# Backward Stackelberg Differential Game with Constraints: a Mixed Terminal-Perturbation and Linear-Quadratic Approach 

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April 6, 2021


#### Abstract

We discuss an open-loop backward Stackelberg differential game involving single leader and single follower. Unlike most Stackelberg game literature, the state to be controlled is characterized by a backward stochastic differential equation (BSDE) for which the terminal- instead initial-condition is specified as a priori; the decisions of leader consist of a static terminal-perturbation and a dynamic linear-quadratic control. In addition, the terminal control is subject to (convex-closed) pointwise and (affine) expectation constraints. Both constraints are arising from real applications such as mathematical finance. For information pattern: the leader announces both terminal and open-loop dynamic decisions at the initial time while takes account the best response of follower. Then, two interrelated optimization problems are sequentially solved by the follower (a backward linear-quadratic (BLQ) problem) and the leader (a mixed terminal-perturbation and backward-forward LQ (BFLQ) problem). Our open-loop Stackelberg equilibrium is represented by some coupled backward-forward stochastic differential equations (BFSDEs) with mixed initial-terminal conditions. Our BFSDEs also involve nonlinear projection operator (due to pointwise constraint) combining with a Karush-Kuhn-Tucker (KKT) system (due to expectation constraint) via Lagrange multiplier. The global solvability of such BFSDEs is also discussed in some nontrivial cases. Our results are applied to one financial example.


Key words: Backward stochastic differential equation, Karush-Kuhn-Tucker (KKT) system, pointwise and affine constraints, Stackelberg game, backward linear-quadratic control, terminal perturbation.

## 1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W=\{W(t), 0 \leq t<\infty\}$ is defined, where $\mathbb{F}=\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ is the natural filtration of $W$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following controlled linear backward stochastic differential equation (BSDE) on a finite time horizon $[0, T]$ :

$$
\begin{equation*}
d X(s)=\left[A(s) X(s)+B_{1}(s) u_{1}(s)+B_{2}(s) u_{2}(s)+C(s) Z(s)\right] d s+Z(s) d W(s), \quad X(T)=\xi \tag{1}
\end{equation*}
$$

where $A(\cdot), B_{1}(\cdot), B_{2}(\cdot), C(\cdot)$ are $\mathbb{F}$-progressively measurable processes defined on $\Omega \times[0, T]$ with proper dimensions. Unlike forward stochastic differential equation (SDE), solution of BSDE (1) consists of a pair of adapted processes $(X(\cdot), Z(\cdot)) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ where the second component $Z(\cdot)$ is necessary to ensure the adaptiveness of $X(\cdot)$ when propagating from terminal- backward to initial-time. In (11), $u_{1}(\cdot)$ and $u_{2}(\cdot)$ are dynamic decision processes employed by Player 1 (the leader, denoted by $\mathcal{A}_{L}$ ) and Player 2 (the follower, denoted by $\mathcal{A}_{F}$ ) in the game with values in $\mathbb{R}^{m_{1}}$ and $\mathbb{R}^{m_{2}}$ respectively. Moreover, unlike SDE , the terminal condition $\xi$ is specified in BSDE (1) by the leader $\mathcal{A}_{L}$ at the initial time, and committed to be steered together with the follower by dynamic $u_{2}(\cdot)$. For some illustrating example, $\xi$ acts as some terminal hedging payoff on $T$, while $u_{1}(\cdot), u_{2}(\cdot)$ represent the possible dynamic portfolio selection or

[^0]consumption process on $[0, T]$. The terminal $\xi$ to be steered may capture some appropriate approximation for quadratic deviation $K\left|X_{T}-\xi\right|^{2}$ with penalty index $K \longrightarrow+\infty$ (see [31]).

Furthermore, let $\mathcal{K}$ be a nonempty closed convex subset in $\mathbb{R}^{n}$. Then, for a deterministic scalar $\beta$ and vector $\alpha \in \mathbb{R}^{n}$, we can define the following two constraints on admissible terminal payoff $\xi$ :

$$
\left\{\begin{array}{c}
\text { Pointwise constraint: } \mathcal{U}_{\mathcal{K}}=L_{\mathcal{F}_{T}}^{2}(\Omega ; \mathcal{K}) ;  \tag{2}\\
\text { Affine expectation constraint: } \mathcal{U}_{\alpha, \beta}=\left\{\xi \mid \xi \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right),\langle\alpha, \mathbb{E} \xi\rangle \geq \beta\right\}
\end{array}\right.
$$

Constraints of such kinds arise naturally in financial applications (e.g., see 44 for expectation constraint, [14, 17, 28 for pointwise one). In particular, the mean-variance portfolio selection with no-shorting yield such constraints both. Now, we define $\mathcal{U}(\mathcal{K}, \alpha, \beta) \triangleq \mathcal{U}_{\mathcal{K}} \bigcap \mathcal{U}_{\alpha, \beta}$ for the admissible terminal control set. Detailed discussion on feasibility of $\mathcal{U}(\mathcal{K}, \alpha, \beta)$ is deferred in Section 4.2 In addition, the following Hilbert spaces are introduced for dynamic admissible controls:
$\mathcal{U}_{i}[0, T] \triangleq\left\{u_{i}:[0, T] \times \Omega \rightarrow \mathbb{R}^{m_{i}} \mid u_{i}(\cdot)\right.$ is $\mathbb{F}$-progressively measurable, $\left.\mathbb{E} \int_{0}^{T}\left|u_{i}(s)\right|^{2} d s<\infty\right\}, \quad i=1,2$.
Any element $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$ is called an admissible control of $\mathcal{A}_{L}$, and any element $u_{2}(\cdot) \in$ $\mathcal{U}_{2}[0, T]$ is called an admissible (dynamic) control of $\mathcal{A}_{F}$. Under some mild conditions on coefficients, for any $\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T] \times \mathcal{U}_{2}[0, T]$, state equation (11) admits a unique square-integrable adapted solution $(X(\cdot), Z(\cdot)) \equiv\left(X\left(\cdot ; \xi, u_{1}(\cdot), u_{2}(\cdot)\right), Z\left(\cdot ; \xi, u_{1}(\cdot), u_{2}(\cdot)\right)\right)$. To evaluate the performance of decisions $\xi, u_{1}(\cdot)$ and $u_{2}(\cdot)$, we introduce the following cost functionals:

$$
\left\{\begin{align*}
J_{1}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right) \triangleq \frac{1}{2} \mathbb{E}\{ & \int_{0}^{T}\left[\left\langle Q_{1}(s) X(s), X(s)\right\rangle+\left\langle S_{1}(s) Z(s), Z(s)\right\rangle+\left\langle R_{11}^{1}(s) u_{1}(s), u_{1}(s)\right\rangle\right] d s  \tag{3}\\
& \left.+\left\langle G_{1} \xi, \xi\right\rangle+\left\langle H_{1} X(0), X(0)\right\rangle\right\} \\
J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right) \triangleq \frac{1}{2} \mathbb{E}\{ & \int_{0}^{T}\left[\left\langle Q_{2}(s) X(s), X(s)\right\rangle+\left\langle S_{2}(s) Z(s), Z(s)\right\rangle+\left\langle R_{22}^{2}(s) u_{2}(s), u_{2}(s)\right\rangle\right] d s \\
& \left.+\left\langle H_{2} X(0), X(0)\right\rangle\right\}
\end{align*}\right.
$$

where $Q_{1}(\cdot), Q_{2}(\cdot), S_{1}(\cdot), S_{2}(\cdot), R_{11}^{1}(\cdot)$, and $R_{22}^{2}(\cdot)$ are all $\mathbb{F}$-progressively measurable symmetric matrix valued processes, defined on $\Omega \times[0, T]$, of proper dimensions, $G_{1}$ is $\mathcal{F}_{T}$-measurable symmetric matrix valued random variable of proper dimension and $H_{1}, H_{2}$ are deterministic symmetric matrices of proper dimensions. For $i=1,2, J_{i}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)$ is the cost functional for agent $i$.

Let us now explain the Stackelberg differential game in some mixed backward linear quadratic (BLQ) and terminal-perturbation pattern.

At initial time, leader $\mathcal{A}_{L}$ announces some terminal (random) target $\xi \in \mathcal{U}(\mathcal{K}, \alpha, \beta)$ (to be reachable at terminal time $T$ ) and his planned dynamic strategy $u_{1}(\cdot) \in \mathcal{U}_{1}[0, T]$ over entire horizon $[0, T] . \xi$ is treated in a hard-constraint case, or in a limiting soft-constraint case (see [2]) when the soft-penalty on quadratic deviation $K\left|X_{T}-\xi\right|^{2}$ is endowed with sufficiently large attenuation level $K>0$. In both cases, the state dynamics becomes (1) (see [31). Actually, $\xi$ may be interpreted as specific requirement of contractual or regulatory nature to reflect some risky position concern at terminal time $T$. Then, given the knowledge of leader's strategy, the follower $\mathcal{A}_{F}$ determines his best response strategy $\bar{u}_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$ over entire horizon to minimize $J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)$. Noticing state $X$ is steered imperatively towards the predetermined random target $\xi$ at maturity $T$. Since the follower's optimal response depends on the leader's strategy, the leader can take it into account as a priori before announcing his committed strategy to minimize $J_{1}\left(\xi, u_{1}(\cdot), \bar{u}_{2}(\cdot)\right)$ over $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$.

A principal-agent framework. The above procedure might fit into some principal-agent problem (see [8]) but in a backward framework: $\mathcal{A}_{L}$ is the principal (owner of given firm) who specifies, at initial contract concluding time, some terminal achievement target $\xi$ to be realized by the agent in contractual manner together with his decision process $u_{1}(\cdot)$. Noticing $u_{1}$ may be interpreted as his committed consumption/capital withdraw process, an outflow on state dynamics $X$ as firm's wealth process. Meanwhile, $\mathcal{A}_{F}$ acts as the agent (manager) who is stimulated to reach such target by utilizing his investment/management/wage process $u_{2}(\cdot)$. When setting contract, $\mathcal{A}_{L}$ may set some constraints on $\xi$ with business concerns, while $\mathcal{A}_{F}$ is pushed to realize the terminal level $\xi$ once contract is executed due to some guarantee or breach clause. Thus, a BSDE state with $\xi$ follows through the contractual force.

Rigorously speaking, $\mathcal{A}_{F}$ aims to find a map $\bar{\alpha}: \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T] \rightarrow \mathcal{U}_{2}[0, T]$ and $\mathcal{A}_{L}$ aims to find a control $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$ such that

$$
\left\{\begin{array}{l}
J_{2}\left(\xi, u_{1}(\cdot), \bar{\alpha}\left[\xi, u_{1}(\cdot)\right](\cdot)\right)=\min _{u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]} J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right), \quad \forall\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T], \\
J_{1}\left(\bar{\xi}, \bar{u}_{1}(\cdot), \bar{\alpha}\left[\bar{\xi}, \bar{u}_{1}\right](\cdot)\right)=\min _{\xi \in \mathcal{U}(\mathcal{K}, \alpha, \beta), u_{1} \in \mathcal{U}_{1}[0, T]} J_{1}\left(\xi, u_{1}(\cdot), \bar{\alpha}\left[\xi, u_{1}(\cdot)\right](\cdot)\right) .
\end{array}\right.
$$

If the above pair $\left(\bar{\xi}, \bar{u}_{1}(\cdot), \bar{\alpha}\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)\right)$ exists, we refer to it as an open-loop Stackelberg equilibrium.
The setup in (1)-(3) above is especially motivated by optimal trading and quadratic hedging problem in financial mathematics when combining with terminal payoff subject to pointwise and integral constraints (see example in Section 6). Accordingly, the main novelties of our contribution are triple: (i) introduction of a new class of backward Stackelberg differential games with (pointwise and expectation affine) constraints and a mixed combination of terminal-perturbation and linear quadratic (LQ) control (both in backward sense); (ii) the characterization of open-loop Stackelberg equilibrium via new class of backward-forward stochastic differential equations (BFSDEs) with Karush-Kuhn-Tucker (KKT) qualification condition; (iii) global solvability for above BFSDEs and some related Riccati equations.

To highlight above novelties, it is helpful to have some literature review comparing to some relevant existing works, especially to BLQ control, (forward) Stackelberg differential games, and various control problems with constraints imposed.

LQ control and game of backward state dynamics. Nonlinear BSDE was initially introduced in 35 and is a well-formulated stochastic system hence it has been found various applications, for example, on stochastic recursive utility in economics by [9. Interested readers may refer [11] for more BSDE applications in financial mathematics. Moreover, the relationship between BSDE and forward LQ optimal control is studied in [25. Based on it, 31] discussed a BLQ optimal control problem motivated by quadratic hedging. [26] studied the BLQ optimal control problem with mean-field type. [19] studied BLQ optimal control with partial information and gave some applications in pension fund optimization problems. Furthermore, some recent literature on games of BSDE can be found in 43, 20,

Stackelberg game. The Stackelberg game (also termed as leader-follower game) was first introduced by [39. It differs from Nash game in its decision hierarchy of involved agents. Stackelberg games have been extensively explored from various settings. We list few works more relevant to ours: for deterministic Stackelberg game, see [2, 32], etc. For stochastic cases, 1] studied LQ Stackelberg differential game, but the state and control variables do not enter the diffusion coefficient. 44 studied a more general Stackelberg game with random coefficients, control enters diffusion terms and control weight may be indefinite. [3] investigated Stackelberg differential game in various different information structures, whereas the diffusion coefficient does not contain the control variables. 34 studied stochastic Stackelberg differential game with time-delayed information. Notice that all above Stackelberg game works are framed in forward sense with underlying state as a forward SDE that differs substantially from our backward one here.

Constrained control and game. Naturally, control or game problems are always subject to possible constraints during its decision making. Such constraints may be posed on underlying state indirectly or decision input directly, or both in some mixed sense. From another viewpoint, these constraints may be structured as soft- or hard-constraint. In soft-constraint, a penalization depending on the deviation from constraints should be implemented in cost functional with some attenuation parameter indicating the softness. Hard-constraint might be viewed as limiting case of soft-constraint with attenuation index tends to infinity. Thus, hard-constraint should be strictly followed in decision process to avoid any cost blow-up. There exist considerable works on constrained stochastic control or games and we name a few more relevant. For example, [17] studied stochastic LQ control constrained in general convex-closed cone, and some extended Riccati method is proposed; 6] extends [17] to infinite time horizon case. [17, 6] are both structured as hard constraint and include no-shorting of mean-variance problem as their special case. Moreover, 27] studied LQ control problems with general input constraint and its applications in financial portfolio selection with no-shorting constraints. Some linear constraints are also treated therein. [30, 29] studied various classes of integral affine and quadratic constraints.

Terminal-perturbation with constraints. There arise various scenarios from mathematical finance with constraints on terminal payoffs that are static, e.g., the Markowitz mean-variance model poses some expectation constraint on terminal return. Thereby, it can convert to a family of indefinite stochastic LQ optimal controls with terminal constraints (46, 27]). 4] first employed backward approach to solve mean-variance problem by Lagrange method and obtained the optimal replicating portfolio strategy by solving some BSDE. To deal with state constraints of dynamic optimization problem, [12] (see
also [37) introduced the backward perturbation method and terminal variable of BSDE is regarded as some "control variable". The terminal-perturbation method is well studied in financial mathematics and stochastic control (see e.g. [21, 22, 23]).

Compared with the above literature reviewed, main contributions of the present paper maybe summarized along the following lines:

- We introduce a new class of backward stochastic Stackelberg differential games featured by a mixed terminal-perturbation and BLQ control pattern. Other technical features include: backwardforward state system, random coefficients and Riccati equations, indefinite control weights.
- Terminal-perturbation is subject to two (pointwise and affine expectation) constraints, some duality approach is invoked to tackle such constraints.
- The open-loop Stackelberg equilibrium is represented by a coupled BFSDEs with mixed initialterminal conditions, projection operator and constraint qualification conditions. To our knowledge, it is the first time to derive such constrained forward-backward systems. Related global wellposedness is also studied in some special but nontrivial cases.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and formulate the Stackelberg game in backward sense. The BLQ problem for follower is studied in Section 3, the mixed terminal-perturbation/backward-forward linear-quadratic (BFLQ) problem for leader is discussed in Section 4. In particular, Stackelberg equilibrium strategy is represented by some coupled BFSDEs with mixed initial-terminal conditions and constrained Karush-Kuhn-Tucker (KKT) system. The global solvability of such BFSDEs is further discussed in Sections 5 in nontrivial cases. As the application, one example is discussed in Section 6.

## 2 Preliminary and BLQ Stackelberg game formulation

The following notations will be used throughout the paper. Let $\mathbb{R}^{n}$ denote the $n$-dimensional Euclidean space with standard Euclidean norm $|\cdot|$ and standard Euclidean inner product $\langle\cdot, \cdot\rangle$. The transpose of a vector (or matrix) $x$ is denoted by $x^{\top} . \operatorname{Tr}(A)$ denotes the trace of a square matrix $A$. Let $\mathbb{R}^{n \times m}$ be the Hilbert space consisting of all $(n \times m)$-matrices with the inner product $\langle A, B\rangle \triangleq \operatorname{Tr}\left(A B^{\top}\right)$ and the norm $\|A\| \triangleq\langle A, A\rangle^{\frac{1}{2}}$. Denote the set of symmetric $n \times n$ matrices with real elements by $\mathbb{S}^{n}$ and $n \times n$ identity matrices by $I_{n}$. If $M \in \mathbb{S}^{n}$ is positive (semi-)definite, we write $M>(\geq) 0$. If there exists a constant $\delta>0$ such that $M \geq \delta I$, we write $M \gg 0$. Let $\mathbb{S}_{+}^{n}$ be the space of all positive semi-definite matrices of $\mathbb{S}^{n}$ and $\hat{\mathbb{S}}_{+}^{n}$ be the space of all positive definite matrices of $\mathbb{S}^{n}$.

Consider a finite time horizon $[0, T]$ for a fixed $T>0$. Let $H$ be a given Hilbert space. The set of $H$-valued continuous functions is denoted by $C([0, T] ; H)$. If $N(\cdot) \in C\left([0, T] ; \mathbb{S}^{n}\right)$ and $N(t)>(\geq) 0$ for every $t \in[0, T]$, we say that $N(\cdot)$ is positive (semi-)definite, which is denoted by $N(\cdot)>(\geq) 0$. For any $t \in[0, T)$ and Euclidean space $\mathbb{H}$, let(for the deterministic process, the subscripts $\mathcal{F}_{t}$ or $\mathbb{F}$ will be omitted)
$L_{\mathcal{F}_{t}}^{2}(\Omega ; \mathbb{H})=\left\{\xi: \Omega \rightarrow \mathbb{H} \mid \xi\right.$ is $\mathcal{F}_{t}$-measurable, $\left.\mathbb{E}|\xi|^{2}<\infty\right\}$,
$L_{\mathcal{F}_{t}}^{\infty}(\Omega ; \mathbb{H})=\left\{\xi: \Omega \rightarrow \mathbb{H} \mid \xi\right.$ is $\mathcal{F}_{t}$-measurable, $\left.\operatorname{esssup}_{\omega \in \Omega}|\xi(\omega)|<\infty\right\}$,
$L_{\mathbb{F}}^{2}(0, T ; \mathbb{H})=\left\{\phi:[0, T] \times \Omega \rightarrow \mathbb{H} \mid \phi\right.$ is $\mathbb{F}$-progressively measurable, $\left.\mathbb{E} \int_{0}^{T}|\phi(s)|^{2} d s<\infty\right\}$,
$L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{H})=\left\{\phi:[0, T] \times \Omega \rightarrow \mathbb{H} \mid \phi\right.$ is $\mathbb{F}$-progressively measurable, $\left.\operatorname{esssup}_{s \in[0, T]} \operatorname{esssup}_{\omega \in \Omega}|\phi(s)|<\infty\right\}$,
$L_{\mathbb{F}}^{2}(\Omega ; C([0, T] ; \mathbb{H}))=\left\{\phi:[0, T] \times \Omega \rightarrow \mathbb{H} \mid \phi\right.$ is $\mathbb{F}$-adapted, continuous, $\left.\mathbb{E}\left[\sup _{s \in[0, T]}|\phi(s)|^{2}\right]<\infty\right\}$.
Recall the sets $\mathcal{U}_{i}[0, T]=L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m_{i}}\right)$. For notational simplicity, let $m=m_{1}+m_{2}$ and denote

$$
B(\cdot)=\left(B_{1}(\cdot), B_{2}(\cdot)\right), \quad R_{1}(\cdot)=\left(\begin{array}{cc}
R_{11}^{1}(\cdot) & 0 \\
0 & 0
\end{array}\right), \quad R_{2}(\cdot)=\left(\begin{array}{cc}
0 & 0 \\
0 & R_{22}^{2}(\cdot)
\end{array}\right) .
$$

Naturally, we identify $u(\cdot)=\left(u_{1}(\cdot)^{\top}, u_{2}(\cdot)^{\top}\right)^{\top} \in \mathcal{U}[0, T]=\mathcal{U}_{1}[0, T] \times \mathcal{U}_{2}[0, T]$. With such notations, the state equation (1) becomes

$$
\begin{equation*}
d X(s)=[A(s) X(s)+B(s) u(s)+C(s) Z(s)] d s+Z(s) d W(s), \quad X(T)=\xi \tag{4}
\end{equation*}
$$

where the terminal condition $\xi$ is a control variable with the constraints (2). The cost functionals become

$$
\left\{\begin{array}{l}
\quad J_{1}(\xi, u(\cdot))=\frac{1}{2} \mathbb{E}\left\{\int_{0}^{T}\left[\left\langle Q_{1}(s) X(s), X(s)\right\rangle+\left\langle S_{1}(s) Z(s), Z(s)\right\rangle+\left\langle R_{1}(s) u(s), u(s)\right\rangle\right] d s\right. \\
\left.\quad+\left\langle G_{1} \xi, \xi\right\rangle+\left\langle H_{1} X(0), X(0)\right\rangle\right\}, \\
J_{2}(\xi, u(\cdot))=\frac{1}{2} \mathbb{E}\left\{\int_{0}^{T}\left[\left\langle Q_{2}(s) X(s), X(s)\right\rangle+\left\langle S_{2}(s) Z(s), Z(s)\right\rangle+\left\langle R_{2}(s) u(s), u(s)\right\rangle\right] d s+\left\langle H_{2} X(0), X(0)\right\rangle\right\} .
\end{array}\right.
$$

Let us introduce the following assumptions, which will be used later.
(H1) The coefficients of the state equation satisfy the following:

$$
A(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{n \times n}\right), \quad B(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{n \times m}\right), \quad C(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{n \times n}\right)
$$

(H2) The weighting coefficients of cost functional satisfy the following:

$$
G_{1} \in L_{\mathcal{F}_{T}}^{\infty}\left(\Omega ; \mathbb{S}^{n}\right), H_{1}, H_{2} \in \mathbb{S}^{n}, Q_{1}(\cdot), Q_{2}(\cdot), S_{1}(\cdot), S_{2}(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}^{n}\right), R_{1}(\cdot), R_{2}(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}^{m}\right)
$$

Under (H1), by 35, Theorem 3.1], for any $\xi \in \mathcal{U}(\mathcal{K}, \alpha, \beta)$ and $u(\cdot) \in \mathcal{U}[0, T]$, (41) admits a unique strong solution $(X(\cdot), Z(\cdot)) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$. Moreover, the following estimation holds:

$$
\begin{equation*}
\mathbb{E}\left[\sup _{s \in[0, T]}|X(s)|^{2}+\int_{0}^{T}|Z(s)|^{2} d s\right] \leq L \mathbb{E}\left[|\xi|^{2}+\int_{0}^{T}|u(s)|^{2} d s\right] \tag{5}
\end{equation*}
$$

where $L>0$ is a constant which depends on the coefficients of (44). Therefore, under (H1)-(H2), the functionals $J_{i}(\xi, u(\cdot))=J_{i}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)$ are well-defined for all $\xi \in \mathcal{U}(\mathcal{K}, \alpha, \beta)$ and $u_{i}(\cdot) \in \mathcal{U}_{i}[0, T]$, $i=1,2$. If the coefficients in (4) are deterministic, by (41, Proposition 2.1], (4) admits a unique strong solution under the following relaxed assumption:
$\left(\mathbf{H 1}^{\prime}\right)$ The coefficients of the state equation satisfy the following:

$$
A(\cdot) \in L^{1}\left(0, T ; \mathbb{R}^{n \times n}\right), \quad B(\cdot) \in L^{\infty}\left(0, T ; \mathbb{R}^{n \times m}\right), \quad C(\cdot) \in L^{2}\left(0, T ; \mathbb{R}^{n \times n}\right)
$$

Moreover, (5) still holds. Hereafter, time variable $s$ will often be suppressed to simplify notations. We briefly state the procedure of finding an open-loop Stackelberg equilibrium: first, for any given $\left(\xi, u_{1}(\cdot)\right)$, $\mathcal{A}_{F}$ should solve a BLQ control problem with $\bar{\alpha}\left(\xi, u_{1}(\cdot)\right)$ as the best response; second, given best response, $\mathcal{A}_{L}$ then solves a BFLQ control and terminal-perturbation with optimal $\bar{\xi}$ and $\bar{u}_{1}(\cdot)$. The Stackelberg equilibrium follows by $\left(\bar{\xi}, \bar{u}_{1}(\cdot), \bar{\alpha}\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)\right)$.

## 3 Backward LQ problem for $\mathcal{A}_{F}$

For given $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$, the follower $\mathcal{A}_{F}$ should solve the following BLQ Problem:
(BLQ): Minimize $\quad J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right) \quad$ subject to (4), $\quad u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$.

Definition 3.1 (a) For given $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$, problem (BLQ) is said to be finite if cost functional $J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)$ is bounded from below, that is, $\inf _{u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]} J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)>-\infty$;
(b) Problem (BLQ) is said to be (uniquely) solvable if there exists a (unique) $u_{2}^{*}(\cdot) \in \mathcal{U}_{2}[0, T]$ such that $J_{2}\left(\xi, u_{1}(\cdot), u_{2}^{*}(\cdot)\right)=\inf _{u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]} J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)$. In this case, $u_{2}^{*}(\cdot)$ is called minimizer of (BLQ).

We now give a representation of cost functional for (BLQ) which helps us to study its solvability. Its proof is straightforward based on duality theory thus we omit details here.

