Quasi-Monte Carlo finite element analysis for wave propagation in heterogeneous random media

M. Ganesh[†], Frances Y. Kuo[‡] and Ian H. Sloan[‡]

January 2021

Abstract

We propose and analyze a quasi-Monte Carlo (QMC) algorithm for efficient simulation of wave propagation modeled by the Helmholtz equation in a bounded region in which the refractive index is random and spatially heterogenous. Our focus is on the case in which the region can contain multiple wavelengths. We bypass the usual sign-indefiniteness of the Helmholtz problem by switching to an alternative sign-definite formulation recently developed by Ganesh and Morgenstern (Numerical Algorithms, **83**, 1441–1487, 2020). The price to pay is that the regularity analysis required for QMC methods becomes much more technical. Nevertheless we obtain a complete analysis with error comprising stochastic dimension truncation error, finite element error and cubature error, with results comparable to those obtained for the diffusion problem.

Keywords: quasi-Monte Carlo method, finite element method, wave propagation, heterogeneous, random media, Helmholtz equation, coercive AMS Subject Classification (2010): 35J05, 35R60, 65D30, 65D32, 65N30

1 Introduction

This paper is concerned with a new algorithm and associated numerical analysis for efficient simulation of wave propagation modeled by the Helmholtz equation in a bounded region in which the refractive index is random and spatially heterogenous. The wave is induced by an impinging incident wave, and our focus is on the case in which the region can contain multiple wavelengths. The main aim of this article is to compute the expected value of a linear functional of the resulting wave field by the use of a well designed *Quasi-Monte Carlo (QMC)* method [9, 11, 31, 34], and to bound the resulting error.

The design and analysis of QMC methods has been well studied for the classical diffusion problem, see for example [19, 20, 21, 25, 27, 28]. However, it is well known that the standard Galerkin variational formulation for the Helmholtz partial differential equation (PDE) lacks positive definiteness unless the wavelength is relatively large compared to the region. The resulting lack of coercivity (or sign-definiteness) rules out the standard QMC analysis that has recently been used successfully for strongly elliptic diffusion problems with random input. The analysis of QMC methods has also been extended to a general class of operator equations, see [33] and subsequent papers, e.g., [7, 10, 17, 18]. These papers include the case of the Helmholtz equation, under an appropriate inf-sup condition on the standard Galerkin variational

[†]Department of Applied Mathematics & Statistics, Colorado School of Mines, Golden, Colorado 80401, USA (mganesh@mines.edu).

[‡]School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia (f.kuo@unsw.edu.au, i.sloan@unsw.edu.au).

formulation and assumptions on the wavelength and the random component of the refractive index.

In this paper we bypass the sign-indefiniteness problem in a different way, by using the recently developed sign-definite deterministic formulation of Ganesh and Morgenstern [16]. In the present work that analysis is extended to include randomness in the heterogeneous refractive index. In the resulting QMC analysis there is a price to pay for using the sign-definite formulation of [16], in that the analysis of regularity with respect to the stochastic variables becomes complicated, and a new approach is needed. On the other hand it has the advantage that the space discretization can be carried out with the standard Galerkin scheme without any threat of instability.

Precisely, we study the wave propagation problem in a bounded domain $D \subset \mathbb{R}^d$, for d = 2, 3 with Lipschitz boundary ∂D . The incident wave is of wavelength $\lambda = 2\pi/k$, where k is the positive wavenumber, and our interest extends to wavelengths λ smaller than L, where L is a characteristic length of D, or equivalently to $kL > 2\pi$. The square of the refractive index, $n(\boldsymbol{x}, \omega)$, in the interior of D may be spatially varying, and is also random, as described below.

For a deterministic forcing function $f \in L^2(D)$ and boundary data $g \in L^2(\partial D)$, and for almost all elementary events ω in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the unknown field $u(\cdot, \omega) \in$ $H^1(D)$ is assumed to satisfy the Helmholtz PDE and an absorbing boundary condition

$$(\mathcal{L}u)(\boldsymbol{x},\omega) = -f(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \quad \text{and} \quad \frac{\partial u}{\partial \vec{\mathbf{n}}}(\widetilde{\boldsymbol{x}},\omega) - \mathrm{i} \ k \ u(\widetilde{\boldsymbol{x}},\omega) = g(\widetilde{\boldsymbol{x}}), \quad \widetilde{\boldsymbol{x}} \in \partial D, \quad (1.1)$$

where the stochastic Helmholtz operator is given by

$$(\mathcal{L}u)(\boldsymbol{x},\omega) := \Delta u(\boldsymbol{x},\omega) + k^2 n(\boldsymbol{x},\omega) u(\boldsymbol{x},\omega), \qquad \boldsymbol{x} \in D, \quad \omega \in (\Omega, \mathcal{A}, \mathbb{P}).$$
(1.2)

Here $\vec{\mathbf{n}} = \vec{\mathbf{n}}(\tilde{x})$ is the outward-pointing unit normal vector, defined almost everywhere on the surface ∂D of the Lipschitz domain D. The system (1.1) is a well known model for a wide class of applications, including acoustic, electromagnetic, and seismic wave propagation in heterogeneous media [5, 26, 30]. The boundary condition in (1.1) is standard for the interior wave propagation model and, as described in [16] and references therein, it can be either considered as an approximation of the Sommerfeld radiation condition occurring in the unbounded medium counterpart of our model, or can be used as an interface condition in the heterogeneous-homogeneous coupled wave propagation model [6, 14].

The random coefficient $n(\boldsymbol{x},\omega), \boldsymbol{x} \in D$, is taken in this article to be parameterized by an infinite-dimensional vector $\boldsymbol{y}(\omega) = (y_1(\omega), y_2(\omega), \ldots)$. For a fixed realization $\omega \in \Omega$, we denote the corresponding deterministic parametric coefficient by $n(\boldsymbol{x}, \boldsymbol{y})$ and the associated solution to the above PDE model by $u(\boldsymbol{x}, \boldsymbol{y})$. We assume that the parameter \boldsymbol{y} is uniformly distributed on

$$U := [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$$

with the uniform probability measure $\mu(d\mathbf{y}) = \bigotimes_{j \ge 1} dy_j = d\mathbf{y}$, where \mathbb{N} is the set of positive integers.

The non-negative, uncertain, coefficient n(x, y) is assumed to be expressible as a mean field $n_0(x)$ plus a perturbation,

$$n(\boldsymbol{x},\boldsymbol{y}) = n_0(\boldsymbol{x}) + \sum_{j\geq 1} y_j \psi_j(\boldsymbol{x}), \qquad \boldsymbol{x} \in D, \quad \boldsymbol{y} \in U,$$
(1.3)

where the functions $\psi_j(\mathbf{x})$ are given. For example, the functions ψ_j may belong to the Karhunen-Loève eigensystem of a covariance operator, or other suitable function systems in $L^2(D)$. We note that n and n_0 represent the square of the non-zero physical refractive index, and hence n and n_0 are positive.

The paper [13] studied a different computational scheme for wave propagation in random media, using the standard sign-indefinite formulation of the Helmholtz equation. In that paper the squared refractive index n was taken to be of the form

$$n(\boldsymbol{x},\omega) = [1 + \epsilon \,\eta(\boldsymbol{x},\omega)]^2, \qquad \boldsymbol{x} \in D, \quad \omega \in \Omega, \tag{1.4}$$

with ϵ a small perturbation parameter controlling the magnitude of the random fluctuation, η being a random process satisfying the constraint that $\mathbb{P}\left\{\omega \in \Omega : \|\eta(\cdot,\omega)\|_{L^{\infty}(D)} \leq 1\right\} = 1$, and Ω being a sample space. With D being star-shaped with respect to the origin, the authors established in [13, Theorem 2.15] well-posedness of the continuous stochastic Helmholtz model, under the restriction that the parameter ϵ is of the order 1/(kL). A discrete form of the stochastic Helmholtz model was then developed in [13] by writing the stochastic solution as a series in powers of $\epsilon^{j}, j = 0, 1, 2, \ldots$. The coefficients in the series expansion were approximated in [13] using the interior-penalty discontinuous Galerkin (IPDG) discretization method in space, and Monte Carlo (MC) cubature in the stochastic variables, the IPDG method being chosen because of its unconditional stability. Because of the low-order convergence of MC approximations, the approach in [13] requires substantially more sampling points in Ω (and hence more Helmholtz system solves) compared to the higher-order QMC method and finite element method (FEM) Galerkin scheme used in the present paper.

The general operator-theory approach in [7] and the expansion in powers of ϵ approach in [13] both require that, roughly speaking,

$kL \times$ (some norm of the stochastic variables) be not large.

(For the case of [7] see Appendix A.) We shall see in (1.5) below that the same is true of the present method. Thus all three approaches have this feature, but with the difference that in [7] the requirement is absolute (see (A.3)), whereas in the present work the consequence of taking larger values of kL is only to increase the constants in our error bounds.

Yet another approach to the Helmholtz problem with random refractive index has been proposed recently in [32]. There well-posedness and stability of the sign-indefinite formulation of the stochastic continuous problem has been proved. However, that article [32] does not consider any form of numerical discretization. Other recent papers concerned with the Helmholtz problem with variable coefficients are [1, 3, 4, 23, 24].

The main challenge in the present article lies in the design and numerical analysis of a highorder QMC-FEM for the evaluation of expected values (that is, of integrals with respect to y over a hypercube of length 1) of linear functionals of the solution u. As with the earlier applications of QMC-FEM to diffusion problems, the key is to find computable bounds on appropriate mixed partial derivatives of u with respect to components of y. The difference in this case is that finding such bounds is now very much harder. The reason for the additional difficulty lies in the much greater complexity of the coercive formulation [16]. In particular, unlike the situation with the diffusion problem, both the trial and test functions of the QMC-FEM analysis have stochastic components.

More precisely, for each $\boldsymbol{y} \in U$, we seek a continuous wave field solution in a special subspace V of $H^1(D)$, see (2.2) below. We fix $u(\cdot, \boldsymbol{y}) \in V$ to be the unique solution of a sign-definite weak formulation of (1.1)–(1.3), and we consider the quantity of interest (QoI) to be a bounded linear functional $G \in V^*$ of $u(\cdot, \boldsymbol{y})$, denoted in this article by $[G(u)](\boldsymbol{y}) = G(u(\cdot, \boldsymbol{y}))$, where V^* denotes the dual space of V, with norm given by (2.10) below. An example of $G \in V^*$ is the average wave field in the heterogenous medium: $G(u(\cdot, \boldsymbol{y})) = \int_D u(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x}$. The aim of this

article is to design and analyze efficient QMC-FEM algorithms to compute approximations to the expected value of $G(u(\cdot, y))$, expressed as an infinite-dimensional integral over y:

$$\int_{[-\frac{1}{2},\frac{1}{2}]^{\mathbb{N}}} G(u(\cdot,\boldsymbol{y})) \,\mathrm{d}\boldsymbol{y}.$$

Key ingredients of our strategy are: (i) truncating the infinite series in (1.3) to finitely many s terms; (ii) discretizing the solution in the spatial variable using FEM based on a mesh parameter h; and (iii) approximating the expected value integral by an N-point QMC cubature rule.

For $kL \ge 1$, we prove that the combined error for the QMC-FEM approximation is of the order

$$s^{-\frac{2}{p_0}+1} + kLh^p + \begin{cases} N^{-\min\left(\frac{1}{p_1}-\frac{1}{2},1-\delta\right)}, \ \delta \in (0,\frac{1}{2}), & \text{for first order randomized QMC}, \\ N^{-\frac{1}{p_1}}, & \text{for higher order deterministic QMC}, \end{cases}$$

where p is the degree of the finite element spline basis functions constructed using a tessellation of D with mesh-width h, and $p_0, p_1 \in (0, 1)$ satisfy the summability and wavenumber decay conditions

$$\sum_{j\geq 1} \left(kL \, \|\psi_j\|_{L^{\infty}(D)} \right)^{p_0} \leq K_0 \quad \text{and} \quad \sum_{j\geq 1} \left(kL \, \|\psi_j\|_{W^{1,\infty}(D)} \right)^{p_1} \leq K_1, \quad (1.5)$$

with $K_0, K_1 \in \mathbb{R}$ independent of the wavenumber k. In particular, the order constant in the error bound depends on f, g, G, but is independent of k.

The rest of this article is organized as follows. In Section 2, for each fixed $\mathbf{y} \in U$, we introduce the coercive formulation of the stochastic model, and recall from [16] a wavenumberexplicit spatial regularity bound on the unique solution. In Section 3 we provide an overview of the analysis needed to obtain the final combined error bound. In Section 4 we derive explicit bounds on partial derivatives with respect to components of \mathbf{y} of the solution u, as needed for the QMC analysis and the construction of QMC points. In Section 5 we quantify the effect of truncation of the infinite series for $n(\mathbf{x}, \mathbf{y})$. In Section 6 we describe the error associated with high-order FEM discretization. In Section 7 we focus on the efficient choice of the randomized and deterministic QMC quadrature rules. In Appendix A we describe the alternative small perturbation QMC-FEM approach. In Appendix B we prove a technical lemma.

2 A coercive reformulation of the stochastic Helmholtz model

A coercive variational formulation was developed and analyzed recently in [16] for a deterministic wave propagation model with an inhomogeneous absorbing boundary condition. Here we extend the method to our stochastic model.

