# On the Number of Circuits in Regular Matroids (with Connections to Lattices and Codes)

Rohit Gurjar<sup>1</sup> and Nisheeth K. Vishnoi<sup>2</sup>

<sup>1</sup>Indian Institute of Technology Bombay, India <sup>2</sup>École Polytechnique Fédérale de Lausanne (EPFL), Switzerland

#### Abstract

We show that for any regular matroid on m elements and any  $\alpha \geq 1$ , the number of  $\alpha$ -minimum circuits, or circuits whose size is at most an  $\alpha$ -multiple of the minimum size of a circuit in the matroid is bounded by  $m^{O(\alpha^2)}$ . This generalizes a result of Karger for the number of  $\alpha$ -minimum cuts in a graph. As a consequence, we obtain similar bounds on the number of  $\alpha$ -shortest vectors in "totally unimodular" lattices and on the number of  $\alpha$ -minimum weight codewords in "regular" codes.

### Contents

1	Introduction	3
	1.1 Our results	5
	1.2 Techniques	6
	1.3 Future directions	7
2	Overview of our proof and comparison with previous work	7
3	Matroid preliminaries	13
	3.1 Matroids and circuits	13
	3.2 $k$ -sums and Seymour's Theorem	15
4	Number of $\alpha$ -shortest vectors in a totally unimodular lattice	17
5	A strengthening of Seymour's Theorem	18
	5.1 Associativity of the $k$ -sum	19
	5.2 Unordered decomposition tree	21
6	Bounding the number of circuits in a regular matroid	24
	6.1 Decomposing the tree $T$ into smaller weight subtrees	24
	6.2 Classifying the circuits using signatures	27
	6.3 Bounding the number of circuits for a given signature	28
7	Max-flow min-cut matroids	32
$\mathbf{A}$	Bounding the number of circuits in a graphic or cographic matroid	38

### 1 Introduction

We study a general question about the number of certain structures, with respect to their sizes, arising in three different settings: matroids, codes, and lattices. More precisely, we are interested in the growth of the number of circuits in a matroid, the number codewords in a code, and the number of vectors in an integral lattice with respect to their size, weight, and length, respectively. These questions have been extensively studied in various forms in areas such as combinatorial optimization, information and coding theory, and discrete geometry (e.g., see [5, 16, 18, 23]).

In all of these cases, a trivial (and rough) upper bound on the number of these objects of size k is  $m^{\text{poly}(k)}$ , where m is the underlying ground set size in the case of matroids, length for codes, and dimension in the case of lattices. There are also elementary constructions of matroids/codes/lattices where these upper bounds are tight (e.g., graphic matroid of the complete graph, the trivial code with distance 1, the lattice  $\mathbb{Z}^m$ ). However, consider the setting where the shortest size of such an object is r. The question of interest is, when the size k is close to the shortest size r, whether the number of objects still grows like  $m^{\text{poly}(k)}$ . Since circuits are well-studied objects in matroids, this is a natural question in the context of matroids. In coding theory, the motivation to study this question comes from "list decodability" (see [29]), and in lattices, this relates to the widely studied question of the "kissing number" of a lattice packing (see [5]). As we subsequently explain, these three questions turn out to be intimately connected in certain cases.

Circuits in matroids. A circuit of a matroid is a minimal dependent set of its ground elements. A motivation for the above question is a seminal result of Karger [19] who showed that for a cographic matroid, the number of near-minimum circuits – circuits whose sizes are at most a constant multiple of the minimum size of a circuit– is bounded by poly(m), that is, independent of the minimum size. The circuits of a cographic matroid are the simple cut-sets of the associated graph, and Karger's result was actually presented in terms of the number of near-minimum cuts in a graph.

An analogous result is also known in the "dual" setting of graphic matroids. The circuits of a graphic matroid are simple cycles in a graph. Subramanian [28] (building on [30]) showed a poly(m) bound on the number of near-minimum cycles. Quantitatively, the results of Karger and Subramanian show that in a graphic/cographic matroid, if the shortest circuit has size r, then the number of circuits of size at most  $\alpha r$ , or  $\alpha$ -minimum circuits, is bounded by  $(2m)^{2\alpha}$ . Subramanian raised the question of identifying other matroids that have only polynomially many near-minimum circuits.

Do all matroids have such a property? The answer is no: the uniform matroid can have exponentially many shortest circuits.<sup>1</sup> Since a uniform matroid is representable (by a family of vectors over some field), one can also rule out the possibility of an affirmative answer for all representable matroids.

The next natural candidate to consider would be the class of binary matroids – matroids representable over GF(2) – that also contains graphic and cographic matroids. Circuits of

<sup>&</sup>lt;sup>1</sup>Consider the uniform matroid  $U_{r,m}$ , a matroid of ground set size m where every subset of size at most r is independent. A circuit of  $U_{r,m}$  is any subset of size r+1. Thus, the number of shortest circuits is  $\binom{m}{r+1}$ , i.e., exponential in r.

binary matroids are closely connected to codewords in binary linear codes and have received considerable attention from this perspective.

Binary linear codes. Let A be a matrix over GF(2) representing a binary matroid M on the ground set [m], i.e., A has m columns, and a set is independent in M if and only if the corresponding set of columns in A is linearly independent. Consider the linear code C whose parity check matrix is A. The codewords of C are the vectors in nullspace(A) and thus, are precisely the disjoint unions of circuits of M. More importantly, the minimum weight of a codeword in C and the minimum size of a circuit in M are same. And thus, any  $\alpha$ -minimum weight codeword of C comes from a union of  $\alpha$ -minimum circuits of M. Thus, the question arises: do all binary linear codes have a small number of minimum or near-minimum weight codewords?

This question derives interest from the perspective of list decoding. Alon [2] gave a construction of a binary linear code where there are  $2^{\Omega(\sqrt{m})}$  codewords of minimum weight. Kalai and Linial [16] studied distance distributions of codes and conjectured that the above number should be  $2^{o(m)}$  for all binary linear codes. However, Ashikhmin, Barg and Vlăduţ [3] disproved this conjecture by giving an explicit binary code with  $2^{\Omega(m)}$  minimum weight codewords. The question is actually much easier to answer when we consider near-minimum weight codewords. Most binary linear codes have exponentially many near-minimum weight codewords<sup>2</sup>.

In short, we cannot get the desired polynomial bound for all binary matroids or binary linear codes. Can we identify a subclass of binary matroids where this is true? Let us first briefly see the history of the analogous question in lattices.

**Lattices.** The question of the number of shortest vectors in a lattice has attracted a lot of attention in mathematics. This number is also referred to as the "kissing number" of a lattice packing of spheres. Consider, for example, the lattice  $\mathbb{Z}^m$ . The number of shortest vectors in this lattice is simply 2m. Moreover, the number of near-shortest vectors – whose length is at most a constant multiple of the shortest length – is bounded by poly(m).

Such a bound does not hold for general lattices. It is widely conjectured that there exists a lattice packing with an exponentially large kissing number. However, the best known lower bound on the kissing number of an m-dimensional lattice is only  $2^{\Omega(\log^2 m)}$  ([22], also see [6]). On the other hand, if we consider the number of near-shortest vectors in lattices, much higher bounds are known. Ajtai [1] showed that for some constants  $\epsilon, \delta > 0$  and infinitely many integers m, there exists an m-dimensional lattice that has at least  $2^{m^{\epsilon}}$  vectors of length at most  $(1+2^{-m^{\delta}})$  times the length of the shortest vector. A polynomial bound on the number of near-shortest vectors could still hold for some special class of lattices. It is an interesting question to characterize such lattices.

<sup>&</sup>lt;sup>2</sup>For a binary code with a random parity check matrix of dimensions  $\lambda m \times m$ , all its codewords have weights in the range [m/3, 2m/3] with a high probability, when  $\lambda > h(1/3) = -(1/3)\log(1/3) - (2/3)\log(2/3)) \approx 0.918$  (see [4]).

#### 1.1 Our results

We make progress on the above questions about matroids, lattices, and codes by showing that, for a large class of each, the number of  $\alpha$ -minimum circuits, vectors, or codewords grow as  $m^{\text{poly}(\alpha)}$ . Our main result concerns matroids and the others are derived via connections between matroids and lattices and matroids and codes.

Near-minimum circuits in matroids. We answer the above question about  $\alpha$ -minimum circuits in affirmative for an extensively studied subclass of binary matroids – called regular matroids. These are matroids that can be represented by a family of vectors over every field.

Theorem 1.1 (Number of near-minimum circuits in a regular matroid). Let M be a regular matroid with ground set size m. Suppose that M has no circuits of size at most r. Then for any  $\alpha \geq 1$ , the number of circuits in M of size at most  $\alpha r$  is bounded by  $m^{O(\alpha^2)}$ .

Since graphic and cographic matroids are the two simplest cases of regular matroids, our result significantly generalizes the results of Karger [19] and Subramanian [28]. Moreover, our result also holds for a class, more general than regular matroids, namely max-flow mincut matroids (see Section 7). This is the most general class of binary matroids, where a natural generalization of max-flow min-cut theorem continues to hold.

A recent work [14] took a step towards answering this question for regular matroids. They showed a polynomial upper bound on the number of circuits whose size is less than 3/2 times the minimum size of a circuit. However, a serious shortcoming of their work is that the proof breaks down for  $\alpha \geq 3/2$  (see Section 2).

List decodability of codes. Since binary matroids have close connections with binary linear codes, our Theorem 1.1 implies a list decodability result for certain special binary linear codes, called regular codes. A regular code – defined in [20] – is a binary linear code such that the columns of its parity check matrix represent a regular matroid. We get that for a regular code with distance d and for any constant  $\alpha$ , the number of codewords with Hamming weight at most  $\alpha d$  is polynomially bounded. This means that C is  $(\alpha d, \text{poly}(m))$ -list decodable, for any constant  $\alpha$ . This is in contrast to general binary linear codes, which are not list-decodable beyond the minimum distance.

Corollary 1.2 (List decodability of regular codes). For a regular code  $C \subseteq GF(2)^m$  with distance d, and any  $\alpha \geq 1$ , the number of codewords with Hamming weight at most  $\alpha d$  is bounded by  $m^{O(\alpha^3)}$ .

To see this, observe that any codeword of weight at most  $\alpha d$  comes from a combination of at most  $\alpha$  circuits of M, since each circuit has size at least d. Since we have a bound of  $m^{O(\alpha^2)}$  on the number of circuits from Theorem 1.1, the bound on the number of codewords follows.

Near-shortest vectors in lattices. In general, matroids are not related to lattices. However, since regular matroids are also representable over the real field, they happen to be connected to certain lattices. A result in matroid theory (see [25]) states that a matroid

is regular if and only if it can be represented by the set of columns of a totally unimodular matrix (TUM). A matrix (over reals) is a TUM if each of its minors is either 0, 1, or -1. TUM are of fundamental importance in discrete optimization, as they are related to the integrality of polyhedra.

We define a lattice corresponding to a TUM, called totally unimodular lattice. For a TUM A, the lattice L(A) of the set of integral vectors in nullspace(A) is said to be a totally unimodular lattice.

$$L(A) := \{ v \in \mathbb{Z}^m \mid Av = 0 \}.$$

It turns out that the near-shortest vectors in a totally unimodular lattice can be related to the near-minimum circuits of the associated regular matroid. And thus, our Theorem 1.1 implies a polynomial upper bound on the number of near-shortest vectors in totally unimodular lattices.

Theorem 1.3 (Number of near-shortest vectors in TU lattices). Let A be an  $n \times m$  totally unimodular matrix. Suppose any nonzero vector  $u \in L(A)$  has a length more than  $\lambda$ . Then for any  $\alpha \geq 1$ , the number of vectors in L(A) of length at most  $\alpha\lambda$  is  $m^{O(\alpha^6)}$ .

Here by the length of a vector we mean its  $\ell_2$ -norm.<sup>3</sup>

### 1.2 Techniques

A landmark result in the study of regular matroids was Seymour's decomposition theorem [26], that also implied the first polynomial time algorithm for testing total unimodularity of a matrix. The theorem states that every regular matroid can be decomposed into simpler matroids, each of which is graphic, cographic, or a special 10-element matroid  $R_{10}$ . The theorem has found many uses in discrete optimization and also used to prove some structural results about regular matroids. To prove any result about regular matroids, a generic approach can be to first prove the corresponding results for the above simpler matroids and then "piece" them together using Seymour's theorem to "lift" the result to regular matroids. The following results about regular matroids are some examples where this approach has been successful: extended formulations for independent set polytope [15], finding minimum cycle basis [13], and deciding first order logic properties [10].

Although Seymour's theorem gives a meta-strategy, it does not automatically imply a result for regular matroids, given the result for graphic/cographic/ $R_{10}$  matroids. Each setting, where one wants to apply Seymour's theorem to prove something about regular matroids, requires some new ideas. In fact, in many settings, Seymour's theorem does not work as it is and a strengthening of its statement is required. Indeed, the recent work [14] on near-minimum circuits uses a refined version of Seymour's theorem (given by [32]). There are a few other results on regular matroids that have used a more refined version (see [32]): faster algorithm to test total unimodularity [31], upper bounding the cycle cover ratio [21], and approximating the partition function of the ferromagnetic Ising model [12].

Some recent works, solving discrete optimization problems for regular matroids, have used a further stronger version of Seymour's theorem. The strongest form was presented recently by Dinitz and Kortsarz [7], which gives the flexibility to decompose a regular

<sup>&</sup>lt;sup>3</sup>Our proof works for any  $\ell_p$  norm  $(p \ge 1)$ , with appropriate dependence on p.

matroid in many different possible sequences. They used it to solve the matroid secretary problem for regular matroids [7]. Later, Fomin, Golovach, Lokshtanov, and Saurabh utilized it in designing parameterized algorithms for the space cover problem [8] and the spanning circuits problem [9].

Our main result (Theorem 1.1) also takes advantage of this strongest version of Seymour's theorem. One of our novel ideas is to use different sequences of decompositions for the same matroid to upper bound different classes of circuits. In contrast, the result of [14], which only works for a multiplicative factor smaller than 3/2, uses a fixed decomposition tree. In the next section, we give an overview of our proof and explain why the techniques of [14] fail to generalize to an arbitrary multiplicative factor.

#### 1.3 Future directions

One natural question motivated by our work is to find out which other matroids have only polynomially many near-minimum circuits. Analogous to Seymour's decomposition theorem, Geelen, Gerards, and Whittle [11] have proposed a structure theorem for any proper minor-closed class of matroids representable over a finite field. Can we use this structure theorem to upper bound the number of near-minimum circuits in these matroids.

Similarly, for what all lattices can we prove a polynomial bound on the number of near-shortest vectors. An interesting candidate to examine would be the lattice L(A) for any matrix A whose entries are O(1).