Proposition 3.1 Let (H1)-(H2) hold. There exist two bounded self-adjoint linear operators $M_{2}$ : $\mathcal{U}_{2}[0, T] \rightarrow \mathcal{U}_{2}[0, T], M_{1}: L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times \mathcal{U}_{1}[0, T] \rightarrow \mathcal{U}_{2}[0, T]$ and some $M_{0} \in \mathbb{R}$ depending on $\left(\xi, u_{1}(\cdot)\right)$ such that

$$
J_{2}\left(\xi, u_{1}(\cdot), u_{2}(\cdot)\right)=\frac{1}{2}\left[\mathbb{E}\left\langle M_{2}\left(u_{2}\right)(\cdot), u_{2}(\cdot)\right\rangle+2 \mathbb{E}\left\langle M_{1}\left(\xi, u_{1}\right)(\cdot), u_{2}(\cdot)\right\rangle+M_{0}\right]
$$

with

$$
\begin{align*}
& M_{2}\left(u_{2}\right)(\cdot)=R_{22}^{2}(\cdot) u_{2}(\cdot)-B_{2}^{\top}(\cdot) Y_{1}(\cdot), \quad M_{1}\left(\xi, u_{1}\right)(\cdot)=-B_{2}^{\top}(\cdot) Y_{2}(\cdot)-B_{2}^{\top}(\cdot) Y_{3}(\cdot), \\
& M_{0}=-\mathbb{E} \int_{0}^{T}\left\langle B_{1}^{\top}(s) Y_{3}(s), u_{1}(s)\right\rangle d s+\mathbb{E}\left\langle Y_{2}(T), \xi\right\rangle+2 \mathbb{E}\left\langle Y_{3}(T), \xi\right\rangle \tag{6}
\end{align*}
$$

where $Y_{1}, Y_{2}, Y_{3}$ satisfy the following backward-forward systems:

$$
\begin{align*}
& \left\{\begin{array}{l}
d Y_{1}(s)=\left[-A^{\top}(s) Y_{1}(s)+Q_{2}(s) X_{1}(s)\right] d s+\left[-C^{\top}(s) Y_{1}(s)+S_{2}(s) Z_{1}(s)\right] d W(s), \\
d X_{1}(s)=\left[A(s) X_{1}(s)+B_{2}(s) u_{2}(s)+C(s) Z_{1}(s)\right] d s+Z_{1}(s) d W(s), \\
X_{1}(T)=0, \quad Y_{1}(0)=H_{2} X_{1}(0),
\end{array}\right.  \tag{7}\\
& \left\{\begin{array}{l}
d Y_{2}(s)=\left[-A^{\top}(s) Y_{2}(s)+Q_{2}(s) X_{2}(s)\right] d s+\left[-C^{\top}(s) Y_{2}(s)+S_{2}(s) Z_{2}(s)\right] d W(s), \\
d X_{2}(s)=\left[A(s) X_{2}(s)+C(s) Z_{2}(s)\right] d s+Z_{2}(s) d W(s), \\
X_{2}(T)=\xi, \quad Y_{2}(0)=H_{2} X_{2}(0),
\end{array}\right. \\
& \left\{\begin{array}{l}
d Y_{3}(s)=\left[-A^{\top}(s) Y_{3}(s)+Q_{2}(s) X_{3}(s)\right] d s+\left[-C^{\top}(s) Y_{3}(s)+S_{2}(s) Z_{3}(s)\right] d W(s), \\
d X_{3}(s)=\left[A(s) X_{3}(s)+B_{1}(s) u_{1}(s)+C(s) Z_{3}(s)\right] d s+Z_{3}(s) d W(s), \\
X_{3}(T)=0, \quad Y_{3}(0)=H_{2} X_{3}(0) .
\end{array}\right.
\end{align*}
$$

In the above, we use $\langle\cdot, \cdot\rangle$ to denote inner products in different Hilbert spaces, which can be identified from the context. Based on Proposition 3.1, we have the following result for the solvability of problem (BLQ), whose proof is similar to that of [45, Theorem 6.2.2].

Proposition 3.2 Let (H1)-(H2) hold.
(a) Problem (BLQ) is finite only if (BLQ) is convex (i.e., $M_{2} \geq 0$ );
(b) Problem (BLQ) is (uniquely) solvable if and only if (iff) ( $\mathbf{B L Q}$ ) is convex $\left(M_{2} \geq 0\right)$ and the following stationary condition holds true: there exists a (unique) $\bar{u}_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$ such that

$$
\begin{equation*}
M_{2}\left(\bar{u}_{2}\right)(\cdot)+M_{1}\left(\xi, u_{1}\right)(\cdot)=0 \tag{8}
\end{equation*}
$$

Moreover, (8) implies that $\mathcal{R}\left(M_{1}\left(\xi, u_{1}\right)\right) \subset \mathcal{R}\left(M_{2}\left(\bar{u}_{2}\right)\right)$, where $\mathcal{R}(\mathcal{S})$ stands for the range of operator (matrix) $\mathcal{S}$.
(c) If (BLQ) is uniformly convex (i.e., $M_{2} \gg 0$ ), then problem ( $\mathbf{B L Q}$ ) admits a unique optimal control given by

$$
\bar{u}_{2}(\cdot)=-M_{2}^{-1}\left(M_{1}\left(\xi, u_{1}\right)\right)(\cdot) .
$$

(a)-(c) in Proposition 3.2 can be summarized by the following inclusion relation diagram:
uniform convexity $\Longrightarrow$ unique solvability $\Longrightarrow$ solvability $(\Longleftrightarrow$ convexity, stationary condition)
$\Longrightarrow$ finiteness $\Longrightarrow$ convexity.
Given representation (6), (18) takes the following form:

$$
R_{22}^{2}(\cdot) \bar{u}_{2}(\cdot)-B_{2}^{\top}(\cdot) Y_{1}(\cdot)-B_{2}^{\top}(\cdot) Y_{2}(\cdot)-B_{2}^{\top}(\cdot) Y_{3}(\cdot)=0
$$

Therefore, if we define $\bar{Y}=Y_{1}+Y_{2}+Y_{3}, \bar{X}=X_{1}+X_{2}+X_{3}, \bar{Z}=Z_{1}+Z_{2}+Z_{3}$, we have the following solvability result in terms of BFSDEs.

Theorem 3.1 Under (H1)-(H2), for any $u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$, suppose that

$$
\begin{equation*}
\mathbb{E}\left\langle M_{2}\left(u_{2}\right)(\cdot), u_{2}(\cdot)\right\rangle=\mathbb{E} \int_{0}^{T}\left\langle R_{22}^{2}(s) u_{2}(s)-B_{2}^{\top}(s) Y_{1}(s), u_{2}(s)\right\rangle d s \geq 0 \tag{9}
\end{equation*}
$$

where $\left(Y_{1}, X_{1}, Z_{1}\right)$ is the solution of (77) with respect to $u_{2}(\cdot)$. Then problem (BLQ) is (uniquely) solvable with an (the) optimal pair $\left(\bar{X}(\cdot), \bar{Z}(\cdot), \bar{u}_{2}(\cdot)\right)$ iff there (uniquely) exists a 4-tuple $\left(\bar{Y}(\cdot), \bar{X}(\cdot), \bar{Z}(\cdot), \bar{u}_{2}(\cdot)\right)$ satisfying BFSDEs

$$
\left\{\begin{array}{l}
d \bar{Y}(s)=\left[-A^{\top}(s) \bar{Y}(s)+Q_{2}(s) \bar{X}(s)\right] d s+\left[-C^{\top}(s) \bar{Y}(s)+S_{2}(s) \bar{Z}(s)\right] d W(s)  \tag{10}\\
d \bar{X}(s)=\left[A(s) \bar{X}(s)+B_{1}(s) u_{1}(s)+B_{2}(s) \bar{u}_{2}(s)+C(s) \bar{Z}(s)\right] d s+\bar{Z}(s) d W(s) \\
\bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\xi
\end{array}\right.
$$

such that

$$
\begin{equation*}
R_{22}^{2}(s) \bar{u}_{2}(s)-B_{2}^{\top}(s) \bar{Y}(s)=0, \quad s \in[0, T], \quad \mathbb{P}-\text { a.s } \tag{11}
\end{equation*}
$$

Let us give the following inverse assumption.
(H3) $R_{22}^{2}(\cdot)$ is invertible and $\left(R_{22}^{2}(\cdot)\right)^{-1} \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{m_{2}}\right)$.
Clearly, under (H3), optimal control $\bar{u}_{2}(\cdot)$ can be further represented as

$$
\begin{equation*}
\bar{u}_{2}(s)=\left(R_{22}^{2}(s)\right)^{-1} B_{2}(s)^{\top} \bar{Y}(s), \tag{12}
\end{equation*}
$$

and (10)-(11) are equivalent to the following BFSDEs:

$$
\left\{\begin{array}{l}
d \bar{Y}(s)=\left[-A^{\top}(s) \bar{Y}(s)+Q_{2}(s) \bar{X}(s)\right] d s+\left[-C^{\top}(s) \bar{Y}(s)+S_{2}(s) \bar{Z}(s)\right] d W(s)  \tag{13}\\
d \bar{X}(s)=\left[A(s) \bar{X}(s)+B_{1}(s) u_{1}(s)+B_{2}(s)\left(R_{22}^{2}(s)\right)^{-1} B_{2}^{\top}(s) \bar{Y}(s)+C(s) \bar{Z}(s)\right] d s+\bar{Z}(s) d W(s), \\
\bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\xi
\end{array}\right.
$$

BFSDEs (13) differs from classical forward-backward stochastic differential equations (FBSDEs) because forward state $\bar{Y}(\cdot)$ depends on backward state $\bar{X}(\cdot)$ via initial $\bar{X}(0)$ instead terminal $\bar{X}(T)$. Unlike Yong [44, the state (13) is not decoupled thus its global solvability is not straightforward. In Section 5.3, we will establish global solvability under some suitable conditions on the coefficients. Moreover, regarding the relation between (13) and (BLQ), we have the following statement:

Corollary 3.1 Under (H1)-(H3), let (19) hold. Then Problem (BLQ) is (pathwise uniquely) solvable iff BFSDEs (13) admits a (unique) strong solution $(\bar{Y}(\cdot), \bar{X}(\cdot), \bar{Z}(\cdot)) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times$ $L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$.

If uniformly convexity holds, i.e., there exists a constant $\gamma>0$ such that for any $u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$,

$$
\begin{equation*}
\mathbb{E}\left\langle M_{2}\left(u_{2}\right)(\cdot), u_{2}(\cdot)\right\rangle=\mathbb{E} \int_{0}^{T}\left\langle R_{22}^{2}(s) u_{2}(s)-B_{2}^{\top}(s) Y_{1}(s), u_{2}(s)\right\rangle d s \geq \gamma \mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s \tag{14}
\end{equation*}
$$

then (BLQ) is uniquely solvable. Therefore, it follows from Corollary 3.1 that BFSDEs (13) admits a unique strong solution $(\bar{Y}(\cdot), \bar{X}(\cdot), \bar{Z}(\cdot))$. Next, we will study the uniformly convex condition (14) of (BLQ). First, introduce the following auxiliary BLQ problem (ABLQ):
$\left\{\begin{array}{l}\text { Minimize } \mathcal{J}\left(u_{2}(\cdot)\right)=\mathbb{E}\left\{\int_{0}^{T}\left[\left\langle Q_{2} x(s), x(s)\right\rangle+\left\langle S_{2} z(s), z(s)\right\rangle+\left\langle R_{22}^{2} u_{2}(s), u_{2}(s)\right\rangle\right] d s+\left\langle H_{2} x(0), x(0)\right\rangle\right\}, \\ \text { subject to } d x(s)=\left[A(s) x(s)+B_{2}(s) u_{2}(s)+C(s) z(s)\right] d s+z(s) d W(s), \quad x(T)=0, \quad s \in[0, T] .\end{array}\right.$
Note that for (ABLQ), its functional $\mathcal{J}\left(u_{2}(\cdot)\right)=\mathbb{E}\left\langle M_{2}\left(u_{2}\right)(\cdot), u_{2}(\cdot)\right\rangle$, which is the left hand side of (9). Therefore, convexity condition (19) holds iff (ABLQ) is well-posed with a necessarily nonnegative minimal cost. Moreover, if there exists a constant $\gamma>0$ such that $\mathcal{J}\left(u_{2}(\cdot)\right)>\gamma \mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s$ for any
$u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$, the uniformly convexity condition (14) holds. Now we introduce the following standard assumptions

$$
\text { (SA-1): } \quad H_{2} \geq 0, \quad Q_{2}(\cdot) \geq 0, \quad S_{2}(\cdot) \geq 0, \quad R_{22}^{2}(\cdot) \gg 0
$$

For any given nonsingular symmetric matrix $M$, we introduce the following Riccati equation (denoted by (SRE-1)):

$$
\left\{\begin{array}{l}
d P=-\left[Q_{2}+P A+A^{\top} P-P B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P-(P C+K)\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right)\right] d s+K d W(s) \\
P(T)=M \\
P(s)+S_{2}(s)>0, \quad 0 \leq s \leq T
\end{array}\right.
$$

Proposition 3.3 Under (H1)-(H3), if $R_{22}^{2}(\cdot)>0$ and Riccati equation (SRE-1) has a solution $(P(\cdot), K(\cdot)) \in$ $L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}^{n}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{S}^{n}\right)$ such that $P(0)+H_{2} \geq 0$. Then for any $u_{2}(\cdot) \in \mathcal{U}_{2}[0, T]$,

$$
\mathcal{J}\left(u_{2}(\cdot)\right) \geq 0
$$

and in this case, ( $\mathbf{B L Q}$ ) is convex on $u_{2}(\cdot)$. Moreover, if there exists a constant $\delta>0$ and $R_{22}^{2}(\cdot) \geq \delta I$, then there exists a constant $\gamma>0$ such that

$$
\mathcal{J}\left(u_{2}(\cdot)\right) \geq \delta \gamma \mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s
$$

and in this case, ( $\mathbf{B L Q}$ ) is uniformly convex on $u_{2}(\cdot)$. In particular, under ( $\mathbf{S A} \mathbf{- 1}$ ), ( $\mathbf{B L Q}$ ) is uniformly convex on $u_{2}(\cdot)$.

The proof of Proposition 3.3 is given in the Appendix, Section 7.1 ,

## 4 Terminal-perturbation and BFLQ problem of $\mathcal{A}_{L}$

Considering (12), the corresponding state process for $\mathcal{A}_{L}$ becomes the following BFSDEs:

$$
\left\{\begin{array}{l}
d Y(s)=\left[-A^{\top}(s) Y(s)+Q_{2}(s) X(s)\right] d s+\left[-C^{\top}(s) Y(s)+S_{2}(s) Z(s)\right] d W(s)  \tag{15}\\
d X(s)=\left[A(s) X(s)+B_{1}(s) u_{1}(s)+B_{2}(s)\left(R_{22}^{2}(s)\right)^{-1} B_{2}^{\top}(s) Y(s)+C(s) Z(s)\right] d s+Z(s) d W(s) \\
Y(0)=H_{2} X(0), \quad X(T)=\xi
\end{array}\right.
$$

which is controlled by $\xi$ (terminal-perturbation) and $u_{1}(\cdot)$ with the following cost functional

$$
\begin{aligned}
& J_{1}\left(\xi, u_{1}(\cdot)\right)=\frac{1}{2} \mathbb{E}\left\{\int_{0}^{T}\left[\left\langle Q_{1}(s) X(s), X(s)\right\rangle+\left\langle S_{1}(s) Z(s), Z(s)\right\rangle+\left\langle R_{11}^{1}(s) u_{1}(s), u_{1}(s)\right\rangle\right] d s\right. \\
& \left.+\left\langle G_{1} \xi, \xi\right\rangle+\left\langle H_{1} X(0), X(0)\right\rangle\right\}
\end{aligned}
$$

The existence and uniqueness of BFSDEs (15) is established in Corollary 3.1. Now, $\mathcal{A}_{L}$ should solve the following mixed terminal-perturbation and BFLQ problem for above system:

$$
(\mathbf{P}): \text { Minimize } \quad J_{1}\left(\xi, u_{1}(\cdot)\right) \quad \text { subject to } \quad \text { (15) },\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T] .
$$

We denote above problem as (P) for primal problem, to be compared with the dual problem that will be introduced later. Now, it is necessary to set some definitions pertinent to its solvability.

Definition 4.1 (a) Problem ( $\mathbf{P}$ ) is said to be finite if cost functional $J_{1}$ is bounded from below, that is, $\mu_{p} \triangleq \inf _{\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]} J_{1}\left(\xi, u_{1}(\cdot)\right)>-\infty . \mu_{p}$ is called the value of (primal) problem ( $\mathbf{P}$ );
(b) Problem ( $\mathbf{P}$ ) is said to be (uniquely) solvable if there exists a (unique) $\left(\xi^{*}, u_{1}^{*}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times$ $\mathcal{U}_{1}[0, T]$ such that $\mu_{p}=J_{1}\left(\xi^{*}, u_{1}^{*}(\cdot)\right)$. In this case, $\left(\xi^{*}, u_{1}^{*}(\cdot)\right)$ is called minimizer of problem ( $\mathbf{P}$ ).
For solvability, a related definition is the convexity. Considering $\mathcal{U}(\mathcal{K}, \alpha, \beta)$ is closed-convex, we formulate the following trivial definition.
Definition 4.2 Problem ( $\mathbf{P}$ ) is said to be convex if its cost functional $J_{1}$ is convex on $\left(\xi, u_{1}(\cdot)\right)$. Its strictly- and uniformly-convexity can be defined similarly.

### 4.1 Convexity and solvability of primal problem

For primal problem (P), the following representation of $J_{1}$ may help to characterize its solvability and convexity in a direct manner.
Proposition 4.1 Let (H1)-(H3) hold. There exist two bounded self-adjoint linear operators $\mathcal{M}_{2}$ : $\mathcal{U}_{1}[0, T] \rightarrow \mathcal{U}_{1}[0, T], \mathcal{M}_{1}: L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \rightarrow L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ and a bounded linear operator $\mathcal{M}_{0}: L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \rightarrow$ $\mathcal{U}_{1}[0, T]$ such that

$$
\begin{equation*}
J_{1}\left(\xi, u_{1}(\cdot)\right)=\frac{1}{2} \mathbb{E}\left[\left\langle\mathcal{M}_{2}\left(u_{1}\right)(\cdot), u_{1}(\cdot)\right\rangle+\left\langle\mathcal{M}_{1}(\xi), \xi\right\rangle+2\left\langle\mathcal{M}_{0}(\xi)(\cdot), u_{1}(\cdot)\right\rangle\right] \tag{16}
\end{equation*}
$$

with

$$
\mathcal{M}_{2}\left(u_{1}\right)(\cdot)=R_{11}^{1}(\cdot) u_{1}(\cdot)-B_{1}^{\top}(\cdot) g_{1}\left(u_{1}\right)(\cdot), \mathcal{M}_{1}(\xi)=G_{1} \xi+g_{2}(T), \mathcal{M}_{0}(\xi)(\cdot)=-B_{1}^{\top}(\cdot) g_{2}(\cdot),
$$

where $g_{1}, g_{2}$ depending on $u_{1}$ and $\xi$ respectively, are defined through the following BFSDEs

$$
\begin{align*}
& \left\{\begin{array}{l}
d Y_{1}=\left[-A^{\top} Y_{1}+Q_{2} X_{1}\right] d s+\left[-C^{\top} Y_{1}+S_{2} Z_{1}\right] d W(s), \\
d X_{1}=\left[A X_{1}+B_{1} u_{1}+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} Y_{1}+C Z_{1}\right] d s+Z_{1} d W(s), \\
d g_{1}=-\left[A^{\top} g_{1}-Q_{2} h_{1}-Q_{1} X_{1}\right] d s-\left[C^{\top} g_{1}-S_{2} q_{1}-S_{1} Z_{1}\right] d W(s), \\
d h_{1}=\left[A h_{1}+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g_{1}+C q_{1}\right] d s+q_{1} d W(s), \\
Y_{1}(0)=H_{2} X_{1}(0), \quad X_{1}(T)=0, \quad g_{1}(0)=H_{1} X_{1}(0)+H_{2} h_{1}(0), \quad h_{1}(T)=0,
\end{array}\right.  \tag{17}\\
& \left\{\begin{array}{l}
d Y_{2}=\left[-A^{\top} Y_{2}+Q_{2} X_{2}\right] d s+\left[-C^{\top} Y_{2}+S_{2} Z_{2}\right] d W(s), \\
d X_{2}=\left[A X_{2}+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} Y_{2}+C Z_{2}\right] d s+Z_{2} d W(s), \\
d g_{2}=-\left[A^{\top} g_{2}-Q_{2} h_{2}-Q_{1} X_{2}\right] d s-\left[C^{\top} g_{2}-S_{2} q_{2}-S_{1} Z_{2}\right] d W(s), \\
d h_{2}=\left[A h_{2}+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g_{2}+C q_{2}\right] d s+q_{2} d W(s), \\
Y_{2}(0)=H_{2} X_{2}(0), \quad X_{2}(T)=\xi, \quad g_{2}(0)=H_{1} X_{2}(0)+H_{2} h_{2}(0), \quad h_{2}(T)=0 .
\end{array}\right. \tag{18}
\end{align*}
$$

The proof of Proposition 4.1 follows from duality of BFSDEs and readers may refer 45 for similar representation. It follows from (16) that $J_{1}$ is quadratic functional on $\left(\xi, u_{1}(\cdot)\right)$ and we have the following result concerning its convexity on constrained admissible set $\mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$.
Proposition 4.2 Let (H1)-(H3) hold. Then (P) is convex iff
block operator: $\mathcal{M} \triangleq\left[\begin{array}{c|c}\mathcal{M}_{1} & \mathcal{M}_{0}^{*} \\ \hline \mathcal{M}_{0} & \mathcal{M}_{2}\end{array}\right] \geq 0 \Longleftrightarrow J_{1}(\zeta, v(\cdot)) \geq 0, \quad \forall(\zeta, v(\cdot)) \in \mathcal{U}_{\tilde{\mathcal{K}}} \times \mathcal{U}_{1}[0, T]$,
where $\mathcal{M}_{0}^{*}\left(u_{1}\right)=g_{1}(T): \mathcal{U}_{1}(0, T) \longmapsto L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ is the adjoint operator of $\mathcal{M}_{0}(\xi)$ and $\widetilde{\mathcal{K}} \triangleq \mathcal{K}-\mathcal{K}=$ $\{x-y: x \in \mathcal{K}, y \in \mathcal{K}\}$ is the algebra difference of $\mathcal{K}$ (it is also convex but not necessary to be closed unless $\mathcal{K}$ is compact). Moreover, ( $\mathbf{P}$ ) is uniformly convex iff for some $\delta>0$,

$$
\begin{equation*}
J_{1}(\zeta, v(\cdot)) \geq \delta\left[\mathbb{E}|\zeta|^{2}+\mathbb{E} \int_{0}^{T}|v(s)|^{2} d s\right], \quad \forall(\zeta, v(\cdot)) \in \mathcal{U}_{\tilde{\mathcal{K}}} \times \mathcal{U}_{1}[0, T] . \tag{20}
\end{equation*}
$$

Proof For $\forall\left(\xi, u_{1}\right),\left(\xi^{\prime}, u_{1}^{\prime}\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$, denote $\xi^{\lambda}=\lambda \xi+(1-\lambda) \xi^{\prime}, u_{1}^{\lambda}=\lambda u_{1}+(1-\lambda) u_{1}^{\prime}$ for $\lambda \in[0,1]$, then $\zeta=\xi-\xi^{\prime} \in \mathcal{U}_{\tilde{\mathcal{K}}}, v=u_{1}-u_{1}^{\prime} \in \mathcal{U}_{1}[0, T]$. Then, by (16), $J_{1}$ should be convex iff

$$
\begin{aligned}
0 & \geq J_{1}\left(\xi^{\lambda}, u_{1}^{\lambda}\right)-\lambda J_{1}\left(\xi, u_{1}\right)-(1-\lambda) J_{1}\left(\xi^{\prime}, u_{1}^{\prime}\right), \\
& =\frac{1}{2} \lambda(\lambda-1)\left[\left\langle\mathcal{M}_{2}(v)(\cdot), v(\cdot)\right\rangle+\left\langle\mathcal{M}_{1}(\zeta),(\zeta)\right\rangle+2\left\langle\mathcal{M}_{0}(\zeta)(\cdot), v(\cdot)\right\rangle\right] \\
& =\lambda(\lambda-1) J_{1}(\zeta, v(\cdot)) .
\end{aligned}
$$

Hence the result (19) follows. Similar arguments apply to uniformly convexity leading to (20).

Remark 4.1 Similar to Schur lemma, we have $J_{1}(\cdot, \cdot)$ is strictly convex iff

$$
\mathcal{M}>0 \Longleftrightarrow \mathcal{M}_{2}>0, \quad \mathcal{M}_{1}-\mathcal{M}_{0}^{*} \mathcal{M}_{2}^{-1} \mathcal{M}_{0}>0 \Longleftrightarrow \mathcal{M}_{1}>0, \quad \mathcal{M}_{2}-\mathcal{M}_{0} \mathcal{M}_{1}^{-1} \mathcal{M}_{0}^{*}>0
$$

It follows that the convexity on $\left(\xi, u_{1}(\cdot)\right)$ jointly is stronger than convexity on $\xi$ and $u_{1}(\cdot)$ marginally. As a consequence, we have the following result when $\mathcal{K}$ is further conic:
Corollary 4.1 Let (H1)-(H3) hold and $\mathcal{K}$ is closed-convex cone. Then, ( $\mathbf{(})$ is convex iff

$$
J_{1}\left(\zeta, v_{1}(\cdot)\right) \geq 0, \quad \forall\left(\zeta, v_{1}(\cdot)\right) \in \mathcal{U}_{\text {aff }(\mathcal{K})} \times \mathcal{U}_{1}[0, T]
$$

where aff $(\mathcal{K})=\mathcal{K}-\mathcal{K}$ is the affine subspace generated by $\mathcal{K}$.
Noticing a closed cone always contains 0 thus $\widetilde{\mathcal{K}}=\operatorname{aff}(\mathcal{K})$ that may be a proper subset of full space $\mathbb{R}^{n}$.
In standard LQ control literature, when the admissible controls are from full linear space, then finiteness of problem implies its convexity. Alternatively, when admissible controls are only from some closedconvex proper subset, we have the following different results.

Lemma 4.1 Suppose $\mathbf{( H 1 ) - ( \mathbf { H } 3 ) ~ h o l d ~ a n d ~} \mathcal{K}$ is a closed-convex set containing origin 0 . Then, problem $(\mathbf{P})$ is finite only if $J_{1}$ is nonnegative functional on $U_{C_{\infty}(\mathcal{K})} \times \mathcal{U}_{1}[0, T]$ where $C_{\infty}(\mathcal{K})$ is the asymptotic (recession) cone of $\mathcal{K}$.

Proof First, recall that $C_{\infty}(\mathcal{K}) \subseteq \mathcal{K}$ if origin $0 \in \mathcal{K}$ hence $\mathcal{U}_{C_{\infty}(\mathcal{K})} \subseteq \mathcal{U}_{\mathcal{K}}$. If the statement is not true, then $J_{1}$ is finite but there exists a pair $\left(\xi^{0}, u_{1}^{0}\right) \in \mathcal{U}_{C_{\infty}(\mathcal{K})} \times \mathcal{U}_{1}[0, T]$ such that $J_{1}\left(\xi^{0}, u_{1}^{0}(\cdot)\right)<0$. So, for any $k>0,\left(k \xi^{0}, k u_{1}^{0}\right)$ is also admissible $\left(\mathcal{K}\right.$ contains 0 thus $\left.k \xi^{0} \in \mathcal{U}_{C_{\infty}(\mathcal{K})}\right)$. Thus, $J_{1}\left(k \xi^{0}, k u_{1}^{0}(\cdot)\right)=$ $k^{2} J_{1}\left(\xi^{0}, u_{1}^{0}(\cdot)\right) \longrightarrow-\infty$ as $k \longrightarrow+\infty$. Contradiction thus arises.
We do not discuss if above result can be strengthen to be sufficient, with some additional conditions. However, in case $\mathcal{K}$ is conic, we do have the following equivalent result.

