Bounded Semigroup Wellposedness for a Linearized Compressible Flow Structure PDE Interaction with Material Derivative

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Abstract

We consider a compressible flow structure interaction (FSI) PDE system which is linearized about some reference rest state. The deformable interface is under the effect of an ambient field generated by the underlying and unbounded material derivative term which further contributes to the non-dissipativity of the FSI system, with respect to the standard energy inner product. In this work we show that, on an appropriate subspace, only one dimension less than the entire finite energy space, the FSI system is wellposed, and is moreover associated with a continuous semigroup which is *uniformly bounded* in time. Our approach involves establishing maximal dissipativity with respect to a special inner product which is equivalent to the standard inner product for the given finite energy space. Among other technical features, the necessary PDE estimates require the invocation of a multiplier which is intrinsic to the given compressible FSI system.

Key terms: Flow-structure interaction, compressible flows, wellposedness, uniformly bounded semigroup, material derivative

1 Introduction

Compressible flow phenomena arise in fluid mechanics, particularly in the modeling of gas dynamics. The motion of such flows is typically described via the Navier Stokes equations by way of providing qualitative information on the three basic physical variables: the pressure of the fluid p = p(x, t), the mass density $\rho = \rho(x, t)$, the fluid velocity field u = u(x, t). Unlike the case of incompressible flows wherein density ρ is a constant, the pressure associated with compressible flow has a non-local character and is an unknown function determined (implicitly) by the fluid motion. Moreover, in compressible flow dynamics the density of the fluid is considered to be an additional variable component, the resolution of which represents substantial difficulties in the associated mathematical analysis.

In this work, we consider the linearization of a coupled flow-structure-interaction (FSI) PDE system, with compressible fluid flow PDE component. In the context of real world applications,

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this FSI finds its key application in aeroelasticity: this PDE system involves the strong coupling between a dynamically deforming structure (e.g. the wing) and the air flow which streams past it. In short, this system describes the interaction between plate and flow dynamics through a deformable interface.

The description of our FSI PDE model is given as follows: Let the flow domain $\mathcal{O} \subset \mathbb{R}^3$ with boundary $\partial \mathcal{O}$. We assume that $\partial \mathcal{O} = \overline{S} \cup \overline{\Omega}$, with $S \cap \Omega = \emptyset$, and with (structure) domain $\Omega \subset \mathbb{R}^3$ being a *flat* portion of $\partial \mathcal{O}$. In particular, $\partial \mathcal{O}$ has the following specific configuration:

$$\Omega \subset \{x = (x_1, x_2, 0)\} \text{ and surface } S \subset \{x = (x_1, x_2, x_3) : x_3 \le 0\}.$$
(1)

Let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to $\partial \mathcal{O}$, and $\mathbf{n}|_{\Omega} = [0, 0, 1]$. Also, we denote the unit outward normal vector to $\partial \Omega$ by $\nu(\mathbf{x})$. Additional geometric assumptions on \mathcal{O} will be specified later. Also, we assume that the pressure is a linear function of the density; $p(x,t) = C\rho(x,t)$ as mostly done in the compressible fluid literature and it is chosen as a primary variable to solve.

With respect to some equilibrium point of the form $\{p_*, \mathbf{U}, \varrho_*\}$ where the pressure and density components p_*, ϱ_* are assumed to be scalars, and the arbitrary ambient field $\mathbf{U} : \mathcal{O} \to \mathbb{R}^3$

$$\mathbf{U}(x_1, x_2, x_3) = [U_1(x_1, x_2, x_3), U_2(x_1, x_2, x_3), U_3(x_1, x_2, x_3)]$$

is given, this linearization produces the following system of equations, in solution variables $u(x_1, x_2, x_3, t)$ (flow velocity), $p(x_1, x_2, x_3, t)$ (pressure), $w_1(x_1, x_2, t)$ (elastic plate displacement) and $w_2(x_1, x_2, t)$ (elastic plate velocity):

$$\begin{cases} p_t + \mathbf{U} \cdot \nabla p + \operatorname{div} u + \operatorname{div}(\mathbf{U})p = 0 & \operatorname{in} \ \mathcal{O} \times (0, \infty) \\ u_t + \mathbf{U} \cdot \nabla u - \operatorname{div}\sigma(u) + \eta u + \nabla p = 0 & \operatorname{in} \ \mathcal{O} \times (0, \infty) \\ (\sigma(u)\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 & \operatorname{on} \ \partial \mathcal{O} \times (0, \infty) \\ u \cdot \mathbf{n} = 0 & \operatorname{on} \ S \times (0, \infty) \\ u \cdot \mathbf{n} = w_2 + \mathbf{U} \cdot \nabla w_1 & \operatorname{on} \ \Omega \times (0, \infty) \\ w_{1t} - w_2 - \mathbf{U} \cdot \nabla w_1 = 0 & \operatorname{on} \ \Omega \times (0, \infty) \\ w_{2t} + \Delta^2 w_1 + [2\nu \partial_{x_3}(u)_3 + \lambda \operatorname{div}(u) - p]_{\Omega} = 0 & \operatorname{on} \ \Omega \times (0, \infty) \\ w_1 = \frac{\partial w_1}{\partial \nu} = 0 & \operatorname{on} \ \partial \Omega \times (0, \infty) \\ p(0), u(0), w_1(0), w_2(0)] = [p_0, u_0, w_a, w_b] \in H_N^{\perp}. \end{cases}$$
(4)

where the space H_N^{\perp} is defined in (16). The quantity $\eta > 0$ represents a drag force of the domain on the viscous flow. In addition, the quantity τ in (2) is in the space $TH^{1/2}(\partial \mathcal{O})$ of tangential vector fields of Sobolev index 1/2; that is,

$$\tau \in TH^{1/2}(\partial \mathcal{O}) = \{ \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O}) : \mathbf{v}_{\partial \mathcal{O}} \cdot \mathbf{n} = 0 \text{ on } \partial \mathcal{O} \}.$$
(5)

(See e.g., p.846 of [15].) In addition, we take ambient field $\mathbf{U} \in \mathbf{V}_0 \cap W$ where

$$\mathbf{V}_0 = \{ \mathbf{v} \in \mathbf{H}^1(\mathcal{O}) : \mathbf{v}|_{\partial \mathcal{O}} \cdot \mathbf{n} = 0 \text{ on } \partial \mathcal{O} \}$$
(6)

and

$$W = \{ v \in \mathbf{H}^{1}(\mathcal{O}) : v \in L^{\infty}(\mathcal{O}), \quad div(v) \in L^{\infty}(\mathcal{O}), \quad \text{and} \quad \mathbf{U}|_{\Omega} \in C^{2}(\overline{\Omega}) \}$$
(7)

(This vanishing of the boundary for ambient fields is a standard assumption in compressible flow literature; see [19], [33], [26], [1].) Moreover, the *stress and strain tensors* in the flow PDE component of (2)-(4) are defined respectively as

$$\sigma(\mu) = 2\nu\epsilon(\mu) + \lambda[I_3 \cdot \epsilon(\mu)]I_3; \quad \epsilon_{ij}(\mu) = \frac{1}{2} \left(\frac{\partial\mu_j}{\partial x_i} + \frac{\partial\mu_i}{\partial x_j}\right), \quad 1 \le i, j \le 3,$$

where Lamé Coefficients $\lambda \geq 0$ and $\nu > 0$.

Remark 1 As will be seen below, the appearance of the term $-w_2 - \mathbf{U} \cdot \nabla w_1$, in the mechanical displacement equation (3), will induce an invariance with respect to the space H_N^{\perp} defined in (16). We will ultimately establish that solutions of (2)-(4), with initial data in H_N^{\perp} , are associated with a bounded semigroup, for \mathbf{U} sufficiently small with respect to an appropriate measurement (see 24)). In addition, if we set $w(t) = w_1(t)$, $w_t = w_2 + \mathbf{U} \cdot \nabla w_1$, then we have that $[p, u, w, w_t]$ solves

$$\begin{cases} p_t + \mathbf{U} \cdot \nabla p + div \ u + div(\mathbf{U})p = 0 \quad in \quad \mathcal{O} \times (0, \infty) \\ u_t + \mathbf{U} \cdot \nabla u - div\sigma(u) + \eta u + \nabla p = 0 \quad in \quad \mathcal{O} \times (0, \infty) \\ (\sigma(u)\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 \quad on \quad \partial \mathcal{O} \times (0, \infty) \\ u \cdot \mathbf{n} = 0 \quad on \quad S \times (0, \infty) \\ u \cdot \mathbf{n} = w_t \quad on \quad \Omega \times (0, \infty) \\ \end{cases} \\ \begin{cases} w_{tt} + \Delta^2 w - \mathbf{U} \cdot \nabla w_t + [2\nu\partial_{x_3}(u)_3 + \lambda div(u) - p]_{\Omega} = 0 \quad on \quad \Omega \times (0, \infty) \\ w = \frac{\partial w}{\partial \nu} = 0 \quad on \quad \partial \Omega \times (0, \infty) \\ w = \frac{\partial w}{\partial \nu} = 0 \quad on \quad \partial \Omega \times (0, \infty) \end{cases} \\ [p(0), u(0), w(0), w_t(0)] = [p_0, u_0, w_a, w_b + \mathbf{U} \cdot \nabla w_a] \in H_N^{\perp}. \end{cases}$$

where $w(0) = w_1(0) = w_a$ and $w_t(0) = w_2(0) + \mathbf{U} \cdot \nabla w_1(0) = w_b + \mathbf{U} \cdot \nabla w_a$.

Here, as usually done for viscous fluids, we impose the so called *impermeability condition* on Ω ; namely, we assume that no fluid passes through the elastic portion of the boundary during deflection [14, 23]. At this point, we emphasize that the FSI problem under consideration has present a *material derivative* term on the deflected interaction surface. This material derivative computes the time rate of change of any quantity such as temperature or velocity (and hence also acceleration) for a portion of a material in motion. Since our material is a fluid, then the movement is simply the flow field and any particle of fluid speeds up and down as it flows along the specified spatial domain. With respect to the change of the speed of the said fluid, the material derivative effectively gives a true rate of change of the velocity. Hence, we describe the interface Ω in Lagrangian coordinates in \mathbb{R}^3 with $S(a_1, a_2, a_3) = 0$; also let $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ be the Eulerian position inside \mathcal{O} . Then, letting $w(x_1, x_2, t)$ represent the transverse (x_3) displacement of the plate on Ω , we have that

$$S(x_1, x_2, x_3 - w(x_1, x_2; t)) \equiv S(x_1, x_2, x_3; t) = 0,$$

describes the time-evolution of the boundary. The impermeability condition requires that the material derivative $(\partial_t + \tilde{u} \cdot \nabla_{\mathbf{x}})$ vanishes on the deflected surface [14, 16, 23]:

$$(\partial_t + \tilde{u} \cdot \nabla_{\mathbf{x}}) \mathcal{S} = 0, \qquad \tilde{u} = u + \mathbf{U}$$

Applying the chain rule and rearranging, we obtain

$$\nabla_{\mathbf{x}} S \cdot \langle 0, 0, -w_t \rangle + \mathbf{U} \cdot [\nabla_{\mathbf{x}} S + \langle -S_{x_3} w_{x_1}, -S_{x_3} w_{x_2}, 0 \rangle] = -u \cdot [\nabla_{\mathbf{x}} S + \langle -S_{x_3} w_{x_1}, -S_{x_3} w_{x_2}, 0 \rangle].$$
(8)



Figure 1: Polyhedral Flow-Structure Geometries

We identify $\nabla_{\mathbf{x}} S$ as the normal to the deflected surface; assuming small deflections and restricting to $(x_1, x_2) \in \Omega$, we can identify $\nabla_{\mathbf{x}} S|_{\Omega}$ with $\mathbf{n}|_{\Omega} = \langle 0, 0, 1 \rangle$. Making use of (8), imposing that $\mathbf{U} \cdot \mathbf{n} = 0$ on $\partial \mathcal{O}$ (see (6) and discussion), and discarding quadratic terms, this relation allows us to write for $(x_1, x_2) \in \Omega$:

$$\mathbf{n} \cdot \langle 0, 0, w_t \rangle + \mathbf{U} \cdot \langle w_{x_1}, w_{x_2}, 0 \rangle = u \cdot \mathbf{n}.$$

This yields the desired flow boundary condition

$$u \cdot \mathbf{n} \big|_{\mathbf{O}} = w_t + \mathbf{U} \cdot \nabla w \tag{9}$$

in $(2)_5$ via the material derivative of the deflected elastic interaction surface.

We note that the flow linearization is taken with respect to a general inhomogeneous compressible Navier-Stokes system. However, unlike the papers [7,9] where some forcing and energy level terms in the pressure and flow equations have been neglected, due to their relative unimportance therein, in this present study, the particular energy level term $\operatorname{div}(\mathbf{U})p$ in $(2)_1$ can not be neglected, inasmuch as it plays a part in establishing that the associated FSI semigroup is uniformly bounded (and invariant) with respect to the subspace H_N^{\perp} . Accordingly, the term $\operatorname{div}(\mathbf{U})p$ is one of the ingredients in the "feedback" operator B defined in (14).

In addition to the properties given for the fluid domain \mathcal{O} before, we impose additional conditions which will be necessary for the application of some elliptic regularity results for solutions of second order boundary value problems on corner domains [20, 22]:

Condition 2 Flow domain \mathcal{O} should be curvilinear polyhedral domain which satisfies the following condition:

- Each corner of the boundary ∂O -if any- is diffeomorphic to a convex cone,
- Each point on an edge of the boundary $\partial \mathcal{O}$ is diffeomorphic to a wedge with opening $< \pi$.

Some examples of geometries can be seen in Figure 1. In reference to problem (2)-(4), the associated finite energy space will be

$$\mathcal{H} \equiv L^2(\mathcal{O}) \times \mathbf{L}^2(\mathcal{O}) \times H^2_0(\Omega) \times L^2(\Omega)$$
⁽¹⁰⁾

which is a Hilbert space, topologized by the following standard inner product:

$$(\mathbf{y}_1, \mathbf{y}_2)_{\mathcal{H}} = (p_1, p_2)_{L^2(\mathcal{O})} + (u_1, u_2)_{\mathbf{L}^2(\mathcal{O})} + (\Delta w_1, \Delta w_2)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)}$$
(11)

for any $\mathbf{y}_i = (p_i, u_i, w_i, v_i) \in \mathcal{H}, \ i = 1, 2.$

1.1 Literature

The PDE's which describe fluid structure interactions have been considered from a variety of viewpoints and with different objectives in mind; [2–13, 17, 18, 24, 30]. Analysis of FSI generally constitutes a broad area of research with applications in aeroelasticity, biomechanics, biomedicine, etc. In particular, the study of wellposedness of various linearized incompressible/compressible FSI models which manifest parabolic-hyperbolic coupling has a large presence in the literature: see e.g., [2,7–11,13,17,30] wherein the Navier-Stokes equations are coupled with the wave/plate equation along a fixed interface. The parabolic-hyperbolic nature of the system generally results in major mathematical difficulties, principally because the coupling mechanisms between the fluid and the solid PDE components inevitably involves boundary terms which are strictly above the level of finite energy. In the case of a *compressible* flow component in the FSI system, the analysis is further complicated: whereas for incompressible flows the density of the fluid is assumed to be a constant and pressure an unknown function determined by the fluid motion, for compressible flows the main difficulty in the analysis of the density or pressure term, arises from the fact that the density variable is no longer constant. Although in most of the works in the literature, the motion of an isentropic compressible fluid - i.e., the density is a linear function of pressure - is solely considered, still, having to contend with this additional density (pressure) variable presents a mathematical challenge, even at the level of well-posedness.

In contrast to the growing literature on incompressible fluids the knowledge about compressible fluids interacting with elastic solids is relatively limited. In fact, the very first contribution to this problem is the pioneering paper [17], where both well-posedness and the existence of global attractors were shown. In [17], the author addresses the simple case that the ambient vector field $\mathbf{U} = 0$, i.e., i.e., the linearization takes place about the trivial flow steady state. For this canonical situation, he used Galerkin approximations to prove the wellposedness result. However, the author duly noted that the case $\mathbf{U} \neq 0$ can not be handled in a similar fashion due to the existence of the troublesome – i.e., unbounded – term $\mathbf{U} \cdot \nabla p$ in the pressure equation (2)₁.