### 2 Overview of our proof and comparison with previous work

As mentioned earlier, Theorem 1.1 was already known in the special cases of graphic and cographic matroids.

Theorem 2.1 (Number of near-minimum circuits in a graphic or cographic matroid [19, 28]). Let M be a graphic or cographic matroid with ground set size  $m \geq 2$ . If every circuit in M has size more than r, then for any  $\alpha \geq 1$ , the number of circuits in M of size at most  $\alpha r$  is bounded by  $(2m)^{2\alpha}$ .

A key component of our proof of Theorem 1.1 is a deep result of Seymour [26] about decomposition of regular matroids. Seymour's Theorem states that every regular matroid can be built from piecing together some simpler matroids, each of which is graphic, cographic or a special matroid with 10 elements,  $R_{10}$ . These building blocks are composed together via a binary operation on matroids, called "k-sum" for k = 1, 2, 3. One can visualize this in terms of a "decomposition tree" – a binary tree where each node represents a matroid such that each internal node is a k-sum of its two children, each leaf node is a graphic, cographic or the  $R_{10}$  matroid, and the root node is the desired regular matroid. It is important to note that this decomposition tree is not unique, and one can perform the decomposition in different ways, optimizing various parameters, e.g., the tree depth.

The k-sum  $M = M_1 \triangle M_2$  of two matroids  $M_1$  and  $M_2$  is a matroid where  $M_1$  and  $M_2$  interact through a common set of elements S (of size 0, 1 or 3) and each circuit of M is obtained by picking a circuit each from  $M_1$  and  $M_2$ , and taking their sum via elements in S. Thus, one can hope to upper bound the number of circuits in M using such bounds for  $M_1$ 

and  $M_2$ . We know a polynomial upper bound on the number of near-minimum circuits for the matroids that are building blocks of Seymour's decomposition theorem: Theorem 2.1 shows it for graphic and cographic matroids; and for the matroid  $R_{10}$ , the bound holds trivially since it has only constantly many circuits. The challenge is to show that the polynomial bound still holds when we compose these matroids together via a sequence of an arbitrary number of k-sum operations.

A natural attempt to get this bound would be to use an induction based on the decomposition tree of a regular matroid. This was precisely the approach [14] took and showed a polynomial upper bound on the number of circuits in a regular matroid whose size is at most 3/2 times the shortest circuit size. However, their proof technique does not generalize to an arbitrary multiplicative factor  $\alpha$ . The use of such an induction severely restricts the power of the argument and it cannot be made to work for arbitrary  $\alpha$ .

**Theorem 2.2** ([14]). Let M be a regular matroid with ground set size m. Suppose that M has no circuits of size less than r. Then the number of circuits in M of size less than 3r/2 is bounded by  $O(m^5)$ .

**Limitations of the arguments in [14].** To see why the arguments in [14] fail to generalize to an arbitrary multiplicative factor, we need to take a closer look at the k-sum operations. If M is a k-sum of two matroids  $M_1$  and  $M_2$  with the set S of common elements, then any circuit C of M

- is a circuit in  $M_1$  or  $M_2$  that avoids any element from S, or
- is the same as  $(C_1 \cup C_2) \setminus \{e\}$ , where  $C_1$  and  $C_2$  are circuits of  $M_1$  and  $M_2$ , respectively that both contain a single element e from S.

Let us say, the shortest circuit size in M is more than r and we want to upper bound the number of circuits in it of size at most  $\alpha r$ . The starting point would be to get such bounds for  $M_1$  and  $M_2$ , using induction. However, we do not know the shortest circuit sizes of  $M_1$  and  $M_2$ . We only know something weaker: that any circuit in  $M_1$  or  $M_2$  that avoids elements from S has size more than r (since it is also a circuit of M).

The idea of [14] is to divide the circuits C in M into two classes according to their decomposition into circuits of  $M_1$  and  $M_2$ :

- (i) when the size of  $C_1$  is small, that is, less than r/2, and
- (ii) when the size of  $C_1$  is at least r/2.

It turns out that if we assume that the size of  $C_1$  is less than r/2, then there is a unique possibility of  $C_1$  (this is the reason for such a classification). In this case, they just absorb  $C_1$  into  $M_2$  by assigning appropriate weights to the common elements. And thus, the question just reduces to bounding the number of circuits of weight at most  $\alpha r$  in  $M_2$  under the assumption that the shortest circuit weight in  $M_2$  is more than r. This is done by induction.

In case (ii), such a trick does not work. Here, one has to work only with a weaker assumption that any circuit in  $M_2$  that avoids the common element e has size more than

r. But on the positive side, the size of  $C_2$  can be at most  $(\alpha - 1/2)r$  and thus, we have to bound a smaller set of circuits.

The problem arises with this weaker assumption. We can try to prove the desired bound with the weaker assumption, by using the same inductive proof methodology. However, as we go deeper into the case (ii) induction, the set of elements (say R) that needs be avoided in the assumption grows in size. We can no longer prove the uniqueness of  $C_1$  (case (i)), as it might or might not contain elements from R. The problem can be fixed in a limited case when R has size 1, by ensuring that R is completely contained in  $M_2$  and thereby, is disjoint from  $C_1$ . This follows from a slightly stronger decomposition theorem [32], which works only when |R| = 1. This restricts the case (ii) induction depth to a single level. Recall that for every level we go down in the case (ii) induction, we get a decrement of 1/2 in  $\alpha$ . Moreover, we do not need to enter case (ii) when circuits C have a size less than r since here either  $C_1$  or  $C_2$  has a size less than r/2. Hence, we have

$$(\alpha - 1/2)r < r$$
, which implies  $\alpha < 3/2$ .

This is the reason why the proof of [14] breaks down beyond  $\alpha \geq 3/2$ .

#### Proof overview of Theorem 1.1

As we have seen, the induction based proof of [14] does not extend for a multiplicative factor  $\alpha$  larger than 3/2. On the other hand, such an inductive approach seems unavoidable if we work with a given decomposition tree, since it gives a fixed sequence of k-sum operations. To overcome this issue, we need several new ideas. As noted earlier, the decomposition tree of a regular matroid is not unique. Our first main idea is to use different decomposition trees to upper bound different kinds of circuits. For instance, we can use a decomposition  $M = (M_1 \triangle M_2) \triangle M_3$  for one kind of circuits and use another decomposition  $M = M_1 \triangle (M_2 \triangle M_3)$  for the rest.

However, to switch between different decompositions, we need an "associativity property" for the k-sum operations. As defined, the k-sum operations do not seem to be associative. However, a closer inspection tells us that in fact, they can be made associative. Using this associativity, Seymour's Theorem can be adapted to give, what we call, an unordered decomposition tree (UDT) of a regular matroid M, which allows us to construct many different possible decomposition trees for M. This form of Seymour's Theorem seems to be a folklore knowledge, but is first formally given in [7] (also see [17]). We believe that using this more structured version of Seymour's theorem is necessary to obtain a result corresponding to an arbitrary multiplicative factor  $\alpha$ .

For a regular matroid M, its UDT is an undirected tree such that each of its nodes corresponds to a graphic/cographic/ $R_{10}$  matroid. Each subtree T of the UDT corresponds to a unique regular matroid  $M_T$ . Moreover, if  $T_1$  and  $T_2$  are the two subtrees obtained by deleting an arbitrary edge from a subtree T, then

$$M_T = M_{T_1} \triangle M_{T_2}$$
.

Thus, a UDT gives us many possible ways of decomposing M, depending on the order we choose for deleting edges from the UDT.

In the proof of Theorem 1.1, instead of an inductive argument, we directly use the structure of the unordered decomposition tree. We first give a brief outline of the steps in our proof. Assume that the shortest size of a circuit in M is more than r, and we would like to bound the number of circuits of size at most  $\alpha r$ .

- Circuit decomposition. We first argue that any circuit C in M can be written as a sum of its *projections* on the nodes of the UDT, which themselves are circuits in the corresponding matroids (Observation 6.1).
- Balanced division of UDT. For any given circuit C of size  $\leq \alpha r$  in M, we divide the UDT into a balanced set of subtrees, i.e., we find a set of  $O(\alpha)$  center nodes in the UDT such that each subtree obtained by deleting the center nodes has a projection of C of size at most r/2 (Claim 6.3).
- Classifying the circuits. We then classify these circuits based on the division of the UDT they produce and argue that the number of different classes is polynomially bounded. To bound the number of circuits in a given class, we work with the corresponding division of the UDT (Claim 6.4).
- Uniqueness of small projections. The number of circuits in a class is bounded by the *product* of the numbers of possible projections on the center nodes and the remaining subtrees. We first show that if a subtree has a projection of size at most r/2, then the projection is *unique* for all the circuits in the class (Lemma 6.7).
- Number of projections on the center nodes. The main technical step is to bound the number of possible projections on the center nodes. Since there are only constantly many center nodes, it suffices to do it for a given center node. This task is reduced to bounding the number of weighted circuits in a graphic/cographic/ $R_{10}$  matroid, through various technical ideas (Lemma 6.9) such as
  - absorbing small projections via common elements,
  - avoiding common elements that connect to large projections.

Some of our lower level techniques are inspired from [14]: uniqueness of small projections, assigning weights, and avoiding a set of elements. We elaborate on each of our proof steps.

Recall that k-sum operations are defined in a way such that any circuit of a matroid  $M = M_1 \triangle M_2$  is either completely contained in  $M_1$  or  $M_2$ , or it can be written as a sum of two circuits, one coming from each  $M_1$  and  $M_2$ , which we refer as projections of the given circuit. When a circuit of M is completely contained in  $M_1$ , we say it has an empty projection on  $M_2$ . A crucial property of the UDT is that any division of the UDT into subtrees gives us a valid decomposition of the matroid into smaller matroids corresponding to the subtrees. Thus, we can also obtain the projection of a circuit on any subtree of the UDT.

A balanced division of the UDT. Our next idea is based on the observation that any weighted tree can be broken down into a balanced set of subtrees, i.e., it has a node such that its removal produces subtrees that have weights at most half of the total weight of the

tree. For a given circuit C of M, we consider its projection size on a node of the UDT as the weight of node and use this balanced division recursively on the UDT. By this, we obtain certain *center* nodes of the UDT such that if we delete these center nodes, then each of the obtained subtrees has only a small projection of C. Here, by a small projection we mean that its size should be at most r/2. We argue that if the size of C is at most  $\alpha r$ , then the number of centers would be at most  $4\alpha$ .

Next, we classify the circuits (of size  $\leq \alpha r$ ) according to the set of centers they produce. Since  $\alpha$  is a constant, the number of possible classes is polynomially bounded. Hence, we just need to upper bound the number of circuits in a given class.

Bounding the number of circuits in a class. To upper bound the number of circuits in a class, let us fix a set of centers. We divide the UDT into the center nodes, and the subtrees we obtain if we delete centers, and write M as a sum of the matroids corresponding to these centers and subtrees. Further, we write any circuit C as a sum of its projections on the matroids associated with the centers and the subtrees. If we upper bound the number of distinct possible projections on each of these smaller matroids, then their product bounds the number of all the desired circuits. Since there can be  $\Omega(m)$  subtrees, this product can easily become exponentially large even if we have just two possible projections for each subtree. We sidestep this exponential blow-up as follows.

If a projection is small, it is unique. The first important step is to show that there is only a unique possibility of the projection on a subtree (to a certain extent), besides the empty projection, if we assume that the size of the projection is at most r/2. It is true because, if there are more than one such projections then we can show that any two of them combine to give a circuit of size at most r in M, which contradicts the initial assumption. This takes care of the projections on the subtrees in the above division of the UDT.

Upper bounding the number of projections on a center node. The next step is the most technically involved part of the proof: bounding the number of projections on the center nodes. As there are only a constant number of center nodes, it suffices to polynomially bound the number of distinct projections on a given center node. Recall that each projection is a circuit in the respective matroid. One might think that since the matroid associated with a single node is graphic, cographic or  $R_{10}$ , it would be easy to bound the number of circuits in it, but it is not that straightforward. The main problem is that we do not know the size of the shortest circuit in this component matroid; in fact, it could be arbitrarily small. And thus, we cannot just get a polynomial bound on the number of circuits whose sizes can be up to  $\alpha r$ . Here, we need to use a more sophisticated argument. There are two main technical ingredients involved.

Technique I: Absorbing small projections on subtrees via common elements. To see the first ingredient, let us consider a relatively simpler case when there is only one center node. In this case, all the subtrees attached to the center node have projections of sizes at most r/2. Let  $M_0$  be the matroid associated with the center node. The idea is to give some weights to the elements of  $M_0$  and define a map from circuits of  $M_0$  to circuits of M in a way that a circuit of  $M_0$  with weight s is mapped to a circuit in M with size s. If

we can design such a weight assignment then we can assume that the smallest weight of a circuit in  $M_0$  is more than r. Then we would just need to bound the number of near-smallest weight circuits in a graphic/cographic matroid, which can be done by a weighted version of Theorem 2.1.

The next question is: how to design such weights? Let T be any subtree attached to the center node and  $M_T$  be the matroid associated with it. Let  $e_T$  be one of the common elements between  $M_T$  and  $M_0$ . As discussed above, there is a unique projection on T that has size at most r/2 and contains  $e_T$ . Let that projection be  $C_T$ . We put a weight  $|C_T \setminus e_T|$  on the element  $e_T$  of  $M_0$ , which is supposed to be a representative of the projection on T. That is, if a circuit  $C_0$  of  $M_0$  takes the element  $e_T$  then it means that  $C_0$  will be summed up with  $C_T$  to form a circuit C of M, otherwise it will not be. We put such weights for every subtree T attached to the center node, and every element common between  $M_T$  and  $M_0$ . It turns out that this gives us a weighting scheme with exactly the desired property: a circuit of  $M_0$  with weight s is mapped to a circuit in M with size s.

We need to consider the case when there are more than one center nodes. The problem with the above argument would be that if we pick any one of the center nodes and delete it from UDT, some of the obtained subtrees can have projection sizes more than r/2, and thus, there is no uniqueness of the projection (recall that we get sizes less than r/2 only when we delete *all* center nodes). This is where we require the second technical ingredient.

Technique II: Avoiding common elements that connect to large projections. Let us pick one of the center nodes and say, the associated matroid is  $M_0$ . Observe that there can be many subtrees attached to the center node that have projection sizes larger than r/2, but the crucial fact is that their number can be at most  $2\alpha$  (since the total size of the circuit C is at most  $\alpha r$ ).

Let T be one such subtree and  $e_T$  be one of the common elements between  $M_T$  and  $M_0$ . Unlike what we did in the first technique, there is no unique way of assigning a weight to  $e_T$ . Instead, we just consider circuits  $C_0$  in  $M_0$  that avoid the element  $e_T$ . In short, here again, we can have a weight-preserving map from circuits in  $M_0$  to circuits in M, but the domain is restricted to those circuits in  $M_0$  which avoid common elements  $e_T$ , for any subtree T with a large projection.