Corollary 4.2 Suppose (H1)-(H3) hold and $\mathcal{K}$ is closed-convex cone. Then, problem (P) is finite iff $J_{1}$ is nonnegative on $\mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T]$.

Proof The necessary part follows from Lemma 4.1 by noticing $C_{\infty}(\mathcal{K})=\mathcal{K}$ when $\mathcal{K}$ is conic. The sufficient part is obvious.
We point out closed-convex cone arises naturally from real applications, for example, $\mathcal{K}$ is positive orthant for no shorting constraint in finance portfolio selection (see [14, 17, 28]). Combining Corollary 4.1 and Corollary 4.2, we have the following more explicit result:
Corollary 4.3 Suppose (H1)-(H3) hold and $\mathcal{K}$ is closed-convex cone. Then, $\mathbf{( P )}$ is finite if it is convex.
We present some related remarks.
Remark 4.2 The result of Corollary 4.3 differs from standard LQ problem (see [45] pp. 287) where finiteness implies convexity, but converse is not true. Also, by Proposition 4.2 and Lemma 4.1. for general convex set $\mathcal{K}$ (not conic), the convexity and finiteness of problem ( P ) have no direct relation. This also differs from standard $L Q$ control where finiteness always implies convexity.
As implied by above, for $(\mathrm{P})$ with general closed-convex set $\mathcal{K}$, it seems lacking tractable equivalent condition to characterize its finiteness. However, on the other hand, convexity is necessary to be established when we plan to apply Lagrange multiplier to tackle the involved constraints in (P). Thus, we primarily focus on convexity and then discuss the related solvability (that in turn implies finiteness).

By representation (16), the mapping $\left(\xi, u_{1}(\cdot)\right) \longmapsto J_{1}\left(\xi, u_{1}(\cdot)\right)$ is Fréchet differentiable with Fréchet derivative $\partial J_{1}=\left(\partial_{\xi} J_{1}, \partial_{u} J_{1}\right)$ given respectively by

$$
\begin{equation*}
\partial_{\xi} J_{1}\left(\xi, u_{1}(\cdot)\right)=\mathcal{M}_{1}(\xi)+\mathcal{M}_{0}^{*}\left(u_{1}\right), \quad \partial_{u} J_{1}\left(\xi, u_{1}(\cdot)\right)=\mathcal{M}_{2}\left(u_{1}\right)+\mathcal{M}_{0}(\xi) \tag{21}
\end{equation*}
$$

When (P) is convex, we have the following solvability result.
Lemma 4.2 If $(\mathbf{P})$ is convex, then it is (uniquely) solvable iff there exists a (unique) minimizer $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ satisfying

$$
\left\langle\partial J_{1}\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right),\left(\xi-\bar{\xi}, u_{1}-\bar{u}_{1}\right)\right\rangle \geq 0 \Longleftrightarrow\left\{\begin{array}{l}
\left\langle\mathcal{M}_{1}(\bar{\xi})+\mathcal{M}_{0}^{*}\left(\bar{u}_{1}\right), \quad \xi-\bar{\xi}\right\rangle \geq 0  \tag{22}\\
\mathcal{M}_{2}\left(\bar{u}_{1}\right)+\mathcal{M}_{0}(\bar{\xi})=0
\end{array}\right.
$$

$\forall\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}(\mathcal{K}, \alpha, \beta) \times \mathcal{U}_{1}[0, T]$. If $(\mathbf{P})$ is further strictly convex, then its minimizer $(s)$, if exist, should be unique.

The above criteria is called first-order regularity condition for (global) optimality which is rather constructive. A more direct and checkable condition for existence is as follows.

Proposition 4.3 If $(\mathbf{P})$ is uniformly convex on $\left(\xi, u_{1}\right)$, then it admits an unique minimizer.
Proof We assume $\mathcal{U}(\widetilde{\mathcal{K}}, \alpha, \beta)$ is not empty (otherwise, ( P ) becomes trivial), thus there exists $\left(\xi^{0}, u_{1}^{0}\right)$ satisfying $-\infty<J_{1}\left(\xi^{0}, u_{1}^{0}\right)$. If $J_{1}$ is uniformly convex, it should also be coercive, that is, $J_{1}\left(\xi, u_{1}\right) \longrightarrow+\infty$ as $\left\|\left(\xi, u_{1}\right)\right\| \longrightarrow+\infty$. To see this point, actually we have

$$
\begin{aligned}
& J_{1}\left(\xi, u_{1}\right)=J_{1}\left(\xi^{0}, u_{1}^{0}\right)+J_{1}\left(\xi-\xi^{0}, u_{1}-u_{1}^{0}\right) \\
& \left.+\left[\left\langle\mathcal{M}_{2}\left(u_{1}-u_{1}^{0}\right)(\cdot), u_{1}^{0}(\cdot)\right\rangle+\left\langle\mathcal{M}_{1}\left(\xi-\xi^{0}\right), \xi^{0}\right\rangle+\left\langle\mathcal{M}_{0}\left(\xi^{0}\right), u_{1}-u_{1}^{0}\right)\right\rangle+\left\langle u_{1}^{0}, \mathcal{M}_{0}\left(\xi-\xi^{0}\right)\right\rangle\right] \\
& \geq J_{1}\left(\xi^{0}, u_{1}^{0}\right)+\delta\left\|\left(\xi-\xi^{0}, u_{1}-u_{1}^{0}\right)\right\|^{2}-\frac{\left\|\mathcal{M}_{2}\right\|^{2}+\left\|\mathcal{M}_{1}\right\|^{2}+\left\|\mathcal{M}_{0}\right\|^{2}}{\mu}\left\|\left(\xi-\xi^{0}, u_{1}-u_{1}^{0}\right)\right\|^{2}-\frac{\mu}{2}\left\|\left(\xi^{0}, u_{1}^{0}\right)\right\|^{2} \\
& \geq J_{1}\left(\xi^{0}, u_{1}^{0}\right)+\frac{\delta}{2}\left\|\left(\xi-\xi^{0}, u_{1}-u_{1}^{0}\right)\right\|^{2}-\frac{\mu}{2}\left\|\left(\xi^{0}, u_{1}^{0}\right)\right\|^{2}
\end{aligned}
$$

for sufficiently large $\mu>0$. Therefore, $J_{1}\left(\xi, u_{1}\right) \longrightarrow+\infty$ as $\left\|\left(\xi, u_{1}\right)\right\| \longrightarrow+\infty$. Note that Proposition 4.3 can only applied to $\left(\xi-\xi^{0}, u_{1}-u_{1}^{0}\right) \in \mathcal{U}_{\tilde{\mathcal{K}}} \times \mathcal{U}_{1}[0, T]$ for uniformly convexity. In general, $\left(\xi, u_{1}\right)$ or $\left(\xi^{0}, u_{1}^{0}\right) \notin \mathcal{U}_{\widetilde{\mathcal{K}}} \times \mathcal{U}_{1}[0, T]$.

Moreover, because $J_{1}$ is a proper quadratic functional with $\mathcal{M}_{0}, \mathcal{M}_{1}, \mathcal{M}_{2}$ being linear bounded operators thus $J_{1}(\cdot, \cdot)$ is also continuous (thus, lower semi-continuous (lsc)). By [13], a lsc convex coercive functional admits at least one minimizer. Moreover, the uniform convexity of $J_{1}$ implies strict convexity thus (P) admits a unique minimizer.
We now discuss condition under which problem (P) becomes convex. First introduce the following standard assumption

$$
\text { (SA-2): } \quad G_{1} \gg 0, \quad H_{1} \geq 0, \quad Q_{1}(\cdot) \geq 0, \quad S_{1}(\cdot) \geq 0, \quad R_{11}^{1}(\cdot) \gg 0
$$

Second, a more general sufficient condition to convexity is via the following stochastic Riccati equation (denoted by (SRE-2)):
$($ SRE-2 $):\left\{\begin{aligned} & d P_{L}=- {\left[\mathbb{A}^{\top} P_{L}+P_{L} \mathbb{A}+\mathbb{C}^{\top} P_{L} \mathbb{C}+\mathbb{Q}+\Lambda_{L} \mathbb{C}+\mathbb{C}^{\top} \Lambda_{L}-\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} P_{L} \mathbb{C}+\mathbb{D}^{\top} \Lambda_{L}\right)^{\top}\right.} \\ &\left.\mathbb{K}^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} P_{L} \mathbb{C}+\mathbb{D}^{\top} \Lambda_{L}\right)\right] d s+\Lambda_{L} d W(s), \\ & P_{L}(T)=\left(\begin{array}{cc}0 & 0 \\ 0 & G_{1}\end{array}\right), \\ & \mathbb{K}(s) \triangleq \mathbb{R}(s)+\mathbb{D}^{\top}(s) P_{L}(s) \mathbb{D}(s)>0,\end{aligned}\right.$
where

$$
\mathbb{A}=\left(\begin{array}{cc}
-A^{\top} & Q_{2} \\
B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & A
\end{array}\right), \mathbb{B}=\left(\begin{array}{cc}
0 & 0 \\
B_{1} & C
\end{array}\right), \mathbb{C}=\left(\begin{array}{cc}
-C^{\top} & 0 \\
0 & 0
\end{array}\right), \mathbb{D}=\left(\begin{array}{cc}
0 & S_{2} \\
0 & I
\end{array}\right), \mathbb{Q}=\left(\begin{array}{cc}
0 & 0 \\
0 & Q_{1}
\end{array}\right), \mathbb{R}=\left(\begin{array}{cc}
R_{11}^{1} & 0 \\
0 & S_{1}
\end{array}\right)
$$

We have the following result concerning convexity and its proof is given in the Appendix, Section 7.2,
Proposition 4.4 Suppose (SRE-2) has a solution $\left(P_{L}(\cdot), \Lambda_{L}(\cdot)\right) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}^{n}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{S}^{n}\right)$ such that

$$
\left(\begin{array}{cc}
0 & 0 \\
0 & H_{1}
\end{array}\right)+P_{L}(0) \geq 0
$$

Then, $J_{1}(\cdot, \cdot)$ is a convex functional with $\left(\xi, u_{1}(\cdot)\right)$ over $L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times U_{1}[0, T]$. In particular, under (SA-2), $J_{1}(\cdot, \cdot)$ is uniformly convex over $L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times \mathcal{U}_{1}[0, T]$.
Proposition 4.3 only specifies the existence of optimal solution to ( P ) but does not discuss how to characterize such solution. This will be discussed below through some Lagrange multiplier method to (P). Our target is to remove the affine-expectation constraint and only keep pointwise constraint.

Further study of (P) involves some Lagrange duality for which we need first address the relevant feasibility, as given below.

### 4.2 Feasibility of problem (P) constraints

Recall problem (P) involves two (pointwise, affine-expectation) constraints, thus it is necessary to discuss their joint feasibility. To start, for any convex-closed proper subset $\mathcal{K} \subset \mathbb{R}^{n}$, we can introduce its support functional: $h_{\mathcal{K}}^{*}(p) \triangleq \sup _{x \in \mathcal{K}}\langle p, x\rangle \in[0,+\infty]$. Its effective domain (i.e., $\left.\left\{p: h_{\mathcal{K}}^{*}(p)<+\infty\right\}\right)$ is $B(\mathcal{K})$, the barrier cone of $\mathcal{K}$. In particular, when $\mathcal{K}$ is convex-closed cone, then $B(\mathcal{K})$ is negative polar cone of $\mathcal{K}$.

Moreover, $-h_{\mathcal{K}}^{*}(-p)=\inf _{x \in \mathcal{K}}\langle p, x\rangle$ and $h_{\mathcal{K}}^{*}(p)+h_{\mathcal{K}}^{*}(-p) \in[0,+\infty]$ is called the breadth for nonempty $\mathcal{K}$ along direction $p$. The breadth takes value 0 iff $\mathcal{K}$ is subset of affine hyperplane $\left\{y:\langle y, p\rangle=h_{\mathcal{K}}^{*}(p)\right\}$ which is orthogonal to $p$. Now, we can discuss the feasibility of constrained $\mathcal{U}(\mathcal{K}, \alpha, \beta)$.

We first claim the following fundamental result that is obvious in its scalar case ( $n=1$ ) but not straightforward in vector case. A similar result may be found in [7] pp. 44.

Lemma $4.3 \forall \xi \in \mathcal{U}_{\mathcal{K}}, \mathbb{E} \xi \in \mathcal{K}$.
Proof Recall that any convex-closed set $\mathcal{K} \subset \mathbb{R}^{n}$ can be equivalently defined as the intersection of all closed half-spaces containing it, thus for a.s. $\omega,\left\langle s_{j}, \xi(\omega)\right\rangle \leq r_{j}$ for some data $\left(s_{j}, r_{j}\right) \in \mathbb{R}^{n} \times \mathbb{R}$ from some index set $j \in J$. By linearity of expectation, $\left\langle s_{j}, \mathbb{E} \xi\right\rangle \leq r_{j}$ for all $j \in J$ also, thus $\mathbb{E} \xi \in \mathcal{K}$. Another proof is based on support functional as follows. $x \in \mathcal{K}$ iff $\langle x, p\rangle \leq h_{\mathcal{K}}^{*}(p)$ for each vector $p$. Again, by linearity of expectation, $\langle\mathbb{E} \xi, p\rangle \leq h_{\mathcal{K}}^{*}(p)$ for each vector $p$, hence $\mathbb{E} \xi \in \mathcal{K}$.

By Lemma 4.3, a necessary condition for $\mathcal{U}(\mathcal{K}, \alpha, \beta)$ being non-empty is $\mathcal{K}_{\alpha, \beta}^{+} \triangleq \mathcal{K} \cap H_{\alpha, \beta}^{+} \neq \emptyset$ where $H_{\alpha, \beta}^{+}=\left\{x \in \mathbb{R}^{n}:\langle\alpha, x\rangle \geq \beta\right\}$ is one half-space delimited by the affine hyperplane $H_{\alpha, \beta}:\langle\alpha, x\rangle=\beta$. Further discussion of feasibility to $\mathcal{U}(\mathcal{K}, \alpha, \beta)$, may depend on the following alternative assumptions.
$($ F1 $)($ positive breadth along $\alpha): \quad h_{\mathcal{K}}^{*}(\alpha)+h_{\mathcal{K}}^{*}(-\alpha)>0$.
(F2) (degenerated breadth along $\alpha$ ): $\quad h_{\mathcal{K}}^{*}(\alpha)+h_{\mathcal{K}}^{*}(-\alpha)=0$.
Depending on (F1) or (F2), we have the following feasibility results respectively.
Proposition 4.5 Under (F1), the terminal admissible set $\mathcal{U}(\mathcal{K}, \alpha, \beta) \triangleq \mathcal{U}_{\mathcal{K}} \bigcap \mathcal{U}_{\alpha, \beta}$ is

- (i) nontrivial (non-empty and admitting two constraints both), if $-h_{\mathcal{K}}^{*}(-\alpha)<\beta<h_{\mathcal{K}}^{*}(\alpha)$;
- (ii) trivial (being reduced to pointwise constraint $\mathcal{U}_{\mathcal{K}}$ only), if $\beta \leq-h_{\mathcal{K}}^{*}(-\alpha)$;
- (iii) trivial (empty set), if $\beta>h_{\mathcal{K}}^{*}(\alpha)$;
- (iv) trivial (degenerated to the exposed face of $\mathcal{K}$ ), if $\beta=h_{\mathcal{K}}^{*}(\alpha)$.

Proposition 4.6 Under (F2), the terminal admissible set $\mathcal{U}(\mathcal{K}, \alpha, \beta) \triangleq \mathcal{U}_{\mathcal{K}} \bigcap \mathcal{U}_{\alpha, \beta}$ is

- (ii') trivial (being reduced to pointwise constraint $\mathcal{U}_{\mathcal{K}}$ only), if $\beta \leq h_{\mathcal{K}}^{*}(\alpha)$;
- (iii') trivial as being empty, if $\beta>h_{\mathcal{K}}^{*}(\alpha)$.

The proofs of Propositions 4.54 .6 follow from standard convex analysis, and readers may refer 38 . Chapters 4 and 5. Of course, we are more interested to the nontrivial case (i). Some related remarks are as follows.

Remark 4.3 (a) When $\mathcal{K}$ is bounded (hence compact), $B(\mathcal{K})=\mathbb{R}^{n}$ thus $-\infty<-h_{\mathcal{K}}^{*}(-\alpha)<h_{\mathcal{K}}^{*}(\alpha)<$ $+\infty$ and (i) always holds true for all affine-expectation constraint pairs $(\alpha, \beta) \in \mathbb{R}^{n} \times\left(-h_{\mathcal{K}}^{*}(-\alpha), h_{\mathcal{K}}^{*}(\alpha)\right)$.
(b) For unbounded $\mathcal{K}$, its asymptotic cone provides more explicit representation of $B(\mathcal{K})$ and the range qualification to $(\alpha, \beta)$ jointly. We omit details here.
(c) Notice that (iv) above involves the exposed face. Recall for convex set $K$, a set $F$ is called its exposed face if there is a supporting hyperplane $H_{s, r}$ of $K$ such that $F=H_{s, r} \cap K$. For unbounded $\mathcal{K}$, there have some subtle difference between exposed face and boundary of $\mathcal{K}$.

It is obvious that $\mathcal{K}_{\alpha, \beta}^{+}$is convex-closed set. We can introduce $\mathcal{U}_{\mathcal{K}_{\alpha, \beta}^{+}}=L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathcal{K}_{\alpha, \beta}^{+}\right)$that satisfies $\mathcal{U}_{\mathcal{K}_{\alpha, \beta}^{+}} \subset \mathcal{U}(\mathcal{K}, \alpha, \beta)$ by Lemma 4.3. Noticing the inclusion here is strictly proper subset by noting, say, in scalar case, it is not very hard to construct a random variable with support on $\mathcal{K}=[0,1]$ but with expectation on $\left[\frac{1}{2},+\infty\right)$ (i.e., $\alpha=1, \beta=\frac{1}{2}$ ).

We continue to discuss the strict feasibility that relates to Slater qualification to be invoked. To start, we first present some relative interior point result for pointwise constraint $\mathcal{U}_{\mathcal{K}}$.

Proposition 4.7 The constrained set $\mathcal{U}_{\mathcal{K}}$ admits no relative interior point.

Proof In case $\operatorname{dim} \mathcal{K}=n$, then $\operatorname{aff}(\mathcal{K})=\operatorname{aff}\left(\mathcal{K}_{\alpha, \beta}^{+}\right)=\mathbb{R}^{n}$, and $\operatorname{aff}\left(\mathcal{U}_{\mathcal{K}}\right) \supseteq \operatorname{aff}\left(\mathcal{U}_{\mathcal{K}_{\alpha, \beta}^{+}}\right)=L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \operatorname{aff}\left(\mathcal{K}_{\alpha, \beta}^{+}\right)\right)=$ $L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$. Then, aff $\left(\mathcal{U}_{\mathcal{K}}\right)=L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$. On the other hand, for any $\xi \in \mathcal{U}_{\mathcal{K}}$, we can always construct $\xi^{\prime} \in B(\xi, \varepsilon) \subset L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$, a small ball centered at $\xi$ with radius $\varepsilon>0$, but support $\xi^{\prime} \in \mathcal{K}^{c}$ with positive probability. Similar arguments can be applied to the case of $\operatorname{dim} \mathcal{K}<n$.

Based on Proposition 4.7 to apply the Lagrange multiplier method, its Slater qualification condition holds true iff $\mathcal{K}_{\alpha, \beta}^{++} \triangleq \mathcal{K} \cap H_{\alpha, \beta}^{++} \neq \emptyset$ with $H_{\alpha, \beta}^{++}=\left\{x \in \mathbb{R}^{n}:\langle\alpha, x\rangle>\beta\right\}$ being the strict half-space (noticing a crucial point here is that $\mathcal{U}_{\alpha, \beta}$ is an inequality constraint on (linear) affine expectation). Actually, for any $y \in \mathcal{K}_{\alpha, \beta}^{++}, \xi(\omega) \equiv y$ a.s. $\in \mathcal{U}(\mathcal{K}, \alpha, \beta)$ and satisfies the affine-expectation constraint strictly. Conversely, if there has any random variable $\xi$ satisfying affine-expectation constraint strictly, and $\xi \in \mathcal{U}_{\mathcal{K}}$, it is necessary to have non-empty $\mathcal{K}_{\alpha, \beta}^{++}$for $\mathbb{E} \xi$ by Lemma 4.3,

In summary to Propositions 4.5-4.7, we set the following assumption under which $\mathcal{U}(\mathcal{K}, \alpha, \beta)$ is nontrivial, strictly feasible and Slater constraint qualification holds true.
$(\mathbf{F})$ The triple $(\mathcal{K}, \alpha, \beta)$ of terminal constraint parameter satisfy: $\quad-h_{\mathcal{K}}^{*}(-\alpha)<\beta<h_{\mathcal{K}}^{*}(\alpha)$.

### 4.3 Solution of primal problem (P) via duality

We introduce the following dual problem (D) associated to the primal (P):

$$
\text { (D) : Maximize } \quad K(\lambda) \triangleq \inf _{\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T]} L\left(\lambda ; \xi, u_{1}(\cdot)\right) \quad \text { subject to } \quad \lambda \geq 0,
$$

where $L\left(\lambda ; \xi, u_{1}(\cdot)\right) \triangleq J_{1}\left(\xi, u_{1}(\cdot)\right)+\lambda(\beta-\mathbb{E}\langle\alpha, \xi\rangle)$ is called the Lagrange functional, $K(\cdot)$ is called dual function which is parallel to primal functional $J_{1}(\cdot, \cdot)$. Dual function $K(\cdot)$ is always concave (even $J_{1}(\cdot, \cdot)$ is not convex) since it is defined by infimum operation on a family of affine functionals.

We can introduce an auxiliary problem (KT) for given $\lambda_{0} \geq 0$ :

$$
\text { (KT) : Minimize } \quad L\left(\lambda_{0} ; \xi, u_{1}(\cdot)\right) \quad \text { subject to } \quad \text { (15), }\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T] .
$$

We stress that here $\xi \in \mathcal{U}_{\mathcal{K}}$ instead $\mathcal{U}(\mathcal{K}, \alpha, \beta)$ as in (P). Now, we can introduce the following definitions based on 38 .
Definition 4.3 (Kuhn-Tucker coefficient) A Kuhn-Tucker coefficient (KT-coefficient) for problem (P) is any $\lambda_{0} \geq 0$ satisfying $-\infty<K\left(\lambda_{0}\right)=\mu_{p}$.
(KT-admissible) Problem (P) is said to be KT-admissible if it has at least one KT-coefficient.
Definition 4.3 imposes no assumption on existence of optimal solutions to primal (P), dual (D) and (KT). Similar to (P), we can further introduce the following definitions.
Definition 4.4 (a) Problem (D) is said to be finite if $\mu_{d} \triangleq \sup _{\lambda \geq 0} K(\lambda)<+\infty$, and $\mu_{d}$ is called the value of (D);
(b) Problem (D) is said to be (uniquely) solvable if there exists a (unique) $\lambda^{*} \geq 0$ such that $\mu_{p}=$ $K\left(\lambda^{*}\right)$ and $\lambda^{*}$ is called maximizer of (D);
(c) Problem (KT) is said to be finite if $K\left(\lambda_{0}\right)>-\infty$, and $K\left(\lambda_{0}\right)$ is the value of (KT);
(d) Problem (KT) is said to be (uniquely) solvable if there exists a (unique) $\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T]$ such that $K\left(\lambda_{0}\right)=L\left(\lambda_{0} ; \xi^{*}, u_{1}^{*}(\cdot)\right)$ and $\left(\xi^{*}, u_{1}^{*}(\cdot)\right)$ is called minimizer of (KT).
The following relations among problem (P), (D) and (KT) are obvious.
Proposition 4.8 (a) If Problem (P) is KT-admissible, then it is finite.
(b) The values of problem (P), (D) and (KT) parameterized by $\lambda_{0} \geq 0$, always satisfy: $K\left(\lambda_{0}\right) \leq$ $\mu_{d} \leq \mu_{p}$ where $\mu_{p}-\mu_{d} \geq 0$ is called the duality gap.

Note that (P) and (KT) in Proposition 4.8 need not to be convex. Moreover, we have the following solvability relations among (P), (D) and (KT), which follow from convex analysis (e.g., see [38 Part VI) and proof details are omitted here:
Lemma 4.4 (a) If Problem (P) is KT-admissible, then duality gap is 0 (namely, strong duality holds) and problem (D) is solvable. Note here, (P) may not be convex.
(b) If ( $\mathbf{P}$ ) is KT-admissible, convex and related (KT) problem with KT-coefficient $\lambda_{0}$ is solvable with optimal solution set $D=\left\{\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right): K\left(\lambda_{0}\right)=L\left(\lambda_{0} ; \bar{\xi}, \bar{u}_{1}(\cdot)\right)\right\}$. Then, the subset $D_{p}$ of $D$ satisfying complementary slackness condition: $\lambda_{0}(\beta-\mathbb{E}\langle\alpha, \bar{\xi}\rangle)=0$, is the optimal solution set to primal (P).

Remark 4.4 We remark that in (a) above, problem (P) and (KT) may not be solvable even (D) is solvable. Also, in (b), (KT) solvability does not imply solvability of $(\mathbf{P})$, conversely, solvability of primal (P) does not imply it is KT-solvable or even KT-admissible.

Part (b) of Lemma 4.4 specifies some sufficient condition to find all optimal solutions to primal problem (P). In usual cases, we are more interested to equivalent condition for ( P ) solvability, and we thus report the following result which proof can be referred from [38] Part VI.

Theorem 4.1 Assume (H1)-(H3) and suppose (P) is convex, then the following three statements: (i), (ii), and (iii) are equivalent:
(i): (P) is KT-admissible with coefficient $\lambda_{0}$, and (P) is solvable with minimizer $\left(\xi^{*}, u_{1}^{*}(\cdot)\right)$;
(ii): The triple $\left(\lambda_{0} ; \xi^{*}, u_{1}^{*}(\cdot)\right) \in[0,+\infty) \times \mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T]$ satisfies the following Karush-Kuhn-Tucker (KKT) system:

$$
\begin{equation*}
\beta \leq \mathbb{E}\langle\alpha, \bar{\xi}\rangle, \quad \bar{\lambda}(\beta-\mathbb{E}\langle\alpha, \bar{\xi}\rangle)=0 ; \quad K\left(\lambda_{0}\right)=L\left(\lambda_{0} ; \bar{\xi}, \bar{u}_{1}(\cdot)\right) \tag{23}
\end{equation*}
$$

(iii): The triple $\left(\lambda_{0} ; \xi^{*}, u_{1}^{*}(\cdot)\right)$ is a saddle point for Lagrange functional $L$ :

$$
L\left(\lambda ; \bar{\xi}, \bar{u}_{1}(\cdot)\right) \leq L\left(\bar{\lambda} ; \bar{\xi}, \bar{u}_{1}(\cdot)\right) \leq L\left(\bar{\lambda} ; \xi, u_{1}(\cdot)\right)
$$

In Theorem 4.1, the KT-admissible and its coefficient $\lambda_{0}$ plays some crucial role. Thus, we present some sufficient condition ensuring them.

