# $L^{\infty}$ norm error estimates for HDG methods applied to the Poisson equation with an application to the Dirichlet boundary control problem 

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#### Abstract

We prove quasi-optimal $L^{\infty}$ norm error estimates (up to logarithmic factors) for the solution of Poisson's problem by the standard Hybridizable Discontinuous Galerkin (HDG) method. Although such estimates are available for conforming and mixed finite element methods, this is the first proof for HDG. The method of proof is motivated by known $L^{\infty}$ norm estimates for mixed finite elements. We show two applications: the first is to prove optimal convergence rates for boundary flux estimates, and the second is to prove that numerically observed convergence rates for the solution of a Dirichlet boundary control problem are to be expected theoretically. Numerical examples show that the predicted rates are seen in practice.


## 1 Introduction

In this paper we derive $L^{\infty}$ norm estimates for the standard hybridizable discontinuous Galerkin (HDG) method applied to a diffusion problem. The problem is posed on a bounded convex polyhedral domain $\Omega \subset \mathbb{R}^{2}$. We assume the data is given as follows: the diffusivity $c \in\left(W^{1, \infty}(\Omega)\right)^{2 \times 2}$ is a uniformly bounded positive definite symmetric matrix-valued function, and the functions $f \in L^{2}(\Omega)$ and $g \in H^{1 / 2}(\partial \Omega)$. Then we seek to approximate the solution $(u, \boldsymbol{q})$ of the following elliptic system:

$$
\begin{align*}
c \boldsymbol{q}+\nabla u & =0 & & \text { in } \Omega,  \tag{1.1a}\\
\nabla \cdot \boldsymbol{q} & =f & & \text { in } \Omega,  \tag{1.1b}\\
u & =g & & \text { on } \partial \Omega . \tag{1.1c}
\end{align*}
$$

In particular, we shall prove quasi-optimal $L^{\infty}$ error estimates (up to logarithmic factors) for the HDG approximation to $u$ and $\boldsymbol{q}$. We also verify that, after a standard procedure, the post-processed solution denoted $u_{h}^{\star}$ is super-convergent in the $L^{\infty}$ norm.

Quasi-optimal $L^{\infty}$ norm estimates on general quasi-uniform meshes for the conforming finite element method were first proved by Scott [31] in 1976. The method of proof is based on weighted $L^{2}$ norms and was extended in [10, 13, 30, 33] to mixed methods for elliptic equations and in [14, 15] to the Stokes equations. Another technique was developed in the series of papers by Schatz and Wahlbin [27-29]. They use dyadic decomposition of the domain and require local energy estimates

[^0]together with sharp pointwise estimates for the corresponding components of the Greens matrix. For smooth domains such a technique was successfully used in [6] for mixed methods, in [18] for discontinuous Galerkin (DG) methods and in [5] for local DG methods. This technique was also applied on a non-smooth domain for the Stokes equations, see Guzmán and Leykekhman [19].

The HDG method for elliptic equations was devised by Cockburn et al. [8] and was analyzed using a special projection in 9 . Since there is a strong relation between the HDG method and mixed finite element methods (see [8]), it is reasonable to ask if similar $L^{\infty}(\Omega)$ norm estimates on general quasi-uniform meshes could be obtained for the HDG method. To the best of our knowledge, there is no such result in the literature. In Section 3, we give quasi-optimal $L^{\infty}$ norm estimates for the flux variable $\boldsymbol{q}$ and scalar variable $u$ (see Theorem 2). One advantage of the HDG method is that we can obtain superconvergent rates of convergence for a post processed approximation $u_{h}^{\star}$ to $u$ in the $L^{2}(\Omega)$ norm [8]. In Theorem 2, we show that the postprocessed solution also enjoys superconvergent rates in the $L^{\infty}$ norm. We present a numerical test in Example 1 (see Table 1) to confirm our theoretical results from Theorem 2. As mentioned in [24], we can use our $L^{\infty}$ norm estimates to improve $L^{2}(\Gamma)$ norm estimates on an interface $\Gamma$, see Theorem 3. The numerical test in Example 1 (see Table 2) confirms the theoretical result from Theorem 3, It is worthwhile to mention that a standard analysis of convergence on an interface (usually via the trace theorem) only gives a suboptimal convergence rate.

The optimal $L^{2}(\Gamma)$ norm estimates on an interface $\Gamma$ or on the boundary of the domain have many applications. One example [24] is where some complex problems require the use of a variety of models in different parts of the computational domain, which in turn are coupled through the normal flux across common interfaces. On the level of numerical methods, this entails a need to understand and quantify the discretization error in the normal flux at interfaces [24]. Another example appears in the problem of Dirichlet boundary control (DBC) of PDEs with $L^{2}(\partial \Omega)$-regularization, where the normal derivative naturally arises in the discrete optimality system. Hence, the estimation of the error in the normal derivative plays an essential role in the error analysis of the DBC of PDEs, see [1, 20, 25, 26, 34] for more details. In recent papers where HDG methods have been sucessful applied to the DBC of PDEs ( $4,16,17,21,22])$, the analysis for the control is optimal in the sense of regularity and suboptimal for other variables. Furthermore, numerical experiments show that the discrete control can achieve optimal convergence with respect to the polynomial degree if the control is smooth enough. However, the analysis in the above mentioned HDG papers is suboptimal in this situation. In Section 5, we use the improved $L^{2}$ norm estimates on the boundary in Theorem 3 to obtain an optimal convergence rate for both the control and the other variables, see Theorem 4, The numerical test in Example 2 confirms our theoretical result.

## 2 HDG formulation and preliminary material

In this section, we shall give the HDG formulation of Equation (1.1), and introduce some standard auxiliary projections. Our main result in this section is to extend the $L^{2}$ norm estimates for the auxiliary projections used in the error analysis of HDG to $L^{p}$ norms $(1 \leq p \leq \infty)$, see Theorem 1. This is one essential step of the paper. Although our final $L^{\infty}$ norm estimates require the domain to be two dimensional and convex, it is worth mentioning that we do not need these restrictions in Theorem 1. Hence, in the present section, we assume that $\Omega \subset \mathbb{R}^{d},(d=2,3)$, and do not assume convexity.

Throughout the paper we adopt the standard notation $W^{m, p}(D)$ for Sobolev spaces on a
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bounded domain $D \subset \mathbb{R}^{d}(d=2,3)$ with norm $\|\cdot\|_{W^{m, p}(D)}$ and seminorm $|\cdot|_{W^{m, p}(D)}$ :

$$
\begin{aligned}
\|u\|_{W^{m, p}(D)}^{p} & =\sum_{|i| \leq m} \int_{D}\left|D^{i} u\right|^{p} \mathrm{~d} \boldsymbol{x}, \\
|u|_{W^{m, p}(D)}^{p} & =\sum_{|i|=m} \int_{D}\left|D^{i} u\right|^{p} \mathrm{~d} \boldsymbol{x},
\end{aligned}
$$

where $i$ is a multi-index and $D^{i}$ is the corresponding partial differential operator of order $|i|$. We denote $W^{m, 2}(D)$ by $H^{m}(D)$ with norm $\|\cdot\|_{H^{m}(D)}$ and seminorm $|\cdot|_{H^{m}(D)}$. Specifically, $H_{0}^{1}(D)=$ $\left\{v \in H^{1}(D): v=0\right.$ on $\left.\partial D\right\}$. We denote the $L^{2}$-inner products on $L^{2}(D)$ and $L^{2}(S)$ by

$$
\begin{array}{ll}
(v, w)_{D}=\int_{D} v w & \forall v, w \in L^{2}(D) \\
\langle v, w\rangle_{S}=\int_{S} v w & \forall v, w \in L^{2}(S)
\end{array}
$$

where $S \subset \partial D$. Finally, we define the space $\boldsymbol{H}(\operatorname{div}, \Omega)$ as

$$
\boldsymbol{H}(\operatorname{div}, \Omega)=\left\{\boldsymbol{v} \in\left[L^{2}(\Omega)\right]^{d}: \nabla \cdot \boldsymbol{v} \in L^{2}(\Omega)\right\} .
$$

Let $\mathcal{T}_{h}$ be a collection of disjoint simplices that partition $\Omega$ and satisfy the usual finite element conditions. We denote by $\partial \mathcal{T}_{h}$ the set $\left\{\partial K: K \in \mathcal{T}_{h}\right\}$. For an element $K$ of the mesh $\mathcal{T}_{h}$, let $F=\partial K \cap \partial \Omega$ denotes the boundary face of $K$ having non-zero $d-1$ dimensional Lebesgue measure. Let $\mathcal{F}_{h}^{\partial}$ be the set of boundary faces and $\mathcal{F}_{h}$ denote the set of all faces. We define the following mesh dependent norms and spaces by

$$
\begin{aligned}
(w, v)_{\mathcal{T}_{h}}=\sum_{K \in \mathcal{T}_{h}}(w, v)_{K}, & \langle\zeta, \rho\rangle_{\partial \mathcal{T}_{h}}=\sum_{K \in \mathcal{T}_{h}}\langle\zeta, \rho\rangle_{\partial K}, \\
H^{1}\left(\mathcal{T}_{h}\right)=\prod_{K \in \mathcal{T}_{h}} H^{1}(K), & L^{2}\left(\partial \mathcal{T}_{h}\right)=\prod_{K \in \mathcal{T}_{h}} L^{2}(\partial K) .
\end{aligned}
$$

Let $\mathcal{P}^{k}(D)$ denote the set of polynomials of degree at most $k$ on a domain $D$. We introduce the discontinuous finite element spaces used in the HDG method as follows:

$$
\begin{aligned}
\boldsymbol{V}_{h} & :=\left\{\boldsymbol{v}_{h} \in\left[L^{2}(\Omega)\right]^{d}:\left.\boldsymbol{v}_{h}\right|_{K} \in\left[\mathcal{P}^{k}(K)\right]^{d}, \forall K \in \mathcal{T}_{h}\right\}, \\
W_{h} & :=\left\{w_{h} \in L^{2}(\Omega):\left.w_{h}\right|_{K} \in \mathcal{P}^{k}(K), \forall K \in \mathcal{T}_{h}\right\}, \\
\widehat{W}_{h} & :=\left\{\widehat{w}_{h} \in L^{2}\left(\mathcal{F}_{h}\right):\left.\widehat{w}_{h}\right|_{F} \in \mathcal{P}^{k}(F), \forall F \in \mathcal{F}_{h} ;\left.\widehat{w}_{h}\right|_{F}=0, \forall F \in \mathcal{F}_{h}^{\partial}\right\} .
\end{aligned}
$$

### 2.1 HDG formulation

To simplify the presentation, we assume the Dirichlet boundary condition is homogeneous, i.e., $g=0$. Then the HDG method of Cockburn et al. [8] seeks the flux $\boldsymbol{q}_{h} \in \boldsymbol{V}_{h}$, the scalar variable $u_{h} \in W_{h}$ and its numerical trace $\widehat{u}_{h} \in \widehat{W}_{h}$ satisfying

$$
\begin{align*}
\left(c \boldsymbol{q}_{h}, \boldsymbol{v}_{h}\right)_{\mathcal{T}_{h}}-\left(u_{h}, \nabla \cdot \boldsymbol{v}_{h}\right)_{\mathcal{T}_{h}}+ & \left\langle\widehat{u}_{h}, \boldsymbol{v}_{h} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=  \tag{2.1a}\\
-\left(\boldsymbol{q}_{h}, \nabla w_{h}\right)_{\mathcal{T}_{h}}+ & \left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, w_{h}\right\rangle_{\partial \mathcal{T}_{h}}=\left(f, w_{h}\right)_{\mathcal{T}_{h}},  \tag{2.1b}\\
& \left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, \widehat{w}_{h}\right\rangle_{\partial \mathcal{T}_{h}}=0, \tag{2.1c}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$. The numerical traces on $\partial \mathcal{T}_{h}$ are defined by [9]

$$
\begin{equation*}
\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}=\boldsymbol{q}_{h} \cdot \boldsymbol{n}+\tau\left(u_{h}-\widehat{u}_{h}\right) \quad \text { on } \partial \mathcal{T}_{h}, \tag{2.1d}
\end{equation*}
$$

where the stabilization parameter $\tau \in L^{\infty}\left(\mathcal{F}_{h}\right)$ is uniformly positive and bounded. For simplicity, we consider the stabilization function $\tau$ to be constant on the boundary of each element.

After computing the solution $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right)$ of (2.1), we can use the following element-by-element postprocessing to find $\left.u_{h}^{\star}\right|_{K} \in \mathcal{P}^{k+1}(K)$ such that for all $\left(z_{h}, w_{h}\right) \in\left[\mathcal{P}^{k+1}(K)\right]^{\perp} \times \mathcal{P}^{0}(K)$

$$
\begin{align*}
\left(\nabla u_{h}^{\star}, \nabla z_{h}\right)_{K} & =-\left(c \boldsymbol{q}_{h}, \nabla z_{h}\right)_{K},  \tag{2.2a}\\
\left(u_{h}^{\star}, w_{h}\right)_{K} & =\left(u_{h}, w_{h}\right)_{K}, \tag{2.2b}
\end{align*}
$$

where $\left[\mathcal{P}^{k+1}(K)\right]^{\perp}=\left\{z_{h} \in \mathcal{P}^{k+1}(K):\left(z_{h}, 1\right)_{K}=0\right\}$.
To shorten lengthy equations, we define the following HDG bilinear form $\mathscr{B}: \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right) \times H^{1}\left(\mathcal{T}_{h}\right) \times$ $L^{2}\left(\partial \mathcal{T}_{h}\right) \times \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right) \times H^{1}\left(\mathcal{T}_{h}\right) \times L^{2}\left(\partial \mathcal{T}_{h}\right) \rightarrow \mathbb{R}$ by

$$
\begin{align*}
\mathscr{B}(\boldsymbol{q}, u, \widehat{u} ; \boldsymbol{v}, w, \widehat{w})= & (c \boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_{h}}-(u, \nabla \cdot \boldsymbol{v})_{\mathcal{T}_{h}}+\langle\widehat{u}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_{h}} \\
& -(\nabla \cdot \boldsymbol{q}, w)_{\mathcal{T}_{h}}-\langle\tau(u-\widehat{u}), w-\widehat{w}\rangle_{\partial \mathcal{T}_{h}}+\langle\boldsymbol{q} \cdot \boldsymbol{n}, \widehat{w}\rangle_{\partial \mathcal{T}_{h}} . \tag{2.3}
\end{align*}
$$

By the definition of $\mathscr{B}$ in (2.3), we can rewrite the HDG formulation of system (2.1), as follows: find $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$ such that

$$
\begin{equation*}
\mathscr{B}\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=-\left(f, w_{h}\right)_{\mathcal{T}_{h}} \tag{2.4}
\end{equation*}
$$

for all $\left(\boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$. Moreover, the exact solution $(\boldsymbol{q}, u)$ also satisfies equation (2.4), i.e.,

$$
\begin{equation*}
\mathscr{B}\left(\boldsymbol{q}, u, u ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=-\left(f, w_{h}\right) \mathcal{T}_{h} \tag{2.5}
\end{equation*}
$$

for all $\left(\boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$.
From [3, Lemma 2] we recall the following stability result.
Lemma 1. For any $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$, we have

$$
\mathscr{B}\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h} ; \boldsymbol{q}_{h},-u_{h},-\widehat{u}_{h}\right)=\left(c \boldsymbol{q}_{h}, \boldsymbol{q}_{h}\right)_{\mathcal{T}_{h}}+\left\langle\tau\left(u_{h}-\widehat{u}_{h}\right), u_{h}-\widehat{u}_{h}\right\rangle_{\partial \mathcal{T}_{h}} .
$$