Subsequently, the linearized model in [17] with $\mathbf{U} \neq 0$ was considered in [7]. The linearization in [7], about an arbitrary non-zero state, gives rise to terms which induce a non-dissipativity of the resulting FSI system. For this non-dissipative FSI in [7], a *pure* velocity matching condition is imposed at the interface (i.e., no material derivative is present in this boundary condition). In contrast to the Galerkin approach applied in [17], the authors in [7] invoke a certain Lumer-Phillips methodology, with a view of associating solutions of the fluid-structure dynamics with a continuous semigroup which is not uniformly bounded. Subsequently, a more convoluted FSI model was considered in [9] where, in addition to the aforesaid non-dissipative and unbounded terms brought about by ambient field $\mathbf{U} \neq 0$, the associated flow-structure interface is also under the effect of this ambient field $\mathbf{U} \neq 0$. In particular, the flow and structure velocity matching boundary condition also contains the *material derivative* of the structure, which again refers to the rate of change of the velocity on the deflected interaction surface. In [9] semigroup wellposedness is established by an appropriate invocation of the Lumer-Phillips Theorem; this semigroup generation is posed with respect to the *entire* phase space \mathcal{H} , as defined in (10) above.

However, this wellposedness result in [9] is not totally satisfactory, from the standpoint of future studies into the long time behavior of FSI solutions: while [9] does provided existence and uniqueness of solutions to the FSI system in the entire finite energy space \mathcal{H} , the resulting semigroup is not uniformly bounded. In particular, the semigroup estimate obtained in [9] is $\mathcal{O}\left(e^{C(\mathbf{U})t}\right)$, for t > 0, where $C(\mathbf{U}) = \frac{1}{2} \|\operatorname{div}(\mathbf{U})\|_{\infty} + \epsilon$. This lack of FSI semigroup boundedness in [9] will therefore forestall any subsequent discussion of FSI stability. Accordingly, with a mind toward future investigations of the asymptotic behavior of FSI solutions, we are led to the following question: Is it possible to obtain a semigroup wellposedness result, with the semigroup being bounded uniformly in time, at least in some (inherently invariant) subspace of the finite energy space?

Motivated by this question, in the present work we consider the linearized compressible flowstructure interaction model (2)-(4), where $\mathbf{U} \neq 0$ and the material derivative term $\mathbf{U} \cdot \nabla w_1$ is in place in the matching velocity boundary condition. Since our main objective here is to obtain a *uniformly bounded* semigroup, our departure point is to find an appropriate subspace for the analysis. In order to have semigroup generation on this sought-after subspace, the prospective generator of the PDE system (2)-(4) should be invariant with respect to it. In this connection, it was shown in [24] that if operator $\mathcal{A}_0 : \mathcal{H} \to \mathcal{H}$ is the FSI generator in [7], which models the "material derivative" free FSI PDE interaction, then zero is an eigenvalue of \mathcal{A}_0 . (In particular, the action of \mathcal{A}_0 is given by \mathcal{A} of (13), with the appropriate domain of definition [which includes the pure matching velocity boundary condition]; see [24] and [7]). In fact, the null space of \mathcal{A}_0 is one dimensional, denoted here by H_N , and given explicitly in (15) below. The point of our mentioning \mathcal{A}_0 in the present problem is that, by way of obtaining a uniformly bounded semigroup, we will take our candidate space of wellposedness to be the orthogonal complement H_N^{\perp} , which is characterized by (16) below.

The necessity of finding an appropriate invariant subspace for uniformly bounded FSI semigroup analysis motivates the presence of the additional (and unbounded) term $w_2 + \mathbf{U} \cdot \nabla w_1$ in (2)-(4). Let $\mathcal{A}_1 : \mathcal{H} \to \mathcal{H}$ be the FSI generator which gives rise to the wellposedness result in [9]; the action of \mathcal{A}_1 is given by \mathcal{A} of (13) with the appropriate domain of definition, which includes the material derivative term matching velocity boundary condition; see p. 342 of [9]. As thus constituted, H_N^{\perp} is *not* invariant with respect to \mathcal{A}_1 . However, if we define an operator B which abstractly models the unbounded term $w_2 + \mathbf{U} \cdot \nabla w_1$ in (2)-(4), as well as the energy level term div(\mathbf{U})p, then with the appropriate domain of definition, H_N^{\perp} is -invariant with respect to the modeling operator ($\mathcal{A} + B$) of (2)-(4. (This is Lemma 3 below).

Having established said invariance, we will subsequently proceed to show that, with respect to a certain inner product which is equivalent to the standard \mathcal{H} -inner product, $(\mathcal{A} + B)$ generates a contraction semigroup on H_N^{\perp} , for ambient field **U** small enough in norm (and so the semigroup will be uniformly bounded with respect to the standard \mathcal{H} -norm). In consequence, the PDE system (2)-(4) is wellposed for initial data $[p_0, u_0, w_a, w_b]$ taken from H_N^{\perp} .

1.2 Challenges encountered and Novelty

In the present work, we establish a result of semigroup wellposedness so as to ascertain the existence and uniqueness of solutions to (2)-(4), for Cauchy data in H_N^{\perp} . Moreover, we find this FSI

semigroup is uniformly bounded in time. This boundedness will have implications in our future analysis of long time behavior of the solutions to the PDE system (2)-(4). The main challenging points and improvements in our treatment are as follows:

(a) Uniformly bounded semigroup in $H_N^{\perp} \subset \mathcal{H}$: By way of fulfilling our objective of obtaining a uniformly bounded semigroup, we adopt a Lumer-Phillips approach, in an appropriate inner product. To wit, to establish dissipativity we topologize the $(\mathcal{A} + B)$ -invariant space H_N^{\perp} with an inner product which is equivalent to the standard \mathcal{H} -inner product. In this construction, we make use of a multiplier $\nabla \psi$ introduced in [17] (defined in (19) below) and previously used in [24]; the multiplier exploits the characterization of H_N^{\perp} in (16) below. In addition, inasmuch as we are after a FSI solution semigroup which is uniformly bounded in time, we give a proof for the maximality (or the range condition) of the operator $(\mathcal{A} + B)$ which is quite different than that in [9]. Unlike [9] where the theory of linear perturbations is used so as to yield a semigroup whose bound is of said exponential order, in the present we totally eschew the Lax-Milgram approach of [9] and instead invoke functional analytical and PDE methods to show that $[\lambda I - (\mathcal{A} + B)]$ is invertible for any $\lambda > 0$. This entails to show that $[\lambda I - (\mathcal{A} + B)]$ is a closed linear operator that has a dense range in H_N^{\perp} and enjoys the inverse estimate (97) below. By these means we establish that $(\mathcal{A} + B)$ is maximal dissipative with respect to said appropriate inner product, and so then a uniformly bounded semigroup on the standard \mathcal{H} -inner product. Our uniformly bounded semigroup result is valid under the assumption that ambient vector field \mathbf{U} is small enough with respect to an appropriate measurement; see (24) below. However, one should bear in mind that the present of $\mathbf{U} \neq 0$ gives rise to terms – namely, $\mathbf{U} \cdot \nabla p$ and $\mathbf{U} \cdot \nabla w_1$ (as it appears twice) – which are unbounded with respect to the underlying finite energy of the FSI system. Thus, our method of proof does not at all involve some bounded perturbation result which exploits the smallness of **U**.

(b) H_N^{\perp} - invariant generator: Subsequent to our work [9], our original immediate objective was to analyze the stability properties of the material derivative FSI system in [9]. However, because of the presence of the zero eigenvalue, as mentioned above, it is problematic to consider the strong or exponential decay problem in the entire phase space \mathcal{H} . Accordingly, we are led here to consider wellposedness (and future stability) analysis on H_N^{\perp} as given in (16) below. (Since H_N of (15) is only one dimensional, -see [24, Lemma 6]- we would not lose too much.) However, as we said above, H_N^{\perp} is not invariant with respect to the material derivative FSI generator $\mathcal{A}_1 : \mathcal{H} \to \mathcal{H}$ in [9]. (The unbounded material derivative term in particular contributes to the non-invariance.) However, the presence of the terms $-w_2 - \mathbf{U} \cdot \nabla w_1$ and $\operatorname{div}(\mathbf{U})p$ in the respective structural displacement and pressure equations in (2)-(4) gives rise to an invariance on H_N^{\perp} . (Actually, the term div(U)p was blithely disgarded during the linearization process in [9], since it is a benign energy level term.) Thus, these two terms are captured abstractly by the "feedback" operator B in (14) below. We say feedback, since B is incorporated so as to beneficiently provide the pre-requisite that H_N^{\perp} is $(\mathcal{A}+B)$ -invariant. We note that the presence of B does not at all give rise to a fortuitous cancellation of terms so as to have dissipativity with respect to the standard \mathcal{H} -inner product. The operator B allows only for said invariance property, so that our wellposedness and uniform bounded semigroup problem can be considered on the slightly smaller subspace H_N^{\perp} . As we said, our finding that the FSI semigroup is uniformly bounded in time in H_N^{\perp} will constitute a departure point in our future work on stability properties of the FSI PDE model.

(c) Less regularity required on the ambient vector field \mathbf{U} : The presence of the nontrivial ambient flow field \mathbf{U} causes substantial difficulties in the wellposedness analysis. In this case $\mathbf{U} \neq 0$, the desired result for a FSI system – with material derivative present in the matching velocities BC – on the entire phase space \mathcal{H} was obtained in the earlier work [9] (with recall, the semigroup estimate $\mathcal{O}\left(e^{C(\mathbf{U})t}\right)$, for t > 0, where $C(\mathbf{U}) = \frac{1}{2} \|\operatorname{div}(\mathbf{U})\|_{\infty} + \epsilon$). In the course of applying the Lax-Milgram Theorem in [9], there is the need to deal with the pressure PDE component of an associated static compressible FSI system. In this regard, a methodology, based upon a treatment of (uncoupled) transport equations in [19], was applied to solve for the pressure and fluid velocity components of said static FSI system. However this approach compelled the authors in [9] to impose that $\mathbf{U} \in \mathbf{H}^3(\mathcal{O})$. In the present work, we require that small enough ambient field $\mathbf{U} \in \mathbf{H}^1(\mathcal{O})$ obey the less stringent regularity assumptions in (7).

1.3 Notation

Throughout, for a given domain D, the norm of corresponding space $L^2(D)$ will be denoted as $||\cdot||_D$ (or simply $||\cdot||$ when the context is clear). Inner products in $L^2(\mathcal{O})$ or $\mathbf{L}^2(\mathcal{O})$ will be denoted by $(\cdot, \cdot)_{\mathcal{O}}$, whereas inner products $L^2(\partial \mathcal{O})$ will be written as $\langle \cdot, \cdot \rangle_{\partial \mathcal{O}}$. We will also denote pertinent duality pairings as $\langle \cdot, \cdot \rangle_{X \times X'}$ for a given Hilbert space X. The space $H^s(D)$ will denote the Sobolev space of order s, defined on a domain D; $H^s_0(D)$ will denote the closure of $C_0^{\infty}(D)$ in the $H^s(D)$ -norm $\|\cdot\|_{H^s(D)}$. We make use of the standard notation for the boundary trace of functions defined on \mathcal{O} , which are sufficiently smooth: i.e., for a scalar function $\phi \in H^s(\mathcal{O}), \frac{1}{2} < s < \frac{3}{2}, \gamma(\phi) = \phi|_{\partial \mathcal{O}}$, which is a well-defined and surjective mapping on this range of s, owing to the Sobolev Trace Theorem on Lipschitz domains (see e.g., [31], or Theorem 3.38 of [29]).

1.4 Plan of the paper

The paper is organized as follows: In Section 2, we first provide the framework which will be required for our proof of semigroup wellposedness. In particular, we carefully describe the FSI generator $(\mathcal{A} + B)$ and its domain, as well as the equivalent inner product which will be used for our proof of wellposedness on subspace H_N^{\perp} of (16) below. Moreover, we show that H_N^{\perp} is $(\mathcal{A} + B)$ invariant. In Section 3, we establish the maximal dissipativity of $(\mathcal{A} + B)$ with respect to said special inner product, thereby allowing for an appeal to the Lumer-Phillips Theorem. In the course of our work, we will have need of a classic lemma of functional analysis, as well as the adjoint of $(\mathcal{A} + B)$. These ingredients are given in Section 4, the Appendix.

2 Functional Setting and Preliminaries

With respect to the above setting, the PDE system given in (2)-(4) can be written as an ODE in Hilbert space \mathcal{H} . That is, if $\Phi(t) = [p, u, w_1, w_2] \in C([0, T]; \mathcal{H})$ solves the problem (2)-(4), then there is a modeling operator $\mathcal{A} + B : D(\mathcal{A} + B) \subset \mathcal{H} \to \mathcal{H}$ such that $\Phi(\cdot)$ satisfies

$$\frac{d}{dt}\Phi(t) = (\mathcal{A} + B)\Phi(t);$$

$$\Phi(0) = \Phi_0$$
(12)

Here the operators \mathcal{A} and the feedback operator B are defined as follows:

$$\mathcal{A} = \begin{bmatrix} -\mathbf{U} \cdot \nabla(\cdot) & -\operatorname{div}(\cdot) & 0 & 0\\ -\nabla(\cdot) & \operatorname{div}\sigma(\cdot) - \eta I - \mathbf{U} \cdot \nabla(\cdot) & 0 & 0\\ 0 & 0 & 0 & I\\ [\cdot]|_{\Omega} & -[2\nu\partial_{x_3}(\cdot)_3 + \lambda \operatorname{div}(\cdot)]_{\Omega} & -\Delta^2 & 0 \end{bmatrix};$$
(13)

and

$$B = \begin{bmatrix} -\operatorname{div}(\mathbf{U})(\cdot) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{U} \cdot \nabla(\cdot) & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
 (14)

Then, $D(\mathcal{A} + B) \subset \mathcal{H}$ is given by

$$D(\mathcal{A}+B) = \{(p_0, u_0, w_1, w_2) \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O}) \times H^2_0(\Omega) \times L^2(\Omega) : \text{ properties } (A.i) - (A.vi) \text{ hold} \},\$$
where

(A.i)
$$\mathbf{U} \cdot \nabla p_0 \in L^2(\mathcal{O})$$

(A.ii) div $\sigma(u_0) - \nabla p_0 \in \mathbf{L}^2(\mathcal{O})$ (So, $[\sigma(u_0)\mathbf{n} - p_0\mathbf{n}]_{\partial\mathcal{O}} \in \mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O})$)
(A.iii) $-\Delta^2 w_1 - [2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)]_{\Omega} + p_0|_{\Omega} \in L^2(\Omega)$ (by elliptic regularity theory $w_1 \in H^3(\Omega)$)
(A.iv) $(\sigma(u_0)\mathbf{n} - p_0\mathbf{n}) \perp TH^{1/2}(\partial\mathcal{O})$. That is,

$$\langle \sigma(u_0)\mathbf{n} - p_0\mathbf{n}, \tau \rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O})} = 0 \text{ in } \mathcal{D}'(\mathcal{O}) \text{ for every } \tau \in TH^{1/2}(\partial \mathcal{O})$$

(A.v) $w_2 + \mathbf{U} \cdot \nabla w_1 \in H^2_0(\Omega)$ (and so $w_2 \in H^1_0(\Omega)$)

(A.vi) The flow velocity component $u_0 = \mathbf{f}_0 + \widetilde{\mathbf{f}}_0$, where $\mathbf{f}_0 \in \mathbf{V}_0$ and $\widetilde{\mathbf{f}}_0 \in \mathbf{H}^1(\mathcal{O})$ satisfies¹

$$\widetilde{\mathbf{f}}_0 = \begin{cases} 0 & \text{on } S \\ (w_2 + \mathbf{U} \cdot \nabla w_1) \mathbf{n} & \text{on } \Omega \end{cases}$$

(and so $\mathbf{f}_0|_{\partial \mathcal{O}} \in TH^{1/2}(\partial \mathcal{O})$).

Moreover, we denote

$$H_N = Span \left\{ \begin{bmatrix} 1\\0\\\mathring{A}^{-1}(1)\\0 \end{bmatrix} \right\},\tag{15}$$

¹The existence of an $\mathbf{H}^{1}(\mathcal{O})$ -function $\tilde{\mathbf{f}}_{0}$ with such a boundary trace on Lipschitz domain \mathcal{O} is assured; see e.g., Theorem 3.33 of [29].

where $\mathring{A}: L^2(\Omega) \to L^2(\Omega)$ is the elliptic operator

$$\mathring{A}\varpi = \Delta^2 \varpi, \text{ with } D(\mathring{A}) = \{ w \in H^2_0(\Omega) : \Delta^2 w \in L^2(\Omega) \},\$$

and

$$H_N^{\perp} = \{ [p_0, u_0, w_1, w_2] \in \mathcal{H} : \int_{\mathcal{O}} p_0 d\mathcal{O} + \int_{\Omega} w_1 d\Omega = 0 \}$$
(16)

(see [24, Lemma 6]).