Proposition 4.9 Assume (H1)-(H3), and suppose problem (P) is convex, finite. Moreover, suppose feasibility condition ( $\mathbf{F}$ ) holds true, then ( $\mathbf{P}$ ) is KT-admissible for some $\lambda_{0} \geq 0$.
Proof When (F) holds true, then (P) satisfies the Slater qualification condition hence it is also KTadmissible by [38, Corollary 28.2.1], considering ( P ) is finite and convex. Hence the result.
Noticing assumption (F) is crucial in above and the following example indicates it can usually be expected. We just present its scalar case for illustration, and the vector case can be constructed similarly.
Example 4.1 In case $n=1$, suppose $\frac{\beta}{\alpha} \in \mathcal{K}^{o}$, where $\mathcal{K}^{o}$ is the interior of $\mathcal{K}$. Then, ( $\mathbf{F}$ ) holds.
Introduce the following assumption:
$(\mathbf{H} 4) G_{1}>0 . R_{11}^{1}(\cdot)$ is invertible and $\left(R_{11}^{1}(\cdot)\right)^{-1} \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{m_{1}}\right)$.
Lemma 4.5 Let (H1)-(H4) hold and (P) is convex. Then, (KT) parameterized by coefficient $\lambda \geq 0$ is (uniquely) solvable iff the following BFSDEs

$$
(\mathbf{B F S D E - 1}):\left\{\begin{array}{l}
d g=-\left[A^{\top} g-Q_{1} \bar{X}-Q_{2} h\right] d s-\left[C^{\top} g-S_{1} \bar{Z}-S_{2} q\right] d W(s) \\
d \bar{Y}=\left[-A^{\top} \bar{Y}+Q_{2} \bar{X}\right] d s+\left[-C^{\top} \bar{Y}+S_{2} \bar{Z}\right] d W(s) \\
d \bar{X}=\left[A \bar{X}+B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} g+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \bar{Y}+C \bar{Z}\right] d s+\bar{Z} d W(s) \\
d h=\left[A h+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g+C q\right] d s+q d W(s) \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0) \\
\bar{X}(T)=\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right], \quad h(T)=0
\end{array}\right.
$$

admits a (unique) solution $(\bar{Y}, g, \bar{X}, \bar{Z}, h, q) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right) \times$ $L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(\Omega ; C\left([0, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$, where $\operatorname{Proj}_{\mathcal{K}}(\cdot)$ is the projection mapping from $\mathbb{R}^{n}$ to closed-convex set $\mathcal{K}$ under the norm $|x|_{G_{1}}^{2} \triangleq\left\langle G_{1}^{\frac{1}{2}} x, G_{1}^{\frac{1}{2}} x\right\rangle$. In this case, the (unique) minimizer $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ to $(\mathbf{K T})$ with coefficient $\lambda$ is given by

$$
\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right], \quad\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot)\right)
$$

Proof Note that (P) is convex, then for any $\lambda \geq 0$, the Lagrange functional $L\left(\lambda ; \xi, u_{1}(\cdot)\right)$ thus (KT) are also convex. Similar to Proposition 4.1, we have that

$$
L\left(\lambda ; \xi, u_{1}(\cdot)\right)=\frac{1}{2} \mathbb{E}\left[\left\langle\mathcal{M}_{2}\left(u_{1}\right)(\cdot), u_{1}(\cdot)\right\rangle+\left\langle\mathcal{M}_{1}(\xi)-2 \lambda \alpha, \xi\right\rangle+2\left\langle\xi, \mathcal{M}_{0}^{*}\left(u_{1}(\cdot)\right)\right\rangle+2 \lambda \beta\right]
$$

Consequently, similar to Lemma 4.2 problem (KT) is solvable iff there exists a pair $\left(\xi^{*}, u_{1}^{*}(\cdot)\right)$ satisfying

$$
\left\{\begin{array}{l}
\left\langle\mathcal{M}_{1}\left(\xi^{*}\right)-\lambda \alpha+\mathcal{M}_{0}^{*}\left(u_{1}^{*}\right), \xi^{*}-\xi_{1}\right\rangle \leq 0, \quad \forall \xi_{1} \in \mathcal{U}_{\mathcal{K}}  \tag{24}\\
\mathcal{M}_{2}\left(u_{1}^{*}\right)+\mathcal{M}_{0}\left(\xi^{*}\right)=0
\end{array}\right.
$$

Let $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ be an optimal control, by (24), we have

$$
\left\{\begin{array}{l}
\mathbb{E}\left\langle-g_{1}(T)-g_{2}(T)+\lambda \alpha-G_{1} \bar{\xi}, \xi_{1}-\bar{\xi}\right\rangle \leq 0, \quad \forall \xi_{1} \in \mathcal{U}_{\mathcal{K}}  \tag{25}\\
R_{11}^{1}(s) \bar{u}_{1}(s)-B_{1}^{\top}(s) g_{1}(s)-B_{1}^{\top}(s) g_{2}(s)=0
\end{array}\right.
$$

where $\left(Y_{1}, g_{1}, X_{1}, Z_{1}, h_{1}, q_{1}\right)$ and $\left(Y_{2}, g_{2}, X_{2}, Z_{2}, h_{2}, q_{2}\right)$ are the solutions of (17) and (18) corresponding to $\left(\bar{\xi}, \bar{u}_{1}\right)$, respectively. Let

$$
\bar{Y}=Y_{1}+Y_{2}, \quad \bar{X}=X_{1}+X_{2}, \quad \bar{Z}=Z_{1}+Z_{2}, \quad g=g_{1}+g_{2}, \quad h=h_{1}+h_{2}, \quad q=q_{1}+q_{2}
$$

and it follows that $(\bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ satisfying (BFSDE-1). Under (H4), it follows from (25) that

$$
\bar{u}_{1}(\cdot)=\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot),
$$

and

$$
\mathbb{E}\left\langle G_{1}^{\frac{1}{2}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)-\bar{\xi}\right], G_{1}^{\frac{1}{2}}\left(\xi_{1}-\bar{\xi}\right)\right\rangle \leq 0, \quad \forall \xi_{1} \in \mathcal{U}_{\mathcal{K}}
$$

Note that $|\cdot|_{G_{1}}$ is equivalent to the Euclidean norm. Let $\xi_{1}=\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right]$, then by Propositions 4.1 and 4.3 in [15], we have

$$
\mathbb{E}\left|\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right]-\bar{\xi}\right|_{G_{1}}^{2} \leq \mathbb{E}\left\langle G_{1}^{\frac{1}{2}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)-\bar{\xi}\right], G_{1}^{\frac{1}{2}}\left(\xi_{1}-\bar{\xi}\right)\right\rangle \leq 0
$$

Thus, we get

$$
\bar{\xi}=\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right]
$$

The uniqueness follows from the uniqueness of the solution of (BFSDE-1).
Combing Theorem 4.1. Proposition 4.9 and Lemma 4.5, we have
Theorem 4.2 Let (H1)-(H4) hold. Suppose (F) hold and (P) is convex and finite, then (P) is KTadmissible with some coefficient $\lambda_{0} \geq 0$. Moreover, $(\mathbf{P})$ is solvable with an optimal solution $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ iff there exist a 7 -tuple $(\lambda ; \bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ satisfying both (BFSDE-1) and (KKT) system:

$$
\begin{cases}\text { complimentary slackness: } & \lambda\left(\beta-\mathbb{E}\left\langle\alpha, \operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right]\right\rangle\right)=0  \tag{26}\\ \text { primal- and dual-constraint: } & \lambda \geq 0 ; \quad \beta \leq \mathbb{E}\left\langle\alpha, \operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right]\right\rangle\end{cases}
$$

In this case, $\lambda$ is a KT-coefficient of $(\mathbf{P})$, and an optimal solution to problem $(\mathbf{P})$ is given by

$$
\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right],\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot)\right)
$$

As a corollary, we have
Corollary 4.4 Let (H1)-(H4) and (F) hold true. Suppose (P) is uniformly convex, then it admits a unique optimal solution $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[G_{1}^{-1}(-g(T)+\lambda \alpha)\right],\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot)\right)$ with $(\lambda ; \bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ is a solution for system (BFSDE-1) and (KKT) system.

### 4.4 Some special cases

This subsection will consider two special cases of problem ( P ) with more detailed analysis.

### 4.4.1 Pointwise constraint

This subsection considers the case with only pointwise constraint $\mathcal{U}_{\mathcal{K}}$. In this special case, Problem (P) now assumes the following form

$$
\left(\mathbf{P}_{1}\right): \text { Minimize } \quad J_{1}\left(\xi, u_{1}(\cdot)\right) \quad \text { subject to } \quad \text { (15), }\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}_{\mathcal{K}} \times \mathcal{U}_{1}[0, T]
$$

By Lemma 4.5, we have the following result.
Corollary 4.5 Let ( $\mathbf{H} 1)-(\mathbf{H} 4)$ hold and $\left(\mathbf{P}_{1}\right)$ is convex. Then $\left(\mathbf{P}_{1}\right)$ admits an (unique) optimal control $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ iff the following BFSDEs
$\left(\right.$ BFSDE-2) $:\left\{\begin{array}{l}d g=-\left[A^{\top} g-Q_{1} \bar{X}-Q_{2} h\right] d s-\left[C^{\top} g-S_{1} \bar{Z}-S_{2} q\right] d W(s), \\ d \bar{Y}=\left[-A^{\top} \bar{Y}+Q_{2} \bar{X}\right] d s+\left[-C^{\top} \bar{Y}+S_{2} \bar{Z}\right] d W(s), \\ d \bar{X}=\left[A \bar{X}+B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} g+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \bar{Y}+C \bar{Z}\right] d s+\bar{Z} d W(s), \\ d h(s)=\left[A h+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g+C q\right] d s+q d W(s), \\ g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\operatorname{Proj}_{\mathcal{K}}\left[-G_{1}^{-1} g(T)\right], \quad h(T)=0,\end{array}\right.$
admits a (unique) solution $(\bar{Y}, g, \bar{X}, \bar{Z}, h, q)$. Moreover, a (the) minimizer of $\left(\mathbf{P}_{1}\right)$ is given by

$$
\begin{equation*}
\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[-G_{1}^{-1} g(T)\right], \quad\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot)\right) \tag{27}
\end{equation*}
$$

### 4.4.2 Affine constraint

This subsection focus on the case with only constraint $\mathcal{U}_{\alpha, \beta}$ for terminal variable $\xi$. In this case, (P) takes the following form:

$$
\left(\mathbf{P}_{2}\right): \text { Minimize } \quad J_{1}\left(\xi, u_{1}(\cdot)\right) \quad \text { subject to } \quad \text { (15), }\left(\xi, u_{1}(\cdot)\right) \in \mathcal{U}_{\alpha, \beta} \times \mathcal{U}_{1}[0, T]
$$

By Theorem 4.2, we have the following result.
Corollary 4.6 Let (H1)-(H4) hold and suppose $\left(\mathbf{P}_{2}\right)$ is convex and finite, then $\left(\mathbf{P}_{2}\right)$ is KT-admissible with some coefficient $\lambda_{0} \geq 0$. Moreover, $\left(\mathbf{P}_{2}\right)$ is solvable with an optimal solution $\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)$ iff there exist a 7-tuple $(\lambda ; g, \bar{Y}, \bar{X}, \bar{Z}, h, q)$ satisfying the following BFSDEs

$$
\left(\text { BFSDE-3) }:\left\{\begin{array}{l}
d g=-\left[A^{\top} g-Q_{1} \bar{X}-Q_{2} h\right] d s-\left[C^{\top} g-S_{1} \bar{Z}-S_{2} q\right] d W(s) \\
d \bar{Y}=\left[-A^{\top} \bar{Y}+Q_{2} \bar{X}\right] d s+\left[-C^{\top} \bar{Y}+S_{2} \bar{Z}\right] d W(s), \\
d \bar{X}=\left[A \bar{X}+B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} g+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \bar{Y}+C \bar{Z}\right] d s+\bar{Z} d W(s) \\
d h=\left[A h+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g+C q\right] d s+q d W(s) \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0) \\
\bar{X}(T)=G_{1}^{-1}(-g(T)+\lambda \alpha), \quad h(T)=0, \\
\lambda\left(\beta-\mathbb{E}\left\langle\alpha, G_{1}^{-1}(-g(T)+\lambda \alpha)\right\rangle\right)=0, \quad \lambda \geq 0, \quad \beta \leq \mathbb{E}\left\langle\alpha, G_{1}^{-1}(-g(T)+\lambda \alpha)\right\rangle
\end{array}\right.\right.
$$

In this case, $\lambda$ is a KT-coefficient of $\left(\mathbf{P}_{2}\right)$, and an optimal solution to problem $\left(\mathbf{P}_{2}\right)$ is

$$
\begin{equation*}
\left(\bar{\xi}, \bar{u}_{1}(\cdot)\right)=\left(G_{1}^{-1}(-g(T)+\lambda \alpha),\left(R_{11}^{1}(\cdot)\right)^{-1} B_{1}^{\top}(\cdot) g(\cdot)\right) . \tag{28}
\end{equation*}
$$

For Corollaries 4.5 and 4.6, it follows that (BFSDE-2) and (BFSDE-3) play some key roles in determining the optimal solution. Specifically, (BFSDE-2) is a nonlinear (because of the projection operator) fullycoupled BFSDEs; (BFSDE-3) is a linear but constrained (because of (KKT) condition) fully-coupled BFSDEs. Both are non-standard in BFSDEs theory. Thus, it remains a challenge to show the global solvability of them, together with (SRE-1), (SRE-2). To this end, we study the wellposedness (existence, uniqueness) of (BFSDE-2), (BFSDE-3), Riccati equations in Sections 5.2, 5.4 and 5.5, respectively.

## 5 Existence and uniqueness of BFSDEs and Riccati equations

### 5.1 Discounting method

In this subsection, we will use the discounting method (see [36) to study the wellposedness of BFSDEs. To begin with, we first give some results for general nonlinear mean-field BFSDEs:

$$
\left\{\begin{array}{l}
d Y(s)=b(s, Y(s), X(s), Z(s), \mathbb{E} Z(s)) d s+\sigma(s, Y(s), X(s), Z(s)) d W(s)  \tag{29}\\
-d X(s)=f(s, Y(s), X(s), Z(s), \mathbb{E} Z(s)) d s-Z d W(s) \\
Y(0)=h(X(0)), \quad X(T)=g(Y(T), \mathbb{E} Y(T))
\end{array}\right.
$$

Accordingly, the following assumptions are imposed:
(H5) There exist $\rho_{1}, \rho_{2} \in \mathbb{R}$ and positive constants $k_{i}, i=1,2, \cdots, 10$ such that for all $s \in[0, T]$, $y, y_{1}, y_{2}, \bar{y}_{1}, \bar{y}_{2} \in \mathbb{R}^{n_{1}}, x, x_{1}, x_{2}, z, z_{1}, z_{2}, \bar{z}_{1}, \bar{z}_{2} \in \mathbb{R}^{n_{2}}$ a.s.,
(i) $\left\langle b\left(s, y_{1}, x, z, \bar{z}\right)-b\left(s, y_{2}, x, z, \bar{z}\right), y_{1}-y_{2}\right\rangle \leq \rho_{1}\left|y_{1}-y_{2}\right|^{2}$,

$$
\left|b\left(s, y, x_{1}, z_{1}, \bar{z}_{1}\right)-b\left(s, y, x_{2}, z_{2}, \bar{z}_{2}\right)\right| \leq k_{1}\left|x_{1}-x_{2}\right|+k_{2}\left|z_{1}-z_{2}\right|+k_{3}\left|\bar{z}_{1}-\bar{z}_{2}\right|
$$

(ii) $\left\langle f\left(s, y, x_{1}, z, \bar{z}\right)-f\left(s, y, x_{2}, z, \bar{z}\right), x_{1}-x_{2}\right\rangle \leq \rho_{2}\left|x_{1}-x_{2}\right|^{2}$, $\left|f\left(s, y_{1}, x, z_{1}, \bar{z}_{1}\right)-f\left(s, y_{2}, x, z_{2}, \bar{z}_{2}\right)\right| \leq k_{4}\left|y_{1}-y_{2}\right|+k_{5}\left|z_{1}-z_{2}\right|+k_{6}\left|\bar{z}_{1}-\bar{z}_{2}\right|$,
(iii) $\left|\sigma\left(s, y_{1}, x_{1}, z_{1}\right)-\sigma\left(s, y_{2}, x_{2}, z_{2}\right)\right|^{2} \leq k_{7}^{2}\left|y_{1}-y_{2}\right|^{2}+k_{8}^{2}\left|x_{1}-x_{2}\right|^{2}+k_{9}^{2}\left|z_{1}-z_{2}\right|^{2}$,
(iv) $\left|h\left(x_{1}\right)-h\left(x_{2}\right)\right| \leq k_{10}\left|x_{1}-x_{2}\right|,\left|g\left(y_{1}, \bar{y}_{1}\right)-g\left(y_{2}, \bar{y}_{2}\right)\right| \leq k_{11}\left|y_{1}-y_{2}\right|+k_{12}\left|\bar{y}_{1}-\bar{y}_{2}\right|$,
(v) $\mathbb{E}\left\{|h(0)|^{2}+|g(0,0)|^{2}+\int_{0}^{T}\left(|b(s, 0,0,0,0)|^{2}+|\sigma(s, 0,0,0)|^{2}+|f(s, 0,0,0,0)|^{2}\right) d s\right\}<\infty$.

Now we present the main result of this subsection on wellposedness of mean-field BFSDEs (29). Its proof is in the appendix.
Theorem 5.1 Under (H5), there exists a $\delta_{1}>0$, which depends on $\rho_{1}, \rho_{2}, T, k_{i}, i=5,6,7$, such that when $k_{i} \in\left[0, \delta_{1}\right), i=1,2,3,4,8,9,10$, there exists a unique adapted solution $(Y(\cdot), X(\cdot), Z(\cdot)) \in$ $L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{1}}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{2}}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{2}}\right)$ to mean-field BFSDEs (29). Further, if $2\left(\rho_{1}+\rho_{2}\right)<$ $-2 k_{5}^{2}-2 k_{6}^{2}-k_{7}^{2}$, there exists a $\delta_{2}>0$, which depends on $\rho_{1}, \rho_{2}, k_{i}, i=5,6,7$, and is independent of $T$, such that when $k_{i} \in\left[0, \delta_{1}\right), i=1,2,3,4,8,9,10$, there exists a unique adapted solution $(Y(\cdot), X(\cdot), Z(\cdot)) \in$ $L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{1}}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{2}}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n_{2}}\right)$ to mean-field BFSDEs (29).

### 5.2 Solvability of (BFSDE-2)

In order to apply Theorem 5.1, denote $\mathbb{Y}=\left(g^{\top}, \bar{Y}^{\top}\right)^{\top}, \mathbb{X}=\left(\bar{X}^{\top}, h^{\top}\right)^{\top}, \mathbb{Z}=\left(\bar{Z}^{\top}, q^{\top}\right)^{\top}$. Rewrite (BFSDE-2) as the following $2 n \times 2 n$-BFSDEs:
$\left(\right.$ BFSDE-2 $\left.^{\prime}\right):\left\{\begin{array}{l}d \mathbb{Y}=\left[-\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right)^{\top} \mathbb{Y}+\left(\begin{array}{cc}Q_{1} & Q_{2} \\ Q_{2} & 0\end{array}\right) \mathbb{X}\right] d s+\left[-\left(\begin{array}{cc}C & 0 \\ 0 & C\end{array}\right)^{\top} \mathbb{Y}+\left(\begin{array}{cc}S_{1} & S_{2} \\ S_{2} & 0\end{array}\right) \mathbb{Z}\right] d W(s), \\ d \mathbb{X}=\left[\begin{array}{cc}\left.\left(\begin{array}{cc}B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\ B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0\end{array}\right) \mathbb{Y}+\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right) \mathbb{X}+\left(\begin{array}{cc}C & 0 \\ 0 & C\end{array}\right) \mathbb{Z}\right] d s+\mathbb{Z} d W(s), \\ \mathbb{Y}(0)=\left(\begin{array}{cc}H_{1} & H_{2} \\ H_{2} & 0\end{array}\right) \mathbb{X}(0), \quad \mathbb{X}(T)=\operatorname{Proj}_{\mathcal{K}}\left[\left(\begin{array}{cc}-G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right) \mathbb{Y}(T)\right],\end{array}\right.\end{array}\right.$
where $\operatorname{Proj}_{\mathcal{K}}(\cdot)=\binom{\operatorname{Proj}_{\mathcal{K}}(\cdot)}{\operatorname{Proj}_{\mathbb{R}^{n}}(\cdot)}$. Now let $\rho^{*}=\operatorname{esssup}_{0 \leq s \leq T} \operatorname{esssup}_{\omega \in \Omega} \Lambda_{\max }\left(-\frac{1}{2}\left(A(s)+A(s)^{\top}\right)\right)$, where $\Lambda_{\max }(M)$ is the largest eigenvalue of the matrix $M$. Comparing (BFSDE-2') with (29), by the Proposition 4.2 in [15], we can check that the parameters of (H5) can be chosen as follows:

$$
\begin{aligned}
& \rho_{1}=\rho_{2}=\rho^{*}, k_{2}=k_{3}=k_{6}=k_{8}=k_{12}=0, k_{1}=\left\|\left(\begin{array}{cc}
Q_{1} & Q_{2} \\
Q_{2} & 0
\end{array}\right)\right\|, k_{5}=\sqrt{2}\|C\|, k_{7}=2\|C\| \\
& k_{9}=\sqrt{2}\left\|\left(\begin{array}{cc}
S_{1} & S_{2} \\
S_{2} & 0
\end{array}\right)\right\|, k_{4}=\left\|\left(\begin{array}{cc}
B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\
B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0
\end{array}\right)\right\|, k_{10}=\left\|\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\right\|, \\
& k_{11}=\left\|\left(\begin{array}{cc}
-G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)\right\|,
\end{aligned}
$$

where for $M(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{n \times n}\right),\|M(\cdot)\| \triangleq \underset{0 \leq s \leq T}{\operatorname{esssup}} \underset{\omega \in \Omega}{\operatorname{esssup}}\|M(s)\|$. By Theorem 5.11, we have
Theorem 5.2 Suppose that $\rho^{*}<-4\|C(\cdot)\|^{2}$. There exists a $\delta_{1}>0$, which depends on $\rho^{*}, k_{i}, i=5,7$, such that when $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{1}\right)$, there exists a unique adapted solution to (BFSDE- $2^{\prime}$ ).

Remark 5.1 By the definition of $\rho^{*}$, Theorem 5.2 establishes the existence and uniqueness of (BFSDE-2) under some condition on the matrix $A(\cdot)$.

Combining Corollary 4.5 and Theorem 5.2, we have the following result.
Theorem 5.3 Let ( $\mathbf{H} 1)-(\mathbf{H} 4)$ and $\left(\mathbf{P}_{1}\right)$ is convex. Suppose that $\rho^{*}<-4\|C(\cdot)\|^{2}$ and there exists a $\delta_{1}>0$ depending on $\rho^{*}, k_{i}, i=5,7$, such that $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{1}\right)$. Then $\left(\mathbf{P}_{1}\right)$ admits a unique optimal control given by (27) where $(\bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ is the unique solution of (BFSDE-2).

### 5.3 Wellposedness of (13)

In this subsection, we will give a direct result on wellposedness of (13) by Theorem 5.1 Let $\rho^{*}=$ $\operatorname{esssup}_{0 \leq s \leq T} \operatorname{esssup}_{\omega \in \Omega} \Lambda_{\max }\left(-\frac{1}{2}\left(A(s)+A(s)^{\top}\right)\right)$,

$$
\begin{aligned}
& \rho_{1}=\rho_{2}=\rho^{*}, k_{2}=k_{3}=k_{6}=k_{8}=k_{11}=k_{12}=0, k_{1}=\left\|Q_{2}\right\|, k_{5}=\|C\|, k_{7}=\sqrt{2}\|C\| \\
& k_{9}=\sqrt{2}\left\|S_{2}\right\|, k_{10}=\left\|H_{2}\right\|, k_{4}=\left\|B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top}\right\|
\end{aligned}
$$

Theorem 5.4 Suppose that $\rho^{*}<-2\|C(\cdot)\|^{2}$. There exists a $\delta_{1}>0$ depending on $\rho^{*}, k_{i}, i=5,7$, such that when $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{1}\right)$, there exists a unique adapted solution to (13).

### 5.4 Solvability of (BFSDE-3)

Now, we consider the solvability of (BFSDE-3) which is a standard fully-coupled BFSDEs but combining with the (KKT) qualification condition. Hence, it becomes non-standard BFSDEs with constraint on its terminal expectation via Lagrange variable $\lambda$ involved. In this sense, we may call it terminal-meanconstrained BFSDEs. To our knowledge, such class of BFSDEs has not been well studied and this sections aims some essential endeavor to it. To this end, we may first rewrite (BFSDE-3) as the following $2 n \times 2 n$-BFSDEs (with same notations to (BFSDE-2)):
$\left(\begin{array}{l}\text { BFSDE-3' }): \\ d \mathbb{Y}=-\left[\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right)^{\top} \mathbb{Y}-\left(\begin{array}{cc}Q_{1} & Q_{2} \\ Q_{2} & 0\end{array}\right) \mathbb{X}\right] d t-\left[\left(\begin{array}{cc}C & 0 \\ 0 & C\end{array}\right)^{\top} \mathbb{Y}-\left(\begin{array}{cc}S_{1} & S_{2} \\ S_{2} & 0\end{array}\right) \mathbb{Z}\right] d W(s), \\ d \mathbb{X}=\left[\left(\begin{array}{cc}B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\ B_{2} R_{22}^{2-1} B_{2}^{\top} & 0\end{array}\right) \mathbb{Y}+\left(\begin{array}{cc}A & 0 \\ 0 & A\end{array}\right) \mathbb{X}+\left(\begin{array}{cc}C & 0 \\ 0 & C\end{array}\right) \mathbb{Z}\right] d s+\mathbb{Z} d W(s), \\ \mathbb{Y}(0)=\left(\begin{array}{cc}H_{1} & H_{2} \\ H_{2} & 0\end{array}\right) \mathbb{X}(0), \quad \mathbb{X}(T)=\left(\begin{array}{cc}G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right)\left(-\mathbb{Y}(T)+\lambda\binom{\alpha}{0}\right), \\ \lambda\left(\beta-\mathbb{E}\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right)\left(-\mathbb{Y}(T)+\lambda\binom{\alpha}{0}\right)\right\rangle\right)=0, \quad \lambda \geq 0, \\ \beta-\mathbb{E}\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right)\left(-\mathbb{Y}(T)+\lambda\binom{\alpha}{0}\right)\right\rangle \leq 0 .\end{array}\right.$
By the first slackness condition of (KKT) system, there arise two cases with $\lambda=0$ or $\lambda=(\beta+$ $\left.\mathbb{E}\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right) \mathbb{Y}(T)\right\rangle\right)\left(\mathbb{E}\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle\right)^{-1}$. We have the following more detailed analysis along these two cases.

