# Bounded Semigroup Wellposedness for a Linearized Compressible Flow Structure PDE Interaction with Material Derivative 

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#### Abstract

We consider a compressible flow structure interaction (FSI) PDE system which is linearized about some reference rest state. The deformable interface is under the effect of an ambient field generated by the underlying and unbounded material derivative term which further contributes to the non-dissipativity of the FSI system, with respect to the standard energy inner product. In this work we show that, on an appropriate subspace, only one dimension less than the entire finite energy space, the FSI system is wellposed, and is moreover associated with a continuous semigroup which is uniformly bounded in time. Our approach involves establishing maximal dissipativity with respect to a special inner product which is equivalent to the standard inner product for the given finite energy space. Among other technical features, the necesssary PDE estimates require the invocation of a multiplier which is intrinsic to the given compressible FSI system.


Key terms: Flow-structure interaction, compressible flows, wellposedness, uniformly bounded semigroup, material derivative

## 1 Introduction

Compressible flow phenomena arise in fluid mechanics, particularly in the modeling of gas dynamics. The motion of such flows is typically described via the Navier Stokes equations by way of providing qualitative information on the three basic physical variables: the pressure of the fluid $p=p(x, t)$, the mass density $\rho=\rho(x, t)$, the fluid velocity field $u=u(x, t)$. Unlike the case of incompressible flows wherein density $\rho$ is a constant, the pressure associated with compressible flow has a nonlocal character and is an unknown function determined (implicitly) by the fluid motion. Moreover, in compressible flow dynamics the density of the fluid is considered to be an additional variable component, the resolution of which represents substantial difficulties in the associated mathematical analysis.

In this work, we consider the linearization of a coupled flow-structure-interaction (FSI) PDE system, with compressible fluid flow PDE component. In the context of real world applications,

[^0]this FSI finds its key application in aeroelasticity: this PDE system involves the strong coupling between a dynamically deforming structure (e.g. the wing) and the air flow which streams past it. In short, this system describes the interaction between plate and flow dynamics through a deformable interface.

The description of our FSI PDE model is given as follows: Let the flow domain $\mathcal{O} \subset \mathbb{R}^{3}$ with boundary $\partial \mathcal{O}$. We assume that $\partial \mathcal{O}=\bar{S} \cup \bar{\Omega}$, with $S \cap \Omega=\emptyset$, and with (structure) domain $\Omega \subset \mathbb{R}^{3}$ being a flat portion of $\partial \mathcal{O}$. In particular, $\partial \mathcal{O}$ has the following specific configuration:

$$
\begin{equation*}
\Omega \subset\left\{x=\left(x_{1}, x_{2}, 0\right)\right\} \text { and surface } S \subset\left\{x=\left(x_{1}, x_{2}, x_{3}\right): x_{3} \leq 0\right\} \tag{1}
\end{equation*}
$$

Let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to $\partial \mathcal{O}$, and $\left.\mathbf{n}\right|_{\Omega}=[0,0,1]$. Also, we denote the unit outward normal vector to $\partial \Omega$ by $\nu(\mathbf{x})$. Additional geometric assumptions on $\mathcal{O}$ will be specified later. Also, we assume that the pressure is a linear function of the density; $p(x, t)=C \rho(x, t)$ as mostly done in the compressible fluid literature and it is chosen as a primary variable to solve.

With respect to some equilibrium point of the form $\left\{p_{*}, \mathbf{U}, \varrho_{*}\right\}$ where the pressure and density components $p_{*}, \varrho_{*}$ are assumed to be scalars, and the arbitrary ambient field $\mathbf{U}: \mathcal{O} \rightarrow \mathbb{R}^{3}$

$$
\mathbf{U}\left(x_{1}, x_{2}, x_{3}\right)=\left[U_{1}\left(x_{1}, x_{2}, x_{3}\right), U_{2}\left(x_{1}, x_{2}, x_{3}\right), U_{3}\left(x_{1}, x_{2}, x_{3}\right)\right]
$$

is given, this linearization produces the following system of equations, in solution variables $u\left(x_{1}, x_{2}, x_{3}, t\right)$ (flow velocity), $p\left(x_{1}, x_{2}, x_{3}, t\right)$ (pressure), $w_{1}\left(x_{1}, x_{2}, t\right)$ (elastic plate displacement) and $w_{2}\left(x_{1}, x_{2}, t\right)$ (elastic plate velocity):