As stated before, in order to be able to obtain a uniformly bounded (contraction) semigroup, we analyze the wellposedness of problem (2)-(4) in the reduced space H_N^{\perp} . This will require us to re-topologize the phase space \mathcal{H} with a new inner product to be used in H_N^{\perp} and equivalent to the natural inner product given in (10). Now, with the above notation let us take $\varphi = [p_0, u_0, w_1, w_2] \in$ $H_N^{\perp}, \tilde{\varphi} = [\tilde{p}_0, \tilde{u}_0, \tilde{w}_1, \tilde{w}_2] \in H_N^{\perp}$. Then the new inner product is given as

$$((\varphi,\widetilde{\varphi}))_{H_N^{\perp}} = (p_0, p_0)_{\mathcal{O}} + (u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1), \widetilde{u}_0 - \alpha D(g \cdot \nabla \widetilde{w}_1)e_3 + \xi \nabla \psi(\widetilde{p}_0, \widetilde{w}_1))_{\mathcal{O}}$$

 $+ (\Delta w_1, \Delta \widetilde{w}_1)_{\Omega} + (w_2 + h_{\alpha} \cdot \nabla w_1 + \xi w_1, \widetilde{w}_2 + h_{\alpha} \cdot \nabla \widetilde{w}_1 + \xi \widetilde{w}_1)_{\Omega},$ (17)

and in turn the norm

$$\||\varphi|\|_{H_N^\perp} = \sqrt{((\varphi,\varphi))_{H_N^\perp}}$$

 $= \|p_0\|_{\mathcal{O}}^2 + \|u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1)\|_{\mathcal{O}}^2 + \|\Delta w_1\|_{\Omega}^2 + \|w_2 + h_\alpha \cdot \nabla w_1 + \xi w_1\|_{\Omega}^2$ (18) for every $\varphi = [p_0, u_0, w_1, w_2] \in H_N^{\perp}$. Here,

(i) the function $\psi = \psi(f,g) \in H^1(\mathcal{O})$ is considered to solve the following BVP for data $f \in L^2(\mathcal{O})$ and $g \in L^2(\Omega)$

$$\begin{cases}
-\Delta \psi = f & \text{in } \mathcal{O} \\
\frac{\partial \psi}{\partial n} = 0 & \text{on } S \\
\frac{\partial \psi}{\partial n} = g & \text{on } \Omega
\end{cases}$$
(19)

with the compatibility condition

$$\int_{\mathcal{O}} f d\mathcal{O} + \int_{\Omega} g d\Omega = 0.$$
⁽²⁰⁾

We should note that by known elliptic regularity results for the Neumann problem on Lipschitz domains—see e.g; [25]— we have

$$\|\psi(f,g)\|_{H^{\frac{3}{2}}(\mathcal{O})} \le [\|f\|_{\mathcal{O}} + \|g\|_{\partial\mathcal{O}}].$$
 (21)

(ii) the map $D(\cdot)$ is the Dirichlet map that extends boundary data φ defined on Ω to a harmonic function in \mathcal{O} satisfying:

$$D\varphi = f \Leftrightarrow \begin{cases} \Delta f = 0 & \text{in } \mathcal{O} \\ f|_{\partial \mathcal{O}} = \varphi|_{ext} & \text{on } \partial \mathcal{O} \end{cases}$$

where

$$\varphi|_{ext} = \begin{cases} 0 & \text{on } S \\ \phi & \text{on } \Omega \end{cases}$$

Then by, e.g., [29, Theorem 3.3.8], and Lax-Milgram, we deduce that

$$D \in \mathcal{L}\big(H_0^{1/2+\epsilon}(\Omega); H^1(\mathcal{O})\big).$$
(22)

(iii) the vector field $h_{\alpha}(\cdot)$ is defined as $h_{\alpha}(\cdot) = \mathbf{U}|_{\Omega} - \alpha g$, where $g(\cdot)$ is a C^2 extension of the normal vector $\mathbf{n}(x)$ (with respect to Ω) and we specify the parameter α to be

$$\alpha = 2 \left\| \mathbf{U} \right\|_{*},\tag{23}$$

where

$$\|\mathbf{U}\|_{*} = \|\mathbf{U}\|_{L^{\infty}(\mathcal{O})} + \|\operatorname{div}(\mathbf{U})\|_{L^{\infty}(\mathcal{O})} + \|\mathbf{U}|_{\Omega}\|_{C^{2}(\overline{\Omega})}.$$
(24)

Also, ξ is eventually specified in (60). Since the main goal of this manuscript is to have the semigroup wellposedness in the subspace H_N^{\perp} , in what follows, for the sake of simplicity, we will use the notation

$$(\mathcal{A}+B)|_{H^{\perp}_{\mathcal{M}}} = (\mathcal{A}+B).$$

Before beginning our wellposedness analysis, we firstly need to justify that the semigroup generator is indeed H_N^{\perp} - invariant. This is given in the following lemma:

Lemma 3 The operator $(\mathcal{A}+B)$ is H_N^{\perp} -invariant; that is $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$.

Proof. Let $\varphi = [p_0, u_0, w_1, w_2] \in H_N^{\perp}$, $\tilde{\varphi} = [\tilde{p}_0, \tilde{u}_0, \tilde{w}_1, \tilde{w}_2] \in H_N$. Recalling the adjoint operator \mathcal{A}^* in (106) we have

$$(\mathcal{A}\varphi,\widetilde{\varphi})_{\mathcal{H}} = (\varphi,\mathcal{A}^*\widetilde{\varphi})_{\mathcal{H}} = (\varphi,L_1\widetilde{\varphi})_{\mathcal{H}} + (\varphi,L_2\widetilde{\varphi})_{\mathcal{H}} = 0 + (\varphi,L_2\widetilde{\varphi})_{\mathcal{H}}$$

$$= \int_{\mathcal{O}} p_{0} \operatorname{div}(\mathbf{U}) 1 d\mathcal{O} + \int_{\Omega} \Delta w_{1} \Delta \mathring{A}^{-1} \left\{ \operatorname{div}[U_{1}, U_{2}] \right\} 1 d\Omega$$
$$= \int_{\mathcal{O}} p_{0} \operatorname{div}(\mathbf{U}) 1 d\mathcal{O} + \int_{\Omega} w_{1} \operatorname{div}[U_{1}, U_{2}] 1 d\Omega$$
$$= \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) p_{0} 1 d\mathcal{O} - \int_{\Omega} (\nabla w_{1} \cdot \mathbf{U}) 1 d\Omega$$
$$= \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) p_{0} 1 d\mathcal{O} - \int_{\Omega} \Delta (\nabla w_{1} \cdot \mathbf{U}) \Delta \mathring{A}^{-1}(1) d\Omega$$
$$= \left(\left[\begin{array}{c} \operatorname{div}(\mathbf{U}) p_{0} \\ 0 \\ -\nabla w_{1} \cdot \mathbf{U} \\ 0 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \\ \mathring{A}^{-1}(1) \\ 0 \end{array} \right] \right)_{\mathcal{H}}$$

$$= - \left(B\varphi, \widetilde{\varphi} \right)_{\mathcal{H}}$$

which yields that

$$(\mathcal{A}\varphi,\widetilde{\varphi})_{\mathcal{H}} = -(B\varphi,\widetilde{\varphi})_{\mathcal{H}}$$

or

$$((\mathcal{A}+B)\varphi,\widetilde{\varphi})_{\mathcal{H}}=0$$

for every $\varphi = [p_0, u_0, w_1, w_2] \in H_N^{\perp}$. Hence, $(\mathcal{A} + B)$ is H_N^{\perp} -invariant.

3 Wellposedness

This section is devoted to showing the semigroup wellposedness of the PDE system (2)-(4). The main result of this paper is given as follows:

Theorem 4 Let Condition 2 hold. Moreover, let $\|\mathbf{U}\|_*$ be sufficiently small. Then the operator $(\mathcal{A} + B) : D(\mathcal{A} + B) \cap H_N^{\perp} \to H_N^{\perp}$, as defined via (13) and (14), generates a strongly continuous semigroup $\{e^{(\mathcal{A}+B)t}\}_{t\geq 0}$ on H_N^{\perp} . Hence, for every initial data $[p_0, u_0, w_{1_0}, w_{2_0}] \in H_N^{\perp}$, the solution $[p(t), u(t), w_1(t), w_2(t)]$ of problem (2)-(4) is given continuously by

$$\begin{bmatrix} p(t) \\ u(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} = e^{(\mathcal{A}+B)t} \begin{bmatrix} p_0 \\ u_0 \\ w_{1_0} \\ w_{2_0} \end{bmatrix} \in C([0,T]; H_N^{\perp}).$$
(25)

Moreover, this semigroup is uniformly bounded in time with respect to the standard \mathcal{H} -inner product. (With respect to the special norm in (18), the semigroup is in fact a contraction.)

Remark 5 In point of fact, for ambient field U smooth enough, the operator $(\mathcal{A} + B)$ generates a continuous semigroup in the entire phase space \mathcal{H} . This conclusion can be straightforwardly obtained by invoking the machinery of [9]. However, this wellposedness on all of \mathcal{H} has its downsides: (i) The ambient field requires the stronger regularity $\mathbf{H}^3(\mathcal{O})$ (ii) the argumentation in [7, 9], which partly involves linear perturbation theory, will culminate in the semigroup of $(\mathcal{A} + B)$ not having a uniform bound; in fact the semigroup estimate on all of \mathcal{H} will be of exponential order.

To prove Theorem 4, we will appeal to Lumer-Phillips Theorem that requires the analysis of the dissipativity and maximality properties of the semigroup generator $(\mathcal{A} + B)$. We start with the dissipativity for which our main tool will be the use of the inner product defined in (17):

3.1 Dissipativity of the Generator $(\mathcal{A} + B)$

We show the dissipativity property of the generator operator $(\mathcal{A} + B)$ in the following lemma:

Lemma 6 With reference to problem (2)-(4), the semigroup generator $(\mathcal{A}+B) : D(\mathcal{A}+B) \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$ is dissipative with respect to inner product $((\cdot, \cdot))_{H_N^{\perp}}$ for $\|\mathbf{U}\|_*$ (defined in (24)) small enough. In particular, for $\varphi = [p_0, u_0, w_1, w_2] \in D(\mathcal{A}+B) \cap H_N^{\perp}$,

$$Re(([\mathcal{A}+B]\varphi,\varphi))_{H_{N}^{\perp}} \leq -\frac{(\sigma(u_{0}),\epsilon(u_{0}))_{\mathcal{O}}}{4} - \frac{\eta \|u_{0}\|_{\mathcal{O}}^{2}}{4} - \frac{\xi \|p_{0}\|_{\mathcal{O}}^{2}}{2} - \frac{\xi \|\Delta w_{1}\|_{\Omega}^{2}}{2},$$
(26)

where ξ is specified in (60).

Proof. Given $\varphi = [p_0, u_0, w_1, w_2] \in D(\mathcal{A} + B) \cap H_N^{\perp}$, we have

$$\begin{split} (([\mathcal{A}+B]\varphi,\varphi))_{H_{N}^{\perp}} &= (-\mathbf{U}\nabla p_{0} - \operatorname{div}(u_{0}) - \operatorname{div}(\mathbf{U})p_{0},p_{0})\wp \\ &+ (-\nabla p_{0} + \operatorname{div}\sigma(u_{0}) - \eta u_{0} - \mathbf{U}\nabla u_{0}, u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3})\wp \\ &+ (-\nabla p_{0} + \operatorname{div}\sigma(u_{0}) - \eta u_{0} - \mathbf{U}\nabla u_{0}, \xi\nabla\psi(p_{0},w_{1}))\wp \\ &- \alpha (D(g \cdot \nabla[w_{2} + \mathbf{U}\nabla w_{1}])e_{3}, u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi\nabla\psi(p_{0},w_{1}))\wp \\ &+ \xi (\nabla\psi(-\mathbf{U}\nabla p_{0} - \operatorname{div}(u_{0}) - \operatorname{div}(\mathbf{U})p_{0}, w_{2} + \mathbf{U}\nabla w_{1}), u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3})\wp \\ &+ \xi^{2} (\nabla\psi(-\mathbf{U}\nabla p_{0} - \operatorname{div}(u_{0}) - \operatorname{div}(\mathbf{U})p_{0}, w_{2} + \mathbf{U}\nabla w_{1}), \nabla\psi(p_{0},w_{1}))\wp \\ &+ (\Delta w_{2}, \Delta w_{1})_{\Omega} + (\Delta(\mathbf{U}\nabla w_{1}), \Delta w_{1})_{\Omega} \\ &+ (p_{0}|_{\Omega} - [2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda\operatorname{div}(u_{0})]|_{\Omega}, w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1})_{\Omega} \\ &+ (h_{\alpha} \cdot \nabla[w_{2} + \mathbf{U}\nabla w_{1}], w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1})_{\Omega} \\ &+ \xi(w_{2} + \mathbf{U}\nabla w_{1}, w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1})_{\Omega}. \end{split}$$

After integration by parts we then arrive at

$$(([\mathcal{A} + B]\varphi, \varphi))_{H_{N}^{\perp}} = -(\sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} - \eta \|u_{0}\|_{\mathcal{O}}^{2} + \frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{U})[|u_{0}|^{2} - |p_{0}|^{2}] d\mathcal{O}$$

+2*i*Im[(*p*₀, div(*u*₀))_{\mathcal{O}} + (\Delta w_{2}, \Delta w_{1})_{\Omega}] - *i*Im[(\mathbf{U}\nabla p_{0}, p_{0})_{\mathcal{O}} + (\mathbf{U}\nabla u_{0}, u_{0})_{\mathcal{O}}]
+ $\sum_{j=1}^{8} I_{j},$ (27)

where above the I_j are given by:

$$I_{1} = (\nabla p_{0} - \operatorname{div}\sigma(u_{0}) + \eta u_{0} + \mathbf{U}\nabla u_{0}, \alpha D(g \cdot \nabla w_{1})e_{3})_{\mathcal{O}}$$
$$- \alpha (p_{0}|_{\Omega} - [2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0})]|_{\Omega}, g \cdot \nabla w_{1})_{\Omega}, \qquad (28)$$

$$I_{2} = (-\nabla p_{0} + \operatorname{div}\sigma(u_{0}) - \eta u_{0} - \mathbf{U}\nabla u_{0}, \xi\nabla\psi(p_{0}, w_{1}))_{\mathcal{O}} - \xi(\Delta^{2}w_{1}, w_{1})_{\Omega} + (p_{0}|_{\Omega} - [2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda\operatorname{div}(u_{0})]|_{\Omega}, \xiw_{1})_{\Omega},$$
(29)

$$I_3 = -\alpha (D(g \cdot \nabla [w_2 + \mathbf{U} \nabla w_1])e_3, u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1))_{\mathcal{O}},$$
(30)

$$I_4 = \xi (\nabla \psi (-\mathbf{U}\nabla p_0 - \operatorname{div}(u_0) - \operatorname{div}(\mathbf{U})p_0, w_2 + \mathbf{U}\nabla w_1), u_0 - \alpha D(g \cdot \nabla w_1)e_3)_{\mathcal{O}},$$
(31)

$$I_5 = \xi^2 (\nabla \psi (-\mathbf{U}\nabla p_0 - \operatorname{div}(u_0) - \operatorname{div}(\mathbf{U})p_0, w_2 + \mathbf{U}\nabla w_1), \nabla \psi (p_0, w_1))_{\mathcal{O}},$$
(32)

$$I_6 = (\Delta(\mathbf{U}\nabla w_1), \Delta w_1)_{\Omega} - (\Delta^2 w_1, h_{\alpha} \cdot \nabla w_1)_{\Omega},$$
(33)

$$I_7 = (h_\alpha \cdot \nabla [w_2 + \mathbf{U} \nabla w_1], w_2)_\Omega, \tag{34}$$

$$I_8 = (h_{\alpha} \cdot \nabla [w_2 + \mathbf{U} \nabla w_1], h_{\alpha} \cdot \nabla w_1 + \xi w_1)_{\Omega} + \xi (w_2 + \mathbf{U} \nabla w_1, w_2 + h_{\alpha} \cdot \nabla w_1 + \xi w_1)_{\Omega}.$$
(35)

where we also recall the definition $h_{\alpha} = \mathbf{U}|_{\Omega} - \alpha g$. In the course of estimating the terms (28)-(35) above, we will invoke the polynomial

$$r(a) = a + a^2 + a^3. ag{36}$$

We start with I_1 ; integrating by parts, we have

$$I_{1} = -\alpha(p_{0}, \operatorname{div}[D(g \cdot \nabla w_{1})e_{3}])_{\mathcal{O}} + \alpha(\sigma(u_{0}), \epsilon(D(g \cdot \nabla w_{1})e_{3})_{\mathcal{O}} + \alpha\eta(u_{0}, D(g \cdot \nabla w_{1})e_{3})_{\mathcal{O}} + \alpha(\mathbf{U}\nabla u_{0}, D(g \cdot \nabla w_{1})e_{3})_{\mathcal{O}}$$
(37)

