|_{\Gamma_D^m} = \text{const}, \quad 1 \le m \le L_D,\\ \langle \partial_{\boldsymbol{n}}(\Delta\,\chi_\ell), 1\rangle_{\Gamma_D^0} = -1 \quad \text{and} \quad \langle \partial_{\boldsymbol{n}}(\Delta\,\chi_\ell), 1\rangle_{\Gamma_D^m} = \delta_{\ell m}, \quad 1 \le m \le L_D.\end{cases} \tag{4.25}$$

Proof. We look for $\chi_\ell \in \Theta^2(\Omega)$ such that

$$\forall r \in \Theta^2(\Omega), \quad \int_\Omega \Delta\chi_\ell \Delta r \, dx = -r|_{\Gamma_D^\ell}. \tag{4.26}$$

This problem admits a unique solution because the form

$$a(\chi_\ell, r) = \int_\Omega \Delta\chi_\ell \Delta r \, dx$$

is coercive on $\Theta^2(\Omega)$ according to the fact that $\|r\|_{H^2(\Omega)} \leq C\|\Delta r\|_{L^2(\Omega)}$ when $\partial_{\boldsymbol{n}} r = 0$ on Γ. Moreover, due to the density of $\mathcal{D}(\overline{\Omega})$ in the space of functions which belong to $H^2(\Omega)$ and their bi-laplacian operator belongs to $L^2(\Omega)$, we can prove the following Green's formula, for any χ_ℓ and r in $\Theta^2(\Omega)$ such that $\Delta^2\chi_\ell \in L^2(\Omega)$

$$\int_\Omega (\Delta^2\chi_\ell) r \, dx = \int_\Omega \Delta\chi_\ell \Delta r \, dx + \sum_{\ell=1}^{L_D} r|_{\Gamma_D^\ell} \langle \partial_{\boldsymbol{n}}(\Delta\chi_\ell), 1\rangle_{\Gamma_D^\ell}.$$

It is readily checked that if $\chi_\ell \in \Theta^2(\Omega)$ satisfies (4.26) then χ_ℓ is solution of (4.25). By taking $r \in \Theta^2(\Omega)$ and by using Green's formula and the fact that $\Delta^2\chi_\ell = 0$ in Ω, we deduce that

$$\langle \partial_{\boldsymbol{n}}\Delta\chi_\ell, r\rangle_\Gamma = -r|_{\Gamma_D^\ell},$$

Hence, $\partial_{\boldsymbol{n}}\Delta\chi_\ell = 0$ on Γ_N.

Furthermore, the functions $\nabla\chi_\ell$ are linearly independent for any $1 \leq \ell \leq L_D$. One has to prove that they span $\mathbf{B}_0^2(\Omega)$. Let $\boldsymbol{w} \in \mathbf{B}_0^2(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{\ell=1}^{L_D} \langle \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{w}), 1\rangle_{\Gamma_D^\ell} \nabla \chi_\ell.$$

which remains in $\mathbf{B}_0^2(\Omega)$ and satisfies $< \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{u}), 1 >_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$. We follow the same approach as in the fourth step of the proof of Proposition 4.8 to show that $\boldsymbol{u} = \mathbf{0}$ in Ω. Indeed, there exists a function $q \in H^2(\Omega^o)$ such that $\boldsymbol{u} = \nabla q$ in Ω^o due to the fact that $\mathbf{curl}\, \boldsymbol{u} = \mathbf{0}$ in Ω and thus in Ω^o. The remainder of the proof is exactly the same because $\Delta^2 q = 0$ in Ω. ☐

The following theorem is an extension of Theorem 3.20 of [2] when $\Gamma_N \neq \emptyset$.

THEOREM 4.12. *If $\partial\Sigma \subset \Gamma_D$, a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\begin{aligned} &\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D, \\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0, \quad 0 \leq \ell \leq L_N, \\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \end{aligned} \tag{4.27}$$

if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}, \quad \operatorname{div}(\Delta\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega, \\ &\boldsymbol{\psi} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}}(\operatorname{div} \boldsymbol{\psi}) = 0 \quad \text{on} \quad \Gamma_N, \\ &\langle \partial_n(\operatorname{div} \boldsymbol{\psi}), 1\rangle_{\Gamma_D^\ell} = 0, \quad 0 \leq \ell \leq L_D. \end{aligned} \tag{4.28}$$

This function $\boldsymbol{\psi}$ is unique.

Proof. The uniqueness of the vector potential is deduced from the characterization of the kernel $\mathbf{B}_0^2(\Omega)$ and the necessary conditions are proved in the same way as in the proof of Theorem 4.3. Note that a function $\boldsymbol{u}$ satisfies (4.27) if and only if there exists a unique vector potential $\boldsymbol{\psi}$ defined in Theorem 4.3 that we will denote hereinafter by $\overline{\boldsymbol{\psi}}$. We consider now the following problem

$$\begin{cases} \Delta^2\lambda = 0 \quad \text{in} \quad \Omega, \\ \lambda = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \partial_n(\Delta\lambda) = 0 \quad \text{on} \quad \Gamma_N, \\ \frac{\partial\lambda}{\partial\boldsymbol{n}} = \overline{\boldsymbol{\psi}}\cdot\boldsymbol{n} \quad \text{on} \quad \Gamma. \end{cases}$$

This problem admits a solution in $H^2(\Omega)$ and the following function

$$\boldsymbol{\psi} = \overline{\boldsymbol{\psi}} - \nabla\lambda + \sum_{\ell=1}^{L_D} \langle \partial_{\boldsymbol{n}}(\Delta\,\chi), 1\rangle_{\Gamma_D^\ell}\,\nabla\,\chi_\ell$$

satisfies the properties (4.28) of Theorem 4.12. ◻

The next result is an extension of Theorem 3.20 in [2] when $\Gamma_D \neq \emptyset$. We skip the proof in this paper.

THEOREM 4.13. *If $\partial\Sigma \subset \Gamma_N$, a function $\boldsymbol{u} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\begin{aligned} &\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_D\cup\Gamma_N, \\ &\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 0 \le j \le J, \end{aligned} \tag{4.29}$$

if and only if there exists a vector potential $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \textit{and} \quad \operatorname{div}(\Delta\boldsymbol{\psi}) = 0 \quad \textit{in} \quad \Omega, \\ &\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \quad \textit{on} \quad \Gamma_D \quad \textit{and} \quad \boldsymbol{\psi} = 0 \quad \textit{on} \quad \Gamma_N, \\ &\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Gamma_D^\ell} = 0, \quad 0 \le \ell \le L_D, \\ &\langle \partial_n(\operatorname{div}\boldsymbol{\psi}), 1\rangle_{\Gamma_N^\ell} = 0, \quad 0 \le \ell \le L_N. \end{aligned} \tag{4.30}$$

4.3. $\mathbf{L}^p$-theory. In this subsection, we investigate the L^p-theory of the vector potentials obtained in Theorems 4.1 and 4.3 for any $1 < p < \infty$. The general case $p \neq 2$ is not as easy as the case $p = 2$ and requires extra work. The following theorems are about the case where $p > 2$ which is a straightforward consequence of Theorems 4.1 and 4.3.