$$
\begin{align*}
& \left\{\begin{array}{l}
p_{t}+\mathbf{U} \cdot \nabla p+\operatorname{div} u+\operatorname{div}(\mathbf{U}) p=0 \text { in } \mathcal{O} \times(0, \infty) \\
u_{t}+\mathbf{U} \cdot \nabla u-\operatorname{div} \sigma(u)+\eta u+\nabla p=0 \text { in } \mathcal{O} \times(0, \infty) \\
(\sigma(u) \mathbf{n}-p \mathbf{n}) \cdot \boldsymbol{\tau}=0 \text { on } \partial \mathcal{O} \times(0, \infty) \\
u \cdot \mathbf{n}=0 \text { on } S \times(0, \infty) \\
u \cdot \mathbf{n}=w_{2}+\mathbf{U} \cdot \nabla w_{1} \text { on } \Omega \times(0, \infty)
\end{array}\right.  \tag{2}\\
& \left\{\begin{array}{l}
w_{1_{t}}-w_{2}-\mathbf{U} \cdot \nabla w_{1}=0 \text { on } \Omega \times(0, \infty) \\
w_{2_{t}}+\Delta^{2} w_{1}+\left[2 \nu \partial_{x_{3}}(u)_{3}+\lambda \operatorname{div}(u)-p\right]_{\Omega}=0 \text { on } \Omega \times(0, \infty) \\
w_{1}=\frac{\partial w_{1}}{\partial \nu}=0 \text { on } \partial \Omega \times(0, \infty)
\end{array}\right.  \tag{3}\\
& {\left[p(0), u(0), w_{1}(0), w_{2}(0)\right]=\left[p_{0}, u_{0}, w_{a}, w_{b}\right] \in H_{N}^{\perp} .} \tag{4}
\end{align*}
$$

where the space $H_{N}^{\perp}$ is defined in 16. The quantity $\eta>0$ represents a drag force of the domain on the viscous flow. In addition, the quantity $\tau$ in (2) is in the space $T H^{1 / 2}(\partial \mathcal{O})$ of tangential vector fields of Sobolev index $1 / 2$; that is,

$$
\begin{equation*}
\tau \in T H^{1 / 2}(\partial \mathcal{O})=\left\{\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O}): \mathbf{v}_{\partial \mathcal{O}} \cdot \mathbf{n}=0 \text { on } \partial \mathcal{O}\right\} \tag{5}
\end{equation*}
$$

(See e.g., p. 846 of 15.) In addition, we take ambient field $\mathbf{U} \in \mathbf{V}_{0} \cap W$ where

$$
\begin{equation*}
\mathbf{V}_{0}=\left\{\mathbf{v} \in \mathbf{H}^{1}(\mathcal{O}):\left.\mathbf{v}\right|_{\partial \mathcal{O}} \cdot \mathbf{n}=0 \text { on } \partial \mathcal{O}\right\} \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
W=\left\{v \in \mathbf{H}^{1}(\mathcal{O}): v \in L^{\infty}(\mathcal{O}), \quad \operatorname{div}(v) \in L^{\infty}(\mathcal{O}), \quad \text { and }\left.\quad \mathbf{U}\right|_{\Omega} \in C^{2}(\bar{\Omega})\right\} \tag{7}
\end{equation*}
$$

(This vanishing of the boundary for ambient fields is a standard assumption in compressible flow literature; see [19], [33], [26], [1].) Moreover, the stress and strain tensors in the flow PDE component of (22)-(4) are defined respectively as

$$
\sigma(\mu)=2 \nu \epsilon(\mu)+\lambda\left[I_{3} \cdot \epsilon(\mu)\right] I_{3} ; \quad \epsilon_{i j}(\mu)=\frac{1}{2}\left(\frac{\partial \mu_{j}}{\partial x_{i}}+\frac{\partial \mu_{i}}{\partial x_{j}}\right), \quad 1 \leq i, j \leq 3,
$$

where Lamé Coefficients $\lambda \geq 0$ and $\nu>0$.
Remark 1 As will be seen below, the appearance of the term $-w_{2}-\mathbf{U} \cdot \nabla w_{1}$, in the mechanical displacement equation (3), will induce an invariance with respect to the space $H_{N}^{\perp}$ defined in (16). We will ultimately establish that solutions of (2)-(4), with initial data in $H_{N}^{\perp}$, are associated with a bounded semigroup, for $\mathbf{U}$ sufficiently small with respect to an appropriate measurement (see 24)). In addition, if we set $w(t)=w_{1}(t), w_{t}=w_{2}+\mathbf{U} \cdot \nabla w_{1}$, then we have that $\left[p, u, w, w_{t}\right]$ solves

$$
\begin{aligned}
& \left\{\begin{array}{l}
p_{t}+\mathbf{U} \cdot \nabla p+\operatorname{div} u+\operatorname{div}(\mathbf{U}) p=0 \text { in } \mathcal{O} \times(0, \infty) \\
u_{t}+\mathbf{U} \cdot \nabla u-\operatorname{div\sigma }(u)+\eta u+\nabla p=0 \text { in } \mathcal{O} \times(0, \infty) \\
(\sigma(u) \mathbf{n}-p \mathbf{n}) \cdot \boldsymbol{\tau}=0 \text { on } \partial \mathcal{O} \times(0, \infty) \\
u \cdot \mathbf{n}=0 \text { on } S \times(0, \infty) \\
u \cdot \mathbf{n}=w_{t} \text { on } \Omega \times(0, \infty)
\end{array}\right. \\
& \left\{\begin{array}{l}
w_{t t}+\Delta^{2} w-\mathbf{U} \cdot \nabla w_{t}+\left[2 \nu \partial_{x_{3}}(u)_{3}+\lambda \operatorname{div}(u)-p\right]_{\Omega}=0 \text { on } \Omega \times(0, \infty) \\
w=\frac{\partial w}{\partial \nu}=0 \text { on } \partial \Omega \times(0, \infty)
\end{array}\right. \\
& {\left[p(0), u(0), w(0), w_{t}(0)\right]=\left[p_{0}, u_{0}, w_{a}, w_{b}+\mathbf{U} \cdot \nabla w_{a}\right] \in H_{N}^{\perp} .}
\end{aligned}
$$

