# Maximum size intersecting families of bounded minimum positive co-degree

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#### Abstract

Let  $\mathcal{H}$  be an *r*-uniform hypergraph. The minimum positive co-degree of  $\mathcal{H}$ , denoted by  $\delta_{r-1}^+(\mathcal{H})$ , is the minimum k such that if S is an (r-1)-set contained in a hyperedge of  $\mathcal{H}$ , then S is contained in at least k hyperedges of  $\mathcal{H}$ . For  $r \geq k$  fixed and n sufficiently large, we determine the maximum possible size of an intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq k$  and characterize the unique hypergraph attaining this maximum. This generalizes the Erdős-Ko-Rado theorem which corresponds to the case k = 1. Our proof is based on the delta-system method.

#### 1 Introduction

A hypergraph  $\mathcal{H}$  is *intersecting* if for every pair of hyperedges  $h, h' \in E(\mathcal{H})$  we have  $h \cap h' \neq \emptyset$ . The celebrated theorem of Erdős, Ko and Rado [3] gives that for  $n \geq 2r$ , the maximum size of an intersecting *r*-uniform *n*-vertex hypergraph is  $\binom{n-1}{r-1}$ . The Erdős-Ko-Rado theorem is a cornerstone of extremal combinatorics and has many proofs, extensions and generalizations, see the excellent survey of Frankl and Tokushige [11] for a history of extremal problems for intersecting hypergraphs. We call the unique hypergraph achieving the maximum in the Erdős-Ko-Rado theorem a *maximal star*, i.e., the hypergraph of all hyperedges containing a given vertex.

The degree of a set of vertices S in a hypergraph  $\mathcal{H}$  is the number of hyperedges containing S, i.e.,  $|\{h \in E(\mathcal{H}) : S \subseteq h\}|$ . Denote by  $\delta_s(\mathcal{H})$  the minimum degree of an *s*-element subset of the vertices of  $\mathcal{H}$ . In this way,  $\delta_1(\mathcal{H})$  is the standard minimum degree of a vertex in  $\mathcal{H}$ .

Huang and Zhao [16] considered a minimum degree version of the Erdős-Ko-Rado theorem. In particular, they proved that for  $n \ge 2r+1$ , if  $\mathcal{H}$  is an intersecting r-uniform

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*n*-vertex hypergraph, then  $\mathcal{H}$  has minimum degree  $\delta_1(\mathcal{H}) \leq \binom{n-2}{r-2}$ . The Huang-Zhao [16] proof uses the linear algebra method and later a combinatorial proof was given by Frankl and Tokushige [10] for  $n \geq 3r$ . Kupavskii [19] gave an extension of this result and showed that for t < r and  $n \geq 2r + 3t/(1-t/r)$ , every intersecting *r*-uniform *n*-vertex hypergraph  $\mathcal{H}$  satisfies  $\delta_t(\mathcal{H}) \leq \binom{n-t-1}{r-t-1}$ .

In the more general hypergraph setting, Mubayi and Zhao [22] introduced the notion of co-degree Turán numbers, i.e., the maximum possible value of  $\delta_{r-1}(\mathcal{H})$  among all *r*uniform *n*-vertex hypergraphs  $\mathcal{H}$  not containing a specified subhypergraph  $\mathcal{F}$ . In their paper they give several results that show that the co-degree extremal problem behaves differently from the classical Turán problem.

Motivated by the degree versions of the Erdős-Ko-Rado theorem and co-degree Turán numbers we propose studying the following hypergraph degree condition.

**Definition 1.** Let  $\mathcal{H}$  be a non-empty r-uniform hypergraph. The minimum positive codegree of  $\mathcal{H}$ , denoted  $\delta^+_{r-1}(\mathcal{H})$ , is the maximum k such that if S is an (r-1)-set contained in a hyperedge of  $\mathcal{H}$ , then S is contained in at least k distinct hyperedges of  $\mathcal{H}$ .

Note that the empty hypergraph is a degenerate case; for simplicity we define its positive co-degree to be zero.

As an example, let us examine hypergraphs that contain no  $F_5 = \{abc, abd, cde\}$  to compare the co-degree and positive co-degree settings. Frankl and Füredi [9] (see [17] for a strengthening) showed that the complete balanced tripartite 3-uniform hypergraph has the maximum number of hyperedges among all 3-uniform *n*-vertex  $F_5$ -free hypergraphs, for *n* sufficiently large. This construction has minimum co-degree 0 and it is easy to see that minimum co-degree at least 2 guarantees the existence of an  $F_5$ . On the other hand, the balanced tripartite hypergraph is  $F_5$ -free and has minimum positive co-degree n/3and it can be shown that minimum positive co-degree strictly greater than n/3 implies the existence of an  $F_5$ .

Note that for ordinary graphs (i.e. 2-uniform hypergraphs), the minimum positive codegree is simply the minimum degree of the non-isolated vertices, which in many extremal problems we may assume is equal to the minimum degree. This suggests positive co-degree as a reasonable notion of "minimum degree" in a hypergraph.

The positive co-degree condition has appeared in several other contexts. For example, in [18] the term d-full was used and the authors gave some simple lemmas for hypergraphs with minimum positive co-degree (in the course of proving theorems about extremal numbers for hypergraphs).

In this paper we investigate the maximum size of an intersecting r-uniform n-vertex hypergraph with positive co-degree at least k. As the condition  $\delta_{r-1}^+(\mathcal{H}) \geq 1$  is vacuous, the maximum in this case is  $\binom{n-1}{r-1}$  as given by the Erdős-Ko-Rado theorem. The unique construction achieving this bound has minimum positive co-degree 1. On the other hand, as shown in Proposition 4, in an intersecting hypergraph the uniformity gives an upper bound on the minimum positive co-degree, i.e.,  $r \geq k$ . Thus the range of interest for our problem is  $2 \leq k \leq r$ . In this range we prove that for n sufficiently large the maximumsize intersecting hypergraph with minimum positive co-degree k is given by the following hypergraph.

**Definition 2.** Given integers  $r \ge k \ge 1$  an (r-uniform) k-kernel system is a hypergraph

 $\mathcal{H}$  on vertex set V with edges  $\mathcal{E} = \{E \in \binom{V}{r} : |E \cap X| \ge k\}$ , were X is a distinguished subset of V of size 2k - 1. The set X is called the kernel of  $\mathcal{H}$ .