The first step is to recognize that given data $f \in L^2(D), g \in L^2(\partial D)$, for each fixed $\boldsymbol{y} \in U$, any sufficiently regular solution $u(\cdot, \boldsymbol{y}) \in H^1(D)$ of our model boundary value problem (BVP)

$$\begin{bmatrix} \Delta + k^2 n(\boldsymbol{x}, \boldsymbol{y}) \end{bmatrix} u(\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \quad \text{and}$$

$$\frac{\partial u}{\partial \vec{\mathbf{n}}} (\widetilde{\boldsymbol{x}}, \boldsymbol{y}) - i \ k \ u(\widetilde{\boldsymbol{x}}, \boldsymbol{y}) = g(\widetilde{\boldsymbol{x}}), \quad \widetilde{\boldsymbol{x}} \in \partial D,$$
(2.1)

has three additional smoothness properties: (i) $\Delta u(\cdot, \boldsymbol{y}) \in L^2(D)$; (ii) $\frac{\partial u}{\partial \mathbf{\tilde{n}}}(\cdot, \boldsymbol{y}) \in L^2(\partial D)$; and (iii) $u(\cdot, \boldsymbol{y}) \in H^1(\partial D)$. The first two properties follow directly from (2.1) and the third property follows from the fact that $\nabla u(\cdot, \boldsymbol{y}) \in L^2(\partial D)$, since $\nabla u(\cdot, \boldsymbol{y}) = \vec{\mathbf{n}} \frac{\partial u}{\partial \vec{\mathbf{n}}}(\cdot, \boldsymbol{y}) + \nabla_{\partial D}u(\cdot, \boldsymbol{y})$ and we have the regularity result from [29, Theorem 4.2] that surface gradient $\nabla_{\partial D}u(\cdot, \boldsymbol{y}) \in L^2(\partial D)$. We incorporate such natural smoothness properties of the Helmholtz PDE model (2.1) in the following Hilbert space:

$$V := \left\{ w : w \in H^1(D), \ \Delta w \in L^2(D), \ w \in H^1(\partial D), \ \frac{\partial w}{\partial \vec{\mathbf{n}}} \in L^2(\partial D) \right\}.$$
(2.2)

Following [15, 16], for the stochastic heterogenous model we equip V with the following norm

$$\|w\|_{V}^{2} := k^{2} \|w\|_{L^{2}(D)}^{2} + \|\nabla w\|_{L^{2}(D)}^{2} + \frac{1}{k^{2}} \|\Delta w\|_{L^{2}(D)}^{2} + L\left(k^{2} \|w\|_{L^{2}(\partial D)}^{2} + \|\nabla_{\partial D} w\|_{L^{2}(\partial D)}^{2} + \left\|\frac{\partial w}{\partial \vec{\mathbf{n}}}\right\|_{L^{2}(\partial D)}^{2}\right), \qquad (2.3)$$

where L is a characteristic length of the Lipschitz domain $D \subset \mathbb{R}^d$, d = 2, 3. Note that each term in (2.3) scales in the same way under a change of length scale. Throughout this article, when considering the trace of a function $w \in H^s(D)$ as a function in $H^{s-1/2}(\partial D)$, for notational convenience we drop the Dirichlet trace operator γ . (That is, we drop γ and write w instead of γw whenever it is considered as a function on ∂D .)

For each fixed $y \in U$, to prove the unique solvability of the BVP (2.1) we ensure the coercivity property of the variational formulation by assuming the following three conditions on the geometry and medium of the wave propagation:

(A0) The domain $D \subset \mathbb{R}^d$, for d = 2, 3, with diameter L, is star-shaped with respect to a ball centered at the origin. That is, there exist constants $\widehat{\gamma}$, $\widehat{\mu}$ with $0 < \widehat{\gamma} \leq \widehat{\mu} \leq 1$ such that

$$\widehat{\gamma} L \leq \widetilde{x} \cdot \vec{\mathbf{n}}(\widetilde{x}) \leq \widehat{\mu} L, \qquad \widetilde{x} \in \partial D.$$

We now fix L by defining $L := \sup_{x \in D} ||x||$, where ||x|| is the Euclidean norm of x.

(A1) For $\boldsymbol{y} \in U$ and $\boldsymbol{x} \in D$, there exist constants $n_{\max}, n_{\min}, b_{\max}$ and b_{\min} such that almost everywhere

$$0 < n_{\min} \le n(\boldsymbol{x}, \boldsymbol{y}) \le n_{\max}, \qquad (2.4)$$

$$0 < b_{\min} \leq \nabla \cdot (\boldsymbol{x} \, n(\boldsymbol{x}, \boldsymbol{y})) \leq b_{\max}, \qquad b_{\min} > (d-2) \, n_{\max}. \tag{2.5}$$

(A2) The mean field and perturbation functions satisfy $n_0 \in W^{1,\infty}(D)$, $\psi_j \in W^{1,\infty}(D)$ and $\sum_{j>1} \|\psi_j\|_{W^{1,\infty}(D)} < \infty$, where throughout the article

$$\|w\|_{W^{1,\infty}(D)} := \max\left\{\|w\|_{L^{\infty}(D)}, \ L \|\nabla w\|_{L^{\infty}(D)}\right\}.$$

The positivity and boundedness of the refractive index in (2.4) is well known for all practical heterogeneous wave propagation media. As described in detail in [16, Remark 2.1], the two inequalities on b_{\min} in (2.5) are necessary to ensure the physical constraint that the (geometric-optical) rays are *non-trapping* [12, Page 191]. For a detailed geometric interpretation related to the positivity condition in (2.5), see [22, Section 7].

To develop the sign-definite variational formulation of the BVP, we consider the following operators [15, 16]

$$\mathcal{M}_{\ell} w := \boldsymbol{x} \cdot \nabla w - \mathrm{i} \, k \, L \, \beta_{\ell} \, w + \alpha_{\ell} \, w, \quad \ell = 1, 2 \,.$$

$$(2.6)$$

The four parameters $\alpha_1, \alpha_2, \hat{\beta}_1, \hat{\beta}_2 \in \mathbb{R}$ and an additional parameter $A \in \mathbb{R}$ will subsequently play a crucial role. To explain the notation, the three parameters without the "hat" tag are independent of the geometry, while the parameters tagged with a "hat" will occur in this article in combination with the "acoustic size" k L.

Next, for each fixed $\boldsymbol{y} \in U$ and for $f \in L^2(D)$ and $g \in L^2(\partial D)$, with \mathcal{L} and $n(\boldsymbol{x}, \boldsymbol{y})$ given by (1.2)–(1.3), we recall a sesquilinear form $\mathscr{B}_{\boldsymbol{y}} : V \times V \to \mathbb{C}$ and an antilinear functional $\mathscr{G}_{\boldsymbol{y},f,g} = \mathscr{G}_{\boldsymbol{y}} : V \to \mathbb{C}$, introduced in [16]:

$$\mathscr{B}_{\boldsymbol{y}}(v,w) := \int_{D} \left[\left(\mathcal{M}_{2}v + \frac{A}{k^{2}}\mathcal{L}v \right) \overline{\mathcal{L}w} + \left(2 - d + \alpha_{1} + \alpha_{2} + \mathrm{i} \ k \ L(\widehat{\beta}_{1} - \widehat{\beta}_{2}) \right) \nabla v \cdot \overline{\nabla w} \right. \\ \left. + \left(-\alpha_{1} - \alpha_{2} - \mathrm{i} \ k \ L(\widehat{\beta}_{1} - \widehat{\beta}_{2}) \right) k^{2} \ n \ v \ \overline{w} + k^{2} \left(\nabla \cdot (\boldsymbol{x}n) \right) v \ \overline{w} \right] \mathrm{d}\boldsymbol{x} \\ \left. - \int_{\partial D} \left[\overline{\mathcal{M}_{1}w} \ \mathrm{i} \ k \ v + \left(\boldsymbol{x} \cdot \nabla_{\partial D}v - \mathrm{i} \ k \ L(\widehat{\beta}_{2} \ v + \alpha_{2} \ v) \frac{\overline{\partial w}}{\partial \mathbf{n}} \right. \\ \left. + \left(\boldsymbol{x} \cdot \mathbf{n} \right) \left(k^{2} \ n \ v \ \overline{w} - \nabla_{\partial D}v \cdot \overline{\nabla_{\partial D}w} \right) \right] \mathrm{d}\boldsymbol{S} \,, \tag{2.7}$$

and

$$\mathscr{G}_{\boldsymbol{y}}(w) := \int_{D} \left(\overline{\mathcal{M}_{1}w} - \frac{A}{k^{2}} \overline{\mathcal{L}w} \right) f \,\mathrm{d}\boldsymbol{x} + \int_{\partial D} \overline{\mathcal{M}_{1}w} \,g \,\mathrm{d}S \,.$$
(2.8)

Using the technical details in the proof of [16, Section 2], we have the following consistency result connecting the PDE model and the variational formulation determined by the above sesquilinear form and antilinear functional: For each $\boldsymbol{y} \in U$, if $u(\cdot, \boldsymbol{y}) \in H^1(D)$ solves the wave propagation PDE model (2.1), then $u(\cdot, \boldsymbol{y}) \in V$ satisfies the variational equation

$$\mathscr{B}_{\boldsymbol{y}}(u,w) = \mathscr{G}_{\boldsymbol{y}}(w) \quad \text{for all} \quad w \in V.$$
 (2.9)

The following coercivity, continuity, and unique solvability of (2.9) with wavenumber-explicit bounds follow from similar results proved in [16]. In particular, for acoustic size $kL \ge 1$, the V-norm spatial regularity bound of the unique solution of the wave propagation model is independent of the wavenumber. Such wavenumber-explicit bounds play a crucial role in the analysis and construction of QMC approximations. Below we use the standard norm for the dual V^* of V:

$$||G||_{V^*} := \sup\left\{\frac{|G(w)|}{||w||_V} : w \in V, w \neq 0\right\}.$$
(2.10)

Theorem 2.1 ([16, Theorems 2.1, 3.1, 3.2, and 4.1]). Let the assumptions (A0) and (A1) hold. If the three parameters $A, \alpha_1, \hat{\beta}_1$ are chosen such that

$$\frac{d-2}{2} < \alpha_1 < \frac{b_{\min}}{2n_{\max}}, \quad 0 < A < \frac{b_{\min}-2\alpha_1 n_{\max}}{2n_{\max}^2}, \quad \widehat{\beta}_1 \ge \frac{n_{\max}\widehat{\mu}}{2} + \frac{2\widehat{\mu}^2}{\gamma} + \frac{\widehat{\gamma}}{2}, \quad (2.11)$$

then for all $\mathbf{y} \in U$, $f \in L^2(D)$ and $g \in L^2(\partial D)$ we have

$$\operatorname{Re}[\mathscr{B}_{\boldsymbol{y}}(w,w)] \ge C_{\operatorname{coer}} \|w\|_{V}^{2} \qquad \qquad \text{for all} \quad w \in V, \qquad (2.12)$$

$$|\mathscr{B}_{\boldsymbol{y}}(v,w)| \leq C_{\text{cont}}(kL) \|v\|_{V} \|w\|_{V} \qquad \qquad \text{for all } v,w \in V, \qquad (2.13)$$

$$\|\mathscr{G}_{\boldsymbol{y},f,g}\|_{V^*} \leq C_{\text{func}}(kL) \left(L \|f\|_{L^2(D)} + L^{1/2} \|g\|_{L^2(\partial D)} \right), \tag{2.14}$$

with

$$\begin{split} C_{\rm coer} &:= \frac{1}{2} \min \left\{ 2 - d + 2\alpha_1 \,, \, b_{\min} - 2\,\alpha_1 \, n_{\max} - 2\,A \, n_{\max}^2 \,, \, A, \, \frac{\widehat{\gamma}}{2} \right\}, \\ C_{\rm cont}(kL) &:= \sqrt{3} \max \left\{ \begin{array}{l} |2 - d + \alpha_1 + \alpha_2| + kL \, |\widehat{\beta}_1 - \widehat{\beta}_2| \,, \\ A \, n_{\max} + |\alpha_2 - {\rm i} \, kL \, \widehat{\beta}_2| + kL + A \,, \\ \frac{\alpha_1}{kL} + \widehat{\beta}_1 + n_{\max} \, \widehat{\mu} \,, \quad \frac{|\alpha_2|}{kL} + |\widehat{\beta}_2| + 2 \, \widehat{\mu} \,, \quad 2 \,, \\ \left(|\alpha_1 + \alpha_2| + b_{\max} + kL \, |\widehat{\beta}_1 - \widehat{\beta}_2| \right) n_{\max} \\ &+ \left(A \, n_{\max}^2 + n_{\max} |\alpha_2 - {\rm i} \, kL \, \widehat{\beta}_2| \right) + kL \, n_{\max} + A \, n_{\max} \right\}, \\ C_{\rm func}(kL) &:= \sqrt{3} \max \left\{ 1 \,, \, \frac{A}{kL} \,, \, \frac{\alpha_1 + A \, n_{\max}}{kL} + \widehat{\beta}_1 \right\}. \end{split}$$

The coercivity constant C_{coer} is independent of the wavenumber. The continuity constant satisfies $C_{\text{cont}}(kL) = \mathcal{O}(kL + (kL)^{-1})$. The functional constant satisfies $C_{\text{func}}(kL) = \mathcal{O}(1 + (kL)^{-1})$, and so is bounded independently of the wavenumber if $kL \ge 1$.

Consequently, for each $\mathbf{y} \in U$, the variational formulation (2.9) has a unique solution $u(\cdot, \mathbf{y}) \in V$ and satisfied the regularity bound

$$\|u(\cdot, \boldsymbol{y})\|_{V} \leq \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left(L \|f\|_{L^{2}(D)} + L^{1/2} \|g\|_{L^{2}(\partial D)} \right) \quad \text{for all} \quad \boldsymbol{y} \in U,$$
(2.15)

which is bounded independently of the wavenumber if $kL \ge 1$.

We note that (2.13) and (2.14) hold even without the weak non-trapping condition (2.5). In particular, as described in [16], only the proof of coercivity requires all assumptions mentioned in Theorem 2.1.

3 Overview of our method and error analysis

The main aim of this article is to design and analyze efficient algorithms to compute approximations to the expected value of $G(u(\cdot, y))$, expressed as an infinite-dimensional integral over y:

$$I(G(u)) := \int_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathbb{N}}} G(u(\cdot, \boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} := \lim_{s \to \infty} I_s(G(u)), \tag{3.1}$$

with

$$I_s(G(u)) := \int_{[-\frac{1}{2},\frac{1}{2}]^s} G(u(\cdot, (y_1, \dots, y_s, 0, 0, \dots))) \, \mathrm{d}y_1 \cdots \mathrm{d}y_s \,. \tag{3.2}$$

Key ingredients of our strategy are: (i) truncating the infinite series in (1.3) to finitely many s terms, yielding the dimensionally-truncated solution u_s ; (ii) discretizing u_s in the spatial variable using FEM based on a mesh parameter h, leading to the discrete solution $u_{s,h}$; and (iii) approximating the s-dimensional expected value integral of $G(u_{s,h})$ by an N-point QMC cubature rule $Q_{s,N}$. The precise details regarding u_s , $u_{s,h}$ and the QMC rule $Q_{s,N}$ are given in later sections. For now it suffices to say that we can write the combined error using the triangle inequality as a sum of three terms: the dimension truncation error, the FEM discretization error, and the QMC cubature error:

$$|I(G(u)) - Q_{s,N}(G(u_{s,h}))| \le |(I - I_s)(G(u))| + |I_s(G(u_s - u_{s,h}))| + |I_s(G(u_{s,h})) - Q_{s,N}(G(u_{s,h}))|.$$

Alternatively, if the QMC rule is randomized then we have the mean-square error

$$\mathbb{E}_{\text{rqmc}} \Big[|I(G(u)) - Q_{s,N}(G(u_{s,h}); \cdot)|^2 \Big]$$

 $\leq 2 |(I - I_s)(G(u))|^2 + 2 |I_s(G(u_s - u_{s,h}))|^2 + \mathbb{E}_{\text{rqmc}} \Big[|I_s(G(u_{s,h})) - Q_{s,N}(G(u_{s,h}); \cdot)|^2 \Big],$

where the expectation \mathbb{E}_{rqmc} is taken with respect to the random element in the QMC rule (see Section 7).

