A game theoretical approach to homothetic robust forward investment performance processes in stochastic factor models

Juan Li* Wenqiang Li† Gechun Liang‡

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Abstract

This paper studies an optimal forward investment problem in an incomplete market with model uncertainty, in which the underlying stocks depend on the correlated stochastic factors. The uncertainty stems from the probability measure chosen by an investor to evaluate the performance. We obtain directly the representation of the homothetic robust forward performance processes in factor-form by combining the zero-sum stochastic differential game and ergodic BSDE approach. We also establish the connections with the risk-sensitive zero-sum stochastic differential games over an infinite horizon with ergodic payoff criteria, as well as with the classical robust expected utilities for long time horizons. Finally, we give an example to illustrate that our approach can be applied to address a type of robust forward investment performance processes with negative realization processes.

Keywords: Forward performance process; model uncertainty; self-generating stochastic differential game; ergodic BSDE; ergodic risk-sensitive stochastic differential game.

*School of Mathematics and Statistics, Shandong University, Weihai, Weihai, China. Partially supported by National Key R and D Program of China (No.2018YFA0703900), by the NSF of P.R. China (No. 11871037) and by NSFC-RS (No. 11661130148; NA150344). Email: juanli@sdu.edu.cn.

†School of Mathematics and Information Sciences, Yantai University, Yantai, China. Partially supported by Natural Science Foundation of Shandong Province (No.ZR2017MA015) and by Doctoral Scientific Research Fund of Yantai University (No. SX17B09). Email: wenqiangli@ytu.edu.cn. Corresponding author.

‡Department of Statistics, The University of Warwick, Coventry, U.K. Partially supported by NNSF of China (Grant No. 11771158, 11801091). Email: g.liang@warwick.ac.uk.

1 Introduction

The aim of this paper is to study optimal investment evaluated by a forward performance criterion in a stochastic factor market model, in which the probability measure that models future stock price evolutions is ambiguous. The forward performance process, as an adapted stochastic dynamic utility evolving forward in time, has been introduced and developed in [41]-[45] (see also [24] and [55], and more recently [2], [3], [6], [12], [23], [28], [33], [37], [39] and [51]). This new concept differs from the classical expected utility function, in which the objective is to solve a stochastic control problem in a backward way via dynamic programming principle. One of the advantages of forward performance processes is allowing the investor to consider optimal investment problems with arbitrary horizons. As such, it provides a useful complement and a natural extension to the classical expected utility function.

Recall the classical expected utility theory for the optimal portfolio selection is

$$\sup_{\pi} E_{\mathbb{P}}[U(X_T^{\pi})],$$

where π is the portfolio choice, $\mathbb{P}$ is a probability measure that is used to measure the evolutions of stock prices, T is the terminal horizon, and U is a *fixed* utility function at time T. In spite of the popularity of expected utility theory, there has been some criticism of it. One of them is the fact that it is not satisfactory in dealing with model uncertainty (also called Knightian uncertainty) as predicted by the famous Ellsberg paradox. In fact, an investor frequently faces significant ambiguity about the probability measure $\mathbb{P}$ to evaluate the investment performance. In finance, [40] argued that the (perceived) failures of the dominant paradigm, for example, in the context of the recent crisis, are due to inadequate attention paid to the kind of uncertainty faced by agents and modelers.

One possible way to address this problem is to use the concept of robust utility, which was introduced to account for uncertain aversion. It can be numerically represented by the following form

$$X \to \inf_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[U(X)],$$

where $\mathcal{P}$ is a family of probability measures describing all the possible probabilities of future scenarios and the infimum means the worst-case scenario is implemented. Robust utility maximization in the optimal investment problems has been widely investigated under different situations with different approaches, among others, a stochastic control method in [10, 26], a stochastic differential game approach in [48], a duality method in [49]. For more details on various portfolio selection problems, we refer to [1, 14, 20, 21, 27, 53, 54] and the references therein. In particular, we refer to [30, 31, 32] for the review of the recent advancements in robust investment management.

In this paper, we consider the ambiguity of the probability measure under the framework of forward performance processes in incomplete markets. We propose a framework that solves directly the above problem in a unified manner, combining the *zero-sum stochastic differential game* and *ergodic backward stochastic differential equation* (BSDE) theory. The concept of robust forward performance processes was recently introduced in [34], by using a penalty function to weight relatively the probability models such that they are more in line with the actual market. They obtained the characterization of the robust forward criteria via a duality approach. See also [13] for an extension to uncertain parameters. However, both papers only consider robust forward performance processes with zero volatility, in particular, the *Markovian* case for the stochastic factor model is not covered. In this paper, we consider the *Markovian* robust forward performance process in a stochastic factor model. The approach is different from the duality approach used in [34] and the saddle point method used in [13]. Next, we briefly introduce our framework and explain our major contributions.

We construct the robust forward performance process via a two-player zero-sum stochastic differential game. In our model, the ambiguity of the probability measure is described via a family of equivalent probability measures parameterized by a density process u in a compact and convex set (see (6)). We parameterize the robust forward performance process by the density process u. This generalizes the original definition of the robust forward performance process introduced in [34] with penalty functions as a special class of parametrization. We refer to Definition 4 and Remarks 5-6 for more details.

To robustify the optimal investment, the investor will select the best investment portfolio that is least affected by the model uncertainty, whereas the nature of the market acts to minimize the expected forward preference by choosing the worst-case scenario. This leads to a two-player zero-sum stochastic differential game between the investor and the market, where each player's decision (strategy) depends on the counterparty's action (control) she has observed. Therefore, the concept of "strategy" corresponding to the "control" will play a key role in analyzing the game (see [11, 19]).

Utilizing the idea of "strategy", we give a new characterization of the robust forward performance process. Specifically, both the worst-case scenario "strategy" corresponding to each portfolio selection and the optimal investment policy under the worst-case scenario "strategy" are given in our characterization (see (15)-(16)). Moreover, if the game value exists, the optimal investment "strategy" corresponding to each scenario and the worst-case scenario under the optimal investment "strategy" are further given in (18)-(19). Compared to the saddle point argument used in [26, 27, 53] in the classical framework and [13] in the forward framework, our characterization (15)-(16) and (18)-(19) relies on the investor's response to each scenario and portfolio choice. Moreover, it is often relatively easy to compute the optimal strategies, as they only involve maximization/minimization problems rather than maxmin/minmax problems. On the other hand, our stochastic differential game approach may also provide

an alternative way to study the robust forward performance process when saddle point does not exist. We present an example for the case $\tau < 0$ (i.e. with a negative realization process) in Section 7 to illustrate this point.

The second component to construct the robust forward performance process in factor form is an ergodic BSDE. The stochastic PDE (SPDE) approach, introduced in [45] to characterize the forward performance processes (without model ambiguity), may not be applied directly to our model. First, the form of the related SPDE is not easy to derive due to the presence of model uncertainty. Second, it is difficult to obtain the solution existence and uniqueness of the SPDE for the general case even if we know the form of the equation.

In order to get the representation of the homothetic robust forward performance process in stochastic factor form, we apply directly the ergodic BSDE approach, which was first proposed in [22] to study ergodic control problems. The ergodic BSDE approach was first exploited in [38] to study the representation of the homothetic forward performance process in the absence of model uncertainty. We first characterize the power robust forward performance process in terms of the solution of an ill-posed Isaacs type equation. Although the solution of this Isaacs equation can not be obtained directly, it offers (i) the construction of the optimal portfolio "strategy", the worst-case scenario "strategy", and the related optimal portfolio choice and the worst-case scenario; (ii) the hint of the driver form of the corresponding ergodic BSDE. Then, we obtain the representation of the robust power forward performance process by using the Markovian solution of the ergodic BSDE. The associated optimal portfolio and worst-case scenario "strategy" and "control" are also obtained in feedback form of the stochastic factor. We can also obtain other type of homothetic robust forward performance processes (logarithmic and exponential) using the same approach.

The third contribution of this paper is establishing a connection between the constant λ appearing in the solution of the ergodic BSDE (38) and a class of zero-sum risk-sensitive stochastic differential game over an infinite horizon with ergodic payoff criteria. Risk-sensitive optimal control has been widely applied to optimal investment problems (see, [8, 17, 18, 25] and references therein). The corresponding risk-sensitive stochastic differential games are studied in [5, 7, 9, 35] via PDE approach and in [15] via BSDE approach.

In this paper, we apply directly the ergodic BSDE approach to address the zero-sum risk-sensitive stochastic differential game with ergodic payoff criteria over an infinite horizon. Thus, we provide a new method to obtain the value of the risk-sensitive game problem and give the robust optimal investment policy which generalizes the results in [17, 18] to the stochastic factor model with uncertainty. To obtain this connection, we prove a comparison result for a class of ergodic BSDE whose drivers are only locally Lipschitz continuous. With the help of this connection, the constant λ can be interpreted as the *optimal long-term growth rate of the expected utility with model uncertainty*, and can also be applied to study the related "robust large deviations" criteria for long-term investment problems.

In addition, we develop a connection between the robust forward performance process and classical robust expected utility. Optimal investment problems with classical robust expected utilities have been studied via different methods, among others, by the duality approach [20, 49], the stochastic control approach based on BSDE [10, 26] and stochastic differential game approach based on PDE [52]. With the help of the relation established in [29] on the solution of finite horizon BSDE and the solution of associated ergodic BSDE, we prove that an appropriately discounted lower value function associated with the classical power robust expected utility will converge to the power robust forward performance process as the trading horizon tends to infinity.

This paper is organized as follows. In section 2, we introduce the market model with uncertainty and the notion of robust forward performance processes. The stochastic differential game approach is given in subsections 2.1 and 2.2 for different situations. In section 3, we focus on the power case and construct the robust forward performance process in factor-form. Two examples are given in section 4 to illustrate the applications in incomplete markets. Then, we present the connections with the risk-sensitive game problem and classical expected utility in sections 5 and 6, respectively. In section 7, we further provide an example with a negative realization process for which saddle point does not exist. Finally, section 8 concludes.

2 The stochastic factor model with uncertainty and its robust forward performance process

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, on which the process $W = (W^1, \cdots, W^d)^T$ is a standard d-dimensional Brownian motion. Here, the superscript T denotes the matrix transpose. Suppose the market consists of a risk-free bond and n risky stocks. The bond is assumed to be zero interest rate and the discounted (by the bond) individual stock price S_t^i, $t \geq 0$, affected by the stochastic factor process V, has the following form, for $i = 1, ..., n$,

$$\frac{dS_t^i}{S_t^i} = b^i(V_t)dt + \sum_{j=1}^{d} \sigma^{ij}(V_t)dW_t^j, \tag{1}$$

with $S_0^i > 0$, where the factor process $V = (V^1, \cdots, V^d)^T$ satisfies, for $i = 1, ..., d$,

$$dV_t^i = \eta^i(V_t)dt + \sum_{j=1}^{d} \kappa^{ij} dW_t^j, \tag{2}$$

with $V_0^i \in \mathbb{R}$. We introduce the basic assumptions on the above model.

Assumption 1 *(H1) The coefficients*

$$b : \mathbb{R}^d \to \mathbb{R}^n,\ \sigma : \mathbb{R}^d \to \mathbb{R}^{n\times d},$$

are uniformly bounded and the volatility matrix $\sigma(v)$ has full row rank n.

(H2) The drift coefficient η satisfies the following dissipative condition: there exists some positive constant $C_\eta > 0$ such that

$$(\eta(v) - \eta(\bar{v}))^T(v - \bar{v}) \leq -C_\eta |v - \bar{v}|^2, \tag{3}$$

for any $v, \bar{v} \in \mathbb{R}^d$. The volatility matrix $\kappa = (\kappa^{ij})$, $1 \leq i, j \leq d$, is a constant matrix with $\kappa\kappa^T$ positive definite and normalized to $||\kappa|| = 1$[1].