Clearly a k-kernel system is intersecting. Observe that the number of hyperedges in an r-uniform n-vertex k-kernel system  $\mathcal{H}$  is

$$|E(\mathcal{H})| = \sum_{i=k}^{\max\{r,2k-1\}} \binom{2k-1}{i} \binom{n-2k+1}{r-i} \ge \binom{2k-1}{k} \binom{n-2k+1}{r-k} = \Omega(n^{r-k}).$$

Note that a 1-kernel system is the hypergraph consisting of all hyperedges containing a fixed vertex x, i.e., the maximal hypergraph in the Erdős-Ko-Rado theorem. Interestingly, k-kernel systems appear as solutions to maximum degree versions of the Erdős-Ko-Rado theorem. Let us give three examples.

First, a special case of a more general theorem of Frankl [7] implies that if  $\mathcal{H}$  is a maximum-size intersecting *r*-uniform *n*-vertex hypergraph with maximum degree at most  $2\binom{n-3}{r-2} + \binom{n-3}{r-3}$ , then  $\mathcal{H}$  is a 2-kernel system, provided *n* is large enough.

Second, Erdős, Rothschild and Szemerédi (see [2]) posed the following problem: determine the maximum size of an intersecting r-uniform n-vertex hypergraph  $\mathcal{H}$  such that each vertex contained in at most  $c|E(\mathcal{H})|$  hyperedges for  $r \geq 3$  and 0 < c < 1. They proved when c = 2/3 and n large, then a 2-kernel system is the unique hypergraph attaining this maximum. Frankl [5] showed that for  $2/3 \leq c < 1$  and n large enough,  $\mathcal{H}$ has no more hyperedges than a 2-kernel system. For 3/5 < c < 2/3 and n large enough, Füredi [5] showed that a 3-kernel system is one of six non-isomorphic hypergraphs attaining this maximum. In the case when  $1/2 < c \leq 3/5$  and n large enough, Frankl [5] showed that  $\mathcal{H}$  has no more hyperedges than a 3-kernel system, although the unique hypergraph attaining this maximum is not isomorphic to a 3-kernel system.

Third, Lemons and Palmer [21] proved that 3-kernel systems are the *r*-uniform *n*-vertex hypergraphs with the largest *diversity*, i.e., the difference between the number of hyperedges and the maximum degree for n large enough (see [8, 20] for improvements to the threshold on n).

The main result of our paper is as follows:

**Theorem 3.** Let  $\mathcal{H}$  be an intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq k$  where  $1 \leq k \leq r$ . If  $\mathcal{H}$  has the maximum number of hyperedges, then  $\mathcal{H}$  is a k-kernel system for n sufficiently large.

Theorem 3 holds for n large, roughly double exponential in r. In Section 3 we give two results that suggest that Theorem 3 should hold for n at least  $cr^{k+2}$ , where c is a polynomial in k. It would be interesting to further refine the range of n as a function of rand k where our results hold. Also, we only considered the positive co-degree of (r-1)sets. We can define  $\delta_s^+(\mathcal{H})$  to be the minimum k such that if S is an s-set contained in a hyperedge of  $\mathcal{H}$ , then S is contained in at least k distinct hyperedges. There may be interesting problems to be considered under this more general condition.

#### 2 Proof of Theorem 3

First, let us observe that the uniformity of an intersecting hypergraph is always at least the minimum positive co-degree. **Proposition 4.** If  $\mathcal{H}$  is a non-empty intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq k$ , then  $r \geq k$ .

Proof. Assume, for the sake of a contradiction, that k > r. Let  $h = \{x_1, x_2, \ldots, x_r\}$  be a hyperedge of  $\mathcal{H}$ . The (r-1)-set  $h \setminus \{x_1\}$  has co-degree at least k, so there is a vertex  $x_{r+1} \notin h$  such that  $(h \setminus \{x_1\}) \cup \{x_{r+1}\}$  is a hyperedge of  $\mathcal{H}$ . Similarly, the (r-1)-set  $(h \setminus \{x_1, x_2\}) \cup \{x_{r+1}\}$  has co-degree at least k, so there is a vertex  $x_{r+2} \notin h \cup \{x_{r+1}\}$  such that  $(h \setminus \{x_1, x_2\}) \cup \{x_{r+1}, x_{r+2}\}$  is a hyperedge of  $\mathcal{H}$ . Because k > r, we can repeat this process to obtain a hyperedge  $(h \setminus \{x_1, \ldots, x_r\}) \cup \{x_{r+1}, \ldots, x_{2r}\} = \{x_{r+1}, \ldots, x_{2r}\}$  that is in  $\mathcal{H}$ . Now we have disjoint hyperedges h and  $\{x_{r+1}, \ldots, x_{2r}\}$  in  $\mathcal{H}$  which contradicts the intersecting property.

An r-uniform hypergraph S is a *sunflower* if every pairwise intersection of the hyperedges is the same set Y, called the *core* of the sunflower. We call the sets  $h \setminus Y$  for  $h \in E(S)$  the *petals* of the sunflower S. Note that the petals are pairwise disjoint. Denote the size of the core of a sunflower S by c(S).

Let f(r, p) denote the minimum integer such that an *r*-uniform hypergraph with f(r, p) hyperedges contains a sunflower with p petals. The Sunflower Lemma of Erdős and Rado [4] claims that  $f(r, p) \leq r!(p-1)^r$ . The determination of f(r, p) is a well-known open problem in combinatorics. A recent breakthrough by Alweiss, Lovett, Wu and Zhang [1] gives a bound on f(r, p) of about  $(\log r)^{r(1+o(1))}$ .

In general we cannot force a sunflower to have a core of a specified size unless we increase the number of hyperedges in the host hypergraph. Mubayi and Zhao (Lemma 6 in [23]) gives conditions for the existence of a sunflower with a core of bounded size.

**Lemma 5** (Mubayi and Zhao, [23]). Fix integers  $r \ge 3$ ,  $k \ge 1$  and  $p \ge 1$  and let C = C(r, p) be a large enough constant. If  $\mathcal{G}$  is an r-uniform n-vertex hypergraph with

$$|E(\mathcal{G})| \ge Cn^{r-k-1},$$

then  $\mathcal{G}$  contains a sunflower with p petals and core of size at most k.