Using the fact that Dirichlet map $D \in L(H_0^{\frac{1}{2}+\epsilon}(\Omega), H^1(\mathcal{O}))$, we have

$$I_{1} \leq r(\|\mathbf{U}\|_{*})C\left\{\|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2}\right\}$$
(38)

We continue with I_2 ; using the definition of the map $\psi(\cdot, \cdot)$ in (19) and integrating by parts we get

$$\begin{split} I_2 &= -\xi \int_{\mathcal{O}} |p_0|^2 \, d\mathcal{O} - \xi(\sigma(u_0), \epsilon(\nabla \psi(p_0, w_1)))_{\mathcal{O}} \\ &+ \xi \, \langle \sigma(u_0)n - p_0 n, (\nabla \psi(p_0, w_1), n)n \rangle_{\partial \mathcal{O}} - \eta(u_0, \xi \nabla \psi(p_0, w_1))_{\mathcal{O}} \\ &\quad (-\mathbf{U} \nabla u_0, \xi \nabla \psi(p_0, w_1))_{\mathcal{O}} - (\Delta^2 w_1, \xi w_1)_{\Omega} \\ &\quad + (p_0|_{\Omega} - [2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)] \, |_{\Omega}, \xi w_1)_{\Omega}, \end{split}$$

whence we obtain

$$I_{2} \leq -\xi \|p_{0}\|_{\mathcal{O}}^{2} - \xi \|\Delta w_{1}\|_{\Omega}^{2} + \xi r(\|\mathbf{U}\|_{*})C\left\{\|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2}\right\} + \xi C\left\{\|u_{0}\|_{H^{1}(\mathcal{O})}^{2} [\|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}]\right\}.$$
(39)

For I_3 : recalling the boundary condition

$$(u_0)_3|_{\Omega} = w_2 + \mathbf{U}\nabla w_1,$$

making use of Lemma 6.1 of [9] and considering the assumptions made on the geometry in Condition 2, we have

$$I_{3} \leq \alpha C \left\| g \cdot \nabla(u_{0})_{3} \right\|_{H^{-\frac{1}{2}}(\Omega)} \left\| u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1}) \right\|_{\mathcal{O}}$$

$$\leq C \left[r(\|\mathbf{U}\|_{*}) \left\{ \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\} + \xi^{2} \left\{ \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\} \right]$$
(40)

where we have also implicitly used the Sobolev Embedding Theorem. To continue with I_4 :

$$I_4 = \xi(\nabla \psi(-\mathbf{U}\nabla p_0 - \operatorname{div}(\mathbf{U})p_0, 0), u_0 - \alpha D(g \cdot \nabla w_1)e_3)_{\mathcal{O}}$$

$$+\xi(\nabla\psi(-\operatorname{div}(u_0), u_0 \cdot \mathbf{n}), u_0 - \alpha D(g \cdot \nabla w_1)e_3)_{\mathcal{O}}$$
$$= I_{4a} + I_{4b}$$
(41)

Since $\mathbf{U} \cdot \mathbf{n}|_{\partial \mathcal{O}} = \mathbf{0}$, we have that $(\mathbf{U} \nabla p_0 + \operatorname{div}(\mathbf{U}) p_0) \in [H^1(\mathcal{O})]'$ with

$$\|\mathbf{U}\nabla p_{0} + \operatorname{div}(\mathbf{U})p_{0}\|_{[H^{1}(\mathcal{O})]'} \leq C \|\mathbf{U}\|_{*} \|p_{0}\|_{\mathcal{O}}.$$
(42)

By Lax-Milgram Theorem, we then have

$$I_{4a} \leq C\xi \|\nabla\psi(-\mathbf{U}\nabla p_0 - \operatorname{div}(\mathbf{U})p_0, 0)\|_{\mathcal{O}} \|u_0 - \alpha D(g \cdot \nabla w_1)e_3\|_{\mathcal{O}}$$
$$\leq C\xi r(\|\mathbf{U}\|_*) \left\{ \|u_0\|_{H^1(\mathcal{O})}^2 + \|p_0\|_{\mathcal{O}}^2 + \|\Delta w_1\|_{\Omega}^2 \right\}$$
(43)

and similarly

$$I_{4b} \le C\xi r(\|\mathbf{U}\|_{*}) \left\{ \|u_0\|_{H^1(\mathcal{O})}^2 + \|\Delta w_1\|_{\Omega}^2 \right\}.$$
(44)

Now, applying (43)-(44) to (41) gives

$$I_{4} \leq C\xi r(\|\mathbf{U}\|_{*}) \left\{ \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\}.$$

$$(45)$$

Estimating I_5 : we proceed as before done for I_4 and invoke (42), Lax Milgram Theorem and the estimate (21) to have

$$I_{5} \leq C\xi^{2} \left[\|\mathbf{U}\|_{*} \left\{ \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\} + \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} \right]$$
(46)

For I_6 , in order to estimate the second term in (33), we follow the standard calculations used for the flux multipliers and the commutator symbol given by

$$[P,Q]f = P(Qf) - Q(Pf)$$
(47)

for the differential operators P and Q. Hence,

$$-(\Delta^2 w_1, h_\alpha \cdot \nabla w_1)_\Omega = (\nabla \Delta w_1, \nabla (h_\alpha \cdot \nabla w_1))_\Omega$$
(48)

$$= -(\Delta w_1, \Delta (h_{\alpha} \cdot \nabla w_1))_{\Omega} + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega, \qquad (49)$$

where, in the first identity we have directly invoked the clamped plate boundary conditions, and in the second we have used the fact that $w_1 = \partial_{\nu} w_1 = 0$ on $\partial \Omega$ which yields that

$$\frac{\partial}{\partial \nu} (h_{\alpha} \cdot \nabla w_1) = (h_{\alpha} \cdot \nu) \frac{\partial^2 w_1}{\partial \nu} = (h_{\alpha} \cdot \nu) (\Delta w_1 \big|_{\partial \Omega}).$$

(See [27] or [28, p.305]). Using the commutator bracket $[\cdot, \cdot]$, we can rewrite the latter relation as

$$-(\Delta^2 w_1, h_{\alpha} \cdot \nabla w_1)_{\Omega} = -(\Delta w_1, [\Delta, h_{\alpha} \cdot \nabla] w_1)_{\Omega} - (\Delta w_1, h_{\alpha} \cdot \nabla (\Delta w_1))_{\Omega} + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega = -(\Delta w_1, h_{\alpha} \cdot \nabla] w_1 + \int_{\partial \Omega} (h_{\alpha} \cdot \nabla) (h_{$$

With Green's relations once more:

$$-(\Delta^{2}w_{1}, h_{\alpha} \cdot \nabla w_{1})_{\Omega} = -(\Delta w_{1}, [\Delta, h_{\alpha} \cdot \nabla]w_{1})_{\Omega} - \frac{1}{2} \int_{\partial\Omega} (h_{\alpha} \cdot \nu) |\Delta w_{1}|^{2} d\partial\Omega + \frac{1}{2} \int_{\Omega} \left[\operatorname{div}(h_{\alpha}) \right] |\Delta w_{1}|^{2} d\Omega - i \operatorname{Im}(\Delta w_{1}, h_{\alpha} \cdot \nabla(\Delta w_{1}))_{\Omega} + \int_{\partial\Omega} (h_{\alpha} \cdot \nu) |\Delta w_{1}|^{2} d\partial\Omega.$$
(50)

Thus,

$$-(\Delta^2 w_1, h_{\alpha} \cdot \nabla w_1)_{\Omega} = -(\Delta w_1, [\Delta, h_{\alpha} \cdot \nabla] w_1)_{\Omega} + \frac{1}{2} \int_{\partial \Omega} (h_{\alpha} \cdot \nu) |\Delta w_1|^2 d\partial \Omega + \frac{1}{2} \int_{\Omega} \left[\operatorname{div}(h_{\alpha}) \right] |\Delta w_1|^2 d\Omega - i \operatorname{Im}(\Delta w_1, h_{\alpha} \cdot \nabla(\Delta w_1)).$$
(51)

Since $h_{\alpha} = \mathbf{U}|_{\Omega} - \alpha g$, where g is an extension of $\nu(\mathbf{x})$, we will have then

$$-\operatorname{Re}(\Delta^{2}w_{1}, h_{\alpha} \cdot \nabla w_{1})_{\Omega} = \frac{1}{2} \int_{\partial\Omega} (\mathbf{U} \cdot \boldsymbol{\nu} - \alpha) |\Delta w_{1}|^{2} d\partial\Omega + \frac{1}{2} \int_{\Omega} \operatorname{div}(h_{\alpha}) |\Delta w_{1}|^{2} d\Omega - \operatorname{Re}(\Delta w_{1}, [\Delta, h_{\alpha} \cdot \nabla] w_{1})_{\Omega}$$

$$\tag{52}$$

Since we can explicitly compute the commutator

$$\begin{split} [\Delta, h_{\alpha} \cdot \nabla] w_1 = & (\Delta h_1)(\partial_{x_1} w_1) + 2(\partial_{x_1} h_1)(\partial_{x_1}^2 w_1) + 2(\partial_{x_2} h_2)(\partial_{x_2}^2 w_1) + (\Delta h_2)(\partial_{x_2} w_1) \\ &+ 2 \mathrm{div}(h_{\alpha})(\partial_{x_1} \partial_{x_2} w_1), \end{split}$$

and

$$\left|\left|\left[\Delta, h_{\alpha} \cdot \nabla\right]w_{1}\right|\right|_{L^{2}(\Omega)} \leq Cr(\left\|\mathbf{U}\right\|_{*})\left|\left|\Delta w_{1}\right|\right|_{L^{2}(\Omega)}.$$
(53)

combining (52)-(53) we eventually get

$$-\operatorname{Re}(\Delta^{2}w_{1}, h_{\alpha} \cdot \nabla w_{1})_{\Omega} \leq \frac{1}{2} \int_{\partial\Omega} [\mathbf{U} \cdot \nu - \alpha] |\Delta w_{1}|^{2} d\partial\Omega + Cr(\|\mathbf{U}\|_{*}) \|\Delta w_{1}\|_{\Omega}^{2}.$$
(54)

Moreover, for the first term of (33), we have

$$(\Delta(\mathbf{U}\nabla w_1), \Delta w_1)_{\Omega} = (\mathbf{U}\nabla w_1), \Delta w_1)_{\Omega} - ([\mathbf{U}\cdot\nabla, \Delta]w_1, \Delta w_1)_{\Omega}$$
$$= \int_{\partial\Omega} (\mathbf{U}\cdot\nu) |\Delta w_1|^2 d\partial\Omega - \int_{\partial\Omega} \operatorname{div}(\mathbf{U}) |\Delta w_1|^2 d\partial\Omega$$
$$-([\mathbf{U}\cdot\nabla, \Delta]w_1, \Delta w_1)_{\Omega} - \int_{\Omega} \Delta w_1 \mathbf{U}\cdot\nabla(\Delta w_1) d\Omega$$

where we also use the commutator expression in (47). This gives us

$$\operatorname{Re}(\Delta(\mathbf{U}\nabla w_{1}), \Delta w_{1})_{\Omega} \leq \frac{1}{2} \int_{\partial\Omega} (\mathbf{U} \cdot \nu) \left| \Delta w_{1} \right|^{2} d\partial\Omega + Cr(\|\mathbf{U}\|_{*}) \left\| \Delta w_{1} \right\|_{\Omega}^{2}.$$
(55)

Now applying (54)-(55) to (33), we obtain

$$\operatorname{Re}I_{6} \leq \int_{\partial\Omega} \left[\mathbf{U} \cdot \boldsymbol{\nu} - \frac{\alpha}{2} \right] \left| \Delta w_{1} \right|^{2} d\partial\Omega + Cr(\|\mathbf{U}\|_{*}) \left\| \Delta w_{1} \right\|_{\Omega}^{2}.$$
(F)

To estimate I_7 : since $w_2 \in H_0^1(\Omega)$, we have

$$\operatorname{Re}(h_{\alpha} \cdot \nabla w_{2}, w_{2})_{\Omega} = -\frac{1}{2} \int_{\Omega} \operatorname{div}(h_{\alpha}) |w_{2}|^{2} d\Omega$$
$$= -\frac{1}{2} \int_{\Omega} \operatorname{div}(h_{\alpha}) |(u_{0})_{3} - \mathbf{U} \nabla w_{1}|^{2} d\Omega$$

after using the boundary condition in (A.v). Applying the last relation to RHS of (34) and recalling that $h_{\alpha} = \mathbf{U}|_{\Omega} - \alpha g$, we get

$$\operatorname{Re} I_{7} = \operatorname{Re}(h_{\alpha} \cdot \nabla w_{2}, w_{2})_{\Omega} + \operatorname{Re}(h_{\alpha} \cdot \nabla (\mathbf{U} \nabla w_{1}), (u_{0})_{3} - \mathbf{U} \nabla w_{1})_{\mathcal{O}}$$
$$\leq Cr(\|\mathbf{U}\|_{*}) \left\{ \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\}$$
(56)

where we also implicitly use Sobolev Trace Theorem. Lastly, for the term I_8 , we proceed in a manner similar to that adopted for I_7 and we have

$$I_{8} = (h_{\alpha} \cdot \nabla(u_{0})_{3}, h_{\alpha} \cdot \nabla w_{1} + \xi w_{1})_{\Omega}$$

+ $\xi((u_{0})_{3}, (u_{0})_{3} - \mathbf{U} \cdot \nabla w_{1} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1})_{\Omega}$
$$\leq C \left[r(\|\mathbf{U}\|_{*}) + \xi^{2} \right] \left\{ \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\}$$

+ $C\xi \left[\|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + r(\|\mathbf{U}\|_{*}) \left\{ \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\} \right]$ (57)

Now, if we apply (38)-(57) to RHS of (27), we obtain

$$\operatorname{Re}(([\mathcal{A} + B]\varphi, \varphi))_{H_{N}^{\perp}} \leq -(\sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} - \eta \|u_{0}\|_{\mathcal{O}}^{2} - \xi \|p_{0}\|_{\mathcal{O}}^{2} - \xi \|\Delta w_{1}\|_{\Omega}^{2}$$

$$+ \int_{\partial\Omega} [\mathbf{U} \cdot \nu - \frac{\alpha}{2}] |\Delta w_{1}|^{2} d\partial\Omega$$

$$+ C \left[r_{\mathbf{U}} + \xi r_{\mathbf{U}} + \xi^{2} + \xi \right] \|u_{0}\|_{H^{1}(\mathcal{O})}^{2}$$

$$+ C \left[r_{\mathbf{U}} + \xi r_{\mathbf{U}} + \xi^{2} + \xi^{2} r_{\mathbf{U}} \right] \left\{ \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\}$$

$$+ C \xi \|u_{0}\|_{H^{1}(\mathcal{O})}^{2} \{\|p_{0}\|_{\mathcal{O}} + \|\Delta w_{1}\|_{\Omega} \}$$
(58)

where, for the simplicity, we have set $r_{\mathbf{U}} = r(\|\mathbf{U}\|_*)$. We recall now the value of $\alpha = 2 \|\mathbf{U}\|_*$ to get

$$\operatorname{Re}(([\mathcal{A}+B]\varphi,\varphi))_{H_N^{\perp}} \leq -(\sigma(u_0),\epsilon(u_0))_{\mathcal{O}} - \eta \|u_0\|_{\mathcal{O}}^2 - \xi \|p_0\|_{\mathcal{O}}^2 - \xi \|\Delta w_1\|_{\Omega}^2$$

$$+ \left[(C_{1} + C_{2}r_{\mathbf{U}})\xi^{2} + C_{2}r_{\mathbf{U}}\xi + C_{2}r_{\mathbf{U}} \right] \left\{ \|p_{0}\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} \right\} \\ + \frac{1}{2} \left\{ (\sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} \right\} \\ + C_{3} \left[r_{\mathbf{U}} + \xi r_{\mathbf{U}} + \xi^{2} + \xi \right] \|u_{0}\|_{H^{1}(\mathcal{O})}^{2}$$
(59)

where the positive constants C_1, C_2 and C_3 are obtained with the application of Holder-Young and Korn's inequalities and C_2 depends on the constant in Korn's inequality. We now specify ξ be a zero of the equation

$$(C_1 + C_2 r_{\mathbf{U}})\xi^2 + (C_2 r_{\mathbf{U}} - \frac{1}{2})\xi + C_2 r_{\mathbf{U}} = 0.$$

Namely,

$$\xi = \frac{\frac{1}{2} - C_2 r_{\mathbf{U}}}{2(C_1 + C_2 r_{\mathbf{U}})} - \frac{\sqrt{(\frac{1}{2} - C_2 r_{\mathbf{U}})^2 - 4C_2(C_1 + C_2 r_{\mathbf{U}})r_{\mathbf{U}}}}{2(C_1 + C_2 r_{\mathbf{U}})}$$
(60)

where the radicand is nonnegative for $\|\mathbf{U}\|_*$ sufficiently small. Then (59) becomes

$$\operatorname{Re}(([\mathcal{A}+B]\varphi,\varphi))_{H_{N}^{\perp}} \leq -\frac{(\sigma(u_{0}),\epsilon(u_{0}))_{\mathcal{O}}}{4} - \eta \frac{\|u_{0}\|_{\mathcal{O}}^{2}}{4} - \frac{\xi}{2} \|p_{0}\|_{\mathcal{O}}^{2} - \frac{\xi}{2} \|\Delta w_{1}\|_{\Omega}^{2} - \frac{(\sigma(u_{0}),\epsilon(u_{0}))_{\mathcal{O}}}{4} - \eta \frac{\|u_{0}\|_{\mathcal{O}}^{2}}{4} + C_{K} \left[r_{\mathbf{U}} + \xi r_{\mathbf{U}} + \xi^{2} + \xi\right] \left\{ (\sigma(u_{0}),\epsilon(u_{0}))_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} \right\}.$$

With ξ as prescribed in (60), we now have the dissipativity estimate (26), for $||\mathbf{U}||_*$ small enough. (Here we also implicitly re-use Korn's inequality and C_K is the constant there). This concludes the proof of Lemma 6.