THEOREM 4.14. *If $\partial\Sigma$ is included in Γ_N and $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ with $p > 2$ satisfies (4.1), then the vector potential $\boldsymbol{\psi}$ given in Theorem 4.1 belongs to $\mathbf{W}^{1,p}(\Omega)$ and satisfies the following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}.$$

Proof. The proof of this theorem is immediately deduced from Theorem 4.1 and Theorem 3.2. ◻

In the same way, we generalize the results of Theorem 4.3 for any $p > 2$

THEOREM 4.15. *If $\partial\Sigma$ is included in Γ_D and $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ with $p > 2$ satisfies (4.10), then the vector potential $\boldsymbol{\psi}$ given in Theorem 4.3 belongs to $\mathbf{W}^{1,p}(\Omega)$ and satisfies the following estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}.$$

We will later on see how to extend the previous results to the case $p < 2$ in Theorems 4.18 and 4.21. The major task consists on proving two Inf-Sup conditions.

LEMMA 4.16. *If $\partial\Sigma \subset \Gamma_N$, there exists a constant $\beta > 0$ depending only on Ω and p, such that the following Inf-Sup condition holds*

$$\inf_{\substack{\boldsymbol{\varphi}\in\widetilde{\mathbf{V}}_0^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in\widetilde{\mathbf{V}}_0^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\right|}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta. \tag{4.31}$$

Proof. We use here the following Helmholtz decomposition: every $\mathbf{g}\in\mathbf{L}^p(\Omega)$ can be decomposed as $\mathbf{g} = \nabla\chi + \boldsymbol{z}$ where $\boldsymbol{z}\in\mathbf{L}^p(\Omega)$ with $\operatorname{div}\boldsymbol{z} = 0$ and χ belongs to $W^{1,p}(\Omega)$ with $\chi = 0$ on Γ_D and $(\nabla\chi - \mathbf{g})\cdot\boldsymbol{n}$ on Γ_N. Furthermore, it satisfies the estimate

$$\|\nabla\chi\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.$$

Let $\boldsymbol{\varphi}$ be a function of $\widetilde{\mathbf{V}}_0^{p'}(\Omega)$. From (3.17) of Remark 3.12, we deduce that

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{L}^{p'}(\Omega)} = C\sup_{\substack{\mathbf{g}\in\mathbf{L}^p(\Omega)\\ \mathbf{g}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}}.$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{\ell=1}^{L_D}\sum_{j=1}^{J}\left(\frac{1}{L_D}\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j} + \frac{1}{J}\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^\ell}\right)\widetilde{\mathbf{grad}}\,q_j^\ell.$$

Thus $\widetilde{\boldsymbol{z}}\in\mathbf{L}^p(\Omega)$, $\operatorname{div}\widetilde{\boldsymbol{z}} = 0$ in Ω, and satisfies $\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n} = 0$ on Γ_N, $\langle\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^m} = 0$, for any $1\leq m\leq L_D$ and $\langle\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$, for any $1\leq j\leq J$. Due to Theorem 4.15 where Γ_D and Γ_N are switched (see Theorem 4.3), there exists a vector potential $\boldsymbol{\psi}\in\widetilde{\mathbf{V}}_0^p(\Omega)$ with $p\geq 2$ such that $\widetilde{z} = \mathbf{curl}\,\boldsymbol{\psi}$ and satisfying (4.11) where Γ_D and Γ_N are switched. This implies that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,dx,$$

because $\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\chi\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\mathbf{grad}}\,q_j^\ell\,dx = 0$. Furthermore, we have

$$\begin{aligned}
\|\widetilde{\boldsymbol{z}}\|_{\mathbf{L}^p(\Omega)} &\leq \|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} + \sum_{\ell=1}^{L_D}\sum_{j=1}^{J}\left|\frac{1}{L_D}\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^\ell} + \frac{1}{J}\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j}\right|\|\widetilde{\mathbf{grad}}\,q_j^\ell\|_{\mathbf{L}^p(\Omega)}\\
&\leq \|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} + C\left(\sum_{\ell=1}^{L_D}\left|\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Gamma_D^\ell}\right| + \sum_{j=1}^{J}\left|\left\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\right\rangle_{\Sigma_j}\right|\right)\\
&\leq C\|\boldsymbol{z}\|_{\mathbf{L}^p(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}.
\end{aligned}$$

We can write now

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\right|}{\|\widetilde{\boldsymbol{z}}\|_{\mathbf{L}^p(\Omega)}} = C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\right|}{\|\mathbf{curl}\,\boldsymbol{\psi}\|_{\mathbf{L}^p(\Omega)}}.$$

But from (3.17) of Remark 3.12, we have that $\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \simeq \|\mathbf{curl}\,\boldsymbol{\psi}\|_{\mathbf{L}^p(\Omega)}$. Finally

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}} \leq C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\right|}{\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)}}.$$

Therefore, we obtain the required Inf-Sup condition for $p \geq 2$. By a symmetry argument, it holds also for $p < 2$. ∎

4.4. First elliptic problem with mixed boundary conditions. The role of the first Inf-Sup condition (4.31) is illustrated in the next proposition as it is used to solve the first elliptic problem.