The following lemma shows that the bilinear form $\mathscr{B}$ is symmetric and is proved by integration by parts. We do not provide details.
Lemma 2. For any $(\boldsymbol{q}, u, \widehat{u} ; \boldsymbol{v}, w, \widehat{w}) \in \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right) \times H^{1}\left(\mathcal{T}_{h}\right) \times L^{2}\left(\partial \mathcal{T}_{h}\right) \times \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right) \times H^{1}\left(\mathcal{T}_{h}\right) \times L^{2}\left(\partial \mathcal{T}_{h}\right)$, we have

$$
\begin{equation*}
\mathscr{B}(\boldsymbol{q}, u, \widehat{u} ; \boldsymbol{v}, w, \widehat{w})=\mathscr{B}(\boldsymbol{v}, w, \widehat{w} ; \boldsymbol{q}, u, \widehat{u}) . \tag{2.6}
\end{equation*}
$$

### 2.2 Preliminary material

Recall the HDG projection $\Pi_{h}(\boldsymbol{q}, u):=\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}, \Pi_{W} u\right)$ (see 9, equation (2.1a)-(2.1c)]), that satisfies the following equations:

$$
\begin{align*}
\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}, \boldsymbol{v}_{h}\right)_{K} & =\left(\boldsymbol{q}, \boldsymbol{v}_{h}\right)_{K}, & & \forall \boldsymbol{v}_{h} \in\left[\mathcal{P}^{k-1}(K)\right]^{d},  \tag{2.7a}\\
\left(\Pi_{W} u, w_{h}\right)_{K} & =\left(u, w_{h}\right)_{K}, & & \forall w \in \mathcal{P}^{k-1}(K),  \tag{2.7b}\\
\left\langle\boldsymbol{\Pi}_{V} \boldsymbol{q} \cdot \boldsymbol{n}+\tau \Pi_{W} u, \widehat{w}_{h}\right\rangle_{F} & =\left\langle\boldsymbol{q} \cdot \boldsymbol{n}+\tau u, \widehat{w}_{h}\right\rangle_{F}, & & \forall \widehat{w}_{h} \in \mathcal{P}^{k}(F), \tag{2.7c}
\end{align*}
$$

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for all faces $F$ of the simplex $K$. If $k=0$, then (2.7a) and (2.7b) are vacuous and $\Pi_{h}$ is defined solely by ( 2.7 c C$)$. Note that although we denoted the first component of the projection by $\boldsymbol{\Pi}_{V} \boldsymbol{q}$, it depends not just on $\boldsymbol{q}$, but on both $\boldsymbol{q}$ and $u$, as we see from (2.7). The same is true for $\Pi_{W} u$. Hence the notation $\left(\Pi_{V} \boldsymbol{q}, \Pi_{W} u\right)$ for $\Pi_{h}(\boldsymbol{q}, u)$ is somewhat misleading, but its convenience outweighs this disadvantage.

It is worthwhile mentioning that the domain of the projection $\Pi_{h}$ is a subspace of $\left[L^{2}(\Omega)\right]^{d} \times$ $L^{2}(\Omega)$ on which the right hand sides of (2.7) are well defined. We do not require that the two components ( $\boldsymbol{q}, u$ ) satisfy the equation (1.1a).

The well-posedness of $\left(\Pi_{V}, \Pi_{W}\right)$ and its approximation properties are given in the following Lemma 3. The proof can be found in [9, Appendix].

Lemma 3. Suppose $\left.\tau\right|_{\partial K}$ is a positive constant. Then the system (2.7) is uniquely solvable for $\Pi_{V} \boldsymbol{q}$ and $\Pi_{W} u$. Furthermore, there is a constant $C$ independent of $K$ and $\tau$ such that

$$
\begin{align*}
& \left\|\Pi_{W} u-u\right\|_{L^{2}(K)} \leq C h_{K}^{\ell_{u}+1}|u|_{H^{\ell_{u}+1}(K)}+C \frac{h_{K}^{\ell_{\boldsymbol{q}}+1}}{\tau_{K}^{\max }}|\nabla \cdot \boldsymbol{q}|_{\boldsymbol{H}^{\ell_{q}(K)}},  \tag{2.8a}\\
& \left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(K)} \leq C h_{K}^{\ell_{q}+1}|\boldsymbol{q}|_{\boldsymbol{H}^{\ell_{q}+1}(K)}+C h_{K}^{\ell_{u}+1} \tau_{K}^{\star}|u|_{H^{\ell_{u}+1}(K)}, \tag{2.8b}
\end{align*}
$$

for $\ell_{u}, \ell_{\boldsymbol{q}} \in[0, k]$. Here $\tau_{K}^{\star}:=\left.\max \tau\right|_{\partial K \backslash F^{\star}}$, where $F^{\star}$ is a face of $K$ at which $\left.\tau\right|_{\partial K}$ is maximum.
Besides the projections $\Pi_{V}$ and $\Pi_{W}$, in the analysis we also need to introduce the standard local $L^{2}$ projection operators $\Pi_{\ell}^{o}: L^{2}(K) \rightarrow \mathcal{P}^{\ell}(K)$ and $\Pi_{k}^{\partial}: L^{2}(F) \rightarrow \mathcal{P}^{k}(F)$ satisfying

$$
\begin{array}{ll}
\left(\Pi_{\ell}^{o} w, w_{h}\right)_{K}=\left(w, w_{h}\right)_{K}, & \forall w_{h} \in \mathcal{P}^{\ell}(K), \\
\left\langle\Pi_{k}^{\partial} w, \widehat{w}_{h}\right\rangle_{F}=\left\langle w, \widehat{w}_{h}\right\rangle_{F}, & \forall \widehat{w}_{h} \in \mathcal{P}^{k}(F) . \tag{2.9b}
\end{array}
$$

We use $\boldsymbol{\Pi}_{\ell}^{o}$ to denote the local vector $L^{2}$ projection operator, the definition componentwise is the same as local scalar $L^{2}$ projection operator. The next lemma gives the approximation properties of $\Pi_{\ell}^{o}$ and its proof can be found in [32, Theorem 3.3.3, Theorem 3.3.4].

Lemma 4. Let $\ell \geq 0$ be an integer and $\rho \in[1,+\infty]$. If $(\ell+1) \rho<d$, then we require $d, \rho$ and $\ell$ to also satisfy $2 \leq \frac{d \rho}{d-(\ell+1) \rho}$. For $j \in\{0,1, \ldots, \ell+1\}$, if $s_{j}$ satisfies

$$
\begin{cases}\rho \leq s_{j} \leq \frac{d \rho}{d-(\ell+1-j) \rho} & (\ell+1-j) \rho<d  \tag{2.10}\\ \rho \leq s_{j}<\infty & (\ell+1-j) \rho=d \\ \rho \leq s_{j} \leq \infty & (\ell+1-j) \rho>d\end{cases}
$$

then there exists a constant $C$ which is independent of $K$ such that

$$
\begin{align*}
\left\|\nabla^{j}\left(\Pi_{\ell}^{o} u-u\right)\right\|_{L^{s_{j}}(K)} & \leq C h_{K}^{\ell+1-j+\frac{d}{s_{j}}-\frac{d}{\rho}}|u|_{W^{\ell+1, \rho}(K)}  \tag{2.11a}\\
\left\|\nabla^{j}\left(\Pi_{\ell}^{o} u-u\right)\right\|_{L^{s_{j}}(\partial K)} & \leq C h_{K}^{\ell+1-j+\frac{d-1}{s_{j}}-\frac{d}{\rho}}|u|_{W^{\ell+1, \rho}(K)} \tag{2.11b}
\end{align*}
$$

In the analysis, we also need the following standard inverse inequality [32, Theorem 3.4.1].
Lemma 5 (Inverse inequality). Let $k \geq 0$ be an integer, $\mu, \rho \in[1,+\infty]$ and $\eta>0$, then there exists $C$ depend on $k, \mu, \rho, d$ and $\eta$ such that

$$
\begin{equation*}
\left|u_{h}\right|_{t, \mu, K} \leq C h_{K}^{\frac{d}{\mu}-\frac{d}{\rho}-t+s}\left|u_{h}\right|_{s, \rho, K}, \quad \forall u_{h} \in \mathcal{P}_{k}(K), \quad t \geq s \tag{2.12}
\end{equation*}
$$

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In the analysis, not only the $L^{2}$ approximation properties of $\left(\Pi_{V}, \Pi_{W}\right)$ are important, for us, but the $L^{\infty}$ approximation of these projection operators plays an essential role. We provide these estimates in the next theorem.

Theorem 1. Let $k \geq 0$ be an integer and $\rho \in[1,+\infty]$. If $(k+1) \rho<d$, then we need $d, \rho$ and $k$ to satisfy $2 \leq \frac{d \rho}{d-(k+1) \rho}$. For $j \in\{0,1, \ldots, k+1\}$, if $s_{j}$ satisfies (2.10), then

$$
\begin{align*}
& \left\|\Pi_{W} u-u\right\|_{L^{s_{j}}(K)} \leq \frac{C}{\tau} h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|\nabla \cdot \boldsymbol{q}|_{W^{k, \rho}(K)}+C h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|u|_{W^{k+1, \rho}(K)},  \tag{2.13a}\\
& \left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \leq C h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \rho}(K)}+C h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|u|_{W^{k+1, \rho}(K)} . \tag{2.13b}
\end{align*}
$$

Proof. First, we prove (2.13a). In the proof of 9, Proposition A.2.] we have

$$
\begin{align*}
\left\|\Pi_{W} u-\Pi_{k}^{o} u\right\|_{L^{2}(K)} \leq & C \frac{h_{K}}{\tau}\left(\left\|\nabla \cdot \boldsymbol{q}-\Pi_{k-1}^{o}(\nabla \cdot \boldsymbol{q})\right\|_{L^{2}(K)}\right)  \tag{2.14}\\
& +C\left(\left\|u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}+h_{K}\left\|\nabla\left(u-\Pi_{k}^{o} u\right)\right\|_{L^{2}(K)}\right) .
\end{align*}
$$

Then, using the local inverse estimate in Lemma 5,

$$
\begin{array}{rlr}
\left\|\Pi_{W} u-u\right\|_{L^{s_{j}}(K)} \leq & \left\|\Pi_{W} u-\Pi_{k}^{o} u\right\|_{L^{s_{j}}(K)}+\left\|\Pi_{k}^{o} u-u\right\|_{L^{s_{j}}(K)} \\
\leq & C h_{K}^{\frac{d}{s_{j}} \frac{d}{2}}\left\|\Pi_{W} u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}+\left\|\Pi_{k}^{o} u-u\right\|_{L^{s_{j}}(K)} \\
\leq & C h_{K}^{\frac{d}{s_{j}} \frac{d}{2}}\left(\frac{h_{K}}{\tau}\left\|\nabla \cdot \boldsymbol{q}-\Pi_{k-1}^{o}(\nabla \cdot \boldsymbol{q})\right\|_{L^{2}(K)}\right. & \text { by (2.12) } \\
& \left.\quad+\left\|u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}+h_{K}\left\|\nabla\left(u-\Pi_{k}^{o} u\right)\right\|_{L^{2}(K)}\right) & \text { by (2.14) } \\
\leq & \frac{C}{\tau} h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|\nabla \cdot \boldsymbol{q}|_{W^{k, \rho}(K)}+C h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|u|_{W^{k+1, \rho}(K)} & \text { by (2.11a). }
\end{array}
$$

Next, we prove (2.13b). Because we do not have an estimate like (2.14) for $\Pi_{V}$, we introduce the single face HDG projection $\boldsymbol{B}_{V}$ defined on sufficiently smooth vector functions $\boldsymbol{q}$ such that $\boldsymbol{B}_{V} \boldsymbol{q} \in\left[\mathcal{P}^{k}(K)\right]^{d}$ satisfies [7, equation (3.10)]:

$$
\begin{align*}
\left(\boldsymbol{B}_{V} \boldsymbol{q}, \boldsymbol{v}_{h}\right)_{K} & =\left(\boldsymbol{q}, \boldsymbol{v}_{h}\right)_{K}, & & \forall \boldsymbol{v}_{h} \in\left[\mathcal{P}^{k-1}(K)\right]^{d}  \tag{2.15a}\\
\left\langle\boldsymbol{B}_{V} \boldsymbol{q} \cdot \boldsymbol{n}, \mu_{h}\right\rangle_{F_{i}} & =\left\langle\boldsymbol{q} \cdot \boldsymbol{n}, \mu_{h}\right\rangle_{F_{i}} & & \forall \mu_{h} \in \mathcal{P}^{k}\left(F_{i}\right), i=1 \ldots, d . \tag{2.15b}
\end{align*}
$$

By [7, equation (3.13) of Lemma 3.2, Lemma 3.3] we have

$$
\begin{align*}
\left\|\boldsymbol{B}_{V} \boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(K)} & \leq\|\boldsymbol{q}\|_{\boldsymbol{L}^{2}(K)}+h_{K}^{1 / 2}\|\boldsymbol{q} \cdot \boldsymbol{n}\|_{L^{2}(\partial K)},  \tag{2.16a}\\
\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{H}^{s}(K)} & \leq C h_{K}^{k+1-s}|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(K)} \tag{2.16b}
\end{align*}
$$

for $0 \leq s \leq k+1$. Note that $\boldsymbol{B}_{V} \boldsymbol{q}_{h}=\boldsymbol{q}_{h}$ for any $\boldsymbol{q}_{h} \in\left[\mathcal{P}^{k}(K)\right]^{d}$. Then, using the local inverse
$L^{\infty}$ norm error estimates for HDG methods applied to the Poisson equation with an application to the Dirichlet boundary control problem
estimate in Lemma 5.

$$
\begin{array}{rlrl}
\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \leq & \left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)}+\left\|\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} & \\
\leq & C h_{K}^{\frac{d}{s_{j}}-\frac{d}{2}}\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(K)}+\left\|\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} & & \text { by (2.12) } \\
= & C h_{K}^{\frac{d}{s_{j}}-\frac{d}{2}}\left\|\boldsymbol{B}_{V}\left(\boldsymbol{q}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}\right)\right\|_{\boldsymbol{L}^{2}(K)}+\left\|\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \\
\leq & & \\
& C h_{K}^{\frac{d}{s_{j}}-\frac{d}{2}}\left(\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(K)}+h_{K}^{1 / 2}\left\|\boldsymbol{q}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(\partial K)}\right) & & \text { by (2.16a) } \\
& +\left\|\boldsymbol{\Pi}_{k}^{o} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \rho}(K)} & & \\
\leq & \text { by (2.11a). } .
\end{array}
$$

where in the last inequality we used (2.11a) and (2.11b). In the proof of [9, Proposition A.3.] we find that

$$
\begin{equation*}
\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{2}(K)} \leq C\left(h_{K}^{1 / 2}\left\|u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}+\left\|\Pi_{W} u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}\right) . \tag{2.17}
\end{equation*}
$$

Then, again using the local inverse estimate in Lemma 5,

$$
\begin{aligned}
&\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \leq\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)}+\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \\
& \leq C h_{K}^{\frac{d}{s_{j}}-\frac{d}{2}}\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(K)}+\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \\
& \leq C h_{K}^{\frac{d}{s_{j}}-\frac{d}{2}}\left(h_{K}^{1 / 2}\left\|u-\Pi_{k}^{o} u\right\|_{L^{2}(\partial K)}+\left\|\Pi_{W} u-\Pi_{k}^{o} u\right\|_{L^{2}(K)}\right) \\
&+\left\|\boldsymbol{B}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{s_{j}}(K)} \\
& \leq \text { by (2.12) } \\
& k+1+\frac{d}{s_{j}}-\frac{d}{\rho}
\end{aligned}|\boldsymbol{q}|_{W^{k+1, \rho(K)}}+C h_{K}^{k+1+\frac{d}{s_{j}}-\frac{d}{\rho}}|u|_{W^{k+1, \rho}(K)}, \quad \text { by }, ~
$$

where in the last inequality we split $\Pi_{W} u-\Pi_{k}^{o} u=\Pi_{W} u-u+u-\Pi_{k}^{o} u$ and used (2.11a), (2.11b) and (2.13b).