### 5.4.1 Multiplier $\lambda=0$

In this case, (BFSDE-3') takes the following form:

$$
\left\{\begin{array}{l}
d \mathbb{Y}=-\left[\left(\begin{array}{cc}
A & 0 \\
0 & A
\end{array}\right)^{\top} \mathbb{Y}-\left(\begin{array}{cc}
Q_{1} & Q_{2} \\
Q_{2} & 0
\end{array}\right) \mathbb{X}\right] d t-\left[\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top} \mathbb{Y}-\left(\begin{array}{cc}
S_{1} & S_{2} \\
S_{2} & 0
\end{array}\right) \mathbb{Z}\right] d W(t),  \tag{30}\\
d \mathbb{X}=\left[\left(\begin{array}{cc}
B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\
B_{2} R_{22}^{2}-1 B_{2}^{\top} & 0
\end{array}\right) \mathbb{Y}+\left(\begin{array}{cc}
A & 0 \\
0 & A
\end{array}\right) \mathbb{X}+\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) \mathbb{Z}\right] d s+\mathbb{Z} d W(s), \\
\mathbb{Y}(0)=\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \mathbb{X}(0), \quad \mathbb{X}(T)=-\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right) \mathbb{Y}(T), \\
\beta+\mathbb{E}\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right) \mathbb{Y}(T)\right\rangle \leq 0 . \quad \text { primal constraint in (KKT) }
\end{array}\right.
$$

We will use Riccati decoupling method to study the wellposedness of (30). Define $\widetilde{\mathbb{Y}}=\mathbb{Y}-\left(\begin{array}{cc}H_{1} & H_{2} \\ H_{2} & 0\end{array}\right) \mathbb{X}$, therefore, $\widetilde{\mathbb{Y}}(0)=0$ and

$$
\mathbb{X}(T)=-\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right) \mathbb{Y}(T)=-\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right) \widetilde{\mathbb{Y}}(T)-\left(\begin{array}{cc}
G_{1}^{-1} H_{1} & G_{1}^{-1} H_{2} \\
0 & 0
\end{array}\right) \mathbb{X}(T)
$$

If $\operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$, then the matrix $\left(\begin{array}{cc}I+G_{1}^{-1} H_{1} & G_{1}^{-1} H_{2} \\ 0 & I\end{array}\right)$ is invertible, and consequently,

$$
\mathbb{X}(T)=\widetilde{G} \widetilde{\mathbb{Y}}(T),
$$

where

$$
\widetilde{G}=-\left(\begin{array}{cc}
I+G_{1}^{-1} H_{1} & G_{1}^{-1} H_{2} \\
0 & I
\end{array}\right)^{-1}\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)=-\left(\begin{array}{cc}
\left(I+G_{1}^{-1} H_{1}\right)^{-1} G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right) .
$$

Therefore, if $\operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$, after some manipulations, we have

$$
\left\{\begin{align*}
d \widetilde{\mathbb{Y}} & =-[\widetilde{A} \widetilde{\mathbb{Y}}+\widetilde{B} \mathbb{X}+\widetilde{C} \mathbb{Z}] d t-\left[\widetilde{A}_{1} \widetilde{\mathbb{Y}}+\widetilde{B}_{1} \mathbb{X}+\widetilde{C}_{1} \mathbb{Z}\right] d W(t),  \tag{31}\\
d \mathbb{X} & =[\widetilde{A} \widetilde{\mathbb{Y}}+\widehat{B} \mathbb{X}+\widehat{C} \mathbb{Z}] d t+\mathbb{Z} d W(t) \\
\widetilde{\mathbb{Y}}(0) & =0, \quad \mathbb{X}(T)=\widetilde{G} \widetilde{\mathbb{Y}}(T)
\end{align*}\right.
$$

where

$$
\begin{align*}
& \widetilde{A}=\left(\begin{array}{cc}
A^{\top}+H_{1} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top}+H_{2} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & H_{1} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\
H_{2} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & A^{\top}+H_{2} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top}
\end{array}\right), \\
& \widetilde{B}=\left(\begin{array}{ll}
\widetilde{B}_{11} & \widetilde{B}_{12} \\
\widetilde{B}_{21} & \widetilde{B}_{22}
\end{array}\right), \\
& \widetilde{B}_{11}=-Q_{1}+H_{1} A+A^{\top} H_{1}+H_{1} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{1}+H_{2} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{1}+H_{1} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{2}, \\
& \widetilde{B}_{12}=-Q_{2}+H_{2} A+A^{\top} H_{2}+H_{1} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{2}+H_{2} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{2}, \\
& \widetilde{B}_{21}=-Q_{2}+H_{2} A+A^{\top} H_{2}+H_{2} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{1}+H_{2} B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{2},  \tag{32}\\
& \widetilde{B}_{22}=H_{2} B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{2}, \\
& \widetilde{C}=\left(\begin{array}{cc}
H_{1} C & H_{2} C \\
H_{2} C & 0
\end{array}\right), \quad \widetilde{A}_{1}=\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}, \quad \widetilde{B}_{1}=\left(\begin{array}{cc}
C^{\top} H_{1} & C^{\top} H_{2} \\
C^{\top} H_{2} & 0
\end{array}\right), \\
& \widetilde{C}_{1}=-\left(\begin{array}{cc}
S_{1}-H_{1} & S_{2}-H_{2} \\
S_{2}-H_{2} & 0
\end{array}\right), \quad \widehat{A}=\left(\begin{array}{cc}
B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\
B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0
\end{array}\right), \\
& \widehat{B}=\left(\begin{array}{cc}
A+B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{1}+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{2} & B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} H_{2} \\
B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{1} & A+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} H_{2}
\end{array}\right), \quad \widehat{C}=\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) .
\end{align*}
$$

Note that $\widehat{A}, \widetilde{B}$ are symmetric and $\widehat{B}=\widetilde{A}^{\top}, \quad \widehat{C}=\widetilde{A}_{1}^{\top}, \quad \widetilde{C}=\widetilde{B}_{1}^{\top}$.

Remark 5.2 Since $G_{1}^{-1}$ is symmetric, it follows from 44] that $\left(I+G_{1}^{-1} H_{1}\right)^{-1} G_{1}^{-1}$ is symmetric, i.e., $\widetilde{G}$ is symmetric.

Suppose the following linear relation holds true,

$$
\begin{equation*}
\mathbb{X}(s)=\widetilde{P}(s) \widetilde{\mathbb{Y}}(s)+\widetilde{p}(s), \quad s \in[0, T], \quad \text { a.s. } \tag{33}
\end{equation*}
$$

If $\operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0,(30)$ is solvable if the following stochastic Riccati equation and BSDE are solvable

$$
\left\{\begin{array}{l}
d \widetilde{P}=\left\{\widehat{A}+\widehat{B} \widetilde{P}+\widetilde{P} \widetilde{A}+\widetilde{P} \widetilde{B} \widetilde{P}+\widetilde{\Lambda}\left(\widetilde{A}_{1}+\widetilde{B}_{1} \widetilde{P}\right)+\left(\widehat{C}+\widetilde{P} \widetilde{C}+\widetilde{\Lambda} \widetilde{C}_{1}\right)\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1}\right.  \tag{34}\\
\\
\left.\quad\left[\widetilde{\Lambda}-\widetilde{P}\left(\widetilde{A}_{1}+\widetilde{B}_{1} \widetilde{P}\right)\right]\right\} d s+\widetilde{\Lambda} d W(s) \\
\widetilde{P}(T)=\widetilde{G} \\
\operatorname{det}\left[I+\widetilde{P} \widetilde{C}_{1}\right] \neq 0
\end{array}\right.
$$

and

$$
\left\{\begin{align*}
& d \widetilde{p}=\left\{\left[\widehat{B}+\widetilde{P} \widetilde{B}+\widetilde{\Lambda} \widetilde{B}_{1}-\left(\widehat{C}+\widetilde{P} \widetilde{C}+\widetilde{\Lambda} \widetilde{C}_{1}\right)\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{P} \widetilde{B}_{1}\right] \widetilde{p}\right.  \tag{35}\\
&\left.+\left(\widehat{C}+\widetilde{P} \widetilde{C}+\widetilde{\Lambda} \widetilde{C}_{1}\right)\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{q}\right\} d s+\widetilde{q} d W(s) \\
& \widetilde{p}(T)=0
\end{align*}\right.
$$

such that (KKT) in (30) is satisfied. It is easy to check that

$$
\begin{equation*}
Z=\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1}\left[\left(\widetilde{\Lambda}-\widetilde{P} \widetilde{A}_{1}-\widetilde{P} \widetilde{B}_{1} \widetilde{P}\right) \widetilde{\mathbb{Y}}-\widetilde{P} \widetilde{B}_{1} \widetilde{p}+\widetilde{q}\right] \tag{36}
\end{equation*}
$$

Next we introduce another assumption under which we will obtain some new form of (34) and (35),
(H6) $\operatorname{det}\left[S_{2}-H_{2}\right] \neq 0$.
Under (H6), we have $\operatorname{det}\left[\widetilde{C}_{1}\right] \neq 0$, hence

$$
\begin{aligned}
& \left(\widehat{C}+\widetilde{P} \widetilde{C}+\widetilde{\Lambda} \widetilde{C}_{1}\right)\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1}\left[\widetilde{\Lambda}-\widetilde{P}\left(\widetilde{A}_{1}+\widetilde{B}_{1} \widetilde{P}\right)\right] \\
& =\left(\widetilde{\Lambda}+\widehat{C} \widetilde{C}_{1}^{-1}+\widetilde{P} \widetilde{C} \widetilde{C}_{1}^{-1}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left[\widetilde{\Lambda}-\widetilde{P}\left(\widetilde{A}_{1}+\widetilde{B}_{1} \widetilde{P}\right)\right] \\
& =\left(\widetilde{\Lambda}+\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) \widetilde{C}_{1}^{-1}+\widetilde{P}\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) \widetilde{C}_{1}^{-1}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1} \\
& \left(\widetilde{\Lambda}-\widetilde{P}\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}-\widetilde{P}\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{P}\right) \\
& =\left(\widetilde{\Lambda}+\left(I+\widetilde{P}\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\right)\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) \widetilde{C}_{1}^{-1}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P}\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}\left(I+\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{P}\right)\right) \\
& =\left(\widetilde{\Lambda}-\left(I+\widetilde{P}\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\right)\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right) \widetilde{P}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P}\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}\left(I+\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{P}\right)\right) \\
& +\left(I+\widetilde{P}\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\right)\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)\left(\widetilde{\Lambda}-\widetilde{P}\left(\begin{array}{cc}
C & 0 \\
0 & C
\end{array}\right)^{\top}\left(I+\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{P}\right)\right) \\
& =(\widetilde{\Lambda}-(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{P})\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)\right)+(\widehat{C}+\widetilde{P} \widetilde{C})\left(\widetilde{\Lambda}-\widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)\right) \\
& =(\widetilde{\Lambda}-(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{P})\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)\right)+\widehat{C} \widetilde{\Lambda}+\widetilde{P} \widetilde{C} \widetilde{\Lambda}-(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right) \text {. }
\end{aligned}
$$

Therefore, (34) and (35) take the following forms:

$$
\left\{\begin{align*}
& d \widetilde{P}=\left\{\widehat{A}+\widehat{B} \widetilde{P}+\widetilde{P} \widehat{B}^{\top}+\widetilde{P} \widetilde{B} \widetilde{P}+\widetilde{\Lambda}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)+(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{\Lambda}-(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)\right. \\
&\left.\quad+(\widetilde{\Lambda}-(\widehat{C}+\widetilde{P} \widetilde{C}) \widetilde{P})\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P}\left(\widehat{C}^{\top}+\widetilde{C}^{\top} \widetilde{P}\right)\right)\right\} d s+\widetilde{\Lambda} d W(s) \\
& \widetilde{P}(T)=\widetilde{G}  \tag{37}\\
& \operatorname{det}\left[I+\widetilde{P} \widetilde{C}_{1}\right] \neq 0
\end{align*}\right.
$$

and

$$
\left\{\begin{align*}
d \widetilde{p}= & \left\{\left[\widehat{B}+\widetilde{P} \widetilde{B}+\widetilde{\Lambda} \widetilde{B}_{1}-\left(\widehat{C} \widetilde{C}_{1}^{-1}+\widetilde{P} \widetilde{C} \widetilde{C}_{1}^{-1}+\widetilde{\Lambda}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1} \widetilde{P}_{B}\right] \widetilde{p}\right.  \tag{38}\\
& \left.+\left(\widehat{C} \widetilde{C}_{1}^{-1}+\widetilde{P} \widetilde{C} \widetilde{C}_{1}^{-1}+\widetilde{\Lambda}\right)\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1} \widetilde{q}\right\} d s+\widetilde{q} d W(s) \\
\widetilde{p}(T)= & 0
\end{align*}\right.
$$

Finally, plugging (33) and (36) into (31), we have

$$
d \widetilde{\mathbb{Y}}=-[\widetilde{\mathbb{A}} \widetilde{\mathbb{Y}}+\widetilde{b}] d t-\left[\widetilde{\mathbb{A}_{1}} \widetilde{\mathbb{Y}}+\widetilde{\sigma}\right] d W(t), \quad \widetilde{\mathbb{Y}}(0)=0
$$

where

$$
\begin{gathered}
\widetilde{\mathbb{A}}=\widetilde{A}+\widetilde{B} \widetilde{P}+\widetilde{C}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P} \widetilde{A}_{1}-\widetilde{P} \widetilde{B}_{1} \widetilde{P}\right) \\
\widetilde{b}=\widetilde{B}+\widetilde{P} \widetilde{p}-\widetilde{C}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{P} \widetilde{B}_{1} \widetilde{p}+\widetilde{C}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{q} \\
\widetilde{\mathbb{A}}_{1}=\widetilde{A}_{1}+\widetilde{B}_{1} \widetilde{P}+\widetilde{C}_{1}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1}\left(\widetilde{\Lambda}-\widetilde{P} \widetilde{A}_{1}-\widetilde{P} \widetilde{B}_{1} \widetilde{P}\right) \\
\widetilde{\sigma}=\widetilde{B}_{1}+\widetilde{P} \widetilde{p}-\widetilde{C}_{1}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{P} \widetilde{B}_{1} \widetilde{p}+\widetilde{C}_{1}\left(I+\widetilde{P} \widetilde{C}_{1}\right)^{-1} \widetilde{q}
\end{gathered}
$$

Therefore,

$$
\widetilde{\mathbb{Y}}(t)=\Phi(t) \int_{0}^{t} \Phi(s)^{-1}\left[\widetilde{b}(s)-\widetilde{\mathbb{A}}_{1}(s) \widetilde{\sigma}(s)\right] d s+\Phi(t) \int_{0}^{t} \Phi(s)^{-1} \widetilde{\sigma}(s) d W(s), \quad t \in[0, T]
$$

where

$$
d \Phi(t)=\widetilde{\mathbb{A}}(t) \Phi(t) d t+\widetilde{\mathbb{A}}_{1} \Phi(t) d W(t), \quad \Phi(0)=I
$$

Hence,

$$
\mathbb{Y}(T)=\widetilde{\mathbb{Y}}(T)-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \mathbb{X}(T)=\widetilde{\mathbb{Y}}(T)-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G} \widetilde{\mathbb{Y}}(T)=\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \widetilde{\mathbb{Y}}(T)
$$

and the (KKT) condition becomes

$$
\begin{align*}
& \beta+\left\langle\binom{\alpha}{0}, \mathbb{E}\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \Phi(T) \int_{0}^{T} \Phi(s)^{-1}\left[\widetilde{b}(s)-\widetilde{\mathbb{A}}_{1}(s) \widetilde{\sigma}(s)\right] d s\right\rangle \\
& +\left\langle\binom{\alpha}{0}, \mathbb{E}\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \Phi(T) \int_{0}^{T} \Phi(s)^{-1} \widetilde{\sigma}(s) d W(s)\right\rangle  \tag{39}\\
= & \beta+\left\langle\binom{\alpha}{0}, \mathbb{E}\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \int_{0}^{T}[\widetilde{\mathbb{A}}(s) \mathbb{Y}(s)+\widetilde{b}(s)] d s\right\rangle \leq 0 .
\end{align*}
$$

Proposition 5.1 Under (H1)-(H4) and (H6), suppose $\operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$. If (37) and (38) admit solutions such that (39) hold, then terminal-mean-constrained BFSDEs (30) is solvable.

In case with deterministic coefficients, (39) takes the following form

$$
\beta+\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}
G_{1}^{-1} & 0 \\
0 & 0
\end{array}\right)\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \int_{0}^{T}[\widetilde{\mathbb{A}}(s) \mathbb{E} \widetilde{\mathbb{Y}}(s)+\widetilde{b}(s)] d s\right\rangle \leq 0
$$

Let the fundamental solution matrices of ordinary differential equation (ODE)

$$
d \widetilde{\varphi}=-\widetilde{\mathbb{A}} \widetilde{\varphi} d t, \quad \widetilde{\varphi}(0)=I
$$

be $\widetilde{\Phi}(t, 0)$. Then

$$
\mathbb{E} \widetilde{Y}(t)=-\widetilde{\Phi}(t, 0) \int_{0}^{t} \widetilde{\Phi}(s, 0) \widetilde{b}(s) d s
$$

Therefore, the condition (39) becomes

$$
\beta+\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}
G_{1}^{-1} & 0  \tag{40}\\
0 & 0
\end{array}\right)\left[I-\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right) \widetilde{G}\right] \int_{0}^{T}\left[-\widetilde{\mathbb{A}}(s) \widetilde{\Phi}(s, 0) \int_{0}^{s} \widetilde{\Phi}(r, 0) \widetilde{b}(r) d r+\widetilde{b}(s)\right] d s\right\rangle \leq 0
$$

Corollary 5.1 Under (H1)-(H4) and (H6), suppose $\operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$. If (37) and (38) admit solutions such that (40) hold, then terminal-mean-constrained BFSDEs (30) is solvable.

Remark 5.3 Besides the Riccati equation decoupling method, wellposedness of (30) can be established by some direct method. For example, under the conditions of Theorem 5.2, we know that there exists a unique adapted solution to (BFSDE-2'). Moreover, if $\beta+\mathbb{E}\left\langle\binom{\alpha}{0},\left(\begin{array}{cc}G_{1}^{-1} & 0 \\ 0 & 0\end{array}\right) \mathbb{Y}(T)\right\rangle \leq 0$, then (30) admits a unique solution.

### 5.4.2 Multiplier $\lambda>0$

In this section, we need to assume that the coefficients are deterministic, i.e., $A, B_{1}, B_{2}, C, G_{1}, Q_{1}, Q_{2}, S_{1}, S_{2}, R_{11}^{1}$ and $R_{22}^{2}$ are deterministic because the BFSDEs now takes some mean-field type form and its expectation is required to be computed. In this case, (BFSDE-3') take the following form:

$$
\left\{\begin{array}{l}
d g=-\left[A^{\top} g-Q_{1} \bar{X}-Q_{2} h\right] d s-\left[C^{\top} g-S_{1} \bar{Z}-S_{2} q\right] d W(s)  \tag{41}\\
d \bar{Y}=\left[-A^{\top} \bar{Y}+Q_{2} \bar{X}\right] d s+\left[-C^{\top} \bar{Y}+S_{2} \bar{Z}\right] d W(s), \\
d \bar{X}=\left[A \bar{X}+B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} g+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \bar{Y}+C \bar{Z}\right] d s+\bar{Z} d W(s), \\
d h=\left[A h+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} g+C q\right] d s+q d W(s), \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \\
\bar{X}(T)=-G_{1}^{-1} g(T)+G_{1}^{-1} \frac{\beta+\left\langle\alpha, G_{1}^{-1} \mathbb{E} g(T)\right\rangle}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle} \alpha, \quad h(T)=0 \\
\beta+\left\langle\alpha, G_{1}^{-1} \mathbb{E} g(T)\right\rangle>0 .
\end{array}\right.
$$

Note that (41) is solvable if and only if the following BFSDEs is solvable

$$
\left\{\begin{aligned}
d \mathbb{E} g=-\left[A^{\top} \mathbb{E} g-Q_{1} \mathbb{E} \bar{X}-Q_{2} \mathbb{E} h\right] d s, \\
d(g-\mathbb{E} g)=-\left[A^{\top} \mathbb{E}(g-\mathbb{E} g)-Q_{1}(\bar{X}-\mathbb{E} \bar{X})-Q_{2}(h-\mathbb{E} h)\right] d s \\
\quad-\left[C^{\top} \mathbb{E} g+C^{\top}(g-\mathbb{E} g)-S_{1} \bar{Z}-S_{2} q\right] d W(s), \\
d \mathbb{E} \bar{Y}=\left[-A^{\top} \mathbb{E} \bar{Y}+Q_{2} \mathbb{E} \bar{X}\right] d s, \\
d(\bar{Y}-\mathbb{E} \bar{Y})=\left[-A^{\top}(\bar{Y}-\mathbb{E} \bar{Y})+Q_{2}(\bar{X}-\mathbb{E} \bar{X})\right] d s+\left[-C^{\top} \mathbb{E} \bar{Y}-C^{\top}(\bar{Y}-\mathbb{E} \bar{Y})+S_{2} \bar{Z}\right] d W(s), \\
d \mathbb{E} \bar{X}=\left[B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} \mathbb{E} g+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \mathbb{E} \bar{Y}+A \mathbb{E} \bar{X}+C \mathbb{E} \bar{Z}\right] d s, \\
d(\bar{X}-\mathbb{E} \bar{X})=\left[B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top}(g-\mathbb{E} g)+B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top}(\bar{Y}-\mathbb{E} \bar{Y})+A(\bar{X}-\mathbb{E} \bar{X})+C \bar{Z}-C \mathbb{E} \bar{Z}\right] d s+\bar{Z} d W(s), \\
d \mathbb{E} h=\left[B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \mathbb{E} g+A \mathbb{E} h+C \mathbb{E} q\right] d s, \\
d(h-\mathbb{E} h)=\left[B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top}(g-\mathbb{E} g)+A(h-\mathbb{E} h)+C q-C \mathbb{E} q\right] d s+q d W(s), \\
\mathbb{E} g(0)=H_{1} \mathbb{E} \bar{X}(0)+H_{2} \mathbb{E} h(0), \quad g(0)-\mathbb{E} g(0)=H_{1}(\bar{X}(0)-\mathbb{E} \bar{X}(0))+H_{2}(h(0)-\mathbb{E} h(0)), \\
\mathbb{E} \bar{Y}(0)=H_{2} \mathbb{E} \bar{X}(0), \quad \bar{Y}(0)-\mathbb{E} \bar{Y}(0)=H_{2}(\bar{X}(0)-\mathbb{E} \bar{X}(0)) \\
\mathbb{E} \bar{X}(T)=-G_{1}^{-1} \mathbb{E} g(T)+\frac{G_{1}^{-1} \alpha \alpha^{\top} G_{1}^{-1}}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle} \mathbb{E} g(T)+\frac{G_{1}^{-1} \alpha \beta}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle}, \quad \bar{X}(T)-\mathbb{E} \bar{X}(T)=-G_{1}^{-1}(g(T)-\mathbb{E} g(T)), \\
\mathbb{E} h(T)=0, \quad h(T)-\mathbb{E} h(T)=0, \\
\beta+\left\langle\alpha, G_{1}^{-1} \mathbb{E} g(T)\right\rangle>0 .
\end{aligned}\right.
$$

Let $\check{Y}=\left(\mathbb{E} g^{\top},(g-\mathbb{E} g)^{\top}, \mathbb{E} \bar{Y}^{\top},(\bar{Y}-\mathbb{E} \bar{Y})^{\top}\right)^{\top}, \check{X}=\left(\mathbb{E} \bar{X}^{\top},(\bar{X}-\mathbb{E} \bar{X})^{\top}, \mathbb{E} h^{\top},(h-\mathbb{E} h)^{\top}\right)^{\top}$ and $\check{Z}=$ $\left(0, \bar{Z}^{\top}, 0, q^{\top}\right)^{\top}$, we have

$$
\left\{\begin{array}{l}
d \check{Y}=-[\check{A} \check{Y}+\check{B} \check{X}] d t-\left[\check{A}_{1} \check{Y}+\check{B}_{1} \check{Z}\right] d W \\
d \check{X}=\left[\check{A}_{2} \check{Y}+\check{B}_{2} \check{X}+\check{C}_{2} \check{Z}+\check{D}_{2} \mathbb{E} Z\right] d t+\check{Z} d W \\
\check{Y}(0)=\check{H} \check{X}(0), \quad \check{X}(T)=\check{G} \check{Y}(T)+\check{f} \\
\beta+\left\langle\alpha,\left(G_{1}^{-1} 0000\right) \mathbb{E} \check{Y}(T)\right\rangle>0
\end{array}\right.
$$

where

$$
\begin{align*}
& \check{A}=\left(\begin{array}{cccc}
A^{\top} & 0 & 0 & 0 \\
0 & A^{\top} & 0 & 0 \\
0 & 0 & A^{\top} & 0 \\
0 & 0 & 0 & A^{\top}
\end{array}\right), \quad \check{B}=\left(\begin{array}{cccc}
-Q_{1} & 0 & -Q_{2} & 0 \\
0 & -Q_{1} & 0 & -Q_{2} \\
-Q_{2} & 0 & 0 & 0 \\
0 & -Q_{2} & 0 & 0
\end{array}\right), \quad \check{A}_{1}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
C^{\top} & C^{\top} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & C^{\top} & C^{\top}
\end{array}\right), \\
& \check{B}_{1}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & -S_{1} & 0 & -S_{2} \\
0 & 0 & 0 & 0 \\
0 & -S_{2} & 0 & 0
\end{array}\right), \quad \check{A}_{2}=\left(\begin{array}{cccc}
B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & 0 & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0 \\
0 & B_{1}\left(R_{11}^{1}\right)^{-1} B_{1}^{\top} & 0 & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} \\
B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0 & 0 & 0 \\
0 & B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} & 0 & 0
\end{array}\right), \\
& \check{B}_{2}=\left(\begin{array}{cccc}
A & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & A
\end{array}\right), \quad \check{C}_{2}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & C & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & C
\end{array}\right), \quad \check{D}_{2}=\left(\begin{array}{cccc}
0 & C & 0 & 0 \\
0 & -C & 0 & 0 \\
0 & 0 & 0 & C \\
0 & 0 & 0 & -C
\end{array}\right), \\
& \check{H}=\left(\begin{array}{cccc}
H_{1} & 0 & H_{2} & 0 \\
0 & H_{1} & 0 & H_{2} \\
H_{2} & 0 & 0 & 0 \\
0 & H_{2} & 0 & 0
\end{array}\right), \quad \check{G}=\left(\begin{array}{cccc}
-G_{1}^{-1}+\frac{G_{1}^{-1} \alpha \alpha^{\top} G_{1}^{-1}}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle} & 0 & 0 & 0 \\
0 & -G_{1}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right), \quad \check{f}=\left(\begin{array}{c}
\frac{G_{1}^{-1} \alpha \beta}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle} \\
0 \\
0 \\
0
\end{array}\right) . \tag{42}
\end{align*}
$$