4 Stochastic parameter regularity of random wave field

For the error analysis it is crucial to understand the behavior of multi-index high-order derivatives of the solution of (2.9) with respect to the stochastic variables y_j , $j \ge 1$. To this end, we first introduce some notation. For a multi-index $\boldsymbol{\nu} = (\nu_j)_{j\ge 1}$ with $\nu_j \in \{0, 1, 2, ...\}$, we write its "order" as $|\boldsymbol{\nu}| := \sum_{j\ge 1} \nu_j$ and its "support" as $\operatorname{supp}(\boldsymbol{\nu}) := \{j \ge 1 : \nu_j \ge 1\}$. Furthermore, we write $\boldsymbol{\nu}! := \prod_{j\ge 1} (\nu_j !)$, which is different from $|\boldsymbol{\nu}|! = (\sum_{j\ge 1} \nu_j)!$. We denote by \mathfrak{F} the (countable) set of all "finitely supported" multi-indices: $\mathfrak{F} := \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : \operatorname{supp}(\boldsymbol{\nu}) < \infty\}$. For $\boldsymbol{\nu} \in \mathfrak{F}$, we denote the $\boldsymbol{\nu}$ -th partial derivative with respect to the parametric variables \boldsymbol{y} by

$$\partial^{\boldsymbol{\nu}} = \partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} = \frac{\partial^{|\boldsymbol{\nu}|}}{\partial y_1^{\nu_1} \partial y_2^{\nu_2} \cdots}.$$

For any sequence of real numbers $\boldsymbol{b} = (b_j)_{j\geq 1}$, we write $\boldsymbol{b}^{\boldsymbol{\nu}} := \prod_{j\geq 1} b_j^{\nu_j}$. By $\boldsymbol{m} \leq \boldsymbol{\nu}$ we mean that the multi-index \boldsymbol{m} satisfies $m_j \leq \nu_j$ for all j. Moreover, $\boldsymbol{\nu} - \boldsymbol{m}$ denotes a multi-index with the elements $\nu_j - m_j$, and $\binom{\boldsymbol{\nu}}{\boldsymbol{m}} := \prod_{j\geq 1} \binom{\nu_j}{m_j}$. We denote by \boldsymbol{e}_j the multi-index whose jth component is 1 and whose other components are 0. We will make repeated use of the Leibniz product rule

$$\partial^{\boldsymbol{\nu}}(PQ) = \sum_{\boldsymbol{m} \leq \boldsymbol{\nu}} {\boldsymbol{\nu} \choose \boldsymbol{m}} (\partial^{\boldsymbol{m}} P) (\partial^{\boldsymbol{\nu}-\boldsymbol{m}} Q).$$
(4.1)

For a general multi-index derivative, ∂^{ν} , we obtain the following result.

Lemma 4.1. Let the assumptions and parameter restrictions in Theorem 2.1 hold. For each $\mathbf{y} \in U$ let $u(\cdot, \mathbf{y}) \in V$ be the unique solution of (2.9). Then for any $\mathbf{\nu} \in \mathfrak{F}$ (including $\mathbf{\nu} = \mathbf{0}$) and any $u, w, z \in V$,

$$\mathscr{B}_{\boldsymbol{y}}(\partial^{\boldsymbol{\nu}}u,w) = \sum_{j\in\operatorname{supp}(\boldsymbol{\nu})} \nu_j R_j(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_j}u,w) + S_{\boldsymbol{\nu}}(u,w) + T_{\boldsymbol{\nu}}(w), \qquad (4.2)$$

where

$$R_{j}(z,w) := -\int_{D} \left[A \psi_{j} z \overline{\mathcal{L}w} + \left(\mathcal{M}_{2}z + \frac{A}{k^{2}} \mathcal{L}z \right) k^{2} \psi_{j} \overline{w} + \left(-\alpha_{1} - \alpha_{2} - i k L \left(\widehat{\beta}_{1} - \widehat{\beta}_{2} \right) \right) k^{2} \psi_{j} z \overline{w} + k^{2} \left(\nabla \cdot (\boldsymbol{x}\psi_{j}) \right) z \overline{w} \right] d\boldsymbol{x} + k^{2} \int_{\partial D} (\boldsymbol{x} \cdot \vec{\mathbf{n}}) \psi_{j} z \overline{w} dS,$$

$$(4.3)$$

 $S_{\boldsymbol{\nu}}(u,w)$

$$:= \begin{cases} 0 & \text{if } |\boldsymbol{\nu}| = 0, 1, \\ -A k^2 \sum_{j \in \text{supp}(\boldsymbol{\nu})} \sum_{\ell \in \text{supp}(\boldsymbol{\nu} - \boldsymbol{e}_j)} \nu_j (\boldsymbol{\nu} - \boldsymbol{e}_j)_\ell \int_D \psi_j \, \psi_\ell \, (\partial^{\boldsymbol{\nu} - \boldsymbol{e}_j - \boldsymbol{e}_\ell} u) \, \overline{w} & \text{otherwise}, \end{cases}$$
(4.4)

and

$$T_{\boldsymbol{\nu}}(w) := \begin{cases} \mathscr{G}_{\boldsymbol{y}}(w) & \text{if } \boldsymbol{\nu} = \boldsymbol{0}, \\ -A \int_{D} \psi_{j} \, \overline{w} \, f \, \mathrm{d} \boldsymbol{x} & \text{if } \boldsymbol{\nu} = \boldsymbol{e}_{j}, \\ 0 & \text{otherwise.} \end{cases}$$
(4.5)

Proof. For any $\boldsymbol{y} \in U$, let $u(\cdot, \boldsymbol{y}) \in V$ be the unique solution of (2.9). For any $w \in V$ (independent of \boldsymbol{y}) and any $\boldsymbol{\nu} \in \mathfrak{F}$, we will prove the lemma by differentiating and equating the two sides of (2.9), that is,

$$\partial^{\boldsymbol{\nu}}(\mathscr{B}_{\boldsymbol{y}}(u(\cdot,\boldsymbol{y}),w)) = \partial^{\boldsymbol{\nu}}(\mathscr{G}_{\boldsymbol{y}}(w)).$$
(4.6)

Starting with the left-hand side of (4.6), we note from the sesquilinear form (2.7) that the factors which depend on \boldsymbol{y} are $u(\boldsymbol{x}, \boldsymbol{y})$ and $n(\boldsymbol{x}, \boldsymbol{y})$ as well as (cf. (1.2))

$$(\mathcal{L}u)(\boldsymbol{x},\boldsymbol{y}) = \Delta u(\boldsymbol{x},\boldsymbol{y}) + k^2 n(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{x},\boldsymbol{y}) \text{ and } (\mathcal{L}w)(\boldsymbol{x}) = \Delta w(\boldsymbol{x}) + k^2 n(\boldsymbol{x},\boldsymbol{y}) w(\boldsymbol{x}).$$

Using the definition of n(x, y) in (1.3), we have

$$\partial^{\boldsymbol{m}} n(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} n(\boldsymbol{x}, \boldsymbol{y}) & \text{if } \boldsymbol{m} = \boldsymbol{0}, \\ \psi_j(\boldsymbol{x}) & \text{if } \boldsymbol{m} = \boldsymbol{e}_j, \\ 0 & \text{otherwise}, \end{cases}$$
(4.7)

It follows that (suppressing from here on the dependence on x and y)

$$\partial^{\boldsymbol{m}}(\mathcal{L}w) = \begin{cases} \mathcal{L}w & \text{if } \boldsymbol{m} = \boldsymbol{0}, \\ k^2 \,\psi_j \, w & \text{if } \boldsymbol{m} = \boldsymbol{e}_j, \\ 0 & \text{otherwise,} \end{cases}$$
(4.8)

and using (4.1) we obtain

$$\partial^{\boldsymbol{\nu}}(\mathcal{L}u) = \Delta(\partial^{\boldsymbol{\nu}}u) + k^{2} \sum_{\boldsymbol{m} \leq \boldsymbol{\nu}} {\boldsymbol{\nu} \choose \boldsymbol{m}} (\partial^{\boldsymbol{m}}n) (\partial^{\boldsymbol{\nu}-\boldsymbol{m}}u)$$

$$= \Delta(\partial^{\boldsymbol{\nu}}u) + k^{2} n (\partial^{\boldsymbol{\nu}}u) + k^{2} \sum_{j \in \text{supp}(\boldsymbol{\nu})} \nu_{j} \psi_{j} (\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u)$$

$$= \mathcal{L}(\partial^{\boldsymbol{\nu}}u) + k^{2} \sum_{j \in \text{supp}(\boldsymbol{\nu})} \nu_{j} \psi_{j} (\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u).$$
(4.9)

To ease our derivation below, we split the sesquilinear form (2.7) into three terms, $\mathscr{B}_{\boldsymbol{y}}(u, w) = \mathscr{B}_1(u, w) + \mathscr{B}_2(u, w) + \mathscr{B}_3(u, w)$, based on the level of dependency on \boldsymbol{y} :

$$\begin{aligned} \mathscr{B}_{1}(u,w) &:= \int_{D} \left(\mathcal{M}_{2}u + \frac{A}{k^{2}}\mathcal{L}u \right) \overline{\mathcal{L}w} \, \mathrm{d}x \\ \mathscr{B}_{2}(u,w) &:= \int_{D} k^{2} \left(\xi_{2} \, n + \nabla \cdot (\boldsymbol{x} \, n) \right) u \, \overline{w} \, \mathrm{d}x - \int_{\partial D} k^{2} \left(\boldsymbol{x} \cdot \vec{\mathbf{n}} \right) n \, u \, \overline{w} \, \mathrm{d}S \,, \\ \mathscr{B}_{3}(u,w) &:= \int_{D} \xi_{1} \, \nabla u \cdot \overline{\nabla w} \, \mathrm{d}x \\ &- \int_{\partial D} \left(\overline{\mathcal{M}_{1}w} \, \mathrm{i} \, k \, u + \left(\boldsymbol{x} \cdot \nabla_{\partial D}u + \xi_{3} \, u \right) \overline{\frac{\partial w}{\partial \vec{\mathbf{n}}}} - (\boldsymbol{x} \cdot \vec{\mathbf{n}}) \nabla_{\partial D}u \cdot \overline{\nabla_{\partial D}w} \right) \mathrm{d}S \,, \end{aligned}$$

with the abbreviations $\xi_1 := 2 - d + \alpha_1 + \alpha_2 + i k L(\widehat{\beta}_1 - \widehat{\beta}_2), \ \xi_2 := -\alpha_1 - \alpha_2 - i k L(\widehat{\beta}_1 - \widehat{\beta}_2),$ and $\xi_3 := -i k L \widehat{\beta}_2 + \alpha_2.$

It is easy to see that

$$\partial^{\boldsymbol{\nu}}(\mathscr{B}_3(u,w)) = \mathscr{B}_3(\partial^{\boldsymbol{\nu}}u,w).$$

Using (4.1) and (4.7) we obtain

$$\partial^{\boldsymbol{\nu}}(\mathscr{B}_{2}(u,w)) = \sum_{\boldsymbol{m}\leq\boldsymbol{\nu}} {\boldsymbol{\nu} \choose \boldsymbol{m}} \left[\int_{D} k^{2} \left(\xi_{2} \left(\partial^{\boldsymbol{m}} n \right) + \nabla \cdot \left(\boldsymbol{x} \left(\partial^{\boldsymbol{m}} n \right) \right) \right) \left(\partial^{\boldsymbol{\nu}-\boldsymbol{m}} u \right) \overline{\boldsymbol{w}} \, \mathrm{d}\boldsymbol{x} - \int_{\partial D} k^{2} (\boldsymbol{x} \cdot \vec{\mathbf{n}}) (\partial^{\boldsymbol{m}} n) (\partial^{\boldsymbol{\nu}-\boldsymbol{m}} u) \, \overline{\boldsymbol{w}} \, \mathrm{d}\boldsymbol{S} \right]$$
$$= \mathscr{B}_{2}(\partial^{\boldsymbol{\nu}} u, w)$$

$$+\sum_{j\in\operatorname{supp}(\boldsymbol{\nu})} \nu_j \left[\int_D k^2 (\xi_2 \,\psi_j + \nabla \cdot (\boldsymbol{x} \,\psi_j)) \,(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_j} u) \,\overline{w} \,\mathrm{d}\boldsymbol{x} - \int_{\partial D} k^2 (\boldsymbol{x} \cdot \vec{\mathbf{n}}) \psi_j (\partial^{\boldsymbol{\nu}-\boldsymbol{e}_j} u) \,\overline{w} \,\mathrm{d}\boldsymbol{S} \right].$$

Using (4.1) and (4.8), followed by applying (4.9) with ν replaced by $\nu - e_j$ and index j replaced by ℓ , we obtain

$$\begin{split} \partial^{\boldsymbol{\nu}}(\mathscr{B}_{1}(u,w)) &= \sum_{m \leq \boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{m} \int_{D} \left[\partial^{\boldsymbol{\nu}-\boldsymbol{m}} \Big(\mathcal{M}_{2}u + \frac{A}{k^{2}} \mathcal{L}u \Big) \right] \left[\partial^{\boldsymbol{m}}(\overline{\mathcal{L}w}) \Big] \, \mathrm{d}\boldsymbol{x} \\ &= \int_{D} \left[\partial^{\boldsymbol{\nu}} \Big(\mathcal{M}_{2}u + \frac{A}{k^{2}} \mathcal{L}u \Big) \right] \overline{\mathcal{L}w} \, \mathrm{d}\boldsymbol{x} + \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \nu_{j} \int_{D} \left[\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}} \Big(\mathcal{M}_{2}u + \frac{A}{k^{2}} \mathcal{L}u \Big) \right] k^{2} \psi_{j} \, \overline{w} \, \mathrm{d}\boldsymbol{x} \\ &= \mathscr{B}_{1}(\partial^{\boldsymbol{\nu}}u,w) + \int_{D} \frac{A}{k^{2}} k^{2} \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \nu_{j} \psi_{j} \left(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u \right) \overline{\mathcal{L}w} \, \mathrm{d}\boldsymbol{x} \\ &+ \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \nu_{j} \int_{D} \left[\mathcal{M}_{2}(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u) + \frac{A}{k^{2}} \mathcal{L}(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u) \\ &+ \frac{A}{k^{2}} k^{2} \sum_{\ell \in \mathrm{supp}(\boldsymbol{\nu}-\boldsymbol{e}_{j})\ell} (\boldsymbol{\nu}-\boldsymbol{e}_{j})_{\ell} \psi_{\ell} \left(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}-\boldsymbol{e}_{\ell}}u \right) \right] k^{2} \psi_{j} \, \overline{w} \, \mathrm{d}\boldsymbol{x} \\ &= \mathscr{B}_{1}(\partial^{\boldsymbol{\nu}}u,w) \\ &+ \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \nu_{j} \int_{D} \left[A \psi_{j} \left(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u \right) \overline{\mathcal{L}w} + \left(\mathcal{M}_{2}(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u) + \frac{A}{k^{2}} \mathcal{L}(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}}u) \right] k^{2} \psi_{j} \, \overline{w} \right] \, \mathrm{d}\boldsymbol{x} \\ &+ A k^{2} \sum_{j \in \mathrm{supp}(\boldsymbol{\nu})} \sum_{\ell \in \mathrm{supp}(\boldsymbol{\nu}-\boldsymbol{e}_{j})\ell} \nu_{j} \left(\boldsymbol{\nu}-\boldsymbol{e}_{j} \right)_{\ell} \int_{D} \psi_{j} \psi_{\ell} \left(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}-\boldsymbol{e}_{\ell}u \right) \overline{w} \, \mathrm{d}\boldsymbol{x} \, . \end{split}$$

We note that for $|\boldsymbol{\nu}| = 1$ and $j \geq 1$, the set $\operatorname{supp}(\boldsymbol{\nu} - \boldsymbol{e}_j)$ is empty; in this case we take $\partial^{\boldsymbol{\nu}-\boldsymbol{e}_j-\boldsymbol{e}_\ell}$ to be the zero operator. Thus using (4.3) and (4.4) we obtain,

$$\partial^{\boldsymbol{\nu}}(\mathscr{B}_{\boldsymbol{y}}(u,w)) = \mathscr{B}_{\boldsymbol{y}}(\partial^{\boldsymbol{\nu}}u,w) - \sum_{j\in\operatorname{supp}(\boldsymbol{\nu})}\nu_j R_j(\partial^{\boldsymbol{\nu}-\boldsymbol{e}_j}u,w) - S_{\boldsymbol{\nu}}(u,w).$$
(4.10)

Now for the right-hand side of (4.6) we use (2.8), (4.7) and (4.5) to obtain

$$\partial^{\boldsymbol{\nu}}(\mathscr{G}_{\boldsymbol{y}}w) = T_{\boldsymbol{\nu}}(w). \tag{4.11}$$

The required result is obtained by equating (4.10) and (4.11).