## $3 \quad L^{\infty}$ norm estimates

In the rest of this paper, we restrict the domain $\Omega$ to two dimensional space, i.e., $d=2$. Furthermore, we assume:
(A) The domain is convex and the triangular mesh $\mathcal{T}_{h}$ is quasi-uniform.

Now, we state the main result of our paper:
Theorem 2. Let $(\boldsymbol{q}, u)$ and $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right)$ be the solution of (1.1) and (2.1), respectively. We assume that (A) holds. First, if $u \in L^{\infty}(\Omega), \boldsymbol{q} \in \boldsymbol{L}^{\infty}(\Omega)$ and $f \in L^{2}(\Omega)$, then we have the following stability bounds:

$$
\begin{align*}
\left\|u_{h}\right\|_{L^{\infty}(\Omega)} \leq\|u\|_{L^{\infty}(\Omega)}+C h^{\min \{k, 1\}}\|f\|_{L^{2}(\Omega)} & & \text { for all } k \geq 0  \tag{3.1a}\\
\left\|\boldsymbol{q}_{h}\right\|_{L^{\infty}(\Omega)} \leq\|\boldsymbol{q}\|_{L^{\infty}(\Omega)}+C\|f\|_{L^{2}(\Omega)} & & \text { for all } k \geq 1 . \tag{3.1b}
\end{align*}
$$

Second, if $(\boldsymbol{q}, u) \in \boldsymbol{W}^{k+1, \infty}(\Omega) \times W^{k+1, \infty}(\Omega)$, then we have the following error estimates:

$$
\begin{array}{ll}
\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(\Omega)} \leq C h^{k+1}\left(|\log h|^{1 / 2}+1\right)\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right), & \text { for all } k \geq 1, \\
\left\|u-u_{h}\right\|_{L^{\infty}(\Omega)} \leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{q}|_{W^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right) & \text { for all } k \geq 1 . \tag{3.1d}
\end{array}
$$

Furthermore, if $(\boldsymbol{q}, u) \in \boldsymbol{W}^{k+1, \infty}(\Omega) \times W^{k+2, \infty}(\Omega)$, then we have the following error estimate for the postprocessed solution:

$$
\begin{equation*}
\left\|u-u_{h}^{\star}\right\|_{L^{\infty}(\Omega)} \leq C h^{k+2}(1+|\log h|)\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+2, \infty}(\Omega)}\right) \text { for all } k \geq 1, \tag{3.1e}
\end{equation*}
$$

where $u_{h}^{\star}$ was defined in (2.2).
The remainder of this section will be devoted to proving the above result.

### 3.1 Proof of Theorem 2

We start the proof of Theorem 2 by defining suitable regularized Green's functions. We follow the notation of Girault, Nochetto and Scott [15] to define by $\delta_{\star} \geq 0$ the usual mollifier in $\mathcal{D}\left(\mathbb{R}^{2}\right)$ such that $\operatorname{supp}\left(\delta_{\star}\right) \subset B(0,1)$ and $\int_{\mathbb{R}^{2}} \delta_{\star}(x) \mathrm{d} \boldsymbol{x}=1$. Then for any point $\boldsymbol{x}_{0} \in \Omega$ and real number $\rho_{0}>0$ such that the ball $B\left(\boldsymbol{x}_{0}, \rho_{0}\right)$ is contained in $\Omega$, we define the mollifier by

$$
\begin{equation*}
\delta(\boldsymbol{x})=\frac{1}{\rho_{0}^{2}} \delta_{\star}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_{0}}{\rho_{0}}\right) \tag{3.2}
\end{equation*}
$$

Lemma 6 ( [15, Lemma 1.1]). Suppose the triangular mesh $\mathcal{T}_{h}$ is quasi-uniform. Let $\varphi_{h}$ be a polynomial in $\mathcal{P}^{k}$ on each $K, \boldsymbol{x}_{M}$ be a point of $\bar{\Omega}$ where $\left|\varphi_{h}(\boldsymbol{x})\right|$ attains its maximum, $K$ be an element containing $\boldsymbol{x}_{M}$ and $B \subset K$ be the disk of radius $\rho_{K}$ inscribed in $K$. Then there exists a smooth function $\delta_{M}$ supported in $B$ such that

$$
\begin{gather*}
\int_{\Omega} \delta_{M} \mathrm{~d} \boldsymbol{x}=1  \tag{3.3a}\\
\left\|\varphi_{h}\right\|_{L^{\infty}(\Omega)}=\left|\int_{B} \delta_{M} \varphi_{h} \mathrm{~d} \boldsymbol{x}\right|, \tag{3.3b}
\end{gather*}
$$

and for any number $t$ with $1<t \leq \infty$, there exists a constant $C$, such that

$$
\begin{align*}
\left\|\delta_{M}\right\|_{L^{t}(B)} & \leq h^{2 / t-2},  \tag{3.3c}\\
\left\|\nabla \delta_{M}\right\|_{L^{t}(B)} & \leq h^{2 / t-3} . \tag{3.3~d}
\end{align*}
$$

Proof. The proof of (3.3a)-(3.3c) can be found in [15, Lemma 1.1] where it is shown that there exists polynomial $P_{M} \in \mathcal{P}^{k}(K)$ such that

$$
\delta_{M}=\delta P_{M},
$$

Since $\|\delta\|_{L^{\infty}\left(\mathbb{R}^{2}\right)} \leq C / \rho_{K}^{2}$ and $\|\nabla \delta\|_{L^{\infty}\left(\mathbb{R}^{2}\right)} \leq C / \rho_{K}^{3}$, by (2.12) and the assumption that the triangle mesh is quasi-uniform, we have

$$
\begin{aligned}
\left\|\nabla \delta_{M}\right\|_{L^{t}(B)} & =\left\|P_{M} \nabla \delta+\delta \nabla P_{M}\right\|_{L^{t}(B)} \\
& \leq\|\nabla \delta\|_{L^{\infty}(B)}\left\|P_{M}\right\|_{L^{t}(B)}+\|\delta\|_{L^{\infty}(B)}\left\|\nabla P_{M}\right\|_{L^{t}(B)} \\
& \leq C h^{2 / t-3} .
\end{aligned}
$$

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The main idea behind the proof of $L^{\infty}$ norm estimates is to use the so called smooth $\delta_{M}$ function, which was described in Lemma 6. Given a scalar function $\delta_{1}$ and a vector $\boldsymbol{\delta}_{2}$ of the above type, we define two regularized Green's functions for problem (1.1) in mixed form:

$$
\begin{align*}
c \boldsymbol{\Phi}_{1}+\nabla \Psi_{1} & =0 & & \text { in } \Omega, \\
\nabla \cdot \boldsymbol{\Phi}_{1} & =\delta_{1} & & \text { in } \Omega,  \tag{3.4}\\
\Psi_{1} & =0 & & \text { on } \partial \Omega,
\end{align*}
$$

and

$$
\begin{align*}
c \boldsymbol{\Phi}_{2}+\nabla \Psi_{2} & =\boldsymbol{\delta}_{2} & & \text { in } \Omega, \\
\nabla \cdot \boldsymbol{\Phi}_{2} & =0 & & \text { in } \Omega,  \tag{3.5}\\
\Psi_{2} & =0 & & \text { on } \partial \Omega .
\end{align*}
$$

We need two auxiliary results before starting the proof of Theorem 2. The first concerns bounds on the regularized Green's function $\left(\Psi_{1}, \Psi_{2}\right)$ :

Lemma 7. Let $\Psi_{1}$ and $\Psi_{2}$ be the solution of (3.4) and (3.5), respectively. If assumption (A) holds, then we have:

$$
\begin{align*}
& \left\|D^{2} \Psi_{1}\right\|_{L^{2}(\Omega)} \leq C h^{-1},\left\|\sigma D^{2} \Psi_{1}\right\|_{L^{2}(\Omega)} \leq C|\log h|^{1 / 2},\left\|D^{2} \Psi_{1}\right\|_{L^{1}(\Omega)} \leq C|\log h|  \tag{3.6a}\\
& \left\|D^{2} \Psi_{2}\right\|_{L^{2}(\Omega)} \leq C h^{-2},\left\|\sigma D^{2} \Psi_{2}\right\|_{L^{2}(\Omega)} \leq C h^{-1},\left\|D^{2} \Psi_{2}\right\|_{L^{1}(\Omega)} \leq C h^{-1}|\log h|^{1 / 2} \tag{3.6b}
\end{align*}
$$

Proof. The proof of (3.6a) can be found in [33, equation (3.6)]. By the elliptic regularity of Poisson problem and (3.3d) we have

$$
\left\|D^{2} \Psi_{2}\right\|_{L^{2}(\Omega)} \leq C\left\|\nabla \cdot \boldsymbol{\delta}_{2}\right\|_{L^{2}(\Omega)} \leq C h^{-2} .
$$

The remaining estimations in (3.6b) can be found in [10, Lemma 3.2] and [33, equation (3.12d)].
Next, we define the weight $\sigma$ by:

$$
\begin{equation*}
\sigma(\boldsymbol{x})=\left(\left|\boldsymbol{x}-\boldsymbol{x}_{0}\right|^{2}+h^{2}\right)^{1 / 2}, \tag{3.7}
\end{equation*}
$$

where $\boldsymbol{x}_{0}$ is a point close to that where the relevant maximum is attained (the center of the inscribed circle in the triangle containing the maximum). In the second auxiliary lemma we summarize some properties of the function $\sigma$, which will be use later.

Lemma 8 ( [33, Equation (2.13)]). For any $\alpha \in \mathbb{R}$ there is a constant $C$ independent of $\alpha$ such that the function $\sigma$ has the following properties:

$$
\begin{gather*}
\frac{\max _{\boldsymbol{x} \in K} \sigma(\boldsymbol{x})^{\alpha}}{\min _{\boldsymbol{x} \in K} \sigma(\boldsymbol{x})^{\alpha}} \leq C, \forall K \in \mathcal{T}_{h},  \tag{3.8a}\\
\left|\nabla^{k}\left(\sigma(\boldsymbol{x})^{\alpha}\right)\right| \leq C \sigma(\boldsymbol{x})^{\alpha-k},  \tag{3.8b}\\
\int_{\Omega} \sigma(\boldsymbol{x})^{-2} \mathrm{~d} \boldsymbol{x} \leq C|\log h| . \tag{3.8c}
\end{gather*}
$$

We split the proof of Theorem 2 into four steps. First, we shall obtain the $L^{1}$ norm approximations error of the solution of (3.4) and (3.5). Second, we prove the $L^{\infty}$ norm stability of $\boldsymbol{q}_{h}$ and $u_{h}$. Next, we obtain the $L^{\infty}$ norm error estimates of $\boldsymbol{q}-\boldsymbol{q}_{h}$ and $u-u_{h}$. Finally, we obtain the $L^{\infty}$ norm error estimates of the postprocessed solution $u_{h}^{\star}$.

Step 1: $L^{1}$ norm error estimates for the regularized Green's functions Let ( $\left.\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right)$ and $\left(\boldsymbol{\Phi}_{2, h}, \Psi_{2, h}, \widehat{\Psi}_{2, h}\right)$ be the HDG solution of (3.4) and (3.5), respectively, i.e.,

$$
\begin{align*}
\mathscr{B}\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) & =\left(\delta_{1}, w_{h}\right)_{\mathcal{T}_{h}},  \tag{3.9a}\\
\mathscr{B}\left(\boldsymbol{\Phi}_{2, h}, \Psi_{2, h}, \widehat{\Psi}_{2, h} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) & =\left(\boldsymbol{\delta}_{2}, \boldsymbol{v}_{h}\right)_{\mathcal{T}_{h}} \tag{3.9b}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times \widehat{W}_{h}$. The existence and uniqueness of these solutions follow by standard HDG theory [8].

Our goal in this step is to prove the upcoming Lemma 11. To start we summarize some relevant results in the following:

$$
\begin{gather*}
\mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}, \Pi_{W} \Psi_{1}, \Pi_{k}^{\partial} \Psi_{1} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{v}_{h}\right) \mathcal{T}_{h}-\left(\delta_{1}, w_{h}\right)_{\mathcal{T}_{h}},  \tag{3.10a}\\
\mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{2}, \Pi_{W} \Psi_{2}, \Pi_{k}^{\partial} \Psi_{2} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{2}-\boldsymbol{\Phi}_{2}, \boldsymbol{v}_{h}\right) \mathcal{T}_{h}+\left(\boldsymbol{\delta}_{2}, \boldsymbol{v}_{h}\right)_{\mathcal{T}_{h}},  \tag{3.10b}\\
\mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1, h}, \Pi_{W} \Psi_{1}-\Psi_{1, h}, \Pi_{k}^{\partial} \Psi_{1}-\widehat{\Psi}_{1, h} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{v}_{h}\right) \mathcal{T}_{h},  \tag{3.10c}\\
\mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{2}-\boldsymbol{\Phi}_{2, h}, \Pi_{W} \Psi_{2}-\Psi_{2, h}, \Pi_{k}^{\partial} \Psi_{2}-\widehat{\Psi}_{2, h} ; \boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{2}-\boldsymbol{\Phi}_{2}, \boldsymbol{v}_{h}\right)_{\mathcal{T}_{h}},  \tag{3.10d}\\
\left\|\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1, h}\right\|_{L^{2}(\Omega)} \leq C, \quad\left\|\Pi_{W} \Psi_{1}-\Psi_{1, h}\right\|_{L^{2}(\Omega)} \leq C h^{\min \{k, 1\}},  \tag{3.10e}\\
\left\|\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{2}-\boldsymbol{\Phi}_{2, h}\right\|_{L^{2}(\Omega)} \leq C h^{-1}, \quad\left\|\Pi_{W} \Psi_{2}-\Psi_{2, h}\right\|_{L^{2}(\Omega)} \leq C h^{\min \{k, 1\}-1} \tag{3.10f}
\end{gather*}
$$

The proof of (3.10a) and (3.10c) can be found in [2, Lemma 3.6] and the proof of (3.10b) and (3.10d) is similar. The proof of (3.10e) and (3.10f) can be found in [9, Theorem 3.1 and Theorem 4.1] and the regularity of of regularized Greens functions in Lemma 7 .