Let $\check{Y}=\check{Y}-\check{H} \check{X}$, then $\check{Y}(0)=0$ and $(I-\check{G} \check{H}) \check{X}(T)=\check{G} \check{Y}(T)+\check{f}$. Suppose $\operatorname{det}\left[I+\left(G_{1}^{-1}-\right.\right.$ $\left.\left.\frac{G_{1}^{-1} \alpha \alpha^{\top} G_{1}^{-1}}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle}\right) H_{1}\right] \neq 0, \operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$, then $\operatorname{det}[I-\check{G} \check{H}] \neq 0$. Hence

$$
\left\{\begin{array}{l}
d \check{Y}=-[\check{\mathbb{A}} \check{Y}+\check{\mathbb{B}} \check{X}+\check{\mathbb{C}} \check{Z}+\check{\mathbb{D}} \mathbb{E} \check{Z}] d t-\left[\check{\mathbb{A}}_{1} \check{\mathbb{Y}}+\check{\mathbb{B}}_{1} \check{X}+\check{\mathbb{C}}_{1} \check{Z}\right] d W, \\
d \check{X}=\left[\check{\mathbb{A}} 2 \check{\mathbb{Y}}+\check{\mathbb{B}}_{2} \check{X}+\check{\mathbb{C}}_{2} \check{Z}+\check{\mathbb{D}}_{2} \mathbb{E} \check{Z}\right] d t+\check{Z} d W, \\
\check{\mathbb{Y}}(0)=0, \quad \check{X}(T)=(I-\check{G} \check{H})^{-1} \check{G} \check{Y}(T)+(I-\check{G} \check{H})^{-1} \check{f}, \\
\beta+\left\langle\alpha,\left(G_{1}^{-1} 000\right)(\mathbb{E} \check{Y}(T)+\check{H} \mathbb{E} \check{X}(T)\rangle>0,\right.
\end{array}\right.
$$

where

$$
\begin{align*}
& \check{\mathbb{A}}=\check{A}+\check{H} \check{A}_{2}, \quad \check{\mathbb{B}}=\check{A} \check{H}+\check{B}+\check{H} \check{A}_{2} \check{H}+\check{H} \check{B}_{2}, \quad \check{\mathbb{C}}=\check{H} \check{C}_{2}, \quad \check{\mathbb{D}}=\check{H} \check{D}_{2}, \quad \check{\mathbb{A}}_{1}=\check{A}_{1}, \quad \check{\mathbb{B}}_{1}=\check{A}_{1} \check{H}, \\
& \check{\mathbb{C}}_{1}=\check{B}_{1}+\check{H}, \quad \check{\mathbb{A}}_{2}=\check{A}_{2}, \quad \check{\mathbb{B}}_{2}=\check{A}_{2} \check{H}+\check{B}_{2}, \quad \check{\mathbb{C}}_{2}=\check{C}_{2}, \quad \check{\mathbb{D}}_{2}=\check{D}_{2} . \tag{43}
\end{align*}
$$

Suppose $\check{X}=\check{P} \check{Y}+\check{p}$, applying Itô's formula, we have
$d \check{X}=[-\check{P} \check{\mathbb{A}} \check{Y}-\check{P} \check{\mathbb{B}} \check{P} \check{\mathbb{Y}}-\check{P} \check{\mathbb{B}} \check{p}-\check{P} \check{\mathbb{C}} \check{Z}-\check{P} \check{\mathbb{D}} \mathbb{E} \check{Z}] d t+\left[-\check{P} \check{\mathbb{A}}_{1} \check{\mathbb{Y}}-\check{P} \check{\mathbb{B}}_{1} \check{P} \check{\mathbb{Y}}-\check{P} \check{\mathbb{B}}_{1} \check{p}-\check{P} \check{\mathbb{C}}_{1} \check{Z}\right] d W+(d \check{P}) \check{\mathbb{Y}}+d \check{p}$.
Comparing the coefficients of the diffusion term, we have

$$
-\check{P} \check{\mathbb{A}}_{1} \check{\mathbb{Y}}-\check{P} \check{\mathbb{B}}_{1} \check{P} \check{\mathbb{Y}}-\check{P} \check{\mathbb{B}}_{1} \check{p}-\check{P} \check{\mathbb{C}}_{1} \check{Z}=\check{Z}
$$

If $\operatorname{det}\left[I+\check{P} \check{\mathbb{C}}_{1}\right] \neq 0$,

$$
\mathbb{E} \check{Z}=-\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1}\left(\check{P} \check{\mathbb{A}}_{1}+\check{P} \check{\mathbb{B}}_{1} \check{P}\right) \mathbb{E} \check{\mathbb{Y}}-\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1} \check{P} \check{\mathbb{B}}_{1} \check{p}
$$

By taking expectation and comparing the coefficients of the drift term, we have the following Riccati equation

$$
\left\{\begin{array}{l}
\dot{P}-\check{P} \check{\mathbb{A}}-\check{P} \check{\mathbb{B}} \check{P}+\left(\check{P} \check{\mathbb{C}}+\check{P} \check{\mathscr{D}}+\check{\mathbb{C}}_{2}+\check{\mathbb{D}}_{2}\right)\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1}\left(\check{P} \check{\mathbb{A}}_{1}+\check{P} \check{\mathbb{B}}_{1} \check{P}\right)-\check{\mathbb{A}}_{2}-\check{\mathbb{B}}_{2} \check{P}=0  \tag{44}\\
\check{P}(T)=(I-\check{G} \check{H})^{-1} \check{G}, \\
\operatorname{det}\left[I+\check{P} \check{\mathbb{C}}_{1}\right] \neq 0
\end{array}\right.
$$

and the following backward ODE

$$
\left\{\begin{array}{l}
\dot{p}-\check{P} \check{\mathbb{B}} \check{p}+\left(\check{P} \check{\mathbb{C}}+\check{P} \check{\mathbb{D}}+\check{\mathbb{C}}_{2}+\check{\mathbb{D}}_{2}\right)\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1} \check{P} \check{\mathbb{B}}_{1} \check{p}-\check{\mathbb{B}}_{2} \check{p}=0,  \tag{45}\\
\check{p}(T)=(I-\check{G} \check{H})^{-1} \check{f}
\end{array}\right.
$$

Moreover, we have

$$
d \mathbb{E} \check{Y}=[\mathbf{A} \mathbb{E} \check{Y}+\mathbf{b}] d t, \quad \mathbb{E} \check{Y}(0)=0
$$

where

$$
\mathbf{A}=-\check{\mathbb{A}}-\check{\mathbb{B}} \check{P}+(\check{\mathbb{C}}+\check{\mathbb{D}})\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1}\left(\check{P} \check{\mathbb{A}}_{1}+\check{P} \check{\mathbb{B}}_{1} \check{P}\right), \quad \mathbf{b}=-\check{\mathbb{B}} \check{p}+(\check{\mathbb{C}}+\check{\mathbb{D}})\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1} \check{P} \check{\mathbb{B}}_{1} \check{p}
$$

Let the fundamental solution matrices of ODE

$$
d \check{\varphi}=\mathbf{A} \check{\varphi} d t, \quad \check{\varphi}(0)=I
$$

be $\check{\Phi}(t, 0)$. Then

$$
\mathbb{E} \check{Y}(t)=\check{\Phi}(t, 0) \int_{0}^{t} \check{\Phi}(s, 0) \mathbf{b}(s) d s
$$

Hence,

$$
\mathbb{E} \check{Y}(t)=(I+\check{H} \check{P}) \check{\Phi}(t, 0) \int_{0}^{t} \check{\Phi}(s, 0) \mathbf{b}(s) d s+\check{H} \check{p}(t)
$$

Therefore, the (KKT) condition becomes

$$
\begin{equation*}
\beta+\left\langle\alpha,\left(G_{1}^{-1} 0000\right)(I+\check{H} \check{P}) \check{\Phi}(T, 0) \int_{0}^{T} \check{\Phi}(s, 0) \mathbf{b}(s) d s\right\rangle+\left\langle\alpha,\left(G_{1}^{-1} 0000\right) \check{H}(I-\check{G} \check{H})^{-1} \check{f}\right\rangle>0 \tag{46}
\end{equation*}
$$

Proposition 5.2 Under (H1)-(H4), suppose $\operatorname{det}\left[I+\left(G_{1}^{-1}-\frac{G_{1}^{-1} \alpha \alpha^{\top} G_{1}^{-1}}{\left\langle\alpha, G_{1}^{-1} \alpha\right\rangle}\right) H_{1}\right] \neq 0, \operatorname{det}\left[I+G_{1}^{-1} H_{1}\right] \neq 0$. If (44) and (45) admit solutions such that (46) hold, then (41) is solvable.

Remark 5.4 Now let

$$
\begin{aligned}
& \rho_{1}=\operatorname{esssup}_{0 \leq s \leq T} \operatorname{esssup}_{\omega \in \Omega} \Lambda_{\max }\left(-\frac{1}{2}\left(\check{\mathbb{A}}(s)+\check{\mathbb{A}}(s)^{\top}\right)\right), \\
& \rho_{2}=\operatorname{esssup}_{0 \leq s \leq T} \operatorname{esssup}_{\omega \in \Omega} \Lambda_{\max }\left(-\frac{1}{2}\left(\check{\mathbb{B}}_{2}(s)+\check{\mathbb{B}}_{2}(s)^{\top}\right)\right) \\
& k_{10}=k_{12}=0, k_{1}=\|\check{\mathbb{B}}\|, k_{2}=\|\check{\mathbb{C}}\|, k_{3}=\|\check{\mathbb{D}}\|, k_{4}=\left\|\check{\mathbb{A}}_{2}\right\|, k_{5}=\left\|\check{\mathbb{C}}_{2}\right\|, k_{6}=\left\|\check{\mathbb{D}}_{2}\right\|, \\
& k_{7}=\sqrt{3}\left\|\check{\mathbb{A}}_{1}\right\|, k_{8}=\sqrt{3}\left\|\check{\mathbb{B}}_{1}\right\|, k_{9}=\sqrt{3}\left\|\check{\mathbb{C}}_{1}\right\|, k_{11}=\left\|(I-\check{G} \check{H})^{-1} \check{G}\right\| .
\end{aligned}
$$

If $2\left(\rho_{1}+\rho_{2}\right)<-2\left\|\check{\mathbb{C}}_{2}\right\|^{2}-2\left\|\check{\mathbb{D}}_{2}\right\|^{2}-3\left\|\check{\mathbb{A}}_{1}\right\|^{2}$, there exists a $\delta_{2}>0$, which depends on $\rho_{1}, \rho_{2}, k_{i}, i=5,6,7$, and is independent of $T$, such that when $k_{i} \in\left[0, \delta_{1}\right), i=1,2,3,4,8,9$, (??) admits a unique adapted solution. Moreover, if $\beta+\left\langle\alpha,\left(G_{1}^{-1} 000\right)(\mathbb{E} \check{Y}(T)+\check{H} \mathbb{E} \check{X}(T)\rangle>0\right.$, then (41) admits a unique solution.

### 5.4.3 Solvability of (5.9) and (5.16)

In Section 5.4.1 and Section 5.4.2, we have discussed the solvability of (BFSDE-3) through Riccati equations (37) and (44). Note that (37) and (44) are not standard Riccati equations and the general solvability remain widely open. We will present the solvability for some special but nontrivial cases. Suppose the coefficients are deterministic and $C=0$, in this case, (37) and (44) reduce to

$$
\begin{equation*}
\dot{\widetilde{P}}-\widehat{A}-\widehat{B} \widetilde{P}-\widetilde{P} \widehat{B}^{\top}-\widetilde{P} \widetilde{B} \widetilde{P}=0, \quad \widetilde{P}(T)=\widetilde{G} \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\mathscr{P}}-\check{P} \check{\mathbb{A}}-\check{P} \check{\mathbb{B}} \check{P}-\check{\mathbb{A}}_{2}-\check{\mathbb{B}}_{2} \check{P}=0, \quad \check{P}(T)=(I-\check{G} \check{H})^{-1} \check{G} \tag{48}
\end{equation*}
$$

Proposition 5.3 For any $s \in[0, T]$, let $\Psi_{1}(\cdot, s)$ and $\Psi_{2}(\cdot, s)$ be the solutions of the following ODEs:

$$
\frac{d}{d t} \Psi_{1}(t, s)=\widehat{\mathbf{A}}_{1}(t) \Psi_{1}(t, s), \quad t \in[s, T], \quad \Psi_{1}(s, s)=I
$$

and

$$
\frac{d}{d t} \Psi_{2}(t, s)=\widehat{\mathbf{A}}_{2}(t) \Psi_{2}(t, s), \quad t \in[s, T], \quad \Psi_{2}(s, s)=I
$$

respectively, where

$$
\widehat{\mathbf{A}}_{1}(\cdot)=\left(\begin{array}{cc}
-\widehat{B}^{\top} & -\widetilde{B} \\
\widehat{A} & \widehat{B}
\end{array}\right), \quad \widehat{\mathbf{A}}_{2}(\cdot)=\left(\begin{array}{cc}
-\check{\mathbb{A}} & -\check{\mathbb{B}} \\
\widetilde{\mathbb{A}}_{2} & \widetilde{\mathbb{B}}_{2}
\end{array}\right) .
$$

Suppose

$$
\begin{aligned}
& {\left[\left(\begin{array}{ll}
0 & I
\end{array}\right) \Psi_{1}(T, t)\binom{0}{I}\right]^{-1} \in L^{1}\left(0, T ; \mathbb{R}^{2 n \times 2 n}\right)} \\
& {\left[\left(\begin{array}{ll}
0 & I
\end{array}\right) \Psi_{2}(T, t)\binom{0}{I}\right]^{-1} \in L^{1}\left(0, T ; \mathbb{R}^{4 n \times 4 n}\right)}
\end{aligned}
$$

Then Riccati equation (47) and (48) admit unique solutions $\widetilde{P}(\cdot)$ and $\check{P}(\cdot)$, which are given by

$$
\widetilde{P}(t)=-\left[\left(\begin{array}{cc}
0 & I
\end{array}\right) \Psi_{1}(T, t)\binom{0}{I}\right]^{-1}\left(\begin{array}{cc}
0 & I \tag{49}
\end{array}\right) \Psi_{1}(T, t)\binom{I}{0}, \quad t \in[0, T]
$$

and

$$
\check{P}(t)=-\left[\left(\begin{array}{ll}
0 & I
\end{array}\right) \Psi_{2}(T, t)\binom{0}{I}\right]^{-1}\left(\begin{array}{ll}
0 & I \tag{50}
\end{array}\right) \Psi_{2}(T, t)\binom{I}{0}, \quad t \in[0, T]
$$

respectively.

Remark 5.5 In general, (44) is asymmetric matric Riccati equation hence its solvability is more challenging than that of (37). For example, if $H_{1}=H_{2}=Q_{1}=Q_{2}=0$, (37) reduces to

$$
\left\{\begin{aligned}
d \widetilde{P}= & \left\{\widehat{A}+\widehat{B} \widetilde{P}+\widetilde{P} \widehat{B}^{\top}+\widetilde{\Lambda} \widehat{C}^{\top}+\widehat{C} \widetilde{\Lambda}-\widehat{C} \widetilde{P} \widehat{C}^{\top}\right. \\
& \left.+(\widetilde{\Lambda}-\widehat{C} \widetilde{P})\left(\widetilde{C}_{1}^{-1}+\widetilde{P}\right)^{-1}\left(\widetilde{\Lambda}-\widehat{C}^{\top}\right)\right\} d s+\widetilde{\Lambda} d W(s) \\
\widetilde{P}(T)= & \widetilde{G}, \quad \operatorname{det}\left[I+\widetilde{P} \widetilde{C}_{1}\right] \neq 0
\end{aligned}\right.
$$

which is the type of Riccati equation studied in [42]. For this kind of Riccati equations, Please refer Section 5.5 for more information.

### 5.5 Solvability of Riccati equations

In this subsection, we will give the general solvability of (SRE-1) and (SRE-2). For $a, c \in L_{\mathbb{F}}^{\infty}\left([0, T] ; \mathbb{R}^{n \times n}\right)$, $b, d \in L_{\mathbb{F}}^{\infty}\left([0, T] ; \mathbb{R}^{n \times k}\right), q \in L_{\mathbb{F}}^{\infty}\left([0, T] ; \mathbb{S}^{n}\right), s \in L_{\mathbb{F}}^{\infty}\left([0, T] ; \mathbb{S}^{m}\right), M \in L_{\mathcal{F}_{T}}^{\infty}\left(\Omega ; \mathbb{R}^{n \times n}\right)$, consider the following Riccati equation

$$
\left\{\begin{array}{l}
d P=-\left\{a^{\top} P+P a+q-[P b+K d]\left[s+d^{\top} P d\right]^{-1}[P b+K d]^{\top}\right\} d t+K d W  \tag{51}\\
P(T)=M
\end{array}\right.
$$

If $q(\cdot) \geq 0, M \geq 0, s(\cdot) \gg 0$, it follows from 42, Theorem 5.3] that (51) admits a unique solution $(P, K) \in L_{\mathbb{F}}^{\infty}\left([0, T] ; \mathbb{S}^{n}\right) \times L_{\mathbb{F}}^{2}\left([0, T] ; \mathbb{S}^{n}\right)$ such that $P(\cdot) \geq 0$. Let

$$
k=n+m, d=\left(\begin{array}{ll}
I & 0
\end{array}\right)_{n \times(n+m)}, b=\left(\begin{array}{ll}
C & B_{2}
\end{array}\right)_{n \times(n+m)}, s=\left(\begin{array}{cc}
S_{2} & 0 \\
0 & R_{22}^{2}
\end{array}\right)_{(n+m) \times(n+m)}
$$

we have

$$
\left.\left.\begin{array}{rl} 
& (P b+K d)\left(s+d^{\top} P d\right)^{-1}(P b+K d)^{\top} \\
= & \left(\begin{array}{ll}
P C+K & P B_{2}
\end{array}\right)\left(\begin{array}{cc}
\left(P+S_{2}\right)^{-1} & 0 \\
0 & \left(R_{22}^{2}\right)^{-1}
\end{array}\right)(P C+K \\
P B_{2}
\end{array}\right)^{\top}\right)
$$

Therefore, we have the following result.
Proposition 5.4 If $Q_{2}(\cdot) \geq 0, M \geq 0, S_{2}(\cdot) \gg 0$ and $R_{22}^{2}(\cdot) \gg 0$, then (SRE-1) admits a unique solution $(P(\cdot), K(\cdot)) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}_{+}^{n}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{S}^{n}\right)$.
Furthermore, for (SRE-1) with scalar value, i.e., $n=m_{1}=m_{2}=1$, we have a better result as follows.
Proposition 5.5 Let $S_{2}(\cdot) \geq 0$ and $Q_{2}(\cdot) \geq 0$, then Riccati equation (SRE-1) admits a unique solution $(P(\cdot), \Lambda(\cdot)) \in L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$.

Proof For simplicity, we only consider the case $S_{2}(\cdot)=0$ since the proof of $S_{2}(\cdot)>0$ is similar. Consider the following equation:

$$
\begin{equation*}
d y=-\left[\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}+\left(C^{2}-2 A\right) y-Q_{2}(s) y^{2}+2 C z\right] d s+z d W(s), \quad y(T)=M^{-1} \tag{52}
\end{equation*}
$$

We will show that (52) admits a unique solution $(y(s), z(s)) \in L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$. First we will prove the uniqueness. Let $(\check{y}(s), \check{z}(s))$ and $(\widetilde{y}(s), \widetilde{z}(s))$ be two solutions of (52) such that $\check{z} \cdot W \triangleq \int_{0}^{\dot{z}} \check{z} d W(s)$ and $\widetilde{z} \cdot W$ are bounded-mean-oscillation (BMO) martingles (see [16]). Set $\hat{y}=\check{y}-\widetilde{y}, \hat{z}=\check{z}-\widetilde{z}$. Then

$$
d \hat{y}=\left[Q_{2}(\check{y}+\widetilde{y}) \hat{y}+\left(2 A-C^{2}\right) \hat{y}-2 C \hat{z}\right] d s+\hat{z} d W, \quad \hat{y}(T)=0
$$

Applying Itô's formula to $|\hat{y}|^{2}$ and taking conditional expectation, we deduce that there exists a constant $k>0$ such that

$$
\begin{aligned}
|\hat{y}(s)|^{2}+\mathbb{E}_{s} \int_{s}^{T}|\hat{z}(r)|^{2} d r & =\mathbb{E}\left[\int_{s}^{T}\left(-2 Q_{2}(\check{y}+\widetilde{y}) \hat{y}^{2}-(2 A-C) \hat{y}^{2}+4 C \hat{y} \hat{z}\right) d r \mid \mathcal{F}_{s}\right] \\
& \leq k \mathbb{E}\left[\int_{s}^{T}|\hat{y}|^{2} d r \mid \mathcal{F}_{s}\right]+\frac{1}{2} \mathbb{E}\left[\int_{s}^{T}|\hat{z}|^{2} d r \mid \mathcal{F}_{s}\right]
\end{aligned}
$$

Therefore,

$$
\check{y}(s)=\widetilde{y}(s), \quad \check{z}(s)=\widetilde{z}(s), \quad \text { a.e. } s \in[0, T], \mathbb{P}-\text { a.s. }
$$

Hence, BSDE (52) admits at most one solution in $L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$.
Let us now prove the existence. For $h(\cdot) \in L_{\mathbb{F}}^{\infty}([0, T] ; \mathbb{R})$, define $\|h(\cdot)\|_{\infty}=\underset{0<s<T}{\operatorname{esssup}} \underset{\omega \in \Omega}{\operatorname{esssup}}|h(s)|$. First, introduce the following equation:

$$
\begin{equation*}
d \bar{y}(s)=-\left[\left\|\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}\right\|_{\infty}+\left\|C^{2}-2 A\right\|_{\infty} \bar{y}+2 C \bar{z}\right] d s+\bar{z} d W, \quad \bar{y}(T)=M^{-1} . \tag{53}
\end{equation*}
$$

BSDE (53) is a standard BSDE with Lipschitz continuous generator, therefore there exists a unique solution $(\bar{y}, \bar{z}) \in L_{\mathbb{F}}^{2}(\Omega ; C([t, T] ; \mathbb{R})) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$ and $\bar{z} \cdot W$ is a BMO martingale. Rewrite BSDE (53) as

$$
d \bar{y}(s)=-\left[\left\|\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}\right\|_{\infty}+\left\|C^{2}-2 A\right\|_{\infty} \bar{y}\right] d s+\bar{z}(d W-2 C d s), \quad \bar{y}(T)=M^{-1} .
$$

Note that $2 C(s) \cdot W$ is a BMO martingale, there exists a new probability measure $\widetilde{\mathbb{P}}$ such that $W_{s}^{\widetilde{\mathbb{P}}} \triangleq$ $W_{s}-\int_{0}^{s} 2 C(s) d s$ is a Brownian motion under $\widetilde{\mathbb{P}}$. Therefore,

$$
\bar{y}(s)=\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{\left\|C^{2}-2 A\right\|_{\infty}(T-s)}+\left\|\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}\right\|_{\infty} \int_{s}^{T} e^{\left\|C^{2}-2 A\right\|_{\infty}(s-v)} d v \mid \mathcal{F}_{s}\right],
$$

from which we deduce that $\bar{y}(s) \leq c_{1}$ where $c_{1}=e^{\left\|C^{2}-2 A\right\|_{\infty} T}+\left\|\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}\right\|_{\infty} T e^{\left\|C^{2}-2 A\right\|_{\infty} T}$. Next, introduce the following BSDE:

$$
\begin{equation*}
d \underline{y}(s)=-\left[-\left\|C^{2}-2 A\right\|_{\infty} \underline{y}(s)-c_{1} Q_{2} \underline{y}(s)+2 C \underline{z}(s)\right] d s+\underline{z}(s) d W(s), \quad \underline{y}(T)=M^{-1} . \tag{54}
\end{equation*}
$$

BSDE (544) is a standard BSDE with Lipschitz continuous generator, therefore there exists a unique solution $(\underline{y}, \underline{z}) \in L_{\mathbb{F}}^{2}(\Omega ; C([t, T] ; \mathbb{R})) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$ and $\underline{z} \cdot W$ is a BMO martingale. Rewrite BSDE (54]) as

$$
d \underline{y}(s)=-\left[-\left\|C^{2}-2 A\right\|_{\infty} \underline{y}(s)-c_{1} Q_{2} \underline{y}(s)\right] d s+\underline{z}(d W-2 C d s), \quad \underline{y}(T)=M^{-1} .
$$

Therefore, $\underline{y}(s)=\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-2\left\|C^{2}-2 A\right\|_{\infty}(T-s)-c_{1} Q_{2}(T-s)} \mid \mathcal{F}_{s}\right]$, from which we deduce that $\underline{y}(s) \geq c_{2}$, where $c_{2}=$ $e^{-2\left\|C^{2}-2 A\right\| \infty T-c_{1} Q_{2} T}$. Moreover, by comparison theorem for BSDE with Lipschitz continuous generator, for $s \in[0, T]$ we have $c_{2} \leq \underline{y}(s) \leq \bar{y}(s) \leq c_{1}, \mathbb{P}-a . s$. Define $\Theta_{c_{1}, c_{2}}(y) \triangleq c_{1} I\left\{y<c_{1}\right\}+p I\left\{c_{1} \leq y \leq\right.$ $\left.c_{2}\right\}+c_{2} I\left\{y>c_{2}\right\}$, and introduce the following BSDE

$$
d y=-\left[\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}+\left(C^{2}-2 A\right) y-Q_{2} \Theta_{c_{1}, c_{2}}(y) y+2 C z\right] d s+z d W(s), \quad y(T)=M^{-1} .
$$

The above BSDE is a standard quadratic BSDE and by [24. Theorem 2.3], it admits at most one solution $\left(y^{c_{1}, c_{2}}(s), z^{c_{1}, c_{2}}(s)\right) \in L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$. Furthermore, let

$$
\left\{\begin{array}{l}
f_{1}(y, z)=\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}+\left(C^{2}-2 A\right) y-Q_{2} \Theta_{c_{1}, c_{2}}(y) y+2 C z, \\
f_{2}(y, z)=\left\|\left(B_{2}\right)^{2}\left(R_{22}^{2}\right)^{-1}\right\|_{\infty}+\left\|C^{2}-2 A\right\|_{\infty} y+2 C z, \\
f_{3}(y, z)=-\left\|C^{2}-2 A\right\|_{\infty} y-c_{1} Q_{2} y+2 C z .
\end{array}\right.
$$

It is easy to check that there exist positive constants $k_{1}, k_{2}, k_{3}$ such that

$$
\left|f_{1}(y, z)\right| \leq k_{1}|y|+k_{2} z^{2}+k_{3}, \quad \frac{\partial f_{1}}{\partial z}=2 C, \quad \frac{\partial f_{1}}{\partial y} \leq C^{2}-2 A-Q_{2} c_{2}, \quad \mathbb{P}-a . s .
$$

Moreover, we have $\forall s \in[0, T], f_{1}(\bar{y}(s), \bar{z}(s)) \leq f_{2}(\bar{y}(s), \bar{z}(s)), f_{1}(\underline{y}(s), \underline{z}(s)) \geq f_{3}(\underline{y}(s), \underline{z}(s)), \mathbb{P}-$ a.s. Hence, it follows from [24, Theorem 2.6] that $\forall s \in[0, T], \underline{y}(s) \leq y(s) \leq \bar{y}(s), \mathbb{P}$ - a.s. Therefore, (52) admits a solution $(y(s), z(s)) \in L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$ and there exist two positive constants $c_{1}, c_{2}$ such that $\forall s \in[0, T], c_{2} \leq y(s) \leq c_{1}, \mathbb{P}-a . s$. Let $P(s)=y^{-1}(s), K(s)=-z(s) y^{-2}(s)$, we have

$$
d P=-\left[Q_{2}+2 A P-B_{2}^{2}\left(R_{22}^{2}\right)^{-1} P^{2}-(P C+K)^{2} P^{-1}\right] d s+K d W(s), \quad P(T)=M,
$$

i.e., (SRE-1) admits a solution $(P(s), K(s)) \in L_{\mathbb{F}}^{\infty}(0, T ; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T ; \mathbb{R})$. Moreover, the uniqueness of solution of (SRE-1) follows from that of (52).