Herein, the dissipative condition (3) is introduced to ensure the existence of a unique invariant measure of the stochastic factor process V, i.e., V is ergodic. To simplify the notation, we introduce the market price of risk vector $\theta(v)$, which is defined as

$$\theta(v) = \sigma(v)^T[\sigma(v)\sigma(v)^T]^{-1}b(v),\ v \in \mathbb{R}^d, \tag{4}$$

so it solves the market price of risk equation $\sigma(v)\theta(v) = b(v)$. In addition, we suppose the market price of risk vector $\theta(v)$, $v \in \mathbb{R}^d$, is uniformly bounded and Lipschitz continuous with bound K_θ and Lipschitz constant C_θ.

We consider an investor starting at time $t = 0$ with initial wealth level $x > 0$ and trading among the bond and the stocks. Let $\tilde{\pi} = (\tilde{\pi}^1, \cdots, \tilde{\pi}^n)^T$ be the proportions of her total wealth in the individual stock accounts. Then, due to the self-financing policy, the cumulative wealth process X^π satisfies

$$dX_t^\pi = \sum_{i=1}^n \frac{\tilde{\pi}_t^i X_t^\pi}{S_t^i} dS_t^i = X_t^\pi \tilde{\pi}_t^T \left(b(V_t)dt + \sigma(V_t)dW_t\right).$$

As in [38], using the investment proportions rescaled by the volatility of stock prices, namely, $\pi_t^T = \tilde{\pi}_t^T \sigma(V_t)$, we get

$$dX_t^\pi = X_t^\pi \pi_t^T(\theta(V_t)dt + dW_t), \tag{5}$$

with $X_0^\pi = x \in \mathbb{R}_+$.

Next, we consider model uncertainty, i.e., the ambiguity of the probability measure which evaluates the performance. We denote by $\tilde{u} = (\tilde{u}^1, \cdots, \tilde{u}^d)^T$ the parameters reflecting the possible future scenarios. For convenience, we will work throughout with the scenario parameters rescaled by the volatility of stochastic factors, i.e.,

$$u = \kappa^T(\kappa\kappa^T)^{-1}\tilde{u}.$$

We introduce admissible spaces $\tilde{\Pi}$ and $\mathcal{U}$ for the rescaled investment proportions π and scenario parameters u, respectively.

[1] Herein, $||\cdot||$ represents the trace norm for the matrix in $\mathbb{R}^{d\times d}$, whereas $|\cdot|$ in (3) represents the norm for the vectors in $\mathbb{R}^d$.

Definition 1 *Let $\Pi \subset \mathbb{R}^d$ be convex and closed and include the origin 0. For any $t \geq 0$, a process $\pi : \Omega \times [0,t] \to \Pi$ is an admissible investment proportion for an investor in the trading interval $[0,t]$, if $\pi \in \mathcal{L}^2_{BMO}[0,t]$, where*

$$\mathcal{L}^2_{BMO}[0,t] = \Big\{(\pi_s)_{s\in[0,t]} : \pi \text{ is } \mathbb{F}\text{-progressively measurable,}$$
$$E_{\mathbb{P}}(\int_\tau^t |\pi_s|^2 ds|\mathcal{F}_\tau) \leq C, a.s., \text{ for some constant } C \text{ and all } \mathbb{F}\text{-stopping times } \tau \leq t\Big\}.$$

The set of all admissible investment proportions in the trading interval $[0,t]$ is denoted by $\Pi_{[0,t]}$. Moreover, we define the set of admissible proportions for all time horizons as $\tilde{\Pi} := \cup_{t\geq 0}\Pi_{[0,t]}$.

Definition 2 *Suppose that $U \subseteq \mathbb{R}^d$ is convex and compact. For any $s \geq t \geq 0$, a process $u : [t,s] \times \Omega \to U$ is an admissible scenario parameter if it is $\mathbb{F}$-progressively measurable and essentially bounded. We denote by $\mathcal{U}_{t,s}$ and $\mathcal{U}$ the set of all admissible scenario parameter in the time interval $[t,s]$ and for all time horizons, respectively.*

In a market with model uncertainty, the investor will apply $\mathbb{P}^u$ to measure her preference instead of the probability measure $\mathbb{P}$, where the probability measure $\mathbb{P}^u$ is an equivalent probability measure with respect to $\mathbb{P}$ and introduced by the following measure transformation

$$\frac{d\mathbb{P}^u}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t u_s^T dW_s\right) := \exp\{\int_0^t u_s^T dW_s - \frac{1}{2}\int_0^t |u_s|^2 ds\},\ u \in \mathcal{U}. \quad (6)$$

In fact, this characterization of model uncertainty, admitting an entire class $\{\mathbb{P}^u | u \in \mathcal{U}\}$ of possible prior models, is a common approach applied in the classical robust expected utility, see [26].

For every $u \in \mathcal{U}$, the process W^u defined as

$$dW_t^u = -u_t dt + dW_t, \quad (7)$$

is a Brownian motion under the probability measure $\mathbb{P}^u$. Moreover, if $\pi \in \Pi_{[0,t]}$, then under $\mathbb{P}^u$, we also have

$$ess\sup_\tau E_{\mathbb{P}^u}\left(\int_\tau^t |\pi_s|^2 ds\Big|\mathcal{F}_\tau\right) < \infty.$$

The investor will evaluate her investment via a forward performance process, the concept of which was first introduced and developed in [41]-[45]. Since the investor is uncertain about the probability measure she uses, she will seek for an optimal investment proportion that is least affected by model uncertainty. This leads to the so called *robust forward performance processes* as first introduced

in [34] and later extended to the case with uncertain parameters in [13]. In [34] a class of convex penalty functions was introduced to represent the weighting/likehood of $\mathbb{P}^u$. To generalize the idea of penalty functions, we parameterize robust forward performance processes by the investor's prediction of the probability measure $\mathbb{P}^u$ (or the scenario parameter u), so penalty functions become a special class of parametrization. For this, we first give a precise meaning of parametrization which we will call *a realization* of the model $\mathbb{P}^u$ (or the scenario parameter u) hereafter.

Definition 3 *For $0 \leq t \leq s < \infty$, a mapping $\gamma : \Omega \times [t,s] \times \mathcal{U}_{t,s} \to L^0(\mathcal{F}_s; \mathbb{R}^+)$ is a realization process, if for each r with $t \leq r \leq s$ and $u \in \mathcal{U}_{t,s}$, it holds*

$$\gamma_{t,s}(u) = \gamma_{t,s}(u_1 \oplus u_2) = \gamma_{t,r}(u_1) + \gamma_{r,s}(u_2), \ \ a.s., \tag{8}$$

where u_1 and u_2 is the restriction of u to trading interval $[t,r]$ and $[r,s]$, respectively, and we denote $u = u_1 \oplus u_2$.

We will use a realization process $\gamma_{t,s}(u)$ to parameterize the original utility on trading horizon $[t,s]$ because of the chosen model $\mathbb{P}^u$. The condition (8) is essentially a time-additivity property, which states that along the same model $\mathbb{P}^u$, the realization process on interval $[t,s]$ is accumulated by the realization processes estimated on $[t,r]$ and $[r,s]$.

The following generalises the definition of robust forward performance processes.

Definition 4 *A process $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, is a robust forward performance process associated with a realization process γ and a parameter $\tau \in \mathbb{R}$ if*

i) *for each $x \in \mathbb{R}_+$, $U(x,t)$ is $\mathbb{F}$-progressively measurable;*

ii) *for each $t \geq 0$, the mapping $x \mapsto U(x,t)$ is strictly increasing and strictly concave;*

iii) *the process $U(x,t)$ satisfies the self-generating property (dynamic programming principle), i.e., for all $s \geq t \geq 0$ and $\bar{u} \in \mathcal{U}_{0,t}$,*

$$\operatorname*{ess\,sup}_{\pi \in \tilde{\Pi}} \operatorname*{ess\,inf}_{u \in \mathcal{U}_{t,s}} E_{\mathbb{P}^u}[\widetilde{U}(X_s^\pi, s, \bar{u} \oplus u) | \mathcal{F}_t, X_t^\pi = x] = \widetilde{U}(x,t,\bar{u}), \quad a.s. \tag{9}$$

where

$$\widetilde{U}(x,t,u) = U(x,t) + \tau \gamma_{0,t}(u). \tag{10}$$

Remark 5 *It is easy to check that the process $\widetilde{U}$ defined in* (10) *also satisfies properties* i) $-$ ii) *in Definition 4 and thus $\widetilde{U}$ parameterized by the scenario parameter u can also be regarded as a robust forward performance process, where $\gamma_{0,t}(u)$ represents a realization of the model $\mathbb{P}^u$ up to time t. In this situation,*

the robust forward performance process $\widetilde{U}(x,t,u)$ at any given time t constitutes of the original utility $U(x,t)$ and the realization process $\gamma_{0,t}(u)$ reflecting the historical cumulative impact of the model $\mathbb{P}^u$ from 0 to t. Considering the utility may be increasing or decreasing along with the model $\mathbb{P}^u$ which depends on the investor's attitude to this model, we introduce a parameter τ in (10) *with its sign indicating the varying trend of the utility and its absolute value $|\tau|$ modeling the sensitively of the utility with respect to the model $\mathbb{P}^u$ (or the scenario u.)*

Remark 6 *It turns out that our definition of robust forward performance processes is a generalization of robust forward performance processes studied in [34], although they are proposed based on different ideas. In fact, it is easy to check from* (8) *that property* (9) *is equivalent to the following form*

$$\operatorname*{ess\,sup}_{\pi\in\tilde{\Pi}} \operatorname*{ess\,inf}_{u\in\mathcal{U}_{t,s}} E_{\mathbb{P}^u}[U(X_s^\pi,s)+\tau\gamma_{t,s}(u)|\mathcal{F}_t, X_t^\pi=x] = U(x,t), \quad a.s. \tag{11}$$

If $\tau\geq 0$ and the realization process $\gamma_{t,s}(u)$ is convex in u, then $\tau\gamma_{t,s}(u)$ becomes a penalty function. In this situation, we can verify from (11) *that our Definition 4 is consistent with [34]. Due to the convexity of $\gamma_{t,s}(u)$ on u, a duality method is developed in [34] to construct $U(x,t)$ and its associated optimal investment proportion π^*. Moreover, a saddle point method is employed to further find the worst case scenario u^* in [13] (with $\tau=0$).*

On the other hand, if $\tau<0$, the economic interpretation of $\tau\gamma_{0,t}(u)$ in (10) *is as follows. From* (10) *the investor's utility at time t includes two terms $U(x,t)$ and $\tau\gamma_{0,t}(u)$ when using the model $\mathbb{P}^u$. Since τ is negative, the utility $U(x,t)+\tau\gamma_{0,t}(u)$ is smaller than the original utility $U(x,t)$ and the term $|\tau|\gamma_{0,t}(u)$ can be regarded as the utility's loss from time 0 to t under model $\mathbb{P}^u$. As a result, the utility at time t becomes $U(x,t)+\tau\gamma_{0,t}(u)$. Following this viewpoint, the term $\tau\gamma_{0,t}(u)$ represents the degree of utility loss based on the investor's prediction over the probability measure $\mathbb{P}^u$ stemming from model uncertainty.*

We remark that we do not assume $\gamma_{t,s}(u)$ is convex with respect to u, which implies that the duality method may not be suitable to our framework even for $\tau>0$. In addition, the classical saddle point argument used in [13] is not valid either because the saddle point in general does not exist (for example, when the term $\tau\gamma_{t,s}(u)$ is concave in u).