We now present a series of lemmas providing convergence estimates for the projections used in our analysis.
Lemma 9. For any integer $k \geq 0, K \in \mathcal{T}_{h}$ and $\alpha \in \mathbb{R}$, let $\Pi_{k}^{o}$ be the standard $L^{2}$ projection (see (2.9a)), then for $\boldsymbol{v} \in \boldsymbol{H}^{k+1}(K)$ we have

$$
\begin{equation*}
\left\|\sigma^{\alpha}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{k}^{o} \boldsymbol{v}\right)\right\|_{\boldsymbol{L}^{2}(K)} \leq C h_{K}^{k+1}\left\|\sigma^{\alpha} \nabla^{k+1} \boldsymbol{v}\right\|_{\boldsymbol{L}^{2}(K)} \tag{3.11a}
\end{equation*}
$$

Furthermore, let $w \in H^{k+1}(K)$, then $\left(\sigma^{2} \boldsymbol{v}, \sigma^{2} w\right)$ is in the domain of $\Pi_{h}$ and we have

$$
\begin{equation*}
\left\|\sigma^{\alpha}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{V} \boldsymbol{v}\right)\right\|_{\boldsymbol{L}^{2}(K)} \leq C h_{K}^{k+1}\left(\left\|\sigma^{\alpha} \nabla^{k+1} \boldsymbol{v}\right\|_{\boldsymbol{L}^{2}(K)}+\left\|\sigma^{\alpha} \nabla^{k+1} w\right\|_{L^{2}(K)}\right) . \tag{3.11b}
\end{equation*}
$$

Proof. We only prove (3.11b) because the proof of (3.11a) is similar.

$$
\begin{aligned}
\left\|\sigma^{\alpha}\left(\boldsymbol{v}-\boldsymbol{\Pi}_{V} \boldsymbol{v}\right)\right\|_{\boldsymbol{L}^{2}(K)} & \leq \max _{\boldsymbol{x} \in K}\left\{\sigma^{\alpha}\right\}\left\|\boldsymbol{v}-\boldsymbol{\Pi}_{V} \boldsymbol{v}\right\|_{\boldsymbol{L}^{2}(K)} \\
& \leq C h_{K}^{k+1} \max _{\boldsymbol{x} \in K}\left\{\sigma^{\alpha}\right\}\left(\left|\nabla^{k+1} \boldsymbol{v}\right|_{L^{2}(K)}+\left|\nabla^{k+1} w\right|_{L^{2}(K)}\right) \\
& \leq C h_{K}^{k+1} \min _{\boldsymbol{x} \in K}\left\{\sigma^{\alpha}\right\}\left(\left|\nabla^{k+1} \boldsymbol{v}\right|_{\boldsymbol{L}^{2}(K)}+\left|\nabla^{k+1} w\right|_{L^{2}(K)}\right) \quad \text { by (2.8b) } \\
& \leq C h_{K}^{k+1}\left(\left\|\sigma^{\alpha} \nabla^{k+1} \boldsymbol{v}\right\|_{L^{2}(K)}+\left\|\sigma^{\alpha} \nabla^{k+1} w\right\|_{L^{2}(K)}\right) .
\end{aligned}
$$

Lemma 10. Let $\left(\boldsymbol{v}_{h}, w_{h}\right) \in \boldsymbol{V}_{h} \times W_{h}$, then for any integer $k \geq 0$, we have

$$
\begin{align*}
\left\|\sigma^{-1}\left(\sigma^{2} \boldsymbol{v}_{h}-\boldsymbol{\Pi}_{k}^{o}\left(\sigma^{2} \boldsymbol{v}_{h}\right)\right)\right\|_{\boldsymbol{L}^{2}(K)} & \leq C h\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{L}^{2}(K)}  \tag{3.12a}\\
\left\|\sigma^{-1}\left(\sigma^{2} \boldsymbol{v}_{h}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \boldsymbol{v}_{h}\right)\right)\right\|_{\boldsymbol{L}^{2}(K)} & \leq C h\left(\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{L}^{2}(K)}+\left\|w_{h}\right\|_{\boldsymbol{L}^{2}(K)}\right) \tag{3.12b}
\end{align*}
$$

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Proof. Notice that $\left.\boldsymbol{v}_{h}\right|_{K} \in\left[P^{k}(K)\right]^{2}$, i.e., $\nabla^{k+1} \boldsymbol{v}_{h}=0$. Then by Lemma 9 we have

$$
\begin{aligned}
\left\|\sigma^{-1}\left(\sigma^{2} \boldsymbol{v}_{h}-\boldsymbol{\Pi}_{k}^{o}\left(\sigma^{2} \boldsymbol{v}_{h}\right)\right)\right\|_{\boldsymbol{L}^{2}(K)} & \leq C h^{k+1}\left\|\sigma^{-1}\left(\nabla^{k+1}\left(\sigma^{2} \boldsymbol{v}_{h}\right)\right)\right\|_{\boldsymbol{L}^{2}(K)} \\
& =C h^{k+1}\left\|\sigma^{-1} \sum_{j=1}^{k+1} \nabla^{j}\left(\sigma^{2}\right) \nabla^{k+1-j} \boldsymbol{v}_{h}\right\|_{\boldsymbol{L}^{2}(K)} \\
& \leq C h\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{L}^{2}(K)}
\end{aligned}
$$

where we applied (3.8b) and Lemma 5 to the above inequality. This proves (3.12a) and the proof of (3.12b) is the same, hence we omit the details here.

Lemma 11. Let $\left(\boldsymbol{\Phi}_{1}, \Psi_{1}\right)$ and $\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right)$ be the solution of (3.4) and (3.9a) respectively, and $\left(\boldsymbol{\Phi}_{2}, \Psi_{2}\right)$ and $\left(\boldsymbol{\Phi}_{2, h}, \Psi_{2, h}, \widehat{\Psi}_{2, h}\right)$ be the solutions of (3.5) and (3.9b). If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{align*}
& \left\|\sigma\left(c \boldsymbol{\Phi}_{1}-c \boldsymbol{\Phi}_{1, h}\right)\right\|_{\boldsymbol{L}^{2}(\Omega)} \leq C h\left(|\log h|^{1 / 2}+1\right)  \tag{3.13a}\\
& \| \sigma\left(c \boldsymbol{\Phi}_{2}-c \boldsymbol{\Phi}_{2, h} \|_{\boldsymbol{L}^{2}(\Omega)} \leq C\right. \tag{3.13b}
\end{align*}
$$

Proof. Let $\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}=\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1, h}, \mathcal{E}_{h}^{\Psi_{1}}=\Pi_{W} \Psi_{1}-\Psi_{1, h}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}}=\Pi_{k}^{\partial} \Psi_{1}-\widehat{\Psi}_{1, h}$. On the one hand, by the definition of $\mathscr{B}$ in (2.3) we obtain:

$$
\begin{aligned}
& \mathscr{B}\left.\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}},-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}},-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right) \\
& \quad=\left.\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right) \mathcal{T}_{h}-\left(\mathcal{E}_{h}^{\Psi_{1}}, \nabla \cdot\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)\right)_{\mathcal{T}_{h}}+\left\langle\mathcal{E}_{h}^{\widehat{\Psi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} \\
& \quad+\left(\nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)_{\mathcal{T}_{h}}+\left\langle\tau\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right), \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right\rangle_{\partial \mathcal{T}_{h}}-\left\langle\mathcal{E}_{h}^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}, \sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right\rangle_{\partial \mathcal{T}_{h}} \\
& \quad=\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}-\left(\mathcal{E}_{h}^{\Psi_{1}}, \sigma^{2} \nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}-\left(\mathcal{E}_{h}^{\Psi_{1}}, 2 \sigma \nabla \sigma \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+\left\langle\mathcal{E}_{h}^{\widehat{\Psi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} \\
& \quad+\left(\nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)_{\mathcal{T}_{h}}+\left\langle\tau\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right), \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right\rangle_{\partial \mathcal{T}_{h}}-\left\langle\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}} \cdot \boldsymbol{n}, \sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right\rangle_{\partial \mathcal{T}_{h}} .
\end{aligned}
$$

This gives

$$
\begin{align*}
& \mathscr{B}\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}},-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}},-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)  \tag{3.14}\\
& \quad=\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right) \mathcal{T}_{h}+\left\|\sigma \sqrt{\tau}\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right\|_{L^{2}\left(\partial \mathcal{T}_{h}\right)}^{2}-\left(\mathcal{E}_{h}^{\Psi_{1}}, 2 \sigma \nabla \sigma \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}} .
\end{align*}
$$

We use definition (2.7) with $\boldsymbol{q}=\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, u=-\sigma^{2} \mathcal{E}_{h}^{\psi_{1}}$. On the other hand, by the error equation (3.10c) are get

$$
\begin{aligned}
& \mathscr{B}\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}},-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}},-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right) \\
& =\mathscr{B}\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\Pi_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right),-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}-\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right),-\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right)\right) \\
& \left.\quad+\mathscr{B}_{\mathcal{E}_{h}}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right), \Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right),-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right) \\
& =\mathscr{B}\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right),-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}-\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right),-\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right)\right) \\
& \quad+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) \mathcal{T}_{h} .
\end{aligned}
$$

Next, we use the definition of $\mathscr{B}$ in (2.3) again to get:

$$
\begin{aligned}
\mathscr{B} & \left.\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right),-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}-\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right),-\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right)\right) \\
= & \left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}}-\left(\mathcal{E}_{h}^{\Psi_{1}}, \nabla \cdot\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)\right)_{\mathcal{T}_{h}} \\
& +\left\langle\mathcal{E}_{h}^{\widehat{\Psi}_{1}},\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}+\left(\nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right)_{\mathcal{T}_{h}} \\
& +\left\langle\tau\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right),\left(\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right)-\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right)\right\rangle_{\partial \mathcal{T}_{h}} \\
& -\left\langle\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}} \cdot \boldsymbol{n}, \sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}-\Pi_{k}^{\partial}\left(\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right\rangle_{\partial \mathcal{T}_{h}}+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) \\
= & \left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}}+\left(\nabla \mathcal{E}_{h}^{\Psi_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}} \\
& -\left\langle\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}},\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}+\left(\nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right)_{\mathcal{T}_{h}} \\
& +\left\langle\tau\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right), \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right\rangle_{\partial \mathcal{T}_{h}}+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}},
\end{aligned}
$$

where we used integration by parts in the above equation. Notice that $\left(\sigma^{2} \mathcal{E}_{h}^{\Phi_{1}},-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)$ is in the domain of $\Pi_{h}$ (see (2.7)), then by (2.7c) and the fact that $\left.\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right|_{F} \in \mathcal{P}^{k}(F)$ for all $F \in \mathcal{F}_{h}$, we have

$$
\left\langle\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}},\left(\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)-\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\tau\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right), \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right\rangle_{\partial \mathcal{T}_{h}}=0 .
$$

Furthermore, by (2.7a) -(2.7b) and the fact that $\left.\nabla \mathcal{E}_{h}^{\Psi_{1}}\right|_{K} \in\left[\mathcal{P}^{k-1}(K)\right]^{2}$ and $\left.\nabla \cdot \mathcal{E}_{h}^{\Phi_{1}}\right|_{K} \in \mathcal{P}^{k-1}(K)$, then we have

$$
\begin{aligned}
\left(\nabla \mathcal{E}_{h}^{\Psi_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\Pi_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}} & =0 \\
\left(\nabla \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}+\Pi_{W}\left(-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}}\right)\right)_{\mathcal{T}_{h}} & =0
\end{aligned}
$$

This gives

$$
\begin{align*}
& \mathscr{B}\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \mathcal{E}_{h}^{\Psi_{1}}, \mathcal{E}_{h}^{\widehat{\Psi}_{1}} ; \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}},-\sigma^{2} \mathcal{E}_{h}^{\Psi_{1}},-\sigma^{2} \mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)  \tag{3.15}\\
& \quad=\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) \mathcal{T}_{h}+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right) .
\end{align*}
$$

Comparing with (3.14) and (3.15) we have

$$
\begin{aligned}
& \left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+\left\|\sigma \sqrt{\tau}\left(\mathcal{E}_{h}^{\Psi_{1}}-\mathcal{E}_{h}^{\widehat{\Psi}_{1}}\right)\right\|_{L^{2}\left(\partial \mathcal{T}_{h}\right)}^{2} \\
& \quad=\left(\mathcal{E}_{h}^{\Psi_{1}}, 2 \sigma \nabla \sigma \cdot \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}} \\
& \quad+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)-\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right) \mathcal{T}_{h} \\
& \quad=I_{1}+I_{2}+I_{3}+I_{4} .
\end{aligned}
$$

For the first term $I_{1}$, we use (3.8b), Young's inequality, (3.10e) and $k \geq 1$ to get

$$
\left|I_{1}\right| \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C\left\|\mathcal{E}_{h}^{\Psi_{1}}\right\|_{L^{2}(\Omega)}^{2} \leq \frac{1}{4}\left\|\sigma \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right\|_{L^{2}(\Omega)}^{2}+C h^{2} .
$$

For the second term $I_{2}$, we use Young's inequality, (3.12b) and (3.10e) to get

$$
\begin{aligned}
\left|I_{2}\right| & =\left|\left(\sigma \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{-1}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)\right)_{\mathcal{T}_{h}}\right| \\
& \leq \frac{1}{4}\left(\left(\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C \| \sigma^{-1}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)\right)_{\mathcal{T}_{h}} \|_{L^{2}(\Omega)}^{2} \\
& \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C h^{2}\left(\left\|\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right\|_{\boldsymbol{L}^{2}(\Omega)}^{2}+\left\|\mathcal{E}_{h}^{\Psi_{1}}\right\|_{L^{2}(\Omega)}^{2}\right) \\
& \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C h^{2} .
\end{aligned}
$$

$L^{\infty}$ norm error estimates for HDG methods applied to the Poisson equation with an application to the Dirichlet boundary control problem

For the third term $I_{3}$, we use Young's inequality, (3.12b), (3.10e), (3.11b) and (3.6a) to get

$$
\begin{aligned}
\left|I_{3}\right| & =\left|\left(\sigma\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}\right), \sigma^{-1}\left(\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)-\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)_{\mathcal{T}_{h}}\right| \\
& \left.\leq \frac{1}{2}\left\|\sigma\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}\right)\right\|_{\boldsymbol{L}^{2}(\Omega)}^{2}+\frac{1}{2} \| \sigma^{-1}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}-\boldsymbol{\Pi}_{V}\left(\sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)\right)\right)_{\mathcal{T}_{h}} \|_{L^{2}(\Omega)}^{2} \\
& \leq C h^{2}\left\|\sigma D^{2} \Psi_{1}\right\|_{L^{2}(\Omega)}^{2}+C h^{2}\left(\left\|\mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right\|_{\boldsymbol{L}^{2}(\Omega)}^{2}+\left\|\mathcal{E}_{h}^{\Psi_{1}}\right\|_{L^{2}(\Omega)}^{2}\right) \\
& \leq C h^{2}(1+|\log h|) .
\end{aligned}
$$