PROPOSITION 4.17. *Assume that $\partial\Sigma \subset \Gamma_N$ and $\boldsymbol{v}$ belongs to $\mathbf{L}^p(\Omega)$. Then the following problem*

$$(4.32)\begin{cases} -\Delta\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0, \quad (\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\xi}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N, \\ \langle \boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0, \quad 1 \leq \ell \leq L_N, \end{cases}$$

has a unique solution in $\mathbf{W}^{1,p}(\Omega)$ and satisfies

$$(4.33) \qquad \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}.$$

Proof. **i)** We consider the following problem:

$$(4.34) \qquad \begin{array}{l} \text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^p(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{V}}_0^{p'}(\Omega), \\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx. \end{array}$$

Using the Inf-Sup condition (4.31), Problem (4.34) admits a unique solution $\boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. Next, we want to extend (4.34) to any test function in $\widetilde{\mathbf{X}}_0^{p'}(\Omega)$. Let $\boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^{p'}(\Omega)$ and $\chi \in W^{1,p}(\Omega)$ be the unique solution of the following mixed problem

$$\Delta\chi = \operatorname{div}\boldsymbol{\varphi} \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{on} \quad \Gamma_N \quad \text{and} \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma_D.$$

We set

$$(4.35) \qquad \widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\chi - \sum_{\ell=1}^{L_N} \langle(\boldsymbol{\varphi} - \nabla\chi)\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell}\,\nabla s_\ell.$$

Note that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\widetilde{\mathbf{V}}_0^{p'}(\Omega)$ and $\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}} = \mathbf{curl}\,\boldsymbol{\varphi}$, so Problem (4.34) is equivalent to

$$(4.36) \qquad \begin{array}{l} \text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^p(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \\ \displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx. \end{array}$$

ii) Now, we will give the interpretation of Problem (4.36). More precisely, we will prove that Problem (4.36) is equivalent to find $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ solution of (4.32). By choosing $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$, we deduce that $-\Delta\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v}$ in Ω. Moreover, because $\boldsymbol{\xi} \in \widetilde{\mathbf{V}}_0^p(\Omega)$ then $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and it satisfies $\boldsymbol{\xi}\times\boldsymbol{n} = \mathbf{0}$ on Γ_N, $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ_D, $\langle\boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0$ for any $1 \leq \ell \leq L_N$. The last point to prove is that $(\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0}$ on Γ_D. The function $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$ belongs to $\widetilde{\mathbf{X}}^p(\Omega)$ and $\mathbf{curl}\,\boldsymbol{z} = \mathbf{0}$ in Ω. Therefore, for any $\boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^{p'}(\Omega)$ we have

$$\int_\Omega \boldsymbol{z}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \langle\boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1/p,p}(\Gamma)\times\mathbf{W}^{1/p,p'}(\Gamma)} = \int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\boldsymbol{\varphi}\,dx = 0.$$

Using (4.36), we deduce that

$$\forall \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}_0^{p'}(\Omega), \quad \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D)\times \mathbf{W}^{1/p,p'}(\Gamma_D)} = 0$$

Let $\boldsymbol{\mu}$ any element of $\mathbf{W}^{1-1/p',p'}(\Gamma_D)$. So, there exists $\boldsymbol{\varphi}$ of $\mathbf{W}^{1,p'}(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau$ on Γ_D and $\boldsymbol{\varphi} = \mathbf{0}$ on Γ_N. It is obvious that $\boldsymbol{\varphi}$ belongs to $\widetilde{\mathbf{X}}_0^p(\Omega)$ and it satisfies

$$\langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\Gamma_D} = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu}_\tau \rangle_{\Gamma_D} = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\Gamma_D} = 0.$$

This implies that $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D, which is the required property.

iii) Let $\mathbf{B} \in \mathcal{L}(\widetilde{\mathbf{V}}_0^p(\Omega), (\widetilde{\mathbf{V}}_0^{p'}(\Omega))')$ be the following operator:

$$\forall \boldsymbol{\psi} \in \widetilde{\mathbf{V}}_0^p(\Omega), \quad \forall \boldsymbol{\varphi} \in \widetilde{\mathbf{V}}_0^{p'}(\Omega), \quad \langle \mathbf{B}\boldsymbol{\psi}, \boldsymbol{\varphi} \rangle = \int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx.$$

Thanks to (4.31), the operator $\mathbf{B}$ is an isomorphism from $\widetilde{\mathbf{V}}_0^p(\Omega)$ into $(\widetilde{\mathbf{V}}_0^{p'}(\Omega))'$ and

$$\|\boldsymbol{\psi}\|_{\widetilde{\mathbf{X}}_0^p(\Omega)} \simeq \|\mathbf{B}\boldsymbol{\psi}\|_{(\widetilde{\mathbf{V}}_0^{p'}(\Omega))'}.$$

Hence, since $\boldsymbol{\xi}$ is solution of Problem (4.32), we have

$$\|\mathbf{B}\boldsymbol{\xi}\|_{(\widetilde{\mathbf{V}}_0^{p'}(\Omega))'} = \sup_{\substack{\boldsymbol{\varphi}\in\widetilde{\mathbf{V}}_0^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{|\langle \mathbf{B}\boldsymbol{\xi}, \boldsymbol{\varphi}\rangle|}{\|\boldsymbol{\varphi}\|_{\widetilde{\mathbf{X}}_0^p(\Omega)}} = \sup_{\substack{\boldsymbol{\varphi}\in\widetilde{\mathbf{V}}_0^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \frac{\left|\displaystyle\int_\Omega \boldsymbol{v}\cdot \mathbf{curl}\,\boldsymbol{\varphi}\, dx\right|}{\|\boldsymbol{\varphi}\|_{\widetilde{\mathbf{X}}_0^p(\Omega)}}$$

Therefore

$$\|\mathbf{B}\boldsymbol{\xi}\|_{(\widetilde{\mathbf{V}}_0^{p'}(\Omega))'} \leq \|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}$$

Thus the estimate (4.33) holds. □

We are now in position to extend Theorem 4.14 to the case $1 < p < 2$. In fact, the proof of the following theorem is given for any $1 < p < \infty$.

THEOREM 4.18. *Suppose that $\partial\Sigma$ is included in Γ_N and $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ satisfies (4.1) with $1 < p < \infty$. Then there exists a unique vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$ satisfying (4.2) with the estimate*

$$\|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}. \tag{4.37}$$

Proof. **Step 1. Uniqueness**. Let $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ be two vector potentials and $\boldsymbol{\psi} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2$. Then $\boldsymbol{\psi}$ belongs to $\mathbf{K}_0^p(\Omega)$ and $\langle \boldsymbol{\psi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^m} = 0$ for any $1 \leq m \leq L_D$ and $\langle \boldsymbol{\psi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$ for any $1 \leq j \leq J$. Hence, from the characterization of the kernel $\mathbf{K}_0^p(\Omega)$ we deduce that $\boldsymbol{\psi} = \mathbf{0}$.