Next we derive a bound on the parametric derivatives of the solution of (2.9) in the V-norm.

Theorem 4.2. Let the assumptions and parameter restrictions in Theorem 2.1 hold, and assume additionally that (A2) holds. For each $\mathbf{y} \in U$, let $u(\cdot, \mathbf{y}) \in V$ be the unique solution of (2.9). Then for all $\mathbf{\nu} \in \mathfrak{F}$ (including $\mathbf{\nu} = \mathbf{0}$),

$$\|\partial^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y})\|_{V} \leq \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left(L \|f\|_{L^{2}(D)} + L^{1/2} \|g\|_{L^{2}(\partial D)} \right) |\boldsymbol{\nu}|! \, \boldsymbol{\Upsilon}^{\boldsymbol{\nu}} , \qquad (4.12)$$

where

$$\Upsilon^{\boldsymbol{\nu}} := \prod_{j \ge 1} \Upsilon_j^{\nu_j}, \qquad \Upsilon_j := C_{\operatorname{regu}}(kL) \, \|\psi_j\|_{W^{1,\infty}(D)},$$

$$C_{\text{regu}}(kL) := \max\left\{\frac{C_R(kL)}{C_{\text{coer}}} + \frac{A}{kL C_{\text{func}}(kL)}, \frac{2C_R(kL)}{C_{\text{coer}}}, \sqrt{\frac{2A}{C_{\text{coer}}}}\right\},$$
(4.13)

$$C_R(kL) := 2A(1+n_{\max}) + kL(1+\hat{\beta}_2) + |\alpha_2| + |-\alpha_1 - \alpha_2 - ikL(\hat{\beta}_1 - \hat{\beta}_2)| + d + 1 + \hat{\mu}.$$
(4.14)

We have $C_R(kL) = \mathcal{O}(kL+1), \ C_{\text{func}}(kL) = \mathcal{O}(1+(kL)^{-1}), \ \text{so } C_{\text{regu}}(kL) = \mathcal{O}(kL+(kL)^{-1}).$

Proof. For the $\nu = 0$ case, (4.12) follows from (2.15). Let $|\nu| \ge 1$. We recall (4.2) and bound each term on the RHS of (4.2). Using the definition of the V-norm in (2.3), for any $w \in V$ we have

$$\|w\|_{L^{2}(D)} \leq \frac{1}{k} \|w\|_{V}, \ \|\nabla w\|_{L^{2}(D)} \leq \|w\|_{V}, \ \|\Delta w\|_{L^{2}(D)} \leq k \|w\|_{V}, \ \|w\|_{L^{2}(\partial D)} \leq \frac{1}{k\sqrt{L}} \|w\|_{V},$$

and hence using the definition of \mathcal{L} in (1.2) and \mathcal{M}_2 in (2.6) we obtain

 $\|\mathcal{L}w\|_{L^{2}(D)} \leq k (1 + n_{\max}) \|w\|_{V}, \qquad \|\mathcal{M}_{2}w\|_{L^{2}(D)} \leq \left(L + L |\widehat{\beta}_{2}| + \frac{|\alpha_{2}|}{k}\right) \|w\|_{V}.$

In addition, for all $j \ge 1$, we have

$$\begin{aligned} \|\nabla \cdot (\boldsymbol{x} \, \psi_j)\|_{L^{\infty}(D)} &= \|(\nabla \cdot \boldsymbol{x}) \psi_j + \boldsymbol{x} \cdot \nabla \psi_j\|_{L^{\infty}(D)} \\ &\leq d \, \|\psi_j\|_{L^{\infty}(D)} + L \, \|\nabla \psi_j\|_{L^{\infty}(D)} \leq (d+1) \, \|\psi_j\|_{W^{1,\infty}(D)}. \end{aligned}$$

For $z, w \in V$, using the above bounds in (4.3), the definition of $C_R(kL)$ in (4.14), applying the

triangle and Cauchy-Schwarz inequalities, we obtain for $j\geq 1,$

$$\begin{split} |R_{j}(z,w)| \\ &\leq A \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{L^{2}(D)} \|\mathcal{L}w\|_{L^{2}(D)} \\ &+ k^{2} \|\psi_{j}\|_{L^{\infty}(D)} \|\mathcal{M}_{2}z\|_{L^{2}(D)} \|w\|_{L^{2}(D)} + A \|\psi_{j}\|_{L^{\infty}(D)} \|\mathcal{L}z\|_{L^{2}(D)} \|w\|_{L^{2}(D)} \\ &+ |-\alpha_{1} - \alpha_{2} - ik L \left(\widehat{\beta}_{1} - \widehat{\beta}_{2}\right)| k^{2} \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{L^{2}(D)} \|w\|_{L^{2}(D)} \\ &+ k^{2} \|\nabla \cdot (\boldsymbol{x}\psi_{j})\|_{L^{\infty}(D)} \|z\|_{L^{2}(D)} \|w\|_{L^{2}(D)} \\ &+ k^{2} \|\boldsymbol{x} \cdot \vec{\mathbf{n}}\|_{L^{\infty}(D)} \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{L^{2}(\partial D)} \|w\|_{L^{2}(\partial D)} \\ &\leq A (1 + n_{\max}) \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{V} \|w\|_{V} \\ &+ k \left(L + L \left|\widehat{\beta}_{2}\right| + \frac{|\alpha_{2}|}{k}\right) \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{V} \|w\|_{V} + A (1 + n_{\max}) \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{V} \|w\|_{V} \\ &+ |-\alpha_{1} - \alpha_{2} - ik L \left(\widehat{\beta}_{1} - \widehat{\beta}_{2}\right)| \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{V} \|w\|_{V} \\ &+ (d + 1) \|\psi_{j}\|_{W^{1,\infty}} \|z\|_{V} \|w\|_{V} + \hat{\mu} \|\psi_{j}\|_{L^{\infty}(D)} \|z\|_{V} \|w\|_{V} \\ &\leq \left[2A (1 + n_{\max}) + kL \left(1 + |\widehat{\beta}_{2}|\right) + |\alpha_{2}| + |-\alpha_{1} - \alpha_{2} - ik L \left(\widehat{\beta}_{1} - \widehat{\beta}_{2}\right)| + d + 1 + \hat{\mu}\right] \\ &\times \|\psi_{j}\|_{W^{1,\infty}} \|z\|_{V} \|w\|_{V} \\ &= C_{R}(kL) \|\psi_{j}\|_{W^{1,\infty}(D)} \|z\|_{V} \|w\|_{V}. \end{split}$$

Similarly, with $u(., \mathbf{y}) \in V$ being the solution of (2.9) and applying (2.14) and the above bounds in (4.4) and (4.5), for any $\mathbf{\nu} \in \mathfrak{F}$, including $\mathbf{\nu} = \mathbf{0}$, we obtain

$$|S_{\boldsymbol{\nu}}(u,w) + T_{\boldsymbol{\nu}}(w)| \qquad \text{if } \boldsymbol{\nu} = \mathbf{0}, \\ A \| \psi_{j} \|_{L^{\infty}(D)} \| w \|_{L^{2}(D)} \| f \|_{L^{2}(D)} \qquad \text{if } \boldsymbol{\nu} = \mathbf{e}_{j}, \\ A k^{2} \sum_{j \in \text{supp}(\boldsymbol{\nu})} \sum_{\ell \in \text{supp}(\boldsymbol{\nu} - \mathbf{e}_{j})} \nu_{j}(\boldsymbol{\nu} - \mathbf{e}_{j})_{\ell} \| \psi_{j} \|_{L^{\infty}(D)} \| \psi_{\ell} \|_{L^{\infty}(D)} \| \partial^{\boldsymbol{\nu} - \mathbf{e}_{j} - \mathbf{e}_{\ell}} u \|_{L^{2}(D)} \| w \|_{L^{2}(D)} \\ \leq \mathbb{S}_{\boldsymbol{\nu}}(u) \| w \|_{V}, \qquad (4.16)$$

where

Taking now $w = \partial^{\nu} u(\cdot, \boldsymbol{y})$ in (4.2), using the coercivity property (2.12) as lower bound, and using (4.15) and (4.16) as upper bounds, we obtain

$$C_{\text{coer}} \|\partial^{\boldsymbol{\nu}} u\|_{V}^{2} \leq |\mathscr{B}_{\boldsymbol{y}}(\partial^{\boldsymbol{\nu}} u, \partial^{\boldsymbol{\nu}} u)| \leq \left(\sum_{j \in \text{supp}(\boldsymbol{\nu})} \nu_{j} C_{R}(kL) \|\psi_{j}\|_{W^{1,\infty}(D)} \|\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}} u\|_{V} + \mathbb{S}_{\boldsymbol{\nu}}(u)\right) \|\partial^{\boldsymbol{\nu}} u\|_{V},$$

and hence (now showing dependence on \boldsymbol{y})

$$\|\partial^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y})\|_{V} \leq \frac{1}{C_{\text{coer}}} \left(C_{R}(kL) \sum_{j \in \text{supp}(\boldsymbol{\nu})} \nu_{j} \|\psi_{j}\|_{W^{1,\infty}(D)} \|\partial^{\boldsymbol{\nu}-\boldsymbol{e}_{j}} u(\cdot, \boldsymbol{y})\|_{V} + \mathbb{S}_{\boldsymbol{\nu}}(u(\cdot, \boldsymbol{y})) \right).$$

$$(4.18)$$

The desired result now follows from (4.17) and (4.18) by applying Lemma B.1 with

$$\begin{aligned} \mathbb{A}_{\boldsymbol{\nu}} &= \|\partial^{\boldsymbol{\nu}} u(\cdot, \boldsymbol{y})\|_{V}, \quad B = \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left(L \|f\|_{L^{2}(D)} + L^{1/2} \|g\|_{L^{2}(\partial D)}\right), \\ \Psi_{j} &= \|\psi_{j}\|_{W^{1,\infty}(D)}, \quad c_{0} = \frac{C_{R}(kL)}{C_{\text{coer}}} + \frac{A}{kL C_{\text{func}}(kL)}, \quad c_{1} = \frac{C_{R}(kL)}{C_{\text{coer}}}, \quad c_{2} = \frac{A}{C_{\text{coer}}}. \end{aligned}$$

The value of *B* is determined by taking $\boldsymbol{\nu} = \mathbf{0}$ in (4.17) and (4.18). The values of c_1 and c_2 follow easily by taking $|\boldsymbol{\nu}| \geq 2$ in (4.17) and (4.18). The remaining case of $|\boldsymbol{\nu}| = 1$ is slightly more complicated: taking $\boldsymbol{\nu} = \boldsymbol{e}_j$ in (4.17) and (4.18) yields

$$\begin{split} \|\partial^{\boldsymbol{e}_{j}}u(\cdot,\boldsymbol{y})\|_{V} &\leq \frac{1}{C_{\text{coer}}} \left(C_{R}(kL) \, \|\psi_{j}\|_{W^{1,\infty}(D)} \, \|u(\cdot,\boldsymbol{y})\|_{V} + \frac{A}{k} \, \|\psi_{j}\|_{L^{\infty}(D)} \, \|f\|_{L^{2}(D)} \right) \\ &\leq \left(\frac{C_{R}(kL)}{C_{\text{coer}}} + \frac{A}{kL \, C_{\text{func}}(kL)} \right) \Psi_{j} \, B \,, \end{split}$$

which gives the value of c_0 . With these values we obtain $C_{\text{regu}}(kL) = \max \{c_0, 2c_1, \sqrt{2c_2}\}$ as given in (4.13). This completes the proof.

5 Stochastic refractive index dimension truncation

For simulation of the stochastic wave propagation induced by the refractive index, we need to truncate the infinitely many terms in the ansatz (1.3). To analyze the dimension truncation error, it is convenient to introduce an operator theoretical framework which incorporates the boundary condition.

Recalling (2.1), for each $\boldsymbol{y} \in U$ we now define the operator $\mathcal{B}(\boldsymbol{y}) : V \to L^2(D) \times L^2(\partial D)$ by

$$[\mathcal{B}(\boldsymbol{y})w](\boldsymbol{x},\widetilde{\boldsymbol{x}}) := \begin{pmatrix} \left[\Delta + k^2 n(\boldsymbol{x},\boldsymbol{y})\right] w(\boldsymbol{x}) \\ \frac{\partial w}{\partial \vec{\mathbf{n}}}(\widetilde{\boldsymbol{x}}) - i \ k \ w(\widetilde{\boldsymbol{x}}) \end{pmatrix}, \qquad \boldsymbol{x} \in D, \quad \widetilde{\boldsymbol{x}} \in \partial D.$$
(5.1)

Then (2.1) can be expressed as

$$[\mathcal{B}(oldsymbol{y})u(\cdot,oldsymbol{y})](oldsymbol{x},\widetilde{oldsymbol{x}}) = egin{pmatrix} -f(oldsymbol{x})\ g(\widetilde{oldsymbol{x}}) \end{pmatrix}, \qquad oldsymbol{x}\in D, \quad \widetilde{oldsymbol{x}}\in \partial D.$$

We equip $L^2(D) \times L^2(\partial D)$ with the weighted product space norm

$$\left\|\binom{f}{g}\right\|_{L^{2}(D) \times L^{2}(\partial D)} := L \left\|f\right\|_{L^{2}(D)} + \sqrt{L} \left\|g\right\|_{L^{2}(\partial D)}, \qquad f \in L^{2}(D), \quad g \in L^{2}(\partial D).$$

It is easy to check that $\mathcal{B}(\boldsymbol{y})$ is a bounded linear operator.