For the last term $I_{4}$, we use Young's inequality, (3.11b) and (3.6a) to get

$$
\begin{aligned}
\left|I_{4}\right| & =\left|\left(\sigma\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\boldsymbol{\Phi}_{1}\right), \sigma \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}\right| \\
& \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C\left\|\sigma\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}_{1}-\mathbf{\Phi}_{1}\right)\right\|_{L^{2}(\Omega)}^{2} \\
& \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C h^{2}\left\|\sigma D^{2} \Psi\right\|_{L^{2}(\Omega)}^{2} \\
& \leq \frac{1}{4}\left(c \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}, \sigma^{2} \mathcal{E}_{h}^{\boldsymbol{\Phi}_{1}}\right)_{\mathcal{T}_{h}}+C h^{2}(1+|\log h|) .
\end{aligned}
$$

Lemma 12. Let $\left(\boldsymbol{\Phi}_{1}, \Psi_{1}\right)$ and $\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right)$ be the solutions of (3.4) and (3.9a) respectively, $\left(\boldsymbol{\Phi}_{2}, \Psi_{2}\right)$ and let $\left(\boldsymbol{\Phi}_{2, h}, \Psi_{2, h}, \widehat{\Psi}_{2, h}\right)$ be the solution of (3.5) and (3.9b). If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{align*}
& \left\|\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right\|_{\boldsymbol{L}^{1}(\Omega)}+\left\|c \boldsymbol{\Phi}_{1}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{1}\right)\right\|_{\boldsymbol{L}^{1}(\Omega)} \leq C h(|\log h|+1),  \tag{3.16a}\\
& \left\|\boldsymbol{\Phi}_{2, h}-\boldsymbol{\Phi}_{2}\right\|_{\boldsymbol{L}^{1}(\Omega)}+\left\|c \boldsymbol{\Phi}_{2}-\mathbf{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{2}\right)\right\|_{\boldsymbol{L}^{1}(\Omega)} \leq C\left(|\log h|^{1 / 2}+1\right) . \tag{3.16b}
\end{align*}
$$

Proof. By the Cauchy-Schwarz inequality and (3.13a) we have

$$
\left\|\mathbf{\Phi}_{1}-\mathbf{\Phi}_{1, h}\right\|_{\boldsymbol{L}^{1}(\Omega)}=\int_{\Omega} \sigma^{-1}\left(\sigma\left|\mathbf{\Phi}_{1}-\mathbf{\Phi}_{1, h}\right|\right) \mathrm{d} \boldsymbol{x} \leq C h(1+|\log h|) .
$$

Next, by (2.11a) we have

$$
\begin{equation*}
\left\|c \boldsymbol{\Phi}_{1}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{1}\right)\right\|_{\mathbf{L}^{1}(\Omega)} \leq C h\left\|\nabla^{2} \Psi_{1}\right\|_{L^{1}(\Omega)} \leq C h|\log h| . \tag{3.17}
\end{equation*}
$$

Then, (3.16a) follows. The proof of (3.16b) is similar to the proof of (3.16a).

## Step 2: Proof of (3.1a)- (3.1b) in Theorem 2

Proof. We only prove (3.1a) since the proof of (3.1b) is similar. We choose $\delta_{1}$ so that $\left\|u_{h}\right\|_{L^{\infty}(\Omega)}=$ $\left(\delta_{1}, u_{h}\right) \mathcal{T}_{h}$, then

$$
\begin{array}{rlrl}
-\left(\delta_{1}, u_{h}\right) & =\mathscr{B}\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h} ; \boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right) & & \text { by ((3.9a) } \\
& =\mathscr{B}\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h} ; \boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right) & & \text { by (2.6) } \\
& =\mathscr{B}\left(\boldsymbol{q}, u, u ; \boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right) & & \text { by (2.5) } \\
& =\mathscr{B}\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h} ; \boldsymbol{q}, u, u\right) & & \text { by (2.6) } \\
& =\mathscr{B}\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}, \Psi_{1, h}-\Psi_{1}, \widehat{\Psi}_{1, h}-\Psi_{1} ; \boldsymbol{q}, u, u\right)+\mathscr{B}\left(\boldsymbol{\Phi}_{1}, \Psi_{1}, \Psi_{1} ; \boldsymbol{q}, u, u\right) & \\
& =\mathscr{B}\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}, \Psi_{1, h}-\Psi_{1}, \widehat{\Psi}_{1, h}-\Psi_{1} ; \boldsymbol{q}, u, u\right)-\left(\delta_{1}, u\right) & & \text { by (3.9a). }
\end{array}
$$

By the definition of $\mathscr{B}$ in (2.3) we have

$$
\begin{aligned}
\mathscr{B} & \left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}, \Psi_{1, h}-\Psi_{1}, \widehat{\Psi}_{1, h}-\Psi_{1} ; \boldsymbol{q}, u, u\right) \\
= & \left(c\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right), \boldsymbol{q}\right)_{\mathcal{T}_{h}}-\left(\Psi_{1, h}-\Psi_{1}, \nabla \cdot \boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\Psi}_{1, h}-\Psi_{1}, \boldsymbol{q} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} \\
& -\left(\nabla \cdot\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right), u\right)_{\mathcal{T}_{h}}+\left\langle\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right) \cdot \boldsymbol{n}, u\right\rangle_{\partial \mathcal{T}_{h}} \\
= & -\left(\Psi_{1, h}-\Psi_{1}, \nabla \cdot \boldsymbol{q}\right)_{\mathcal{T}_{h}}=-\left(\Psi_{1, h}-\Psi_{1}, f\right)_{\mathcal{T}_{h}},
\end{aligned}
$$

where we used integration by parts and the fact that $\left\langle\widehat{\Psi}_{1, h}, \boldsymbol{q} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=0$ and $\left\langle\Psi_{1}, \boldsymbol{q} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=0$ in the last equality. Combining the above two equations, and using the fact that $\left\|\delta_{1}\right\|_{L^{1}(\Omega)}=1$ together with the estimates in (3.10e) gives

$$
\left\|u_{h}\right\|_{L^{\infty}(\Omega)} \leq\|u\|_{L^{\infty}(\Omega)}+\left(\Psi_{1, h}-\Psi_{1}, f\right)_{\mathcal{T}_{h}} \leq\|u\|_{L^{\infty}(\Omega)}+C h^{\min \{k, 1\}}\|f\|_{L^{2}(\Omega)}
$$

This completes the proof of (3.1a).
Step 3: Proof of (3.1c)-(3.1d) in Theorem 2 We choose $\delta_{1}$ and $\boldsymbol{\delta}_{2}$ such that $\| \Pi_{W} u$ $u_{h}\left\|_{L^{\infty}(\Omega)}=\left(\delta_{1}, \Pi_{W} u-u_{h}\right) \mathcal{T}_{h},\right\| \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h} \|_{\boldsymbol{L}^{\infty}(\Omega)}=\left(\boldsymbol{\delta}_{2}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right) \mathcal{T}_{h}$.
Lemma 13. Let $\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right)$ and $\left(\boldsymbol{\Phi}_{2, h}, \Psi_{2, h}, \widehat{\Psi}_{2, h}\right)$ be the HDG solutions of (3.9a) and (3.9b), respectively. Then we have

$$
\begin{aligned}
-\left(\delta_{1}, \Pi_{W} u-u_{h}\right)_{\mathcal{T}_{h}} & =\left(c\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(c \boldsymbol{\Phi}_{1}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{1}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}} \\
\left(\boldsymbol{\delta}_{2}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right)_{\mathcal{T}_{h}} & =\left(c\left(\boldsymbol{\Phi}_{2, h}-\boldsymbol{\Phi}_{2}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(c \boldsymbol{\Phi}_{2}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{2}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}
\end{aligned}
$$

Proof. We take $\left(\boldsymbol{v}_{h}, w_{h}, \widehat{w}_{h}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}, \Pi_{W} u-u_{h}, \Pi_{k}^{\partial} u-\widehat{u}_{h}\right)$ in (3.9a) to get

$$
\begin{array}{rlrl}
-\left(\delta_{1}, \Pi_{W} u-u_{h}\right) \mathcal{T}_{h}= & \mathscr{B}\left(\boldsymbol{\Phi}_{h}, \Psi_{h}, \widehat{\Psi}_{h} ; \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}, \Pi_{W} u-u_{h}, \Pi_{k}^{\partial} u-\widehat{u}_{h}\right) & \\
= & \mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}, \Pi_{W} u-u_{h}, \Pi_{k}^{\partial} u-\widehat{u}_{h} ; \boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right) & & \text { by (2.6) } \\
= & \mathscr{B}\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}, \Pi_{W} u-u, \Pi_{k}^{\partial} u-u ; \boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h}\right) & \text { by (2.5) } \\
= & \mathscr{B}\left(\boldsymbol{\Phi}_{1, h}, \Psi_{1, h}, \widehat{\Psi}_{1, h} ; \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}, \Pi_{W} u-u, \Pi_{k}^{\partial} u-u\right) & \text { by (2.6) } \\
= & \left(c \boldsymbol{\Phi}_{1, h}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}-\left(\Psi_{1, h}, \nabla \cdot\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)\right)_{\mathcal{T}_{h}} & \\
& +\left\langle\widehat{\Psi}_{1, h},\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}-\left(\nabla \cdot \boldsymbol{\Phi}_{1, h}, \Pi_{W} u-u\right)_{\mathcal{T}_{h}} & \\
& -\left\langle\tau\left(\Psi_{1, h}-\widehat{\Psi}_{1, h}\right), \Pi_{W} u-u\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\boldsymbol{\Phi}_{1, h} \cdot \boldsymbol{n}, \Pi_{k}^{\partial} u-u\right\rangle_{\partial \mathcal{T}_{h}}, &
\end{array}
$$

where we used the definition of $\mathscr{B}$ in the last step. Next, by (2.7a) -(2.7c) we have

$$
\begin{aligned}
-\left(\delta_{1}, \Pi_{W} u-u_{h}\right)_{\mathcal{T}_{h}}= & \left(c \boldsymbol{\Phi}_{1, h}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}-\left(\Psi_{1, h}, \nabla \cdot\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)\right)_{\mathcal{T}_{h}} \\
& +\left\langle\widehat{\Psi}_{1, h},\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}-\left\langle\tau\left(\Psi_{1, h}-\widehat{\Psi}_{1, h}\right), \Pi_{W} u-u\right\rangle_{\partial \mathcal{T}_{h}} \\
= & \left(c \boldsymbol{\Phi}_{1, h}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(\nabla \Psi_{h}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}} \\
& -\left\langle\Psi_{1, h}-\widehat{\Psi}_{1, h},\left(\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}-\left\langle\tau\left(\Psi_{1, h}-\widehat{\Psi}_{1, h}\right), \Pi_{W} u-u\right\rangle_{\partial \mathcal{T}_{h}} \\
= & \left(c \boldsymbol{\Phi}_{1, h}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}} \\
= & \left(c\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(c \boldsymbol{\Phi}, \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}} \\
= & \left(c\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(c \boldsymbol{\Phi}-\boldsymbol{\Pi}_{k-1}(c \boldsymbol{\Phi}), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}} .
\end{aligned}
$$

This gives the proof of the first identity, we omit the proof of the second identity since it follows along the same lines.
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Lemma 14. Let $(\boldsymbol{q}, u)$ and $\left(\boldsymbol{q}_{h}, u_{h}\right)$ be the solution of (1.1) and (2.1), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{align*}
& \left\|\Pi_{W} u-u_{h}\right\|_{L^{\infty}(\Omega)} \leq C h^{k+2}(|\log h|+1)\left(|\boldsymbol{q}|_{W^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right),  \tag{3.18a}\\
& \left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(\Omega)} \leq C h^{k+1}\left(|\log h|^{1 / 2}+1\right)\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right) . \tag{3.18b}
\end{align*}
$$

Proof. By Lemmas 12 and 13 we have

$$
\begin{aligned}
\left\|\Pi_{W} u-u_{h}\right\|_{L^{\infty}(\Omega)} & =\left|\left(c\left(\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}+\left(c \boldsymbol{\Phi}_{1}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{1}\right), \boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right)_{\mathcal{T}_{h}}\right| \\
& \leq C\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{\infty}(\Omega)}\left(\left\|\boldsymbol{\Phi}_{1, h}-\boldsymbol{\Phi}_{1}\right\|_{\boldsymbol{L}^{1}(\Omega)}+\left\|c \boldsymbol{\Phi}_{1}-\boldsymbol{\Pi}_{k-1}\left(c \boldsymbol{\Phi}_{1}\right)\right\|_{\boldsymbol{L}^{1}(\Omega)}\right) \\
& \leq C h^{k+2}(|\log h|+1)\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right) .
\end{aligned}
$$

As a consequence, a simple application of the triangle inequality and Lemma 14 and theorem 1 gives convergence rates for $\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(\Omega)}$ and $\left\|u-u_{h}\right\|_{L^{\infty}(\Omega)}$. This completes the proof of (3.1a)(3.1b) in Theorem 2.