Step 2. Existence. Let $\boldsymbol{\psi}_0 \in \mathbf{W}^{1,p}(\Omega)$ such that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0$ and $\operatorname{div}\boldsymbol{\psi}_0 = 0$ in Ω (see Lemma 4.1 of [5]). Let $\chi \in W^{1,p}(\Omega)$ such that

$$\Delta\chi = 0 \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = \boldsymbol{\psi}_0\cdot\boldsymbol{n} \quad \text{on} \quad \Gamma_N,$$

with

$$\|\chi\|_{W^{1,p}(\Omega)} \leq C\|\boldsymbol{\psi}_0\cdot\boldsymbol{n}\|_{W^{-1/p,p}(\Gamma_N)} \leq C\|\boldsymbol{\psi}_0\|_{\mathbf{L}^p(\Omega)}.$$

Setting now $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla\chi$, then $\mathbf{curl}\,\boldsymbol{\psi}_1 = \boldsymbol{u}$ and $\operatorname{div}\boldsymbol{\psi}_1 = 0$ in Ω with $\boldsymbol{\psi}_1\cdot\boldsymbol{n} = 0$ on Γ_N. Due to the Inf-Sup condition (4.31), the following problem

$$(4.38)\qquad \begin{aligned}&\text{Find}\quad \boldsymbol{\xi}\in\widetilde{\mathbf{V}}^p_0(\Omega)\quad\text{such that for any}\quad \boldsymbol{\varphi}\in\widetilde{\mathbf{V}}^{p'}_0(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx,\end{aligned}$$

admits a unique solution in $\widetilde{\mathbf{V}}^p_0(\Omega)$ and this solution belongs to $\mathbf{W}^{1,p}(\Omega)$. As previously in the proof of Theorem 4.1, Problem (4.38) is equivalent to

$$(4.39)\qquad \begin{aligned}&\text{Find}\quad \boldsymbol{\xi}\in\widetilde{\mathbf{V}}^p_0(\Omega)\quad\text{such that for any}\quad \boldsymbol{\varphi}\in\widetilde{\mathbf{X}}^{p'}_0(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx.\end{aligned}$$

The rest of the proof is similar to that of Theorem 4.1. The required vector potential $\boldsymbol{\psi}$ given by (4.9) belongs to $\mathbf{W}^{1,p}(\Omega)$ since $\mathbf{curl}\,\boldsymbol{\xi}$, $\boldsymbol{\psi}_1$ and $\widetilde{\mathbf{grad}}\,q^\ell_j \in \mathbf{W}^{1,p}(\Omega)$. Furthermore, it satisfies the estimate (4.37). ∎

In the case where $\partial\Sigma\subset\Gamma_D$, we also need to establish an Inf-Sup condition in order to solve the second elliptic problem.

LEMMA 4.19. *If $\partial\Sigma\subset\Gamma_D$, there exists a constant $\beta > 0$ depending only on Ω and p, such that the following Inf-Sup condition holds*

$$(4.40)\qquad \inf_{\substack{\boldsymbol{\varphi}\in\widetilde{\mathbf{W}}^{p'}_\Sigma(\Omega)\\ \boldsymbol{\varphi}\neq\mathbf{0}}}\ \sup_{\substack{\boldsymbol{\xi}\in\widetilde{\mathbf{W}}^{p}_\Sigma(\Omega)\\ \boldsymbol{\xi}\neq\mathbf{0}}} \frac{\left|\displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\right|}{\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)}} \geq \beta.$$

Proof. We use here the same Helmholtz decomposition as in the proof of Lemma 4.16. Let $\boldsymbol{\varphi}$ be a function of $\widetilde{\mathbf{W}}^{p'}_\Sigma(\Omega)$. From (3.18) of Remark 3.12, we deduce that

$$\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1,p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{L}^{p'}(\Omega)} = C\sup_{\substack{\mathbf{g}\in\mathbf{L}^p(\Omega)\\ \mathbf{g}\neq\mathbf{0}}}\frac{\left|\displaystyle\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\right|}{\|\mathbf{g}\|_{\mathbf{L}^p(\Omega)}}.$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{\ell=1}^{L_D}\langle\boldsymbol{z}\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_D}\nabla q_\ell.$$

Thus $\widetilde{\boldsymbol{z}}\in\mathbf{L}^p(\Omega)$, $\operatorname{div}\widetilde{\boldsymbol{z}} = 0$ in Ω, and satisfies $\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n} = 0$ on Γ_N and $\langle\widetilde{\boldsymbol{z}}\cdot\boldsymbol{n},1\rangle_{\Gamma^m_D} = 0$, for any $1\leq m\leq L_D$. Due to Theorem 4.18 when Γ_D and Γ_N are switched, there exists a vector potential $\boldsymbol{\psi}\in\widetilde{\mathbf{W}}^p_\Sigma(\Omega)$ such that $\widetilde{z} = \mathbf{curl}\,\boldsymbol{\psi}$. This implies that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{g}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,dx,$$

because $\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\chi\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla q_\ell\,dx = 0$. The rest of the proof is similar to that of Lemma 4.16. ∎

4.5. Second elliptic problem with mixed boundary conditions.

PROPOSITION 4.20. *Assume that $\partial\Sigma \subset \Gamma_D$ and $\boldsymbol{v}$ belongs to $\mathbf{L}^p(\Omega)$. Then the following problem*

$$(4.41)\quad \begin{cases} -\Delta\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0, \quad (\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \boldsymbol{\xi}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_N, \\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0, \quad 1 \le \ell \le L_N \quad \text{and} \quad \langle\boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{cases}$$

has a unique solution in $\mathbf{W}^{1,p}(\Omega)$ and satisfies

$$(4.42)\qquad \|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \le C\|\boldsymbol{v}\|_{\mathbf{L}^p(\Omega)}.$$

Proof. **i)** We consider the following problem:

$$(4.43)\qquad \begin{aligned} &\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}^p_\Sigma(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}^{p'}_\Sigma(\Omega), \\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx. \end{aligned}$$

Using the Inf-Sup condition (4.40), Problem (4.43) admits a unique solution $\boldsymbol{\xi} \in \widetilde{\mathbf{W}}^p_\Sigma(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. As in Theorem 4.3, we show that Problem (4.43) is equivalent to the following one

$$(4.44)\qquad \begin{aligned} &\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}^p_\Sigma(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{X}}^{p'}_0(\Omega), \\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx. \end{aligned}$$

ii) By taking $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$, we deduce that $-\Delta\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v}$. It is clear that since $\boldsymbol{\xi} \in \widetilde{\mathbf{W}}^p_\Sigma(\Omega)$ then $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and it satisfies $\boldsymbol{\xi}\times\boldsymbol{n} = \mathbf{0}$ on Γ_N, $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ_D, $\langle\boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0$ for any $1 \le \ell \le L_N$ and $\langle\boldsymbol{\xi}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$ for any $1 \le j \le J$. To prove that $(\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0}$ on Γ_D and that the estimate (4.42) holds, we use the same argument as in Proposition 4.17. ∎

By using the existence and uniqueness result of the second elliptic problem with mixed boundary conditions, we prove the existence and uniqueness of the following vector potential for any $1 < p < \infty$

THEOREM 4.21. *Suppose that $\partial\Sigma$ is included in Γ_D and $\boldsymbol{u} \in \mathbf{L}^p(\Omega)$ satisfies (4.10). Then there exists a unique vector potential $\boldsymbol{\psi} \in \mathbf{W}^{1,p}(\Omega)$ satisfying (4.11) and the estimates*

$$(4.45)\qquad \|\boldsymbol{\psi}\|_{\mathbf{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\mathbf{L}^p(\Omega)}.$$

Proof. **Step 1. Uniqueness**. It is based on the characterization of the kernel $\mathbf{K}^p_0(\Omega)$ when $\partial\Sigma \subset \Gamma_D$.