where $w(0)=w_{1}(0)=w_{a}$ and $w_{t}(0)=w_{2}(0)+\mathbf{U} \cdot \nabla w_{1}(0)=w_{b}+\mathbf{U} \cdot \nabla w_{a}$.
Here, as usually done for viscous fluids, we impose the so called impermeability condition on $\Omega$; namely, we assume that no fluid passes through the elastic portion of the boundary during deflection 14,23 . At this point, we emphasize that the FSI problem under consideration has present a material derivative term on the deflected interaction surface. This material derivative computes the time rate of change of any quantity such as temperature or velocity (and hence also acceleration) for a portion of a material in motion. Since our material is a fluid, then the movement is simply the flow field and any particle of fluid speeds up and down as it flows along the specified spatial domain. With respect to the change of the speed of the said fluid, the material derivative effectively gives a true rate of change of the velocity. Hence, we describe the interface $\Omega$ in Lagrangian coordinates in $\mathbb{R}^{3}$ with $S\left(a_{1}, a_{2}, a_{3}\right)=0$; also let $\mathbf{x}=\left\langle x_{1}, x_{2}, x_{3}\right\rangle$ be the Eulerian position inside $\mathcal{O}$. Then, letting $w\left(x_{1}, x_{2}, t\right)$ represent the transverse $\left(x_{3}\right)$ displacement of the plate on $\Omega$, we have that

$$
S\left(x_{1}, x_{2}, x_{3}-w\left(x_{1}, x_{2} ; t\right)\right) \equiv \mathcal{S}\left(x_{1}, x_{2}, x_{3} ; t\right)=0
$$

describes the time-evolution of the boundary. The impermeability condition requires that the material derivative $\left(\partial_{t}+\tilde{u} \cdot \nabla_{\mathbf{x}}\right)$ vanishes on the deflected surface 14, 16, 23]:

$$
\left(\partial_{t}+\tilde{u} \cdot \nabla_{\mathbf{x}}\right) \mathcal{S}=0, \quad \tilde{u}=u+\mathbf{U}
$$

Applying the chain rule and rearranging, we obtain

$$
\begin{equation*}
\nabla_{\mathbf{x}} S \cdot\left\langle 0,0,-w_{t}\right\rangle+\mathbf{U} \cdot\left[\nabla_{\mathbf{x}} S+\left\langle-S_{x_{3}} w_{x_{1}},-S_{x_{3}} w_{x_{2}}, 0\right\rangle\right]=-u \cdot\left[\nabla_{\mathbf{x}} S+\left\langle-S_{x_{3}} w_{x_{1}},-S_{x_{3}} w_{x_{2}}, 0\right\rangle\right] . \tag{8}
\end{equation*}
$$



Figure 1: Polyhedral Flow-Structure Geometries

We identify $\nabla_{\mathbf{x}} S$ as the normal to the deflected surface; assuming small deflections and restricting to $\left(x_{1}, x_{2}\right) \in \Omega$, we can identify $\left.\nabla_{\mathbf{x}} S\right|_{\Omega}$ with $\left.\mathbf{n}\right|_{\Omega}=\langle 0,0,1\rangle$. Making use of (8), imposing that $\mathbf{U} \cdot \mathbf{n}=0$ on $\partial \mathcal{O}$ (see (6) and discussion), and discarding quadratic terms, this relation allows us to write for $\left(x_{1}, x_{2}\right) \in \Omega$ :

$$
\mathbf{n} \cdot\left\langle 0,0, w_{t}\right\rangle+\mathbf{U} \cdot\left\langle w_{x_{1}}, w_{x_{2}}, 0\right\rangle=u \cdot \mathbf{n} .
$$

This yields the desired flow boundary condition

$$
\begin{equation*}
\left.u \cdot \mathbf{n}\right|_{\Omega}=w_{t}+\mathbf{U} \cdot \nabla w \tag{9}
\end{equation*}
$$

in $(2)_{5}$ via the material derivative of the deflected elastic interaction surface.
We note that the flow linearization is taken with respect to a general inhomogeneous compressible Navier-Stokes system. However, unlike the papers [7,9] where some forcing and energy level terms in the pressure and flow equations have been neglected, due to their relative unimportance therein, in this present study, the particular energy level term $\operatorname{div}(\mathbf{U}) p$ in $(2)_{1}$ can not be neglected, inasmuch as it plays a part in establishing that the associated FSI semigroup is uniformly bounded (and invariant) with respect to the subspace $H_{N}^{\perp}$. Accordingly, the term $\operatorname{div}(\mathbf{U}) p$ is one of the ingredients in the "feedback" operator $B$ defined in (14).

In addition to the properties given for the fluid domain $\mathcal{O}$ before, we impose additional conditions which will be necessary for the application of some elliptic regularity results for solutions of second order boundary value problems on corner domains [20, 22]:

Condition 2 Flow domain $\mathcal{O}$ should be curvilinear polyhedral domain which satisfies the following condition:

- Each corner of the boundary $\partial \mathcal{O}$-if any- is diffeomorphic to a convex cone,
- Each point on an edge of the boundary $\partial \mathcal{O}$ is diffeomorphic to a wedge with opening $<\pi$.