2.1 A stochastic differential game approach

In contrast to [34] and [13], which do not cover the case of the stochastic factor model, we aim to construct a class of robust forward performance processes with explicit dependency on the stochastic factor process V. Our approach is based on *stochastic differential games*, which is an alternative and more direct approach for the case $\tau=0$, and may also work for the case $\tau\neq 0$ when the realization process γ has some specific forms (see Section 7 for the case $\tau<0$).

The basic idea of the stochastic differential game approach is as follows. To robustify the optimal investment, the inner part of the above optimization problem (11) is played by the market minimizes the expected forward utility by choosing the worst-case scenario, whereas the investor aims to select the best investment proportion that is least affected by the market's choice. This leads to a stochastic differential game between the investor and market.

In addition to the representation of the robust forward performance process, we also aim to provide both the optimal investment proportion for each scenario and the worst-case scenario for each investment proportion. The investment proportion (resp. worst-case scenario) responding to each scenario (resp. investment proportion) can be exactly expressed as the "strategy to control" in the setup of stochastic differential games (see [11, 19]). Thus, we next give the definitions of two admissible "strategies" associated with their respective "controls".

Definition 7 *An admissible investment strategy responding to each scenario parameter for an investor is a mapping* $\alpha : [0,\infty) \times \Omega \times \mathcal{U} \to \tilde{\Pi}$ *satisfying the following two properties:*
i) *For each* $u \in \mathcal{U}$, α *is* $\mathbb{F}$*-progressively measurable;*
ii) *Non-anticipative property, that is, for all* $t > 0$ *and all* $u_1, u_2 \in \mathcal{U}$, *with* $u_1 = u_2$, $dsd\mathbb{P}$*-a.e., on* $[0,t]$, *it holds that* $\alpha(\cdot, u_1) = \alpha(\cdot, u_2)$, $dsd\mathbb{P}$*-a.e., on* $[0,t]$.

An admissible scenario parameter strategy responding to each investment proportion for the market, $\beta : [0,\infty) \times \Omega \times \tilde{\Pi} \to \mathcal{U}$, *is defined similarly. The set of all admissible investment strategies for the investor is denoted by* $\mathcal{A}$, *while the set of all admissible scenario parameter strategies is denoted by* $\mathcal{B}$.

Herein, the concept *non-anticipative property* is widely used in the definition of *admissible strategies* in differential games to characterize that each player's decision, depending on the other's action, will not change if the other one chooses the same control (see [11]). We introduce this concept here to enforce that an investor will take the same investment action if the scenario does not change.

We consider a zero-sum stochastic differential game, where the state dynamic is given by the wealth equation (5). Furthermore, let $U(x,t)$ be a stochastic process satisfying i) and ii) in Definition 4 and γ be a realization process with parameter $\tau \in \mathbb{R}$. For any $s \geq t$, the objective functional is given by

$$J(x,t;s,\pi,u) = E_{\mathbb{P}^u}[U(X_s^\pi, s) + \tau\gamma_{t,s}(u)|\mathcal{F}_t, X_t^\pi = x].$$

The lower and upper values of the game are then defined as

$$\underline{U}(x,t;s) = \operatorname*{ess\,inf}_{\beta\in\mathcal{B}} \operatorname*{ess\,sup}_{\pi\in\tilde{\Pi}} J(x,t;s,\pi,\beta(\cdot,\pi)), \quad a.s., \tag{12}$$

and

$$\overline{U}(x,t;s) = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \operatorname*{ess\,inf}_{u\in\mathcal{U}} J(x,t;s,\alpha(\cdot,u),u), \quad a.s., \tag{13}$$

respectively.

Note that if $\underline{U}(x,t;s) = U(x,t)$, for all $s \geq t$, which implies that the objective functional of the stochastic differential game "self generates" the lower value of the game, then it is clear that $U(x,t)$ becomes a robust forward performance process satisfying i)-iii) in Definition 4. Thus, we say the game is *self-generating* if

$$\underline{U}(x,t;s) = U(x,t), \text{ for all } s \geq t, \tag{14}$$

which will in turn provide a robust forward performance process. To this end, we will construct a control $\pi^* \in \tilde{\Pi}$, a strategy $\beta^* \in \mathcal{B}$, and a process $U(x,t)$ satisfying the martingale properties: For any $\pi \in \tilde{\Pi}$, $s \geq t$, a.s.,

$$\operatorname*{ess\,inf}_{\beta \in \mathcal{B}} J(x,t;s,\pi,\beta(\cdot,\pi)) = J(x,t;s,\pi,\beta^*(\cdot,\pi)) \leq U(x,t); \tag{15}$$

$$J(x,t;s,\pi^*,\beta^*(\cdot,\pi^*)) = U(x,t). \tag{16}$$

2.2 Further discussion on the stochastic differential game approach when the game value exists

In this subsection, according to the sign of the parameter τ in (11), we further explain the application of our stochastic differential game method in robust forward investment problems.

If $\tau \geq 0$ and $\gamma_{t,s}(u)$ is convex in u, a saddle point in general exists at least for a special class of penalty functions $\gamma_{t,s}(u)$ (as shown in [34] and [13]). In contrast to the saddle point method, the advantage of the stochastic differential game approach is to provide, in explicit form, the optimal investment choice for the investor not only under the worst-case scenario but also for each scenario, as well as the worst case scenario for each investment choice not only the optimal one. Moreover, it is often relatively easy to compute the optimal strategy pair (α^*, β^*), as they only involve maximization/minimization problems rather than maxmin/minmax problems.

From Sections 3 to 6, we consider a robust forward problem without realization process, i.e., $\tau = 0$. In this situation, the game value exists, and we will construct the associated forward performance process by the value of the stochastic differential game. Recall that the value of the game exists if

$$\underline{U}(x,t;s) = \overline{U}(x,t;s), \text{ for all } s \geq t, \tag{17}$$

which further implies that both equal to $U(x,t)$ if the self-generating condition (14) also holds. In this situation, with the help of the upper value function $\overline{U}(x,t;s)$, we will construct a control pair $(\pi^*, u^*) \in \tilde{\Pi} \times \mathcal{U}$, a strategy pair $(\alpha^*, \beta^*) \in \mathcal{A} \times \mathcal{B}$, and a process $U(x,t)$ satisfying the martingale properties: (15), (16), and for any $u \in \mathcal{U}$, $s \geq t$, a.s.,

$$\operatorname*{ess\,sup}_{\alpha \in \mathcal{A}} J(x,t;s,\alpha(\cdot,u),u) = J(x,t;s,\alpha^*(\cdot,u),u) \geq U(x,t); \tag{18}$$

$$J(x,t;s,\alpha^*(\cdot,u^*),u^*) = U(x,t)\,. \tag{19}$$

Note that (15) and (16) are the martingale characterization of the lower value of the game in (12), whereas (18) and (19) characterize the upper value of the game in (13).

Moreover, if it also holds that $\pi^* = \alpha^*(\cdot,u^*)$ and $u^* = \beta^*(\cdot,\pi^*)$, then the martingale conditions (15)-(16) and (18)-(19) further imply that, for all $s \geq t$,

$$\begin{aligned} J(x,t;s,\pi^*,u) &\geq J(x,t;s,\pi^*,\beta^*(\cdot,\pi^*)) \\ &= J(x,t;s,\alpha^*(\cdot,u^*),u^*) \geq J(x,t;s,\pi,u^*), \end{aligned}$$

so the control pair (π^*,u^*) is a saddle point for the stochastic differential game with the value $J(x,t;s,\pi^*,u^*) = U(x,t)$.

Remark 8 *When $\tau \neq 0$, it is unclear how to construct a robust forward performance process with a general realization process γ (see, for example, the discussion of the time consistency issue of penalty functions in Section 4 of [34]). Moreover, a saddle point may even fail to exist if $\tau < 0$. Nevertheless, we will show in section 7 that our stochastic differential game approach may still work for $\tau < 0$, at least for a special class of quadratic form realization processes. A more general case for $\tau \neq 0$ is still left open.*

3 Power robust forward performance processes with zero realization processes

In this section, we focus on a class of homothetic robust forward performance processes that are homogenous in the degree of $\delta \in (0,1)$, and has the factor-form

$$U(x,t) = \frac{x^\delta}{\delta} e^{f(V_t,t)}, \tag{20}$$

where $f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is a deterministic function to be specified, and the parameter τ of the realization process is equal to zero. We call such a robust forward performance process a *power robust forward performance process*.

Proposition 9 *Assume that $f(v,t)$, $(v,t) \in \mathbb{R}^d \times [0,\infty)$, is a classical solution (with enough regularity) of the semilinear PDE*

$$f_t + \frac{1}{2} Trace\left(\kappa\kappa^T \nabla^2 f\right) + \eta(v)^T \nabla f + G(v,\kappa^T \nabla f) = 0, \tag{21}$$

where

$$G(v,z) = \inf_{u\in U} \sup_{\pi\in\Pi} F(v,z,\pi,u), \tag{22}$$

with

$$F(v,z,\pi,u)=-\frac{1}{2}\delta(1-\delta)|\pi|^2+\delta\pi^T(\theta(v)+z+u)+z^Tu+\frac{1}{2}|z|^2. \tag{23}$$

Then, $U(x,t)=\frac{x^\delta}{\delta}e^{f(V_t,t)}$ *is a power robust forward performance process.*

Proof. Since $U(x,t)$ obviously satisfies i) and ii) in Definition 4, it is sufficient to examine iii) in Definition 4.

Step 1. From (22) and (23), we have

$$G(v,z)=\inf_{u\in U}\sup_{\pi\in\Pi}F(v,z,\pi,u)=\inf_{u\in U}F(v,z,\alpha^*(v,z,u),u), \tag{24}$$

with

$$\alpha^*(v,z,u)=argmax_{\pi\in\Pi}F(v,z,\pi,u)=Proj_\Pi(\frac{\theta(v)+z+u}{1-\delta}). \tag{25}$$

Using the Lipschitz continuity of the projection operator on the convex set Π, there exists a Borel measurable mapping $u^*:\mathbb{R}^d\times\mathbb{R}^d\to U$ such that

$$u^*(v,z)=argmin_{u\in U}F(v,z,\alpha^*(v,z,u),u). \tag{26}$$

Then, from (24) and (26), we have

$$G(v,z)=F(v,z,\pi^*(v,z),u^*(v,z)), \tag{27}$$

with

$$\pi^*(v,z):=\alpha^*(v,z,u^*(v,z)). \tag{28}$$

We claim that, for any $u\in U$,

$$F(v,z,\pi^*(v,z),u)\geq F(v,z,\pi^*(v,z),u^*(v,z)). \tag{29}$$

If (29) holds, then

$$\begin{aligned}\sup_{\pi\in\Pi}\inf_{u\in U}F(v,z,\pi,u)&\geq\inf_{u\in U}F(v,z,\pi^*(v,z),u) &(30)\\ &\geq F(v,z,\pi^*(v,z),u^*(v,z)) &(31)\\ &=G(v,z)=\inf_{u\in U}\sup_{\pi\in\Pi}F(v,z,\pi,u),\end{aligned}$$

so both (30) and (31) become equalities. In turn, $\pi^*(v,z)$ in (28) and $u^*(v,z)$ in (26) satisfy, respectively,

$$\pi^*(v,z)=argmax_{\pi\in\Pi}\inf_{u\in U}F(v,z,\pi,u),$$

and

$$u^*(v,z)=argmin_{u\in U}F(v,z,\pi^*(v,z),u). \tag{32}$$

On the other hand, there exists a U-valued Borel measurable mapping $\bar{\beta}^*(v,z,\pi)$ such that $F(v,z,\pi,u)$ attains the minimum, i.e.