Step 2. Existence. Setting again $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0 - \nabla\chi$ with the same $\boldsymbol{\psi}_0$ and χ as in the proof of Theorem 4.1. Due to the Inf-Sup condition (4.40), the following problem

$$(4.46)\qquad \begin{aligned} &\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}^p_\Sigma(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}^{p'}_\Sigma(\Omega), \\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx, \end{aligned}$$

admits a unique solution in $\widetilde{\mathbf{W}}^p_\Sigma(\Omega)$ and this solution belongs to $\mathbf{W}^{1,p}(\Omega)$. Next, as previously Problem (4.46) is equivalent to the following one

$$\text{(4.47)}\qquad \begin{aligned}&\text{Find}\quad \boldsymbol{\xi}\in\widetilde{\mathbf{W}}^p_\Sigma(\Omega)\quad\text{such that for any}\quad \boldsymbol{\varphi}\in\widetilde{\mathbf{X}}^{p'}_0(\Omega),\\ &\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx=\int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx-\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,dx.\end{aligned}$$

Finally, the potential we take is given by

$$\boldsymbol{\psi}=\boldsymbol{\psi}_1-\mathbf{curl}\,\boldsymbol{\xi}-\sum_{\ell=1}^{L_D}\langle(\boldsymbol{\psi}_1-\mathbf{curl}\,\xi)\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_D}\,\nabla q_\ell,$$

and it satisfies the properties (4.11) together with the estimate (4.45). ∎

Remark 4.22. As we managed to generalize the first vector potentials for any $1<p<\infty$, we can handle the L^p theory of the less standard ones mentioned in Theorems 4.9 and 4.12. We omit the proofs in this paper.

Remark 4.23. In some particular geometries, one part of $\partial\Sigma$ may be included in Γ_D and the other part in Γ_N, the existence and uniqueness of vector potentials is still an open question in this case.

5. Stokes problem. We consider the Stokes problem subjected to Navier-type boundary condition on some part of the boundary and a pressure boundary condition on the other part. Assume that $\partial\Sigma\subset\Gamma_D$

$$(S)\begin{cases}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f},\quad \operatorname{div}\boldsymbol{u}=0\quad\text{in}\quad\Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\mathbf{0},\quad \pi=\pi_0\quad\text{on}\quad\Gamma_N,\\ \boldsymbol{u}\cdot\boldsymbol{n}=0,\quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}\quad\text{on}\quad\Gamma_D,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_N}=0,\quad 1\le\ell\le L_N,\quad \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\quad 1\le j\le J,\end{cases}$$

where $\boldsymbol{f}$, $\boldsymbol{h}$, π_0 are given functions or distributions. Our aim is to prove the existence and uniqueness of weak solutions of the system (S). To achieve this result, we solve the following auxiliary problem where $\partial\Sigma\subset\Gamma_D$:

$$(S_1)\begin{cases}-\Delta\boldsymbol{\xi}=\boldsymbol{f},\quad \operatorname{div}\boldsymbol{\xi}=0\quad\text{in}\quad\Omega,\\ \boldsymbol{\xi}\times\boldsymbol{n}=\mathbf{0}\quad\text{on}\quad\Gamma_N,\\ \boldsymbol{\xi}\cdot\boldsymbol{n}=0,\quad \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}\quad\text{on}\quad\Gamma_D,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},1\rangle_{\Gamma^\ell_N}=0,\quad 1\le\ell\le L_N,\quad \langle\boldsymbol{\xi}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0,\quad 1\le j\le J.\end{cases}$$

We define $r(p)$ by

$$\frac{1}{r(p)}=\begin{cases}1/p+\frac13\quad\text{if}\quad p>\frac32\\ 1-\varepsilon\quad\text{if}\quad p=\frac32\\ 1\quad\text{if}\quad 1\le p<\frac32.\end{cases}$$

PROPOSITION 5.1. *Assume that* $\partial\Sigma\subset\Gamma_D$. *Let* $\boldsymbol{f}\in\mathbf{L}^{r(p)}(\Omega)$, $\operatorname{div}\boldsymbol{f}=0$, $\boldsymbol{h}\times\boldsymbol{n}\in\mathbf{W}^{-1/p,p}(\Gamma_D)$ *satisfying the following compatibility conditions for any* $\boldsymbol{\varphi}\in\widetilde{\mathbf{K}}^{p'}_0(\Omega)$:

$$\text{(5.1)}\qquad \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx+\langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D)\times\mathbf{W}^{1/p,p'}(\Gamma_D)}=0,$$

$$\text{(5.2)}\qquad \boldsymbol{f}\cdot\boldsymbol{n}=\operatorname{div}_{\Gamma_D}(\boldsymbol{h}\times\boldsymbol{n})\quad\textit{on}\quad\Gamma_D,$$

where $\operatorname{div}_{\Gamma_D}$ *is the surface divergence on* Γ_D*. Then Problem* (S_1) *has a unique solution* $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ *satisfying the estimate*

$$\|\boldsymbol{\xi}\|_{\mathbf{W}^{1,p}(\Omega)} \leq C\left(\|\boldsymbol{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)}\right). \tag{5.3}$$

Furthermore, if Ω *is of class* $\mathcal{C}^{2,1}$*,* $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ *and* $\boldsymbol{h}\times\boldsymbol{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$*, then* $\boldsymbol{\xi}$ *belongs to* $\mathbf{W}^{2,p}(\Omega)$*.*

Proof. **i) Uniqueness.** To prove the uniqueness of $\boldsymbol{\xi}$, we take $\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{h} = \mathbf{0}$ in (S_1). Then the function $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$ belongs to $\mathbf{L}^p(\Omega)$ and

$$\operatorname{div}\boldsymbol{z} = 0, \quad \mathbf{curl}\,\boldsymbol{z} = \mathbf{0} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{z}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma_D, \ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N.$$

This implies that $\boldsymbol{z} \in \mathbf{K}_0^p(\Omega)$. Thus, we can write $\boldsymbol{z}$ as

$$\boldsymbol{z} = \sum_{\ell=1}^{L_D} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} \nabla q_\ell.$$