From Theorem 2.1 we conclude that $\mathcal{B}(\boldsymbol{y})$ is boundedly invertible for all $\boldsymbol{y} \in U$. Indeed, for any $\binom{f}{g} \in L^2(D) \times L^2(\partial D)$ we can write

$$u(\cdot, \boldsymbol{y}) = [\mathcal{B}(\boldsymbol{y})]^{-1} {\binom{-f}{g}}, \quad \text{with} \quad \left\| [\mathcal{B}(\boldsymbol{y})]^{-1} {\binom{-f}{g}} \right\|_{V} \leq \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left\| {\binom{f}{g}} \right\|_{L^{2}(D) \times L^{2}(\partial D)},$$

and therefore

$$\left\| \left[\mathcal{B}(\boldsymbol{y}) \right]^{-1} \right\| \leq \frac{C_{\text{func}}(kL)}{C_{\text{coer}}},$$
(5.2)

which is bounded independently of the wavenumber if $kL \ge 1$.

Corresponding to (1.3), for a truncation parameter s we consider a truncated refractive index (essentially by setting $y_j = 0$ for j > s)

$$n_s({m x},{m y})\,=\,n_s({m x},{m y}_{\{1:s\}})\,=\,n_0({m x})+\sum_{j=1}^s y_j\,\psi_j({m x})$$

and define the operator $\mathcal{B}_s(\boldsymbol{y}) = \mathcal{B}_s(\boldsymbol{y}_{\{1:s\}}) : V \to L^2(D) \times L^2(\partial D)$ as in (5.1) but with *n* replaced by n_s . Then we have also

$$u_s(\cdot, \boldsymbol{y}) = [\mathcal{B}_s(\boldsymbol{y})]^{-1} {\binom{-f}{g}} \quad \text{and} \quad \left\| [\mathcal{B}_s(\boldsymbol{y})]^{-1} \right\| \le \frac{C_{\text{func}}(kL)}{C_{\text{coer}}}.$$
(5.3)

For a fixed truncated dimension s, in the following theorem we will estimate the approximation error $(I - I_s)(G(u))$, where the infinite-dimensional integral I and the finite-dimensional integral I_s are as defined in (3.1)–(3.2). The proof of the estimate is based on the dimension truncation error of the integrand

$$u(\cdot, \boldsymbol{y}) - u_s(\cdot, \boldsymbol{y}) = ([\mathcal{B}(\boldsymbol{y})]^{-1} - [\mathcal{B}_s(\boldsymbol{y})]^{-1}) {\binom{-f}{g}}$$

by a Neumann series argument. The first critical step is to recogonize that we can write the difference operator $[\mathcal{B}(\boldsymbol{y}) - \mathcal{B}_s(\boldsymbol{y})]: V \to L^2(D) \times L^2(\partial D)$ as

$$[\mathcal{B}(\boldsymbol{y}) - \mathcal{B}_s(\boldsymbol{y})]w = \sum_{j \ge s+1} y_j \mathcal{T}_j w, \qquad (5.4)$$

with operators $\mathcal{T}_j: V \to L^2(D) \times L^2(\partial D)$ defined as

$$\mathcal{T}_{j}w := k^{2} \begin{pmatrix} \psi_{j}w\\ 0 \end{pmatrix}, \qquad j \ge 1.$$
(5.5)

The proof follows the general argument of [17] but there are some key differences which mean that we do not need to impose the kind of small perturbation assumption discussed in Appendix A.

For developing the dimension truncation and QMC-FEM analysis in this article, we will impose the following assumptions on the perturbation functions ψ_j in (1.3):

- (A3) The sequence ψ_j is ordered: $\|\psi_1\|_{L^{\infty}(D)} \ge \|\psi_2\|_{L^{\infty}(D)} \ge \cdots$.
- (A4) There exists $p_0 \in (0, 1)$ and $K_0 \in \mathbb{R}$ independently of k such that

$$\sum_{j\geq 1} \left[(kL+1) \, \|\psi_j\|_{L^{\infty}(D)} \right]^{p_0} \leq K_0 < \infty.$$
(5.6)

(A5) There exists $p_1 \in (0, 1)$ and $K_1 \in \mathbb{R}$ independently of k such that

$$\sum_{j\geq 1} \left[\left(kL + (kL)^{-1} \right) \|\psi_j\|_{W^{1,\infty}(D)} \right]^{p_1} \leq K_1 < \infty.$$
(5.7)

These conditions are similar to counterpart conditions assumed for the diffusion model in [28] and also for general class of operator equations in [7, 8], but now with explicit dependence on kL. We use the assumption (A5) in the next section to obtain QMC error bounds.

Theorem 5.1. Let the assumptions (A0)–(A4) and parameter restrictions in Theorem 2.1 hold. For every $\mathbf{y} \in U$, $f \in L^2(D)$ and $g \in L^2(\partial D)$, let $u(\cdot, \mathbf{y}) \in V$ be the unique solution of (2.1), and for each $s \in \mathbb{N}$ let $u_s(\cdot, \mathbf{y})$ denote the solution of the truncated version of (2.1) with n replaced by n_s . Then for every linear functional $G \in V^*$, there exist a constant C independent of s, f, g, G and kL such that

$$|(I - I_s)(G(u))| = |I(G(u - u^s))| \le C \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \|\binom{f}{g}\|_{L^2(D) \times L^2(\partial D)} \|G\|_{V^*} s^{-\frac{2}{p_0}+1}, \quad (5.8)$$

which is bounded independently of the wavenumber if $kL \ge 1$.

Proof. In this proof we will suppress the dependence on \boldsymbol{y} to simplify our notation where possible. We will begin by expanding $u - u_s = (\mathcal{B}^{-1} - \mathcal{B}_s^{-1}) {\binom{-f}{g}}$ in a Neumann series for sufficient large s. Writing $\mathcal{B}^{-1} = (\mathcal{I} + \mathcal{B}_s^{-1}(\mathcal{B} - \mathcal{B}_s))^{-1}\mathcal{B}_s^{-1}$, we need to first ensure that $\| - \mathcal{B}_s^{-1}(\mathcal{B} - \mathcal{B}_s) \| < 1$.

For each $j \ge 1$ and $w \in V$, we have from (5.5) that

$$\begin{aligned} \left\| \mathcal{B}_{s}^{-1} \mathcal{T}_{j} w \right\|_{V} &= \left\| k^{2} \mathcal{B}_{s}^{-1} \begin{pmatrix} \psi_{j} w \\ 0 \end{pmatrix} \right\|_{V} \\ &\leq k^{2} \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left\| \begin{pmatrix} \psi_{j} w \\ 0 \end{pmatrix} \right\|_{L^{2}(D) \times L^{2}(\partial D)} \leq kL \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \left\| \psi_{j} \right\|_{L^{\infty}(D)} \left\| w \right\|_{V}, \end{aligned}$$

where we used $\|\binom{w}{0}\|_{L^2(D)\times L^2(\partial D)} = L \|w\|_{L^2(D)} \leq \frac{L}{k} \|w\|_V$. Thus $\mathcal{B}_s^{-1} \mathcal{T}_j$ is a bounded operator from V to V, with norm

$$\left\|\mathcal{B}_{s}^{-1}\mathcal{T}_{j}\right\| \leq kL \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \|\psi_{j}\|_{L^{\infty}(D)} =: b_{j}.$$
(5.9)

Hence from (5.4) we have for all $\boldsymbol{y} \in U$,

$$\left\| - \mathcal{B}_s^{-1}(\mathcal{B} - \mathcal{B}_s) \right\| = \left\| \sum_{j \ge s+1} y_j \mathcal{B}_s^{-1} \mathcal{T}_j \right\| \le \frac{1}{2} \sum_{j \ge s+1} b_j.$$

Since $kL C_{\text{func}}(kL) = \mathcal{O}(kL+1)$, from Assumptions (A3) and (A4) we know that the sequence $\{b_j\}_{j\geq 1}$ is nonincreasing, and that

$$\sum_{j\geq 1} b_j^{p_0} \le r_0 K_0 < \infty, \tag{5.10}$$

for some constant r_0 independent of the wavenumber k.

Let s^* be such that $\sum_{j \ge s^*+1} b_j \le \frac{1}{2}$, implying that $\|-\mathcal{B}_s^{-1}(\mathcal{B}-\mathcal{B}_s)\| \le \frac{1}{4}$. Then for all $s \ge s^*$, by the bounded invertibility of $\mathcal{B}(\boldsymbol{y})$ and $\mathcal{B}_s(\boldsymbol{y}_{\{1:s\}})$ for all $\boldsymbol{y} \in U$, we can write the inverse of \mathcal{B} in terms of the Neumann series, as

$$\mathcal{B}^{-1} = \left(\mathcal{I} + \mathcal{B}_s^{-1}(\mathcal{B} - \mathcal{B}_s)\right)^{-1} \mathcal{B}_s^{-1} = \sum_{\ell \ge 0} \left(-\mathcal{B}_s^{-1}(\mathcal{B} - \mathcal{B}_s)\right)^{\ell} \mathcal{B}_s^{-1}.$$

Then, using representations (5.2), (5.3) and (5.5), we obtain

$$\begin{aligned} u - u_s &= \left(\mathcal{B}^{-1} - \mathcal{B}_s^{-1}\right) {\binom{-f}{g}} = \sum_{\ell \ge 1} \left(-\mathcal{B}_s^{-1} (\mathcal{B} - \mathcal{B}_s)\right)^{\ell} \mathcal{B}_s^{-1} {\binom{-f}{g}} \\ &= \sum_{\ell \ge 1} (-1)^{\ell} \left(\sum_{j \ge s+1} y_j \, \mathcal{B}_s^{-1} \, \mathcal{T}_j\right)^{\ell} u_s \\ &= \sum_{\ell \ge 1} (-1)^{\ell} \sum_{\eta \in \{s+1:\infty\}^{\ell}} \prod_{i=1}^{\ell} \left(y_{\eta_i} \, \mathcal{B}_s^{-1} \, \mathcal{T}_{\eta_i}\right) u_s \end{aligned}$$

where we use the shorthand notation $\{s+1:\infty\}^{\ell} = \{s+1, s+2, \dots, \infty\}^{\ell}$.

Thus we can write

$$\begin{split} &\int_{U} G(u-u_{s}) \,\mathrm{d}\boldsymbol{y} \,=\, \sum_{\ell \geq 1} (-1)^{\ell} \sum_{\boldsymbol{\eta} \in \{s+1:\infty\}^{\ell}} \int_{U} G\left[\left(\prod_{i=1}^{\ell} (y_{\eta_{i}} \,\mathcal{B}_{s}^{-1} \,\mathcal{T}_{\eta_{i}}) \right) u_{s} \right] \,\mathrm{d}\boldsymbol{y} \\ &=\, \sum_{\ell \geq 1} (-1)^{\ell} \sum_{\boldsymbol{\eta} \in \{s+1:\infty\}^{\ell}} \left(\int_{U_{s+}} \prod_{i=1}^{\ell} y_{\eta_{i}} \,\mathrm{d}\boldsymbol{y}_{\{s+1:\infty\}} \right) \left(\int_{U_{s}} G\left[\left(\prod_{i=1}^{\ell} (\mathcal{B}_{s}^{-1} \,\mathcal{T}_{\eta_{i}}) \right) u_{s} \right] \,\mathrm{d}\boldsymbol{y}_{\{1:s\}} \right), \end{split}$$

where we separated the integrals for $\boldsymbol{y}_{\{1:s\}} \in U_s := \left[-\frac{1}{2}, \frac{1}{2}\right]^s$ and $\boldsymbol{y}_{\{s+1:\infty\}} := (y_j)_{j \ge s+1} \in U_{s+1} := \{(y_j)_{j \ge s+1} : y_j \in \left[-\frac{1}{2}, \frac{1}{2}\right], j \ge s+1\}$, which is an essential step of this proof. The integral over $\boldsymbol{y}_{\{s+1:\infty\}}$ is nonnegative due to the simple yet crucial observation that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} y_j^n \, \mathrm{d}y_j = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{2^n(n+1)} & \text{if } n \text{ is even.} \end{cases}$$
(5.11)

The integral over $y_{\{1:s\}}$ can be estimated, using (5.3) and (5.9), as

$$\begin{split} \left| \int_{U_s} G \left[\left(\prod_{i=1}^{\ell} (\mathcal{B}_s^{-1} \, \mathcal{T}_{\eta_i}) \right) u_s \right] \mathrm{d} \boldsymbol{y}_{\{1:s\}} \right| &\leq \|G\|_{V^*} \sup_{\boldsymbol{y}_{\{1:s\}} \in U_s} \left\| \left(\prod_{i=1}^{\ell} (\mathcal{B}_s^{-1} \, \mathcal{T}_{\eta_i}) \right) \right\| \|u_s\|_V \\ &\leq \frac{C_{\mathrm{func}}(kL)}{C_{\mathrm{coer}}} \, \| \binom{f}{g} \|_{L^2(D) \times L^2(\partial D)} \, \|G\|_{V^*} \prod_{i=1}^{\ell} b_{\eta_i} \, . \end{split}$$

Hence, with the abbreviation

$$C_1 := \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \| \binom{f}{g} \|_{L^2(D) \times L^2(\partial D)} \| G \|_{V^*},$$
(5.12)

we obtain

$$\left| \int_{U} G(u-u_{s}) \,\mathrm{d}\boldsymbol{y} \right| \leq C_{1} \sum_{\ell \geq 1} \sum_{\boldsymbol{\eta} \in \{s+1:\infty\}^{\ell}} \left(\int_{U_{s+}} \prod_{i=1}^{\ell} y_{\eta_{i}} \,\mathrm{d}\boldsymbol{y}_{\{s+1:\infty\}} \right) \prod_{i=1}^{\ell} b_{\eta_{i}}$$
$$= C_{1} \sum_{\ell \geq 1} \int_{U_{s+}} \sum_{\boldsymbol{\eta} \in \{s+1:\infty\}^{\ell}} \left(\prod_{i=1}^{\ell} y_{\eta_{i}} b_{\eta_{i}} \right) \,\mathrm{d}\boldsymbol{y}_{\{s+1:\infty\}} = C_{1} \sum_{\ell \geq 1} \int_{U_{s+}} \left(\sum_{j \geq s+1} y_{j} \, b_{j} \right)^{\ell} \,\mathrm{d}\boldsymbol{y}_{\{s+1:\infty\}}.$$

Using the multinomial theorem with multi-index ν and $\binom{\ell}{\nu} = \ell!/(\prod_{j\geq 1}\nu_j!)$, we can write

$$\begin{aligned} \left| \int_{U} G(u-u_{s}) \,\mathrm{d}\boldsymbol{y} \right| &\leq C_{1} \sum_{\ell \geq 1} \int_{U_{s+}} \sum_{\substack{|\boldsymbol{\nu}| = \ell \\ \nu_{j} = 0 \ \forall j \leq s}} \binom{\ell}{\boldsymbol{\nu}} \prod_{j \geq s+1} (y_{j} \, b_{j})^{\nu_{j}} \,\mathrm{d}\boldsymbol{y}_{\{s+1:\infty\}} \\ &= C_{1} \sum_{\ell \geq 1} \sum_{\substack{|\boldsymbol{\nu}| = \ell \\ \nu_{j} = 0 \ \forall j \leq s}} \binom{\ell}{\boldsymbol{\nu}} \left(\prod_{j \geq s+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} y_{j}^{\nu_{j}} \,\mathrm{d}y_{j} \right) \prod_{j \geq s+1} b_{j}^{\nu_{j}} \\ &\leq C_{1} \sum_{\ell \geq 2 \text{ even}} \sum_{\substack{|\boldsymbol{\nu}| = \ell \\ \nu_{j} = 0 \ \forall j \leq s}} \binom{\ell}{\boldsymbol{\nu}} \prod_{j \geq s+1} b_{j}^{\nu_{j}} = C_{1} \sum_{\substack{\ell' \geq 1 \\ \nu_{j} = 0 \ \forall j \leq s}} \binom{2\ell'}{\boldsymbol{\nu}} \prod_{j \geq s+1} b_{j}^{\nu_{j}}, \end{aligned}$$

where the last inequality follows from (5.11).