$$\inf_{u\in U} F(v,z,\pi,u) = F(v,z,\pi,\bar{\beta}^*(v,z,\pi)).$$

Then, from (32), the mapping $\beta^*(v,z,\pi)$ defines as

$$\beta^*(v,z,\pi) = \begin{cases} u^*(v,z), & \text{if } \pi = \pi^*(v,z); \\ \bar{\beta}^*(v,z,\pi), & \text{otherwise,} \end{cases} \tag{33}$$

also minimizes $F(v,z,\pi,u)$ over $u \in U$, and moreover,

$$\pi^*(v,z) = argmax_{\pi\in\Pi} F(v,z,\pi,\beta^*(v,z,\pi)). \tag{34}$$

Step 2. We are left to prove the inequality (29). We omit the variables (v,z) in $\pi^*(v,z)$ and $u^*(v,z)$, and write them as π^* and u^* in this step. For any $u \in U$ and $\lambda \in (0,1)$ let

$$u_1 := \lambda u + (1-\lambda)u^*.$$

Set $\pi_1 := \alpha^*(v,z,u_1)$ and recall from (28) that $\pi^* = \alpha^*(v,z,u^*)$. Then, it follows from (26) that

$$\begin{aligned} F(v,z,\pi^*,u^*) &\le F(v,z,\pi_1,u_1) \\ &= \lambda F(v,z,\pi_1,u) + (1-\lambda)F(v,z,\pi_1,u^*) \\ &\le \lambda F(v,z,\pi_1,u) + (1-\lambda)F(v,z,\pi^*,u^*). \end{aligned}$$

where we used $F(v,z,\pi,u) \le F(v,z,\alpha^*(v,z,u),u)$ in the last inequality. Thus,

$$F(v,z,\pi^*,u^*) \le F(v,z,\pi_1,u) = F(v,z,\alpha^*(v,z,u_1),u)$$

for any $u \in U$. Sending $\lambda \to 0$ and using the continuity of $\alpha^*(v,z,u)$ in u, we have $\alpha^*(v,z,u_1) \to \alpha^*(v,z,u^*) = \pi^*$. Then, the inequality (29) follows by the continuity of $F(v,z,\pi,u)$ in π.

Step 3. Using the homothetic form (20) and applying Itô's formula to $U(X_s^\pi, s)$, we get

$$\begin{aligned} &dU(X_s^\pi,s) \\ &= U(X_s^\pi,s)\big[f_s + \frac{1}{2}Trace\left(\kappa\kappa^T\nabla^2 f\right) + \eta(V_s)^T\nabla f + F(V_s,\kappa^T\nabla f,\pi_s,u_s)\big]ds \\ &\quad + U(X_s^\pi,s)(\delta\pi_s^T + \nabla f^T\kappa)dW_s^u. \end{aligned}$$

For any $s \ge t \ge 0$, from (21), we further get

$$\begin{aligned} &E_{\mathbb{P}^u}[U(X_s^\pi,s)|\mathcal{F}_t, X_t^\pi = x] - U(x,t) \\ &= J(x,t;s,\pi,u) - U(x,t) \\ &= E_{\mathbb{P}^u}\big[\int_t^s U(X_r^\pi,r)\big(F(V_r,\kappa^T\nabla f,\pi_r,u_r) - G(V_r,\kappa^T\nabla f)\big)dr|\mathcal{F}_t, X_t^\pi = x\big]. \end{aligned} \tag{35}$$

We set

$$\begin{aligned}&\pi_t^* = \pi^*(V_t, \kappa\nabla f(V_t,t)),\ u_t^* = u^*(V_t, \kappa\nabla f(V_t,t)),\\ &\alpha^*(t,u_t) = \alpha^*(V_t, \kappa\nabla f(V_t,t), u_t),\ \beta^*(t,\pi_t) = \beta^*(V_t, \kappa\nabla f(V_t,t), \pi_t),\end{aligned} \tag{36}$$

with the mappings $(\pi^*, u^*, \alpha^*, \beta^*)$ given in (28), (26), (25) and (33), respectively. Then, it is easy to check that $U(x,t)$ satisfies the martingale conditions (15)-(16) and (18)-(19), which implies that $U(x,t) = \frac{x^\delta}{\delta} e^{f(V_t,t)}$ is a power robust forward performance process, with the optimal control pair (π^*, u^*) and the optimal strategy pair (α^*, β^*). ■

Remark 10 *It is worth to point out that the strategies α^* and β^* we constructed in the above proof are also called "counterstrategies"; the reader can refer to Chapter 10, Section 1 in [36] for more details.*

Since, by our construction, $\pi_t^* = \alpha^*(t, u_t^*)$ and $u_t^* = \beta^*(t, \pi_t^*)$, it follows that (π^*, u^*) is actually a saddle point for the associated game. However, compared to the classical saddle point argument such as Sion's Minimax Theorem (see, for example, [13, 53]), our formulae are more explicit and is constructed via their corresponding counterstrategies.

Note that the semi-linear PDE (21) is a new class of Hamilton-Jacobi-Bellman-Isaacs equations, which is ill-posed for the equation is posed forward in time. Due to this "wrong" time direction, one does not expect solutions to exist for all initial conditions or to depend continuously on them, making the problem ill-posed. A similar difficulty also appears in [4], [46], [47] and [50] for the construction of forward processes without model ambiguity, where the Widder's theorem is employed. Nevertheless, the form of PDE (21) motivates us how to construct the optimal investment proportion, worst-case scenario parameter and the related optimal strategies for different situations, which will be used in the following Theorem 12. In order to give the specific form of the process $f(V_t, t)$, we bypass PDE (21) by directly using the Markovian solution of an ergodic BSDE whose driver has the form (22). This approach was first introduced in [38] to study the forward performance process in the absence of model uncertainty. We first give the existence and uniqueness of the Markovian solution of the associated ergodic BSDE. For this, we further strengthen Assumption 1 by requiring the constant C_η given in (3) satisfies

$$C_\eta \geq \frac{3\delta C_\theta}{1-\delta}\left[(K_\theta + K_u) \vee 1\right], \tag{37}$$

where $\delta \in (0,1)$ is the risk aversion degree, C_θ and K_θ are the Lipschitz constant and bound of the market price of risk vector $\theta(v)$ respectively, and K_u is the bound of the scenario parameter u with $K_u = \max_{u\in U} |u|$.

Lemma 11 *Assume the function G has the form* (22)*. Then, the ergodic BSDE*

$$dY_t = (-G(V_t, Z_t) + \lambda)dt + Z_t^T dW_t, \tag{38}$$

admits a unique Markovian solution (Y_t, Z_t, λ), $t \geq 0$, *i.e., there exist a unique constant* λ *and functions* $y : \mathbb{R}^d \to \mathbb{R}$, $z : \mathbb{R}^d \to \mathbb{R}^d$ *such that* $Y_t = y(V_t)$, $Z_t = z(V_t)$. *Here, we say a Markovian solution is unique in the following sense: the function* $y(\cdot)$ *is unique up to a constant and has at most linear growth, and* $z(\cdot)$ *is bounded.*

Proof. Using the Lipschitz continuity of the projection operator, it follows from (23) and (25) that

$$\begin{aligned}
&|F(v,z,\alpha^*(v,z,u),u) - F(\bar{v},z,\alpha^*(\bar{v},z,u),u)| \leq C(1+|z|)\cdot|v-\bar{v}|,\\
&|F(v,z,\alpha^*(v,z,u),u) - F(v,\bar{z},\alpha^*(v,\bar{z},u),u)| \leq C(1+|z|+|\bar{z}|)\cdot|z-\bar{z}|,\\
&|F(v,0,\alpha^*(v,0,u),u)| \leq C.
\end{aligned} \tag{39}$$

Indeed, to show the first inequality, we note from (23) that

$$\begin{aligned}
&|F(v,z,\alpha^*(v,z,u),u) - F(\bar{v},z,\alpha^*(\bar{v},z,u),u)|\\
\leq\ &\frac{\delta(1-\delta)}{2}\,|\alpha^*(v,z,u)+\alpha^*(\bar{v},z,u)|\times|\alpha^*(v,z,u)-\alpha^*(\bar{v},z,u)|\\
&+\delta|\theta(v)+z+u|\times|\alpha^*(v,z,u)-\alpha^*(\bar{v},z,u)|+\delta|\alpha^*(\bar{v},z,u)|\times|\theta(v)-\theta(\bar{v})|.
\end{aligned}$$

Since the projection operator $Proj_\Pi(\cdot)$ is Lipschitz continuous with its Lipschitz constant 1 and $0 \in \Pi$, from (25) we have

$$|\alpha^*(v,z,u)-\alpha^*(\bar{v},z,u)| \leq \frac{1}{1-\delta}|\theta(v)-\theta(\bar{v})| \leq \frac{C_\theta}{1-\delta}|v-\bar{v}|,$$

and

$$|\alpha^*(\bar{v},z,u)| \leq \frac{1}{1-\delta}|\theta(\bar{v})+z+u| \leq \frac{K_\theta+|z|+K_u}{1-\delta}.$$

In turn,

$$\begin{aligned}
&|F(v,z,\alpha^*(v,z,u),u) - F(\bar{v},z,\alpha^*(\bar{v},z,u),u)|\\
\leq\ &3\frac{\delta}{1-\delta}(K_\theta+|z|+K_u)C_\theta|v-\bar{v}|\\
\leq\ &\frac{3\delta C_\theta}{1-\delta}\left[(K_\theta+K_u)\vee 1\right](1+|z|)\cdot|v-\bar{v}| \leq C_\eta(1+|z|)\cdot|v-\bar{v}|,
\end{aligned}$$

with C_η given in (37). The other two inequalities in (39) can be proved in a similar way. Furthermore, we note that the constant C in (39) is independent of $u \in U$. Hence, from (24), we further obtain

$$\begin{aligned}
&|G(v,z)-G(\bar{v},z)| \leq C(1+|z|)\cdot|v-\bar{v}|,\\
&|G(v,z)-G(v,\bar{z})| \leq C(1+|z|+|\bar{z}|)\cdot|z-\bar{z}|,\ |G(v,0)| \leq C.
\end{aligned} \tag{40}$$

Therefore, from Proposition 3.1 and Appendix A in [38] we obtain the desired result. ■

Roughly speaking, the additional "large enough" requirement of the constant C_η in (37) is to guarantee the forward stochastic factor process V converges fast enough to dominate the dissipative nature of the backward equation for Y reflected by the Lipschitz constant C in the first inequality of (40). This additional requirement plays an important role in the study of ergodic BSDE (38). We refer the reader to Appendix A in [38].

We next present the specific form of the process $f(V_t, t)$ by using the solution of the ergodic BSDE (38).

Theorem 12 *Let $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda), t \geq 0$, be the unique Markovian solution of (38). Then, the process $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by*

$$U(x,t) = \frac{x^\delta}{\delta} e^{y(V_t) - \lambda t} \ , \tag{41}$$

is a power robust forward performance process. Moreover, the optimal portfolio weight π^*, the worst-case scenario parameter u^* and the optimal strategies α^*, β^* responding to each scenario parameter u and portfolio weight π are given as follows

$$\begin{aligned} &\pi_t^* = \pi^*(V_t, z(V_t)), \ u_t^* = u^*(V_t, z(V_t)), \\ &\alpha^*(t, u_t) = \alpha^*(V_t, z(V_t), u_t), \ \beta^*(t, \pi_t) = \beta^*(V_t, z(V_t), \pi_t), \end{aligned} \tag{42}$$

where the mappings $(\pi^*, u^*, \alpha^*, \beta^*)$ are given in (28), (26), (25) and (33), respectively.