Using this map, we can argue that the weight of any circuit in  $M_0$ , that avoids elements  $e_T$ , is more than r. This is a weaker assumption than what we require, in the sense that a circuit in  $M_0$  that takes some elements  $e_T$  can have arbitrary small weight. It turns out that such a weaker assumption is sufficient to give a polynomial upper bound on the number of circuits of weight  $\leq \alpha r$  in a graphic/cographic matroid, as long as the number of elements  $e_T$  that we need to avoid remains a constant (Lemma 6.8). We claim that the number of elements  $e_T$  are at most  $O(\alpha)$ , which is a constant. This is true because  $M_T$  and  $M_0$  can have at most 3 elements in common, and as we saw above, the number of subtrees T with a large projection can be at most  $2\alpha$ .

Finishing the proof. In essence, through various technical components, we reduce the problem to a single component matroid, which is graphic/cographic/ $R_{10}$ . However, we have to consider weights on the elements and also have a weaker assumption about the smallest weight of a circuit – that is – we assume a lower bound on the weight of only those circuits

that avoid a fixed set R of elements. It turns out that even with this weaker assumption, we can get an upper bound on the number of desired circuits in a graphic/cographic matroid by cleverly modifying the proofs of [19, 28]. This was partially done in [14], where they only considered the case of |R| = 1. Here, we generalize their proof to an arbitrary set R (Lemma 6.8).

### Organization of the rest of the paper.

In Section 3, we introduce well-known concepts from matroid theory and describe Seymour's Theorem for regular matroids. In Section 4, we relate circuits of a regular matroid with vectors of a totally unimodular lattice and prove Theorem 1.3 using Theorem 1.1. Section 5 talks about a refinement of Seymour's Theorem, where we have more structure in the decomposition of a regular matroid. In Section 6 we use this structured decomposition to upper bound the number of near-minimum circuits in regular matroids. Section 7 describes max-flow min-cut matroids. We argue that the same proof technique gives a polynomial bound on the number of near-minimum circuits in these matroids. For proving Theorem 1.1, we first need that result for graphic and cographic matroids, but in a stronger form. Appendix A is devoted to prove this.

### 3 Matroid preliminaries

#### 3.1 Matroids and circuits

**Definition 3.1** (Matroid). For a finite set E and a nonempty collection  $\mathcal{I}$  of its subsets, the pair  $M = (E, \mathcal{I})$  is called a matroid if

- 1. for every  $I_1 \subseteq I_2$ ,  $I_1 \in \mathcal{I}$  implies  $I_2 \in \mathcal{I}$ ,
- 2. if  $I_1, I_2 \in \mathcal{I}$  with  $|I_1| > |I_2|$  then there exists an element  $e \in I_1 \setminus I_2$  such that  $I_2 \cup \{e\} \in \mathcal{I}$ .

Every set in  $\mathcal{I}$  is said to be an independent set of M.

Every independent set of maximum size is called a *base* of M. A subset of E that is not independent is said to be dependent. Note that since  $\mathcal{I}$  is nonempty, the empty set  $\emptyset$  must be an independent set.

**Definition 3.2** (Circuit). For a matroid M, any inclusion-wise minimal dependent subset is called a circuit.

We define some special classes of matroids that are useful for us. For a matrix A, let E be the set of its columns and  $\mathcal{I}$  be the collection of all linearly independent sets of columns. It is known that  $M(A) = (E, \mathcal{I})$  is a matroid.

**Definition 3.3** (Linear, binary, and regular matroids). A matroid M is called representable over a field  $\mathbb{F}$  if there exists a matrix A over  $\mathbb{F}$  such that M = M(A).

• A matroid representable over some field is called linear.

- A matroid representable over GF(2) is called binary.
- A matroid representable over every field is called regular.

Regular matroids are known to be characterized by totally unimodular matrix.

**Definition 3.4** (Totally unimodular matrix). A matrix A over real numbers is said to be totally unimodular if every square submatrix of A has determinant 0, 1, or -1.

Note that by definition, each entry in a totally unimodular matrix is 0, 1 or -1.

**Theorem 3.5** (Characterization of regular matroids, see [25]). A matroid M is regular if and only if there is a totally unimodular matrix A such that M = M(A).

Two well-known special cases of regular matroids are graphic and cographic matroids.

**Definition 3.6** (Graphic matroids). The graphic matroid M(G) for a graph G is defined as  $(E, \mathcal{I})$ , where E is the set of edges in G and  $\mathcal{I}$  is the collection of all sets of edges without cycles.

For a graph G, its cographic matroid  $M^*(G)$  is the duals the graphic matroid M(G). For a matroid  $M = (E, \mathcal{I})$ , its dual is  $M^* = (E, \mathcal{I}^*)$  where,

$$\mathcal{I}^* = \{ I \subseteq E \mid E \setminus I \text{ contains a base set of } M \}.$$

Observe that a base set of a graphic matroid M(G) is a spanning tree in G (or spanning forest, if G is not connected). Thus, a set of edges is independent in  $M^*(G)$  if and only if its removal keeps G connected.

The circuits of graphic and cographic matroids are easy to characterize in terms of cycles and cut-sets. For a graph G, and any partition  $V_1 \cup V_2$  of its vertices, the set of edges  $E(V_1, V_2)$  connecting a vertex in  $V_1$  to another in  $V_2$  is called a cut-set.

**Fact 3.7** (Circuits in graphic and cographic matroids). For a graph G,

- a circuit of the graphic matroid M(G) is any simple cycle of G and
- a circuit of the cographic matroid  $M^*(G)$  is any inclusion-wise minimal cut-set of G.

Recall that the symmetric difference  $C_1 \triangle C_2$  of two sets  $C_1$  and  $C_2$  is given by  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ . Note that the symmetric difference of two cycles in a graph can be expressed as a disjoint union of cycles in the graph. Same holds true for two cut-sets. These two statements are special cases a more general fact about binary matroids. Recall that graphic and cographic matroids are regular and thus, binary.

**Fact 3.8.** For two circuits  $C_1$  and  $C_2$  of a binary matroid M, their symmetric difference  $C_1 \triangle C_2$  is a disjoint union of circuits of M.

It is known that for a matroid M, one can delete one of its ground set elements e to obtain another matroid  $M \setminus e$  defined as follows.

**Definition 3.9** (Deletion). For a matroid  $M = (E, \mathcal{I})$  and an element  $e \in E$ ,  $M \setminus e$  is defined to be matroid on the ground set  $E \setminus \{e\}$  such that any independent set of M not containing e is an independent set of  $M \setminus e$ .

For a graphic matroid, deletion of an element corresponds to the deletion of the edge from the graph, while for a cographic matroid, it corresponds to the contraction of the edge. It is easy to characterize the circuits of  $M \setminus e$ .

**Fact 3.10.** The circuits of  $M \setminus e$  are those circuits of M that do not contain e.

Another operation one can do on a matroid is to add a new element parallel to an existing element. This new element is essentially a copy of the existing element. Formally, let  $M = (E, \mathcal{I})$  be a matroid with a given element e. One can define a new matroid M' on the ground set  $E \cup \{e'\}$  with a new element e' such that  $\{e, e'\}$  is a circuit. This also implies that for any circuit C of M that contains e, the set  $C \setminus \{e\} \cup \{e'\}$  is a circuit of M'. In the case of a graphic matroid, adding a parallel element means adding a parallel edge in the graph, while in the cographic case, it means splitting an edge into two by adding a new vertex.

Fact 3.11. The classes of regular matroids, graphic matroids, and cographic matroids are closed under the deletion operation and under the addition of a parallel element.

### 3.2 k-sums and Seymour's Theorem

To prove Theorem 1.1, a crucial ingredient is the remarkable decomposition theorem for regular matroids by Seymour. Seymour [26] showed that every regular matroid can be constructed by piecing together three special kinds of matroids – graphic matroids, cographic matroids and a certain matroid  $R_{10}$  of size 10. The operation involved in this composition is called a k-sum, for k = 1, 2, or 3. The k-sum operation is defined for arbitrary binary matroids.

**Definition 3.12** (Sum of two matroids, [25,26]). Let  $M_1 = (E_1, \mathcal{I}_1)$  and  $M_2 = (E_2, \mathcal{I}_2)$  be two binary matroids with  $E_1 \cap E_2 = S$ . The matroid  $M_1 \triangle M_2$  is defined over the ground set  $E_1 \triangle E_2$  such that the circuits of  $M_1 \triangle M_2$  are the minimal non-empty subsets of  $E_1 \triangle E_2$  that are of the form  $C_1 \triangle C_2$ , where  $C_i$  is a (possibly empty) disjoint union of circuits of  $M_i$  for i = 1, 2.

The fact that the above definition indeed gives a matroid can be verified from the circuit characterization of a matroid [25, Theorem 1.1.4]. We are only interested in three special cases of this sum, called 1-sum, 2-sum, and 3-sum.

**Definition 3.13** (1,2,3-sums). The sum  $M_1 \triangle M_2$  of two binary matroids  $M_1 = (E_1, \mathcal{I}_1)$  and  $M_2 = (E_2, \mathcal{I}_2)$  with  $E_1 \cap E_2 = S$  is called

- 1.  $a \ 1$ -sum if |S| = 0,
- 2. a 2-sum if |S|=1, S is not a circuit of  $M_1, M_2$  or their duals, and  $|E_1|, |E_2| \geq 3$ , and
- 3. a 3-sum if |S| = 3, S is a circuit of  $M_1$  and  $M_2$ , S does not contain a circuit of the duals of  $M_1$  and  $M_2$ , and  $|E_1|, |E_2| \ge 7$ .

The conditions on the ground set sizes are there to avoid degenerate cases. The following facts follows from the definition and Fact 3.8.

**Fact 3.14.** For i = 1, 2, if  $C_i$  is a circuit of  $M_i$  that does not contain any elements from S then  $C_i$  is a circuit of  $M_1 \triangle M_2$ .

**Fact 3.15.** Let  $C_i$  be a disjoint union of circuits of  $M_i$  for i = 1, 2. If  $C_1 \triangle C_2$  is a subset of  $E_1 \triangle E_2$  then it is a disjoint union of circuits in  $M_1 \triangle M_2$ .

The operation of 1-sum is the easiest sum operation.

Fact 3.16 (Circuits in a 1-sum). If M is a 1-sum of  $M_1$  and  $M_2$  then any circuit of M is either a circuit of  $M_1$  or a circuit of  $M_2$ .

Characterizing the circuits of a 2-sum or a 3-sum is a bit more non-trivial. The following lemma [26, Lemma 2.7] gives a way to represent circuits of  $M_1 \triangle M_2$  in terms of circuits of  $M_1$  and  $M_2$ .

**Proposition 3.17** (Circuits in a 2-sum or a 3-sum, [26]). Let  $C_1$  and  $C_2$  be the sets of circuits of  $M_1$  and  $M_2$ , respectively. Let M be a 2-sum or a 3-sum of  $M_1$  and  $M_2$  and let  $E_1 \cap E_2 = S$  (|S| = 1 or 3). Then for any circuit C of M, exactly one of the following holds:

- 1.  $C \in \mathcal{C}_1$  with  $S \cap C = \emptyset$
- 2.  $C \in \mathcal{C}_2$  with  $S \cap C = \emptyset$
- 3. there exist unique  $e \in S$ ,  $C_1 \in C_1$ , and  $C_2 \in C_2$  such that

$$S \cap C_1 = S \cap C_2 = \{e\} \text{ and } C = C_1 \triangle C_2.$$

With all the required definitions, we can finally present Seymour's theorem for regular matroids [26, Theorem 14.3].

**Theorem 3.18 (Seymour's Theorem).** Every regular matroid can be obtained by means of 1-sums, 2-sums and 3-sums, starting from matroids which are graphic, cographic or  $R_{10}$ .

The matroid  $R_{10}$ , which forms one of the building blocks for Seymour's Theorem, is represented by the following matrix over GF(2).

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The following fact about  $R_{10}$  is useful.

**Fact 3.19.** The matroid obtained by deleting any element from  $R_{10}$  is a graphic matroid.

### 4 Number of $\alpha$ -shortest vectors in a totally unimodular lattice

In this section, we show how Theorem 1.3 follows from Theorem 1.1. Recall that for an  $n \times m$  matrix A, the lattice L(A) is defined as:

$$L(A) = \{ v \in \mathbb{Z}^m \mid Av = 0 \}.$$

We first define circuits of a matrix A, which are vectors in L(A) and show a correspondence between the circuits of a TU matrix A and the circuits of the associated regular matroid M(A). Thus, an upper bound on number of near-minimum circuits in M(A) (Theorem 1.1) implies an upper bound on number of near-minimum circuits of the matrix A. Finally, we argue that any  $\alpha$ -minimum vector in L(A) comes from a combination of at most  $\alpha^2$ -many  $\alpha$ -minimum circuits of A. A specialized version of this statement was shown in [14] – any 2-minimum vector in L(A) comes from a 2-minimum circuit of A.

**Definition 4.1** (Circuits of a matrix). For a matrix A, a vector  $u \in L(A)$  is a circuit of A if for any vector  $v \in L(A)$  with  $supp(v) \subseteq supp(u)$ , it must be that  $v = \gamma u$  for some integer  $\gamma$ .

Note that the circuits of A come in pairs, in the sense that if u is a circuit of A, then so is -u. The following is well-known for the circuits of a TU matrix (see [24, Lemma 3.18]).

**Fact 4.2.** Every circuit of a TU matrix has its coordinates in  $\{-1, 0, 1\}$ .

We show a correspondence between circuits of A and circuits of M(A) when A is TU.

**Lemma 4.3** (Circuits of a TU matrix and a regular matroid). Let A be a TU matrix and  $M = (E, \mathcal{I})$  be the regular matroid represented by it. Then the circuits of M have a one to one correspondence with the circuits of A (up to change of sign).

*Proof.* By definition, a circuit of A has an inclusion-wise minimal support. Thus for a circuit u of A, the columns in A corresponding to the set  $\operatorname{supp}(u)$  are minimally dependent. We know that a minimal dependent set is a circuit in the associated matroid. Hence, the set  $\operatorname{supp}(u)$  is a circuit of matroid M.

In the other direction, if  $C \subseteq E$  is a circuit of matroid M, then the set of columns of A corresponding to C is minimally linear dependent. Hence, there is a unique linear dependence (up to a multiplicative factor) among the set of columns corresponding to C. This means that there are precisely two circuits  $u, -u \in L(A)$  with their support being C.

Lemma 4.3 together with Theorem 1.1 gives the following corollary. Let  $\|\cdot\|$  denote the  $\ell_2$ -norm of a vector.