3.2 Maximality of the Generator $(\mathcal{A} + B)$

In order to complete the proof of Theorem 4, we also need to show that the semigroup generator $(\mathcal{A} + B) : D(\mathcal{A} + B) \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$ is maximal dissipative. This is given in the following lemma:

Lemma 7 With reference to problem (2)-(4), the semigroup generator $(\mathcal{A}+B)$: $D(\mathcal{A}+B) \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$ is maximal dissipative. In other words, the following range condition holds:

$$Range[\lambda I - (\mathcal{A} + B)] = H_N^{\perp}$$
(61)

for some $\lambda > 0$.

Proof of Lemma 7

Proof of relation (61) is based on showing that $[\lambda I - (\mathcal{A} + B)]^{-1} \in \mathcal{L}(H_N^{\perp})$. For this, we appeal to linear operator theory and exploit Lemma 12 in Appendix as our main tool. So, with respect to

Lemma 12 the requirements to be shown are:

$$\| |[\lambda I - (\mathcal{A} + B)]\varphi| \|_{H_N^\perp} \ge m \, \||\varphi|\|_{H_N^\perp}$$

for all $\varphi \in D([\lambda I - (\mathcal{A} + B)]) \cap H_N^{\perp} = D(\mathcal{A} + B) \cap H_N^{\perp}.$

STEP (M-I): Firstly, to prove that Range[$\lambda I - (\mathcal{A} + B)$] is dense in H_N^{\perp} , we use the fact that

$$\operatorname{Range}[\lambda I - (\mathcal{A} + B)] = Null([\lambda I - (\mathcal{A} + B)]^*)^{\perp}$$

which is given in the following lemma:

Lemma 8 Let parameter $\lambda > 0$ be given. Then for $\|\mathbf{U}\|_*$ sufficiently small,

$$Null[\lambda I - (\mathcal{A} + B)^*] = \{0\}$$

Proof. Suppose that $\varphi = [p_0, u_0, w_1, w_2] \in D((\mathcal{A} + B)^*) \cap H_N^{\perp}$ satisfies

$$[\lambda I - (\mathcal{A} + B)^*]\varphi = 0.$$
⁽⁶²⁾

In PDE terms, this is

$$\begin{cases} \lambda p_{0} - \mathbf{U}\nabla p_{0} - \operatorname{div}(u_{0}) = 0 \quad \text{in } \mathcal{O} \\ \lambda u_{0} - \nabla p_{0} - \operatorname{div}\sigma(u_{0}) + \eta u_{0} - \mathbf{U}\nabla u_{0} + \operatorname{div}(\mathbf{U})u_{0} = 0 \quad \text{in } \mathcal{O} \\ u_{0} \cdot n = 0 \quad \text{on } S \\ u_{0} \cdot n = w_{2} \quad \text{on } \Omega \\ \lambda w_{1} + w_{2} - \mathring{A}^{-1} \left\{ \operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla \right\} \left[p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0}) - \Delta^{2}w_{1} \right]_{\Omega} \\ - \mathbf{U} \cdot \nabla w_{1} - \Delta \mathring{A}^{-1} \nabla^{*} (\nabla \cdot (\mathbf{U} \cdot \nabla w_{1})) = 0 \quad \text{in } \Omega \\ \lambda w_{2} + \left[p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0}) \right] |_{\Omega} - \Delta^{2}w_{1} = 0 \quad \text{in } \Omega \\ w_{1}|_{\partial\Omega} = \frac{\partial w_{1}}{\partial \nu}|_{\partial\Omega} = 0 \end{cases}$$

$$\tag{63}$$

Since we have from (62)

$$0 = \lambda \left\|\varphi\right\|_{\mathcal{H}}^2 - ((\mathcal{A} + B)^* \varphi, \varphi)_{\mathcal{H}}$$
(64)

integrating by parts as usual, we get

$$\begin{split} \lambda \left\|\varphi\right\|_{\mathcal{H}}^{2} + (\sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} + \eta \left\|u_{0}\right\|_{\mathcal{O}}^{2} \\ &= -\frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{U})[|p_{0}|^{2} + 3|u_{0}|^{2}]d\mathcal{O} \\ &+ \left(\{\operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla\} \left[p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0}) - \Delta^{2} w_{1}\right]_{\Omega}, w_{1}\right)_{\Omega} \end{split}$$

$$+ (\Delta [\mathbf{U} \cdot \nabla w_1], \Delta w_1)_{\Omega} + (\nabla^* (\nabla \cdot (\mathbf{U} \cdot \nabla w_1)), \Delta w_1)_{\Omega}$$
(65)

To handle the terms on RHS of (65), we firstly invoke the map given in (19) and apply the multiplier $\nabla \psi(p_0, w_1)$ to the fluid equation (63)₂. This gives

$$\lambda (u_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} - (\nabla p_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} - (\operatorname{div}\sigma(u_0), \nabla \psi(p_0, w_1))_{\mathcal{O}} + \eta (u_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} - (\mathbf{U}\nabla u_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} + (\operatorname{div}(\mathbf{U})u_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} = 0$$
(66)

Let us look at the terms of (66):

$$- (\nabla p_0, \nabla \psi(p_0, w_1))_{\mathcal{O}} = \int_{\partial \mathcal{O}} (p_0 \cdot n) \nabla \psi(p_0, w_1) d\partial \mathcal{O}$$
$$+ \int_{\mathcal{O}} p_0 \operatorname{div}(\nabla \psi(p_0, w_1)) d\mathcal{O}$$
$$= - \int_{\mathcal{O}} |p_0|^2 d\mathcal{O} - \int_{\Omega} p_0 w_1 d\Omega.$$
(67)

Also,

$$- (\operatorname{div}\sigma(u_0), \nabla\psi(p_0, w_1))_{\mathcal{O}} + \eta (u_0, \nabla\psi(p_0, w_1))_{\mathcal{O}}$$
$$= (\sigma(u_0), \epsilon(\nabla\psi(p_0, w_1)))_{\mathcal{O}} - \langle \sigma(u_0) \cdot n, \nabla\psi(p_0, w_1) \rangle_{\partial\mathcal{O}}$$
$$+ \eta (u_0, \nabla\psi(p_0, w_1))_{\mathcal{O}}$$
(68)

Applying (67)-(68) to (66), we then have

$$\int_{\mathcal{O}} |p_0|^2 d\mathcal{O} = \lambda \left(u_0, \nabla \psi(p_0, w_1) \right)_{\mathcal{O}} - \left(\mathbf{U} \nabla u_0, \nabla \psi(p_0, w_1) \right)_{\mathcal{O}} \\
+ \left(\operatorname{div}(\mathbf{U}) u_0, \nabla \psi(p_0, w_1) \right)_{\mathcal{O}} - \left([p_0 + 2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)]_{\Omega}, w_1 \right)_{\Omega} \\
+ \left(\sigma(u_0), \epsilon (\nabla \psi(p_0, w_1)) \right)_{\mathcal{O}} + \eta \left(u_0, \nabla \psi(p_0, w_1) \right)_{\mathcal{O}} \tag{69}$$

Subsequently, we apply the multiplier w_1 to the structural equation in (63)₇, and use (69) to get

$$\int_{\mathcal{O}} |p_0|^2 d\mathcal{O} + \left(\Delta^2 w_1, w_1\right)_{\Omega} = \lambda(w_2, w_1)_{\Omega} + \lambda \left(u_0, \nabla \psi(p_0, w_1)\right)_{\mathcal{O}} \\
+ \left(\sigma(u_0), \epsilon(\nabla \psi(p_0, w_1))\right)_{\mathcal{O}} + \eta \left(u_0, \nabla \psi(p_0, w_1)\right)_{\mathcal{O}} \\
- \left(\mathbf{U}\nabla u_0, \nabla \psi(p_0, w_1)\right)_{\mathcal{O}} + \left(\operatorname{div}(\mathbf{U})u_0, \nabla \psi(p_0, w_1)\right)_{\mathcal{O}}$$
(70)

To estimate the terms on RHS of (70), we appeal to the elliptic regularity results for solutions of second order BVPs on corner domains [21]. At this point, using the geometrical assumptions in Condition 2 and the higher regularity estimate

$$\|\psi(p,w)\|_{H^2(\mathcal{O})} \le C \left[\|p\|_{\mathcal{O}} + \|w_{ext}\|_{H^{\frac{1}{2}+\varepsilon}(\partial\mathcal{O})} \right]$$

$$\leq C[\|p\|_{\mathcal{O}} + \|w\|_{H^2_0(\Omega)}],\tag{71}$$

where

$$w_{ext}(x) = \begin{cases} 0, & x \in S \\ w(x), & x \in \Omega \end{cases}$$

we obtain

$$\int_{\mathcal{O}} |p_0|^2 d\mathcal{O} + \int_{\Omega} |\Delta w_1|^2 d\Omega \le C_{\epsilon} r(\|\mathbf{U}\|_*) \left\{ \sigma(u_0), \epsilon(u_0) \right\}_{\mathcal{O}} + \eta \|u_0\|_{\mathcal{O}}^2 + \lambda \|\varphi\|_{\mathcal{H}}^2 \right\}$$
(72)

Here, we also used Holder-Young Inequalities and $r(\cdot)$ and $\|\mathbf{U}\|_*$ are given as in (36) and (24), respectively. Now, to proceed with the second term on RHS of (65):

$$\left(\left\{ \operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla \right\} \left[p_0 + 2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0) - \Delta^2 w_1 \right]_{\Omega}, w_1 \right)_{\Omega}$$

$$= \left(\left\{ \operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla \right\} \left[p_0 + 2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0) \right]_{\Omega}, w_1 \right)_{\Omega}$$

$$- \left(\left\{ \operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla \right\} \Delta^2 w_1, w_1 \right)_{\Omega}$$

$$= K_1 + K_2$$

$$(73)$$

For K_1 :

$$K_{1} = \left(\{ \operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla \} [p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0})]_{\Omega}, w_{1} \right)_{\Omega}$$

= $- \left([p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0})]_{\Omega}, \mathbf{U} \cdot \nabla w_{1} \right)_{\Omega}$ (74)

To handle the term on RHS of (74): Let $D_{\Omega}: H_0^{\frac{1}{2}+\epsilon}(\Omega) \to H^1(\mathcal{O})$ be defined by

$$D_{\Omega}g = f \Leftrightarrow \begin{cases} -\Delta f = 0 & \text{in } \mathcal{O} \\ f|_{S} = 0 & \text{on } S \\ f|_{\Omega} = g & \text{on } \Omega \end{cases}$$
(75)

Therewith,

$$([p_{0} + 2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0})]_{\Omega}, \mathbf{U} \cdot \nabla w_{1})_{\Omega} = (\sigma(u_{0}), \epsilon(D_{\Omega}(\mathbf{U} \cdot \nabla w_{1})))_{\mathcal{O}}$$
$$+ (\nabla p_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}} + (p_{0}, \operatorname{div}(D_{\Omega}(\mathbf{U} \cdot \nabla w_{1})))_{\mathcal{O}} + (\operatorname{div}\sigma(u_{0}), D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}}$$
$$= (\sigma(u_{0}), \epsilon(D_{\Omega}(\mathbf{U} \cdot \nabla w_{1})))_{\mathcal{O}} + \eta(u_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}} + (p_{0}, \operatorname{div}(D_{\Omega}(\mathbf{U} \cdot \nabla w_{1})))_{\mathcal{O}}$$
$$+ \lambda(u_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}} - (\mathbf{U} \cdot \nabla u_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}} + (\operatorname{div}(\mathbf{U})u_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}}$$
(76)

Now, applying (76) to RHS of (74), and invoking (72) we then have

$$|K_{1}| = \left| \left(\left\{ \operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla \right\} \left[p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0}) \right]_{\Omega}, w_{1} \right)_{\Omega} \right|$$

$$\leq Cr(\|\mathbf{U}\|_{*}) \left\{ \sigma(u_{0}), \epsilon(u_{0}) \right)_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} + \lambda \|\varphi\|_{\mathcal{H}}^{2} \right\}$$
(77)

where again $r(\cdot)$ and $\|\mathbf{U}\|_*$ are given as in (36) and (24), respectively. Let us now continue with K_2 :

$$K_2 = -\left(\{\operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla\} \Delta^2 w_1, w_1\right)_{\Omega}$$

$$= \left(\Delta^2 w_1, \mathbf{U} \cdot \nabla w_1\right)_{\Omega} \tag{78}$$

If we argue as in the estimates (50)-(51) by replacing h_{α} with **U**, we then have

$$\left(\Delta^2 w_1, \mathbf{U} \cdot \nabla w_1\right)_{\Omega} = (\Delta w_1, [\Delta, \mathbf{U} \cdot \nabla] w_1)_{\Omega} - \frac{1}{2} \int_{\partial \Omega} (\mathbf{U} \cdot \nu) |\Delta w_1|^2 d\partial \Omega - \frac{1}{2} \int_{\Omega} \operatorname{div}(\mathbf{U}) |\Delta w_1|^2 d\Omega$$
(79)

For the second term on RHS of (79), let $\gamma(x)$ be a C^2 -extension of the normal vector $\nu(\mathbf{x})$ to the boundary of Ω . Applying the multiplier $\gamma \cdot \nabla w_1$ to the structral equation (63)₇, we get

$$\left(\Delta^2 w_1, \gamma \cdot \nabla w_1\right)_{\Omega} = \left(\left[p_0 + 2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)\right]\right]_{\Omega}, \gamma \cdot \nabla w_1\right)_{\Omega} + \lambda (w_2, \gamma \cdot \nabla w_1)_{\Omega}$$
(80)

Revoking the elliptic map (75), we have

$$([p_{0} + 2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0})]|_{\Omega}, \gamma \cdot \nabla w_{1})_{\Omega}$$