In addition, the associated wealth process X^* under the worst-case scenario is given by

$$X_t^* = X_0 \mathcal{E}\Big(\int_0^t (\pi_s^*)^T \cdot [(\theta(V_s) + u_s^*)ds + dW_s^{u^*}] \Big).$$

It is worth to point out that the constant λ, as a part of the solution of ergodic BSDE (38), can be regarded as the optimal long-term growth rate of the corresponding expected utility of wealth with model uncertainty (see Remark 23).

We now give the proof of Theorem 12.

Proof. It is easy to check that the process given by (41) is $\mathbb{F}$-progressively measurable, strictly increasing and strictly concave in x. We only need to show that the martingale conditions (15)-(16) and (18)-(19) hold. For this, from (5), (7) and (38) we get, for all $s \geq t \geq 0$, $(\pi, u) \in \tilde{\Pi} \times \mathcal{U}$,

$$\begin{aligned} X_s^\pi &= X_t^\pi \cdot \exp\Big\{ \int_t^s \pi_r^T(\theta(V_r) + u_r) - \frac{1}{2}|\pi_r|^2 dr + \int_t^s \pi_r^T dW_r^u \Big\}, \\ (Y_s - \lambda s) &= (Y_t - \lambda t) - \int_t^s G(V_r, Z_r) - Z_r^T u_r dr + \int_t^s Z_r^T dW_r^u. \end{aligned}$$

Thus, we have

$$\begin{aligned}
&U(X_s^\pi, s) = \frac{(X_s^\pi)^\delta}{\delta} e^{Y_s - \lambda s} \\
&= U(X_t^\pi, t) \cdot \mathcal{E}\Big(\int_t^s (\delta \pi_r^T + Z_r^T) dW_r^u \Big) \cdot \exp\Big\{ \int_t^s F(V_r, Z_r, \pi_r, u_r) - G(V_r, Z_r) dr \Big\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E_{\mathbb{P}^u}\left[U(X_s^\pi, s) | \mathcal{F}_t, X_t = x\right] - U(x,t) \\
&= J(x,t;s,\pi,u) - U(x,t) \\
&= U(x,t) \cdot E_{\mathbb{P}^u}\left(\frac{M_s}{M_t} \cdot \exp\Big\{ \int_t^s \Big(F(V_r, Z_r, \pi_r, u_r) - G(V_r, Z_r) \Big) dr \Big\} \Big| \mathcal{F}_t \right) - U(x,t),
\end{aligned}$$

where, for $t \in [0,s]$, $M_t := \mathcal{E}\Big(\int_0^t (\delta \pi_r^T + Z_r^T) dW_r^u \Big)$, is a uniformly integrable exponential martingale (since π satisfies the BMO-condition and $z(\cdot)$ is bounded). Similar to the argument in the proof of Lemma 9, we get the the martingale conditions (15)-(16) and (18)-(19) from the above equality. ■

Remark 13 *The probability measure $\mathbb{P}^{u^*}$ associated with u^* given in Theorem 12 has the following form*

$$\frac{d\mathbb{P}^{u*}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t (u_s^*)^T dW_s \right).$$

Thus, as a byproduct, we obtain a specific formula for the least favorable martingale measure as considered in [20].

Remark 14 *Similar to Proposition 3.4 in [38], it is easy to check that*

$$f(v,t) = y(v) - \lambda t$$

is a classical solution of the semilinear PDE (21) *with the initial condition $f(v,0) = y(v)$, where $(y(V_t), z(V_t), \lambda)$ is the solution of ergodic BSDE* (38).

Next, we build a connection between power robust forward performance processes and the solutions of a family of infinite horizon BSDE. For $\rho > 0$, we consider the following infinite horizon BSDE

$$dY_t^\rho = \left(-G(V_t, Z_t^\rho) + \rho Y_t^\rho \right) dt + \left(Z_t^\rho \right)^T dW_t, \tag{43}$$

where the driver $G(\cdot,\cdot)$ is given in (38). Then, this BSDE admits a unique Markovian solution $(Y_t^\rho, Z_t^\rho) = (y^\rho(V_t), z^\rho(V_t))$. Moreover, there exists a subsequence, denoted by ρ_n, such that

$$y(v) = \lim_{\rho_n \downarrow 0} y^{\rho_n}(v), \ z(v) = \lim_{\rho_n \downarrow 0} z^{\rho_n}(v), \ \lambda = \lim_{\rho_n \downarrow 0} \rho_n y^{\rho_n}(v_0),$$

where $(y(V_t), z(V_t), \lambda)$ is the solution of ergodic BSDE (38) and $v_0 \in \mathbb{R}^d$ is an arbitrary given reference point. These results were first obtained in [22] with Lipschitz driver and then extended to the quadratic driver in [38].

Similar to the proof of Theorem 12, we can examine that the process $U^\rho(x,t)$ given by (44) is still a power robust forward performance process and it converges in an appropriate discounted manner to the process $U(x,t)$ as ρ tends to 0.

Corollary 15 *The process* $U^\rho(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, *given by*

$$U^\rho(x,t) = \frac{x^\delta}{\delta} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds} \tag{44}$$

is a power robust forward performance process and the optimal portfolio strategy $\alpha_t^{*,\rho}$ *for each scenario parameter* u *is given by*

$$\alpha_t^{*,\rho}(u) = Proj_\Pi \left(\frac{\theta(V_t) + z^\rho(V_t) + u_t}{1-\delta} \right).$$

Furthermore, there exists a subsequence $\rho_n \downarrow 0$ *such that, for* $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} \frac{U^{\rho_n}(x,t) e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1. \tag{45}$$

and the associated optimal portfolio strategies α^{*,ρ_n} *and* α^* *satisfy*

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t \left| \alpha^{*,\rho_n}(s,u_s) - \alpha^*(s,u_s) \right|^2 ds = 0, \text{ for } t \geq 0, \ u \in \mathcal{U}. \tag{46}$$

Remark 16 *We have obtained the representation of the power robust forward performance process in factor-form by combining the zero-sum stochastic differential game and ergodic BSDE approach. In fact, this approach can be applied to study other type of homothetic robust forward performance processes, such as logarithmic and exponential cases. More specifically, the processes* $U^1(x,t)$ *and* $U^2(x,t)$ *given by*

$$U^1(x,t) = \ln x + y^1(V_t) - \lambda^1 t, \ (x,t) \in \mathbb{R}_+ \times [0,\infty),$$

$$U^2(x,t) = -e^{-\gamma x + y^2(V_t) - \lambda^2 t}, \ (x,t) \in \mathbb{R} \times [0,\infty)$$

are logarithmic and exponential (with risk aversion parameter $\gamma > 0$*) robust forward performance processes, respectively, where* $(Y_t^i, Z_t^i, \lambda^i) = (y^i(V_t), z^i(V_t), \lambda^i)$, $i = 1,2, \ t \geq 0$, *are the unique Markovian solution of the ergodic BSDE* (38) *with the generator* $G = G_i, \ i = 1,2$, *respectively, with*

$$G_1(v,z) = \inf_{u \in U} \sup_{\pi \in \Pi} \{ -\frac{1}{2}|\pi|^2 + \pi^T \theta(v) + (\pi^T + z^T)u \},$$

$$G_2(v,z) = \sup_{u \in U} \inf_{\pi \in \Pi} \{ \frac{1}{2}|\gamma\pi - z|^2 - r\pi^T(\theta(v) + u) + z^T u \}.$$

4 Examples

We apply Theorem 12 to analyze two specific examples. The first example is driven by the Brownian noise which can be fully hedged. The second example is a single stock model correlated with a single stochastic factor where only partial hedging is possible. In both examples, no constraints on portfolios are imposed and optimal robust investment policies for the power robust forward performance processes are given in the feedback form of stochastic factors.

4.1 Market model I

We consider the case that the set Π is large enough in the sense that the mappings α^* in (25) has the following form

$$\alpha^*(v,z,u) = Proj_{\Pi}(\frac{\theta(v)+z+u}{1-\delta}) = \frac{\theta(v)+z+u}{1-\delta}.$$

Then, the mappings u^* in (26) and π^* in (28) as well as β^* in (33) take the form

$$\begin{aligned}
&u^*(v,z) = argmin_{u\in U} F(v,z,\alpha^*(v,z,u),u) = Proj_U\big(-\theta(v)-\frac{1}{\delta}z\big),\\
&\pi^*(v,z) = \alpha^*(v,z,u^*(v,z)) = \frac{\theta(v)+z+Proj_U\big(-\theta(v)-\frac{1}{\delta}z\big)}{1-\delta},\\
&\beta^*(v,z,\pi) = \begin{cases} Proj_U\big(-\theta(v)-\frac{1}{\delta}z\big), & \text{if } \pi=\pi^*(v,z);\\ argmin_{u\in U}(\delta\pi+z)^T u, & \text{otherwise.}\end{cases}
\end{aligned}$$

In this case, the ergodic BSDE (38) becomes

$$dY_t = \Big(-\frac{1}{2}\frac{\delta}{1-\delta}dist^2\big(U,-\theta(V_t)-\frac{1}{\delta}Z_t\big)+\frac{1}{2\delta}|Z_t|^2+Z_t^T\theta(V_t)+\lambda\Big)dt+Z_t^T dW_t. \tag{47}$$

In turn, from Theorem 12, we obtain the following result.

Proposition 17 *Denote by $(y(V_t),z(V_t),\lambda)$ the Markovian solution of (47). Then, the process $U(x,t)$ given by*

$$U(x,t) = \frac{x^\delta}{\delta}e^{y(V_t)-\lambda t}\ ,$$

is a power robust forward performance process. Moreover, the optimal control pair $(\pi^*,u^*)\in\tilde{\Pi}\times\mathcal{U}$ and optimal strategy pair $(\alpha^*,\beta^*)\in\mathcal{A}\times\mathcal{B}$, have the

following feedback form

$$\pi_t^* = \frac{\theta(V_t) + z(V_t) + Proj_U\big(-\theta(V_t) - \frac{1}{\delta}z(V_t)\big)}{1-\delta},$$
$$u_t^* = Proj_U\big(-\theta(V_t) - \frac{1}{\delta}z(V_t)\big),$$
$$\alpha^*(t,u_t) = \frac{\theta(V_t) + z(V_t) + u_t}{1-\delta},$$
$$\beta^*(t,\pi_t) = \begin{cases} Proj_U\big(-\theta(V_t) - \frac{1}{\delta}z(V_t)\big), & \text{if } \pi = \pi^*(v,z); \\ argmin_{u_t \in U}(\delta\pi_t + z(V_t))^T u_t, & \text{otherwise.} \end{cases}$$

Remark 18 *It is worth to point out that the presence of the uncertainty in our forward setting may lead to extreme prediction and conservative policy implications for an ambiguity-averse investor. In fact, if we consider the situation that the set U is large enough such that*

$$u^*(v,z) = Proj_U\big(-\theta(v) - \frac{1}{\delta}z\big) = -\theta(v) - \frac{1}{\delta}z.$$

Then, ergodic BSDE (47) *has the form*

$$dY_t = \Big(\frac{1}{2\delta}|Z_t|^2 + Z_t^T\theta(V_t) + \lambda\Big)dt + Z_t^T dW_t, \tag{48}$$

and the robust optimal portfolio weight $\pi_t^* = -\frac{1}{\delta}z(V_t)$. Note that $(0,0,0)$ is the unique Markovian solution of ergodic BSDE (48)*, from Proposition 17 we get*

$$\pi_t^* = -\frac{1}{\delta}z(V_t) = 0,\ u_t^* = -\theta(V_t) - \frac{1}{\delta}z(V_t) = -\theta(V_t),$$
$$\alpha^*(t,u_t) = \frac{\theta(V_t) + u_t}{1-\delta},$$
$$\beta^*(t,\pi_t) = \begin{cases} -\theta(V_t), & \text{if } \pi = \pi^*(v,z); \\ argmin_{u_t \in U}(\delta\pi_t)^T u_t, & \text{otherwise.} \end{cases}$$

This implies that the robust investment policy for an investor is no actions to be taken in the market if the degree of the uncertainty is too large for her. At the same time, the worst-case scenario has a simple form and depends only on the market price of risk $\theta(v)$ and the stochastic factor V_t. In addition, even if an investor is forced to invest (or pursuit high profits) in some situations such as the investor has a wrong judgment on the uncertainty of the market, or is influenced by other extreme events, the optimal investment strategy α^* still gives the corresponding action policy for different scenarios.