## Step 4: Proof of (3.1e) in Theorem 2

Proof. First, for all $w_{0} \in \mathcal{P}^{0}(K)$, we have

$$
\begin{equation*}
\left(\Pi_{W} u-\Pi_{k+1}^{o} u, w_{0}\right)_{K}=\left(\Pi_{W} u-u, w_{0}\right)_{K}+\left(u-\Pi_{k+1}^{o} u, w_{0}\right)_{K}=0 . \tag{3.19}
\end{equation*}
$$

Let $e_{h}=u_{h}^{\star}-u_{h}+\Pi_{W} u-\Pi_{k+1}^{o} u$, by (2.2) we obtain

$$
\begin{aligned}
\left\|\nabla e_{h}\right\|_{L^{2}(K)}^{2} & =\left(\nabla\left(u_{h}^{\star}-u_{h}\right), \nabla e_{h}\right)_{K}+\left(\nabla\left(\Pi_{W} u-\Pi_{k+1}^{o} u\right), \nabla e_{h}\right)_{K} \\
& =\left(-\nabla u_{h}-\boldsymbol{q}_{h}, \nabla e_{h}\right)_{K}+\left(\nabla\left(\Pi_{W} u-\Pi_{k+1}^{o} u\right), \nabla e_{h}\right)_{K} \\
& =\left(\nabla\left(\Pi_{W} u-u_{h}\right)_{K}-\left(\boldsymbol{q}_{h}-\boldsymbol{q}\right)+\nabla\left(u-\Pi_{k+1}^{o} u\right), \nabla e_{h}\right)_{K} .
\end{aligned}
$$

Using Lemma 5 this implies that

$$
\begin{equation*}
\left\|\nabla e_{h}\right\|_{L^{2}(K)} \leq C\left(h_{K}^{-1}\left\|\Pi_{W} u-u_{h}\right\|_{L^{2}(K)}+\left\|\boldsymbol{q}_{h}-\boldsymbol{q}\right\|_{L^{2}(K)}+\left\|\nabla\left(u-\Pi_{k+1} u\right)\right\|_{L^{2}(K)}\right) . \tag{3.20}
\end{equation*}
$$

By (2.2b) and (3.19) we get $\left(e_{h}, 1\right)_{K}=0$, i.e., $\Pi_{0}^{o} e_{h}=0$. Then standard estimates for the $L^{2}$ projection given in (3.20) shows that

$$
\begin{aligned}
\left\|e_{h}\right\|_{L^{2}(K)} & =\left\|e_{h}-\Pi_{0}^{o} e_{h}\right\|_{L^{2}(K)} \\
& \leq C h_{K}\left\|\nabla e_{h}\right\|_{L^{2}(K)} \\
& \leq C\left(\left\|\Pi_{W} u-u_{h}\right\|_{L^{2}(K)}+h_{K}\left\|\boldsymbol{q}_{h}-\boldsymbol{q}\right\|_{L^{2}(K)}+h_{K}\left\|\nabla\left(u-\Pi_{k+1} u\right)\right\|_{L^{2}(K)}\right) .
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{2}(K)} \leq & C\left\|\Pi_{W} u-u_{h}\right\|_{L^{2}(K)}+C h\left\|\boldsymbol{q}_{h}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{2}(K)} \\
& +C h\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}(K)}+C h\left\|\nabla\left(u-\Pi_{k+1}^{o} u\right)\right\|_{L^{2}(K)} .
\end{aligned}
$$

We use the above inequality and Lemma 5 to get:

$$
\begin{aligned}
\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{\infty}(K)} \leq & C h^{-1}\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{2}(K)} \\
\leq & C h^{-1}\left\|\Pi_{W} u-u_{h}\right\|_{L^{2}(K)}+C\left\|\boldsymbol{q}_{h}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{2}(K)} \\
& +C\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}(K)}+C\left\|\nabla\left(u-\Pi_{k+1}^{o} u\right)\right\|_{L^{2}(K)} \\
\leq & C\left\|\Pi_{W} u-u_{h}\right\|_{L^{\infty}(K)}+C h\left\|\boldsymbol{q}_{h}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{\infty}(K)} \\
& +C\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}(K)}+C\left\|\nabla\left(u-\Pi_{k+1}^{o} u\right)\right\|_{L^{2}(K)} .
\end{aligned}
$$

Now let $K^{\star}$ denote the element in which $\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{\infty}(\Omega)}=\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{\infty}\left(K^{\star}\right)}$. Then

$$
\begin{aligned}
\left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{\infty}(\Omega)}= & \left\|\Pi_{k+1}^{o} u-u_{h}^{\star}\right\|_{L^{\infty}\left(K^{\star}\right)} \\
\leq & C\left(\left\|\Pi_{W} u-u_{h}\right\|_{L^{\infty}\left(K^{\star}\right)}+h\left\|\boldsymbol{q}_{h}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{\infty}\left(K^{\star}\right)}\right) \\
& +C\left(\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}\left(K^{\star}\right)}+\left\|\nabla\left(u-\Pi_{k+1}^{o} u\right)\right\|_{L^{2}\left(K^{\star}\right)}\right) \\
\leq & C\left(\left\|\Pi_{W} u-u_{h}\right\|_{L^{\infty}(\Omega)}+h\left\|\boldsymbol{q}_{h}-\boldsymbol{\Pi}_{V} \boldsymbol{q}\right\|_{L^{\infty}(\Omega)}\right) \\
& +C\left(\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}\left(K^{\star}\right)}+\left\|\nabla\left(u-\Pi_{k+1}^{o} u\right)\right\|_{L^{2}\left(K^{\star}\right)}\right) .
\end{aligned}
$$

By the estimates in Lemmas 4, 5and 14 and theorem 1 and the triangle inequality we get our claimed result.

## 4 Quasi-optimal estimates on interfaces

Let $\Gamma$ be a finite union of line segments such that $\Omega$ is decomposed into finitely many Lipschitz domains by $\Gamma$. We stress that, while $\Omega$ is assumed to be convex, the subdomains need not be convex. Define $\mathcal{F}_{h}^{\Gamma}$ by

$$
\mathcal{F}_{h}^{\Gamma}=\left\{F \in \mathcal{F}_{h}: \text { measure }(F \cap \Gamma)>0\right\} .
$$

We assume, furthermore, that the triangle mesh $\mathcal{T}_{h}$ resolves $\Gamma$. Hence, $\Gamma$ can be written as the union of $\mathcal{O}\left(h^{-1}\right)$ edges in $\mathcal{F}_{h}$, i.e., $\bar{\Gamma}=\bigcup_{F \in \mathcal{F}_{h}^{\Gamma} \subset \mathcal{F}_{h}} \bar{F}$.

Theorem 3. Assume $\Gamma$ has the above properties and let $(\boldsymbol{q}, u)$ and $\left(\boldsymbol{q}_{h}, u_{h}\right)$ be the solution of (1.1) and (2.1), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{align*}
& \left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{2}(\Gamma)} \leq C h^{k+1}\left(|\log h|^{1 / 2}+1\right)\left(|\boldsymbol{q}|_{W^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right),  \tag{4.1a}\\
& \left\|u-u_{h}\right\|_{L^{2}(\Gamma)} \leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+1, \infty}(\Omega)}\right) \tag{4.1b}
\end{align*}
$$

Furthermore, we have the following error estimate for the postprocessed solution:

$$
\begin{equation*}
\left\|u-u_{h}^{\star}\right\|_{L^{2}(\Gamma)} \leq C h^{k+2}(|\log h|+1)\left(|\boldsymbol{q}|_{W^{k+1, \infty}(\Omega)}+|u|_{W^{k+2, \infty}(\Omega)}\right), \tag{4.1c}
\end{equation*}
$$

where $u_{h}^{\star}$ is defined in (2.2).
Remark 1. The result proves the observation seen in numerical experiments that the flux on $\Gamma$ converges at an optimal rate. The best theoretical estimates known to us before our paper is $\mathcal{O}\left(h^{k+1 / 2}\right)$.
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Proof. We only prove (4.1a) since the proof of (4.1b) and (4.1c) are very similar. We define tubular neighborhoods of $\Gamma$ by

$$
\mathbb{S}_{h}:=\left\{K \in \mathcal{T}_{h} \mid \Gamma \cap \partial K \neq \emptyset\right\}
$$

Then the number of the elements in $\mathbb{S}_{h}$ is order of $\mathcal{O}\left(h^{-1}\right)$.

$$
\begin{array}{rlr}
\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{L^{2}(\Gamma)}^{2} & \leq\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{L^{2}(\Gamma)}^{2}+\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{\boldsymbol{h}}\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq \sum_{K \in \mathbb{S}_{h}}\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}\right\|_{\boldsymbol{L}^{2}(\partial K)}^{2}+\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{2}(\partial K)}^{2} \\
& \leq C \sum_{K \in \mathbb{S}_{h}}\left(h^{2 k+3}\left|\nabla^{k+1} \boldsymbol{q}\right|_{\boldsymbol{L}^{\infty}(\Omega)}^{2}+h^{-1}\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{2}(K)}^{2}\right) & \text { by (2.13b) } \\
& \leq C \sum_{K \in \mathbb{S}_{h}}\left(h^{2 k+3}\left|\nabla^{k+1} \boldsymbol{q}\right|_{\boldsymbol{L}^{\infty}(\Omega)}^{2}+h^{-1} h^{2}\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(K)}^{2}\right) & \text { by (2.12) } \\
& \leq C \sum_{K \in \mathbb{S}_{h}} h^{2 k+3}\left|\nabla^{k+1} \boldsymbol{q}\right|_{\boldsymbol{L}^{\infty}(\Omega)}^{2}+C\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(\Omega)}^{2} \sum_{K \in \mathbb{S}_{h}} h^{-1} h^{2} \\
& \leq C h^{2 k+2}\left|\nabla^{k+1} \boldsymbol{q}\right|_{\boldsymbol{L}^{\infty}(\Omega)}^{2}+C\left\|\boldsymbol{\Pi}_{V} \boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{\boldsymbol{L}^{\infty}(\Omega)}^{2} \\
& \leq C h^{2 k+2}\left(|\log h|^{1 / 2}+1\right)^{2}\left(|\boldsymbol{q}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|u|_{W^{k+2, \infty}(\Omega)}\right)^{2} & \text { by (3.18b). }
\end{array}
$$

This completes the proof of (4.1a).

## 5 Dirichlet Boundary Control Problem

In this section, we consider an elliptic Dirichlet boundary control problem. Let $u_{d} \in L^{2}(\Omega)$ denote a given desired state for the solution, and let $\gamma>0$ be a given regularization parameter. The problem is to solve the following optimization problem:

$$
\begin{equation*}
\min _{g \in L^{2}(\partial \Omega)} J(g), \quad J(g):=\frac{1}{2}\left\|u-u_{d}\right\|_{L^{2}(\Omega)}^{2}+\frac{\gamma}{2}\|g\|_{L^{2}(\partial \Omega)}^{2}, \tag{5.1a}
\end{equation*}
$$

where $u$ is the solution of the Poisson equation with non-homogeneous Dirichlet boundary conditions

$$
\begin{array}{rll}
-\Delta u=f & \text { in } \Omega, \\
u=g & \text { on } \partial \Omega . \tag{5.1c}
\end{array}
$$

The function $g$ is called the control, and computing the optimal $g$ is the desired result of the above problem.

It is well known that the Dirichlet boundary control problem (5.1a)-(1.1b) is equivalent to solving the following optimality system for $(u, z, g)$ given by:

$$
\begin{align*}
-\Delta u & =f & & \text { in } \Omega,  \tag{5.2a}\\
u & =g & & \text { on } \partial \Omega,  \tag{5.2b}\\
-\Delta z & =u-u_{d} & & \text { in } \Omega,  \tag{5.2c}\\
z & =0 & & \text { on } \partial \Omega,  \tag{5.2d}\\
g & =\gamma^{-1} \partial_{\boldsymbol{n}} z & & \text { on } \partial \Omega . \tag{5.2e}
\end{align*}
$$

Define $\boldsymbol{q}=-\nabla u$ and $\boldsymbol{p}=-\nabla z$, then the mixed weak form of (5.2a) -(5.2e) is to find $(\boldsymbol{q}, u, \boldsymbol{p}, z, g) \in$ $\boldsymbol{H}(\operatorname{div}, \Omega) \times L^{2}(\Omega) \times \boldsymbol{H}(\operatorname{div}, \Omega) \times L^{2}(\Omega) \times L^{2}(\partial \Omega)$ such that

$$
\begin{align*}
\left(\boldsymbol{q}, \boldsymbol{v}_{1}\right)-(u, \nabla \cdot \boldsymbol{r})+\left\langle g, \boldsymbol{v}_{1} \cdot \boldsymbol{n}\right\rangle & =0,  \tag{5.3a}\\
\left(\nabla \cdot \boldsymbol{q}, w_{1}\right) & =\left(f, w_{1}\right),  \tag{5.3b}\\
\left(\boldsymbol{p}, \boldsymbol{v}_{2}\right)-\left(z, \nabla \cdot \boldsymbol{v}_{2}\right) & =0,  \tag{5.3c}\\
\left(\nabla \cdot \boldsymbol{p}, w_{2}\right)-\left(u, w_{2}\right) & =-\left(u_{d}, w_{2}\right),  \tag{5.3d}\\
\langle\gamma g+\boldsymbol{p} \cdot \boldsymbol{n}, \xi\rangle & =0 \tag{5.3e}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{1}, w_{1}, \boldsymbol{v}_{2}, w_{2}, \xi\right) \in \boldsymbol{H}(\operatorname{div}, \Omega) \times L^{2}(\Omega) \times \boldsymbol{H}(\operatorname{div}, \Omega) \times L^{2}(\Omega) \times L^{2}(\partial \Omega)$.
To give the HDG formulation of the above mixed system (5.3), we need to introduce the following finite element space for the boundary control $g$ :

$$
\widehat{W}_{h}(\partial):=\left\{\widehat{w}_{h} \in L^{2}\left(\mathcal{F}_{h}^{\partial}\right):\left.\widehat{w}_{h}\right|_{F} \in \mathcal{P}^{k}(F), \forall F \in \mathcal{F}_{h}^{\partial}\right\}
$$

By the definition of $\mathscr{B}$ in (2.3) and setting $c=1$. The HDG formulation of the optimality system (2.1) is to find ( $\left.\boldsymbol{q}_{h}, \boldsymbol{p}_{h}, u_{h}, z_{h}, \widehat{u}_{h}, \widehat{z}_{h}, g_{h}\right) \in \boldsymbol{V}_{h} \times \boldsymbol{V}_{h} \times W_{h} \times W_{h} \times \widehat{W}_{h} \times \widehat{W}_{h} \times \widehat{W}_{h}(\partial)$ such that

$$
\begin{align*}
\mathscr{B}\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h} ; \boldsymbol{v}_{1}, w_{1}, \widehat{w}_{1}\right) & =-\left\langle g_{h}, \boldsymbol{v}_{\mathbf{1}} \cdot \boldsymbol{n}+\tau w_{1}\right\rangle_{\mathcal{F}_{h}^{\partial}}-\left(f, w_{1}\right)_{\mathcal{T}_{h}},  \tag{5.4a}\\
\mathscr{B}\left(\boldsymbol{p}_{h}, z_{h}, \widehat{z}_{h} ; \boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right) & =-\left(u_{h}-u_{d}, w_{2}\right)_{\mathcal{T}_{h}},  \tag{5.4b}\\
\gamma^{-1}\left\langle\boldsymbol{p}_{h} \cdot \boldsymbol{n}+\tau z_{h}, \widehat{w}_{3}\right\rangle_{\mathcal{F}_{h}^{\partial}} & =-\left\langle g_{h}, \widehat{w}_{3}\right\rangle_{\mathcal{F}_{h}^{\partial}} \tag{5.4c}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}, w_{1}, w_{2}, \widehat{w}_{1}, \widehat{w}_{2}, \widehat{w}_{3}\right) \in \boldsymbol{V}_{h} \times \boldsymbol{V}_{h} \times W_{h} \times W_{h} \times \widehat{W}_{h} \times \widehat{W}_{h} \times \widehat{W}_{h}(\partial)$.
We can now state our main result of this section:
Theorem 4. Let ( $\boldsymbol{q}, u, \boldsymbol{p}, z, g$ ) and $\left(\boldsymbol{q}_{h}, u_{h}, \boldsymbol{p}_{h}, z_{h}, g_{h}\right)$ be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{gathered}
\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)}+\left\|u-u_{h}\right\|_{L^{2}(\Omega)}+\left\|z-z_{h}\right\|_{L^{2}(\Omega)}+\left\|\boldsymbol{p}-\boldsymbol{p}_{h}\right\|_{L^{2}(\Omega)}+h^{1 / 2}\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{L^{2}(\Omega)} \\
\leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{p}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right) .
\end{gathered}
$$

Remark 2. Numerical experiments for the Dirichlet boundary control problem given in Equation (5.2) always show optimal order convergence rates if the solution is smooth enough. The first work to prove this observation can be found in [23] by May, Rannacher and Vexler. The proof is based on a duality argument and gives estimates for the control in weaker norms than $L^{2}(\partial \Omega)$. However, this technique is not straightforward for the HDG method, see [4, 16, 17, 21, 22]. Hence, Theorem 4 is the first proof that the HDG method achieves an optimal order convergence rate for the control, state and dual state, provided we assume the solution of the Dirichlet boundary control problem is smooth enough.