Since $\mathbf{K}_0^p(\Omega) \subset \mathbf{W}^{1,q}(\Omega)$ for any $q \geq 1$, in particular $\boldsymbol{z}$ belongs to $\mathbf{L}^2(\Omega)$ and we have

$$\int_\Omega |\boldsymbol{z}|^2\,dx = \int_\Omega \boldsymbol{z}\cdot\mathbf{curl}\,\boldsymbol{\xi}\,dx = \sum_{\ell=1}^{L_D} \langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\nabla q_\ell\,dx = 0,$$

which means that $\mathbf{curl}\,\boldsymbol{\xi} = \mathbf{0}$. Then $\boldsymbol{\xi}$ belongs to $\widetilde{\mathbf{K}}_0^p(\Omega)$. As the fluxes of $\boldsymbol{\xi}$ on the connected components of Γ_N and on the cuts Σ_j, with $1 \leq j \leq J$, are equal to zero, we conclude that $\boldsymbol{\xi} = \mathbf{0}$ and this completes the uniqueness proof.

ii) Compatibility conditions. The weak formulation of (S_1) is given as follow:

$$\begin{aligned}
&\text{Find} \quad \boldsymbol{\xi} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega),\\
&\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx + \langle \boldsymbol{h}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_{\Gamma_D}.
\end{aligned} \tag{5.4}$$

So the first compatibility condition (5.1) appears directly by taking $\boldsymbol{\varphi} \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$. Setting $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$, it is clear that

$$\forall \varphi \in W^{2,p'}(\Omega); \quad \langle \mathbf{curl}\,\boldsymbol{z}\cdot\boldsymbol{n}, \varphi\rangle_{\Gamma_D} = -\langle \boldsymbol{z}\times\boldsymbol{n}, \nabla\varphi\rangle_{\Gamma_D},$$

where $< \cdot,\cdot >_{\Gamma_D}$ denotes the duality product between $W^{1/p,p}(\Gamma_D)$ and $W^{-1/p,p'}(\Gamma_D)$. So since $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$, we have

$$\langle \boldsymbol{f}\cdot\boldsymbol{n}, \varphi\rangle_{\Gamma_D} = -\langle \boldsymbol{h}\times\boldsymbol{n}, \nabla\varphi\rangle_{\Gamma_D} = \langle \operatorname{div}_{\Gamma_D}(\boldsymbol{h}\times\boldsymbol{n}), \varphi\rangle_{\Gamma_D}.$$

Hence $\boldsymbol{f}\cdot\boldsymbol{n} = \operatorname{div}_{\Gamma_D}(\boldsymbol{h}\times\boldsymbol{n})$ in the sense of $W^{-1-1/p,p}(\Gamma_D)$ (and also in the sense of $W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)$).

iii) Existence. Using the Inf-Sup condition (4.40), we know that Problem (5.4) admits a unique solution $\boldsymbol{u} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \hookrightarrow \mathbf{W}^{1,p}(\Omega)$. In order to extend (5.4) to any test function in $\widetilde{\mathbf{X}}_0^{p'}(\Omega)$, we use the same argument as in Proposition 4.20 which enable us to prove that every solution of (5.4) also solves (S_1).

iv) Estimate. The estimate (5.3) is obtained by using the same tools as in Proposition 4.17.

v) Regularity. We set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{\xi}$. Hence $\boldsymbol{z} \in \mathbf{L}^p(\Omega)$, $\mathbf{curl}\, \boldsymbol{z} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ on Γ_D and $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ_N. Due to Corollary 3.4, $\boldsymbol{z}$ belongs to $\mathbf{W}^{1,p}(\Omega)$. Since $\boldsymbol{\xi} \in \mathbf{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω, $\mathbf{curl}\, \boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$, $\boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0}$ on Γ_N and $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$ on Γ_D, then according to Corollary 3.5, we deduce that $\boldsymbol{\xi}$ belongs to $\mathbf{W}^{2,p}(\Omega)$. ∎

Remark 5.2. Assume that $\boldsymbol{h} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D and suppose that (5.1)-(5.2) hold. Then we have $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ on Γ_D with $\langle \boldsymbol{f} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0$ for any $0 \le \ell \le L_N$ and $\langle \boldsymbol{f} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$ for any $1 \le j \le J$ (see the proof of Proposition 3.8). Then due to Theorem 4.21, there exists a unique $\boldsymbol{z} \in \mathbf{W}^{1,r(p)}(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$ such that $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{z}$, $\operatorname{div} \boldsymbol{z} = 0$ in Ω satisfying $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ_D and $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ_N. Moreover, $\langle \boldsymbol{z} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_D^\ell} = 0$ for any $0 \le \ell \le L_D$. Now, according to Theorem 4.18 where we interchange Γ_D and Γ_N, there exists a unique $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\Omega)$ such that $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{\xi}$ and $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω satisfying $\boldsymbol{\xi} \times \boldsymbol{n} = \mathbf{0}$ on Γ_N and $\boldsymbol{\xi} \cdot \boldsymbol{n} = 0$ on Γ_D. Moreover, $\langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0$ for any $0 \le \ell \le L_N$ and $\langle \boldsymbol{\xi} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$ for any $1 \le j \le J$, thus $\boldsymbol{\xi}$ is the unique solution of Problem (S_1).

We state in the following theorem the existence and uniqueness of weak solutions to Problem (S). Furthermore, we give more regularity properties to that solution, which is the last main result of this work.

THEOREM 5.3. *Assume that* $\boldsymbol{f} \in \mathbf{L}^{r(p)}(\Omega)$, $\boldsymbol{h} \times \boldsymbol{n} \in \mathbf{W}^{-1/p,p}(\Gamma_D)$, $\operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n}) \in W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)$ *and* $\pi_0 \in W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)$ *satisfying the compatibility condition for any* $\boldsymbol{\varphi} \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx - \int_{\Gamma_N} \pi_0 \boldsymbol{\varphi} \cdot \boldsymbol{n}\, ds + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} = 0. \tag{5.5}$$

Then Problem (S) has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega) \times W^{1,r(p)}(\Omega)$ *satisfying the estimate*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,r(p)}(\Omega)} \le C\Big(&\|\boldsymbol{f}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)} \\ &+ \|\operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)}\Big). \end{aligned} \tag{5.6}$$