4.2 Market model II

We consider a single stock and single stochastic factor model. In this situation, we suppose $n = 1$ and $d = 2$ in the state equations (1) and (2), i.e.,

$$\begin{aligned} dS_t &= b(V_t)S_t dt + \sigma(V_t)S_t dW_t^1, \\ dV_t^1 &= \eta(V_t)dt + \rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2 \quad \text{and} \quad dV_t^2 = 0, \end{aligned} \tag{49}$$

with constant $\rho \in (0,1)$ and $\sigma(\cdot)$ bounded by a positive constant. Note that the stochastic factor cannot be traded directly so that the market model is typically incomplete.

Here, we consider an optimal portfolio problem with no constraints, i.e., $\Pi = \mathbb{R}\times\{0\}$ (which means $\pi_t^2 \equiv 0$). Let $U = \{(u_1,u_2) : -R \le u_1 \le u_2 \le R\}$ (a triangle domain in $\mathbb{R}^2$) with some given constant $R > 0$. Then, the wealth equation (5) reduces to $dX_t^\pi = X_t^\pi \pi_t^1 \left(\theta(V_t)dt + dW_t^1\right)$ with $\theta(V_t) = b(V_t)/\sigma(V_t)$, and the driver of (38) takes the form

$$\begin{aligned} G(v,z_1,z_2) =& \frac{\delta}{2(1-\delta)} dist^2\Big([-R,R], -\theta(v) - \frac{1}{\delta}z_1 - \frac{1-\delta}{\delta}z_2 I_{\{z_2\ge 0\}}\Big) \\ & - \frac{1}{2\delta}|z_1|^2 - \theta(v)z_1 + \Big(\frac{2\delta-1}{2\delta}z_2 - \frac{1}{\delta}z_1 - \theta(v)\Big) z_2 I_{\{z_2\ge 0\}} \\ & + (\frac{1}{2}z_2 + R) z_2 I_{\{z_2<0\}}. \end{aligned} \tag{50}$$

Then, from Theorem 12, we have the following result.

Proposition 19 *Suppose that* $(Y(t), Z_1(t), Z_2(t), \lambda) = (y(V_t), z^1(V_t), z^2(V_t), \lambda)$ *is the Markovian solution of ergodic BSDE* (38) *with the driver* (50)*. Then, the process* $U(x,t)$ *given by*

$$U(x,t) = \frac{x^\delta}{\delta} e^{y(V_t) - \lambda t} \ ,$$

is a power robust forward performance process. Moreover, the optimal portfolio weights and worst-case scenario parameters are given by

$$\begin{aligned} \pi_1^*(t) =& \frac{1}{1-\delta}\Big(\theta(V_t) + Z_1(t) \\ & + Proj_{[-R,R]}\big(-\theta(V_t) - \frac{1}{\delta}Z_1(t) - \frac{1-\delta}{\delta}Z_2(t) I_{\{Z_2(t)\ge 0\}}\big)\Big), \\ \pi_2^*(t) =& 0, \\ u_1^*(t) =& Proj_{[-R,R]}\Big(-\theta(V_t) - \frac{1}{\delta}Z_1(t) - \frac{1-\delta}{\delta}Z_2(t)\cdot I_{\{Z_2(t)\ge 0\}}\Big), \\ u_2^*(t) =& Proj_{[-R,R]}\Big(-\theta(V_t) - \frac{1}{\delta}Z_1(t) - \frac{1-\delta}{\delta}Z_2(t)\Big)\cdot I_{\{Z_2(t)\ge 0\}} + R\cdot I_{\{Z_2(t)<0\}}. \end{aligned}$$

The optimal portfolio weight strategies for each scenario $u \in \mathcal{U}$ and the worst case scenario strategies for each investment weight $\pi \in \tilde{\Pi}$ are given as follows

$$\alpha_1^*(t,u(t)) = \frac{1}{1-\delta}[\theta(V_t) + Z_1(t) + u_1(t)],\ \alpha_2^*(t,u(t)) = 0,$$

$$\beta_1^*(t,\pi(t)) = \begin{cases} u_1^*(t), & \text{if } \pi_1(t) = \pi_1^*(t), \\ -R \cdot sgn(a(t)), & \text{otherwise,} \end{cases}$$

$$\beta_2^*(t,\pi(t)) = \begin{cases} u_2^*(t), & \text{if } \pi_1(t) = \pi_1^*(t), \\ -R \cdot sgn(a(t)) \cdot I_{\{Z_2(t)\geq 0\}} + R \cdot I_{\{Z_2(t)<0\}}, & \text{otherwise,} \end{cases}$$

where $a(t) := \delta\pi_1(t) + Z_1(t) + Z_2(t) \cdot I_{\{Z_2(t)\geq 0\}}$.

Similar to Remark 18, we can conclude that the best choice for an investor will take no action in an incomplete market if the uncertainty is large enough. In fact, from Proposition 19 (especially, the form of π_1^*) we can derive a boundary (say, M) or a domain of the scenario's value in what degree an ambiguity-averse investor should not take action in the market with uncertainty. Herein, the boundary M depends on the boundedness of the functions $z^1(\cdot)$ and $z^2(\cdot)$ as well as $\theta(\cdot)$. Once the uncertainty exceeds the boundary M, the robust investment opportunity will disappear (in this case, $\pi_1^* \equiv 0$ because $Z_1 = Z_2 \equiv 0$). In turn, this reflects that the uncertainty in our model is essentially associated with the risk price θ and the part Z of the solution of ergodic BSDE.

Moreover, when the stock price is not affected by the stochastic factor in the sense that the coefficients b and σ in (49) are constants, the processes Z_1 and Z_2, as part of the solution of the ergodic BSDE (38), will equal to 0. Then, from Proposition 19, it is easy to check that the worst-case scenario parameters u_1^* and u_2^* will choose the values closest to $-\theta(= -\frac{b}{\sigma})$ for any given $R > 0$ and the optimal portfolio weight π_1^* will have the form

$$\pi_1^*(t) = \frac{1}{1-\delta}\Big(\theta + Proj_{[-R,R]}\big(-\theta\big)\Big).$$

Therefore, for the model that the stock price is not affected by the stochastic factor, there will be no investment action into the stock when the value of the risk θ belongs to the range of the uncertainty (i.e., $|\theta| \leq R$).

In addition, we observe that the sign of $z^2(V_t)(= Z_2(t))$ has an important impact on the the worst-case scenario, albeit not shown explicitly in the form of the power robust forward performance process $U(x,t)$. It seems interesting to observe that the sign of $z^2(V_t)(= Z_2(t))$ only affects the worst-case scenario strategies β_1^* and β_2^*, not the optimal investment policy strategies α_1^* and α_2^* responding to each scenario. A similar situation occurs if one consider a general compact and convex subset $U \subset \mathbb{R}^2$ (e.g. $U = \{(u_1,u_2) : -R \leq u_i \leq R,\ i = 1,2\}$); the only difference is that for this case the form of worst-case scenario parameters depend also on the sign of some process involving $z^1(V_t)$. Therefore, one may deduce that the Z's part of the solution of the ergodic BSDE (38) carries with the important information on the worst-case scenario.

Remark 20 *The above incomplete market model with uncertainty has also been studied in [26] in the framework of classical robust expected utility. They give an explicit PDE characterization for the lower value function of a robust utility maximization problem combining the duality approach and the stochastic control approach.*

On the other hand, when we do not consider the model uncertainty, the above model will reduce to the case that has been studied in [38] (Section 3.1.3 therein). The optimal portfolio weights obtained in [38] have the following form

$$\tilde{\pi}_1^*(t) = \frac{1}{1-\delta}(\theta(V_t) + \tilde{Z}_1(t)),\ \tilde{\pi}_2^*(t) = 0,$$

where $(\tilde{Y}, \tilde{Z}_1, \tilde{Z}_2, \tilde{\lambda})$ *is the Markovian solution of ergodic BSDE* (38) *with the driver*

$$\tilde{G}(v, z_1, z_2) = \frac{1}{2}\frac{\delta}{1-\delta}|z_1 + \theta(v)|^2 + \frac{1}{2}(|z_1|^2 + |z_2|^2).$$

Comparing the form between optimal portfolio weight $\tilde{\pi}_1^*$ *and the robust weight* π_1^* *given in Proposition 19, we observe that the model uncertainty affects the optimal policy in the following two aspects:*
i) π_1^* *has an additional projection term, which can be seen as a direct reflection on the model uncertainty influencing the robust investment policy;*
ii) *The solutions of the ergodic BSDE* (38) *with driver* G *and* $\tilde{G}$*, especially for the* Z*'s part shown in* π_1^* *and* $\tilde{\pi}_1^*$*, are different. Note that the difference between* G *and* $\tilde{G}$ *is mainly caused by the model uncertainty. This reflects indirectly the impact on optimal policy induced by the uncertainty via the associated ergodic BSDE.*

5 Connection with ergodic risk-sensitive stochastic differential games

We establish a connection between the constant λ appearing in the solution of the ergodic BSDE (38) and a zero-sum risk-sensitive stochastic differential game over the infinite horizon with ergodic payoff criteria. It turns out the constant λ is the value of the zero-sum risk-sensitive game and can be interpreted as the optimal long-term growth rate of expected utility of wealth with model uncertainty.

We first give the comparison theorem for ergodic BSDE (38), which will be employed in Theorem 22. Moreover, this result can be applied to compare the robust optimal long-term growth rate of expected utility for the model with different parameters.

Lemma 21 *Suppose that* G_i, $i = 1, 2$, *satisfy the following conditions*

$$\begin{aligned} &|G_i(v, z) - G_i(\bar{v}, z)| \le C(1 + |z|) \cdot |v - \bar{v}|, \\ &|G_i(v, z) - G_i(v, \bar{z})| \le C(1 + |z| + |\bar{z}|) \cdot |z - \bar{z}|,\ |G_i(v, 0)| \le C. \end{aligned} \tag{51}$$

For $i = 1, 2$, let (Y^i, Z^i, λ^i) be the unique Markovian solution of the ergodic BSDE (38) *with driver $G_i(v, z)$. If $G_1(v, z) \geq G_2(v, z)$, then we have*

$$\lambda^1 \geq \lambda^2.$$

We remark that, under the assumptions on the coefficients of our model (mainly the boundedness and Lipschitz assumption on the market price of the risk θ), the function $G(v, z)$ defined in (22) satisfies the condition (51) (see (40)). We next give the proof of Lemma 21.