Observe that Lemma 5 is sharp in the order of magnitude of n. Indeed, the r-uniform n-vertex hypergraph consisting of all hyperedges containing a fixed set Y of k+1 vertices contains  $\binom{n-k-1}{r-k-1}$  hyperedges, but no sunflower with a core of size at most k as any two hyperedges intersect in at least k+1 vertices. We remark that the problem to determine the best constant C in Lemma 5 is interesting in its own right. In the Appendix at the end of the paper we give a new proof of Lemma 5 that gives an improvement to C.

We will need a lower bound on the size of a core of a sunflower in an intersecting hypergraph.

**Lemma 6.** If S is a sunflower with at least r + 1 petals in an intersecting r-uniform hypergraph  $\mathcal{G}$  with  $\delta^+_{r-1}(\mathcal{G}) \geq k$ , then the core Y of S satisfies  $|Y| \geq k$ .

*Proof.* For the sake of contradiction, assume that the core Y of S is small, i.e., |Y| < k. Observe that Y is a transversal of  $\mathcal{G}$ , i.e., every hyperedge of  $\mathcal{G}$  intersects Y. Indeed, as the petals of the sunflower S are pairwise vertex-disjoint, each hyperedge of  $\mathcal{G}$  must intersect the core Y in order to intersect each of the at least r + 1 hyperedges associated with the petals of the sunflower. Now let Y' be a minimum transversal in  $\mathcal{G}$ . Thus  $|Y'| \leq |Y| < k$  and the minimality of Y' guarantees the existence of a hyperedge h that intersects Y' in exactly one element. The (r-1)-set  $h \setminus Y'$  is contained in at most k-1 hyperedges of  $\mathcal{G}$ ; one for each element of Y'. This contradicts the positive co-degree condition on  $\mathcal{G}$ .

Proof of Theorem 3. Let  $\mathcal{H}$  be an intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq k$  where  $1 \leq k \leq r$ . Moreover, suppose that  $\mathcal{H}$  has the maximum number of hyperedges. We will show that  $\mathcal{H}$  is a k-kernel system for n sufficiently large.

We have observed that a k-kernel system has minimum positive co-degree at least k, so we may assume that

$$|E(\mathcal{H})| \ge \binom{2k-1}{k} \binom{n-2k+1}{r-k} = \Omega(n^{r-k}).$$

Therefore, for *n* large enough, Lemmas 5 and 6 guarantees the existence of a sunflower S with  $p = (r+1)r^{k-1}$  petals and core of size *k*. Denote the core of S by  $Y = \{y_1, y_2, \ldots, y_k\}$ .

Note that in order to apply Lemma 5 we need that the following inequality is satisfied:

$$\binom{2k-1}{k}\binom{n-2k+1}{r-k} \ge Cn^{r-k-1}$$

where C = C(r, p) is the constant from Lemma 5. This is satisfied when

$$n \ge \frac{(2r-2k)^{r-k}}{\binom{2k-1}{k}}C$$

The value  $C = (pr2^r)^{2^r}$  given in [23] follows from a theorem of Füredi [15].

**Claim 7.** There is a set of vertices  $Z = \{z_1, z_2, \ldots, z_{k-1}\}$  such that  $Z \cap Y = \emptyset$  and  $Z \cup \{y_k\}$  is the core of a sunflower with r + 1 petals.

Proof. We will prove the following stronger claim: For  $0 \le i \le k-1$ , there is a set of vertices  $Z_i = \{z_1, z_2, \ldots, z_i\}$  such that  $Y \cap Z_i = \emptyset$  and  $Z_i \cup \{y_k, y_{k-1}, \ldots, y_{i+1}\}$  is the core of a sunflower  $S_i$  with  $(r+1)r^{k-1-i}$  petals. The claim follows from the case i = k-1. We proceed by induction on i. The base case i = 0 is immediate as  $Z_0 = \emptyset$  and  $S_0 = S$ 

We proceed by induction on *i*. The base case i = 0 is immediate as  $Z_0 = \emptyset$  and  $S_0 = S$  is a sunflower with core  $Z_0 \cup \{y_k, y_{k-1}, \ldots, y_1\} = Y$  with  $(r+1)r^{k-1}$  petals. Now suppose i > 0 and the statement holds for i - 1. Let  $S_{i-1}$  be a sunflower given by the inductive hypothesis.

For each petal P in  $S_{i-1}$  consider the (r-1)-set  $P \cup Z_{i-1} \cup \{y_k, \ldots, y_{i+1}\} = P \cup Z_{i-1} \cup \{y_k, \ldots, y_i\} \setminus \{y_i\}$ . By the positive co-degree condition on  $\mathcal{H}$ , the set  $P \cup Z_{i-1} \cup \{y_k, \ldots, y_{i+1}\}$  is contained in k hyperedges of  $\mathcal{H}$ . Therefore, as  $i \leq k-1$ , there is a vertex x(P) such that  $x(P) \notin \{y_1, y_2, \ldots, y_i\}$  and  $\{x(P)\} \cup P \cup Z_{i-1} \cup \{y_k, \ldots, y_{i+1}\}$  is a hyperedge of  $\mathcal{H}$ .

Now suppose there are distinct vertices  $x_1, x_2, \ldots, x_{r+1}$  among the vertices in  $\{x(P) : P \text{ is a petal in } \mathcal{S}\}$ . Let  $P_1, P_2, \ldots, P_{r+1}$  be the petals corresponding to these vertices, i.e.,  $\{x_j\} \cup P_j \cup Z_{i-1} \cup \{y_k, \ldots, y_{i+1}\} \in E(\mathcal{H})$  for  $j = 1, 2, \ldots, r+1$ . Then  $Z_{i-1} \cup \{y_k, \ldots, y_{i+1}\}$  is the core of size k-1 of a sunflower with petals  $P_j \cup \{x_j\}$  for  $j = 1, 2, \ldots, r+1$  in  $\mathcal{H}$ . This contradicts Lemma 6. Therefore, there are at most r distinct vertices among the

vertices in  $\{x(P) : P \text{ is a petal in } S_{i-1}\}$ . This implies that there is a vertex x that is the vertex x(P) for at least  $\frac{1}{r}|E(S_{i-1})| \geq (r+1)r^{k-2-(i-1)}$  petals P in  $S_{i-1}$ . Put  $z_i = x$  and  $Z_i = \{z_1, z_2, \ldots, z_i\}$  and let  $S_i$  be the sunflower consisting of  $(r+1)r^{k-1-i}$  hyperedges of  $S_{i-1}$  containing  $x = z_i$ . Observe that  $Z_i \cup \{y_k, \ldots, y_{i+1}\}$  is the core of the sunflower  $S_i$  with  $(r+1)r^{k-1-i}$  petals.