**Corollary 4.4.** Let A be an  $n \times m$  TU matrix. If for every circuit u of A, we have ||u|| > r then the number of its circuits u with  $||u|| \le \alpha r$  is  $m^{O(\alpha^4)}$ .

*Proof.* If u is a circuit of A with  $\sup(u) = \gamma$  then  $||u|| = \sqrt{\gamma}$  (from Fact 4.2). Thus, any circuit u of A with  $||u|| \le \alpha r$  corresponds to a circuit of the regular matroid M(A) of size at most  $\alpha^2 r^2$ . The desired bound follows from Theorem 1.1.

We now show that we can get an upper bound on the number of all short vectors in L(A) from Corollary 4.4. We define a notion of conformality among two vectors and show that every vector in L(A) is a conformal combination of circuits of A.

**Definition 4.5** (Conformal vectors [24]). Let  $u, v \in \mathbb{R}^m$ . We say that u is conformal to v, denoted by  $u \sqsubseteq v$ , if  $u_i v_i \geq 0$  and  $|u_i| \leq |v_i|$ , for each  $1 \leq i \leq m$ .

The following lemma which says that each vector in L(A) is a conformal sum of circuits, follows from [24, Lemma 3.2 and 3.19].

**Lemma 4.6.** Let A be a TU matrix. Then for any nonzero vector  $v \in L(A)$ , we have

$$v = u_1 + u_2 + \dots + u_p$$

where each  $u_i$  is a circuit of A and is conformal to v.

We have the following easy observation for a conformal sum.

**Observation 4.7.** If  $v, u_1, u_2, \ldots, u_p \in \mathbb{Z}^m$  are such that  $v = u_1 + u_2 + \cdots + u_p$  and each  $u_i$  is conformal to v then

$$||v||^2 \ge ||u_1||^2 + ||u_2||^2 + \dots + ||u_p||^2$$
.

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. From the assumption in the theorem, for any circuit u of A, we have  $||u|| > \lambda$ . Consider a vector  $v \in L(A)$  with  $||v|| \le \alpha \lambda$ . From Lemma 4.6, we can write v as a conformal sum of circuits

$$v = u_1 + u_2 + \dots + u_n.$$

We know that  $||u_i|| > \lambda$  for each i. This together with Observation 4.7 implies that

$$(\alpha \lambda)^2 \ge ||v||^2 \ge \sum_{i=1}^p ||u_i||^2 > p\lambda^2.$$

Thus, we get that  $p < \alpha^2$ .

Note that each  $u_i$  is smaller than v and hence,  $||u_i|| \leq \alpha \lambda$ . From Corollary 4.4, the number of circuits u of A with  $||u|| \leq \alpha \lambda$  is  $m^{O(\alpha^4)}$ . Thus, there can be at most  $m^{O(p\alpha^4)}$  vectors of the form  $u_1 + u_2 + \cdots + u_p$ . This gives a bound of  $m^{O(\alpha^6)}$  on the number of vectors v with  $||v|| \leq \alpha \lambda$ .

### 5 A strengthening of Seymour's Theorem

In this section, we look at a stronger version of Seymour's Theorem, which gives a more structured decomposition of a regular matroid. One way to present Seymour's Theorem (Theorem 3.18) can be in terms of a decomposition tree.

**Theorem 5.1** (Seymour's Theorem). For every regular matroid M, there exists a decomposition tree BT(M) – a rooted binary tree whose every vertex is regular matroid such that

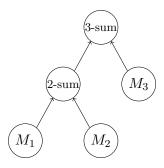


Figure 1: The decomposition tree of a matroid given by  $(M_1 \oplus_2 M_2) \oplus_3 M_3$ .

- every internal vertex is a k-sum of its two children for k = 1, 2 or 3,
- every leaf vertex is a graphic matroid, a cographic matroid or the R<sub>10</sub> matroid,
- the root vertex is the matroid M.

A few observations can be made about such a decomposition tree. Recall from Section 3 that for two matroids with grounds sets  $E_1$  and  $E_2$ , their k-sum is a matroid on the ground set  $E_1 \triangle E_2$ .

**Observation 5.2.** For any vertex  $M_0$  of the decomposition tree BT(M) of a binary matroid M,

- 1. Each element in  $M_0$  belongs to exactly one of its children matroids. Arguing recursively, each element in  $M_0$  belongs to a unique leaf in the subtree rooted at  $M_0$ .
- 2. The ground set of  $M_0$  is the symmetric difference of all the ground sets of the leaf vertices in the subtree rooted at  $M_0$ .

For example, Figure 1 shows the decomposition tree for a matroid  $M = (M_1 \oplus_2 M_2) \oplus_3 M_3$ . Note that the decomposition tree specifies an order of the decomposition or composition, that is, M can be obtained by first taking a 2-sum of  $M_1$  and  $M_2$  and then taking a 3-sum of the resulting matroid with  $M_3$ . It is not clear if the k-sum operations are associative. It turns out that one can strengthen the decomposition theorem such that the k-sum operations involved in the composition are associative up to a certain extent.

### 5.1 Associativity of the k-sum

The operations of 1-sums are trivially associative. It can be shown that the 2-sum operations are always associative and so are 3-sum operations in some special cases. The following lemma gives a criterion when the associativity holds.

**Lemma 5.3** (Associativity of k-sums). Let  $M_1, M_2, M_3, M_4$  be binary matroids with ground sets  $E_1, E_2, E_3, E_4$ , respectively. Let  $M_2 = M_3 \triangle M_4$  be a k-sum of  $M_3$  and  $M_4$  for k = 1, 2, or 3 with  $S_1 := E_3 \cap E_4$ . Let  $M = M_1 \triangle M_2$  be a k-sum of  $M_1$  and  $M_2$  for k = 1, 2, or 3 with

 $S_2 := E_1 \cap E_2$ . Further, we assume that the set  $S_2$ , which is contained in  $E_2 = E_3 \triangle E_4$ , is entirely contained in  $E_3$  (or in  $E_4$ , which is a similar case). Then

$$M = M_1 \triangle (M_3 \triangle M_4) = (M_1 \triangle M_3) \triangle M_4,$$

where  $M_1 \triangle M_3$  is defined via the common set  $S_2$  and  $(M_1 \triangle M_3) \triangle M_4$  is defined via the common set  $S_1$ .

Proof. We will show that the sets of circuits of the two matroids  $M_1\triangle(M_3\triangle M_4)$  and  $(M_1\triangle M_3)\triangle M_4$  are the same. This would imply that the two matroids are the same. Consider a circuit C of  $M_1\triangle(M_3\triangle M_4)$ . From Proposition 3.17, there are two possibilities for the circuit C. First is when C is a circuit of  $M_1$  or  $M_3\triangle M_4$  that avoids the common elements  $S_2$ . We skip this easy case and only consider the other possibility which is non-trivial. In the other possibility, C must be of the form  $C_1\triangle C_2$ , where  $C_1$  and  $C_2$  are circuits in  $M_1$  and  $M_3\triangle M_4$ , respectively and  $S_2\cap C_1=S_2\cap C_2$ . Similarly for  $C_2$ , there exist circuits  $C_3$  and  $C_4$  of  $M_3$  and  $M_4$  respectively such that  $C_2=C_3\triangle C_4$ .

Since  $S_2$  is contained entirely in  $E_3$  we get that

$$S_2 \cap C_1 = S_2 \cap C_2 = S_2 \cap C_3.$$

Thus,  $C_1\triangle C_3$  is a subset of  $E_1\triangle E_3$  and hence, is a disjoint union of circuits of  $M_1\triangle M_3$ , from Fact 3.15. Since  $C_4$  is a circuit of  $M_4$ , it follows that  $(C_1\triangle C_3)\triangle C_4$  is a disjoint union of circuits in  $(M_1\triangle M_3)\triangle M_4$ , again from Fact 3.15. But,

$$(C_1 \triangle C_3) \triangle C_4 = C_1 \triangle (C_3 \triangle C_4) = C.$$

Thus, C is a disjoint union of circuits in  $(M_1 \triangle M_3) \triangle M_4$ .

The other direction is similar. Consider a circuit C of  $(M_1 \triangle M_3) \triangle M_4$ . In the non-trivial case, the circuit C must be of the form  $C' \triangle C_4$ , where C' and  $C_4$  are circuits of  $(M_1 \triangle M_3)$  and  $M_4$ , respectively, with  $S_1 \cap C' = S_1 \cap C_4$  (Proposition 3.17). Similarly,  $C' = C_1 \triangle C_3$ , where  $C_1$  and  $C_3$  are circuits in  $M_1$  and  $M_3$ , respectively. Since  $S_1$  is disjoint from  $E_1$ , it must be that  $S_1 \cap C' = S_1 \cap C_3$ . Thus,  $C_3 \triangle C_4$  is a subset of  $E_3 \triangle E_4$  and hence, is a disjoint union of circuits of  $M_3 \triangle M_4$  (Fact 3.15). Similarly, since  $C_1$  is a circuit in  $M_1$ , it follows that  $C_1 \triangle (C_3 \triangle C_4)$  is a disjoint union of circuits in  $M_1 \triangle (M_3 \triangle M_4)$ . But,

$$C_1 \triangle (C_3 \triangle C_4) = (C_1 \triangle C_3) \triangle C_4 = C.$$

Thus, C is a disjoint union of circuits in  $M_1 \triangle (M_3 \triangle M_4)$ .

We have shown that a circuit of one matroid is a disjoint union of circuits in the other matroid and vice-versa. Consequently, by the minimality of circuits, it follows that their sets of circuits must be the same.  $\Box$ 

To summarize the above lemma, the sequence of two k-sums  $M_1\triangle(M_3\triangle M_4)$  is associative, when the common sets involved in the k-sum operations are completely contained in the starting matroids. Note that this is always true for a 2-sum operation since the common set has a single element in this case. However, this need not be always true in the case of a 3-sum. It is possible that the common set  $S_2$  between  $M_1$  and  $M_3\triangle M_4$  has elements in both  $M_3$  and  $M_4$ . Dinitz and Kortsarz [7] call such a set  $S_2$  as a bad sum-set. More generally, they define a notion of good or bad for a decomposition tree of a regular matroid obtained from Seymour's Theorem.

**Definition 5.4** (Good decomposition tree [7]). The decomposition tree BT(M) of a regular matroid M, as in Theorem 5.1, is said to be good if for every internal vertex  $M_0$  of the tree BT(M), which is a k-sum of its two children  $M_1$  and  $M_2$ , the common set S between  $M_1$  and  $M_2$  is completely contained in one of the leaf vertices of the subtree rooted at  $M_1$  and also in one of the leaf vertices of the subtree rooted at  $M_2$ .

As we see later, if we have a good decomposition tree of a binary matroid, then the ksum operations involved in the decomposition have a suitably defined associative property. Dinitz and Kortsarz [7] showed how to modify a given decomposition tree of a regular matroid to get a good decomposition tree. To do this, their basic step involves 'moving' an element from one matroid to another matroid. Consider the above discussed example of a matroid given by  $M_1 \triangle (M_3 \triangle M_4)$  and assume that the common set of elements between  $M_1$  and  $M_3 \triangle M_4$ , i.e.,  $E_1 \cap (E_3 \triangle E_4)$ , has elements in both  $E_3$  and  $E_4$ . In this case, they delete an element from one of the matroids, say  $M_3$ , and add an element in  $M_4$  parallel<sup>4</sup> to an existing element, to create new matroids  $M'_3$  and  $M'_4$  such that

$$M_1 \triangle (M_3 \triangle M_4) = M_1 \triangle (M_3' \triangle M_4').$$

Doing this repeatedly with the decomposition tree, starting from the leaves and going up to the root vertex, they obtain the desired decomposition tree. Formally, [7, Lemma 3.1] implies the following stronger version of Seymour's Theorem.

**Lemma 5.5.** For any regular matroid, there is a good decomposition tree such that each of the leaf vertices is a graphic, cographic or the  $R_{10}$  matroid.

Proof. [7, Lemma 3.1] says that for any regular matroid M and a given decomposition tree BT(M) of M, one can construct a good decomposition tree with the same tree structure, but each leaf vertex L is possibly replaced with another matroid obtained from L by deleting some elements in it and/or adding elements parallel to some elements in it. Recall that the classes of graphic and cographic matroids are closed under deletion of an element or addition of an element in parallel (Fact 3.11). Since the  $R_{10}$  matroid does not have any circuit with 3 elements, the procedure of Dinitz and Kortsarz [7] can only delete an element from  $R_{10}$  and not add one. The matroid obtained by deleting some elements from  $R_{10}$  is a graphic matroid (Fact 3.19). Thus, all the leaf vertices remain graphic, cographic or  $R_{10}$ .

#### 5.2 Unordered decomposition tree

Next, we define an *unordered decomposition tree (UDT)*, which allows us to decompose a matroid in many different ways. We show that we can obtain an unordered decomposition tree of a matroid from its good decomposition tree.

Let T be a tree with its vertex set V(T) such that each vertex  $v \in V(T)$  has a corresponding binary matroid  $M_v$  with the ground set  $E_v$ . Further, for any two vertices u and v of the tree we have the following.

$$|E_u \cap E_v| = \begin{cases} 0, 1 \text{ or } 3 & \text{if there is an edge between } u \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$
 (1)

<sup>&</sup>lt;sup>4</sup>Two elements a and b of a matroid are in parallel if  $\{a,b\}$  is a circuit.



Figure 2: The unordered decomposition tree (UDT) representing the matroid given by  $(M_1 \triangle M_2) \triangle M_3 = M_1 \triangle (M_2 \triangle M_3)$ .

In particular, this means that any element is a part of at most two ground sets. For any subtree T' of this tree T, we define a binary matroid  $M_{T'}$  with its ground being  $E_{T'} = \triangle_{u \in V(T')} E_u$ . It is defined recursively as follows:

- if T' is a vertex, say v, then  $M_{T'} = M_v$ .
- Otherwise, let e = (u, v) be an edge in the tree T'. Let  $T_1$  and  $T_2$  be the two subtrees obtained by removing the edge e from T'. From (1), it follows that  $E_{T_1} \cap E_{T_2} = E_u \cap E_v$ . Define

$$M_{T'} = M_{T_1} \triangle M_{T_2}.$$

Here, the sum  $M_{T_1} \triangle M_{T_2}$  is a 1-sum, 2-sum or a 3-sum depending on the size of  $E_{T_1} \cap E_{T_2} = E_u \cap E_v$ . At first, it is not clear if  $M_{T'}$  (and  $M_T$ ) is uniquely defined since one can pick any edge e from T' and get different subtrees (see fig. 2). In the following, we argue that  $M_T$  is indeed uniquely defined. For this we need the associativity of the k-sum operations proved in Lemma 5.3.