Some examples of geometries can be seen in Figure 1. In reference to problem (2)-(4), the associated finite energy space will be

$$
\begin{equation*}
\mathcal{H} \equiv L^{2}(\mathcal{O}) \times \mathbf{L}^{2}(\mathcal{O}) \times H_{0}^{2}(\Omega) \times L^{2}(\Omega) \tag{10}
\end{equation*}
$$

which is a Hilbert space, topologized by the following standard inner product:

$$
\begin{equation*}
\left(\mathbf{y}_{1}, \mathbf{y}_{2}\right)_{\mathcal{H}}=\left(p_{1}, p_{2}\right)_{L^{2}(\mathcal{O})}+\left(u_{1}, u_{2}\right)_{\mathbf{L}^{2}(\mathcal{O})}+\left(\Delta w_{1}, \Delta w_{2}\right)_{L^{2}(\Omega)}+\left(v_{1}, v_{2}\right)_{L^{2}(\Omega)} \tag{11}
\end{equation*}
$$

for any $\mathbf{y}_{i}=\left(p_{i}, u_{i}, w_{i}, v_{i}\right) \in \mathcal{H}, i=1,2$.

### 1.1 Literature

The PDE's which describe fluid structure interactions have been considered from a variety of viewpoints and with different objectives in mind; $2,13,17,18,24,30$. Analysis of FSI generally constitutes a broad area of research with applications in aeroelasticity, biomechanics, biomedicine, etc. In particular, the study of wellposedness of various linearized incompressible/compressible FSI models which manifest parabolic-hyperbolic coupling has a large presence in the literature; see e.g., $[2,7,11,13,17,30$ wherein the Navier-Stokes equations are coupled with the wave/plate equation along a fixed interface. The parabolic-hyperbolic nature of the system generally results in major mathematical difficulties, principally because the coupling mechanisms between the fluid and the solid PDE components inevitably involves boundary terms which are strictly above the level of finite energy. In the case of a compressible flow component in the FSI system, the analysis is further complicated: whereas for incompressible flows the density of the fluid is assumed to be a constant and pressure an unknown function determined by the fluid motion, for compressible flows the main difficulty in the analysis of the density or pressure term, arises from the fact that the density variable is no longer constant. Although in most of the works in the literature, the motion of an isentropic compressible fluid - i.e., the density is a linear function of pressure - is solely considered, still, having to contend with this additional density (pressure) variable presents a mathematical challenge, even at the level of well-posedness.

In contrast to the growing literature on incompressible fluids the knowledge about compressible fluids interacting with elastic solids is relatively limited. In fact, the very first contribution to this problem is the pioneering paper [17], where both well-posedness and the existence of global attractors were shown. In [17], the author addresses the simple case that the ambient vector field $\mathbf{U}=0$, i.e., i.e., the linearization takes place about the trivial flow steady state. For this canonical situation, he used Galerkin approximations to prove the wellposedness result. However, the author duly noted that the case $\mathbf{U} \neq 0$ can not be handled in a similar fashion due to the existence of the troublesome - i.e., unbounded $-\operatorname{term} \mathbf{U} \cdot \nabla p$ in the pressure equation $(2)_{1}$.

Subsequently, the linearized model in [17] with $\mathbf{U} \neq 0$ was considered in [7]. The linearization in $[7$, about an arbitrary non-zero state, gives rise to terms which induce a non-dissipativity of the resulting FSI system. For this non-dissipative FSI in [7], a pure velocity matching condition is imposed at the interface (i.e., no material derivative is present in this boundary condition). In contrast to the Galerkin approach applied in [17], the authors in [7] invoke a certain LumerPhillips methodology, with a view of associating solutions of the fluid-structure dynamics with a continuous semigroup which is not uniformly bounded. Subsequently, a more convoluted FSI model was considered in [9] where, in addition to the aforesaid non-dissipative and unbounded terms brought about by ambient field $\mathbf{U} \neq 0$, the associated flow-structure interface is also under the effect of this ambient field $\mathbf{U} \neq 0$. In particular, the flow and structure velocity matching boundary condition also contains the material derivative of the structure, which again refers to the rate of change of the velocity on the deflected interaction surface. In (9) semigroup wellposedness is
established by an appropriate invocation of the Lumer-Phillips Theorem; this semigroup generation is posed with respect to the entire phase space $\mathcal{H}$, as defined in above.

However, this wellposedness result in $[9]$ is not totally satisfactory, from the standpoint of future studies into the long time behavior of FSI solutions: while [9] does provided existence and uniqueness of solutions to the FSI system in the entire finite energy space $\mathcal{H}$, the resulting semigroup is not uniformly bounded. In particular, the semigroup estimate obtained in 9 is $\mathcal{O}\left(e^{C(\mathbf{U}) t}\right)$, for $t>0$, where $C(\mathbf{U})=\frac{1}{2}\|\operatorname{div}(\mathbf{U})\|_{\infty}+\epsilon$. This lack of FSI semigroup boundedness in 9 will therefore forestall any subsequent discussion of FSI stability. Accordingly, with a mind toward future investigations of the asymptotic behavior of FSI solutions, we are led to the following question: Is it possible to obtain a semigroup wellposedness result, with the semigroup being bounded uniformly in time, at least in some (inherently invariant) subspace of the finite energy space?