Let  $S_Z$  be a sunflower with r + 1 petals and core  $Z \cup \{y_k\}$  given by Claim 7. There are at most (r+1)(r-k) + (k-1) vertices disjoint from Y spanned by  $S_Z$ . As S has  $(r+1)r^{k-1}$  petals, we may choose r+1 petals of S that are vertex-disjoint from the vertices of  $S_Z$ . Call the resulting sunflower  $S_Y$ . Note that  $S_Y$  has r+1 petals and core Y.

**Claim 8.** For every petal P in  $S_Z$  and every  $y \in Y$  we have that  $P \cup Z \cup \{y\}$  is a hyperedge in  $\mathcal{H}$ .

Proof. Observe that the (r-1)-set  $P \cup Z$  is contained in the hyperedge  $P \cup Z \cup \{y_k\}$ , so by the positive co-degree condition  $P \cup Z$  is contained in k hyperedges of  $\mathcal{H}$ . Moreover, each of these hyperedges must intersect every hyperedge in the sunflower  $S_Y$ . As  $S_Y$  has at least 2 petals, each of the k hyperedges containing  $P \cup Z$  must contain a distinct vertex of Y.

We now continue with a technical claim that will imply the theorem.

**Claim 9.** For every k-set  $T \subset Y \cup Z$  we have:

- (1)  $Q \cup T \in E(\mathcal{H})$  for every petal Q of  $S_Y$ ,
- (2)  $((Y \cup Z) \setminus T) \cup \{s\} \cup P \in E(\mathcal{H})$  for every  $s \in T$  and petal P of  $\mathcal{S}_Z$ .

*Proof.* We proceed by induction on  $t = |T \cap Z|$ . Note that  $t \leq k - 1$ . When t = 0 we have that T = Y, then (1) is immediate as  $Q \cup Y \in E(\mathcal{S}_Y) \subset \mathcal{H}$  and (2) follows from Claim 8.

Let t > 0 and suppose the statement of the claim holds for all smaller values of t. As  $0 < t \le k - 1$ , there exists a  $z \in Z \cap T$  and a  $y \in Y \setminus T$ . Fix an arbitrary petal Q of  $S_Y$ . Put  $T' = T \cup \{y\} \setminus \{z\}$  and note that  $|T' \cap Z| = t - 1$ . Therefore, by induction, we have  $Q \cup T' \in E(\mathcal{H})$  and  $((Y \cup Z) \setminus T') \cup \{s'\} \cup P \in E(\mathcal{H})$  for every  $s' \in T'$  and petal P of  $S_Z$ .

By the positive co-degree condition, the (r-1)-set  $Q \cup T' \setminus \{y\}$  is contained in at least k hyperedges. Moreover,  $Q \cup T' \setminus \{y\}$  is disjoint from the hyperedges of the form  $((Y \cup Z) \setminus T') \cup \{y\} \cup P$  where P is a petal of  $\mathcal{S}_Z$ . As  $\mathcal{S}_Z$  has r+1 petals and  $\mathcal{H}$  is intersecting, this implies that the k hyperedges containing  $Q \cup T' \setminus \{y\}$  each intersect the k-set  $((Y \cup Z) \setminus T) \cup \{y\}$ . In particular,  $(Q \cup T' \setminus \{y\}) \cup \{z\} = Q \cup T$  is a hyperedge of  $\mathcal{H}$ . This proves (1).

In order to prove (2), let us fix an arbitrary petal P of  $\mathcal{S}_Z$ . Observe that the (r-1)-set

$$((Y \cup Z) \setminus T) \cup P = ((Y \cup Z) \setminus (T' \cup \{z\} \setminus \{y\})) \cup P = ((Y \cup Z) \setminus T') \setminus \{z\} \cup \{y\} \cup P$$

is contained in the hyperedge  $(Y \cup Z) \setminus T' \cup \{y\} \cup P \in E(\mathcal{H})$  whose existence is given by the inductive hypothesis on (2) with  $y = s' \in T'$ . Therefore, the positive co-degree condition guarantees that the (r-1)-set  $((Y \cup Z) \setminus T) \cup P$  is contained in k hyperedges. In order for these hyperedges to intersect the r + 1 hyperedges  $Q \cup T$  for each petal Q of  $S_Y$ , we have that each set of the form  $((Y \cup Z) \setminus T) \cup \{s\} \cup P$  for  $s \in T$  must be a hyperedge of  $\mathcal{H}$ .

We are now ready to complete the proof of Theorem 3. Suppose that there is a hyperedge  $h \in E(\mathcal{H})$  such that  $|h \cap (Y \cup Z)| \leq k-1$ . Then there exists a k-set  $T \subset Y \cup Z$ such that T is disjoint from h. Moreover, as  $\mathcal{S}_Y$  has at least r+1 petals, there is a petal Q in  $\mathcal{S}_Y$  that is disjoint from h. By Claim 9 we have that  $T \cup Q \in E(\mathcal{H})$  which is disjoint from  $h \in E(\mathcal{H})$ . This violates the intersecting property of  $\mathcal{H}$ , a contradiction.

Therefore, every hyperedge  $h \in E(\mathcal{H})$  intersects  $Y \cup Z$  in at least k vertices. This implies that  $\mathcal{H}$  is a subhypergraph of a k-kernel system, i.e., as  $\mathcal{H}$  is edge-maximal, it is exactly a k-kernel system.

**Remark.** Observe that the proof of Theorem 3 gives a stability result. In particular, if  $\mathcal{H}$  has enough edges to apply Lemma 5, then we have that  $\mathcal{H}$  is a subhypergraph of a k-kernel system.

#### **3** Improved thresholds on *n*

We now show that in the case  $k \leq 3$ , Theorem 3 holds for  $n \geq cr^{k+2}$ . In Theorem 3 we need *n* to be at least double exponential in *r*. Recall that two hypergraphs  $\mathcal{A}$  and  $\mathcal{B}$  are cross-intersecting if for every pair of hyperedges  $A \in E(\mathcal{A})$  and  $B \in E(\mathcal{B})$  we have  $A \cap B \neq \emptyset$ . Also, a transversal for a hypergraph  $\mathcal{H}$  is a set of vertices *T* such that  $T \cap h \neq \emptyset$  for every hyperedge  $h \in E(\mathcal{H})$ . The transversal number  $\tau(\mathcal{H})$  is the minimum *t* such that there is a transversal *T* of  $\mathcal{H}$  of size *t*.