Furthermore, if Ω is of class $\mathcal{C}^{2,1}$, $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $\boldsymbol{h} \times \boldsymbol{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_D)$ *and* $\pi_0 \in W^{1-1/p,p}(\Gamma_N)$ *then the solution* $(\boldsymbol{u}, \pi)$ *belongs to* $\mathbf{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ *and the following estimate holds*

$$\begin{aligned} \|\boldsymbol{u}\|_{\mathbf{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C\Big(&\|\boldsymbol{f}\|_{\mathbf{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_D)} \\ &+ \|\pi_0\|_{W^{1-1/p,p}(\Gamma_N)}\Big). \end{aligned} \tag{5.7}$$

Proof. **i)** To get the compatibility condition, we give the weak formulation of (S)

$$\begin{aligned} &\text{Find} \quad \boldsymbol{u} \in \widetilde{\mathbf{W}}_\Sigma^p(\Omega) \quad \text{such that for any} \quad \boldsymbol{\varphi} \in \widetilde{\mathbf{W}}_\Sigma^{p'}(\Omega), \\ &\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, dx - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx + \\ &+ \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\varphi}\rangle_{\mathbf{W}^{-1/p,p}(\Gamma_D) \times \mathbf{W}^{1/p,p'}(\Gamma_D)} - \int_{\Gamma_N} \pi_0 \boldsymbol{\varphi} \cdot \boldsymbol{n}\, ds. \end{aligned} \tag{5.8}$$

By taking $\varphi \in \widetilde{\mathbf{K}}_0^{p'}(\Omega)$, we deduce that (5.5) holds.

ii) Note that applying the divergence operator to the Stokes equation leads to

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in} \quad \Omega.$$

Setting then $\boldsymbol{\psi} = \operatorname{\mathbf{curl}} \boldsymbol{u}$, we have

$$-\Delta \boldsymbol{u} = \operatorname{\mathbf{curl}} \boldsymbol{\psi} \quad \text{in} \quad \Omega,$$

and

$$-\Delta \boldsymbol{u} \cdot \boldsymbol{n} = \operatorname{\mathbf{curl}} \boldsymbol{\psi} \cdot \boldsymbol{n} = (\boldsymbol{f} - \nabla \pi) \cdot \boldsymbol{n}.$$

So the pressure satisfies the following boundary conditions

$$(\nabla \pi - \boldsymbol{f}) \cdot \boldsymbol{n} = \operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n}) \quad \text{on} \quad \Gamma_D, \quad \pi = \pi_0 \quad \text{on} \quad \Gamma_N.$$

We infer that the pressure can be found independently of the velocity field. We solve now the following elliptic problem subjected to Dirichlet and Neumann boundary conditions

$$\text{(5.9)} \qquad \left\{ \begin{array}{l} \Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in} \quad \Omega \\ (\nabla \pi - \boldsymbol{f}) \cdot \boldsymbol{n} = \operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n}) \quad \text{on} \quad \Gamma_D, \quad \pi = \pi_0 \quad \text{on} \quad \Gamma_N. \end{array} \right.$$

Let $\theta \in W^{1,r(p)}(\Omega)$ be the unique solution of

$$\left\{ \begin{array}{l} \Delta \theta = 0 \quad \text{in} \quad \Omega, \\ \theta = \pi_0 \quad \text{on} \quad \Gamma_N, \quad \theta = 0 \quad \text{on} \quad \Gamma_D \end{array} \right.$$

and $\chi \in W^{1,r(p)}(\Omega)$ be the unique solution of

$$\left\{ \begin{array}{l} \Delta \chi = \operatorname{div} \boldsymbol{f} \quad \text{in} \quad \Omega, \\ (\nabla \chi - \boldsymbol{f}) \cdot \boldsymbol{n} = \operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n}) - \frac{\partial \theta}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma_D, \quad \chi = 0 \quad \text{on} \quad \Gamma_N. \end{array} \right.$$

Moreover θ and χ satisfy respectively the following estimates

$$\|\theta\|_{W^{1,r(p)}(\Omega)} \leq C \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)},$$

and

$$\begin{aligned} \|\chi\|_{W^{1,r(p)}(\Omega)} \leq C \Big(& \|f\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)} \\ & + \|\operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} \Big). \end{aligned}$$

Setting $\pi = \chi + \theta$, we have

$$\left\{ \begin{array}{l} \Delta \chi = \operatorname{div} \boldsymbol{f} \quad \text{in} \quad \Omega, \\ (\nabla \chi - \boldsymbol{f}) \cdot \boldsymbol{n} = \operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n}) \quad \text{on} \quad \Gamma_D, \quad \chi = 0 \quad \text{on} \quad \Gamma_N. \end{array} \right.$$

This implies the existence and uniqueness of $\pi \in W^{1,r(p)}(\Omega)$ solution of (5.9) satisfying the estimate

$$\|\pi\|_{W^{1,r(p)}(\Omega)} \leq C \big(\|f\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\pi_0\|_{W^{1-\frac{1}{r(p)},r(p)}(\Gamma_N)} + \|\operatorname{div}_{\Gamma_D}(\boldsymbol{h} \times \boldsymbol{n})\|_{W^{-\frac{1}{r(p)},r(p)}(\Gamma_D)} \big).$$

iii) Setting $\mathbf{F} = \boldsymbol{f} - \nabla\pi$, since the conditions (5.1)-(5.2) hold, we know from Proposition 5.1, that there exists a unique $\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ satisfying $-\Delta\boldsymbol{u} = \boldsymbol{F}$ and $\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ in Γ_N, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, $\quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ_D, $\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_N^\ell} = 0$, for any $0 \le \ell \le L_N$, $\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j} = 0$, for any $1 \le j \le J$. Moreover, we have the following estimate

$$\|\boldsymbol{u}\|_{\mathbf{W}^{1,p}(\Omega)} \le C\big(\|\boldsymbol{F}\|_{\mathbf{L}^{r(p)}(\Omega)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\mathbf{W}^{-1/p,p}(\Gamma_D)}\big).$$

Hence the problem (S) admits a unique solution $(\boldsymbol{u},\pi) \in \mathbf{W}^{1,p}(\Omega)\times W^{1,r(p)}(\Omega)$ satisfying the required estimate (5.6).
According to Proposition 5.1, we know that if Ω is of class $\mathcal{C}^{2,1}$, $\boldsymbol{f}\in\mathbf{L}^p(\Omega)$, $\boldsymbol{h}\times\boldsymbol{n}\in\mathbf{W}^{1-1/p,p}(\Gamma_D)$ and $\pi_0 \in W^{1-1/p,p}(\Gamma_N)$ then $\boldsymbol{u}$ belongs to $\mathbf{W}^{2,p}(\Omega)$ and $\pi\in W^{1,p}(\Omega)$. The estimate (5.7) is readily deduced. ∎

Remark 5.4. We also can consider the case where $\partial\Sigma \subset \Gamma_N$ in the Stokes problem (S) which can be solved by using the first Inf-Sup condition and the first elliptic problem.