For (SRE-2), by [42, Theorem 5.3] again, we have the following result.
Proposition 5.6 Let $Q_{1}(\cdot) \geq 0, G_{1} \geq 0, S_{1}(\cdot) \gg 0, R_{11}^{1}(\cdot) \gg 0$, then Riccati equation (SRE-2) admits a unique solution $\left(P_{L}(\cdot), \Lambda_{L}(\cdot)\right) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{S}_{+}^{n}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{S}^{n}\right)$.
Remark 5.6 The wellposedness of (SRE-1) and (SRE-2) are established under some positive definite assumptions. For the indefinite case, please refer [?] for more information.

## 6 Application

To simplify presentation, we consider a financial market with only one (risk-free) bond and one (risky) stock. Their prices $P_{0}(\cdot), P_{1}(\cdot)$ evolve respectively:

$$
\begin{equation*}
d P_{0}(s)=r(s) P_{0}(s) d s, \quad d P_{1}(s)=P_{1}(s)[\mu(s) d s+\sigma(s) d W(s)], \quad P_{0}(0)=p_{0}, \quad P_{1}(0)=p_{1} \tag{55}
\end{equation*}
$$

Here, random processes $r(\cdot), \mu(\cdot), \sigma(\cdot)$ are respectively interest rate, risky return rate, and instantaneous volatility. Assume that $\mu(s)>r(s)$, a.s. for any $0 \leq s \leq T$, thus the risk premium is positive. Suppose there involve two economic agents formulated in leader-follower decision pattern: one agent acts as leader (it may be interpreted as firm owner or principal) wish to achieve or hedge some terminal wealth objective $\xi$. It can also be interpreted as some payoff target to be replicated in pension planning. In addition, the leader may utilize some continuous consumption process with instantaneous rate $c_{1}(\cdot)$. Another agent is the follower (e.g., pension fund manager) who may implement a dynamic operation (or, wage) process $c_{2}(\cdot)$. Thus, the state process $X(s)$ becomes the following BSDE

$$
\begin{equation*}
d X(s)=\left[r(s) X(s)+\frac{\mu(s)-r(s)}{\sigma(s)} Z(s)-c_{1}(s)-c_{2}(s)\right] d s+Z(s) d W(s), \quad X(T)=\xi \tag{56}
\end{equation*}
$$

where $Z(s)=\pi(s) \sigma(s)$ and $\pi(\cdot)$ is the amount of risky allocation from wealth process. For $i=1,2$, let $\mathcal{U}_{i} \triangleq\left\{c_{i}:[0, T] \times \Omega \rightarrow \mathbb{R} \mid c_{i}(\cdot)\right.$ is $\mathbb{F}$ - progressively measurable, $\left.\mathbb{E} \int_{0}^{T}\left|c_{i}(t)\right|^{2} d t<\infty\right\}$ represent the operation and consumption process. Also, the terminal target $\xi$ is subject to some practical constraints $\mathcal{U}_{\mathcal{K}}, \mathcal{U}_{\alpha, \beta}$ and $\mathcal{U}(\mathcal{K}, \alpha, \beta)$. For quadratic hedging, the following functionals are often employed (see [10]):

$$
\begin{align*}
& J_{1}\left(\xi, c_{1}(\cdot), c_{2}(\cdot)\right) \triangleq \frac{1}{2} \mathbb{E}\left\{G_{1} \xi^{2}+H_{1} X^{2}(0)+\int_{0}^{T}\left[Q_{1}(s) X^{2}(s)+S_{1}(s) Z^{2}(s)+R_{1}(s) c_{1}^{2}(s)\right] d s\right\} \\
& J_{2}\left(\xi, c_{1}(\cdot), c_{2}(\cdot)\right) \triangleq \frac{1}{2} \mathbb{E}\left\{H_{2} X^{2}(0)+\int_{0}^{T}\left[Q_{2}(s) X^{2}(s)+S_{2}(s) Z^{2}(s)+R_{2}(s) c_{2}^{2}(s)\right] d s\right\} \tag{57}
\end{align*}
$$

where $H_{1}, H_{2}$ denote the initial hedging surplus index. Comparing with (1) and (3), we obtain that $A=r, \quad B_{1}=B_{2} \equiv-1, \quad C=\frac{\mu-r}{\sigma}, \quad R_{11}^{1}=R_{1}, \quad R_{22}^{2}=R_{2}$. Thus (SRE-1) takes the following form:

$$
\left\{\begin{array}{l}
d P=-\left[Q_{2}+2 P r-\frac{P^{2}}{R_{2}}-\left(P \frac{\mu-r}{\sigma}+K\right)^{2} \frac{1}{P+S_{2}}\right] d s+K d W(s),  \tag{58}\\
P(T)=M>0, \quad P(s)+S_{2}(s)>0, \quad 0 \leq s \leq T
\end{array}\right.
$$

Now, we give the following assumption:
(H7) All the coefficients in (56) and (57) are bounded. Moreover, $H_{1} \geq 0, Q_{1}(\cdot) \geq 0, G_{1}>0, S_{1}(\cdot) \gg$ $0, R_{1}(\cdot) \gg 0, Q_{2}(\cdot) \geq 0, S_{2}(\cdot) \gg 0, R_{2}(\cdot) \gg 0$.

Note that in (H7), there has no positive (semi-)definite assumption on $H_{2}$. Under (H7), It follows from Proposition 5.4 that (58) admits a unique solution. Moreover, if $P(0)+H_{2} \geq 0$, then by Proposition 3.3 and Theorem 3.1, the optimal consumption $\bar{c}_{2}(\cdot)$ of the follower is given by $\bar{c}_{2}(\cdot)=-\frac{\bar{Y}(\cdot)}{R_{2}(\cdot)}$, where $(\bar{Y}, \bar{X}, \bar{Z})$ is the solution of the following BFSDEs

$$
\left\{\begin{array}{l}
d \bar{Y}=\left(-r \bar{Y}+Q_{2} \bar{X}\right) d s-\left(\frac{\mu-r}{\sigma} \bar{Y}-S_{2} \bar{Z}\right) d W(s)  \tag{59}\\
d \bar{X}=\left[r \bar{X}-c_{1}+\frac{\bar{Y}}{R_{2}}+\frac{\mu-r}{\sigma} \bar{Z}\right] d s+\bar{Z} d W \\
\bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\xi
\end{array}\right.
$$

For the leader, (SRE-2) takes the following form:

$$
\left\{\begin{align*}
& d P_{L}=-\left[\mathbb{A}^{\top} P_{L}+P_{L} \mathbb{A}+\mathbb{C}^{\top} P_{L} \mathbb{C}+\mathbb{Q}+\Lambda_{L} \mathbb{C}+\mathbb{C}^{\top} \Lambda_{L}-\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} P_{L} \mathbb{C}+\mathbb{D}^{\top} \Lambda_{L}\right)^{\top}\right.  \tag{60}\\
&\left.\mathbb{K}^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} P_{L} \mathbb{C}+\mathbb{D}^{\top} \Lambda_{L}\right)\right] d s+\Lambda_{L} d W(s), \\
& P_{L}(T)=\left(\begin{array}{cc}
0 & 0 \\
0 & G_{1}
\end{array}\right), \quad \mathbb{K}(s) \triangleq \mathbb{R}(s)+\mathbb{D}^{\top}(s) P_{L}(s) \mathbb{D}(s)>0, \quad 0 \leq s \leq T,
\end{align*}\right.
$$

where $\mathbb{A}=\left(\begin{array}{cc}-r & Q_{2} \\ \frac{1}{R_{2}} & r\end{array}\right), \mathbb{B}=\left(\begin{array}{cc}0 & 0 \\ -1 & \frac{\mu-r}{\sigma}\end{array}\right), \mathbb{C}=\left(\begin{array}{cc}-\frac{\mu-r}{\sigma} & 0 \\ 0 & 0\end{array}\right), \mathbb{D}=\left(\begin{array}{cc}0 & S_{2} \\ 0 & 1\end{array}\right), \mathbb{Q}=\left(\begin{array}{cc}0 & 0 \\ 0 & Q_{1}\end{array}\right), \mathbb{R}=$ $\left(\begin{array}{cc}R_{1} & 0 \\ 0 & S_{1}\end{array}\right)$. Under (H7), it follows from Proposition5.6 that (60) admits a unique solution. Furthermore, suppose that $P_{L}(0)+\left(\begin{array}{cc}0 & 0 \\ 0 & H_{1}\end{array}\right) \geq 0$ and $(\mathrm{F})$ holds, it follows from Proposition4.4 and Theorem4.2 that an optimal control of leader is given by $\left(\bar{\xi}, \bar{c}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[\frac{-g(T)+\lambda \alpha}{G_{1}}\right],-\frac{g(\cdot)}{R_{1}(\cdot)}\right)$, where $(\lambda ; \bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ is the solution of the following BFSDEs

$$
\left\{\begin{array}{l}
d g=\left(-r g+Q_{1} \bar{X}+Q_{2} h\right) d s-\left(\frac{\mu-r}{\sigma} g-S_{1} \bar{Z}-S_{2} q\right) d W(s),  \tag{61}\\
d \bar{Y}=\left(-r \bar{Y}+Q_{2} \bar{X}\right) d s-\left(\frac{\mu-r}{\sigma} \bar{Y}-S_{2} \bar{Z}\right) d W(s), \\
d \bar{X}=\left[r \bar{X}+\frac{g}{R_{1}}+\frac{\bar{Y}}{R_{2}}+\frac{\mu-r}{\sigma} \bar{Z}\right] d s+\bar{Z} d W(s), \quad d h=\left[r h+\frac{g}{R_{2}}+\frac{\mu-r}{\sigma} q\right] d s+q d W(s), \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\operatorname{Proj}_{\mathcal{K}}\left[\frac{-g(T)+\lambda \alpha}{G_{1}}\right], \quad h(T)=0, \\
\lambda\left(\beta-\alpha \mathbb{E} \operatorname{Proj}_{\mathcal{K}}\left[\frac{-g(T)+\lambda \alpha}{G_{1}}\right]\right)=0, \quad \lambda \geq 0, \quad \beta \leq \alpha \mathbb{E} \operatorname{Proj}_{\mathcal{K}}\left[\frac{-g(T)+\lambda \alpha}{G_{1}}\right] .
\end{array}\right.
$$

### 6.1 Pointwise constraint

In case there has only one constraint $\xi \in \mathcal{U}_{\mathcal{K}}$, (161) assumes the following form:

$$
\left\{\begin{array}{l}
d g=\left(-r g+Q_{1} \bar{X}+Q_{2} h\right) d s-\left(\frac{\mu-r}{\sigma} g-S_{1} \bar{Z}-S_{2} q\right) d W(s),  \tag{62}\\
d \bar{Y}=\left(-r \bar{Y}+Q_{2} \bar{X}\right) d s-\left(\frac{\mu-r}{\sigma} \bar{Y}-S_{2} \bar{Z}\right) d W(s), \\
d \bar{X}=\left[r \bar{X}+\frac{g}{R_{1}}+\frac{\bar{Y}}{R_{2}}+\frac{\mu-r}{\sigma} \bar{Z}\right] d s+\bar{Z} d W(s), \quad d h=\left[r h+\frac{g}{R_{2}}+\frac{\mu-r}{\sigma} q\right] d s+q d W(s), \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=\operatorname{Proj}_{\mathcal{K}}\left[-G_{1}^{-1} g(T)\right], \quad h(T)=0 .
\end{array}\right.
$$

Here, the parameters of (H5) can be chosen as follows:

$$
\begin{align*}
& \rho_{1}=\rho_{2}=-\underset{0 \leq s \leq T}{\operatorname{essinf}} \underset{\omega \in \Omega}{\operatorname{essinf}}|r(s)|, k_{1}=\left\|\left(\begin{array}{cc}
Q_{1} & Q_{2} \\
Q_{2} & 0
\end{array}\right)\right\|, k_{2}=k_{3}=k_{6}=k_{8}=k_{12}=0, \\
& k_{4}=\left\|\left(\begin{array}{cc}
R_{1}^{-1}(\cdot) & R_{2}^{-1}(\cdot) \\
R_{2}^{-1}(\cdot) & 0
\end{array}\right)\right\|, k_{5}=\underset{0 \leq s \leq T}{\operatorname{essup}} \underset{\omega \in \Omega}{\operatorname{esssup}} \sqrt{2}\left|\frac{\mu(s)-r(s)}{\sigma(s)}\right|, k_{7}=\underset{0 \leq s \leq T}{\operatorname{esssup}} \underset{\omega \in \Omega}{\operatorname{esssup} 2}\left|\frac{\mu(s)-r(s)}{\sigma(s)}\right|,  \tag{63}\\
& k_{9}=\sqrt{2}\left\|\left(\begin{array}{cc}
S_{1} & S_{2} \\
S_{2} & 0
\end{array}\right)\right\|, k_{10}=\left\|\left(\begin{array}{cc}
H_{1} & H_{2} \\
H_{2} & 0
\end{array}\right)\right\|, k_{11}=\underset{\omega \in \Omega}{\operatorname{esssup}} G_{1}^{-1} .
\end{align*}
$$

Therefore, by Theorem 5.2, we have the following result.
Proposition 6.1 Suppose that $2 \rho_{1}<-2 k_{5}^{2}-k_{7}^{2}$. There exists a $\delta_{1}>0$, which depends on $\rho_{1}, k_{i}, i=5,7$, such that when $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{1}\right)$, there exists a unique adapted solution to (62).

Under (H7), suppose $P_{L}(0)+\left(\begin{array}{cc}0 & 0 \\ 0 & H_{1}\end{array}\right) \geq 0$ and conditions of Proposition 6.1 holds, the optimal control of $\mathcal{A}_{L}$ is given by $\left(\bar{\xi}, \bar{c}_{1}(\cdot)\right)=\left(\operatorname{Proj}_{\mathcal{K}}\left[\frac{-g(T)}{G_{1}}\right],-\frac{g(\cdot)}{R_{1}(\cdot)}\right)$, where $(\bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ is the solution of (62).

Next, we give a more specific condition for wellposedness of (62). For $c_{1}, c_{3}, c_{4}, \bar{\rho}_{1}$ and $\bar{\rho}_{2}$, please refer Lemma 7.2 and Lemma 7.3 in the appendix of [?].
Remark 6.1 For some $\varepsilon>0$, set $c_{1}=\frac{k_{1}}{\varepsilon}$, $c_{4}=\frac{k_{4}}{\varepsilon}, c_{5}=\frac{k_{5}}{2\left(k_{5}^{2}+\varepsilon\right)}$ and $c_{6}=\frac{k_{6}}{2\left(k_{6}^{2}+\varepsilon\right)}$. Suppose $2\left(\rho_{1}+\rho_{2}\right)<$ $-2 k_{5}^{2}-k_{7}^{2}-3 \varepsilon$ and define $d=-2 k_{5}^{2}-k_{7}^{2}-3 \varepsilon-2 \rho_{1}-2 \rho_{2}=-4 k_{5}^{2}-3 \varepsilon-4 \rho_{1}$. Therefore, we can choose $\rho$ such that $\bar{\rho}_{1}=\bar{\rho}_{2}=\frac{d}{2}$. In this case, let

$$
\theta=\left(\frac{2}{-4 \rho_{1}-4 k_{5}^{2}-3 \varepsilon}+5+\frac{2 k_{5}^{2}+2 k_{6}^{2}}{\varepsilon}\right)\left(2 k_{9}^{2}+\frac{2 k_{4}^{2}}{-\varepsilon\left(4 \rho_{1}+4 k_{5}^{2}+3 \varepsilon\right)}\right) .
$$

That is, if $4 \rho_{1}<-4 k_{5}^{2}-3 \varepsilon, k_{9}^{2} \theta<1, k_{10}^{2} \theta<1, \frac{k_{1}^{2} \theta}{\varepsilon}<1$, there exists a unique solution to (62).

### 6.2 Affine constraint

In this subsection, suppose that there is only one constraint $\xi \in \mathcal{U}_{\alpha, \beta}$ and all the coefficients are deterministic. We will study the case $\lambda=0$ and $\lambda \neq 0$ separately.

In case $\lambda=0$, 61) becomes

$$
\left\{\begin{array}{l}
d g=\left(-r g+Q_{1} \bar{X}+Q_{2} h\right) d s-\left(\frac{\mu-r}{\sigma} g-S_{1} \bar{Z}-S_{2} q\right) d W(s)  \tag{64}\\
d \bar{Y}=\left(-r \bar{Y}+Q_{2} \bar{X}\right) d s-\left(\frac{\mu-r}{\sigma} \bar{Y}-S_{2} \bar{Z}\right) d W(s), \\
d \bar{X}=\left[r \bar{X}+\frac{g}{R_{1}}+\frac{\bar{Y}}{R_{2}}+\frac{\mu-r}{\sigma} \bar{Z}\right] d s+\bar{Z} d W(s), d h=\left[r h+\frac{g}{R_{2}}+\frac{\mu-r}{\sigma} q\right] d s+q d W(s), \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=-G_{1}^{-1} g(T), \quad h(T)=0 \\
\beta+\alpha G_{1}^{-1} \mathbb{E} g(T) \leq 0
\end{array}\right.
$$

Here, we present some detailed solution. Note that (64) is linear and homogeneous. Thus if (64) admits an unique solution, it must be $\bar{Y}=g=\bar{X}=\bar{Z}=h=q \equiv 0$. In this case, if $\beta \leq 0$, (KKT) condition holds. Let $\rho_{1}, \rho_{2}, k_{i}, i=1, \cdots, 12$ be defined as in (63). Therefore, by Theorem 5.2, suppose that $2 \rho_{1}<-2 k_{5}^{2}-k_{7}^{2}$ and $\beta \leq 0$, if there exists a $\delta_{2}>0$ depending on $\rho_{1}, k_{i}, i=5,7$, such that $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{2}\right)$, there exists a unique adapted solution to (64). Therefore, under (H7), suppose that $P_{L}(0)+\left(\begin{array}{cc}0 & 0 \\ 0 & H_{1}\end{array}\right) \geq 0,2 \rho_{1}<-2 k_{5}^{2}-k_{7}^{2}$ and $\beta \leq 0$, if there exists a $\delta_{2}>0$ depending on $\rho_{1}, k_{i}, i=5,7$, such that $k_{1}, k_{4}, k_{9}, k_{10} \in\left[0, \delta_{2}\right)$, the optimal control of the leader is given by $\left(\bar{\xi}, \bar{c}_{1}(\cdot)\right)=(0,0)$.

Next we consider the case $\lambda>0$. (61) becomes

$$
\left\{\begin{array}{l}
d g=\left(-r g+Q_{1} \bar{X}+Q_{2} h\right) d s-\left(\frac{\mu-r}{\sigma} g-S_{1} \bar{Z}-S_{2} q\right) d W(s),  \tag{65}\\
d \bar{Y}=\left(-r \bar{Y}+Q_{2} \bar{X}\right) d s-\left(\frac{\mu-r}{\sigma} \bar{Y}-S_{2} \bar{Z}\right) d W(s), \\
d \bar{X}=\left[r \bar{X}+\frac{g}{R_{1}}+\frac{\bar{Y}}{R_{2}}+\frac{\mu-r}{\sigma} \bar{Z}\right] d s+\bar{Z} d W(s), \quad d h=\left[r h+\frac{g}{R_{2}}+\frac{\mu-r}{\sigma} q\right] d s+q d W(s), \\
g(0)=H_{1} \bar{X}(0)+H_{2} h(0), \quad \bar{Y}(0)=H_{2} \bar{X}(0), \quad \bar{X}(T)=-G_{1}^{-1} g(T)+G_{1}^{-1} \mathbb{E} g(T)+\frac{\beta}{\alpha}, \\
h(T)=0, \quad \beta+\alpha G_{1}^{-1} \mathbb{E} g(T)>0
\end{array}\right.
$$

Hence, (44) and (45) take the form

$$
\begin{align*}
& \left\{\begin{array}{l}
\dot{P}-\check{P} \check{\mathbb{A}}-\check{P} \check{\mathbb{B}} \check{P}+\left(\check{P} \check{\mathbb{C}}+\check{P} \check{\mathbb{D}}+\check{\mathbb{C}}_{2}+\check{\mathbb{D}}_{2}\right)\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1}\left(\check{P} \check{\mathbb{A}}_{1}+\check{P} \check{\mathbb{B}}_{1} \check{P}\right)-\check{\mathbb{A}}_{2}-\check{\mathbb{B}}_{2} \check{P}=0, \\
\check{P}(T)=(I-\check{G} \check{H})^{-1} \check{G}, \quad \operatorname{det}\left[I+\check{P} \check{\mathbb{C}}_{1}\right] \neq 0,
\end{array}\right.  \tag{66}\\
& \dot{p}-\check{P} \check{\mathbb{B}} \check{p}+\left(\check{P} \check{\mathbb{C}}+\check{P} \check{\mathbb{D}}+\check{\mathbb{C}}_{2}+\check{\mathbb{D}}_{2}\right)\left(I+\check{P} \check{\mathbb{C}}_{1}\right)^{-1} \check{P} \check{\mathbb{B}}_{1} \check{p}-\check{\mathbb{B}}_{2} \check{p}=0, \quad \check{p}(T)=(I-\check{G} \check{H})^{-1} \check{f}, \tag{67}
\end{align*}
$$

where the notations of the coefficients are defined in (43). Now (KKT) condition (46) becomes

$$
\begin{equation*}
\beta+\alpha\left(G_{1}^{-1} 0000\right)(I+\check{H} \check{P}) \check{\Phi}(T, 0) \int_{0}^{T} \check{\Phi}(s, 0) \mathbf{b}(s) d s+\alpha\left(G_{1}^{-1} 0000\right) \check{H}(I-\check{G} \check{H})^{-1} \check{f}>0 \tag{68}
\end{equation*}
$$

where $\check{\Phi}(t, 0)$ is the fundamental solution matrices of ODE

$$
d \check{\varphi}=\left[-\check{\mathbb{A}}-\check{\mathbb{B}} \check{P}+(\check{\mathbb{C}}+\check{\mathbb{D}})\left(I+\check{P} \check{C}_{1}\right)^{-1}\left(\check{P} \check{\mathbb{A}}_{1}+\check{P} \check{\mathbb{B}}{ }_{1} \check{P}\right)\right] \check{\varphi} d t, \quad \check{\varphi}(0)=1
$$

Under (H7), if $G_{1}^{-1} H_{1} \neq-1$, by Proposition 5.2 if (66) and (67) admit solutions such that (68) holds, then (65) is solvable. Therefore, an optimal control of the leader is given by $\left(\bar{\xi}, \bar{c}_{1}(\cdot)\right)=\left(\frac{-g(T)+\lambda \alpha}{G_{1}},-\frac{g(\cdot)}{R_{1}(\cdot)}\right)$, where $(\lambda ; \bar{Y}, g, \bar{X}, \bar{Z}, h, q)$ is the solution of (65).

## Conclusion

We discuss an open-loop backward Stackelberg differential game where the state is characterized by BSDE and the decisions of leader consist of a static terminal-perturbation and a dynamic linear-quadratic control. The terminal control is subject to pointwise and expectation constraints. Our open-loop Stackelberg equilibrium is represented by some coupled BFSDEs with mixed initial-terminal conditions and the global solvability of such BFSDEs is discussed in some nontrivial cases.