We begin with a simple bound on the size of an intersecting hypergraph  $\mathcal{H}$  with transversal number  $\tau(\mathcal{H}) = t$ . Stronger results for  $\tau(\mathcal{H}) = 3$  and  $\tau(\mathcal{H}) = 4$  are given by Frankl [6] and Frankl, Ota and Tokushige [12], but we include an argument for the sake of completeness and as our argument holds for all n and t.

**Lemma 10.** Fix  $n \ge r \ge t$ . Let  $\mathcal{H}$  be an intersecting r-uniform n-vertex hypergraph with transversal number  $\tau(\mathcal{H}) \ge t$ . Then

$$|E(\mathcal{H})| \le r^t \binom{n-t}{r-t}.$$

Proof. Let us construct a t-uniform hypergraph  $\mathcal{T}$  with  $|E(\mathcal{T})| \leq r^t$  such that for every  $h \in E(\mathcal{H})$  there exists a  $h' \in E(\mathcal{T})$  with  $h' \subset h$ . The existence of  $\mathcal{T}$  immediately implies the lemma as  $|E(\mathcal{H})| \leq |E(\mathcal{T})| {n-t \choose r-t}$ .

We proceed iteratively. First select an arbitrary hyperedge  $h_1 \in E(\mathcal{H})$ . For each vertex  $v_1 \in h_1$ , the set  $\{v_1\}$  is not a transversal of  $\mathcal{H}$ , so there is a hyperedge  $h_2 \in E(\mathcal{H})$ that is disjoint from  $\{v_1\}$ . For each vertex  $v_2 \in h_2$ , the set  $\{v_1, v_2\}$  is not a transversal of  $\mathcal{H}$ , so there is a hyperedge  $h_3 \in E(\mathcal{H})$  that is disjoint from  $\{v_1, v_2\}$ . We continue this process to select a set of t distinct vertices  $v_1, v_2, \ldots, v_t$ . Let  $\mathcal{T}$  be the collection of all t-sets constructed in this way. Note that in each step there are at most r choices for the vertex  $v_i$ , so  $|E(\mathcal{T})| \leq r^t$ .

Now it remains to show that for every  $h \in E(\mathcal{H})$  there exists an  $h' \in E(\mathcal{T})$  with  $h' \subset h$ . Observe that at each step *i*, our hyperedge *h* must intersect  $h_i$ , so there is

a choice of vertex in  $h_i \cap h$ . Therefore, there is at least one *r*-set constructed that is contained in h.

We first consider the case of minimum positive co-degree at least 2.

**Proposition 11.** Fix  $r \geq 3$  and let  $n \geq \frac{1}{3}r^4$ . Let  $\mathcal{H}$  be an intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq 2$ . If  $\mathcal{H}$  has the maximum number of hyperedges, then  $\mathcal{H}$  is a 2-kernel system.

*Proof.* We distinguish three cases based on the minimum transversal size  $\tau(\mathcal{H})$  of  $\mathcal{H}$ . Case 1:  $\tau(\mathcal{H}) = 1$ .

Then there is a vertex x in each hyperedge of  $\mathcal{H}$ . Fix a hyperedge  $h \in E(\mathcal{H})$  and observe that the (r-1)-set  $h \setminus \{x\}$  is contained in exactly one hyperedge which violates the positive co-degree condition.

Case 2:  $\tau(\mathcal{H}) \geq 3$ .

Then Lemma 10 gives

$$|E(\mathcal{H})| \le r^3 \binom{n-3}{r-3}$$

which for  $n \ge \frac{1}{3}r^4$  is smaller than  $3\binom{n-3}{r-2}$ , a contradiction. Case 3:  $\tau(\mathcal{H}) = 2$ .

Let  $\{x, y\}$  be a minimum transversal of  $\mathcal{H}$ . Consider the (r-1)-uniform hypergraphs  $\mathcal{H}_x = \{h \setminus \{x\} : h \in E(\mathcal{H}) \text{ and } h \cap \{x, y\} = \{x\}\}$  and  $\mathcal{H}_y = \{h \setminus \{y\} : h \in E(\mathcal{H}) \text{ and } h \cap \{x, y\} = \{y\}\}$ . First observe that this pair of hypergraphs is cross-intersecting as  $\mathcal{H}$  is intersecting. Now observe that any hyperedge  $h \in E(\mathcal{H}_x)$  is a set of size r-1 that is contained in a hyperedge of  $\mathcal{H}$ . Thus, h has co-degree at least 2, therefore must be a member of  $\mathcal{H}_y$ . This implies that  $\mathcal{H}_x = \mathcal{H}_y$ , therefore  $\mathcal{H}_x$  is intersecting.

Now if  $\mathcal{H}_x = \mathcal{H}_y$  is not a maximal star, then by the Erdős-Ko-Rado theorem we have

$$|E(\mathcal{H})| < 2\binom{n-3}{r-2} + \binom{n-2}{r-2} = 3\binom{n-3}{r-2} + \binom{n-3}{r-3},$$

i.e.,  $\mathcal{H}$  has fewer hyperedges than a 2-kernel system, a contradiction. Therefore, every hyperedge of  $\mathcal{H}_x$  contains a fixed vertex z. This implies that every hyperedge of  $\mathcal{H}$  contains at least two of  $\{x, y, z\}$ , i.e., maximality implies that  $\mathcal{H}$  is a 2-kernel system.  $\Box$ 

We now turn to the case when k = 3. We will need two lemmas. The first is due to Frankl (Proposition 1.4 in [7]).

**Lemma 12** (Frankl, [7]). Let  $\mathcal{A}$  and  $\mathcal{B}$  be cross-intersecting hypergraphs on vertex set [N] such that  $\mathcal{A}$  is a-uniform and  $\mathcal{B}$  is (a + 1)-uniform and intersecting. If N > 2a + 1, then

$$|\mathcal{A}| + |\mathcal{B}| \le \binom{N}{a},$$

with equality if and only if either  $\mathcal{B}$  is empty and  $\mathcal{A}$  has size  $\binom{N}{a}$  or both  $\mathcal{A}$  and  $\mathcal{B}$  are maximal stars containing the same a fixed vertex q.

The next lemma gives the size of a minimum transversal for a hypergraph with minimum co-degree at least k.