Motivated by this question, in the present work we consider the linearized compressible flowstructure interaction model (2)-(4), where $\mathbf{U} \neq 0$ and the material derivative term $\mathbf{U} \cdot \nabla w_{1}$ is in place in the matching velocity boundary condition. Since our main objective here is to obtain a uniformly bounded semigroup, our departure point is to find an appropriate subspace for the analysis. In order to have semigroup generation on this sought-after subspace, the prospective generator of the PDE system (2)-(4) should be invariant with respect to it. In this connection, it was shown in [24] that if operator $\mathcal{A}_{0}: \mathcal{H} \rightarrow \mathcal{H}$ is the FSI generator in [7], which models the "material derivative" free FSI PDE interaction, then zero is an eigenvalue of $\mathcal{A}_{0}$. (In particular, the action of $\mathcal{A}_{0}$ is given by $\mathcal{A}$ of (13), with the appropriate domain of definition [which includes the pure matching velocity boundary condition]; see [24] and [7]). In fact, the null space of $\mathcal{A}_{0}$ is one dimensional, denoted here by $H_{N}$, and given explicitly in (15) below. The point of our mentioning $\mathcal{A}_{0}$ in the present problem is that, by way of obtaining a uniformly bounded semigroup, we will take our candidate space of wellposedness to be the orthogonal complement $H_{N}^{\perp}$, which is characterized by (16) below.

The necessity of finding an appropriate invariant subspace for uniformly bounded FSI semigroup analysis motivates the presence of the additional (and unbounded) term $w_{2}+\mathbf{U} \cdot \nabla w_{1}$ in (2)-(4). Let $\mathcal{A}_{1}: \mathcal{H} \rightarrow \mathcal{H}$ be the FSI generator which gives rise to the wellposedness result in 9 ; the action of $\mathcal{A}_{1}$ is given by $\mathcal{A}$ of 13 ) with the appropriate domain of definition, which includes the material derivative term matching velocity boundary condition; see p. 342 of 9$]$. As thus constituted, $H_{N}^{\perp}$ is not invariant with respect to $\mathcal{A}_{1}$. However, if we define an operator $B$ which abstractly models the unbounded term $w_{2}+\mathbf{U} \cdot \nabla w_{1}$ in (2)-(4), as well as the energy level term $\operatorname{div}(\mathbf{U}) p$, then with the appropriate domain of definition, $H_{N}^{-}$is -invariant with respect to the modeling operator $(\mathcal{A}+B)$ of (2)-4 (This is Lemma 3 below).

Having established said invariance, we will subsequently proceed to show that, with respect to a certain inner product which is equivalent to the standard $\mathcal{H}$-inner product, $(\mathcal{A}+B)$ generates a contraction semigroup on $H_{N}^{\perp}$, for ambient field $\mathbf{U}$ small enough in norm (and so the semigroup will be uniformly bounded with respect to the standard $\mathcal{H}$-norm). In consequence, the PDE system (22)-(4) is wellposed for initial data $\left[p_{0}, u_{0}, w_{a}, w_{b}\right]$ taken from $H_{N}^{\perp}$.