= $(\sigma(u_{0}), \epsilon(D_{\Omega}(\gamma \cdot \nabla w_{1})))_{\mathcal{O}} + \eta (u_{0}, D_{\Omega}(\gamma \cdot \nabla w_{1}))_{\mathcal{O}} + (p_{0}, \operatorname{div}(D_{\Omega}(\gamma \cdot \nabla w_{1})))_{\mathcal{O}}$
+ $\lambda (u_{0}, D_{\Omega}(\gamma \cdot \nabla w_{1}))_{\mathcal{O}} - (\mathbf{U} \cdot \nabla u_{0}, D_{\Omega}(\gamma \cdot \nabla w_{1}))_{\mathcal{O}} + (\operatorname{div}(\mathbf{U})u_{0}, D_{\Omega}(\mathbf{U} \cdot \nabla w_{1}))_{\mathcal{O}}$ (81)

Moreover, proceeding as in (79), we get

$$\left(\Delta^2 w_1, \gamma \cdot \nabla w_1 \right)_{\Omega} = (\Delta w_1, [\Delta, \gamma \cdot \nabla] w_1)_{\Omega} - \frac{1}{2} \int_{\partial \Omega} |\Delta w_1|^2 d\partial \Omega - \frac{1}{2} \int_{\Omega} \operatorname{div}(\gamma) |\Delta w_1|^2 d\Omega$$
 (82)

Now, applying (81), (82) to (80), using (53) (replacing h_{α} with γ) and subsequently re-invoking (72), we obtain

$$\int_{\partial\Omega} |\Delta w_1|^2 d\partial\Omega \le Cr(\|\mathbf{U}\|_*) \left\{ \sigma(u_0), \epsilon(u_0))_{\mathcal{O}} + \eta \|u_0\|_{\mathcal{O}}^2 + \lambda \|\varphi\|_{\mathcal{H}}^2 \right\}$$
(83)

Combining now (78), (79), (83) and (72), we have

$$|K_{2}| = \left| \left(\{ \operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla \} \Delta^{2} w_{1}, w_{1} \right)_{\Omega} \right|$$

$$\leq Cr(\|\mathbf{U}\|_{*}) \left\{ \sigma(u_{0}), \epsilon(u_{0}) \right)_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} + \lambda \|\varphi\|_{\mathcal{H}}^{2} \right\}$$
(84)

Hence, the second term of (65) can be handled by

$$\left(\left\{ \operatorname{div}[U_{1}, U_{2}] + \mathbf{U} \cdot \nabla \right\} \left[p_{0} + 2\nu \partial_{x_{3}}(u_{0})_{3} + \lambda \operatorname{div}(u_{0}) - \Delta^{2} w_{1} \right]_{\Omega}, w_{1} \right)_{\Omega} \right|$$

$$\leq |K_{1}| + |K_{2}|$$

$$\leq Cr(\|\mathbf{U}\|_{*}) \left\{ \sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} + \lambda \|\varphi\|_{\mathcal{H}}^{2} \right\}$$

$$(85)$$

Also, for the third and fourth terms of (65):

$$(\Delta[\mathbf{U}\cdot\nabla w_1],\Delta w_1)_{\Omega} + (\nabla^*(\nabla\cdot(\mathbf{U}\cdot\nabla w_1)),\Delta w_1)_{\Omega}$$

$$= (\mathbf{U} \cdot \nabla (\Delta w_1), \Delta w_1)_{\Omega} + ([\Delta, \mathbf{U} \cdot \nabla] w_1, \Delta w_1)_{\Omega} + (\nabla [\mathbf{U} \cdot \nabla w_1], \nabla (\Delta w_1))_{\Omega}$$

$$= \frac{1}{2} \int_{\partial \Omega} (\mathbf{U} \cdot \nu) |\Delta w_1|^2 d\partial \Omega - \frac{1}{2} \int_{\Omega} \operatorname{div}(\mathbf{U}) |\Delta w_1|^2 d\Omega$$

$$+ ([\Delta, \mathbf{U} \cdot \nabla] w_1, \Delta w_1)_{\Omega} - (\mathbf{U} \cdot \nabla w_1, \Delta^2 w_1)_{\Omega}$$

Proceeding as done above, we then have

$$\left\| (\Delta [\mathbf{U} \cdot \nabla w_1], \Delta w_1)_{\Omega} + (\nabla^* (\nabla \cdot (\mathbf{U} \cdot \nabla w_1)), \Delta w_1)_{\Omega} \right\|$$

$$\leq Cr(\|\mathbf{U}\|_*) \left\{ \sigma(u_0), \epsilon(u_0))_{\mathcal{O}} + \eta \|u_0\|_{\mathcal{O}}^2 + \lambda \|\varphi\|_{\mathcal{H}}^2 \right\}$$
(86)

Finally, if we apply the estimates (72), (85) and (86) to RHS of (65), we arrive at

$$\lambda \|\varphi\|_{\mathcal{H}}^{2} + \sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2}$$
$$\leq C \|\mathbf{U}\|_{*} \left\{ \lambda \|\varphi\|_{\mathcal{H}}^{2} + (\sigma(u_{0}), \epsilon(u_{0}))_{\mathcal{O}} + \eta \|u_{0}\|_{\mathcal{O}}^{2} \right\}$$

For $\|\mathbf{U}\|_*$ small enough-independent of $\lambda > 0$ - we infer that the solution φ of (62) is zero which concludes the proof of Lemma 8.

STEP (M-II): We continue with showing that $[\lambda I - (\mathcal{A} + B)]$ is a closed operator. For this, it will be enough to prove the following lemma:

Lemma 9 The operator $\mathcal{A} + B : D(\mathcal{A} + B) \cap H_N^{\perp} \to H_N^{\perp}$ is closed.

Proof. Let $\{\varphi_n\} = \{[p_{0n}, u_{0n}, w_{1n}, w_{2n}]\} \subseteq D(\mathcal{A} + B) \cap H_N^{\perp}$ satisfy

$$\begin{array}{rccc} \varphi_n & \to & \varphi & \mathrm{in} & H_N^{\perp}, \\ (\mathcal{A} + B)\varphi_n & \to & \varphi^* & \mathrm{in} & H_N^{\perp} \end{array}$$

We must show that $\varphi \in D(\mathcal{A} + B) \cap H_N^{\perp}$, and $(\mathcal{A} + B)\varphi = \varphi^*$. To start, via the relation (26) in Lemma 6, we have

$$\frac{(\sigma(u_{0m}-u_{0n}),\epsilon(u_{0m}-u_{0n}))_{\mathcal{O}}}{4} \le -\operatorname{Re}(([\mathcal{A}+B](\varphi_m-\varphi_n,\varphi_m-\varphi_n))_{H_N^{\perp}})_{\mathbb{A}}$$

from which we infer that

$$u_{0n} \to u \quad \text{in} \quad H^1(\mathcal{O})$$

$$\tag{87}$$

Assume that for $\varphi_n^* = \{ [p_{0n}^*, u_{0n}^*, w_{1n}^*, w_{2n}^*] \} \subseteq H_N^{\perp}$

$$(\mathcal{A} + B)\varphi_n = \varphi_n^* \tag{88}$$

In PDE terms this gives

$$\begin{cases}
-\mathbf{U}\nabla p_{0n} - \operatorname{div}(u_{0n}) - \operatorname{div}(\mathbf{U})p_{0n} = p_{0n}^{*} & \text{in } \mathcal{O} \\
-\nabla p_{0n} + \operatorname{div}\sigma(u_{0n}) - \eta u_{0n} - \mathbf{U}\nabla u_{0n} = u_{0n}^{*} & \text{in } \mathcal{O} \\
w_{2n} + \mathbf{U} \cdot \nabla w_{1n} = w_{1n}^{*} & \text{in } \Omega \\
p_{0n} - [2\nu\partial_{x_{3}}(u_{0n})_{3} + \lambda \operatorname{div}(u_{0n})]|_{\Omega} - \Delta^{2}w_{1n} = w_{2n}^{*} & \text{in } \Omega
\end{cases}$$
(89)

If we read off the first equation in (89) to have

$$\mathbf{U}\nabla p_{0n} = -\operatorname{div}(u_{0n}) - \operatorname{div}(\mathbf{U})p_{0n} - p_{0n}^*$$

and take upon the limit when $n \to \infty$ we get

$$\mathbf{U}\nabla p_0 = \left[-\operatorname{div}(u_0) - \operatorname{div}(\mathbf{U})p_0 - p_0^*\right] \in L^2(\mathcal{O})$$
(90)

Moreover, using the third equation in (89), we have

$$w_2 = \lim_{n \to \infty} w_{2n} = \lim_{n \to \infty} [w_{1n}^* - \mathbf{U} \cdot \nabla w_{1n}] = [w_1^* - \mathbf{U} \cdot \nabla w_1] \in H_0^1(\Omega)$$
(91)

In addition, from the domain criteria for $(\mathcal{A} + B)$, we have $u_{0n} = \mu_{0n} + \tilde{\mu}_{0n}$, where $\mu_{0n} \in \mathbf{V}_0$ and $\tilde{\mu}_{0n} \in H^1(\mathcal{O})$ satisfies

$$\widetilde{\mu}_{0n} = \begin{cases} 0 & \text{on } S \\ (w_{2n} + \mathbf{U} \cdot \nabla w_{1n}) \mathbf{n} & \text{on } \Omega \end{cases}$$

Since V_0 is closed, then by (87), (91) and the Sobolev Trace Theorem, we have

$$u_0 = \mu_0 + \widetilde{\mu}_0,\tag{92}$$

where $\mu_0 \in \mathbf{V}_0$ and $\widetilde{\mu}_0 \in H^1(\mathcal{O})$ satisfies

$$\widetilde{\mu}_0 = \begin{cases} 0 & \text{on } S \\ (w_2 + \mathbf{U} \cdot \nabla w_1) \mathbf{n} & \text{on } \Omega \end{cases}$$

Furthermore, we recall the form of the adjoint $(\mathcal{A} + B)^* : D(\mathcal{A} + B)^* \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$ in (106) and given arbitrary $\Phi \in \mathcal{D}(\mathcal{O})$ we will have then $[0, \Phi, 0, 0] \in D(\mathcal{A} + B)^* \cap H_N^{\perp}$. Therewith, we have

$$\begin{pmatrix} \varphi, (\mathcal{A}+B)^* \begin{bmatrix} 0\\ \Phi\\ 0\\ 0 \end{bmatrix} \end{pmatrix}_{\mathcal{H}} = \lim_{n \to \infty} \begin{pmatrix} \varphi_n, (\mathcal{A}+B)^* \begin{bmatrix} 0\\ \Phi\\ 0\\ 0 \end{bmatrix} \end{pmatrix}_{\mathcal{H}}$$
$$= \lim_{n \to \infty} \begin{pmatrix} (\mathcal{A}+B)\varphi_n, \begin{bmatrix} 0\\ \Phi\\ 0\\ 0 \end{bmatrix} \end{pmatrix}_{\mathcal{H}} = \begin{pmatrix} (\varphi^*, \begin{bmatrix} 0\\ \Phi\\ 0\\ 0 \end{bmatrix} \end{pmatrix}_{\mathcal{H}},$$

or

$$(p_0, \operatorname{div}(\Phi))_{\mathcal{O}} + (u_0, \operatorname{div}\sigma(\Phi) - \eta\Phi + \mathbf{U} \cdot \nabla\Phi + \operatorname{div}(\mathbf{U})\Phi)_{\mathcal{O}} = (u_0^*, \Phi)_{\mathcal{O}}$$

Upon an integration by parts this relation now becomes

$$-(\nabla p_0, \Phi)_{\mathcal{O}} + (\operatorname{div}\sigma(u_0), \Phi)_{\mathcal{O}} - \eta(u_0, \Phi)_{\mathcal{O}} - (\mathbf{U} \cdot \nabla u_0, \Phi)_{\mathcal{O}} = (u_0^*, \Phi)_{\mathcal{O}}, \quad \forall \ \Phi \in \mathcal{D}(\mathcal{O})$$

Applying a density argument to the above relation gives

$$-\nabla p_0 + \operatorname{div}\sigma(u_0) - \eta u_0 - \mathbf{U} \cdot \nabla u_0 = u_0^* \in L^2(\mathcal{O})$$
(93)

A further integration by parts assigns a meaning to the trace $[\sigma(u_0)\mathbf{n}-p_0\mathbf{n}]_{\partial\mathcal{O}}$ in $H^{-\frac{1}{2}}$ -sense. What is more: If $\gamma_0^+(\cdot) \in L(H^{\frac{1}{2}}(\partial\mathcal{O}), H^1(\mathcal{O}))$ is the right inverse of Sobolev Trace Map $\gamma_0(\cdot) = (\cdot)|_{\partial\mathcal{O}}$, then for every $g \in H^{\frac{1}{2}}(\partial\mathcal{O})$, we have

$$\langle [\sigma(u_0)\mathbf{n} - p_0\mathbf{n}]_{\partial\mathcal{O}}, g \rangle_{\partial\mathcal{O}} = (\sigma(u_0), \epsilon(\gamma_0^+(g)))\mathcal{O} + (\operatorname{div}\sigma(u_0), \gamma_0^+(g))\mathcal{O} - (p_0, \operatorname{div}\gamma_0^+(g))\mathcal{O} - (\nabla p_0, \gamma_0^+(g))\mathcal{O} = (\sigma(u_0), \epsilon(\gamma_0^+(g)))\mathcal{O} + \eta(u_0, \gamma_0^+(g))\mathcal{O} + (\mathbf{U} \cdot \nabla u_0, \gamma_0^+(g))\mathcal{O} + (u_0^*, \gamma_0^+(g))\mathcal{O} - (p_0, \operatorname{div}\gamma_0^+(g))\mathcal{O} = \lim_{n \to \infty} [(\sigma(u_{0n}), \epsilon(\gamma_0^+(g)))\mathcal{O} + \eta(u_{0n}, \gamma_0^+(g))\mathcal{O} + (\mathbf{U} \cdot \nabla u_{0n}, \gamma_0^+(g))\mathcal{O} + (u_{0n}^*, \gamma_0^+(g))\mathcal{O} - (p_{0n}, \operatorname{div}\gamma_0^+(g))\mathcal{O}] = \lim_{n \to \infty} \langle [\sigma(u_{0n})\mathbf{n} - p_{0n}\mathbf{n}]_{\partial\mathcal{O}}, g \rangle_{\partial\mathcal{O}}$$

That is

$$[\sigma(u_{0n})\mathbf{n} - p_{0n}\mathbf{n}]_{\partial\mathcal{O}} \to [\sigma(u_0)\mathbf{n} - p_0\mathbf{n}]_{\partial\mathcal{O}} \quad \text{in} \quad H^{\frac{1}{2}}(\partial\mathcal{O})$$
(94)

The last relation in turn allows us to pass to limit in $(89)_4$, and we get

$$[p_0 - (2\nu\partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0))]|_{\Omega} - \Delta^2 w_1 = w_2^* \in L^2(\Omega)$$
(95)

Lastly, from (92) and (93) and the Lax-Milgram Theorem, the flow component $u_0 = \mu_0 + \tilde{\mu}_0$ can be characterized via the solution $\mu_0 \in \mathbf{V}_0$ of the following variational problem for all $\chi \in \mathbf{V}_0$:

 $(\sigma(\mu_0), \epsilon(\chi))_{\mathcal{O}} + \eta(\mu_0, \chi)_{\mathcal{O}} = -(\sigma(\widetilde{\mu}_0), \epsilon(\chi))_{\mathcal{O}} - \eta(\widetilde{\mu}_0, \chi)_{\mathcal{O}} + (p_0, \operatorname{div}(\chi))_{\mathcal{O}} - (\mathbf{U} \cdot \nabla u_0, \chi)_{\mathcal{O}} - (u_0^*, \chi)_{\mathcal{O}}$

An integration by parts with respect to this relation now gives for all $\chi \in V_0$,

$$-(\operatorname{div}\sigma(u_0),\chi)_{\mathcal{O}} + \eta(u_0,\chi)_{\mathcal{O}} + \langle \sigma(u_0)\mathbf{n},\chi \rangle_{\partial\mathcal{O}}$$
$$= -(\nabla p_0,\chi)_{\mathcal{O}} + \langle p_0\mathbf{n},\chi \rangle_{\partial\mathcal{O}} - (\mathbf{U} \cdot \nabla u_0,\chi)_{\mathcal{O}} - (u_0^*,\chi)_{\mathcal{O}}$$

or after using (93)

$$\langle \sigma(u_0)\mathbf{n} - p_0\mathbf{n}, \chi \rangle_{\partial \mathcal{O}} = 0, \text{ for every } \chi \in V_0$$

which gives in the sense of distributions

$$[\sigma(u_0)\mathbf{n} - p_0\mathbf{n}] \cdot \tau = 0, \quad \forall \ \tau \in TH^{\frac{1}{2}}(\partial \mathcal{O})$$
(96)

Hence, the estimates (87)-(96) now give the desired conclusion and completes the proof of Lemma 9. \blacksquare

STEP (M-III): Lastly, we prove the following fact:

Lemma 10 For given $\lambda > 0$, we have the existence of a constant $\rho > 0$ such that for all $\varphi \in D(\mathcal{A} + B) \cap H_N^{\perp}$

$$\||[\lambda I - (\mathcal{A} + B)]\varphi|\|_{H_N^\perp} \ge \varrho \, \||\varphi|\|_{H_N^\perp} \tag{97}$$

where the norm $\||\cdot|\|_{H_N^{\perp}}$ is defined in (18).