**Lemma 13.** Fix  $r \geq 3$ ,  $k \geq 2$  and let  $n \geq 2\binom{2k-1}{k}^{-1}(r-k)r^{k+1}$ . Let  $\mathcal{H}$  be an intersecting *r*-uniform *n*-vertex hypergraph with minimum positive co-degree  $\delta_{r-1}^+(\mathcal{H}) \geq k$ . If  $\mathcal{H}$  has the maximum number of hyperedges, then  $\mathcal{H}$  has transversal number  $\tau(\mathcal{H}) = k$ .

Proof. First suppose that  $\tau(\mathcal{H}) < k$ . As in the proof of Lemma 6, let X be a minimal transversal for  $\mathcal{H}$  and consider a hyperedge h that intersects X in exactly one element. Such a hyperedge exists as otherwise X is not minimal. The (r-1)-set  $h \setminus X$  is contained in at most k - 1 hyperedges of  $\mathcal{H}$ ; one for each element of X. This contradicts the co-degree condition on  $\mathcal{H}$ .

Now suppose that  $\tau(\mathcal{H}) > k$ . Lemma 10 gives  $|E(\mathcal{H})| \leq r^{k+1} \binom{n-k-1}{r-k-1}$ . On the other hand, our construction has at least  $\binom{2k-1}{k} \binom{n-2k+1}{r-k}$  hyperedges. Therefore, for  $n \geq 2\binom{2k-1}{k}^{-1}(r-k)r^{k+1}$  we have a contradiction, thus,  $\tau(\mathcal{H}) = k$ .

Finally, we need a technical definition to construct auxiliary hypergraphs from  $\mathcal{H}$ .

**Definition 14.** Let  $\mathcal{H}$  be an r-uniform hypergraph and let T be a fixed set of vertices in  $\mathcal{H}$ . For a subset  $S \subset T$  define

$$\mathcal{H}_{S}^{T} = \{h \setminus S : h \in E(\mathcal{H}) \text{ and } h \cap T = S\},\$$

*i.e.*,  $\mathcal{H}_{S}^{T}$  is the (r-|S|)-uniform hypergraph constructed by removing S from each hyperedge of  $\mathcal{H}$  that intersects T in exactly S.

For ease of notation we will often denote  $\mathcal{H}_S^T$  by  $\mathcal{H}_{x_1x_2...x_s}^T$  when  $S = \{x_1, x_2, \ldots, x_s\}$ .

**Theorem 15.** Fix  $r \geq 3$  and let  $n \geq 2r^5$ . Let  $\mathcal{H}$  be an intersecting r-uniform n-vertex hypergraph with minimum positive co-degree  $\delta^+_{r-1}(\mathcal{H}) \geq 3$ . If  $\mathcal{H}$  has the maximum number of hyperedges, then  $\mathcal{H}$  is a 3-kernel system.

*Proof.* By Lemma 13 we may assume the minimum transversal size of  $\mathcal{H}$  is  $\tau(\mathcal{H}) = 3$ . Let  $X = \{x, y, z\}$  be a minimum transversal of  $\mathcal{H}$ .

Consider the three (r-1)-uniform hypergraphs  $\mathcal{H}_x^X$ ,  $\mathcal{H}_y^X$  and  $\mathcal{H}_z^X$ . First observe that any pair of these hypergraphs is cross-intersecting as  $\mathcal{H}$  is intersecting. Now observe that any hyperedge  $h \in E(\mathcal{H}_x^X)$  is a set of size r-1 that is contained in a hyperedge of  $\mathcal{H}$ , therefore h has co-degree at least 3. This implies that h is also a member of  $\mathcal{H}_y^X$  and  $\mathcal{H}_z^X$ . Thus, all three hypergraphs  $\mathcal{H}_x^X, \mathcal{H}_y^X, \mathcal{H}_z^X$  are the same. Moreover, this implies that  $\mathcal{H}_x^X$ is intersecting.

We distinguish three cases based on  $\tau(\mathcal{H}_x^X)$ .

Case 1:  $\tau(\mathcal{H}_x^X) = 1.$ 

Let u be a transversal of  $\mathcal{H}_x^X$ . Every hyperedge of  $\mathcal{H}_x^X, \mathcal{H}_y^X, \mathcal{H}_z^X$  contains u, therefore, every hyperedge of  $\mathcal{H}$  contains at least two vertices from  $\{x, y, z, u\}$ . Put  $T = X \cup \{u\} = \{x, y, z, u\}$ .

Claim 16. The six hypergraphs  $\mathcal{H}_{ab}^T$  for  $a, b \in T = \{x, y, z, u\}$  are equal.

*Proof.* It is enough to show that  $E(\mathcal{H}_{ab}^T) \subseteq E(\mathcal{H}_{ac}^T)$  for any three vertices  $a, b, c \in T$ . Let  $h \in E(\mathcal{H}_{ab}^T)$  and consider the (r-1)-set  $h \cup \{a\}$ . By the co-degree condition on  $\mathcal{H}$  we have that  $h \cup \{a\}$  is contained in at least three hyperedges. Each of these hyperedges includes at least two vertices from  $\{x, y, z, u\}$ , so  $h \cup \{a\}$  is contained in the hyperedge  $h \cup \{a, c\}$ , i.e.,  $h \in E(\mathcal{H}_{ac}^T)$ .

Observe that  $\mathcal{H}_{xy}^T$  and  $\mathcal{H}_{zu}^T$  are cross-intersecting, which implies that  $\mathcal{H}_{xy}^T$  is intersecting. Now if  $\mathcal{H}_{xy}^T$  is not a maximal star, then by the Erdős-Ko-Rado theorem we have

$$|E(\mathcal{H})| < 6\binom{n-5}{r-3} + 4\binom{n-4}{r-3} + \binom{n-4}{r-4} = 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5},$$

i.e.,  $\mathcal{H}$  has fewer hyperedges than a 3-kernel system, a contradiction. Therefore, every hyperedge of  $\mathcal{H}_{xy}$  contains a fixed vertex v. As the six hypergraphs  $\mathcal{H}_{ab}^T$  for  $a, b \in$  $T = \{x, y, z, u\}$  are equal, we have that every hyperedge of  $\mathcal{H}$  contains at least three of  $\{x, y, z, u, v\}$ , i.e., maximality implies that  $\mathcal{H}$  is a 3-kernel system. **Case 2:**  $\tau(\mathcal{H}_x^X) = 2$ .