Proof. Denote

$$\gamma_t = \begin{cases} \frac{G_1(V_t, Z_t^1) - G_1(V_t, Z_t^2)}{|Z_t^1 - Z_t^2|^2}(Z_t^1 - Z_t^2), & \text{if } Z_t^1 \neq Z_t^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, from the boundedness of Z^1 and Z^2, we know γ is a bounded process. We define the probability measure Q as follows

$$\frac{dQ}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}(\int_0^t \gamma_r dW_r).$$

Using the notations $\hat{Y} = Y^1 - Y^2$, $\hat{Z} = Z^1 - Z^2$, $\hat{\lambda} = \lambda^1 - \lambda^2$, we get

$$\begin{aligned} \hat{Y}_0 - \hat{Y}_T &= \int_0^T G_1(V_t, Z_t^2) - G_2(V_t, Z_t^2) + \gamma_t^T \hat{Z}_t dt - \hat{\lambda} T - \int_0^T \hat{Z}_t^T dW_t \\ &= \int_0^T G_1(V_t, Z_t^2) - G_2(V_t, Z_t^2) dt - \hat{\lambda} T - \int_0^T \hat{Z}_t^T dW_t^Q, \end{aligned}$$

where W^Q defined via $dW_t^Q = -\gamma_t dt + dW_t$ is a Brownian motion under the probability measure Q. Therefore, we get

$$\frac{1}{T} E_Q[\hat{Y}_0 - \hat{Y}_T] + \hat{\lambda} = \frac{1}{T} E_Q[\int_0^T G_1(V_t, Z_t^2) - G_2(V_t, Z_t^2) dt]. \tag{52}$$

Note that there exist mappings y^i, $i = 1, 2$, such that $Y_t^i = y^i(V_t)$, $i = 1, 2$. Since y^i, $i = 1, 2$, are of linear growth, there exists a constant C independent of T such that

$$E_Q|\hat{Y}_T| \leq C(1 + E_Q|V_T|) \leq C, \tag{53}$$

where the last inequality is derived from the dissipative condition (3). It follows from (52) and $G_1(v, z) \geq G_2(v, z)$ that

$$\hat{\lambda} = \limsup_{T \to \infty} \frac{1}{T} E_Q[\int_0^T G_1(V_t, Z_t^2) - G_2(V_t, Z_t^2) dt] \geq 0,$$

which completes the proof. ■

We start to formulate a two-player zero-sum risk-sensitive stochastic differential game associated with the forward process. The dynamic is given by the stochastic factor model (2) and the running payoff function is given by

$$L(v,\pi,u) = -\frac{1}{2}\delta(1-\delta)|\pi|^2 + \delta\pi^T[\theta(v)+u], \quad (v,\pi,u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d.$$

In line with the forward process, the planning horizon of the differential game is infinite and we study the following ergodic payoff criterion

$$\mathcal{J}(\pi,u) = \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}^{\pi,u}} \left(e^{\int_0^T L(V_s,\pi_s,u_s)ds} \right), \quad (\pi,u) \in \tilde{\Pi} \times \mathcal{U}, \tag{54}$$

where the probability measure $\mathbb{P}^{\pi,u}$ is defined as follows

$$\left.\frac{d\mathbb{P}^{\pi,u}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t (\delta\pi_r^T + u_r^T) dW_r \right). \tag{55}$$

Note that the criterion $\mathcal{J}$ represents the gain for Player 1 and the loss for Player 2. Thus, Player 1 aims to maximize $\mathcal{J}$ by using her control π, whereas Player 2 wants to minimize it via her control u. Intuitively, this model can be applied to describe the long-time investment action of a risk-averse investor in a market with model uncertainty (see Remark 23 on the equivalent form of $\mathcal{J}$), namely, the investor is trying to maximize her long-term portfolio gain rate via choosing the portfolio weight π, whereas the market, by default, aims to minimize the investor's gain rate via hiding the real market model and adding disturbance terms.

Theorem 22 *For any $(\pi,u) \in \tilde{\Pi} \times \mathcal{U}$ with feedback forms, i.e. $(\pi_s,u_s) = (\pi(V_s),u(V_s))$ for some Borel measurable mappings $(\pi(\cdot),u(\cdot))$, let $(y(V_t), z(V_t), \lambda)$, $t \geq 0$, be the unique Markovian solution of the ergodic BSDE (38). Furthermore, if the set Π is also assumed to be bounded, then λ is the value of the associated risk-sensitive game problem, namely,*

$$\lambda = \inf_{u\in\mathcal{U}} \sup_{\pi\in\tilde{\Pi}} \mathcal{J}(\pi,u) = \sup_{\pi\in\tilde{\Pi}} \inf_{u\in\mathcal{U}} \mathcal{J}(\pi,u). \tag{56}$$

Moreover, the supremum and infimum in (56) *can be attainable by choosing π^* and u^* as in* (42).

Proof. From (23), we have

$$\begin{aligned} &|F(v,z,\pi,u) - F(\bar{v},z,\pi,u)| \leq C|\pi| \cdot |v-\bar{v}|, \\ &|F(v,z,\pi,u) - F(v,\bar{z},\pi,u)| \leq C(1+|\pi|+|z|+|\bar{z}|) \cdot |z-\bar{v}|, \\ &|F(v,0,\pi,u)| \leq C|\pi|^2 + C|\pi|. \end{aligned} \tag{57}$$

Then, similar to the proof of Lemma 11, from (39) and (57), the following two ergodic equations

$$\begin{aligned} dY_t^u &= (-\sup_{\pi_t\in\Pi} F(V_t, Z_t^u, \pi_t, u_t) + \lambda^u)dt + (Z_t^u)^T dW_t, \\ dY_t^\pi &= (-\inf_{u_t\in U} F(V_t, Z_t^\pi, \pi_t, u_t) + \lambda^\pi)dt + (Z_t^\pi)^T dW_t, \end{aligned} \tag{58}$$

have unique Markovian solutions (Y^u, Z^u, λ^u) and $(Y^\pi, Z^\pi, \lambda^\pi)$, respectively, for each $(u, \pi) \in \mathcal{U} \times \tilde{\Pi}$ with feedback forms.

Step 1. We first show that

$$\lambda = \inf_{u\in\mathcal{U}} \lambda^u = \sup_{\pi\in\tilde{\Pi}} \lambda^\pi. \tag{59}$$

Since

$$\inf_{u_t\in U} F(V_t, Z_t, \pi_t, u_t) \le G(V_t, Z_t) \le \sup_{\pi_t\in\Pi} F(V_t, Z_t, \pi_t, u_t),$$

Lemma 21 then implies that

$$\lambda^\pi \le \lambda \le \lambda^u, \text{ for all } (\pi, u) \in \tilde{\Pi} \times \mathcal{U} \text{ with feedback forms.} \tag{60}$$

On the other hand, from the uniqueness of the solution of the ergodic BSDE (38), we know $\lambda = \lambda^{u^*} = \lambda^{\pi^*}$ with u^* and π^* given in (42). Thus, we have established (59).

Step 2. We show that, for each $(u, \pi) \in \mathcal{U} \times \tilde{\Pi}$ with feedback forms,

$$\lambda^u = \sup_{\pi\in\tilde{\Pi}} \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}^{\pi,u}} \left(e^{\int_0^T L(V_s,\pi_s,u_s)ds} \right), \tag{61}$$

$$\lambda^\pi = \inf_{u\in\mathcal{U}} \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}^{\pi,u}} \left(e^{\int_0^T L(V_s,\pi_s,u_s)ds} \right). \tag{62}$$

We only prove (61), and the proof of (62) is analogous.

For arbitrary but fixed $u \in \mathcal{U}$, from (58) we get, for every $\tilde{\pi} \in \tilde{\Pi}$,

$$\begin{aligned} dY_t^u &= (-\sup_{\pi_t\in\Pi} F(V_t, Z_t^u, \pi_t, u_t) + \lambda^u)dt + (Z_t^u)^T dW_t \\ &= \Big(-\sup_{\pi_t\in\Pi} F(V_t, Z_t^u, \pi_t, u_t) + \lambda^u + (Z_t^u)^T(\delta\tilde{\pi}_t + u_t) \Big) dt + (Z_t^u)^T dW_t^{\tilde{\pi},u}, \end{aligned} \tag{63}$$

where $W^{\tilde{\pi},u}$ defined via $dW^{\tilde{\pi},u} = -(\delta\tilde{\pi}_t + u_t)dt + dW_t$ is a Brownian motion under probability measure $\mathbb{P}^{\tilde{\pi},u}$ (see (55)). We observe that the function F (see (23)) in (63) can be written as

$$F(V_t, Z_t^u, \pi_t, u_t) = L(V_t, \pi_t, u_t) + (Z_t^u)^T(\delta\pi_t + u_t) + \frac{1}{2}|Z_t^u|^2.$$

Therefore, we rewrite the ergodic BSDE (63) as

$$\begin{aligned}
&Y_0^u - Y_T^u + \lambda^u T \\
&= \int_0^T \sup_{\pi_t \in \tilde{\Pi}} \big(L(V_t, \pi_t, u_t) + (Z_t^u)^T \delta\pi_t \big) - (Z_t^u)^T \delta\tilde{\pi}_t + \frac{1}{2}|Z_t^u|^2 dt - \int_0^T (Z_t^u)^T dW_t^{\tilde{\pi},u},
\end{aligned}$$

which follows that, for arbitrary $\tilde{\pi} \in \tilde{\Pi}$,

$$\begin{aligned}
&e^{\lambda^u T + Y_0^u} e^{-Y_T^u} \mathcal{E}\Big(\int_0^T (Z_t^u)^T dW_t^{\tilde{\pi},u} \Big) \\
&= \exp\Big(\int_0^T \sup_{\pi_t \in \tilde{\Pi}} \big(L(V_t, \pi_t, u_t) + (Z_t^u)^T \delta\pi_t \big) - L(V_t, \tilde{\pi}_t, u_t) - (Z_t^u)^T \delta\tilde{\pi}_t dt \Big) \\
&\quad \cdot e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt} \\
&\geq e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt}.
\end{aligned}$$

Then, we obtain

$$e^{\lambda^u T + Y_0^u} E_{\mathbb{P}^{\tilde{\pi},u}} \Big[e^{-Y_T^u} \mathcal{E}\Big(\int_0^T (Z_t^u)^T dW_t^{\tilde{\pi},u} \Big) \Big] \geq E_{\mathbb{P}^{\tilde{\pi},u}} \Big[e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt} \Big]. \tag{64}$$

We define the probability measure $Q^{\tilde{\pi},u}$ as follows

$$\frac{dQ^{\tilde{\pi},u}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^t (\delta\tilde{\pi}_r + u_r + Z_r^u)^T dW_r \right).$$

Using the measure $Q^{\tilde{\pi},u}$, from (64) we get

$$e^{\lambda^u T + Y_0^u} E_{Q^{\tilde{\pi},u}} \Big[e^{-Y_T^u} \Big] \geq E_{\mathbb{P}^{\tilde{\pi},u}} \Big[e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt} \Big].$$

Thus, it holds

$$\lambda^u + \frac{Y_0^u}{T} + \frac{1}{T} \ln E_{Q^{\tilde{\pi},u}} \Big[e^{-Y_T^u} \Big] \geq \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi},u}} \Big[e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt} \Big]. \tag{65}$$

Similar to the proof of estimate (53), from the boundedness of Π and Jensen's inequality, there exists a constant C independent of T such that

$$\frac{1}{C} \leq e^{-E_{Q^{\tilde{\pi},u}}[Y_T^u]} \leq E_{Q^{\tilde{\pi},u}} \left(e^{-Y_T^u} \right) \leq C, \tag{66}$$

where the last inequality is obtained using Lemma 3.1 in [16]. It follows from (65) and (66) that, for any $\tilde{\pi} \in \tilde{\Pi}$,

$$\lambda^u \geq \limsup_{T \uparrow \infty} \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi},u}} \Big[e^{\int_0^T L(V_t, \tilde{\pi}_t, u_t) dt} \Big].$$

with equality choosing $\tilde{\pi}_t = \pi_t^*$, where π_t^* is given in (42).