Now we split the sum into a sum over $\ell' \geq \ell^*$ (dropping the condition " ν_j even") and the initial terms $1 \leq \ell' < \ell^*$ (substituting $\nu_j = 2\nu'_j$) to obtain the estimate

$$\begin{aligned} \left| \int_{U} G(u-u_{s}) \,\mathrm{d}\boldsymbol{y} \right| &\leq C_{1} \sum_{\substack{\ell' \geq \ell^{*} \\ \nu_{j}=0 \ \forall j \leq s \\ \nu_{j}=0 \ \forall j \leq s + 1}} \sum_{\substack{j \geq s+1 \\ \nu_{j}=0 \ \forall j \leq s \\ \forall j \geq s+1}} \left| \int_{j \geq s+1} b_{j}^{\nu_{j}} + C_{1} \sum_{\substack{1 \leq \ell' < \ell^{*} \\ \ell' \geq \ell^{*} \\ \nu_{j}=0 \ \forall j \leq s}} \left(\frac{2\ell'}{2\nu'} \right) \prod_{j \geq s+1} b_{j}^{\nu_{j}} + C_{1} \sum_{1 \leq \ell' < \ell^{*}} \frac{(2\ell')!}{(\ell')!} \sum_{\substack{|\nu'|=\ell' \\ \nu_{j}'=0 \ \forall j \leq s}} \left(\frac{\ell'}{\nu'} \right) \prod_{j \geq s+1} (b_{j}^{2})^{\nu'_{j}} \\ &\leq C_{1} \sum_{\ell' \geq \ell^{*}} \left(\sum_{j \geq s+1} b_{j} \right)^{2\ell'} + C_{1} \sum_{1 \leq \ell' < \ell^{*}} \frac{(2\ell')!}{(\ell')!} \left(\sum_{j \geq s+1} b_{j}^{2} \right)^{\ell'} \\ &\leq C_{1} \frac{(\sum_{j \geq s+1} b_{j})^{2\ell^{*}}}{1 - (\sum_{j \geq s+1} b_{j})^{2}} + C_{1} \frac{(2\ell'-2)!}{(\ell^{*}-1)!} \frac{\sum_{j \geq s+1} b_{j}^{2}}{1 - (\sum_{j \geq s+1} b_{j}^{2})}, \end{aligned}$$
(5.13)

where we used the multinomial theorem and the geometric series formula, noting that for $s \ge s^*$ we have $\sum_{j\geq s+1} b_j^2 \leq \sum_{j\geq s+1} b_j \leq \frac{1}{2}$. From [28, Theorem 5.1] we know that

$$\sum_{j\geq s+1} b_j \leq \min\left(\frac{1}{\frac{1}{p_0}-1}, 1\right) \left(\sum_{j\geq 1} b_j^{p_0}\right)^{\frac{1}{p_0}} s^{-\frac{1}{p_0}+1}.$$
(5.14)

With a similar argument we can show that

ć

$$\sum_{j\geq s+1} b_j^2 \leq \frac{1}{\frac{2}{p_0} - 1} \left(\sum_{j\geq 1} b_j^{p_0}\right)^{\frac{2}{p_0}} s^{-\frac{2}{p_0} + 1}.$$
(5.15)

Using the estimates (5.14) and (5.15) for the numerators in (5.13) and bounding the sums in the denominators by 1/2, we see that the first term in (5.13) is $\mathcal{O}(s^{-2\ell^*(1/p_0-1)})$ while the second term is $\mathcal{O}(s^{-(2/p_0-1)})$. We therefore choose ℓ^* such that $2\ell^*(1/p_0-1) \geq 2/p_0-1$, i.e., $\ell^* := \lfloor (2-p_0)/(2-2p_0) \rfloor$. Hence, for all $s \ge s^*$ we arrive at

$$\left| \int_{U} G(u - u_{s}) \,\mathrm{d}\boldsymbol{y} \right| \leq C_{1} C_{2} \, s^{-\frac{2}{p_{0}} + 1}, \tag{5.16}$$

where C_2 is a constant depending on p_0 and K_0 , and is independent of k.

It remains to derive the bound for $s < s^*$. Using (5.2) and (5.3) we have the estimate

$$\left| \int_{U} G(u - u_{s}) \,\mathrm{d}\boldsymbol{y} \right| \leq \|G\|_{V^{*}} \sup_{\boldsymbol{y} \in U} \left(\|u(\cdot, \boldsymbol{y})\|_{V} + \|u_{s}(\cdot, \boldsymbol{y})\|_{V} \right)$$

$$\leq \frac{2 C_{\mathrm{func}}(kL)}{C_{\mathrm{coer}}} \|\binom{f}{g}\|_{L^{2}(D) \times L^{2}(\partial D)} \|G\|_{V^{*}} \leq 2 C_{1} \cdot (s^{*})^{\frac{2}{p_{0}} - 1} s^{-\frac{2}{p_{0}} + 1}, \quad (5.17)$$

where we used $s^*/s > 1$ and the definition of C_1 in (5.12). We now use (5.10) to get an upper bound on (5.14) involving K_0 , and choose s^* such that when s replaced by s^* this upper bound is at most 1/2. Consequently s^* is a constant depending on p_0 and K_0 , and is independent of k.

Combining now (5.16) and (5.17), and plugging in the definition (5.12) for C_1 , we obtain the required result for all values of s.

6 Finite element discretizations

In this section first we consider a high-order FEM for computationally solving the sign-definite sesquilinear formulation. For each $\boldsymbol{y} \in U$, having quantified the error resulting from dimension truncation of the stochastic refractive index field $n(\cdot, \boldsymbol{y})$ by $n_s(\cdot, \boldsymbol{y}_{\{1:s\}})$ to approximate the solution $u(\cdot, \boldsymbol{y})$ of (2.9) by the solution $u_s(\cdot, \boldsymbol{y}_{\{1:s\}})$ satisfying

$$\mathscr{B}_{\boldsymbol{y}_{\{1:s\}}}(u_s, v) = \mathscr{G}_{\boldsymbol{y}_{\{1:s\}}}(v), \quad \text{for all } v \in V,$$

$$(6.1)$$

we consider the spatial Galerkin FEM approximation of u_s by $u_{s,h}$. To this end, we choose a finite dimensional subspace $V_h^p \subset H^2(\Omega)$ spanned by splines of degree $p \geq 2$ on a tessellation (of at least C^1 -elements with maximum width h) of D. The space V_h^p is chosen so that the following approximation property holds: for $0 \leq t \leq 2$ and for any $v \in H^{t^*}(\Omega)$ with $t^* \geq t + 1$,

$$\inf_{w_h \in V_h^p} \|v - w_h\|_{H^t} \le C_{\text{appr}} h^{\min\{p+1,t^*\}-t},\tag{6.2}$$

and the constant C_{appr} depends on the chosen norm of v.

For each $\boldsymbol{y} \in U$, the FEM approximation $u_{s,h}(\cdot, \boldsymbol{y}_{\{1:s\}}) \in V_h^p$ to the unique solution $u_s(\cdot, \boldsymbol{y}_{\{1:s\}})$ of (6.1) is required to be computed by solving the linear algebraic system arising from the finitedimensional coercive variational form

$$\mathscr{B}_{\boldsymbol{y}_{\{1:s\}}}(u_{s,h},v) = \mathscr{G}_{\boldsymbol{y}_{\{1:s\}}}(v) \quad \text{for all } v \in V_h^p.$$

$$(6.3)$$

Since $V \subset H^{3/2}(\Omega)$, using (6.2), the coercivity and continuity of the sesquilinear form $\mathscr{B}_{y_{\{1:s\}}}$, Theorem 2.1 and Cea's Lemma, under appropriate spatial regularity assumption of u_s satisfying (6.1) and the degree $p \geq 2$ of the splines, the high-order FEM approximation $u_{s,h} \in V_h^p$ satisfies the following error bound:

$$\|u_{s}(\cdot, \boldsymbol{y}_{\{1:s\}}) - u_{s,h}(\cdot, \boldsymbol{y}_{\{1:s\}})\|_{V} \leq C_{\text{appr}} \frac{C_{\text{cont}}(kL)}{C_{\text{coer}}} \|\binom{f}{g}\|_{L^{2}(D) \times L^{2}(\partial D)} h^{p-1}.$$
(6.4)

Recall that $C_{\text{cont}}(kL) = \mathcal{O}(kL + (kL)^{-1})$. This highlights that the well known *pollution effect* is present in our (and all known) FEM approximations (converging in h) for the Helmholtz PDE in two and higher dimensions. While the pollution effect requires large degrees of freedom (DoF) for large acoustic size kL > 1 using the standard piecewise-linear (p = 1) low-order FEM, we have demonstrated in [15, 16] that the pollution error can be efficiently avoided by using high-order FEM ($p \geq 2$), even for solutions with limited regularity.

In particular, as demonstrated in [16] using an efficient construction of the space V_h^p , the number of DoF do not increase substantially despite imposing higher continuity requirements needed for larger degree splines. In [16], for heterogeneous deterministic models (that is, with $y_j = 0$ in (1.3) for all $j \ge 1$ and spatially dependent mean-field n_0) and for various acoustic size kL values with sufficiently smooth solutions, we have numerically demonstrated p-1 estimated order of convergence (EOC) for p = 2, 3, 4 in the V norm, as stated in (6.4), and also p + 1 and p EOC, respectively, in the H^0 -norm and the H^1 -norm.

For the bounded linear functional $G \in V^*$, based on Nitsche arguments, we obtain for all $\boldsymbol{y} \in U$

$$\left| G(u_s(\cdot, \boldsymbol{y}_{\{1:s\}})) - G(u_{s,h}(\cdot, \boldsymbol{y}_{\{1:s\}})) \right| \leq C_{\text{appr}} \frac{C_{\text{cont}}(kL)}{C_{\text{coer}}} \| \binom{f}{g} \|_{L^2(D) \times L^2(\partial D)} \| G \|_{V^*} h^p, \quad (6.5)$$

and the same upper bound is obtained for for its integral counterpart

$$|I_s(G(u_s - u_{s,h}))| = |I(G(u_s(\cdot, \boldsymbol{y}_{\{1:s\}}))) - I(G(u_{s,h}(\cdot, \boldsymbol{y}_{\{1:s\}})))|.$$

Thus the upper bound is of order $\mathcal{O}((kL + (kL)^{-1})h^p) = \mathcal{O}((1 + (kL)^{-1})(kL + 1)h^p).$

Quasi-Monte Carlo integration 7

For complete details of various QMC integration rules, we refer to the survey [9] and extensive references therein; see also the survey [27] for some QMC theory applied in the context of PDE problems. In the next two subsections we focus on two QMC rules.

7.1Randomly shifted lattice rules (first order convergence)

For a fixed dimension truncation parameter s, we consider the integral of a general complexvalued function F defined over the s-dimensional unit cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^s$

$$I_s(F) = \int_{[-\frac{1}{2},\frac{1}{2}]^s} F(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y},$$

and we approximate this by a randomly shifted lattice rule

$$Q_{s,N}(F; \boldsymbol{\Delta}) = \frac{1}{N} \sum_{i=1}^{N} F\left(\{\boldsymbol{t}_i + \boldsymbol{\Delta}\} - \frac{1}{2}\right), \qquad (7.1)$$

where $t_1, \ldots, t_N \in [0,1]^s$ are deterministic lattice cubature points, and Δ is a random shift which is drawn from the uniform distribution on $[0,1]^s$. The braces in (7.1) indicate that we take the fractional part of each component in a vector, while the subtraction of $\frac{1}{2}$ takes care of the translation from the standard unit cube $[0,1]^s$ to $[-\frac{1}{2},\frac{1}{2}]^s$. The lattice points are given by $t_i = \{\frac{iz}{N}\}$ for i = 1, ..., N, where $z \in \mathbb{Z}^s$ is known as the generating vector and it determines the quality of the lattice rule.