### 1.2 Challenges encountered and Novelty

In the present work, we establish a result of semigroup wellposedness so as to ascertain the existence and uniqueness of solutions to $\sqrt{22}$-(4), for Cauchy data in $H_{N}^{\perp}$. Moreover, we find this FSI
semigroup is uniformly bounded in time. This boundedness will have implications in our future analysis of long time behavior of the solutions to the PDE system (22)-(4). The main challenging points and improvements in our treatment are as follows:
(a) Uniformly bounded semigroup in $H_{N}^{\perp} \subset \mathcal{H}$ : By way of fulfilling our objective of obtaining a uniformly bounded semigroup, we adopt a Lumer-Phillips approach, in an appropriate inner product. To wit, to establish dissipativity we topologize the $(\mathcal{A}+B)$-invariant space $H_{N}^{\perp}$ with an inner product which is equivalent to the standard $\mathcal{H}$-inner product. In this construction, we make use of a multiplier $\nabla \psi$ introduced in [17] (defined in (19) below) and previously used in [24]; the multiplier exploits the characterization of $H_{N}^{\perp}$ in below. In addition, inasmuch as we are after a FSI solution semigroup which is uniformly bounded in time, we give a proof for the maximality (or the range condition) of the operator $(\mathcal{A}+B)$ which is quite different than that in [9]. Unlike [9] where the theory of linear perturbations is used so as to yield a semigroup whose bound is of said exponential order, in the present we totally eschew the Lax-Milgram approach of [9] and instead invoke functional analytical and PDE methods to show that $[\lambda I-(\mathcal{A}+B)]$ is invertible for any $\lambda>0$. This entails to show that $[\lambda I-(\mathcal{A}+B)]$ is a closed linear operator that has a dense range in $H_{N}^{\perp}$ and enjoys the inverse estimate (97) below. By these means we establish that $(\mathcal{A}+B)$ is maximal dissipative with respect to said appropriate inner product, and so then a uniformly bounded semigroup on the standard $\mathcal{H}$-inner product. Our uniformly bounded semigroup result is valid under the assumption that ambient vector field $\mathbf{U}$ is small enough with respect to an appropriate measurement; see (24) below. However, one should bear in mind that the present of $\mathbf{U} \neq 0$ gives rise to terms - namely, $\mathbf{U} \cdot \nabla p$ and $\mathbf{U} \cdot \nabla w_{1}$ (as it appears twice) - which are unbounded with respect to the underlying finite energy of the FSI system. Thus, our method of proof does not at all involve some bounded perturbation result which exploits the smallness of $\mathbf{U}$.
(b) $H_{N}^{\perp}$ - invariant generator: Subsequent to our work $[9$, our original immediate objective was to analyze the stability properties of the material derivative FSI system in [9]. However, because of the presence of the zero eigenvalue, as mentioned above, it is problematic to consider the strong or exponential decay problem in the entire phase space $\mathcal{H}$. Accordingly, we are led here to consider wellposedness (and future stability) analysis on $H_{N}^{\perp}$ as given in 16) below.(Since $H_{N}$ of 15 ) is only one dimensional, -see [24, Lemma 6]- we would not lose too much.) However, as we said above, $H_{N}^{\perp}$ is not invariant with respect to the material derivative FSI generator $\mathcal{A}_{1}: \mathcal{H} \rightarrow \mathcal{H}$ in 9 ]. (The unbounded material derivative term in particular contributes to the non-invariance.) However, the presence of the terms $-w_{2}-\mathbf{U} \cdot \nabla w_{1}$ and $\operatorname{div}(\mathbf{U}) p$ in the respective structural displacement and pressure equations in (2)-(4) gives rise to an invariance on $H_{N}^{\perp}$. (Actually, the term $\operatorname{div}(\mathbf{U}) p$ was blithely disgarded during the linearization process in $[9]$, since it is a benign energy level term.) Thus, these two terms are captured abstractly by the "feedback" operator $B$ in (14) below. We say feedback, since $B$ is incorporated so as to beneficently provide the pre-requisite that $H_{N}^{\perp}$ is $(\mathcal{A}+B)$-invariant. We note that the presence of $B$ does not at all give rise to a fortuitous cancellation of terms so as to have dissipativity with respect to the standard $\mathcal{H}$-inner product. The operator $B$ allows only for said invariance property, so that our wellposedness and uniform bounded semigroup problem can be considered on the slightly smaller subspace $H_{N}^{\perp}$. As we said, our finding that the FSI semigroup is uniformly bounded in time in $H_{N}^{\perp}$ will constitute a departure point in our future work on stability properties of the FSI PDE model.
(c) Less regularity required on the ambient vector field $\mathbf{U}$ : The presence of the nontrivial ambient flow field $\mathbf{U}$ causes substantial difficulties in the wellposedness analysis. In this case $\mathbf{U} \neq 0$, the desired result for a FSI system - with material derivative present in the matching velocities BC - on the entire phase space $\mathcal{H}$ was obtained in the earlier work [9] (with recall, the semigroup estimate $\mathcal{O}\left(e^{C(\mathbf{U}) t}\right)$, for $t>0$, where $C(\mathbf{U})=\frac{1}{2}\|\operatorname{div}(\mathbf{U})\|_{\infty}+\epsilon$. In the course of applying the Lax-Milgram Theorem in [9], there is the need to deal with the pressure PDE component of an associated static compressible FSI system. In this regard, a methodology, based upon a treatment of (uncoupled) transport equations in [19], was applied to solve for the pressure and fluid velocity components of said static FSI system. However this approach compelled the authors in 9 to impose that $\mathbf{U} \in \mathbf{H}^{3}(\mathcal{O})$. In the present work, we require that small enough ambient field $\mathbf{U} \in \mathbf{H}^{1}(\mathcal{O})$ obey the less stringent regularity assumptions in 7 .

### 1.3 Notation

Throughout, for a given domain $D$, the norm of corresponding space $L^{2}(D)$ will be denoted as $\|\cdot\|_{D}$ (or simply $\|\cdot\|$ when the context is clear). Inner products in $L^{2}(\mathcal{O})$ or $\mathbf{L}^{2}(\mathcal{O})$ will be denoted by $(\cdot, \cdot)_{\mathcal{O}}$, whereas inner products $L^{2}(\partial \mathcal{O})$ will be written as $\langle\cdot, \cdot\rangle_{\partial \mathcal{O}}$. We will also denote pertinent duality pairings as $\langle\cdot, \cdot\rangle_{X \times X^{\prime}}$ for a given Hilbert space $X$. The space $H^{s}(D)$ will denote the Sobolev space of order $s$, defined on a domain $D ; H_{0}^{s}(D)$ will denote the closure of $C_{0}^{\infty}(D)$ in the $H^{s}(D)$ norm $\|\cdot\|_{H^{s}(D)}$. We make use of the standard notation for the boundary trace of functions defined on $\mathcal{O}$, which are sufficently smooth: i.e., for a scalar function $\phi \in H^{s}(\mathcal{O}), \frac{1}{2}<s<\frac{3}{2}, \gamma(\phi)=\left.\phi\right|_{\partial \mathcal{O}}$, which is a well-defined and surjective mapping on this range of $s$, owing to the Sobolev Trace Theorem on Lipschitz domains (see e.g., [31], or Theorem 3.38 of [29]).