Claim 5.6. For a tree T as above, the matroid  $M_T$  is uniquely defined.

*Proof.* We want to show that  $M_T$  is the same matroid whatever be our choice of edge e to decompose it into two subtrees. We argue inductively. We assume that for any subtree T' of tree T, the matroid  $M_{T'}$  is uniquely defined. We first consider two neighboring edges  $e_1$  and  $e_2$  in T, and the subtrees obtained after removing them. Let  $e_1 = (u, v)$  and  $T_1$  and  $T_2$  be the two subtrees obtained by removing  $e_1$  such that u is in  $T_1$  and v is in  $T_2$ . Similarly, let  $e_2 = (v, w)$  and  $T_3$  and  $T_4$  be such subtrees with v in  $T_3$  and w in  $T_4$ . We show that

$$M_{T_1} \triangle M_{T_2} = M_{T_3} \triangle M_{T_4}. \tag{2}$$

Let  $S_1$  be the set  $E_{T_1} \cap E_{T_2} = E_u \cap E_v$  and  $S_2$  be the set  $E_{T_3} \cap E_{T_4} = E_v \cap E_w$ . Observe that  $T_1$  is a subtree of  $T_3$  and  $T_4$  is a subtree of  $T_2$  (see fig. 3). In particular, when we remove  $e_2$  from  $T_2$ , one of the two subtrees we get is  $T_4$ . Let the other one be  $T_5$ . Since  $M_{T_2}$  is well defined by the assumption it can be written as  $M_{T_5} \triangle M_{T_4}$  (note that  $E_{T_5} \cap E_{T_4} = E_v \cap E_w = S_2$ ). Thus, we can write

$$M_{T_1} \triangle M_{T_2} = M_{T_1} \triangle (M_{T_5} \triangle M_{T_4}).$$

Since the commons set between  $E_{T_1}$  and  $E_{T_5} \triangle E_{T_4}$  is  $E_u \cap E_v$ , which is completely contained in  $E_{T_5}$ , we can use Lemma 5.3 to get

$$M_{T_1} \triangle M_{T_2} = M_{T_1} \triangle (M_{T_5} \triangle M_{T_4}) = (M_{T_1} \triangle M_{T_5}) \triangle M_{T_4}.$$
 (3)

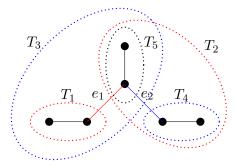


Figure 3: Obtaining two subtrees  $T_1$  and  $T_2$  by removing an edge  $e_1 = (u, v)$ . Obtaining two subtrees  $T_3$  and  $T_4$  by removing an edge  $e_2 = (v, w)$ .

Note that  $T_1$  and  $T_5$  are the subtrees we get by removing  $e_1$  from  $T_3$ . Thus,  $M_{T_1} \triangle M_{T_5}$  is the same as  $M_{T_3}$ . Combining this with (3) we get (2).

We have shown that we get the same matroid whether we remove  $e_1$  or  $e_2$ , in the case when  $e_1$  and  $e_2$  are neighboring edges. It immediately follows that for any choice of edge e in the tree T, we get the same matroid when we take the sum of the two matroids associated with the two subtrees in  $T - \{e\}$ .

We show that for any regular matroid, we can get such an unordered decomposition tree from its good decomposition tree.

**Lemma 5.7.** For any regular matroid M, there is an unordered decomposition tree T such that  $M = M_T$ , where each vertex of the tree T is a graphic, cographic or the  $R_{10}$  matroid.

*Proof.* From Lemma 5.5, we get a good decomposition tree BT(M) of M, where each leaf vertex is a graphic, cographic, or the  $R_{10}$  matroid. The unordered decomposition tree UT(M) for the matroid M has its vertices corresponding to the leaf vertices of BT(M). The edges of UT(M) correspond to the internal vertices of BT(M), and thus, correspond to k-sum operations on two intermediate matroids. For any vertex  $M_0$  of the tree BT(M), we construct its unordered decomposition tree  $UT(M_0)$  recursively.

- If  $M_0$  is a leaf vertex of BT(M), then its unordered decomposition tree  $UT(M_0)$  is defined as a vertex labeled with  $M_0$ .
- If  $M_0$  is an internal vertex of BT(M), then suppose it is a k-sum of its two children  $M_1$  and  $M_2$  with S being the set of common elements in  $M_1$  and  $M_2$ . By the definition of a good decomposition tree, S is contained in a unique leaf  $L_1$  of the subtree rooted at  $M_1$  and also in a unique leaf  $L_2$  of the subtree rooted at  $M_2$ . The tree  $UT(M_0)$  is obtained by taking the union of  $UT(M_1)$  and  $UT(M_2)$  and adding an edge between the vertex  $L_1$  of  $UT(M_1)$  and the vertex  $L_2$  of  $UT(M_2)$ .

In this construction, it is clear that we add an edge between two vertices  $L_1$  and  $L_2$  if and only if they contain a set of common elements. Moreover, by the definitions of a decomposition tree and an unordered decomposition tree, it follows that UT(M) corresponds to the given matroid M.

### 6 Bounding the number of circuits in a regular matroid

This section is devoted to a proof of Theorem 1.1. Using the unordered decomposition tree of a regular matroid from the previous section, we bound the number of near-minimum circuits in a regular matroid M. Recall that in Theorem 1.1 we assume there are no circuits in M of size at most r.

Let T be the unordered decomposition tree from Lemma 5.7 such that  $M = M_T$ . Let  $M_v$  be matroid corresponding to a vertex v in T with ground set  $E_v$ . Recall that  $E = \triangle_{v \in V(T)} E_v$ . Recall from the definition of an unordered decomposition tree that if  $T_1$  and  $T_2$  are subtrees of T obtained by removing an edge then we can write  $M = M_{T_1} \triangle M_{T_2}$ . By Proposition 3.17, any circuit C of M can be written as  $C_{T_1} \triangle C_{T_2}$  such that one of the following holds

- $C_{T_2}$  is empty and  $C_{T_1}$  is a circuit of  $M_{T_1}$  that avoids the common elements  $E_{T_1} \cap E_{T_2}$ ,
- $C_{T_1}$  is empty and  $C_{T_2}$  is a circuit of  $M_{T_2}$  that avoids the common elements  $E_{T_1} \cap E_{T_2}$ ,
- $C_{T_1}$  and  $C_{T_2}$  are circuits of  $M_{T_1}$  and  $M_{T_2}$ , respectively such that both  $C_{T_1}$  and  $C_{T_2}$  contain a single common element coming from  $E_{T_1} \cap E_{T_2}$ .

Recursively, one can define  $C_{T'}$  for any subtree T' of T, including the case when T' is a vertex. And we can write

$$C = \triangle_{v \in V(T)} C_v$$
.

Note that  $C_v$  can be empty for some vertices v. See Figure 4. The following observation follows.

**Observation 6.1** (Projections of a circuit on the vertices of UDT). For any circuit C of M,

- the set of vertices u of the tree T such that  $C_u \neq \emptyset$ , form a connected subgraph of T,
- for any two adjacent vertices u and v in the tree T, if  $C_u$  and  $C_v$  are non-empty then  $C_u$  and  $C_v$  both contain an element from  $E_u \cap E_v$ , which is the same for them,
- for any two non-adjacent vertices u and v in the tree T,  $C_u \cap C_v = \emptyset$ .

Let  $\gamma_v(C) = |C \cap E_v| = |C_v \cap E|$ . Thus,  $|C| = \sum_{v \in V(T)} \gamma_v(C)$ . For any subtree T', we define

$$\gamma_{T'}(C) = \sum_{v \in V(T')} \gamma_v(C).$$

### 6.1 Decomposing the tree T into smaller weight subtrees.

We classify the circuits C of M according to the distribution of  $\gamma_v(C)$  for  $v \in V(T)$ . For this purpose, we create a signature for each circuit. Let C be a circuit of M of size at most  $\alpha r$ . For circuit C, we first find a center vertex of the tree T. That is, a vertex  $u_0$  in the tree T such that if  $T_1, T_2, \ldots, T_k$  are the subtrees obtained by deleting  $u_0$ , then  $\gamma_{T_i}(C) \leq \alpha r/2$  for each  $i \in [k]$ . The following claim shows that such a vertex exists.

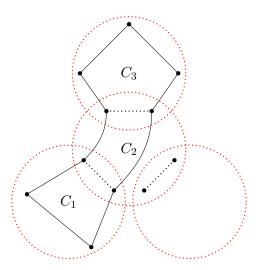


Figure 4: Decomposition of a circuit. Each dotted circle shows a vertex of the unordered decomposition tree. For each vertex of the tree we have a component matroid, which is a graphic matroid in this example, i.e., the elements of the matroid are the graph edges and circuits are cycles. The intersection of two dotted circles contains an element (edge) common between the two corresponding matroids. We have a cycle C which is equal to the symmetric difference  $C_1 \triangle C_2 \triangle C_3$ , where  $C_1, C_2, C_3$  are cycles in the component matroids.

Claim 6.2 (Balanced division of a tree). Let T be a tree and let each vertex  $v \in V(T)$  be associated with a non-negative integer  $\gamma_v$ . Let  $q := \sum_{v \in V(T)} \gamma_v$ . Then there exists a center vertex  $c(T) \in V(T)$  such that for the subtrees  $T_1, T_2, \ldots, T_k$  obtained by deleting c(T) from T, we have

$$\sum_{v \in T_i} \gamma_v \le q/2$$

for each  $1 \le i \le k$ .

*Proof.* We give a procedure to find the vertex c(T). Start with an arbitrary vertex  $v_0$ . Let  $T_1, T_2, \ldots, T_k$  be the subtrees obtained by removing  $v_0$ . Let  $T_i$  be the subtree which maximizes  $\sum_{v \in T_i} \gamma_v$  and let  $q_0$  be the maximum value of this sum. If  $q_0 \leq q/2$  then we output  $v_0$  as the desired vertex. On the other hand, if we have  $q_0 > q/2$  then we get that

$$\sum_{v \in V(T) \backslash V(T_i)} \gamma_v < q/2.$$

Let  $v_1$  be the vertex in  $T_i$  connected with  $v_0$ . If  $v_1$  is a leaf, that is,  $T_i$  is a single vertex  $v_1$  then we output  $v_1$  as the desired vertex c(T). The vertex  $v_1$  has the desired property because its removal gives a single subtree that is  $T - T_i$ , which has weight < q/2.

In the other case when  $v_1$  is not a leaf, we repeat the procedure. That is, check if there is a subtree connected to  $v_1$  with weight > q/2 and if yes, then move to the neighbor of  $v_1$  in this subtree. We claim that this algorithm terminates since we never go back to a previous vertex. To see this, observe that the subtree connected to  $v_1$  which contains  $v_0$  is  $T - T_i$  and its weight is less than q/2. Thus, we do not move back to  $v_0$ .

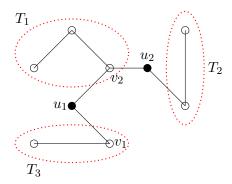


Figure 5: The set U is shown via bold vertices, removing which gives us subtrees  $T_1, T_2$  and,  $T_3$ , indicated by dotted circles.

Note that the center vertex is not uniquely defined. For our purpose, any vertex with the desired property would work. We apply the center finding procedure recursively on the subtrees  $T_1, T_2, \ldots, T_k$ , till we reach a point where for any of the obtained subtrees T', the sum  $\gamma_{T'}$  is at most r/2.

Claim 6.3 (Balanced recursive division of a tree). Let T be a tree with each vertex v associated with a non-negative integer  $\gamma_v$ . Let

$$\gamma_T = \sum_{v \in T} \gamma_v \le 2^p r$$

for some integers  $p, r \ge 0$ . Then there exists a set U of at most  $2^{p+2}$  vertices such that on removing these vertices from the tree T, for each obtained subtree T' (see fig. 5) we have that

$$\gamma_{T'} = \sum_{v \in T'} \gamma_v \le r/2.$$

*Proof.* We construct the set U in p+1 rounds. Let  $U_i \subseteq V(T)$  be the set of vertices we construct after the i-th round and  $\mathcal{T}_i$  be the set of subtrees obtained by deleting the vertices in  $U_i$  from the tree T. We want to ensure that for each subtree  $T' \in \mathcal{T}_i$ , we get

$$\gamma_{T'} \leq 2^{p-i}r.$$

Note that this is true for i=0 with the set  $U_0=\emptyset$  and  $\mathcal{T}_0=\{T\}$ . For  $0 \le i \le p$ , we describe the (i+1)-th round assuming that we have  $U_i$  and  $\mathcal{T}_i$  with the desired properties. Among the subtrees in  $\mathcal{T}_i$ , we need not care about the subtrees T' that already have  $\gamma_{T'} \le 2^{p-i-1}r$ . So, consider the subtrees  $T' \in \mathcal{T}_i$  that have

$$\gamma_{T'} > 2^{p-i-1}r.$$

There can be at most  $2^{i+1}$  such subtrees, as  $\gamma_T \leq 2^p r$ . For each such subtree T' we find a center vertex c(T') using Claim 6.2. This choice of c(T') is such that removing it from T' gives us subtrees T'' having

$$\gamma_{T''} \le 2^{p-i-1}r.$$

Thus, we define the set  $U_{i+1}$  to be the union of  $U_i$  and the set of centers c(T'), i.e.,

$$U_{i+1} = U_i \cup \{c(T') \mid T' \in \mathcal{T}_i, \ \gamma_{T'} > 2^{p-i-1}r\}.$$

By construction,  $U_{i+1}$  has the desired property. For the size bound, observe that

$$|U_{i+1}| \le |U_i| + 2^{i+1}.$$

Solving the recurrence we get

$$|U_{p+1}| \le 2^{p+2}.$$

### 6.2 Classifying the circuits using signatures.

Recall that C is a circuit of M of size at most  $\alpha r$ , which implies  $\gamma_T(C) \leq \alpha r$ . From Claim 6.3, there exists a set  $U(C) \subseteq V(T)$  of at most  $4\alpha$  vertices whose removal ensures that for any obtained subtree T' we have

$$\gamma_{T'}(C) \le r/2. \tag{4}$$

To bound the number of such circuits, we classify the circuits using a signature. Then, we bound the number of circuits associated with a fixed signature and also the number of possible signatures. signature(C) will be defined for any circuit C of M with  $|C| \leq \alpha r$ . The first part of signature(C) is the set U(C) which ensures (4). Note that the set U(C), and thus, the signature is not uniquely defined. We take any arbitrary choice of U(C) for the signature.