Let u, v be a minimal transversal of  $\mathcal{H}_x^X$ , i.e., every hyperedge of  $\mathcal{H}_x^X$  contains at least one of u, v. As  $\mathcal{H}_x^X = \mathcal{H}_y^X = \mathcal{H}_z^X$ , we have that every hyperedge of  $\mathcal{H}$  contains at least two vertices from  $T = \{x, y, z, u, v\}$ . Moreover,  $\mathcal{H}_{xu}^T = \mathcal{H}_{yu}^T = \mathcal{H}_{zu}^T$  and  $\mathcal{H}_{xv}^T = \mathcal{H}_{yv}^T = \mathcal{H}_{zv}^T$ and each of these (r-2)-uniform hypergraphs is non-empty (as otherwise u, v would not be a minimal transversal). Note that there is no hyperedge that intersects T in exactly uand v, so  $\mathcal{H}_{uv}^T$  is empty. For simplicity, we consider the empty hypergraph as intersecting.

Claim 17. The hypergraph  $\mathcal{H}_{ab}^{T}$  is intersecting for every  $a, b \in T = \{x, y, z, u, v\}$ .

Proof. Suppose not. Then there are hyperedges  $A, B \in E(\mathcal{H}_{ab}^T)$  such that  $A \cap B = \emptyset$ . By the co-degree condition, the (r-1)-set  $A \cup \{a\}$  is contained in at least three hyperedges of  $\mathcal{H}$ . Since each hyperedge of  $\mathcal{H}$  contains at least two elements from T, there is a hyperedge  $A \cup \{a, c\}$  where  $c \in T \setminus \{a, b\}$ . Similarly, the (r-1)-set  $B \cup \{b\}$  is contained in some hyperedge  $B \cup \{b, d\}$  where  $d \in T \setminus \{a, b, c\}$ . However, the hyperedges  $A \cup \{a, c\}$  and  $B \cup \{b, d\}$  are disjoint which violates the intersecting property of  $\mathcal{H}$ .

Now for any  $a, b \in T$  we have  $\mathcal{H}_{T \setminus \{a,b\}}^T$  and  $\mathcal{H}_{ab}^T$  are cross-intersecting,  $\mathcal{H}_{T \setminus \{a,b\}}^T$  is (r-3)-uniform and  $\mathcal{H}_{ab}^T$  is (r-2)-uniform and intersecting. Therefore, as n-5 > 2(r-3) + 1, we may apply Lemma 12 to get

$$|E(\mathcal{H}_{ab}^T)| + |E(\mathcal{H}_{T\setminus\{a,b\}}^T)| \le \binom{n-5}{r-3}$$

Thus

$$|E(\mathcal{H})| = \sum_{S \subseteq T} |E(\mathcal{H}_S^T)| \le 10 \binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}.$$

As  $\mathcal{H}$  has the maximum number of hyperedges, we must have equality above. Therefore, we must have that for every  $a, b \in T$ , the hypergraphs  $\mathcal{H}_{T \setminus \{a,b\}}^T$  and  $\mathcal{H}_{ab}^T$  have the form of one of the two extremal constructions in Lemma 12. In particular,  $\mathcal{H}_{ab}^T$  is either empty or a maximal star. As  $\mathcal{H}_{xu}^T = \mathcal{H}_{yu}^T = \mathcal{H}_{zu}^T$  and  $\mathcal{H}_{xv}^T = \mathcal{H}_{yv}^T = \mathcal{H}_{zv}^T$  are non-empty, each is a maximal star. The hypergraphs  $\mathcal{H}_{xu}^T$  and  $\mathcal{H}_{yv}^T$  are cross-intersecting which implies that all six of these these maximal stars share the same fixed vertex q. Therefore, we can replace minimal transversal u, v of  $\mathcal{H}_x^X$  with q, a contradiction.

Case 3:  $\tau(\mathcal{H}_x^X) \geq 3$ .

Then Lemma 10 gives

$$|E(\mathcal{H}_x^X)| \le (r-1)^3 \binom{(n-1)-3}{(r-1)-3} \le r^3 \binom{n-4}{r-4}$$

The remaining hyperedges of  $\mathcal{H}$  are counted by  $\mathcal{H}_{xyz}^X$  and  $\mathcal{H}_{ab}^X$  for  $a, b \in \{x, y, z\}$ . We need a simple claim. Recall that the *shadow* of an *r*-uniform hypergraph  $\mathcal{G}$  is the collection of all (r-1)-sets contained in a hyperedge of  $\mathcal{G}$ . We denote the shadow of  $\mathcal{G}$  by  $\Delta(\mathcal{G})$ .

**Claim 18.** For each hyperedge  $h \in E(\mathcal{H}_{yz}^X)$  there is some hyperedge  $g \in E(\mathcal{H}_x^X)$  that contains h. Thus,

$$|E(\mathcal{H}_{yz}^X)| \le |\Delta(\mathcal{H}_x^X)|.$$

Proof. Let h be an arbitrary hyperedge of  $\mathcal{H}_{yz}^X$ . Consider the (r-1)-set  $A = h \cup \{y\}$ . The set A has co-degree at least 3, so it is contained in three hyperedges of  $\mathcal{H}$ ; one such hyperedge is  $A \cup \{z\}$ , another could be  $A \cup \{x\}$ , so there exists at least one hyperedge of the form  $A \cup \{w\}$  where  $w \notin \{x, y, z\}$ . However,  $A \cap \{x, y, z\} = \{y\}$ , so  $(A \cup \{w\}) \setminus \{y\} \in E(\mathcal{H}_y^X) = E(\mathcal{H}_x^X)$ .

By Claim 18 we have

$$|E(\mathcal{H}_{yz}^X)| \le |\Delta(\mathcal{H}_x^X)| \le (r-1)|E(\mathcal{H}_x^X)| \le r^4 \binom{n-4}{r-4}.$$

Finally,  $|E(\mathcal{H}_{xyz}^X)| \leq \binom{n-3}{r-3}$ . Thus,

$$|E(\mathcal{H})| \le {\binom{n-3}{r-3}} + 3(r^4 + r^3) {\binom{n-4}{r-4}}$$

which is less than  $10\binom{n-5}{r-3}$  for  $n \ge 2r^5$ , a contradiction.

In order to extend the technique used in this section to reprove our theorem for minimum positive co-degree  $k \ge 4$  we would need to distinguish additional cases based on the transversal size of  $\mathcal{H}_x^X$ . Some of these cases can be addressed with Lemmas 10 and 12, but probably new ideas will be needed.