We apply the theory and construction of randomly shifted lattice rules in weighted Sobolev spaces to obtain first order convergence rates. Loosely speaking, these spaces contain functions with square integrable mixed first derivatives. The norm is given by

$$\|F\|_{s,\boldsymbol{\gamma}} = \left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}};\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}) \,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right|^{2} \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2},$$

where $\{1:s\}$ is a shorthand notation for the set of indices $\{1, 2, \ldots, s\}, (\partial^{|\mathbf{u}|}F)/(\partial y_{\mathbf{u}})$ denotes the mixed first derivative of F with respect to the "active" variables $y_{\mathfrak{u}} = (y_j)_{j \in \mathfrak{u}}$, while $y_{\{1:s\}\setminus\mathfrak{u}} =$ $(y_i)_{i \in \{1:s\} \setminus u}$ denotes the "inactive" variables. The weights γ_u moderate the relative importance between subsets of variables. It is known that (see e.g., [9, Theorem 5.1]), given N a prime power and the weights γ_{μ} as input, a generating vector z can be obtained by the componentby-component (CBC) construction to achieve the root-mean-square error (with respect to the random shift)

$$\sqrt{\mathbb{E}_{\mathrm{rqmc}}\left[\left|I_{s}(F)-Q_{s,N}(F;\cdot)\right|^{2}\right]} \leq \left(\frac{2}{N}\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}}\gamma_{\mathfrak{u}}^{\lambda}\left[\varrho(\lambda)\right]^{|\mathfrak{u}|}\right)^{1/(2\lambda)}\|F\|_{s,\boldsymbol{\gamma}} \quad \forall \lambda\in(\frac{1}{2},1], \quad (7.2)$$

where $\rho(\lambda) = \frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}}$, with ζ being the Riemann zeta function. In our Helmholtz PDE problem, the integrand is given by

$$F(\boldsymbol{y}) = G(u_{s,h}(\cdot, \boldsymbol{y})).$$

To apply the relevant QMC theory we need to obtain a bound on the norm $||F||_{s,\gamma} = ||G(u_{s,h})||_{s,\gamma}$. Using linearity and boundedness of G, we have

$$\left|\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}}G(u_{s,h}(\cdot,\boldsymbol{y}))\right| = \left|G\left(\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}}u_{s,h}(\cdot,\boldsymbol{y})\right)\right| \leq \|G\|_{V^*} \left\|\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{y}_{\mathfrak{u}}}u_{s,h}(\cdot,\boldsymbol{y})\right\|_{V}.$$
(7.3)

Now applying Theorem 4.2 with u replaced with $u_{s,h}$ and restricting to multi-indices ν with $\nu_j \leq 1$, we obtain

$$\|G(u_{s,h})\|_{s,\gamma} \leq \frac{C_{\text{func}}(kL)}{C_{\text{coer}}} \|\binom{f}{g}\|_{L^{2}(D) \times L^{2}(\partial D)} \|G\|_{V^{*}} \left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \frac{(|\mathfrak{u}|!)^{2} \prod_{j \in \mathfrak{u}} \Upsilon_{j}^{2}}{\gamma_{\mathfrak{u}}}\right)^{1/2}.$$
 (7.4)

The bound (7.4) takes exactly the same form as in the diffusion case in [28], so we could follow the same line of argument there. Here instead we use a slightly simpler and shorter argument.

Substituting (7.4) into the bound (7.2) and then choosing the weights $\gamma_{\mathfrak{u}}$ to equate the expressions inside the two sums, we obtain

$$\gamma_{\mathfrak{u}} = \left(|\mathfrak{u}|! \prod_{j \in \mathfrak{u}} \frac{\Upsilon_j}{\sqrt{\rho(\lambda)}} \right)^{2/(1+\lambda)}, \qquad \Upsilon_j = C_{\operatorname{regu}}(kL) \, \|\psi_j\|_{W^{1,\infty}(D)}, \tag{7.5}$$

and this yields

$$\sqrt{\mathbb{E}_{\mathrm{rqmc}}\left[\left|I_s(G(u_h^s)) - Q_{s,N}(G(u_h^s); \cdot)\right|^2\right]} \le \frac{C_{s,\gamma}(\lambda)}{N^{1/(2\lambda)}} \frac{C_{\mathrm{func}}(kL)}{C_{\mathrm{coer}}} \|\binom{f}{g}\|_{L^2(D) \times L^2(\partial D)} \|G\|_{V^*},$$

with

$$C_{s,\boldsymbol{\gamma}}(\lambda) := 2^{\frac{1}{2\lambda}} \left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \left(|\mathfrak{u}|! \prod_{j \in \mathfrak{u}} \left(\Upsilon_j \left[\varrho(\lambda) \right]^{1/(2\lambda)} \right) \right)^{\frac{2\lambda}{1+\lambda}} \right)^{\frac{1+\lambda}{2\lambda}}.$$

We proceed to choose the parameter λ such that $C_{s,\gamma}(\lambda)$ is bounded independently of s. Since $C_{\text{regu}}(kL) = \mathcal{O}(kL + (kL)^{-1})$, from (5.7) in Assumption (A5) we know that

$$\sum_{j\geq 1}\Upsilon_j^{p_1}\leq r_1\,K_1\,<\,\infty\,,$$

for some constant r_1 independent of the wavenumber k. Writing $\theta_j := \Upsilon_j [\varrho(\lambda)]^{1/(2\lambda)}$ and $\tau := \frac{2\lambda}{1+\lambda}$, we have

$$\sum_{\mathfrak{u} \subseteq \{1:s\}} \left(|\mathfrak{u}|! \prod_{j \in \mathfrak{u}} \theta_j \right)^{\tau} = \sum_{\ell=0}^s (\ell!)^{\tau} \sum_{\substack{\mathfrak{u} \subseteq \{1:s\}\\ |\mathfrak{u}|=\ell}} \prod_{j \in \mathfrak{u}} \theta_j^{\tau} \le \sum_{\ell=0}^s (\ell!)^{\tau-1} \bigg(\sum_{j=1}^s \theta_j^{\tau} \bigg)^{\ell},$$

where the inequality holds because each term $\prod_{j \in \mathfrak{u}} \theta_j^{\tau}$ from the left-hand side of the inequality appears in the expansion $(\sum_{j=1}^s \theta_j^{\tau})^{\ell}$ exactly $\ell!$ times, and the expansion contains other terms. By the ratio test, the right-hand side is bounded independently of s provided that $\sum_{j=1}^{\infty} \theta_j^{\tau} < \infty$ and $\tau < 1$. Thus in our case we require $p_1 \leq \tau < 1$, i.e.,

$$p_1 \le \frac{2\lambda}{1+\lambda} < 1 \quad \iff \quad \frac{p_1}{2-p_1} \le \lambda < 1.$$

Noting that λ also needs to satisfy $\frac{1}{2} < \lambda \leq 1$, we therefore choose

$$\lambda = \begin{cases} \frac{1}{2 - 2\delta} & \text{for some } \delta \in (0, \frac{1}{2}) & \text{when } p_1 \in (0, \frac{2}{3}], \\ \frac{p_1}{2 - p_1} & \text{when } p_1 \in (\frac{2}{3}, 1), \end{cases}$$

This leads to the convergence rate $\mathcal{O}(N^{-\min(\frac{1}{p_1}-\frac{1}{2},1-\delta)})$, with the implied constant independent of s.

Weights of the form (7.5) are known as POD weights ("product and order dependent weights"). The CBC construction of lattice generating vector can be done for POD weights in $\mathcal{O}(s N \log N + s^2 N)$ operations, see [28].

Combining the estimates from this subsection with (5.8) and (6.5), we obtain the first main conclusion of this paper.

Theorem 7.1. Let the assumptions (A0)–(A5) and parameter restrictions in Theorem 2.1 hold. For each $\mathbf{y} \in U$, let $u(\cdot, \mathbf{y}) \in V$ be the unique solution of (2.9) and $u_{s,h}(\cdot, \mathbf{y}) \in V_h^p$ be the unique solution of (6.3). Then for every $f \in L^2(D)$ and $g \in L^2(\partial D)$, and every linear functional $G \in V^*$, a generating vector can be constructed for a randomly shifted lattice rule such that

$$\begin{split} &\sqrt{\mathbb{E}_{\rm rqmc}} \left[\left| I(G(u)) - Q_{s,N}(G(u_{s,h}); \cdot) \right|^2 \right] \\ &\leq C \cdot \left(1 + (kL)^{-1} \right) \left(s^{-\frac{2}{p_0} + 1} + \left(kL + 1 \right) h^p + N^{-\min(\frac{1}{p_1} - \frac{1}{2}, 1 - \delta)} \right), \quad \delta \in (0, \frac{1}{2}), \end{split}$$

where C depends on f, g, G, but is independent of s, h, N and the wavenumber k.

7.2 Interlaced polynomial lattice rules (higher order convergence)

In this subsection we briefly outline the results when we replace randomly shifted lattice rules by deterministic *interlaced polynomial lattice rules*, which allow us to obtain higher order convergence rates. The description below follows closely [7].

Without giving the full technical details, we simply say here that (7.1) is now replaced by a deterministic quadrature rule

$$Q_{s,N}(F) = \frac{1}{N} \sum_{i=1}^{N} F\left(\boldsymbol{t}_i - \frac{1}{2}\right),$$

where the points $t_i \in [0,1]^s$ are obtained by "interlacing" the points of a "polynomial lattice rule", which are specified by a generating vector of "polynomials" rather than of integers. For the precise details as well as implementation, see e.g., [7, 27] and the references there. The error bound (7.2) is now replaced by

$$\left|I_{s}(F) - Q_{s,N}(F)\right| \leq \left(\frac{2}{N} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \left[\varrho_{\alpha}(\lambda)\right]^{|\mathfrak{u}|}\right)^{1/(2\lambda)} \|F\|_{s,\alpha,\gamma} \quad \forall \lambda \in (\frac{1}{\alpha}, 1],$$

where $\alpha \geq 2$ is an integer smoothness parameter (also known as the "interlacing factor"), N is a power of 2, $\rho_{\alpha}(\lambda) = 2^{\alpha\lambda(\alpha-1)/2} [(1 + \frac{1}{2^{\alpha\lambda}-2})^{\alpha}]$. and the norm is now

$$\|F\|_{s,\alpha,\boldsymbol{\gamma}} := \sup_{\mathfrak{u} \subseteq \{1:s\}} \sup_{\boldsymbol{y}_{\mathfrak{v}} \in [0,1]^{|\mathfrak{v}|}} \frac{1}{\gamma_{\mathfrak{u}}} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u} \setminus \mathfrak{v}} \in \{1:\alpha\}^{|\mathfrak{u} \setminus \mathfrak{v}|}} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{v}|}} (\partial^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u} \setminus \mathfrak{v}},\mathbf{0})} F)(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}_{\{1:s\} \setminus \mathfrak{v}} \right|$$

Using again (7.3) and Theorem 4.2 (this time with general multi-indices), we obtain instead of (7.4),

$$\|G(u_{s,h})\|_{s,\alpha,\boldsymbol{\gamma}} \leq \frac{C_{\mathrm{func}}(kL)}{C_{\mathrm{coer}}} \|\binom{f}{g}\|_{L^{2}(D) \times L^{2}(\partial D)} \|G\|_{V^{*}} \sup_{\mathfrak{u} \subseteq \{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \sum_{\boldsymbol{\nu}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} |\boldsymbol{\nu}_{\mathfrak{u}}|! \prod_{j \in \mathfrak{u}} \left(2^{\delta(\nu_{j},\alpha)} \Upsilon_{j}^{\nu_{j}}\right),$$

where $\delta(\nu_j, \alpha)$ is 1 if $\nu_j = \alpha$ and is 0 otherwise. We now choose $\gamma_{\mathfrak{u}}$ so that the supremum is 1, i.e.,

$$\gamma_{\mathfrak{u}} = \sum_{\boldsymbol{\nu}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} |\boldsymbol{\nu}_{\mathfrak{u}}|! \prod_{j \in \mathfrak{u}} \left(2^{\delta(\nu_{j},\alpha)} \Upsilon_{j}^{\nu_{j}} \right).$$
(7.6)

Using the above weights and following the arguments in [7, Pages 2694–2695], by taking $\lambda = p_1$ and the interlacing factor $\alpha = \lfloor 1/p_1 \rfloor + 1$, we eventually arrive at the convergence rate $\mathcal{O}(N^{-1/p_1})$, with the implied constant independent of s, h, N.

Weights of the form (7.6) are called SPOD weights ("smoothness-driven product and order dependent weights"). The generating vector (of polynomials) can be obtained by a CBC construction in $\mathcal{O}(\alpha s N \log N + \alpha^2 s^2 N)$ operations, see [7].

We summarize our second main conclusion in the following theorem.

Theorem 7.2. Let the assumptions (A0)–(A5) and parameter restrictions in Theorem 2.1 hold. For each $\mathbf{y} \in U$, let $u(\cdot, \mathbf{y}) \in V$ be the unique solution of (2.9) and $u_{s,h}(\cdot, \mathbf{y}) \in V_h^p$ be the unique solution of (6.3). Then for every $f \in L^2(D)$ and $g \in L^2(\partial D)$, and every linear functional $G \in V^*$, a generating vector can be constructed for an interlaced polynomial lattice rule with interlacing factor $\alpha = \lfloor 1/p_1 \rfloor + 1 \geq 2$ such that

$$\left| I(G(u)) - Q_{s,N}(G(u_{s,h})) \right| \le C \cdot \left(1 + (kL)^{-1} \right) \left(s^{-\frac{2}{p_0}+1} + (kL+1)h^p + N^{-\frac{1}{p_1}} \right),$$

where C depends on f, g, G, but is independent of s, h, N and the wavenumber k.

Acknowledgements

We sincerely thank the anonymous referees for insightful comments and suggestions which helped to improve the paper. We gratefully acknowledge the financial support from the Australian Research Council for the project DP180101356.

A Small perturbation approach

In this section, by a partial differential operator (PDO) associated with a boundary value problem (BVP), governed by a PDE and a boundary condition (BC), we mean the PDO in its weak sense. The weak PDO (WPDO) is a linear operator induced by a sesquilinear form associated with an equivalent weak formulation of the BVP (WBVP).

The stochastic wave propagation Helmholtz PDE model introduced in Section 1 can be reformulated, using the celebrated standard weak form, as

$$S_{\boldsymbol{y}}(u,w) = \ell(w) \quad \text{for all} \quad \boldsymbol{y} \in U, \quad w \in H^1(D),$$
 (A.1)

where, for fixed $\boldsymbol{y}, \, \mathcal{S}_{\boldsymbol{y}} : H^1(D) \times H^1(D) \to \mathbb{C}$ is a sesquilinear form, and ℓ is a linear functional. More precisely,

$$\mathcal{S}_{\boldsymbol{y}}(v,w) = a_0(v,w) + \sum_{j \ge 1} y_j a_j(v,w),$$

where for $v, w \in H^1(D)$

$$a_0(v,w) = \int_D \left[\nabla v \cdot \overline{\nabla w} - k^2 \, n_0 \, v \, \overline{w} \right] - \mathrm{i} \, k \int_{\partial D} \gamma v \overline{\gamma w}, \qquad a_j(v,w) = k^2 \int_D \psi_j v \, \overline{w}, \qquad j \ge 1.$$

It is well known that a_0 is sign-indefinite (that is, non-coercive). However, a_0 satisfies the inf-sup condition with inf-sup constant $\mu_0 = \mathcal{O}(1/k)$, see for example [2, Cor. 1.10]. Indeed, till recently,

all known and analyzed variational reformulations of the heterogeneous media deterministic Helmholtz model are sign-indefinite, see for example [16] and references therein.