REFERENCES

[1] P. Acevedo, C. Amrouche, C. Conca, and A. Ghosh, *Stokes and Navier–Stokes equations with Navier boundary condition*, C. R. Math, 357 (2019), pp. 115–119.

[2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, *Vector potentials in three-dimensional non-smooth domains*, Math. Meth. Appl. Sci, 21 (1998), pp. 823–864.

[3] C. Amrouche and A. Rejaibia, *L^p-theory for Stokes and Navier-Stokes equations with Navier boundary condition*, J. Differential Equations, 256 (2014), pp. 1551–1547.

[4] C. Amrouche and N. Seloula, *On the Stokes equations with the Navier-type boundary conditions*, Differ. Equ. Appl, 3 (2011), pp. 581–607.

[5] C. Amrouche and N. Seloula, *L^p-theory for vector potentials and Sobolev's inequalities for vector fields. Applications to the Stokes equations with pressure boundary conditions*, Math. Mod. and Meth. in App, 23 (2013), pp. 37–92.

[6] G. Beavers and D. Joseph, *Boundary conditions at a naturally permeable wall*, J. Fluid Mech, 30 (1967), pp. 197–207.

[7] H. Beirão da Veiga, *Regularity for Stokes and generalized Stokes system under non-homogeneous slip-type boundary conditions*, Adv Differential Equ, 9 (2004), pp. 1079–1114.

[8] F. Ben Belgassem, C. Bernardi, M. Costabel, and M. Dauge, *Un réultat de densité pour les équations de maxwell*, C. R. Acad. Sci. Paris, t. 324, Serie I, (1997), pp. 731–736.

[9] A. Bendali, J. Dominguez, and S. Gallic, *A variational approah for the vector potential formulation of the Stokes and Navier-Stokes problems in three dimensional domains*, J. Math. Anal. App, 107 (1985), pp. 537–560.

[10] M. Beneš and P. Kučera, *Solutions to the Navier-Stokes equations with mixed boundary conditions in two-dimensional bounded domains*, Math. Nachr, 289 (2016), pp. 194–212.

[11] J. Bernard, *Non-standard Stokes and Navier–Stokes problem: existence and regularity in stationary case*, Math. Meth. Appl. Sci, 25 (2002), pp. 627–661.

[12] J. Bernard, *Time-dependent Stokes and Navier-Stokes problems with boundary conditions involving the pressure, existence and regularity*, Nonlinear Anal. Real World Appl, 4 (2003), pp. 805–839.

[13] S. Bernardi, T. Chacon Rebello, and D. Yakoubi, *Finite element discretization of the Stokes and Navier-Stokes equations with boundary condition on pressure*, SIAM J. Numer. Anal, 53 (2015), pp. 826–850.

[14] S. Bertoluzza, V. Chabannes, C. Prud'homme, and M. Szopos, *Boundary conditions involving pressure for the Stokes problem and applications in computational hemodynamics*, Comp. Meth. Appl. Mech. Engin, 322 (2017), pp. 58–80.

[15] M. Boukrouche, I. Boussetouan, and L. Paoli, *Non-isothermal Navier-Stokes system with mixed boundary conditions and friction law: Uniqueness and regularity properties*, Nonlinear Anal Theory Methods Appl, 102 (2014), pp. 168–185.

[16] M. Boukrouche, I. Boussetouan, and L. Paoli, *Existence for non-isothermal fluid flows with Tresca's friction and Cattaneo's heat law*, J. Math. Anal. Appl, 427 (2015), pp. 499–514.

[17] M. Boukrouche, I. Boussetouan, and L. Paoli, *Existence and approximation for Navier-Stokes system with Tresca's friction at the boundary and heat transfer governed by Cattaneo's law*, Math. Mech. solids, 23 (2018), pp. 519–540.

[18] C. Conca, F. Murat, and O. Pironneau, *The Stokes and Navier-Stokes equations with boundary conditions involving the pressure*, Japan. J. Math. New series, 20 (1994), pp. 279–318.

[19] P. A. Durbin, *Considerations on the moving contact-line singularity, with application to frictional drag on a slender drop*, J. Fluid Mech., 197 (1988), pp. 157–169.

[20] J. Foucher-Incaux, *Artificial boundaries and formulations for the incompressible Navier-Stokes equations : applications to air and blood flows*, SeMA Journal, 64 (2014), pp. 1–40.

[21] H. Fujita, *A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions*, Mathematical Fluid Mechanics and Modeling, 888 (1994), pp. 199–216.

[22] V. Girault, *Incompressible finite element methods for Navier-Stokes equations with non-standard boundary conditions in* $\mathbb{R}^3$, Math. Comput, 51 (1988), pp. 55–74.

[23] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations, theory and algorithms*, Springer-Verlag, 1986.

[24] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Publishing Inc, 2nd ed., 1985.

[25] D. Iftimie, E. Raugel, and G. R. Sell, *Navier-Stokes equations in thin 3D domains with Navier boundary conditions*, Indiana Univ. Math. J, 56 (2007), pp. 1083–1156.

[26] J. K. Kelliher, *Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane*, SIAM J. Math. Anal, 38 (2004), pp. 210–232.

[27] J. L. Lions, R. Temam, and S. Wang, *Mathematical theory for the coupled atmosphere-ocean models (CAO III)*, J. Math. Pures Appl., 74 (1995), pp. 105–163.

[28] R. Mazya and J. Rossman, *Elliptic equations in polyhedral domains (mathematical surveys and monographs)*, 162 (2010).

[29] D. Mitrea, M. Mitrea, and J. Pipher, *Vector potential theory on nonsmooth domains in* $\mathbb{R}^3$ *and applications to electromagnetic scattering)*, Fourier Anal. Appl., 3 (1997), pp. 131–192.

[30] C. Navier, *Sur les lois d'équilibre et du mouvement des corps élastiques*, Mém. Acad. Sci., 7 (1827), pp. 375–394.

[31] K. R. Rajagopal and P. N. Kaloni, *Some remarks on boundary conditions for flows of fluids of the differential type*, Cont. Mech. and its Applications, (Hemisphere Press, New York), (1989), pp. 935–942.

[32] V. A. Solonnikov and V. E. Scadilov, *A certain boundary value problem for the stationary system of Navier-Stokes equations, (Russian)*, Trudy Mat. Inst. Steklov, (1973), pp. 1996–2100.

[33] W. Von Wahl, *Estimating* ∇u *by* $\operatorname{div} u$, $\operatorname{curl} u$, Math. Methods Appl. Sci, 15 (1992), pp. 123–143.