## 7 Appendix

### 7.1 Proof of Proposition 3.3:

Before we give the proof the Proposition 3.3, first we prove the following lemma.
Lemma 7.1 For any $u_{2}(s) \in \mathcal{U}_{2}[0, T]$, let $\left(x^{\left(u_{2}\right)}(s), z^{\left(u_{2}\right)}(s)\right)$ be the solution of

$$
d x^{\left(u_{2}\right)}(s)=\left[A(s) x^{\left(u_{2}\right)}(s)+B_{2}(s) u_{2}(s)+C(s) z^{\left(u_{2}\right)}(s)\right] d s+z^{\left(u_{2}\right)}(s) d W(s), \quad x^{\left(u_{2}\right)}(T)=0
$$

Then for any $\Theta(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{m_{2} \times n}\right)$, there exists a constant $L>0$ such that

$$
\begin{equation*}
\mathbb{E} \int_{0}^{T}\left|u_{2}(s)-\Theta(s) x^{\left(u_{2}\right)}(s)\right|^{2} d s \geq L \mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s, \quad \forall u_{2}(\cdot) \in \mathcal{U}_{2}[0, T] \tag{69}
\end{equation*}
$$

Proof Let $\Theta(\cdot) \in L_{\mathbb{F}}^{\infty}\left(0, T ; \mathbb{R}^{m_{2} \times n}\right)$, define a bounded linear operator $\mathcal{L}: \mathcal{U}_{2}[0, T] \rightarrow \mathcal{U}_{2}[0, T]$ by $\mathcal{L} u_{2}=$ $u_{2}-\Theta x^{\left(u_{2}\right)}$. Then $\mathcal{L}$ is a bijection, and its inverse is given by $\mathcal{L}^{-1} u_{2}=u_{2}+\Theta \widetilde{x}^{\left(u_{2}\right)}$, where $\widetilde{X}^{\left(u_{2}\right)}(s)$ is the solution of
$d \widetilde{x}^{\left(u_{2}\right)}(s)=\left[A(s) \widetilde{x}^{\left(u_{2}\right)}(s)+B_{2}(s)\left(\Theta(s) \widetilde{x}^{\left(u_{2}\right)}(s)+u_{2}(s)\right)+C(s) \widetilde{z}^{\left(u_{2}\right)}(s)\right] d s+\widetilde{z}^{\left(u_{2}\right)}(s) d W(s), \quad \widetilde{x}^{\left(u_{2}\right)}(T)=0$.
By the bounded inverse theorem, $\mathcal{L}^{-1}$ is bounded with norm $\left\|\mathcal{L}^{-1}\right\|>0$. Therefore,

$$
\mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s \leq\left\|\mathcal{L}^{-1}\right\| \mathbb{E} \int_{0}^{T}\left|\mathcal{L} u_{2}(s)\right|^{2} d s=\left\|\mathcal{L}^{-1}\right\| \mathbb{E} \int_{0}^{T}\left|u_{2}(s)-\Theta(s) x^{\left(u_{2}\right)}(s)\right|^{2} d s
$$

Now we will give the proof of Proposition 3.3. First, let

$$
\Gamma \triangleq-\left(Q_{2}+P A+A^{\top} P-(P C+K)\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right)-P B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P\right)
$$

Let processes $P(\cdot)$ satisfy the following equations

$$
d P(s)=\Gamma(s) d s+K(s) d W(s), \quad P(T)=M^{-1}
$$

Applying Itô's formula to $\langle P x, x\rangle$, integrating from 0 to $T$, we have

$$
-\mathbb{E}\langle P(0) x(0), x(0)\rangle=\mathbb{E} \int_{0}^{T}\left[\left\langle\left(\Gamma+P A+A^{\top} P\right) x, x\right\rangle+2\left\langle x, P B_{2} u_{2}\right\rangle+\langle P z, z\rangle+2\langle(P C+K) z, x\rangle\right] d s
$$

Therefore,

$$
\begin{aligned}
J\left(u_{2}(\cdot)\right)= & \mathbb{E}\left\langle H_{2} x(0), x(0)\right\rangle+\mathbb{E} \int_{0}^{T}\left[\left\langle Q_{2} x, x\right\rangle+\left\langle S_{2} z, z\right\rangle+\left\langle R_{22}^{2} u_{2}, u_{2}\right\rangle\right] d s+\mathbb{E}\langle P(0) x(0), x(0)\rangle \\
& +\mathbb{E} \int_{0}^{T}\left[\left\langle\left(\Gamma+P A+A^{\top} P\right) x, x\right\rangle+2\left\langle x, P B_{2} u_{2}\right\rangle\langle P z, z\rangle+2\langle(P C+K) z, x\rangle\right] d s
\end{aligned}
$$

First, consider the terms involving $u_{2}$,

$$
\left\langle R_{22}^{2} u_{2}, u_{2}\right\rangle+2\left\langle x, P B_{2} u_{2}\right\rangle=\left\langle R_{22}^{2}\left(u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right), u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right\rangle-\left\langle x, P B_{2}\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right\rangle
$$

Next, consider the terms involving $z$,

$$
\begin{aligned}
& \left\langle S_{2} z, z\right\rangle+\langle P z, z\rangle+2\langle(P C+K) z, x\rangle \\
= & \left\langle\left(P+S_{2}\right)\left(z+\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right) x\right), z+\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right) x\right\rangle \\
& -\left\langle x,(P C+K)\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right) x\right\rangle .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J\left(u_{2}(\cdot)\right)= & \left.\mathbb{E}\left\langle\left(H_{2}+P(0)\right) x(0), x(0)\right\rangle\right\rangle+\mathbb{E} \int_{0}^{T}\left\langle R_{22}^{2}\left(u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right), u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right\rangle d s \\
& +\mathbb{E} \int_{t}^{T}\left\langle\left(P+S_{2}\right)\left(z+\left(P+S_{2}\right)^{-1}\left(C^{\top} P+K\right) x\right), z+\left(P+S_{2}\right)^{-1}\left(\left(C^{\top} P+K\right) x\right\rangle d s \geq 0 .\right.
\end{aligned}
$$

Moreover, if $R_{22}^{2}(\cdot) \geq \delta I$, then it follows from Lemma 7.1 that

$$
\mathcal{J}\left(u_{2}(\cdot)\right) \geq \delta \mathbb{E} \int_{0}^{T}\left\langle u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x, u_{2}+\left(R_{22}^{2}\right)^{-1} B_{2}^{\top} P x\right\rangle d s \geq \delta \gamma \mathbb{E} \int_{0}^{T}\left|u_{2}(s)\right|^{2} d s
$$

### 7.2 Proof of Proposition 4.4:

For simplicity, let

$$
\begin{aligned}
\Gamma=- & \left(\mathbb{Q}+P_{L} \mathbb{A}+\Lambda \mathbb{C}+\mathbb{C}^{\top} \Lambda+\mathbb{A}^{\top} P_{L}+\mathbb{C}^{\top} P_{L} \mathbb{C}\right. \\
& \left.-\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)^{\top}\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\right) .
\end{aligned}
$$

Applying Itô's formula to $\left\langle P_{L}\binom{Y}{X},\binom{Y}{X}\right\rangle$, we have

$$
\begin{aligned}
& d\left\langle P_{L}\binom{Y}{X},\binom{Y}{X}\right\rangle \\
= & \left\langle P_{L} \mathbb{A}\binom{Y}{X}+P_{L} \mathbb{B}\binom{u_{1}}{Z},\binom{Y}{X}\right\rangle d s+\left\langle\Gamma\binom{Y}{X},\binom{Y}{X}\right\rangle d s+\left\langle\Lambda \mathbb{C}\binom{Y}{X}+\Lambda \mathbb{D}\binom{u_{1}}{Z},\binom{Y}{X}\right\rangle d s \\
& +\left\langle P_{L}\binom{Y}{X}, \mathbb{A}\binom{Y}{X}+\mathbb{B}\binom{u_{1}}{Z}\right\rangle d s+\left\langle\Lambda\binom{Y}{X}, \mathbb{C}\binom{Y}{X}+\mathbb{D}\binom{u_{1}}{Z}\right\rangle d s \\
& +\left\langle P_{L}\left(\mathbb{C}\binom{Y}{X}+\mathbb{D}\binom{u_{1}}{Z}\right), \mathbb{C}\binom{Y}{X}+\mathbb{D}\binom{u_{1}}{Z}\right\rangle d s+[\cdots] d W(s) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathbb{E}\left\langle P_{L}(T)\binom{Y(T)}{X(T)},\binom{Y(T)}{X(T)}\right\rangle-\mathbb{E}\left\langle P_{L}(0)\binom{Y(0)}{X(0)},\binom{Y(0)}{X(0)}\right\rangle \\
= & \mathbb{E} \int_{0}^{T}\left\langle\binom{ Y}{X}, P_{L} \mathbb{A}\binom{Y}{X}+\Gamma\binom{Y}{X}+\Lambda \mathbb{C}\binom{Y}{X}+\mathbb{C}^{\top} \Lambda\binom{Y}{X}+\mathbb{A}^{\top} P_{L}\binom{Y}{X}+\mathbb{C}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& +\mathbb{E} \int_{0}^{T}\left\langle\binom{ u_{1}}{Z}, 2 \mathbb{B}^{\top} P_{L}\binom{Y}{X}+2 \mathbb{D}^{\top} \Lambda\binom{Y}{X}+2 \mathbb{D}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& +\mathbb{E} \int_{0}^{T}\left\langle\binom{ u_{1}}{D}, \mathbb{D}^{\top} P_{L} \mathbb{D}\binom{u_{1}}{D}\right\rangle d s .
\end{aligned}
$$

Adding this into the functional, we have

$$
\begin{aligned}
& J\left(\xi, u_{1}(\cdot)\right) \\
& =\frac{1}{2} \mathbb{E}\left\{\left\langle G_{1} \xi, \xi\right\rangle+\left\langle H_{1} X(0), X(0)\right\rangle+\int_{0}^{T}\left[\left\langle Q_{1} X, X\right\rangle+\left\langle S_{1} Z, Z\right\rangle+\left\langle R_{11}^{1} u_{1}, u_{1}\right)\right\rangle\right] d s \\
& -\left\langle P_{L}(T)\binom{Y(T)}{X(T)},\binom{Y(T)}{X(T)}\right\rangle+\left\langle P_{L}(0)\binom{Y(0)}{X(0)},\binom{Y(0)}{X(0)}\right\rangle \\
& +\int_{0}^{T}\left\langle\binom{ Y}{X}, P_{L} \mathbb{A}\binom{Y}{X}+\Gamma\binom{Y}{X}+\Lambda \mathbb{C}\binom{Y}{X}+\mathbb{C}^{\top} \Lambda\binom{Y}{X}+\mathbb{A}^{\top} P_{L}\binom{Y}{X}+\mathbb{C}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& +\int_{0}^{T}\left\langle\binom{ u_{1}}{Z}, 2 \mathbb{B}^{\top} P_{L}\binom{Y}{X}+2 \mathbb{D}^{\top} \Lambda\binom{Y}{X}+2 \mathbb{D}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& \left.+\int_{0}^{T}\left\langle\binom{ u_{1}}{Z}, \mathbb{D}^{\top} P_{L} \mathbb{D}\binom{u_{1}}{Z}\right\rangle d s\right\} \\
& =\frac{1}{2} \mathbb{E}\left\{\left\langle\left[\left(\begin{array}{cc}
0 & 0 \\
0 & H_{1}
\end{array}\right)+P_{L}(0)\right]\binom{Y(0)}{X(0)},\binom{Y(0)}{X(0)}\right\rangle+\frac{1}{2}\left\langle G_{1} \xi, \xi\right\rangle\right. \\
& +\int_{0}^{T}\left\langle\binom{ Y}{X}, \mathbb{Q}\binom{Y}{X}+P_{L} \mathbb{A}\binom{Y}{X}+\Gamma\binom{Y}{X}+\Lambda \mathbb{C}\binom{Y}{X}+\mathbb{C}^{\top} \Lambda\binom{Y}{X}+\mathbb{A}^{\top} P_{L}\binom{Y}{X}+\mathbb{C}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& +\int_{0}^{T}\left\langle\binom{ u_{1}}{Z}, 2 \mathbb{B}^{\top} P_{L}\binom{Y}{X}+2 \mathbb{D}^{\top} \Lambda\binom{Y}{X}+2 \mathbb{D}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& \left.+\int_{0}^{T}\left\langle\binom{ u_{1}}{Z},\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)\binom{u_{1}}{Z}\right\rangle d s\right\} .
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \mathbb{E} \int_{0}^{T}\left\langle\binom{ u_{1}}{Z}, 2 \mathbb{B}^{\top} P_{L}\binom{Y}{X}+2 \mathbb{D}^{\top} \Lambda\binom{Y}{X}+2 \mathbb{D}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s+\mathbb{E} \int_{0}^{T}\left\langle\binom{ u_{1}}{Z},\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)\binom{u_{1}}{Z}\right\rangle d s \\
= & \mathbb{E} \int_{0}^{T}\left\langle\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)\left(\binom{u_{1}}{Z}+\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right),\right. \\
& \left.\binom{u_{1}}{Z}+\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right\rangle d s \\
& -\mathbb{E} \int_{0}^{T}\left\langle\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X},\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right\rangle d s
\end{aligned}
$$

and recall the definition of $\Gamma$, we have

$$
\begin{aligned}
& J\left(\xi, u_{1}(\cdot)\right) \\
= & \frac{1}{2} \mathbb{E}\left\{\left\langle\left[\left(\begin{array}{cc}
0 & 0 \\
0 & H_{1}
\end{array}\right)+P_{L}(0)\right]\binom{Y(0)}{X(0)},\binom{Y(0)}{X(0)}\right\rangle\right. \\
& +\int_{0}^{T}\left\langle\binom{ Y}{X}, \mathbb{Q}\binom{Y}{X}+P_{L} \mathbb{A}\binom{Y}{X}+\Gamma\binom{Y}{X}+\Lambda \mathbb{C}\binom{Y}{X}+\mathbb{C}^{\top} \Lambda\binom{Y}{X}+\mathbb{A}^{\top} P_{L}\binom{Y}{X}+\mathbb{C}^{\top} P_{L} \mathbb{C}\binom{Y}{X}\right\rangle d s \\
& -\int_{0}^{T}\left\langle\binom{ Y}{X},\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)^{\top}\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right\rangle d s \\
& +\int_{0}^{T}\left\langle\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)\left[\binom{u_{1}}{Z}+\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right]\right. \\
= & \frac{1}{2} \mathbb{E}\left\langle\left[\binom{u_{1}}{Z}+\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{0}\right\rangle d s\right\} \\
& +\frac{1}{2} \mathbb{E} \int_{0}^{T}\left\langle\left(\mathbb{R}+P_{L}(0)\right]\binom{Y(0)}{X(0)},\binom{Y(0)}{X(0)}\right\rangle \\
& \left.P_{L} \mathbb{D}\right)\left[\binom{u_{1}}{Z}+\left(\mathbb{R}+\mathbb{D}^{\top} P_{L} \mathbb{D}\right)^{-1}\left(\mathbb{B}^{\top} P_{L}+\mathbb{D}^{\top} \Lambda+\mathbb{D}^{\top} P_{L} \mathbb{C}\right)\binom{Y}{X}\right]
\end{aligned}
$$

### 7.3 Proof of Theorem 5.1:

First, we will give two lemmas. Note that for a given $(X(\cdot), Z(\cdot)) \times X(0) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times$ $L_{\mathcal{F}_{0}}^{2}\left(\Omega ; \mathbb{R}^{m}\right)$, where $X(0)$ is the value of process $X(\cdot)$ at initial time, the forward equation in the BFSDEs (29) has a unique solution $Y(\cdot) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$, thus we introduce a map $\mathbb{M}_{1}: L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times$ $L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathcal{F}_{0}}^{2}\left(\Omega ; \mathbb{R}^{m}\right) \rightarrow L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$, through

$$
\begin{equation*}
Y(t)=h(X(0))+\int_{0}^{t} b(s, Y, X, Z, \mathbb{E} Z) d s+\int_{0}^{t} \sigma(s, Y, X, Z) d W(s) \tag{70}
\end{equation*}
$$

Therefore, $\mathbb{E} \sup _{t \in[0, T]}|Y(t)|^{2}<\infty$. For any $\rho \in \mathbb{R}$, define $\|X\|_{\rho} \triangleq\left(E \int_{0}^{T} e^{-\rho t}|X(t)|^{2} d t\right)^{\frac{1}{2}}$.
Lemma 7.2 Let $Y_{i}(\cdot)$ be the solution of (70) corresponding to $\left(X_{i}(\cdot), Z_{i}(\cdot)\right) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right), i=$ 1,2 . Then for all $\rho \in \mathbb{R}, c_{1}, c_{2}, c_{3}>0$, we have

$$
\begin{align*}
& e^{-\rho t} \mathbb{E}|\widehat{Y}(t)|^{2}+\bar{\rho}_{1} \int_{0}^{t} e^{-\rho s} \mathbb{E}|\widehat{Y}(s)|^{2} d s  \tag{71}\\
& \leq k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right) \int_{0}^{t} e^{-\rho s} \mathbb{E}|\widehat{X}(s)|^{2} d s+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right) \int_{0}^{t} e^{-\rho s} \mathbb{E}|\widehat{Z}(s)|^{2} d s \\
& e^{-\rho t} \mathbb{E}|\widehat{Y}(t)|^{2} \leq  \tag{72}\\
& k_{10}^{2} e^{-\bar{\rho}_{1} t} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right) \int_{0}^{t} e^{-\bar{\rho}_{1}(t-s)-\rho s} \mathbb{E}|\widehat{X}(s)|^{2} d s \\
& \\
& \quad+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right) \int_{0}^{t} e^{-\bar{\rho}_{1}(t-s)-\rho s} \mathbb{E}|\widehat{Z}(s)|^{2} d s
\end{align*}
$$

where $\bar{\rho}_{1}=\rho-2 \rho_{1}-k_{1} c_{1}^{-1}-k_{2} c_{2}^{-1}-k_{3} c_{3}^{-1}-k_{7}^{2}$ and $\widehat{\varphi}=\varphi_{1}-\varphi_{2}, \varphi=Y, X, Z$. Moreover, we have

$$
\begin{gather*}
\|\widehat{Y}(\cdot)\|_{\rho}^{2} \leq \frac{1-e^{-\bar{\rho}_{1} T}}{\bar{\rho}_{1}}\left[k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right)\|\widehat{X}(\cdot)\|_{\rho}^{2}+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right)\|\widehat{Z}(\cdot)\|_{\rho}^{2}\right]  \tag{73}\\
e^{-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2} \leq \max \left\{1, e^{-\bar{\rho}_{1} T}\right\}\left[k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right)\|\widehat{X}(\cdot)\|_{\rho}^{2}+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right)\|\widehat{Z}(\cdot)\|_{\rho}^{2}\right] \tag{74}
\end{gather*}
$$

In particular, if $\bar{\rho}_{1}>0$, we have

$$
e^{-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2} \leq k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right)\|\widehat{X}(s)\|_{\rho}^{2}+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right)\|\widehat{Z}(\cdot)\|_{\rho}^{2}
$$

Proof Under (H5), applying Itô's formula to $e^{-\rho s}|\widehat{Y}(s)|^{2}$ and taking expectation, we obtain (71). Furthermore, applying Itô's formula again to $e^{-\bar{\rho}_{1}(t-s)-\rho s}|\widehat{Y}(s)|^{2}$ for $s \in[0, t]$ and taking expectation, we get (72). Integrating both sides of (72) on $[0, T]$ and noting $\frac{1-e^{-\bar{\rho}_{1}(T-s)}}{\bar{\rho}_{1}} \leq \frac{1-e^{-\bar{\rho}_{1} T}}{\bar{\rho}_{1}}, \forall s \in[0, T]$, we have (73). Letting $t=T$ in (72) and noticing that $e^{-\bar{\rho}_{1}(T-s)} \leq \max \left\{1, e^{-\bar{\rho}_{1} T}\right\}$, we obtain (74).
Similarly, for given $Y(\cdot) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$, the backward equation in the BFSDEs (29) has a unique solution $(X(\cdot), Z(\cdot)) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$, and the corresponding initial value of $X(\cdot)$ is denoted by $X(0) \in L_{\mathcal{F}_{0}}^{2}\left(\Omega ; \mathbb{R}^{m}\right)$. Thus, we can introduce another map $\mathbb{M}_{2}: L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right) \rightarrow L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times$ $L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathcal{F}_{0}}^{2}\left(\Omega ; \mathbb{R}^{m}\right)$, through

$$
\begin{equation*}
X(t)=g(Y(T), \mathbb{E} Y(T))+\int_{0}^{T} f(s, Y, X, Z, \mathbb{E} Z) d s-\int_{0}^{T} Z d W(s) \tag{75}
\end{equation*}
$$

which satisfies $\mathbb{E} \sup _{t \in[0, T]}|X(t)|^{2}+\mathbb{E} \int_{0}^{T}|Z(t)|^{2} d t<\infty$. Similar to Lemma 7.2 we have
Lemma 7.3 Let $\left(X_{i}(\cdot), Z_{i}(\cdot)\right)$ be the solution of (75) corresponding to $Y_{i}(\cdot) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right), i=1,2$. Then for all $\rho \in \mathbb{R}, c_{4}, c_{5}, c_{6}>0$, we have

$$
\begin{aligned}
& e^{-\rho t} \mathbb{E}|\widehat{X}(t)|^{2}+\bar{\rho}_{2} \int_{0}^{T} e^{-\rho s} \mathbb{E}|\widehat{X}(s)|^{2} d s+\left(1-k_{5} c_{5}-k_{6} c_{6}\right) \int_{0}^{T} e^{-\rho s} \mathbb{E}|\widehat{Z}(s)|^{2} d s \\
& \leq 2\left(k_{9}^{2}+k_{10}^{2}\right) \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4} \int_{0}^{T} e^{-\rho s} \mathbb{E}|\widehat{Y}(s)|^{2} d s, \\
& \quad e^{-\rho t} \mathbb{E}|\widehat{X}(t)|^{2}+\left(1-k_{5} c_{5}-k_{6} c_{6}\right) \int_{0}^{T} e^{-\bar{\rho}_{2}(s-t)-\rho s} \mathbb{E}|\widehat{Z}(s)|^{2} d s \\
& \leq \\
& \leq 2\left(k_{9}^{2}+k_{10}^{2}\right) e^{-\bar{\rho}_{2}(T-t)-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4} \int_{0}^{T} e^{-\bar{\rho}_{2}(s-t)-\rho s} \mathbb{E}|\widehat{Y}(s)|^{2} d s,
\end{aligned}
$$

where $\bar{\rho}_{2}=-\rho-2 \rho_{2}-k_{4} c_{4}^{-1}-k_{5} c_{5}^{-1}-k_{6} c_{6}^{-1}$ and $\widehat{\varphi}=\varphi_{1}-\varphi_{2}, \varphi=Y, X, Z$. Moreover, choosing $c_{4} \in\left(0, k_{4}^{-1}\right)$, we have

$$
\begin{gathered}
\|\widehat{X}(\cdot)\|_{\rho}^{2} \leq \frac{1-e^{-\bar{\rho}_{2} T}}{\bar{\rho}_{2}}\left[2\left(k_{9}^{2}+k_{10}^{2}\right) e^{-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4}\|\widehat{Y}(\cdot)\|_{\rho}^{2}\right] \\
\|\widehat{Z}(\cdot)\|_{\rho}^{2} \leq \frac{2\left(k_{9}^{2}+k_{10}^{2}\right) e^{-\left(\bar{\rho}_{2}+\rho\right) T} \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4} \max \left\{1, e^{-\bar{\rho}_{2} T}\right\}\|\widehat{Y}(\cdot)\|_{\rho}^{2}}{\left(1-k_{5} c_{5}-k_{6} c_{6}\right) \min \left\{1, e^{-\bar{\rho}_{2} T}\right\}}, \\
\mathbb{E}|\widehat{X}(0)|^{2} \leq \max \left\{1, e^{-\bar{\rho}_{2} T}\right\}\left[2\left(k_{9}^{2}+k_{10}^{2}\right) e^{-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4}\|\widehat{Y}(\cdot)\|_{\rho}^{2}\right]
\end{gathered}
$$

In particular, if $\bar{\rho}_{2}>0$, we have

$$
\|\widehat{Z}(\cdot)\|_{\rho}^{2} \leq \frac{2\left(k_{9}^{2}+k_{10}^{2}\right) \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4}\|\widehat{Y}(\cdot)\|_{\rho}^{2}}{1-k_{5} c_{5}-k_{6} c_{6}}
$$

Now we will give the proof of Theorem 5.1 Consider the map $\mathbb{M} \triangleq \mathbb{M}_{2} \circ \mathbb{M}_{1}$. It suffices to show that $\mathbb{M}$ is a contraction mapping under $\|\cdot\|_{\rho}$. In fact, for $\left(X_{i}(\cdot), Z_{i}(\cdot)\right) \times X_{i}(0) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right) \times$ $L_{\mathcal{F}_{0}}^{2}\left(\Omega ; \mathbb{R}^{m}\right), i=1,2$, let $Y_{i} \triangleq \mathbb{M}_{1}\left(X_{i}(\cdot), Z_{i}(\cdot), X_{i}(0)\right)$ and $\left(\bar{X}_{i}(\cdot), \bar{Z}_{i}(\cdot), \bar{X}_{i}(0)\right) \triangleq \mathbb{M}\left(\left(X_{i}(\cdot), Z_{i}(\cdot), X_{i}(0)\right)\right)$, by Lemmas 7.2 and 7.3, we have

$$
\begin{aligned}
& \mathbb{E}\left|\bar{X}_{1}(0)-\bar{X}_{2}(0)\right|^{2}+\left\|\bar{X}_{1}(\cdot)-\bar{X}_{2}(\cdot)\right\|_{\rho}^{2}+\left\|\bar{Z}_{1}(\cdot)-\bar{Z}_{2}(\cdot)\right\|_{\rho}^{2} \\
\leq & {\left[\frac{1-e^{-\bar{\rho}_{2} T}}{\bar{\rho}_{2}}+\frac{\max \left\{1, e^{-\bar{\rho}_{2} T}\right\}}{\left(1-k_{5} c_{5}-k_{6} c_{6}\right) \min \left\{1, e^{-\bar{\rho}_{2} T}\right\}}+\max \left\{1, e^{-\bar{\rho}_{2} T}\right\}\right]\left[2\left(k_{9}^{2}+k_{10}^{2}\right) e^{-\rho T} \mathbb{E}|\widehat{Y}(T)|^{2}+k_{4} c_{4}\|\widehat{Y}(\cdot)\|_{\rho}^{2}\right] } \\
\leq & {\left[\frac{1-e^{-\bar{\rho}_{2} T}}{\bar{\rho}_{2}}+\frac{\max \left\{1, e^{-\bar{\rho}_{2} T}\right\}}{\left(1-k_{5} c_{5}-k_{6} c_{6}\right) \min \left\{1, e^{-\bar{\rho}_{2} T}\right\}}+\max \left\{1, e^{-\bar{\rho}_{2} T}\right\}\right]\left[2\left(k_{9}^{2}+k_{10}^{2}\right) \max \left\{1, e^{-\bar{\rho}_{1} T}\right\}+k_{4} c_{4} \frac{1-e^{-\bar{\rho}_{1} T}}{\bar{\rho}_{1}}\right] } \\
& \times\left[k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right)\|\widehat{X}(\cdot)\|_{\rho}^{2}+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right)\|\widehat{Z}(\cdot)\|_{\rho}^{2}\right] .
\end{aligned}
$$

Recalling that $\bar{\rho}_{1}=\rho-2 \rho_{1}-k_{1} c_{1}^{-1}-k_{2} c_{2}^{-1}-k_{3} c_{3}^{-1}-k_{7}^{2}$ and $\bar{\rho}_{2}=-\rho-2 \rho_{2}-k_{4} c_{4}^{-1}-k_{5} c_{5}^{-1}-k_{6} c_{6}^{-1}$. Then by choosing suitable $\rho$, the first assertion is immediate. For the second assertion, since $2\left(\rho_{1}+\rho_{2}\right)<$ $-2 k_{5}^{2}-2 k_{6}^{2}-k_{7}^{2}$, we can choose a $\rho \in \mathbb{R}, 0<c_{5}<\frac{1}{2} k_{5}^{-1}, 0<c_{6}<\frac{1}{2} k_{6}^{-1}$ and sufficient large $c_{1}, c_{2}, c_{3}, c_{4}$ such that $\bar{\rho}_{1}>0, \quad \bar{\rho}_{2}>0, \quad 1-k_{5} c_{5}-k_{6} c_{6}>0$. Then, using a similar method, we get

$$
\begin{aligned}
& \mathbb{E}\left|\bar{X}_{1}(0)-\bar{X}_{2}(0)\right|^{2}+\left\|\bar{X}_{1}(\cdot)-\bar{X}_{2}(\cdot)\right\|_{\rho}^{2}+\left\|\bar{Z}_{1}(\cdot)-\bar{Z}_{2}(\cdot)\right\|_{\rho}^{2} \\
\leq & {\left[\frac{1}{\bar{\rho}_{2}}+\frac{1}{1-k_{5} c_{5}-k_{6} c_{6}}+1\right]\left[2 k_{9}^{2}+2 k_{10}^{2}+\frac{k_{4} c_{4}}{\bar{\rho}_{1}}\right] } \\
& {\left[k_{10}^{2} \mathbb{E}|\widehat{X}(0)|^{2}+\left(k_{1} c_{1}+k_{8}^{2}\right)\|\widehat{X}(\cdot)\|_{\rho}^{2}+\left(k_{2} c_{2}+k_{3} c_{3}+k_{9}^{2}\right)\|\widehat{Z}(\cdot)\|_{\rho}^{2}\right] }
\end{aligned}
$$

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[^0]:    ${ }^{1}$ The work of Ying Hu is partially supported by Lebesgue Center of Mathematics "Investissementsd'avenir" program-ANR-11-LABX-0020-01, by ANR CAESARS (Grant No. 15-CE05-0024) and by ANR MFG (Grant No. 16-CE40-0015-01).

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