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## Appendix

We now give an improvement to Lemma 5 which we believe is of independent interest. Recall that f(r, p) is the minimum integer such that an r-uniform hypergraph with f(r, p) hyperedges contains a sunflower with p petals.

**Lemma 19.** Fix integers  $r \ge 3$ ,  $k \ge 1$  and  $p \ge 1$  and let n be large enough. If  $\mathcal{G}$  is an r-uniform n-vertex hypergraph with

$$|E(\mathcal{G})| \ge 2r^{r-k}f(r, pr^{r-k})\binom{n-k-1}{r-k-1},$$

then  $\mathcal{G}$  contains a sunflower with p petals and core of size at most k.

This replaces the value of  $C = (pr2^r)^{2^r}$  in Lemma 5 with  $C = 2r^{r-k}f(r, pr^{r-k})$  which is significantly smaller when using the bound on  $f(r, pr^{r-k})$  from [1].

*Proof.* For the sake of a contradiction, suppose that  $\mathcal{G}$  contains no sunflower with p petals and core of size at most k.

Iteratively remove from  $\mathcal{G}$  a sunflower  $\mathcal{S}$  with exactly  $pr^{c(\mathcal{S})-k}$  petals such that at each step we choose a sunflower with minimum available core size  $c(\mathcal{S})$ . Let t be the number of steps in this sunflower removal procedure. Note that t grows with n as at each step we remove at most  $pr^{r-k}$  hyperedges from  $\mathcal{G}$  and we only need constant number of hyperedges to guarantee the existence of a sunflower with  $pr^{c(\mathcal{S})-k}$  petals. In particular, we have

$$t \ge \frac{|E(\mathcal{G})| - f(r, pr^{r-k})}{pr^{r-k}} \ge \frac{|E(\mathcal{G})|}{2pr^{r-k}}$$

for n large enough.

The core of each removed sunflower is of size at least k + 1 and at most r - 1. Therefore, there is some integer s such that there are at least t/r cores of size s among the removed sunflowers. Some of these cores may be identical. Let us compute the maximum multiplicity of a core Y. There are at most  $\binom{n-|Y|}{r-|Y|}$  hyperedges containing Y and each removed sunflower with core Y has exactly  $pr^{|Y|-k}$  hyperedges. Therefore, the maximum multiplicity of a core Y is at most

$$\frac{1}{pr^{|Y|-k}} \binom{n-|Y|}{r-|Y|} \le \frac{1}{pr} \binom{n-k-1}{r-k-1}$$

for  $n \ge r$ . Therefore, there is a collection of at least

$$(t/r)pr\binom{n-k-1}{r-k-1}^{-1} \ge p\frac{|E(\mathcal{G})|}{2pr^{r-k}}\binom{n-k-1}{r-k-1}^{-1} \ge f(r, pr^{r-k})$$

distinct cores of size s. Let  $Y_1, Y_2, \ldots, Y_q$  be these cores and let  $S_i$  be the sunflower with core  $Y_i$  for  $i = 1, 2, \ldots, q$ . Note that each of these sunflowers has exactly  $pr^{s-k}$  petals.

Let t be the first step in the sunflower removal procedure in which a sunflower with core of size s is chosen to be removed. This implies that all later cores are of size at least s. Now we will show that there is a sunflower  $\mathcal{B}$  with core of size less than s and  $pr^{c(\mathcal{B})-k}$  petals among the hyperedges in the sunflowers  $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_q$ . Before removing the sunflower in step t, all hyperedges of the sunflowers  $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_q$  are still in  $\mathcal{H}$ . Therefore, the sunflower  $\mathcal{B}$  with core of size less than s could be chosen in step t, this will contradict the choice of t.

We may think of the s-sets  $Y_1, \ldots, Y_q$  as an s-uniform hypergraph on the vertex set of  $\mathcal{H}$ . As  $q \geq f(r, pr^{r-k}) \geq f(s, pr^{r-k}) \geq f(s, pr^{s-k})$ , the s-sets  $Y_1, \ldots, Y_q$  contain an s-uniform sunflower  $\mathcal{A}$  with  $pr^{s-k}$  petals and core  $Y^*$  of size less than s. By relabelling, we may suppose that  $Y_i$  is a member of  $\mathcal{A}$  for  $i = 1, 2, \ldots, pr^{s-r}$ . Note that the petals  $Y_i \setminus Y^*$  of  $\mathcal{A}$  are pairwise disjoint by definition. The sunflower  $\mathcal{A}$  is not in the hypergraph  $\mathcal{H}$  as it is s-uniform. However, each hyperedge of  $\mathcal{A}$  is the core of some sunflower  $\mathcal{S}_i$  in  $\mathcal{H}$ . Therefore, we will use the members of  $\mathcal{A}$  to identify an r-uniform sunflower  $\mathcal{B}$  with core  $Y^*$  in  $\mathcal{H}$ . The main idea will be carefully choose a petal from each sunflower  $\mathcal{S}_i$  whose core is a member of  $\mathcal{A}$ . To this end, define  $\mathcal{B}$  as follows:

First pick any hyperedge of  $S_1$ ; denote it by  $h_1$ . Now suppose we have chosen  $\ell$  hyperedges  $h_1, h_2, \ldots, h_\ell$  that form a sunflower with core  $Y^*$ . The union of these hyperedges contains  $\ell(r - |Y^*|)$  vertices outside of  $Y^*$ . Therefore, as long as

$$pr^{s-k} > \ell(r - |Y^*|),$$
 (1)

there is a petal  $Y_i \setminus Y^*$  of  $\mathcal{A}$  that is disjoint from each of the hyperedges  $h_1, h_2, \ldots, h_\ell$ . The corresponding sunflower  $\mathcal{S}_i$  with core  $Y_i$  has

$$pr^{s-k} > \ell(r - |Y^*|)$$

petals by (1). Therefore, there is a petal P of  $S_i$  that is also disjoint from the hyperedges in  $h_1, h_2, \ldots, h_\ell$ . Let  $h_{\ell+1}$  be the hyperedge  $P \cup Y_i$ . Now we have a sunflower with  $\ell + 1$ petals and core  $Y^*$ . We may repeat this procedure as long as  $\ell$  satisfies (1), i.e., until  $\ell = pr^{s-k-1}$ . This implies that the number of petals in sunflower  $\mathcal{B}$  is at least

$$pr^{s-k-1}$$

As  $\mathcal{B}$  has core  $Y^*$  of size  $c(\mathcal{B}) < s$  we have a contradiction to the choice of sunflower in step t.