Step 3. Finally, we readily obtain (56) from (59) in Step 1 and (61) and (62) in Step 2. ■

Remark 23 *Note that*

$$\begin{aligned}&E_{\mathbb{P}^{\pi,u}}\left(e^{\int_0^T L(V_s,\pi_s,u_s)ds}\right)\\ =\ &E_{\mathbb{P}^u}\left(e^{\int_0^T -\frac{1}{2}\delta|\pi_s|^2+\delta\pi_s^T\theta(V_s)ds+\int_0^T \delta\pi_s^T dW_s}\right)=E_{\mathbb{P}^u}\left[\frac{(X_T^\pi)^\delta}{\delta}\right]\cdot\frac{\delta}{x^\delta}.\end{aligned}$$

In turn, from (56)*, it is easy to check that* λ *is also the value for the following game problem*

$$\begin{aligned}\lambda&=\inf_{u\in\mathcal{U}}\sup_{\pi\in\tilde{\Pi}}\limsup_{T\uparrow\infty}\frac{1}{T}\ln E_{\mathbb{P}^u}\left[\frac{(X_T^\pi)^\delta}{\delta}\right]\\ &=\sup_{\pi\in\tilde{\Pi}}\inf_{u\in\mathcal{U}}\limsup_{T\uparrow\infty}\frac{1}{T}\ln E_{\mathbb{P}^u}\left[\frac{(X_T^\pi)^\delta}{\delta}\right].\end{aligned}$$

Therefore, Theorem 22 can be viewed as an optimal investment model, and the constant λ *is the optimal long-term growth rate of the expected utility of wealth with model uncertainty. Such asymptotic results on robust utility maximization have been treated in [35] using the duality method and are related to "robust large deviations" criteria to optimal long-term investment, that is, the investor aims to maximize the portfolio's growth rate exceeding some threshold* $C\in\mathbb{R}$ *under the worst-case probability*

$$\sup_{\pi\in\tilde{\Pi}}\inf_{u\in\mathcal{U}}\limsup_{T\uparrow\infty}\frac{1}{T}\ln\mathbb{P}^u\big(\frac{1}{T}\ln X_T^\pi\geq C\big).$$

6 Connection with classical expected utility maximization for long time horizons

We establish a link between the power robust forward process $U(x,t)$ and the long-time behaviour of the lower value function of the classical power robust expected utility. For the latter, let $[0,T]$ be an arbitrary trading horizon and we introduce the lower value function as follows

$$w_T(x,v)=\sup_{\pi\in\Pi_{[0,T]}}\inf_{u\in\mathcal{U}_{[0,T]}}E_{\mathbb{P}^u}\left[\frac{(X_T^\pi)^\delta}{\delta}|X_0^\pi=x,V_0=v\right],\ (x,v)\in\mathbb{R}_+\times\mathbb{R}^d, \tag{67}$$

where the wealth process X_s^π, $s\in[0,T]$, solving (5) with $X_0^\pi=x$, the stochastic factor process V_s, $s\in[0,T]$, solving (2) with $V_0=v$, and $u\in\mathcal{U}_{[0,T]}$ means that u belongs to $\mathcal{U}$ and is restricted to the time horizon $[0,T]$.

We recall that the optimal investment problem for the classical robust expected utility has been considered in [10] via the stochastic control approach based on BSDE, in [49] via the duality approach, and in [26] combining these two methods.

Proposition 24 *Let $U(x,t) = \frac{x^\delta}{\delta} e^{y(V_t)-\lambda t}$ be the power robust forward performance process as in (41). Then, there exists a constant $L \in \mathbb{R}$, independent of the initial states $X_0^\pi = x$ and $V_0 = v$, such that, for $(x,v) \in \mathbb{R}_+ \times \mathbb{R}^d$,*

$$\lim_{T\uparrow\infty} \frac{w_T(x,v)e^{-\lambda T - L}}{U(x,0)} = 1.$$

Proof. Since the maxmin problem (67) is standard in the literature (see, for example, [53]), we only demonstrate its main steps briefly. To this end, for each $\pi \in \Pi_{[0,T]}$ and $u \in \mathcal{U}_{[0,T]}$, we introduce the objective functional

$$w_T(x,v,\pi,u) = E_{\mathbb{P}^u}\left[\frac{(X_T^\pi)^\delta}{\delta} | X_0^\pi = x, V_0 = v\right].$$

We aim to find a saddle point $(\pi^*, u^*) \in \Pi_{[0,T]} \times \mathcal{U}_{[0,T]}$ such that

$$w_T(x,v,\pi,u^*) \le w_T(x,v,\pi^*,u^*) \le w_T(x,v,\pi^*,u).$$

Then, it is clear that $w_T(x,v) = w_T(x,v,\pi^*,u^*)$. We claim that

$$w_T(x,v) = \frac{x^\delta}{\delta} e^{\bar{Y}_0}, \tag{68}$$

$$\pi_t^* = \pi^*(V_t, \bar{Z}_t),\ u_t^* = u^*(V_t, \bar{Z}_t),\ t \in [0,T], \tag{69}$$

with the mappings (π^*, u^*) given in (28) and (26), and $(\bar{Y}, \bar{Z})$ being the unique solution of the following BSDE

$$\bar{Y}_t = \int_t^T G(V_r, \bar{Z}_r)dr - \int_t^T \left(\bar{Z}_r\right)^T dW_r, \tag{70}$$

where the driver G is given in (22). The proof follows along similar arguments as in Proposition 9 and Theorem 12, and thus omitted.

From Theorem 4.4 in [29], there exists a constant $L \in \mathbb{R}$ such that

$$\lim_{T\uparrow\infty} (\bar{Y}_0 - \lambda T - Y_0) = L, \tag{71}$$

where (Y, Z, λ) is the solution of ergodic BSDE (38). Finally, from (41), (68) and (71) we have

$$\lim_{T\uparrow\infty} \frac{w_T(x,v)e^{-\lambda T - L}}{U(x,0)} = 1,$$

which completes the proof. ■

We have showed that the discounted classical power robust expected utility converges to the power robust forward performance process as the investment horizon tends to infinite. This result is obtained via the relationship between BSDE and ergodic BSDE. Then it seems natural and interesting to consider the

connection of the optimal strategies between these two investment problems. In fact, from (42) we get that the optimal investment strategy to each scenario

$$\alpha^*(t,u_t) = Proj_{\Pi}(\frac{\theta(V_t)+Z_t+u_t}{1-\delta}),$$

where (Y,Z,λ) is the solution of ergodic BSDE (38). Similar arguments conclude that for classical expected utility problem (67), the optimal investment strategy has the form

$$\alpha_T^*(t,u_t) = Proj_{\Pi}(\frac{\theta(V_t)+\bar{Z}_t+u_t}{1-\delta}),$$

where $(\bar{Y},\bar{Z})$ is the unique solution of BSDE (70). Since the projection operator on a closed convex set is Lipschitz continuous, the convergence of α_T^* to α^* boils down to the convergence of

$$E_{\mathbb{P}}[\int_0^T |Z_t-\bar{Z}_t|^2 dt] \to 0, \text{ as } T\to\infty. \tag{72}$$

However, (72) is yet to be established. Hence, although we have established a connection of the robust forward performance process with the corresponding classical robust expected utility, it is still an open problem to prove the convergence of the associated optimal trading strategies.

7 Logarithmic robust forward performance processes with negative realization processes

In this section, we provide an example to show the advantage of our stochastic differential approach to solve homothetic robust forward investment problems compared with the classical saddle point argument. For simplicity of the calculations, we consider the following situation:

i) a single stock and single stochastic factor model (i.e., $n=d=1$ in state equations (1) and(2)).

ii) $U=[0,1]$, $\Pi=[0,1]$, the market price of the risk $\theta\in[-1,0]$.

iii) a logarithmic robust forward performance process,

$$U(x,t) = \ln x + f(V_t,t)$$

with a quadratic form on the realization process γ in (11), i.e.,

$$\gamma_{t,s}(u) = \int_t^s \frac{1}{2}|u_s|^2 ds, \tag{73}$$

which expresses the cumulative evaluation of the model $\mathbb{P}^u$ predicted by the investor in $[t,s]$. Herein, since we consider negative realization processes, we choose the parameter $\tau=-1$ without loss of generality. Using similar arguments to Theorem 12, we obtain the following results.

Theorem 25 *Let* $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{\lambda}) = (\widetilde{y}(V_t), \widetilde{z}(V_t), \widetilde{\lambda}), t \geq 0$, *be the unique Markovian solution of (38) with the generator*

$$\widetilde{G}(v,z) = \max_{\pi \in \Pi} \min_{u \in U} \widetilde{F}(v,z,\pi,u),$$

where $\widetilde{F}(v,z,\pi,u) = -\frac{1}{2}\pi^2 + \pi\theta(v) + (\pi+z)u - \frac{1}{2}u^2$. *Then, the process* $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, *given by*

$$U(x,t) = \ln x + \widetilde{y}(V_t) - \widetilde{\lambda} t,$$

is a logarithmic robust forward performance process with realization process γ *given in* (73) *and parameter* $\tau = -1$. *Moreover, the optimal portfolio weight* $\widetilde{\pi}^*$, *and the worst-case scenario strategy* $\widetilde{\beta}^*$ *responding to each portfolio weight* π *are given as follows*

$$\widetilde{\pi}^*_t = \widetilde{\pi}^*(V_t, \widetilde{z}(V_t)),\ \widetilde{\beta}^*(t,\pi_t) = \widetilde{\beta}^*(\widetilde{z}(V_t), \pi_t),$$

where the mappings $(\widetilde{\pi}^*, \widetilde{\beta}^*)$ *have the form*

$$\widetilde{\pi}^*(v,z) = \begin{cases} \theta(v)+1, & \text{if } \frac{1}{2} - z \geq \theta(v)+1; \\ \frac{1}{2} - z, & \text{if } \theta(v) \leq \frac{1}{2} - z \leq \theta(v)+1; \\ \theta(v), & \text{if } \frac{1}{2} - z \leq \theta(v), \end{cases}$$

and

$$\widetilde{\beta}^*(z,\pi) = \begin{cases} 1, & \text{if } \pi + z \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that the saddle point for this forward investment problem does not exist since $\widetilde{F}$ is concave both in variables π and u. Therefore, the classical saddle point argument can not be applied directly, whereas our stochastic differential game approach provides an alternative and efficient way to address this problem.

8 Conclusion

This paper provides a stochastic differential game framework to construct a class of forward performance processes with model uncertainty. In particular, the homothetic robust forward process in factor form is represented in terms of the Markovian solution of ergodic BSDE. The approach and results may be extended in several directions. First, one may consider a general realization process γ with parameter τ not necessarily zero. Second, it would be interesting to prove the convergence of the finite time horizon optimal investment strategy to its forward counterpart as time horizon becomes large. Both are left for the future research.

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