### 5.1 Proof of Theorem 4

We follow the strategy in [22] and introduce an auxiliary problem: find $\left(\boldsymbol{q}_{h}(g), \boldsymbol{p}_{h}(g), u_{h}(g), z_{h}(g), \widehat{u}_{h}(g), \widehat{z}_{h}(g)\right) \in$ $\boldsymbol{V}_{h} \times \boldsymbol{V}_{h} \times W_{h} \times W_{h} \times \widehat{W}_{h} \times \widehat{W}_{h}$ such that

$$
\begin{align*}
\mathscr{B}\left(\boldsymbol{q}_{h}(g), u_{h}(g), \widehat{u}_{h}(g) ; \boldsymbol{v}_{1}, w_{1}, \widehat{w}_{1}\right) & =-\left\langle g, \boldsymbol{v}_{\mathbf{1}} \cdot \boldsymbol{n}+\tau w_{1}\right\rangle_{\mathcal{F}_{h}^{\partial}}-\left(f, w_{1}\right)_{\mathcal{T}_{h}},  \tag{5.5a}\\
\mathscr{B}\left(\boldsymbol{p}_{h}(g), z_{h}(g), \widehat{z}_{h}(g) ; \boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right) & =-\left(u-u_{d}, w_{2}\right)_{\mathcal{T}_{h}} \tag{5.5b}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}, w_{1}, w_{2}, \widehat{w}_{1}, \widehat{w}_{2}\right) \in \boldsymbol{V}_{h} \times \boldsymbol{V}_{h} \times W_{h} \times W_{h} \times \widehat{W}_{h} \times \widehat{W}_{h}$, where $g \in L^{2}(\partial \Omega)$ is the exact optimal control.

The proof now proceeds in three steps as follows.
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Step 1 We first bound the error between the solutions of the auxiliary problem Equation (5.5) and the mixed weak form (5.3a)-(5.3e) of the optimality system. The proof of Lemma 15 can be found in [9, Theorem 4.1 Appendix] and Theorem 3.
Lemma 15. Let ( $\boldsymbol{q}, u, \boldsymbol{p}, z, g)$ and $\left(\boldsymbol{q}_{h}(g), \boldsymbol{p}_{h}(g), u_{h}(g), z_{h}(g), \widehat{u}_{h}(g), \widehat{z}_{h}(g)\right)$ be the solution of (5.3) and (5.5), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{align*}
\left\|\boldsymbol{q}-\boldsymbol{q}_{h}(g)\right\|_{\boldsymbol{L}^{2}(\Omega)} & \leq C h^{k+1}\left(|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right),  \tag{5.6a}\\
\left\|u-u_{h}(g)\right\|_{L^{2}(\Omega)} & \leq C h^{k+1}\left(|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right),  \tag{5.6b}\\
\left\|\boldsymbol{p}-\boldsymbol{p}_{h}(g)\right\|_{\boldsymbol{L}^{2}(\Omega)} & \leq C h^{k+1}\left(|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}+|z|_{H^{k+1}(\Omega)}\right),  \tag{5.6c}\\
\left\|z-z_{h}(g)\right\|_{L^{2}(\Omega)} & \leq C h^{k+1}\left(|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}+|z|_{H^{k+1}(\Omega)}\right),  \tag{5.6d}\\
\left\|\boldsymbol{p}_{h}(g)-\boldsymbol{p}\right\|_{L^{2}(\partial \Omega)} & \leq C h^{k+1}\left(|\log h|^{1 / 2}+1\right)\left(|\boldsymbol{p}|_{W^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}\right),  \tag{5.6e}\\
\left\|\tau\left(z_{h}(g)-z\right)\right\|_{L^{2}(\partial \Omega)} & \leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{p}|_{W^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}\right) . \tag{5.6f}
\end{align*}
$$

Step 2 Next, we bound the error between the solutions of the auxiliary problem and the HDG problem (2.4). Note that

$$
\begin{align*}
\mathscr{B}\left(\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h} ; \boldsymbol{v}_{1}, w_{1}, \widehat{w}_{1}\right) & =-\left\langle g-g_{h}, \boldsymbol{v}_{1} \cdot \boldsymbol{n}+\tau w_{1}\right\rangle_{\mathcal{F}_{h}^{\partial}},  \tag{5.7a}\\
\mathscr{B}\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h} ; \boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right) & =-\left(u-u_{h}, w_{2}\right)_{\mathcal{T}_{h}} \tag{5.7b}
\end{align*}
$$

for all $\left(\boldsymbol{v}_{1}, \boldsymbol{v}_{2}, w_{1}, w_{2}, \widehat{w}_{1}, \widehat{w}_{2}\right) \in \boldsymbol{V}_{h} \times \boldsymbol{V}_{h} \times W_{h} \times W_{h} \times \widehat{W}_{h} \times \widehat{W}_{h}$.
Lemma 16. Let $(u, g)$ and $\left(u_{h}, g_{h}\right)$ be the solution of (5.3) and (55.4), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{aligned}
&\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)} \leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{p}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right), \\
&\left\|u-u_{h}\right\|_{L^{2}(\Omega)} \leq C h^{k+1}(|\log h|+1)\left(|\boldsymbol{p}|_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right) .
\end{aligned}
$$

Proof. First, we take $\left(\boldsymbol{v}_{1}, w_{1}, \widehat{w}_{1}\right)=\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h}\right),\left(\boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right)=\left(\boldsymbol{q}_{h}(g)-\right.$ $\left.\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h}\right)$ in (5.7), and use Lemma 2 to get

$$
\left\langle g-g_{h},\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right) \cdot \boldsymbol{n}+\tau\left(z_{h}(g)-z_{h}\right)\right\rangle_{\mathcal{F}_{h}^{\partial}}=\left(u-u_{h}, u_{h}(g)-u_{h}\right) \mathcal{T}_{h} .
$$

Since $g+\gamma^{-1} \boldsymbol{p} \cdot \boldsymbol{n}=0$ on $\mathcal{F}_{h}^{\partial}$ and $g_{h}+\gamma^{-1} \boldsymbol{p}_{h} \cdot \boldsymbol{n}+\gamma^{-1} \tau z_{h}=0$ on $\mathcal{F}_{h}^{\partial}$ we have

$$
\begin{aligned}
\left(u-u_{h}, u_{h}(g)-u_{h}\right)_{\mathcal{T}_{h}} & =\left\langle g-g_{h}, \boldsymbol{p}_{h}(g) \cdot \boldsymbol{n}+\tau z_{h}(g)+\gamma g_{h}\right\rangle_{\mathcal{F}_{h}^{\partial}} \\
& =\left\langle g-g_{h}, \boldsymbol{p}_{h}(g) \cdot \boldsymbol{n}-\boldsymbol{p} \cdot \boldsymbol{n}+\boldsymbol{p} \cdot \boldsymbol{n}+\tau z_{h}(g)+\gamma g_{h}\right\rangle_{\mathcal{F}_{h}^{\partial}} \\
& =\left\langle g-g_{h}, \boldsymbol{p}_{h}(g) \cdot \boldsymbol{n}-\boldsymbol{p} \cdot \boldsymbol{n}+\tau z_{h}(g)+\gamma g_{h}-\gamma g\right\rangle_{\mathcal{F}_{h}^{\partial}} \\
& =\left\langle g-g_{h}, \boldsymbol{p}_{h}(g) \cdot \boldsymbol{n}-\boldsymbol{p} \cdot \boldsymbol{n}+\tau z_{h}(g)\right\rangle_{\mathcal{F}_{h}^{\partial}}-\gamma\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)}^{2} .
\end{aligned}
$$

Since $z=0$ on $\mathcal{F}_{h}^{\partial}$, we rearrange the above equality and obtain

$$
\begin{aligned}
& \gamma\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)}^{2}+\left\|u-u_{h}\right\|_{L^{2}(\Omega)}^{2} \\
& \quad=\left\langle\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}\right) \cdot \boldsymbol{n}+\tau z_{h}(g), g-g_{h}\right\rangle_{\mathcal{F}_{h}^{a}}-\left(u-u_{h}, u_{h}(g)-u\right)_{\mathcal{T}_{h}} \\
& \leq\left(\left\|\boldsymbol{p}_{h}(g)-\boldsymbol{p}\right\|_{L^{2}(\partial \Omega)}+\left\|\tau\left(z_{h}(g)-z\right)\right\|_{L^{2}(\partial \Omega)}\right)\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)} \\
& \quad+\left\|u-u_{h}\right\|_{L^{2}(\Omega)}\left\|u_{h}(g)-u\right\|_{L^{2}(\Omega)} .
\end{aligned}
$$

Our desired result follows by Young's inequality, the triangle inequality and Lemma 15 .

## Step 3

Lemma 17. Let $(\boldsymbol{p}, z)$ and $\left(\boldsymbol{p}_{h}, z_{h}\right)$ be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\begin{aligned}
&\left\|\boldsymbol{p}-\boldsymbol{p}_{h}\right\|_{L^{2}(\Omega)} \leq C h^{k+1}(|\log h|+1)\left(\left|\boldsymbol{p}_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right),\right. \\
&\left\|z-z_{h}\right\|_{L^{2}(\Omega)} \leq C h^{k+1}(|\log h|+1)\left(\left|\boldsymbol{p}_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right) .\right.
\end{aligned}
$$

Proof. By Lemma 1 and letting $\left(\boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right)=\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h}\right)$ in the error equation (5.7b), we have

$$
\begin{align*}
\left\|\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right\|_{L^{2}(\Omega)}^{2} & \leq-\left(u-u_{h}, z_{h}(g)-z_{h}\right) \mathcal{T}_{h} \\
& \leq\left\|u-u_{h}\right\|_{\mathcal{T}_{h}}\left\|z_{h}(g)-z_{h}\right\|_{\mathcal{T}_{h}}  \tag{5.8}\\
& \leq \frac{1}{\rho}\left\|u-u_{h}\right\|_{\mathcal{T}_{h}}^{2}+\rho\left\|z_{h}(g)-z_{h}\right\|_{\mathcal{T}_{h}}^{2}
\end{align*}
$$

Here $\rho$ is a positive constant which will be assigned later. Next, we introduce the dual problem of finding $(\boldsymbol{\Phi}, \Psi)$ such that

$$
\begin{align*}
c \boldsymbol{\Phi}+\nabla \Psi & =0 & & \text { in } \Omega, \\
\nabla \cdot \boldsymbol{\Phi} & =z_{h}(g)-z_{h} & & \text { in } \Omega,  \tag{5.9}\\
\Psi & =0 & & \text { on } \partial \Omega .
\end{align*}
$$

Since the domain $\Omega$ is convex, we have the following regularity estimate

$$
\begin{equation*}
\|\boldsymbol{\Phi}\|_{\boldsymbol{H}^{1}(\Omega)}+\|\Psi\|_{H^{2}(\Omega)} \leq C_{\mathrm{reg}}\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)} . \tag{5.10}
\end{equation*}
$$

On the one hand, we take $\left(\boldsymbol{v}_{2}, w_{2}, \widehat{w}_{2}\right)=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}, \Pi_{W} \Psi, \Pi_{k}^{\partial} \Psi\right)$ in (5.7b) to get

$$
\begin{align*}
& \mathscr{B}\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h} ; \boldsymbol{\Pi}_{V} \boldsymbol{\Phi}, \Pi_{W} \Psi, \Pi_{k}^{\partial} \Psi\right) \\
& \quad=\mathscr{B}^{\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}, \Pi_{W} \Psi, \Pi_{k}^{a} \Psi ; \boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h}\right)} \quad=\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}-\boldsymbol{\Phi}, \boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right)_{\mathcal{T}_{h}}+\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2} . \tag{5.11}
\end{align*}
$$

On the other hand, by the error equation (5.7b), we have

$$
\begin{equation*}
\mathscr{B}\left(\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}, z_{h}(g)-z_{h}, \widehat{z}_{h}(g)-\widehat{z}_{h} ; \boldsymbol{\Pi}_{V} \boldsymbol{\Phi}, \Pi_{W} \Psi, \Pi_{k}^{\partial} \Psi\right)=-\left(u-u_{h}, \Pi_{W} \Psi\right)_{\mathcal{T}_{h}} \tag{5.12}
\end{equation*}
$$

Comparing the above two equalities (5.11), (5.12) and (2.8b) gives

$$
\begin{aligned}
\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2}= & -\left(u-u_{h}, \Pi_{W} \Psi\right)_{\mathcal{T}_{h}}-\left(\boldsymbol{\Pi}_{V} \boldsymbol{\Phi}-\boldsymbol{\Phi}, \boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right)_{\mathcal{T}_{h}} \\
\leq & C\left\|u-u_{h}\right\|_{L^{2}(\Omega)}^{2}+\frac{1}{4}\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{h^{2}}{4}\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2}+C\left\|\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right\|_{L^{2}(\Omega)}^{2} \\
\leq & C\left(1+\frac{1}{\rho}\right)\left\|u-u_{h}\right\|_{L^{2}(\Omega)}^{2}+\frac{1}{2}\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2}+C \rho\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)}^{2} .
\end{aligned}
$$

Taking $\rho=\frac{1}{4 C}$, then we have

$$
\begin{equation*}
\left\|z_{h}(g)-z_{h}\right\|_{L^{2}(\Omega)} \leq C\left\|u-u_{h}\right\|_{L^{2}(\Omega)} \tag{5.13}
\end{equation*}
$$

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Inserting this inequality into (5.8) gives

$$
\begin{equation*}
\left\|\boldsymbol{p}_{h}(g)-\boldsymbol{p}_{h}\right\|_{\boldsymbol{L}^{2}(\Omega)} \leq C\left\|u-u_{h}\right\|_{L^{2}(\Omega)} . \tag{5.14}
\end{equation*}
$$

Then our desired result follows by (5.13), (5.14) and Lemma 17 .
Lemma 18. Let ( $\boldsymbol{q}, u, \boldsymbol{p}, z, g$ ) and ( $\left.\boldsymbol{q}_{h}, u_{h}, \boldsymbol{p}_{h}, z_{h}, g_{h}\right)$ be the solution of (5.3) and (5.4), respectively. If assumption (A) holds and $k \geqslant 1$, then we have:

$$
\left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{L^{2}(\Omega)} \leq C h^{k+\frac{1}{2}}(|\log h|+1)\left(\left|\boldsymbol{p}_{\boldsymbol{W}^{k+1, \infty}(\Omega)}+|z|_{W^{k+1, \infty}(\Omega)}+|\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(\Omega)}+|u|_{H^{k+1}(\Omega)}\right) .\right.
$$

Proof. On the one hand, by the error equation (5.7a), we have

$$
\begin{aligned}
& \mathscr{B}\left(\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h} ; \boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h}\right) \\
& \quad=-\left\langle g-g_{h},\left(\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}\right) \cdot \boldsymbol{n}-\tau\left(u_{h}(g)-u_{h}\right)\right\rangle_{\mathcal{F}_{h}} \\
& \quad \leq\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)}\left(\left\|\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}\right\|_{L^{2}(\partial \Omega)}+\left\|u_{h}(g)-u_{h}\right\|_{L^{2}(\partial \Omega)}\right) \\
& \quad \leq C h^{-1 / 2}\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)}\left(\left\|\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}\right\|_{L^{2}(\Omega)}+\left\|u_{h}(g)-u_{h}\right\|_{L^{2}(\Omega)}\right) .
\end{aligned}
$$

On the other hand, by Lemma 1, we obtain

$$
\begin{aligned}
& \mathscr{B}\left(\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h} ; \boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}, u_{h}(g)-u_{h}, \widehat{u}_{h}(g)-\widehat{u}_{h}\right) \\
& \quad=\left\|\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}\right\|_{\mathcal{T}_{h}}^{2}+\left\|\sqrt{\tau}\left(\left(u_{h}(g)-u_{h}\right)-\left(\widehat{u}_{h}(g)-\widehat{u}_{h}\right)\right)\right\|_{\partial \mathcal{T}_{h}} .
\end{aligned}
$$

Comparing the above two inequalities, using Young's inequality and (16) gives

$$
\begin{equation*}
\left\|\boldsymbol{q}_{h}(g)-\boldsymbol{q}_{h}\right\|_{\mathcal{T}_{h}} \leq C h^{-1 / 2}\left\|g-g_{h}\right\|_{L^{2}(\partial \Omega)} \tag{5.15}
\end{equation*}
$$

Then our desired result follows by Lemma 17, the triangle inequality and (5.6a).