We further expand the signature. Recall that  $C = \triangle_{v \in V(T)} C_v$ . Let u be a vertex in U(C) and let N(u) be the set of its neighboring vertices in the tree T. We define a subset  $N^*(u)$  of neighboring vertices N(u) as follows:

$$N^*(u) := \{v \in N(u) \mid v \text{ is on a path connecting } u \text{ to another vertex } u' \in U(C)\}.$$

For example, in Figure 5, the vertex  $u_1$  has  $N(u_1) = \{v_1, v_2\}$  as neighbor set and  $N^*(u_1) = \{v_2\}$ . We claim that the set  $C_u$  and the set  $C_v$  for any vertex  $v \in N^*(u)$  must be non-empty. This is because (i) from the construction of the set U(C), if a subtree T' of T contains a vertex from U(C), then it must be the case that  $\gamma_{T'}(C) \neq 0$ , i.e.,  $C_{T'}$  is non-empty and (ii) the vertices v with  $C_v \neq \emptyset$  form a connected subgraph of T (Observation 6.1).

Recall from Observation 6.1 that  $C_u$  and  $C_v$  have a single common element coming from  $E_u \cap E_v$ . For each  $u \in U(C)$  and for each  $v \in N^*(u)$ , we put this common element of  $C_u$  and  $C_v$  in signature C. This finishes the construction of the signature.

signature(C) := 
$$U(C) \cup \{(u, E_u \cap C_v) \mid u \in U(C), v \in N^*(v)\}.$$

First, we upper bound the number of possibilities for the signature.

Claim 6.4. There are at most  $(9m)^{4\alpha}$  possibilities for signature (C) with  $|C| < \alpha r$ .

*Proof.* There are at most  $4\alpha$  vertices in U(C) and thus, there at most  $m^{4\alpha}$  possibilities for U(C). We would like to bound the number of tuples  $(u, E_u \cap C_v)$  in signature (C). Observe that this number is bounded by

$$\sum_{u \in U(C)} |N^*(u)|.$$

We claim that this sum is bounded by  $2|U(C)| \le 8\alpha$ . To see this, make the tree T rooted at an arbitrary vertex. For any vertex  $u \in U(C)$ , let  $A(u) \in U(C)$  be its nearest ancestor in the rooted tree T. For each  $u \in U(C)$ , we count two vertices – first, the neighbor of u connecting it to A(u) and second, the neighbor of A(u) connecting it to u. This way, we have counted every vertex in  $\bigcup_{u \in U(C)} N^*(v)$ .

Recall that the element  $E_u \cap C_v$  comes from  $E_u \cap E_v$ , which contains at most 3 elements. Thus, for each tuple  $(u, C_u \cap E_v)$ , there are at most 3 possibilities. This gives us an upper bound of  $3^{8\alpha}$  on the number of possibilities of all tuples.

Multiplying together the bounds for U(C) and all the tuples  $(u, E_u \cap C_v)$ , we get an upper bound on the number of possible signatures.

### 6.3 Bounding the number of circuits for a given signature.

Our aim is to bound the number of circuits with a given signature. Let S be a signature of a circuit of size at most  $\alpha r$ . Let  $C_S$  be the set of all circuits C of size at most  $\alpha r$  such that signature C be the set of vertices in the signature S. Let C be the set of subtrees obtained after removing the vertices in C from C. Then, for the ground set C of matroid C we can write

$$E = (\triangle_{u \in U} E_u) \triangle (\triangle_{T' \in \mathcal{T}} E_{T'}).$$

Also, every circuit C of M can be written as

$$C = (\triangle_{u \in U} C_u) \triangle (\triangle_{T' \in \mathcal{T}} C_{T'}).$$

To bound the number of circuits in C in  $C_S$ , we bound the number of possibilities for the component circuits  $C_u$  and  $C_{T'}$  for  $u \in U$  and  $T' \in \mathcal{T}$ . We know that for any subtree  $T' \in \mathcal{T}$  and for any circuit  $C \in C_S$ ,

$$\gamma_{T'}(C) = |E_{T'} \cap C| \le r/2.$$

For any subtree  $T' \in \mathcal{T}$ , let  $E_{T-T'}$  be the set  $\triangle_{v \in V(T) \setminus V(T')} E_v$ . Then we have  $E = E_{T'} \triangle E_{T-T'}$ . The circuit  $C_{T'}$  also has elements from  $E_{T'} \cap E_{T-T'}$ , which we call its boundary elements. We first show that for all circuits  $C \in \mathcal{C}_{\mathcal{S}}$ , there is a unique possibility of the circuit  $C_{T'}$ , once we fix these boundary elements  $C_{T'} \cap E_{T-T'}$ .

Claim 6.5. Let C and D be two circuits in  $C_S$ . If  $C_{T'} \cap E_{T-T'} = D_{T'} \cap E_{T-T'}$ , then  $C_{T'} = D_{T'}$ .

*Proof.* Let us assume for the sake of contradiction that  $C_{T'}$  and  $D_{T'}$  are different. Consider the symmetric difference  $C_{T'} \triangle D_{T'}$ , which is a disjoint union of circuits in  $M_{T'}$  (Fact 3.8). From the assumption of the claim, the set  $C_{T'} \triangle D_{T'}$  does not have any elements from  $E_{T-T'}$ . In particular, this means that  $C_{T'} \triangle D_{T'}$  is actually a subset of  $E = E_{T'} \triangle E_{T-T'}$ . In this case, the set  $C_{T'} \triangle D_{T'}$  is, in fact, a disjoint union of circuits in M (Fact 3.14). However,

$$|C_{T'} \triangle D_{T'}| \le |C \cap E_{T'}| + |D \cap E_{T'}| \le r/2 + r/2 = r,$$

which contradicts the hypothesis of the theorem that M has no circuits of size at most r.

We argue that for some of the subtrees  $T' \in \mathcal{T}$ , the boundary elements of circuit  $C_{T'}$  indeed get fixed by the signature  $\mathcal{S}$ . We distinguish two types of subtrees in the set  $\mathcal{T}$ . Let  $\mathcal{T}_0 \subseteq \mathcal{T}$  be the set of all the subtrees which have at most one vertex of U adjacent to them. The other set  $\mathcal{T}_1 = \mathcal{T} \setminus \mathcal{T}_0$  has all the subtrees which have at least two vertices of U adjacent to them. For example, in Figure 5, the subtrees  $T_2$  and  $T_3$  are in  $T_0$  and the subtree  $T_1$  is in  $T_1$ .

We first show that for any subtree  $T' \in \mathcal{T}_1$ , all circuits  $C_{T'}$ , for  $C \in \mathcal{C}_{\mathcal{S}}$ , have the same boundary elements.

Claim 6.6. For any two circuits  $C, D \in \mathcal{C}_{\mathcal{S}}$  and any subtree  $T' \in \mathcal{T}_1$ ,

$$C_{T'} \cap E_{T-T'} = D_{T'} \cap E_{T-T'}.$$

Proof. Let u be any vertex in U that is adjacent to T' and let  $v \in V(T')$  be the vertex in T' connected to u. Since T' is adjacent to at least one more vertex  $u' \in U$  that is different from u, the vertex v must be on the path connecting u to u'. In other words,  $v \in N^*(u)$ . Thus, the tuple  $(u, E_u \cap C_v)$  is in signature (C) for any circuit  $C \in \mathcal{C}_S$ . Since C and D have the same signature, we get  $E_u \cap C_v = E_u \cap D_v$ . Since this is true for any vertex  $u \in U$  that is adjacent to T' and its neighbor v in T', we can write

$$E_{T-T'} \cap C_{T'} = E_{T-T'} \cap D_{T'}.$$

Claims 6.5 and 6.6 together imply that for any subtree  $T' \in \mathcal{T}_1$ , there is only one possible circuit  $C_{T'}$  for all  $C \in \mathcal{C}_{\mathcal{S}}$ .

**Lemma 6.7** (Unique projection on a subtree in  $\mathcal{T}_1$ ). For any subtree  $T' \in \mathcal{T}_1$ , the set  $\{C_{T'} \mid C \in \mathcal{C}_{\mathcal{S}}\}$  has cardinality exactly one.

We move on to the subtrees in  $\mathcal{T}_0$ . A claim similar to Claim 6.6 cannot be made about a subtree  $T' \in \mathcal{T}_0$ . This is because the signature  $\mathcal{S}$  does not fix the boundary elements of  $C_{T'}$  for  $T' \in \mathcal{T}_0$ .

Recall that by definition any subtree  $T' \in \mathcal{T}_0$  has a unique vertex of U adjacent to it, say u. This means,  $E_{T'} \cap E_{T-T'} = E_{T'} \cap E_u$ , which has at most three elements in it. Thus, the number of possibilities for the set  $C_{T'} \cap E_{T-T'}$  is at most three. From Claim 6.5, we get that there are at most three distinct possibilities of  $C_{T'}$  for  $C \in \mathcal{C}_S$ . However, there can be a large number of subtrees in  $\mathcal{T}_0$ , possibly O(m). This would mean the total number of possibilities for circuits  $C_{T'}$  for all  $T' \in \mathcal{T}_0$  can be  $3^{O(m)}$ .

Instead, we use a different strategy where we bound the number of possibilities for circuits  $C_u$  for  $u \in U$  and  $C_{T'}$  for  $T' \in \mathcal{T}_0$  simultaneously.

Bounding the number of circuits in a graphic/cographic matroid. To proceed further, we need a result for graphic and cographic matroids, similar to Theorem 1.1, albeit in a stronger form. Here, we have weights assigned to the elements and we consider the weight of a circuit defined as  $w(C) = \sum_{e \in C} w(e)$ . Also, we only have a weaker assumption that circuits avoiding a given subset R have size more than r. A proof can be found in appendix A.

**Lemma 6.8.** Let  $M=(E,\mathcal{I})$  be a graphic or cographic matroid with ground set size  $m\geq 2$ . Let  $R\subseteq E$  be any (possibly empty) set of elements of the ground set. Let  $w\colon E\to\mathbb{N}$  be a weight assignment on the ground set. Assume that there is no circuit C in M such that  $C\cap R=\emptyset$  and  $w(C)\leq r$ . Then, for any integer  $\alpha\geq 1$ , the number of circuits C such that  $R\subseteq C$  and  $w(C)\leq \alpha r$  is at most  $(4\alpha+2|R|)^{|R|}(2m)^{2\alpha}$ .

We return to bounding the number of circuits  $C_u$  for  $u \in U$  and  $C_{T'}$  for  $T' \in \mathcal{T}_0$ . For any vertex  $u \in U$ , let  $\mathcal{T}_{0,u} \subseteq \mathcal{T}_0$  be the set of subtrees in  $\mathcal{T}_0$  that have a vertex adjacent to u.

**Lemma 6.9.** For any vertex  $u \in U$ , the cardinality of the set

$$\{C_u \triangle \left(\triangle_{T' \in \mathcal{T}_{0,u}} C_{T'}\right) \mid C \in \mathcal{C}_{\mathcal{S}}\}$$

is at most  $(12\alpha)^{4\alpha}(2m)^{2\alpha}$ .

*Proof.* The idea is to design a one-one map from the set in the lemma to the set of circuits of the matroid  $M_u$ . Since  $M_u$  is a graphic, cographic, or the  $R_{10}$  matroid, we can bound the number of circuits in  $M_u$  of a certain size using Lemma 6.8. Our map is simply

$$\phi \colon C_u \triangle \left( \triangle_{T' \in \mathcal{T}_{0,u}} C_{T'} \right) \mapsto C_u.$$

First, we argue that the map  $\phi$  is one-one. For a given circuit  $C_u$ , we show that it has a unique pre-image under  $\phi$ . For any subtree  $T' \in \mathcal{T}_{0,u}$ , let  $S_{T'} := E_{T'} \cap E_u$  be the set of common elements between  $E_{T'}$  and  $E_u$ .

If  $C_u \cap S_{T'} = \emptyset$  for some  $T' \in \mathcal{T}_{0,u}$ , then  $C_{T'}$  must be empty (Observation 6.1). In this case, there is a unique choice for  $C_{T'}$ .

Consider the case when  $C_u \cap S_{T'} = \{s\}$  for some  $T' \in \mathcal{T}_{0,u}$ . For the sake of contradiction, let C and D be two circuits in  $C_S$  such that  $C_u = D_u$  but  $C_{T'} \neq D_{T'}$ . Since  $C_u \cap S_{T'} = \{s\}$ , we also have  $C_{T'} \cap E_u = \{s\}$  (Observation 6.1). Similarly,  $D_{T'} \cap E_u = \{s\}$ , which means

$$D_{T'} \cap E_u = C_{T'} \cap E_u$$
.

Recall that since u is the single vertex in T-T' adjacent to T', we have  $E_{T'} \cap E_u = E_{T'} \cap E_{T-T'}$ . Thus, we get that

$$D_{T'} \cap E_{T-T'} = C_{T'} \cap E_{T-T'}.$$

Hence, from Claim 6.5,  $D_{T'} = C_{T'}$ . This proves that given a  $C_u$ , there is a unique choice for  $C_{T'}$  for each  $T' \in \mathcal{T}_{0,u}$ .

We would like the map  $\phi$  to be size preserving in some sense. For this reason, we have to assign some integer weights for elements in  $M_u$  and then consider the weight of a circuit in  $M_u$ . From the above discussion, for any subtree  $T' \in \mathcal{T}_{0,u}$  and element  $s \in S_{T'}$ , there is

a unique circuit  $C_{T'}^s$  of  $M_{T'}$  such that  $C_{T'}^s \cap E_u = \{s\}$  and  $C^s \in \mathcal{C}_{\mathcal{S}}$ . We define the weights in  $M_u$  as follows:

$$w(s) = \begin{cases} |C_{T'}^s| - 1 & \text{if } s \in S_{T'} \text{ for some } T' \in \mathcal{T}_{0,u} \\ 1 & \text{otherwise.} \end{cases}$$

Recall that for any circuit  $C_u$  of  $M_u$ , its weight is defined to be  $w(C_u) = \sum_{e \in C_u} w(e)$ .

Claim 6.10. For any 
$$C \in \mathcal{C}_{\mathcal{S}}$$
,  $|C_u \triangle (\triangle_{T' \in \mathcal{T}_{0,u}} C_{T'})| = w(C_u)$ .

*Proof.* From the arguments above, for a subtree  $T' \in \mathcal{T}_{0,u}$ , if  $C_u \cap S_{T'} = \{s\}$  then  $C_{T'} = C^s_{T'}$  and if  $C_u \cap S_{T'} = \emptyset$  then  $C_{T'} = \emptyset$ . The claim follows directly from the definition of the weight function  $w(\cdot)$ .