### 1.4 Plan of the paper

The paper is organized as follows: In Section 2, we first provide the framework which will be required for our proof of semigroup wellposedness. In particular, we carefully describe the FSI generator $(\mathcal{A}+B)$ and its domain, as well as the equivalent inner product which will be used for our proof of wellposedness on subspace $H_{N}^{\perp}$ of 16 below. Moreover, we show that $H_{N}^{\perp}$ is $(\mathcal{A}+B)$ invariant. In Section 3, we establish the maximal dissipativity of $(\mathcal{A}+B)$ with respect to said special inner product, thereby allowing for an appeal to the Lumer-Phillips Theorem. In the course of our work, we will have need of a classic lemma of functional analysis, as well as the adjoint of $(\mathcal{A}+B)$. These ingredients are given in Section 4 , the Appendix.

## 2 Functional Setting and Preliminaries

With respect to the above setting, the PDE system given in (2)- (4) can be written as an ODE in Hilbert space $\mathcal{H}$. That is, if $\Phi(t)=\left[p, u, w_{1}, w_{2}\right] \in C([0, T] ; \mathcal{H})$ solves the problem (2)-(4), then there is a modeling operator $\mathcal{A}+B: D(\mathcal{A}+B) \subset \mathcal{H} \rightarrow \mathcal{H}$ such that $\Phi(\cdot)$ satisfies

$$
\begin{align*}
\frac{d}{d t} \Phi(t) & =(\mathcal{A}+B) \Phi(t) \\
\Phi(0) & =\Phi_{0} \tag{12}
\end{align*}
$$

Here the operators $\mathcal{A}$ and the feedback operator $B$ are defined as follows:

$$
\mathcal{A}=\left[\begin{array}{cccc}
-\mathbf{U} \cdot \nabla(\cdot) & -\operatorname{div}(\cdot) & 0 & 0  \tag{13}\\
-\nabla(\cdot) & \operatorname{div} \sigma(\cdot)-\eta I-\mathbf{U} \cdot \nabla(\cdot) & 0 & 0 \\
0 & 0 & 0 & I \\
{[\cdot] \|_{\Omega}} & -\left[2 \nu \partial_{x_{3}}(\cdot)_{3}+\lambda \operatorname{div}(\cdot)\right]_{\Omega} & -\Delta^{2} & 0
\end{array}\right] ;
$$

and

$$
B=\left[\begin{array}{cccc}
-\operatorname{div}(\mathbf{U})(\cdot) & 0 & 0 & 0  \tag{14}\\
0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{U} \cdot \nabla(\cdot) & 0 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

Then, $D(\mathcal{A}+B) \subset \mathcal{H}$ is given by
$D(\mathcal{A}+B)=\left\{\left(p_{0}, u_{0}, w_{1}, w_{2}\right) \in L^{2}(\mathcal{O}) \times \mathbf{H}^{1}(\mathcal{O}) \times H_{0}^{2}(\Omega) \times L^{2}(\Omega): \operatorname{properties}(A . i)-(A . v i)\right.$ hold $\}$, where
(A.i) $\mathbf{U} \cdot \nabla p_{0} \in L^{2}(\mathcal{O})$
(A.ii) $\operatorname{div} \sigma\left(u_{0}\right)-\nabla p_{0} \in \mathbf{L}^{2}(\mathcal{O})\left(\right.$ So, $\left.\left[\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right]_{\partial \mathcal{O}} \in \mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O})\right)$
(A.iii) $-\Delta^{2} w_{1}-\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}+\left.p_{0}\right|_{\Omega} \in L^{2}(\Omega)$ (by elliptic regularity theory $w_{1} \in H^{3}(\Omega)$ )
(A.iv) $\left(\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right) \perp T H^{1 / 2}(\partial \mathcal{O})$. That is,

$$
\left\langle\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}, \tau\right\rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O})}=0 \text { in } \mathcal{D}^{\prime}(\mathcal{O}) \text { for every } \tau \in T H^{1 / 2}(\partial \mathcal{O})
$$

(A.v) $w_{2}+\mathbf{U} \cdot \nabla w_{1} \in H_{0}^{2}(\Omega)$ (and so $w_{2} \in H_{0}^{1}(\Omega)$ )
(A.vi) The flow velocity component $u_{0}=\mathbf{f}_{0}+\widetilde{\mathbf{f}}_{0}$, where $\mathbf{f}_{0} \in \mathbf{V}_{0}$ and $\widetilde{\mathbf{f}}_{0} \in \mathbf{H}^{1}(\mathcal{O})$ satisfies $\rrbracket^{1}$

$$
\widetilde{\mathbf{f}}_{0}= \begin{cases}0 & \text { on } S \\ \left(w_{2}+\mathbf{U} \cdot \nabla w_{1}\right) \mathbf{n} & \text { on } \Omega\end{cases}
$$

(and so $\left.\mathbf{f}_{0}\right|_{\partial \mathcal{O}} \in T H^{1 / 2}(\partial \mathcal{O})$ ).