## 6 Numerical Results

In this section, we present two examples to illustrate our theoretical results.
Example 1. We first test the convergence rate of the $L^{\infty}$ norm estimate on a convex domain and the $L^{2}$ norm estimate on the boundary. The data is chosen to be

$$
\Omega=(0,1) \times(0,1), \quad c=1, \quad u(x, y)=\sin (10 x) .
$$

The source term $f$ is chosen to match the exact solution of Equation (1.1) and the approximation errors are listed in Table 1 for the $L^{\infty}(\Omega)$ norm error and Table 2 for the $L^{2}(\partial \Omega)$ norm error. The rates match the theoretical predictions in Theorems 2 and 3.

Our theoretical result needs the domain to be convex, but it is interesting to observe whether the convergence rate can still hold for a non-convex domain. For example, we choose the same data as above except the domain is chosen to be an $L$-shape domain:

$$
\Omega=(0,1) \times(0,1) \backslash[1 / 2,1) \times(0,1 / 2] .
$$

In this case the $H^{2}$ regularity of $\Psi_{1}$ and $\Psi_{2}$ in Lemma 7 does not hold. The approximation errors are listed in Table 3 for the $L^{\infty}(\Omega)$ norm error (the $L^{2}(\partial \Omega)$ norm error also converges at the quasioptimal rate: results are not shown). It is obvious that the quasi-optimal convergence rate is still seen for the $L$-shape domain.

| Degree | $\frac{h}{\sqrt{2}}$ | $\boldsymbol{q}-\boldsymbol{q}_{h} \\|_{\boldsymbol{L}^{\infty}(\Omega)}$ |  | $u-u_{h} \\|_{L^{\infty}(\Omega)}$ |  | $u-u_{h}^{\star} \\|_{L^{\infty}(\Omega)}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error | Rate | Error | Rate | Error | Rate |
| $k=1$ | $2^{-1}$ | $1.8881 \mathrm{E}+01$ | - | $8.1191 \mathrm{E}+00$ | - | $1.4941 \mathrm{E}+00$ | - |
|  | $2^{-2}$ | $1.0384 \mathrm{E}+01$ | 0.86 | $2.9595 \mathrm{E}+00$ | 1.46 | $4.0248 \mathrm{E}-01$ | 1.89 |
|  | $2^{-3}$ | $2.9862 \mathrm{E}+00$ | 1.80 | 7.7800E-01 | 1.93 | $5.9420 \mathrm{E}-02$ | 2.76 |
|  | $2^{-4}$ | 7.5737E-01 | 1.98 | $2.0046 \mathrm{E}-01$ | 1.96 | 7.6187E-03 | 2.96 |
|  | $2^{-5}$ | 1.9487E-01 | 1.96 | 4.9683E-02 | 2.01 | 9.7985E-04 | 2.96 |
| $k=2$ | $2^{-1}$ | $1.8115 \mathrm{E}+01$ | - | $6.5763 \mathrm{E}+00$ | - | 7.2961E-01 | - |
|  | $2^{-2}$ | $3.4370 \mathrm{E}+00$ | 2.40 | $1.0994 \mathrm{E}+00$ | 2.58 | 6.9452E-02 | 3.39 |
|  | $2^{-3}$ | $4.7355 \mathrm{E}-01$ | 2.86 | $1.4548 \mathrm{E}-01$ | 2.92 | $4.6990 \mathrm{E}-03$ | 3.89 |
|  | $2^{-4}$ | 6.2699E-02 | 2.92 | $1.7948 \mathrm{E}-02$ | 3.02 | 3.1054E-04 | 3.92 |
|  | $2^{-5}$ | 7.8798E-03 | 2.99 | $2.2918 \mathrm{E}-03$ | 2.97 | $1.9522 \mathrm{E}-05$ | 3.99 |

Table 1: Example 1: $L^{\infty}(\Omega)$ errors for $\boldsymbol{q}_{h}, u_{h}$ and $u_{h}^{\star}$ on the convex domain $(0,1) \times(0,1)$.

| Degree | $\frac{h}{\sqrt{2}}\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|_{\boldsymbol{L}^{2}(\partial \Omega)}$ | $\left\\|u-u_{h}\right\\|_{L^{2}(\partial \Omega)}$ |  | $\left\\|u-u_{h}^{\star}\right\\|_{L^{2}(\partial \Omega)}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error | Rate | Error | Rate | Error | Rate |
| $k=1$ | $2^{-1}$ | $9.3751 \mathrm{E}+00$ | - | $3.9706 \mathrm{E}+00$ | - | $5.4467 \mathrm{E}-01$ | - |
|  | $2^{-2}$ | $4.1197 \mathrm{E}+00$ | 1.19 | $1.9143 \mathrm{E}+00$ | 1.05 | $1.0446 \mathrm{E}-01$ | 2.38 |
|  | $2^{-3}$ | $1.1791 \mathrm{E}+00$ | 1.80 | $6.1659 \mathrm{E}-01$ | 1.63 | $1.4777 \mathrm{E}-02$ | 2.82 |
|  | $2^{-4}$ | $3.0648 \mathrm{E}-01$ | 1.94 | $1.6398 \mathrm{E}-01$ | 1.91 | $1.9370 \mathrm{E}-03$ | 2.93 |
|  | $2^{-5}$ | $7.7039 \mathrm{E}-02$ | 1.99 | $4.1472 \mathrm{E}-02$ | 1.98 | $2.4450 \mathrm{E}-04$ | 2.99 |
| $k=2$ | $2^{-1}$ | $6.4399 \mathrm{E}+00$ | - | $3.4609 \mathrm{E}+00$ | - | $1.9906 \mathrm{E}-01$ | - |
|  | $2^{-2}$ | $9.3121 \mathrm{E}-01$ | 2.79 | $5.3204 \mathrm{E}-01$ | 2.70 | $1.3075 \mathrm{E}-02$ | 3.93 |
|  | $2^{-3}$ | $1.1602 \mathrm{E}-01$ | 3.00 | $5.7436 \mathrm{E}-02$ | 3.21 | $9.1221 \mathrm{E}-04$ | 3.84 |
|  | $2^{-4}$ | $1.4279 \mathrm{E}-02$ | 3.02 | $6.5665 \mathrm{E}-03$ | 3.13 | $5.8411 \mathrm{E}-05$ | 3.97 |
|  | $2^{-5}$ | $1.7866 \mathrm{E}-03$ | 3.00 | $8.0200 \mathrm{E}-04$ | 3.03 | $3.6752 \mathrm{E}-06$ | 3.99 |

Table 2: Example 1. $L^{2}(\partial \Omega)$ errors for $\boldsymbol{q}_{h}, u_{h}$ and $u_{h}^{\star}$ on the convex domain $(0,1) \times(0,1)$.

| Degree | $\frac{h}{\sqrt{2}}$ | $\boldsymbol{q}-\boldsymbol{q}_{h} \\|_{\boldsymbol{L}^{\infty}(\Omega)}$ |  | $u-u_{h} \\|_{L^{\infty}(\Omega)}$ |  | $\left\\|u-u_{h}^{\star}\right\\|_{L^{\infty}(\Omega)}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error | Rate | Error | Rate | Error | Rate |
| $k=1$ | $2^{-1}$ | $1.9604 \mathrm{E}+01$ | - | $8.1190 \mathrm{E}+00$ | - | $1.4713 \mathrm{E}+00$ | - |
|  | $2^{-2}$ | $9.9832 \mathrm{E}+00$ | 0.97 | $2.9608 \mathrm{E}+00$ | 1.46 | $3.7109 \mathrm{E}-01$ | 1.99 |
|  | $2^{-3}$ | $2.9810 \mathrm{E}+00$ | 1.74 | $7.7748 \mathrm{E}-01$ | 1.93 | $5.9410 \mathrm{E}-02$ | 2.64 |
|  | $2^{-4}$ | 7.5727E-01 | 1.98 | $2.0046 \mathrm{E}-01$ | 1.96 | $7.6187 \mathrm{E}-03$ | 2.96 |
|  | $2^{-5}$ | $1.9487 \mathrm{E}-01$ | 1.96 | $4.9683 \mathrm{E}-02$ | 2.01 | $9.8015 \mathrm{E}-04$ | 2.96 |
| $k=2$ | $2^{-1}$ | $1.6115 \mathrm{E}+01$ |  | $6.4608 \mathrm{E}+00$ |  | $5.6157 \mathrm{E}-01$ |  |
|  | $2^{-2}$ | $3.4372 \mathrm{E}+00$ | 2.23 | $1.0994 \mathrm{E}+00$ | 2.55 | $6.9454 \mathrm{E}-02$ | 3.02 |
|  | $2^{-3}$ | 4.7348E-01 | 2.86 | $1.4548 \mathrm{E}-01$ | 2.92 | $4.7007 \mathrm{E}-03$ | 3.89 |
|  | $2^{-4}$ | 6.2862E-02 | 2.91 | $1.7948 \mathrm{E}-02$ | 3.02 | $3.1283 \mathrm{E}-04$ | 3.91 |
|  | $2^{-5}$ | $7.8980 \mathrm{E}-03$ | 2.99 | $2.2918 \mathrm{E}-03$ | 2.97 | $1.9653 \mathrm{E}-05$ | 3.99 |

Table 3: Example 1 $L^{\infty}(\Omega)$ errors for $\boldsymbol{q}_{h}, u_{h}$ and $u_{h}^{\star}$ on the nonconvex L-shaped domain.
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| $h / \sqrt{2}$ | $1 / 16$ | $1 / 32$ | $1 / 64$ | $1 / 128$ | $1 / 256$ | EO |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|_{\boldsymbol{L}^{2}(\Omega)}$ | $2.1856 \mathrm{E}-02$ | $6.3683 \mathrm{E}-03$ | $1.9677 \mathrm{E}-03$ | $6.3980 \mathrm{E}-04$ | $2.1568 \mathrm{E}-04$ |  |
| order | - | 1.78 | 1.69 | 1.62 | 1.57 | 1.50 |
| $\left\\|\boldsymbol{p}-\boldsymbol{p}_{h}\right\\|_{\boldsymbol{L}^{2}(\Omega)}$ | $6.3866 \mathrm{E}-03$ | $1.5958 \mathrm{E}-03$ | $3.9873 \mathrm{E}-04$ | $9.9650 \mathrm{E}-05$ | $2.4911 \mathrm{E}-05$ |  |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\left\\|u-u_{h}\right\\|_{L^{2}(\Omega)}$ | $8.3560 \mathrm{E}-03$ | $2.1051 \mathrm{E}-03$ | $5.2796 \mathrm{E}-04$ | $1.3218 \mathrm{E}-04$ | $3.3073 \mathrm{E}-05$ |  |
| order | - | 1.99 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\left\\|z-z_{h}\right\\|_{L^{2}(\Omega)}$ | $3.1536 \mathrm{E}-03$ | $7.9650 \mathrm{E}-04$ | $2.0006 \mathrm{E}-04$ | $5.0125 \mathrm{E}-05$ | $1.2545 \mathrm{E}-05$ |  |
| order | - | 1.99 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\left\\|g-g_{h}\right\\|_{L^{2}(\partial \Omega)}$ | $7.2110 \mathrm{E}-03$ | $1.8119 \mathrm{E}-03$ | $4.5412 \mathrm{E}-04$ | $1.1367 \mathrm{E}-04$ | $2.8425 \mathrm{E}-05$ |  |
| order | - | 1.99 | 2.00 | 2.00 | 2.00 | 2.00 |

Table 4: Example 2, $k=1$ : Errors, observed convergence orders, and expected order (EO) for the control $g$, state $u$, adjoint state $z$, and their fluxes $\boldsymbol{q}$ and $\boldsymbol{p}$.

Example 2. Lastly, we test the convergence rate for a smooth solution to the Dirichlet boundary control problem. The data and the exact solution is chosen to be

$$
\Omega=(0,1) \times(0,1), \quad \gamma=1, \quad u(x, y)=-\pi(\sin (\pi x)+\sin (\pi y)), \quad z(x, y)=\sin (\pi x) \sin (\pi y) .
$$

The source term $f$, the desired state $u_{d}$ and the control $g$ are chosen to match the exact solution of Equation (5.2) and the approximation errors are listed in Table 4 when $k=1$. Results (not shown) for $k=2$ also confirm the predicted higher order convergence rate in this case. The rates are matched with Theorem 4.

## 7 Conclusion

We have proved quasi-optimal $L^{\infty}$ norm estimates for the Poisson equation in 2D. Using this result, we obtained quasi-optimal $L^{2}$ estimates on an interface. Moreover, we obtained quasioptimal convergence rates for the Dirichlet boundary control of Poisson's equation, provided the solution is smooth enough.

Our work suggests several interesting directions for further research. First we would like to extend the results to cover $L^{\infty}$ norm estimates in 3D. In addition the quasi-uniformity assumption on our mesh is restrictive for problems that require adaptive mesh refinement, including those on non-convex domains. Finally it would be desirable to prove the optimal convergence rate for the Dirichlet boundary control of PDEs without assuming that the solution is smooth.

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