Note that since |C| is at most  $\alpha r$  for any  $C \in \mathcal{C}_{\mathcal{S}}$ , the size of the set  $C_u \triangle (\triangle_{T' \in \mathcal{T}_{0,u}} C_{T'})$  is also at most  $\alpha r$ . To bound the cardinality of the set in the lemma, we have to bound the number of circuits  $C_u$  of  $M_u$  with  $w(C_u) \leq \alpha r$ . We want to use Lemma 6.8 for this. However, note that we do not have any assumption on the minimum weight of a circuit in  $M_u$ , which is required in Lemma 6.8. What we do have is an assumption that there are no circuits of size r in M. We tackle this problem as follows.

Let  $R \subseteq E_u$  be a set defined as

$$R := \{e \mid \text{the tuple } (u, e) \text{ belongs to the signature } \mathcal{S}\}.$$

Recall from the construction of the signature that each element in R is contained in  $C_u$  for every circuit  $C \in \mathcal{C}_{\mathcal{S}}$ .

Let  $\mathcal{T}_{1,u} \subseteq \mathcal{T}_1$  be the set of subtrees in  $\mathcal{T}_1$  that have a vertex adjacent to u. For a subtree  $T' \in \mathcal{T}_{1,u}$ , let  $S_{T'} := E_{T'} \cap E_u$ . Recall the construction of signature(C) and observe that the set R takes exactly one element from each  $S_{T'}$ , for  $T' \in \mathcal{T}_{1,u}$  (since the subtrees in  $\mathcal{T}_{1,u}$  have more than one adjacent vertices in U). Since the circuit  $C_u$  can take at most one element from  $S_{T'}$  (Observation 6.1), the elements in  $\bigcup_{T' \in \mathcal{T}_{1,u}} S_{T'} \setminus R$  are not part of  $C_u$  for any  $C \in \mathcal{C}_{\mathcal{S}}$ . Hence, we can safely delete these elements from  $M_u$ .

Let  $\overline{M}_u$  be the matroid obtained from  $M_u$  by deleting the elements in  $\bigcup_{T' \in \mathcal{T}_{1,u}} S_{T'} \setminus R$ . Note that  $\overline{M}_u$  remains a graphic, cographic or the  $R_{10}$  matroid (Fact 3.11). Observe that for any circuit  $C \in \mathcal{C}_{\mathcal{S}}$ , the set  $C_u$  is a circuit of  $\overline{M}_u$  which contains all elements of R. We plan to apply Lemma 6.8 on the matroid  $\overline{M}_u$ . We show the following about minimum weight of a circuit in  $\overline{M}_u$ .

**Claim 6.11.** There is no circuit  $D_u$  of  $\overline{M}_u$  such that  $D_u \cap R = \emptyset$  and  $w(D_u) \leq r$ .

*Proof.* For the sake of contradiction, let there be such a circuit  $D_u$ . We show that this implies the existence of a circuit of M of size at most r, which would be a contradiction.

For any subtree  $T' \in \mathcal{T}_{0,u}$ , if  $D_u \cap S_{T'} = \emptyset$  then define  $D_{T'} = \emptyset$  and if  $D_u \cap S_{T'} = \{s\}$  then define  $D_{T'} = C^s_{T'}$ . Consider the set

$$D = D_u \triangle \left( \triangle_{T' \in \mathcal{T}_{0,u}} D_{T'} \right).$$

From the definition of the weight function  $w(\cdot)$ , it is clear that  $|D| = w(D_u) \le r$ .

We would like to argue that D is, in fact, a disjoint union of circuits of M. Let T'' be the subtree obtained by joining u and all the subtrees in  $\mathcal{T}_{0,u}$ . By construction, D is a disjoint union of circuits of the matroid  $M_{T''}$  (Fact 3.15). We claim that D has no elements from  $E_{T''} \cap E_{T-T''}$ , which implies that D is a disjoint union of circuits of M (Fact 3.14). To see this claim, observe that the set of common elements  $E_{T''} \cap E_{T-T''}$  is the set  $\bigcup_{T' \in \mathcal{T}_{1,u}} S_{T'} \setminus R$  have been deleted in  $\overline{M}_u$  and (ii)  $D_u \cap R = \emptyset$  by the assumption in the claim.

Using Claim 6.11, we can apply Lemma 6.8 on the matroid  $\overline{M}_u$ . We get that the number of circuits in  $C_u$  in  $\overline{M}_u$  with  $R \subseteq C_u$  and weight at most  $\alpha r$  is bounded by  $(4\alpha+2|R|)^{|R|}(2m)^{2\alpha}$ . Note that the size of |R| is bounded by the number of tuples (u,e) in signature (C). Recall the construction of signature (C) and observe that there can be at most  $|N^*(u)|$  such tuples, which is bounded by  $|U| \le 4\alpha$ . Thus,  $|R| \le 4\alpha$ . Hence, the bound we get on the cardinality of the set in the lemma is  $(12\alpha)^{4\alpha}(2m)^{2\alpha}$ .

Using Lemma 6.7 and Lemma 6.9 we can bound the number of circuits in  $C_S$ .

**Lemma 6.12.** For any signature S, the number of circuits in the set  $C_S$  is at most  $m^{O(\alpha^2)}$ .

*Proof.* For any circuit  $C \in \mathcal{C}_{\mathcal{S}}$ , we write it as

$$(\triangle_{T'\in\mathcal{T}_1}C_{T'}) \triangle (\triangle_{u\in U} (C_u\triangle (\triangle_{T'\in\mathcal{T}_{0,u}}C_{T'})))$$
.

From Lemma 6.7, the number of possibilities for the set  $\triangle_{T'\in\mathcal{T}_1}C_{T'}$  is 1. From Lemma 6.9, for each  $u\in U$  the number of possibilities for the set  $C_u\triangle\left(\triangle_{T'\in\mathcal{T}_0,u}C_{T'}\right)$  is at most  $(12\alpha)^{4\alpha}(2m)^{2\alpha}$ . The number of vertices in U is at most  $4\alpha$ . Together this gives us a bound of  $((12\alpha)^{4\alpha}(2m)^{2\alpha})^{4\alpha}$  on the number of circuits in  $\mathcal{C}_{\mathcal{S}}$ .

By Claim 6.4, the number of different possible signatures for a circuit in M of size at most  $\alpha r$  is  $m^{O(\alpha)}$ . Together with Lemma 6.12, we get that the number of circuits in M of size at most  $\alpha r$  is  $m^{O(\alpha^2)}$ . This finishes the proof of Theorem 1.1.

### 7 Max-flow min-cut matroids

Max-flow min-cut (MFMC) matroids were defined by Seymour [27], which were inspired from the max-flow min-cut theorem on graphs. The MFMC theorem on graphs can be rephrased as follows: Let G be an undirected graph. Let s and t be two special vertices with an edge  $\ell$  between them. Let every other edge e have a positive integer capacity, say  $c_e$ . Let us say, we want to find a maximum size family F of cycles (not necessarily distinct), such that each cycle contains the edge  $\ell$  and moreover, any edge e is part of at most  $c_e$  cycles in the family. Let us call the maximum size of F as the max-flow. For any cut-set containing the edge  $\ell$ , let us define its capacity as the sum of the capacities of the edges in it except  $\ell$ . Let us call the minimum capacity of any cut-set containing  $\ell$  as the min-cut. It is easy to see that the max-flow cannot be larger than the min-cut. The MFMC theorem for graphs says that these two quantities, max-flow and min-cut are, in fact, always equal.

We can write analogous definitions for any matroid M: we fix a special element  $\ell$  of the ground set, assign a positive integer capacity  $c_e$  to an element e (except  $\ell$ ) and then

use circuits instead of cycles and cocircuits instead of cut-sets for defining max-flow and min-cut. Recall that a cocircuit of a matroid is a circuit of the dual matroid. Let  $\mathcal{C}_{\ell}$  be the set of circuits of M that contain  $\ell$  and  $\mathcal{C}_{\ell}^*$  be the set of cocircuits of M that contain  $\ell$ . Max-flow and min-cut are defined by the following two integer programs, respectively.

$$\begin{aligned} & \text{max-flow} &= & \max\{1^T y : y \in \mathbb{Z}^{\mathcal{C}_\ell}, y \geq 0, \text{ and for all } e, \sum_{\substack{C \in \mathcal{C}_\ell \\ C \text{ contains } e}} y_C \leq c_e\}. \\ & \text{min-cut} &= & \min_{C \in C_\ell^*} \sum_{e \in C \setminus \ell} c_e. \end{aligned}$$

Recall that cocircuits of a graphic matroid are the minimal cut-sets of the graph. Thus, in the case of graphic matroids, the above definitions of max-flow and min-cut coincide with the usual max-flow and min-cut in a graph. Hence, the MFMC theorem holds for any graphic matroid. A natural question arises that whether it holds for any other matroids besides graphic matroids. It is not hard to verify that one side of the inequality is true for any matroid, that is, the max-flow is no greater than the min-cut.

For an element  $\ell$  of a matroid M, we say the pair  $(M,\ell)$  has the MFMC property if the max-flow and min-cut are equal for any given positive integer capacities  $c_e$ , while treating  $\ell$  as the special element in the above sense. We call a matroid max-flow min-cut (MFMC) matroid, if for every element  $\ell$  of its ground set, the pair  $(M,\ell)$  has the MFMC property. There are simple examples of matroids which are not MFMC, e.g.,  $U_4^2$  – rank-2 uniform matroid of 4 elements and  $F_7^*$  – the dual of the 7-element Fano matroid  $F_7$  (see [27]). Still, Seymour [27] showed that MFMC matroids form a large class, which in particular, contains regular matroids. The precise characterization of MFMC matroids given by Seymour [27] is in terms of forbidden minors. He showed that any MFMC matroid must be binary and should not have a  $F_7^*$  minor. Regular matroids form a subclass of these matroids since they are binary and do not have  $F_7$  or  $F_7^*$  as minors (see, for example, [25, Theorem 13.1.2]). The following matrix represents the  $F_7$  matroid over GF(2).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

More importantly for us, in a later work, Seymour also gave a decomposition theorem [26, (7.6)] for MFMC matroid, similar to regular matroids (also see [32]).

**Theorem 7.1** ([26]). Every MFMC matroid can be obtained by means of 1-sums and 2-sums, starting from matroids which are regular or  $F_7$ .

Since a regular matroid itself can be decomposed into graphic, cographic and  $R_{10}$  matroids (Theorem 3.18), one can get a more refined decomposition for MFMC matroids.

Corollary 7.2. Every MFMC matroid can be obtained by means of 1-sums, 2-sums and 3-sums, starting from matroids which are graphic, cographic,  $R_{10}$  or  $F_7$ .

Note that the only difference between the decomposition of MFMC matroids and that of regular matroids is the presence of  $F_7$  matroid as one of the building blocks. Like the  $R_{10}$ 

matroid, the  $F_7$  matroid also has a constant number of elements. Thus, our whole argument about bounding the number of near-minimum circuits in a regular matroid can be applied as it is to MFMC matroids, which would give us the following.

**Theorem 7.3.** Let M be a MFMC matroid with ground set size m. Suppose that M has no circuits of size at most r. Then the number of circuits in M of size at most  $\alpha r$  is bounded by  $m^{O(\alpha^2)}$ .

### Acknowledgements

We thank Rohan Kapadia, Thomas Thierauf for helpful discussions, and Ben Lee Volk for pointing out that random binary linear codes have exponentially many near-minimum weight codewords.

### References

- [1] Miklós Ajtai. The shortest vector problem in L<sub>2</sub> is NP-hard for randomized reductions (extended abstract). In *Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing*, STOC '98, pages 10–19, New York, NY, USA, 1998. ACM.
- [2] Noga Alon. Packings with large minimum kissing numbers. Discrete Mathematics, 175(1):249 251, 1997.
- [3] Alexei Ashikhmin, Alexander Barg, and Serge Vlăduţ. Linear codes with exponentially many light vectors. *Journal of Combinatorial Theory*, Series A, 96(2):396 399, 2001.
- [4] Alexander Barg. Error-correcting codes. Lecture notes, University of Maryland http://www.ece.umd.edu/~abarg/626-2009/notes2005/lecture25.pdf, 2005.
- [5] J. H. Conway, N. J. A. Sloane, and E. Bannai. *Sphere-packings, Lattices, and Groups*. Springer-Verlag, Berlin, Heidelberg, 1987.
- [6] J. H. Conway, N. J. A. Sloane, and E. Bannai. Sphere-packings, Lattices, and Groups. Springer-Verlag New York, Inc., New York, NY, USA, 1987.
- [7] Michael Dinitz and Guy Kortsarz. Matroid secretary for regular and decomposable matroids. In *Proceedings of the Twenty-fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '13, pages 108–118, Philadelphia, PA, USA, 2013. Society for Industrial and Applied Mathematics.
- [8] Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, and Saket Saurabh. Covering vectors by spaces: Regular matroids. In 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland, pages 56:1–56:15, 2017.
- [9] Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, and Saket Saurabh. Spanning circuits in regular matroids. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '17, pages 1433–1441, Philadelphia, PA, USA, 2017. Society for Industrial and Applied Mathematics.
- [10] Tomáš Gavenčiak, Daniel Král, and Sang-il Oum. Deciding first order properties of matroids. In Automata, Languages, and Programming, pages 239–250, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.
- [11] Jim Geelen, Bert Gerards, and Geoff Whittle. The highly connected matroids in minor-closed classes. *Annals of Combinatorics*, 19(1):107–123, January 2015.
- [12] Leslie Ann Goldberg and Mark Jerrum. A polynomial-time algorithm for estimating the partition function of the ferromagnetic ising model on a regular matroid. In *Proceedings of the 38th International Colloquim Conference on Automata, Languages and Programming Volume Part I*, ICALP'11, pages 521–532, Berlin, Heidelberg, 2011. Springer-Verlag.