Proof. Using the estimate (26) in Lemma 6, we have for given $\lambda > 0$,

$$(([\lambda I - (\mathcal{A} + B)]\varphi, \varphi))_{H_{N}^{\perp}} \geq \lambda |||\varphi||_{H_{N}^{\perp}}^{2} + C_{1} ||u_{0}||_{H^{1}(\mathcal{O})}^{2} + \frac{\epsilon}{2} \left[||p_{0}||_{\mathcal{O}}^{2} + ||\Delta w_{1}||_{\Omega}^{2} \right] \geq \lambda |||\varphi||_{H_{N}^{\perp}}^{2} + (C_{1} - \frac{\epsilon}{2}) ||u_{0}||_{H^{1}(\mathcal{O})}^{2} + \frac{\epsilon}{2} \left[||p_{0}||_{\mathcal{O}}^{2} + ||u_{0}||_{\mathcal{O}}^{2} + ||\Delta w_{1}||_{\Omega}^{2} \right]$$
(98)

With respect to the RHS: we firstly add and subtract, so as to have

$$\|u_{0}\|_{\mathcal{O}}^{2} = \|[u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})] + \alpha D(g \cdot \nabla w_{1})e_{3} - \xi \nabla \psi(p_{0}, w_{1})\|_{\mathcal{O}}^{2}$$

$$= \|[u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})]\|_{\mathcal{O}}^{2}$$

$$+ 2\operatorname{Re} (u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1}), \alpha D(g \cdot \nabla w_{1})e_{3} - \xi \nabla \psi(p_{0}, w_{1}))_{\mathcal{O}}$$

$$+ \|\alpha D(g \cdot \nabla w_{1})e_{3} - \xi \nabla \psi(p_{0}, w_{1})\|_{\mathcal{O}}^{2}$$
(99)

By using Holder-Young Inequalities we get

$$\|u_0\|_{\mathcal{O}}^2 \ge (1-\delta) \|u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1)\|_{\mathcal{O}}^2 + (1-C_{\delta}) \|\alpha D(g \cdot \nabla w_1)e_3 - \xi \nabla \psi(p_0, w_1)\|_{\mathcal{O}}^2$$
(100)

Using the boundedness of the maps $D(\cdot)$ and $\psi(\cdot, \cdot)$ defined in (22) and (21), respectively we then have

$$\|u_0\|_{\mathcal{O}}^2 \ge (1-\delta) \|u_0 - \alpha D(g \cdot \nabla w_1)e_3 + \xi \nabla \psi(p_0, w_1)\|_{\mathcal{O}}^2 + C_2(1-C_\delta) \left[\|\mathbf{U}\|_*^2 + \xi^2 \right] \|\Delta w_1\|_{\Omega}^2$$
(101)

Now, applying (101) to the RHS of (98), we get

$$(([\lambda I - (\mathcal{A} + B)]\varphi, \varphi))_{H_{N}^{\perp}} \geq \lambda |||\varphi||_{H_{N}^{\perp}}^{2} + (C_{1} - \frac{\epsilon}{2}) ||u_{0}||_{H^{1}(\mathcal{O})}^{2} + \frac{\epsilon}{2} \{ ||p_{0}||_{\mathcal{O}}^{2} + (1 - \delta) ||u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})||_{\mathcal{O}}^{2} + \left[1 + C_{2}(1 - C_{\delta}) \left[||\mathbf{U}||_{*}^{2} + \xi^{2} \right] \right] ||\Delta w_{1}||_{\Omega}^{2} \}$$
(102)

If we take now $\left\| \mathbf{U} \right\|_*$ so small such that

$$\|\mathbf{U}\|_{*}^{2} + \xi^{2} < \frac{1}{2C_{2}(C_{\delta} - 1)},$$

we then have

$$(([\lambda I - (\mathcal{A} + B)]\varphi, \varphi))_{H_{N}^{\perp}} \geq \lambda |||\varphi|||_{H_{N}^{\perp}}^{2} + (C_{1} - \frac{\epsilon}{2}) ||u_{0}||_{H^{1}(\mathcal{O})}^{2}$$

$$+ \frac{\epsilon}{2} \left\{ ||p_{0}||_{\mathcal{O}}^{2} + (1 - \delta) ||u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})||_{\mathcal{O}}^{2} + \frac{1}{2} ||\Delta w_{1}||_{\Omega}^{2} \right\}$$

$$\geq \frac{\epsilon}{2} \left\{ ||p_{0}||_{\mathcal{O}}^{2} + (1 - \delta) ||u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})||_{\mathcal{O}}^{2} + \frac{1}{2} ||\Delta w_{1}||_{\Omega}^{2} \right\}$$

$$+ \lambda ||w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1}||_{\mathcal{O}}^{2}$$

$$(103)$$

Using Cauchy-Schwarz now we obtain

$$\| |[\lambda I - (\mathcal{A} + B)]\varphi|\|_{H_{N}^{\perp}} \||\varphi|\|_{H_{N}^{\perp}} \leq \frac{\epsilon}{2} \left\{ \|p_{0}\|_{\mathcal{O}}^{2} + (1 - \delta) \|u_{0} - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p_{0}, w_{1})\|_{\mathcal{O}}^{2} + \frac{1}{2} \|\Delta w_{1}\|_{\Omega}^{2} \right\} + \lambda \|w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1}\|_{\mathcal{O}}^{2}$$

$$(104)$$

which gives the desired estimate (97), with therein

$$\varrho = \min\left\{\frac{\epsilon}{4}, \lambda\right\}$$

and finishes the proof of Lemma 10. \blacksquare Now, combining Lemma 8, Lemma 9 and Lemma 10 gives that the map $[\lambda I - (\mathcal{A} + B)]$ satisfies the requirements of Lemma 12 in Appendix which, in turn, yields that

$$[\lambda I - (\mathcal{A} + B)]^{-1} \in \mathcal{L}(H_N^{\perp})$$

and the range condition (61) holds. This finishes the proof of Lemma 7.

By Lemma 6 and Lemma 7, we have the desired contraction semigroup generation with respect to the special inner product $((\cdot, \cdot))_{H_N^{\perp}}$. Hence we have the asserted wellposedness statement of Theorem 4.

Moreover, form the values of the parameters α and ξ in (23) and (60), respectively, as well as the definition of $((\cdot, \cdot))_{H_N^{\perp}}$ in (17), we infer that $e^{(\mathcal{A}+B)t}$ is uniformly bounded in time, in the standard \mathcal{H} -norm. In fact, given $\phi^* = [p^*, u^*, w_1^*, w_2^*] \in H_N^{\perp}$, set

$$\phi(t) = \begin{bmatrix} p(t) \\ u(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} = e^{(\mathcal{A}+B)t} \begin{bmatrix} p^* \\ u^* \\ w_1^* \\ w_2^* \end{bmatrix}$$
(105)

Then,

$$\|\phi(t)\|_{\mathcal{H}}^{2} = \|p\|_{\mathcal{O}}^{2} + \|u\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} + \|w_{2}\|_{\Omega}^{2}$$

$$\leq C \Big[\|p\|_{\mathcal{O}}^{2} + \|u - \alpha D(g \cdot \nabla w_{1})e_{3} + \xi \nabla \psi(p, w_{1})\|_{\mathcal{O}}^{2} + \alpha^{2} \|D(g \cdot \nabla w_{1})e_{3}\|_{\mathcal{O}}^{2}$$

$$+ \xi^{2} \|\nabla \psi(p, w_{1})\|_{\mathcal{O}}^{2} + \|\Delta w_{1}\|_{\Omega}^{2} + \|w_{2} + h_{\alpha} \cdot \nabla w_{1} + \xi w_{1}\|_{\Omega}^{2} + \|h_{\alpha} \cdot \nabla w_{1} + \xi w_{1}\|_{\Omega}^{2} \Big]$$

$$\leq C \Big[\left\| \left\| e^{(\mathcal{A}+B)t} \phi^* \right\| \right\|_{H_N^{\perp}}^2 + \alpha^2 \left\| D(g \cdot \nabla w_1) e_3 \right\|_{\mathcal{O}}^2 + \xi^2 \left\| \nabla \psi(p,w_1) \right\|_{\mathcal{O}}^2 + \left\| h_{\alpha} \cdot \nabla w_1 + \xi w_1 \right\|_{\Omega}^2 \Big]$$

Using the fact that $e^{(\mathcal{A}+B)t}$ is a contraction semigroup on H_N^{\perp} with respect to the norm $\||\cdot|\|_{H_N^{\perp}}$, then combining this fact with (18), we have

$$\|\phi(t)\|_{\mathcal{H}}^{2} \leq C[\|\mathbf{U}\|_{*}^{2} + \xi^{2}] \|\phi(t)\|_{\mathcal{H}}^{2} + C_{1} \|\phi^{*}\|_{\mathcal{H}}^{2}$$

For $\|\mathbf{U}\|_*$ small enough, we then have

$$\|\phi(t)\|_{\mathcal{H}} \le C^* \|\phi^*\|_{\mathcal{H}}, \quad \text{for all } t > 0.$$

This concludes the proof of Theorem 4.

4 Appendix

In this section we will provide some useful lemmas that are critically used in this manuscript. In reference to problem (2)-(4), we start with defining the adjoint operator $(\mathcal{A} + B)^* : D((\mathcal{A} + B)^*) \cap H_N^{\perp} \subset H_N^{\perp} \to H_N^{\perp}$ of the semigroup generator $\mathcal{A} + B$ in the following lemma:

Lemma 11 The adjoint operator of the generator (A + B) (given via (13)-(14)) is defined as

Here, $\nabla^* \in \mathcal{L}(L^2(\Omega), [H^1(\Omega)]')$ is the adjoint of the gradient operator $\nabla \in \mathcal{L}(H^1(\Omega), L^2(\Omega))$ and the domain of $(\mathcal{A} + B)^*|_{H_N^{\perp}}$ is given as

 $D((\mathcal{A}+B)^*)\cap H_N^{\perp} = \{(p_0, u_0, w_1, w_2) \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O}) \times H_0^2(\Omega) \times L^2(\Omega) : \text{ properties } (\mathbf{A}^*.\mathbf{i}) - (\mathbf{A}^*.\mathbf{vii}) \text{ hold}\},$ where

- 1. $(\mathbf{A}^*.\mathbf{i}) \ \mathbf{U} \cdot \nabla p_0 \in L^2(\mathcal{O})$
- 2. $(\mathbf{A}^*.\mathbf{i}\mathbf{i}) \operatorname{div} \sigma(u_0) + \nabla p_0 \in \mathbf{L}^2(\mathcal{O}) (\operatorname{So}, [\sigma(u_0)\mathbf{n} + p_0\mathbf{n}]_{\partial\mathcal{O}} \in \mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O}))$
- 3. $(\mathbf{A}^*.\mathbf{iii}) \ \Delta^2 w_1 [2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)]_{\Omega} p_0|_{\Omega} \in L^2(\Omega)$
- 4. $(\mathbf{A}^*.\mathbf{iv}) \ (\sigma(u_0)\mathbf{n} + p_0\mathbf{n}) \perp TH^{1/2}(\partial \mathcal{O}).$ That is,

$$\langle \sigma(u_0)\mathbf{n} + p_0\mathbf{n}, \tau \rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O})} = 0 \text{ in } \mathcal{D}'(\mathcal{O}) \text{ for every } \tau \in TH^{1/2}(\partial \mathcal{O})$$

5. $(\mathbf{A}^*.\mathbf{v})$ The flow velocity component $u_0 = \mathbf{f}_0 + \tilde{\mathbf{f}}_0$, where $\mathbf{f}_0 \in \mathbf{V}_0$ and $\tilde{\mathbf{f}}_0 \in \mathbf{H}^1(\mathcal{O})$ satisfies

$$\widetilde{\mathbf{f}}_0 = \begin{cases} 0 & \text{on } S \\ w_2 \mathbf{n} & \text{on } \Omega \end{cases}$$

(and so $\mathbf{f}_0|_{\partial \mathcal{O}} \in TH^{1/2}(\partial \mathcal{O})$)

6.
$$(\mathbf{A}^*.\mathbf{vi}) [-w_2 + \mathbf{U}\cdot\nabla w_1 + \Delta \mathring{A}^{-1}\nabla^*(\nabla\cdot(\mathbf{U}\cdot\nabla w_1))] \in H^2_0(\Omega), \text{ (and so } w_2 \in H^1_0(\Omega))$$

7.
$$(\mathbf{A}^*.\mathbf{vii}) \int_{\mathcal{O}} [\mathbf{U} \cdot \nabla p_0 + \operatorname{div} (u_0)] d\mathcal{O}$$

+ $\int_{\Omega} \mathring{A}^{-1} \{ (\operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla) ([p_0 + 2\nu \partial_{x_3}(u_0)_3 + \lambda \operatorname{div}(u_0)]_{\Omega}) \} d\Omega$
- $\int_{\Omega} \mathring{A}^{-1} \{ (\operatorname{div}[U_1, U_2] + \mathbf{U} \cdot \nabla) \Delta^2 w_1 \} d\Omega$
+ $\int_{\Omega} [\mathbf{U} \cdot \nabla w_1 + \Delta \mathring{A}^{-1} \nabla^* (\nabla \cdot (\mathbf{U} \cdot \nabla w_1))] d\Omega$
= $0.$

Proof. Let $\varphi = [p_0, u_0, w_1, w_2] \in D(\mathcal{A} + B) \cap H_N^{\perp}, \, \widetilde{\varphi} = [\widetilde{p}_0, \widetilde{u}_0, \widetilde{w}_1, \widetilde{w}_2] \in D(\mathcal{A} + B)^* \cap H_N^{\perp}.$ Then, we have $(\mathcal{A} \circ \widetilde{\varphi}) = (\mathbf{U} \nabla \pi, \widetilde{\varphi}) = (\operatorname{div}(w_1), \widetilde{\varphi}) = (\nabla \pi, \widetilde{\varphi})$

$$\begin{aligned} (\mathcal{A}\varphi,\varphi)_{\mathcal{H}} &= -(\mathbf{U}\nabla p_{0},p_{0})_{\mathcal{O}} - (\operatorname{div}(u_{0}),p_{0})_{\mathcal{O}} - (\nabla p_{0},u_{0})_{\mathcal{O}} \\ &+ (\operatorname{div}\sigma(u_{0}),\widetilde{u}_{0})_{\mathcal{O}} - \eta(u_{0},\widetilde{u}_{0})_{\mathcal{O}} - (\mathbf{U}\nabla u_{0},\widetilde{u}_{0})_{\mathcal{O}} \\ &+ (\Delta w_{2},\Delta\widetilde{w}_{1})_{\Omega} + (p_{0}|_{\Omega} - [2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda\operatorname{div}(u_{0})]|_{\Omega},\widetilde{w}_{2})_{\Omega} - (\Delta^{2}w_{1},\widetilde{w}_{2})_{\Omega} \\ &= (p_{0},\operatorname{div}(\mathbf{U})\widetilde{p}_{0})_{\mathcal{O}} + (p_{0},\mathbf{U}\nabla\widetilde{p}_{0})_{\mathcal{O}} - \langle u_{0}\cdot\mathbf{n},\widetilde{p}_{0}\rangle_{\partial\mathcal{O}} + (u_{0},\nabla\widetilde{p}_{0})_{\mathcal{O}} \\ &+ (p_{0},\operatorname{div}(\mathbf{U}))_{\mathcal{O}} - \langle p_{0},\widetilde{u}_{0}\cdot\mathbf{n}\rangle_{\partial\mathcal{O}} - (\sigma(u_{0}),\epsilon(\widetilde{u}_{0}))_{\mathcal{O}} \\ &+ \langle\sigma(u_{0})\cdot\mathbf{n},\widetilde{u}_{0}\rangle_{\partial\mathcal{O}} - \eta(u_{0},\widetilde{u}_{0})_{\mathcal{O}} \\ &+ (u_{0},\operatorname{div}(\mathbf{U})\widetilde{u}_{0})_{\mathcal{O}} + (u_{0},\mathbf{U}\nabla\widetilde{u}_{0})_{\mathcal{O}} + (\Delta w_{2},\Delta\widetilde{w}_{1})_{\Omega} \\ &- ([2\nu\partial_{x_{3}}(u_{0})_{3} + \lambda\operatorname{div}(u_{0})]|_{\Omega} - p_{0}|_{\Omega},\widetilde{w}_{2})_{\Omega} - (\Delta w_{1},\Delta\widetilde{w}_{2})_{\Omega}. \end{aligned}$$