The inf-sup property of a_0 has been established [2], using the following weighted (k-dependent) norm in $H^1(D)$ [2]:

$$\|v\|_{H^1_k}^2 = \|\nabla v\|_{L^2(D)}^2 + k^2 \|v\|_{L^2(D)}^2$$

The framework in [7, 8] is established for a general class of operators defined on reflexive Banach spaces X, Y. For our wave propagation model, it is appropriate to consider linear operators $\mathcal{A}_j: X \to Y'$ defined as

$$_{Y}\langle w, \mathcal{A}_{j}v \rangle_{Y'} = a_{j}(v, w), \qquad v \in X, \quad w \in Y, \qquad j \ge 0,$$

with $X = Y = H_k^1(D)$. Consequently the standard WBVP (A.1) based WPDO \mathcal{A} of the BVP (1.1) with PDO \mathcal{L} in (1.2) is:

$$\mathcal{A}(\boldsymbol{y}) = \mathcal{A}_0 + \sum_{j \ge 1} y_j \mathcal{A}_j, \qquad \boldsymbol{y} \in U.$$
(A.2)

Thus, thanks to the inf-sup property of a_0 , we have $\mathcal{A}_0 \in \mathcal{L}(X, Y')$ is boundedly invertible with $\|\mathcal{A}_0^{-1}\| = \mathcal{O}(k)$ and $\|\mathcal{A}_j\| = \mathcal{O}(1)$, since

$$\|\mathcal{A}_{j}\| \leq \sup_{v,w\neq 0} \frac{|a_{j}(v,w)|}{\|v\|_{X} \|w\|_{X}} \leq \frac{\|\psi_{j}\|_{L^{\infty}(D)} k^{2} \|v\|_{L^{2}(D)} \|w\|_{L^{2}(D)}}{\|v\|_{X} \|w\|_{X}} \leq \|\psi_{j}\|_{L^{\infty}(D)}$$

The class of stochastic WPDOs considered in [7, 8] are of the form in (A.2). The framework in [7, 8] starts with the summability assumption [7, Equation (1.3)]

$$\sum_{j\geq 1} \|\mathcal{A}_j\|_{\mathcal{L}(X,Y')}^p < \infty, \quad \text{for some} \quad p \in (0,1],$$

and bounded invertible assumption of \mathcal{A}_0 , as a linear operator from X to Y'.

The analysis in [7, 8] and related papers, while of wide generality, requires that the operator sum in (A.2) be small, in the sense that

$$\mathcal{A}_0 + \sum_{j \ge 1} y_j \mathcal{A}_j = \mathcal{A}_0 \Big(\mathcal{I} + \sum_{j \ge 1} y_j \mathcal{A}_0^{-1} \mathcal{A}_j \Big)$$

should satisfy, using $|y_j| \leq 1/2$,

$$\frac{1}{2} \sum_{j \ge 1} \|\mathcal{A}_0^{-1} \mathcal{A}_j\| < 1,$$
 (A.3)

since if this is satisfied then the Neumann series for the inverse of the operator sum converges in operator norm in the space X. Accordingly, it seems reasonable to say that any argument based on (A.3) is using the "small perturbation" approach. Note that (A.3), when applied to our wave propagation model, requires that a quantity of the order $k \sum_{j\geq 1} \|\psi_j\|_{L^{\infty}(D)}$ be less than 1.

B Technical lemma

Lemma B.1. Given some non-negative real numbers $(\Psi_j)_{j \in \mathbb{N}}$ and constants c_0, c_1, c_2, B , let $(\mathbb{A}_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathfrak{F}}$ be non-negative real numbers satisfying the inequality

$$\mathbb{A}_{\boldsymbol{\nu}} \leq \begin{cases} B & \text{if } \boldsymbol{\nu} = \boldsymbol{0}, \\ c_0 \Psi_j B & \text{if } \boldsymbol{\nu} = \boldsymbol{e}_j, \\ c_1 \sum_{j \in \text{supp}(\boldsymbol{\nu})} \nu_j \Psi_j \mathbb{A}_{\boldsymbol{\nu} - \boldsymbol{e}_j} \\ + c_2 \sum_{j \in \text{supp}(\boldsymbol{\nu})} \sum_{\ell \in \text{supp}(\boldsymbol{\nu} - \boldsymbol{e}_j)} \nu_j (\boldsymbol{\nu} - \boldsymbol{e}_j)_{\ell} \Psi_j \Psi_{\ell} \mathbb{A}_{\boldsymbol{\nu} - \boldsymbol{e}_j - \boldsymbol{e}_{\ell}} & \text{if } |\boldsymbol{\nu}| \ge 2. \end{cases}$$

Then for any $\boldsymbol{\nu} \in \mathfrak{F}$ we have

$$\mathbb{A}_{\boldsymbol{\nu}} \leq |\boldsymbol{\nu}|! \, \boldsymbol{\Upsilon}^{\boldsymbol{\nu}} B , \quad with \quad \boldsymbol{\Upsilon}^{\boldsymbol{\nu}} := \prod_{j \geq 1} \Upsilon_j^{\nu_j}, \qquad \Upsilon_j := \max\left\{c_0, 2c_1, \sqrt{2c_2}\right\} \Psi_j .$$

Proof. Let $\Upsilon_j = C \Psi_j$. We prove this result by induction while determining the multiplying factor C. The cases $|\boldsymbol{\nu}| \leq 1$ hold trivially if $c_0 \leq C$. Suppose that the result holds for all $|\boldsymbol{\nu}| < n$ with some $n \geq 1$. Then for $|\boldsymbol{\nu}| = n \geq 2$, we can split the terms in the inequality into

$$\mathbb{A}_{\boldsymbol{\nu}} \leq c_1 \sum_{j \geq 1} \nu_j \Psi_j \mathbb{A}_{\boldsymbol{\nu}-\boldsymbol{e}_j} + c_2 \sum_{j \geq 1} \nu_j (\nu_j - 1) \Psi_j^2 \mathbb{A}_{\boldsymbol{\nu}-2\boldsymbol{e}_j} + c_2 \sum_{j \geq 1} \sum_{\substack{\ell \geq 1 \\ \ell \neq j}} \nu_j \nu_\ell \Psi_j \Psi_\ell \mathbb{A}_{\boldsymbol{\nu}-\boldsymbol{e}_j-\boldsymbol{e}_\ell}.$$

Applying the induction hypothesis then leads to

$$\begin{split} \mathbb{A}_{\boldsymbol{\nu}} &\leq c_{1} \sum_{j \geq 1} \nu_{j} \Psi_{j} \left(|\boldsymbol{\nu}| - 1 \right)! \Upsilon^{\boldsymbol{\nu}-\boldsymbol{e}_{j}} B + c_{2} \sum_{j \geq 1} \nu_{j} \left(\nu_{j} - 1 \right) \Psi_{j}^{2} \left(|\boldsymbol{\nu}| - 2 \right)! \Upsilon^{\boldsymbol{\nu}-\boldsymbol{2}\boldsymbol{e}_{j}} B \\ &+ c_{2} \sum_{j \geq 1} \sum_{\substack{\ell \geq 1 \\ \ell \neq j}} \nu_{j} \nu_{\ell} \Psi_{j} \Psi_{\ell} \left(|\boldsymbol{\nu}| - 2 \right)! \Upsilon^{\boldsymbol{\nu}-\boldsymbol{e}_{j}-\boldsymbol{e}_{\ell}} B \\ &\leq \frac{c_{1}}{C} \sum_{j \geq 1} \nu_{j} (|\boldsymbol{\nu}| - 1)! \Upsilon^{\boldsymbol{\nu}} B + \frac{c_{2}}{C^{2}} \sum_{j \geq 1} \nu_{j} \left(\nu_{j} - 1 \right) \left(|\boldsymbol{\nu}| - 2 \right)! \Upsilon^{\boldsymbol{\nu}} B \\ &+ \frac{c_{2}}{C^{2}} \sum_{j \geq 1} \sum_{\substack{\ell \geq 1 \\ \ell \neq j}} \nu_{j} \nu_{\ell} \left(|\boldsymbol{\nu}| - 2 \right)! \Upsilon^{\boldsymbol{\nu}} B = \left(\frac{c_{1}}{C} + \frac{c_{2}}{C^{2}} \right) |\boldsymbol{\nu}|! \Upsilon^{\boldsymbol{\nu}} B. \end{split}$$

If $c_1 \leq \frac{C}{2}$ and $c_2 \leq \frac{C^2}{2}$, then $\frac{c_1}{C} + \frac{c_2}{C^2} \leq 1$. So we may choose $C := \max\{c_0, 2c_1, \sqrt{2c_2}\}$ as stated in the lemma.

An alternative bound can be obtained by choosing $C := \max\{c_0, c_3\}$, with $c_3 := \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}$ which satisfies $\frac{c_1}{c_3} + \frac{c_2}{c_3^2} = 1$.

References

- H. Barucq, T. Chaumont-Frelet, and C. Gout, Stability analysis of heterogeneous Helmholtz problems and finite element solution based on propagation media approximation, *Math. Comp.*, 86, 2129–2157, 2017.
- [2] D. Baskin, E. A. Spence, and J. Wunsch, Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations, SIAM J. Math. Anal., 48, 229– 267, 2016.

- [3] T. Chaumont-Frelet, On high order methods for the heterogeneous Helmholtz equation, *Comp. Math. Appls.*, 72, 2203–2225, 2016.
- [4] T. Chaumont-Frelet and S. Nicaise, Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problem, *IMA J. Num. Anal.*, 40, 1503–1543, 2020.
- [5] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer, New York, 2012.
- [6] V. Domínguez, M. Ganesh and F.-J. Sayas. An overlapping decomposition framework for wave propagation in heterogeneous and unbounded media: Formulation, analysis, algorithm, and simulation J. Comput. Phys., 403, 109052, 2020.
- [7] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and Ch. Schwab, Higher order QMC Galerkin discretization for parametric operator equations, *SIAM J. Numer. Anal.*, **52**, 2676–2702, 2014.
- [8] J. Dick, F. Y. Kuo, Q. T. Le Gia, and Ch. Schwab, Multi-level higher order QMC Galerkin discretization for affine parametric operator equations, *SIAM J. Numer. Anal.*, 54, 2541– 2568, 2016.
- [9] J. Dick, F. Y. Kuo, and I. H. Sloan, High-dimensional integration: the Quasi-Monte Carlo way, Acta Numer., 22, 133–288, 2013.
- [10] J. Dick, Q. T. Le Gia, and Ch. Schwab, Higher order Quasi-Monte Carlo integration for holomorphic, parametric operator equations, SIAM/ASA J. Uncertain. Quantifi., 4, 48–79, 2016.
- [11] J. Dick and F. Pillichshammer, *Digital Nets and Sequences*, Cambridge University Press, Cambridge, 2010.
- [12] Y. V. Egorov and M. A. Shubin (Eds.), Partial Differential Equations, Springer, New York, 1993.
- [13] X. Feng, J. Lin, and C. Lorton, An efficient numerical method for acoustic wave scattering in random media, SIAM/ASA J. Uncertain. Quantifi., 3, 790–822, 2015.
- [14] M. Ganesh and C. Morgenstern, High-order FEM-BEM computer models for wave propagation in unbounded and heterogeneous media: application to time-harmonic acoustic horn problem, J. Comput. Appl. Math., 307, 183–203, 2016.
- [15] M. Ganesh and C. Morgenstern, A sign-definite preconditioned high-order FEM part-I: formulation and simulation for bounded homogeneous media wave propagation SIAM J. Sci. Comput., 39, S563–S586, 2017.
- [16] M. Ganesh and C. Morgenstern, A coercive heterogeneous media Helmholtz model: formulation, wavenumber-explicit analysis, and preconditioned high-order FEM, *Numer. Algorithms*, 83, 1441–1487, 2020.
- [17] R. N. Gantner, Dimension truncation in QMC for affine-parametric operator equations, in: Monte Carlo and Quasi-Monte Carlo Methods 2016 (A. B. Owen and P. W. Glynn, eds), Springer, Berlin, pp. 249–264, 2018.

- [18] R. N. Gantner, L. Herrmann and Ch. Schwab, Quasi-Monte Carlo integration for affineparametric, elliptic PDEs: local supports imply product weights, SIAM J. Numer. Anal., 56, 111–135, 2018.
- [19] A. D. Gilbert, I. G. Graham, F. Y. Kuo, R. Scheichl, and I. H. Sloan, Analysis of quasi-Monte Carlo methods for elliptic eigenvalue problems with stochastic coefficients *Numer*. *Math.*, **142**, 863–915, 2019.
- [20] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, Ch. Schwab and I. H. Sloan, Quasi-Monte Carlo finite element methods for elliptic PDEs with log-normal random coefficients, *Numer. Math.*, 131, 329–368, 2015.
- [21] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, Circulant embedding with QMC – analysis for elliptic PDE with lognormal coefficients *Numer. Math.*, 140, 479–511, 2018.
- [22] I. G. Graham, O. R. Pembery, E. A. Spence, The Helmholtz equation in heterogeneous media: a priori bounds, well-posedness, and resonances, J. Diff. Eqns., 266, 2869–2923, 2019.
- [23] I. G. Graham and S. A. Sauter, Stability and finite element error analysis for the Helmholtz equation with variable coefficients, *Math. Comp.*, 89, 105–138, 2020.
- [24] J. Galkowski, E. A. Spence, and J. Wunsch, Optimal constants in nontrapping resolvent estimates and applications in numerical analysis, *Pure Appl. Anal.*, 2, 157–202, 2020.
- [25] H. Harbrecht, M. Peters, and M. Siebenmorgen. On the quasi-Monte Carlo method with Halton points for elliptic PDEs with log-normal diffusion, Math. Comp., 86, 771–797, 2017.
- [26] F. Ihlenburg. Finite element analysis of acoustic scattering, Appl. Math. Sci., 132, Springer-Verlag, New York, 1998.
- [27] F. Y. Kuo and D. Nuyens, Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients – a survey of analysis and implementation, *Found. Comput. Math.*, 16, 1631–1696, 2016.
- [28] F. Y. Kuo, Ch. Schwab, and I. H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficient, SIAM J. Numer. Anal., 50, 3351–3374, 2012.
- [29] W. McLean. Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, Cambridge, 2000.
- [30] J.-C. Nédélec. Acoustic and Electromagnetic Equations. Springer, New York, 2001.
- [31] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, SIAM, Philadelphia, 1992.
- [32] O. R. Pembery and E. A. Spence, The Helmholtz equation in random media: well-posedness and a priori bounds, SIAM/ASA J. Uncertain. Quantifi., 8, 58–87, 2020.
- [33] Ch. Schwab, QMC Galerkin discretizations of parametric operator equations, in: Monte Carlo and Quasi-Monte Carlo Methods 2012 (J. Dick, F. Y. Kuo, G. W. Peters and I. H. Sloan, eds.), Springer Verlag, Heidelberg, pp. 613–629, 2013.
- [34] I. H. Sloan and S. Joe, Lattice Methods for Multiple Integration, Oxford University Press, Oxford, 1994.