- [13] Alexander Golynski and Joseph D. Horton. A polynomial time algorithm to find the minimum cycle basis of a regular matroid. In Martti Penttonen and Erik Meineche Schmidt, editors, Algorithm Theory — SWAT 2002, pages 200–209, Berlin, Heidelberg, 2002. Springer Berlin Heidelberg.
- [14] Rohit Gurjar, Thomas Thierauf, and Nisheeth K. Vishnoi. Isolating a vertex via lattices: Polytopes with totally unimodular faces. In 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic, pages 74:1–74:14, 2018.
- [15] Volker Kaibel, Jon Lee, Matthias Walter, and Stefan Weltge. Extended formulations for independence polytopes of regular matroids. *Graphs and Combinatorics*, 32(5):1931– 1944, 2016.
- [16] Gil Kalai and Nathan Linial. On the distance distribution of codes. *IEEE TRANS*. *INFORM*. *THEORY*, 41:1467–1472, 1995.
- [17] Rohan Kapadia. Finding shortest circuits in binary matroids. The Matroid Union (blog) http://matroidunion.org/?p=1106, 2014.
- [18] David Karger. A randomized fully polynomial approximation scheme for the all terminal network reliability problem. SIAM Journal on Computing, 29:492–514, 11 1996.
- [19] David R. Karger. Global min-cuts in RNC, and other ramifications of a simple min-cut algorithm. In *Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '93, pages 21–30, Philadelphia, PA, USA, 1993. Society for Industrial and Applied Mathematics.
- [20] Navin Kashyap. A decomposition theory for binary linear codes. *IEEE Trans. Information Theory*, 54(7):3035–3058, 2008.
- [21] Hong-Jian Lai and Hoifung Poon. Cycle cover ratio of regular matroids. *European Journal of Combinatorics*, 23(8):1007 1014, 2002.
- [22] John Leech. Some sphere packings in higher space. Canadian Journal of Mathematics (CJM), 16:657–682, Jan 1964.
- [23] Nati Linial and Jonathan Mosheiff. On the weight distribution of random binary linear codes. *CoRR*, abs/1806.08392, 2018.
- [24] S. Onn. *Nonlinear Discrete Optimization: An Algorithmic Theory*. Zurich lectures in advanced mathematics. European Mathematical Society Publishing House, 2010.
- [25] James G. Oxley. Matroid Theory (Oxford Graduate Texts in Mathematics). Oxford University Press, Inc., New York, NY, USA, 2006.
- [26] Paul D. Seymour. Decomposition of regular matroids. J. Comb. Theory, Ser. B, 28(3):305–359, 1980.
- [27] P.D. Seymour. The matroids with the max-flow min-cut property. *Journal of Combinatorial Theory, Series B*, 23(2):189 222, 1977.

- [28] Ashok Subramanian. A polynomial bound on the number of light cycles in an undirected graph. *Information Processing Letters*, 53(4):173 176, 1995.
- [29] Madhu Sudan. List decoding: Algorithms and applications. SIGACT News, 31(1):16–27, March 2000.
- [30] C. P. Teo and K. M. Koh. The number of shortest cycles and the chromatic uniqueness of a graph. *Journal of Graph Theory*, 16(1):7–15, 1992.
- [31] Klaus Truemper. A decomposition theory for matroids. v. testing of matrix total unimodularity. *Journal of Combinatorial Theory, Series B*, 49(2):241 281, 1990.
- [32] Klaus Truemper. Matroid Decomposition. Leibniz, Plano, Texas (USA), 1998.

## A Bounding the number of circuits in a graphic or cographic matroid

In this section, we prove the statement Theorem 1.1 for graphic and cographic matroids, but with a bit stronger version. Here, we are interested in circuits that contain a given subset R of the ground set. Also, we have weights assigned to the elements and we consider weight of a circuit defined as  $w(C) = \sum_{e \in C} w(e)$ . We need this stronger version for graphic and cographic case to prove Theorem 1.1.

**Lemma** (Lemma 6.8). Let  $M = (E, \mathcal{I})$  be a graphic or cographic matroid with ground set size  $m \geq 2$ . Let  $R \subseteq E$  be any (possibly empty) set of elements of the ground set. Let  $w: E \to \mathbb{N}$  be a weight assignment on the ground set. Assume that there is no circuit C in M such that  $C \cap R = \emptyset$  and  $w(C) \leq r$ . Then, for any integer  $\alpha \geq 1$ , the number of circuits C such that  $R \subseteq C$  and  $w(C) \leq \alpha r$  is at most  $(4\alpha + 2|R|)^{|R|}(2m)^{2\alpha}$ .

Proof. When M is a graphic matroid. Let G = (V, E) be the graph corresponding to the graphic matroid M. In this setting, the assumption of the lemma means that for any cycle C in the graph G such that  $C \cap R = \emptyset$ , w(C) > r. Consider any cycle C in G with  $R \subseteq C$  and  $w(C) \le \alpha r/2$ . Let the edge sequence of the cycle C be  $(e_1, e_2, e_3, \ldots, e_q)$  such that if R is nonempty then  $e_1 \in R$ . We choose a subset of edges in the cycle C as follows: Let  $i_1 = 1$  and for  $j = 2, 3, \ldots k = 2\alpha + |R|$ , define  $i_j$  to be the minimum index (if one exists) greater than  $i_{j-1}$  such that either  $e_{i_j} \in R$  or

$$\sum_{a=i_{j-1}+1}^{i_j} w(e_a) > r/2. \tag{5}$$

If such an index does not exists then define  $i_j = q$ . The edges  $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$  divide the cycle C into at most k segments, which are defined as follows: for  $j = 1, 2, \ldots, k-1$ 

$$p_j := (e_{i_j+1}, e_{i_j+2}, \dots, e_{i_{j+1}-1}),$$

and

$$p_k := (e_{i_k+1}, e_{i_k+2}, \dots, e_q).$$

Note that some of these paths  $p_j$  can be empty (for example, when  $i_{j+1} = i_j + 1$ ). By the choice of  $i_j$  we know that  $w(p_j) \le r/2$  for j = 1, 2, ..., k-1. We make a similar claim for the last segment  $p_k$ .

Claim A.1.  $w(p_k) < r/2$ .

Proof. If  $i_k = q$  then  $p_k$  is empty and  $w(p_k) = 0$ . So, we assume that  $i_k < q$  and hence,  $i_j < q$  for each  $1 \le j \le k$ . Hence, by the choice of  $i_j$ , we know that either  $e_{i_j} \in R$  or  $\sum_{a=i_{j-1}+1}^{i_j} w(e_a) > r/2$  for each  $2 \le j \le k$ . Since  $k = 2\alpha + |R|$ , the second possibility must happen at least  $2\alpha - 1$  times. That is, the set  $\{j \mid 2 \le j \le k, \sum_{a=i_{j-1}+1}^{i_j} w(e_a) > r/2\}$  has cardinality at least  $2\alpha - 1$ . Thus, we can write

$$\sum_{a=2}^{i_k} w(e_a) > (2\alpha - 1)r/2 = \alpha r - r/2.$$

This together with the fact that  $w(C) \leq \alpha r$ , gives us  $w(p_k) < r/2$ .

We associate the ordered tuple of oriented edges  $t_C = (e_{i_1}, e_{i_2}, \dots, e_{i_k})$  with the cycle C. Note that depending on the starting edge, there can be many possible tuples associated to a cycle C. We fix an arbitrary such tuple to be  $t_C$ .

Claim A.2. Let C and C' be two distinct cycles in G such that both contain the set R and  $w(C), w(C') \leq \alpha r$ . Then  $t_C \neq t_{C'}$ .

Proof. For the sake of contradiction, let us assume  $t_C = t_{C'}$  and let it be  $(e_{i_1}, e_{i_2}, \dots, e_{i_k})$ . That is, C and C' pass through  $(e_{i_1}, e_{i_2}, \dots, e_{i_k})$  in the same order and with the same orientation of these edges. Let  $p_1, p_2, \dots, p_k$  be the path segments in C connecting the edges  $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ . And let  $p'_1, p'_2, \dots, p'_k$  be these segments in C'. Since C and C' are distinct,  $p_j$  and  $p'_j$  must be distinct for at least one  $j \in \{1, 2, \dots, k\}$ . Since the starting and end vertices of  $p_j$  and  $p'_j$  are same,  $p_j \cup p'_j$  contains a cycle C''. By the construction of the tuple  $t_C$ , we have  $w(p_j), w(p'_j) \leq r/2$ . This implies that  $w(C'') \leq r$ .

Finally, since each edge in R is among the edges  $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ , the segments  $p_j$  and  $p'_j$  do not have any edge from R. This means that  $C'' \cap R = \emptyset$ . This contradicts the assumption that there is no cycle C in G such that  $w(C) \leq r$  and  $C \cap R = \emptyset$ .

From Claim A.2, it follows that the number of possible distinct tuples  $t_C$  upper bounds the number of cycles C in G with  $w(C) \leq \alpha r$  and  $R \subseteq C$ . Hence, we bound the number of possible tuples.

Claim A.3. The number of possible distinct tuples  $t_C$  is at most  $(4\alpha + 2|R|)^{|R|}(2m)^{2\alpha}$ .

*Proof.* Recall that there are  $2\alpha + |R|$  edges in  $t_C$  and it contains each edge of R. There are  $(2\alpha + |R|)!/(2\alpha)!$  ways of choosing |R| indices where the edges of R would appear. And for the rest of the indices there are at most  $m^{2\alpha}$  ways of choosing the edges in G that would appear in these indices. Finally, there are  $2^{2\alpha+|R|}$  ways of orienting the edges in  $t_C$ . The product of these numbers gives us an upper bound on number of tuples  $t_C$ .

$$(2\alpha + |R|)!/(2\alpha)! \cdot 2^{2\alpha + |R|} \cdot m^{2\alpha} \le (4\alpha + 2|R|)^{|R|} (2m)^{2\alpha}.$$

From Claim A.2 and A.3, we get that the number of cycles C in G with  $w(C) \leq \alpha r$  and  $R \subseteq C$  is bounded by  $(4\alpha + 2|R|)^{|R|}(2m)^{2\alpha}$ .

When M is a cographic matroid. Let G = (V, E) be the graph corresponding to the cographic matroid M and let n = |V|. Recall from Fact 3.7 that circuits in cographic matroids are inclusion-wise minimal cut-sets in G. By the assumption of the lemma, any cut-set C in G with  $R \cap C = \emptyset$  has weight w(C) > r. Note that this implies that G is connected, and therefore  $m \geq n-1$ . We want to give a bound on the number of cut-sets  $C \subseteq E$  such that  $w(C) \leq \alpha r$  and  $R \subseteq C$ .

We argue similar to the probabilistic construction of a minimum cut of Karger [19]. The basic idea is to contract randomly chosen edges. Contraction of an edge e = (u, v) means that all edges between u and v are deleted and then u is identified with v. Note that we get a multi-graph that way: if there were two edges (u, w) and (v, w) before the contraction,

they become two parallel edges after identifying u and v. The contracted graph is denoted by G/e. The intuition behind Karger's contraction algorithm is that when a randomly chosen edge is contracted, a small weight cut-set survives with a good probability.

The following algorithm implements the idea. It does  $k \leq n$  contractions in the first phase and then chooses a random cut within the remaining vertices of the contracted graph in the second phase that contains the edges of R. Note that any cut-set of the contracted graph is also a cut-set of the original graph. Here, E(G) denotes the set of edges in graph G.

SMALL CUT  $(G = (V, E), R \subseteq E, \alpha \in \mathbb{N})$ 

Initialize

1  $G_0 \leftarrow G$ ,  $E_0 \leftarrow E$ ,  $R_0 \leftarrow R$ .

Contraction

- 2 For  $i = 1, 2, \dots, k = n 2\alpha |R|$
- 3 randomly choose  $e \in E_{i-1} \setminus R_{i-1}$  with probability  $\frac{w(e)}{w(E_{i-1} \setminus R_{i-1})}$
- 4  $G_i \leftarrow G_{i-1}/e$
- 5  $R_i \leftarrow R_{i-1} \cup \{\text{new parallel edges to the edges in } R_{i-1}\}$

Selection

6 Among all possible cut-sets C in the obtained graph  $G_k$  with  $R_k \subseteq C$ , choose one uniformly at random and return it.

Let  $C \subseteq E$  be a cut-set with  $w(C) \leq \alpha r$  and  $R \subseteq C$ . We want to give a lower bound on the probability that SMALL CUT outputs C.

Note that  $G_i$  has  $n_i = n - i$  vertices since each contraction decreases the number of vertices by 1. Since  $R_i$  is the set of edges parallel to those in R, in the case that R is empty, the set  $R_i$  is also empty. Note that if  $R \subseteq C$  and no edge of C has been contracted till iteration i, then  $R_i \subseteq C$ .

Conditioned on the event that no edge in C has been contracted in iterations 1 to i, the probability that an edge from C is contracted in the (i + 1)-th iteration is

$$\frac{w(C\setminus R_i)}{w(E_i\setminus R_i)}.$$

We know that  $w(C \setminus R_i) \leq w(C) \leq \alpha r$ . For a lower bound on  $w(E_i \setminus R_i)$ , consider the graph  $G_i'$  obtained from  $G_i$  by contracting the edges in  $R_i$ . Note that contracting the edges in  $R_i$  actually involves at most |R| contractions. Thus, the number of vertices  $n_i'$  in  $G_i'$  is at least n-i-|R| and its set of edges is  $E_i \setminus R_i$ . For any vertex v in  $G_i'$ , consider the set  $\delta(v)$  of edges incident on v in  $G_i'$ . The set  $\delta(v)$  forms a cut-set in  $G_i'$  and also in G. Note that  $\delta(v) \cap R = \emptyset$ , as the edges in R have been contracted in  $G_i'$ . Thus, we can deduce that  $w(\delta(v)) > r$ , from the lemma hypothesis. By summing this up for all vertices in  $G_i'$ , we obtain

$$w(E_i \setminus R_i) > r n_i'/2.$$

Hence,

$$w(E_i \setminus R_i) > r(n-i-|R|)/2.$$

Therefore the probability that an edge from C is contracted in the (i+1)-th iteration is

$$\leq \frac{w(C \setminus R_i)}{w(E_i \setminus R_i)} < \frac{\alpha r}{r(n-i-|R|)/2} = \frac{2\alpha}{n-i-|R|}.$$

This bound becomes greater than 1 when  $i > n - 2\alpha - |R|$ . This is the reason why we stop the contraction process after  $k = n - 2\alpha - |R|$  iterations.

The probability that no edge from C is contracted in any of the rounds is

$$\geq \prod_{i=0}^{k-1} \left( 1 - \frac{2\alpha}{n-i-|R|} \right)$$

$$= \prod_{i=0}^{k-1} \left( 1 - \frac{2\alpha}{k+2\alpha-i} \right)$$

$$= \prod_{i=0}^{k-1} \frac{k-i}{k+2\alpha-i}$$

$$= \frac{1}{\binom{k+2\alpha}{k}}$$

$$= \frac{1}{\binom{n-|R|}{2\alpha}}.$$

After  $n-2\alpha-|R|$  contractions we are left with  $2\alpha+|R|$  vertices. The number of possible cut-sets on  $2\alpha+|R|$  vertices that contain R is at most  $2^{2\alpha+|R|-1}$ . The selection phase chooses one of these cuts randomly. Thus, the probability that C survives the *contraction* process and is also chosen in the *selection* phase is at least

$$\frac{1}{2^{2\alpha+|R|-1}\binom{n-|R|}{2\alpha}} \ge \frac{1}{2^{|R|}(n-|R|)^{2\alpha}}.$$

Note that in the end we get exactly one cut-set. Thus, the number of cut-sets C of weight  $\leq \alpha r/2$  and  $R \subseteq C$  must be at most  $2^{|R|}(n-|R|)^{2\alpha}$ , which is smaller than the bound desired in the lemma.