Using the domain criterion (A.vi), we then have from the above equality

$$(\mathcal{A}\varphi,\widetilde{\varphi})_{\mathcal{H}} = (p_0,\operatorname{div}(\mathbf{U})\widetilde{p}_0)_{\mathcal{O}} + (p_0,\mathbf{U}\nabla\widetilde{p}_0)_{\mathcal{O}}$$

$$-(w_{2} + \mathbf{U}\nabla w_{1}, \widetilde{p}_{0})_{\Omega} + (u_{0}, \nabla \widetilde{p}_{0})_{\mathcal{O}} + (p_{0}, \operatorname{div}(\widetilde{u}_{0}))_{\mathcal{O}}$$
$$-(\sigma(u_{0}), \epsilon(\widetilde{u}_{0}))_{\mathcal{O}} - \eta(u_{0}, \widetilde{u}_{0})_{\mathcal{O}} + (u_{0}, \operatorname{div}(\mathbf{U})\widetilde{u}_{0})_{\mathcal{O}} + (u_{0}, \mathbf{U}\nabla \widetilde{u}_{0})_{\mathcal{O}}$$
$$+(w_{2}, \Delta^{2}\widetilde{w}_{1})_{\Omega} - (\Delta w_{1}, \Delta \widetilde{w}_{2})_{\Omega}.$$

Subsequently, integrating by parts in the third line of the last relation, we get

$$(\mathcal{A}\varphi,\widetilde{\varphi})_{\mathcal{H}} = (p_0,\operatorname{div}(\mathbf{U})\widetilde{p}_0)_{\mathcal{O}} + (p_0,\mathbf{U}\nabla\widetilde{p}_0)_{\mathcal{O}}$$
$$-(w_2 + \mathbf{U}\nabla w_1,\widetilde{p}_0)_{\Omega} + (u_0,\nabla\widetilde{p}_0)_{\mathcal{O}} + (p_0,\operatorname{div}(\widetilde{u}_0))_{\mathcal{O}}$$
$$+(u_0,\operatorname{div}\sigma(\widetilde{u}_0))_{\mathcal{O}} - \langle u_0,\sigma(\widetilde{u}_0)\cdot\mathbf{n}\rangle_{\partial\mathcal{O}} - \eta(u_0,\widetilde{u}_0)_{\mathcal{O}}$$
$$+(u_0,\operatorname{div}(\mathbf{U})\widetilde{u}_0)_{\mathcal{O}} + (u_0,\mathbf{U}\nabla\widetilde{u}_0)_{\mathcal{O}}$$
$$+(w_2,\Delta^2\widetilde{w}_1)_{\Omega} - (\Delta w_1,\Delta\widetilde{w}_2)_{\Omega}.$$

Now, integrating by parts in the second line, and using again domain criterion (A.vi), we have

$$(\mathcal{A}\varphi,\widetilde{\varphi})_{\mathcal{H}} = (p_0,\operatorname{div}(\mathbf{U})\widetilde{p}_0)_{\mathcal{O}} + (p_0,\mathbf{U}\nabla\widetilde{p}_0)_{\mathcal{O}}$$
$$-(w_2,[\widetilde{p}_0 + 2\nu\partial_{x_3}(\widetilde{u}_0)_3 + \lambda\operatorname{div}(\widetilde{u}_0)]|_{\Omega})_{\Omega}$$
$$+(w_1,(\operatorname{div}[U_1,U_2] + \mathbf{U}\nabla)[\widetilde{p}_0 + 2\nu\partial_{x_3}(\widetilde{u}_0)_3 + \lambda\operatorname{div}(\widetilde{u}_0)]|_{\Omega})_{\Omega}$$
$$+(u_0,\nabla\widetilde{p}_0)_{\mathcal{O}} + (p_0,\operatorname{div}(\widetilde{u}_0))_{\mathcal{O}} + (u_0,\operatorname{div}\sigma(\widetilde{u}_0))_{\mathcal{O}}$$
$$-\eta(u_0,\widetilde{u}_0)_{\mathcal{O}} + (u_0,\operatorname{div}(\mathbf{U})\widetilde{u}_0)_{\mathcal{O}} + (u_0,\mathbf{U}\nabla\widetilde{u}_0)_{\mathcal{O}}$$
$$+(w_2,\Delta^2\widetilde{w}_1)_{\Omega} - (\Delta w_1,\Delta\widetilde{w}_2)_{\Omega}.$$
(107)

Also we have

$$(B\varphi, \widetilde{\varphi})_{\mathcal{H}} = -(\operatorname{div}(\mathbf{U})p_0, \widetilde{p}_0)_{\mathcal{O}} + (\Delta(\mathbf{U}\nabla w_1), \Delta\widetilde{w}_1)_{\Omega}.$$
(108)

For the second term of the RHS of the above equality: for any $w_1, \widetilde{w}_1 \in H^3(\Omega)$

$$(\Delta(\mathbf{U}\nabla w_1), \Delta \widetilde{w}_1)_{\Omega} = \left\langle \frac{\partial}{\partial \nu} (\mathbf{U}\nabla w_1), \Delta \widetilde{w}_1 \right\rangle_{\partial \Omega}$$
$$-(\nabla(\mathbf{U}\nabla w_1), \nabla \Delta \widetilde{w}_1)_{\Omega}$$
$$= \langle (\mathbf{U} \cdot \nu) \Delta w_1, \Delta \widetilde{w}_1 \rangle_{\partial \Omega} - (\nabla(\mathbf{U}\nabla w_1), \nabla \Delta \widetilde{w}_1)_{\Omega}$$

where we have used the fact that $w_1 = \frac{\partial w_1}{\partial \nu} = 0$ and this yields

$$\frac{\partial}{\partial \nu} (\mathbf{U} \nabla w_1) = (\mathbf{U} \cdot \nu) \frac{\partial^2 w_1}{\partial \nu} = (\mathbf{U} \cdot \nu) (\Delta w_1|_{\partial \Omega}).$$

Then

$$(\Delta(\mathbf{U}\nabla w_1), \Delta \widetilde{w}_1)_{\Omega} = \left\langle \Delta w_1, \frac{\partial}{\partial \nu} (\mathbf{U}\nabla \widetilde{w}_1) \right\rangle_{\partial\Omega} - (\nabla(\mathbf{U}\nabla w_1), \nabla\Delta \widetilde{w}_1)_{\Omega}$$
$$= (\Delta w_1, \Delta(\mathbf{U}\nabla \widetilde{w}_1))_{\Omega} + (\nabla\Delta w_1, \nabla(\mathbf{U}\nabla \widetilde{w}_1))_{\Omega} - (\nabla(\mathbf{U}\nabla w_1), \nabla\Delta \widetilde{w}_1)_{\Omega}$$
$$= (\Delta w_1, \Delta(\mathbf{U}\nabla \widetilde{w}_1))_{\Omega} + (\Delta w_1, \nabla^* [\nabla(\mathbf{U}\nabla \widetilde{w}_1)])_{\Omega} - (\nabla(\mathbf{U}\nabla w_1), \nabla\Delta \widetilde{w}_1)_{\Omega}$$
(109)

where $\nabla^* \in \mathcal{L}(L^2(\Omega), [H^1(\Omega)]')$ is the adjoint of the gradient operator $\nabla \in \mathcal{L}(H^1(\Omega), [L^2(\Omega)])$. To continue with the third term on RHS of (109):

$$-(\nabla(\mathbf{U}\nabla w_{1}), \nabla\Delta\widetilde{w}_{1})_{\Omega} = (\mathbf{U}\nabla w_{1}, \Delta^{2}\widetilde{w}_{1})_{\Omega}$$
$$= -(w_{1}, \{\operatorname{div}[U_{1}, U_{2}] + \mathbf{U}\nabla\} \Delta^{2}\widetilde{w}_{1})_{\Omega}$$
$$= -(\Delta w_{1}, \Delta \mathring{A}^{-1} \{\operatorname{div}[U_{1}, U_{2}] + \mathbf{U}\nabla\} \Delta^{2}\widetilde{w}_{1})_{\Omega}$$
(110)

If we take into account (110) in (109) and invoke the biharmonic operator with clamped homogeneous boundary conditions we take

$$(\Delta(\mathbf{U}\nabla w_1), \Delta \widetilde{w}_1)_{\Omega} = -(\Delta w_1, \Delta \mathring{A}^{-1} \{ \operatorname{div}[U_1, U_2] + \mathbf{U}\nabla \} \Delta^2 \widetilde{w}_1)_{\Omega} + (\Delta w_1, \Delta(\mathbf{U}\nabla \widetilde{w}_1))_{\Omega} + (\Delta w_1, \Delta[\Delta \mathring{A}^{-1}\nabla^*[\nabla(\mathbf{U}\nabla \widetilde{w}_1)]])_{\Omega}.$$
(111)

Now, considering (111) in (108) and combining the result with (107) gives the adjoint operator given in (106) and completes the proof of Lemma 11. \blacksquare

In order to establish the wellposedness result, one of the key tools that we use in our proof is the invertibility criterion of a linear, closed operator which we recall in the following lemma [32, pg.102, Lemma 3.8.18]:

Lemma 12 Let L be a linear and closed operator from the Hilbert space H into H. Then $L^{-1} \in \mathcal{L}(H)$ if and only if R(L) is dense in H and there is an m > 0 such that

$$||Lf|| \ge m ||f|| \quad for \ all \quad f \in D(L).$$

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References

- Aoyama, R. and Kagei, Y., 2016. Spectral properties of the semigroup for the linearized compressible Navier-Stokes equation around a parallel in a cylindrical domain. Advances in Differential Equations, 21(3/4), pp.265–300.
- [2] Avalos, G. and Clark, T., 2014. A Mixed Variational Formulation for the Wellposedness and Numerical Approximation of a PDE Model Arising in a 3-D Fluid-Structure Interaction, *Evolution Equations and Control Theory*, 3(4), pp.557–578.
- [3] Avalos, G. and Dvorak, M., 2008. A New Maximality Argument for a Coupled Fluid-Structure Interaction, with Implications for a Divergence Free Finite Element Method, *Applicationes Mathematicae*, 35(3), pp.259–280.
- [4] Avalos, G. and Bucci, F., 2014. Exponential decay properties of a mathematical model for a certain flow-structure interaction. In New Prospects in Direct, Inverse and Control Problems for Evolution Equations (pp. 49–78). Springer International Publishing.

- [5] Avalos, G. and Bucci, F., 2015. Rational rates of uniform decay for strong solutions to a flow-structure PDE system. *Journal of Differential Equations*, 258(12), pp.4398–4423.
- [6] G. Avalos, P.G. Geredeli, "Exponential stability of a nondissipative, compressible flowstructure PDE model", J. Evol. Equ., https://doi.org/10.1007/s00028-019-00513-9 ,(2019)
- [7] G. Avalos, P. G. Geredeli and J.T. Webster "Semigroup Well-posedness of A Linearized, Compressible flow with An Elastic Boundary", *Discrete and Continuous Dynamical Systems-B*, (2018), 23(3), pp. 1267-1295
- [8] G. Avalos, P. G. Geredeli, B. Muha; "Wellposedness, spectral analysis and asymptotic stability of a multilayered heat-wave-wave system" *Journal of Diff. Equ.*, 269 (2020), 7129-7156.
- [9] George Avalos, Pelin G. Geredeli, Justin T. Webster; "A Linearized Viscous, Compressible Flow-Plate Interaction with Non-dissipative Coupling, "Journal of Math. Anal. And Appl. Vol. 23, No. 3, May 2018.
- [10] Avalos G. and Triggiani R., 2007. The Coupled PDE System Arising in Fluid-Structure Interaction, Part I: Explicit Semigroup Generator and its Spectral Properties, *Contemporary Mathematics*, 440, pp.15–54.
- [11] Avalos G. and Triggiani R., 2009. Semigroup Wellposedness in The Energy Space of a Parabolic-Hyperbolic Coupled Stokes-Lamé PDE of Fluid-Structure Interactions, *Discrete and Continuous Dynamical Systems*, 2(3), pp.417–447.
- [12] G. Avalos, R. Triggiani, and I. Lasiecka, Heat-Wave interaction in 2 or 3 dimensions: optimal decay rates", *Journal of Mathematical Analysis and Applications*, Volume 437, Issue 2, 15 May 2016, Pages 782–815.
- [13] Bociu, L., Toundykov, D. and Zolsio, J.P., 2015. Well-posedness analysis for a linearization of a fluid-elasticity interaction. SIAM Journal on Mathematical Analysis, 47(3), pp.1958-2000.
- [14] Bolotin, V.V., 1963. Nonconservative problems of the theory of elastic stability. Macmillan.
- [15] Buffa, A., Costabel, M. and Sheen, D., 2002. On traces for $\mathbf{H}(\operatorname{curl}, \Omega)$ in Lipschitz domains. Journal of Mathematical Analysis and Applications, 276(2), pp.845–867.
- [16] Chorin, A.J. and Marsden, J.E., 1990. A mathematical introduction to flow mechanics (Vol. 3). New York: Springer.
- [17] Chueshov, I., 2014. Dynamics of a nonlinear elastic plate interacting with a linearized compressible viscous flow. Nonlinear Analysis: Theory, Methods & Applications, 95, pp.650–665.
- [18] Chueshov, I., 2014. Interaction of an elastic plate with a linearized inviscid incompressible fluid. Communications on Pure & Applied Analysis, 13(5), pp.1459–1778.
- [19] da Veiga, H.B., 1985. Stationary Motions and Incompressible Limit for Compressible Viscous flows, Houston Journal of Mathematics, Volume 13, No. 4 (1987), pp. 527-544.

- [20] M. Dauge, January 1989. Stationary Stokes and Navier Stokes Systems on Two or Three Dimensional Domains with Corners, Part I: Linearized Equations, Siam J. Math. Anal., Vol 20, No.1.
- [21] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, Lecture Notes in Mathematics, 1341, Springer-Verlag, New York (1988).
- [22] M. Dauge, Regularity and singularities in polyhedral domains. The case of Laplace and Maxwell equations, Slides d'un mini-cours de 3 heures, Karlsruhe, 7 (avril 2008), https://perso.univrennes1.fr/monique.dauge/publis/Talk-Karlsruhe08.html.
- [23] E. Dowell, 2004. A Modern Course in Aeroelasticity. Kluwer Academic Publishers.
- [24] Pelin G. Geredeli, A Time Domain Approach for the Exponential Stability of a Nondissipative Linearized Compressible Flow-Structure PDE System (https://arxiv.org/pdf/2003.00068.pdf) (submitted, 2020)
- [25] D.S. Jerison and C. E. Kenig, The Neumann Problem on Lipschitz Domains Bulletin (New Series) of the American Mathematical Society, Vol 4 (2), March (1981).
- [26] Kagei, Y., "Decay estimates on solutions of the linearized compressible Navier-Stokes equation around a parallel flow in a cylindrical domain, *Kyusha J. Math.* 69 (2015), pp. 293-343.
- [27] J. Lagnese, 1989. Boundary Stabilization of Thin Plates, SIAM, 1989.
- [28] I. Lasiecka, R. Triggiani; Control Theory for Partial Differential Equations: Volume 1, Abstract Parabolic Systems: Continuous and Approximation Theories, Cambridge University Press, 2000.
- [29] McLean, W.C.H., 2000. Strongly elliptic systems and boundary integral equations, Cambridge university press.
- [30] Muha, B. and Canic, S., 2013. Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. Arch. Rat. Mech. Analy., 207(3), pp.919–968.
- [31] Nečas, 2012. Direct Methods in the Theory of Elliptic Equations (translated by Gerard Tronel and Alois Kufner), Springer, New York.
- [32] Pazy, A., 2012. Semigroups of linear operators and applications to partial differential equations (Vol. 44). Springer Science & Business Media.
- [33] Valli, A., 1987. On the existence of stationary solutions to compressible Navier-Stokes equations, In Annales de l'IHP Analyse non linéaire, Vol.4, No 1, pp. 99-113