Moreover, we denote

$$
H_{N}=\operatorname{Span}\left\{\left[\begin{array}{c}
1  \tag{15}\\
0 \\
\AA^{-1}(1) \\
0
\end{array}\right]\right\}
$$

[^1]where $\AA: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ is the elliptic operator
$$
\AA \varpi=\Delta^{2} \varpi, \text { with } D(\AA)=\left\{w \in H_{0}^{2}(\Omega): \Delta^{2} w \in L^{2}(\Omega)\right\},
$$
and
\[

$$
\begin{equation*}
H_{N}^{\perp}=\left\{\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in \mathcal{H}: \int_{\mathcal{O}} p_{0} d \mathcal{O}+\int_{\Omega} w_{1} d \Omega=0\right\} \tag{16}
\end{equation*}
$$

\]

(see [24, Lemma 6]).
As stated before, in order to be able to obtain a uniformly bounded (contraction) semigroup, we analyze the wellposedness of problem (2)-(4) in the reduced space $H_{N}^{\perp}$. This will require us to re-topologize the phase space $\mathcal{H}$ with a new inner product to be used in $H_{N}^{\perp}$ and equivalent to the natural inner product given in (10). Now, with the above notation let us take $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in$ $H_{N}^{\perp}, \widetilde{\varphi}=\left[\widetilde{p}_{0}, \widetilde{u}_{0}, \widetilde{w}_{1}, \widetilde{w}_{2}\right] \in H_{N}^{\perp}$. Then the new inner product is given as

$$
\begin{gather*}
((\varphi, \widetilde{\varphi}))_{H_{N}^{\prime}}=\left(p_{0}, p_{0}\right)_{\mathcal{O}}+\left(u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right), \widetilde{u}_{0}-\alpha D\left(g \cdot \nabla \widetilde{w}_{1}\right)_{3}+\xi \nabla \psi\left(\widetilde{p}_{0}, \widetilde{w}_{1}\right)\right)_{\mathcal{O}} \\
+\left(\Delta w_{1}, \Delta \widetilde{w}_{1}\right)_{\Omega}+\left(w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}, \widetilde{w}_{2}+h_{\alpha} \cdot \nabla \widetilde{w}_{1}+\xi \widetilde{w}_{1}\right)_{\Omega} \tag{17}
\end{gather*}
$$

and in turn the norm

$$
\begin{gather*}
\||\varphi|\|_{H_{\frac{\perp}{N}}}=\sqrt{((\varphi, \varphi))_{H_{N}^{\perp}}} \\
=\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}+\left\|w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\Omega}^{2} \tag{18}
\end{gather*}
$$

for every $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in H_{N}^{\perp}$. Here,
(i) the function $\psi=\psi(f, g) \in H^{1}(\mathcal{O})$ is considered to solve the following BVP for data $f \in L^{2}(\mathcal{O})$ and $g \in L^{2}(\Omega)$

$$
\left\{\begin{array}{c}
-\Delta \psi=f \quad \text { in } \mathcal{O}  \tag{19}\\
\frac{\partial \psi}{\partial n}=0 \text { on } S \\
\frac{\partial \psi}{\partial n}=g \text { on } \Omega
\end{array}\right.
$$

with the compatibility condition

$$
\begin{equation*}
\int_{\mathcal{O}} f d \mathcal{O}+\int_{\Omega} g d \Omega=0 . \tag{20}
\end{equation*}
$$

We should note that by known elliptic regularity results for the Neumann problem on Lipschitz domains-see e.g; 25]- we have

$$
\begin{equation*}
\|\psi(f, g)\|_{H^{\frac{3}{2}}(\mathcal{O})} \leq\left[\|f\|_{\mathcal{O}}+\|g\|_{\partial \mathcal{O}}\right] \tag{21}
\end{equation*}
$$

(ii) the map $D(\cdot)$ is the Dirichlet map that extends boundary data $\varphi$ defined on $\Omega$ to a harmonic function in $\mathcal{O}$ satisfying:

$$
D \varphi=f \Leftrightarrow\left\{\begin{array}{c}
\Delta f=0 \quad \text { in } \mathcal{O} \\
\left.f\right|_{\partial \mathcal{O}}=\left.\varphi\right|_{\text {ext }} \quad \text { on } \partial \mathcal{O}
\end{array}\right.
$$

where

$$
\left.\varphi\right|_{e x t}=\left\{\begin{array}{lll}
0 & \text { on } & S \\
\phi & \text { on } & \Omega
\end{array}\right.
$$

Then by, e.g., [29, Theorem 3.3.8], and Lax-Milgram, we deduce that

$$
\begin{equation*}
D \in \mathcal{L}\left(H_{0}^{1 / 2+\epsilon}(\Omega) ; H^{1}(\mathcal{O})\right) . \tag{22}
\end{equation*}
$$

(iii) the vector field $h_{\alpha}(\cdot)$ is defined as $h_{\alpha}(\cdot)=\left.\mathbf{U}\right|_{\Omega}-\alpha g$, where $g(\cdot)$ is a $C^{2}$ extension of the normal vector $\mathbf{n}(x)$ (with respect to $\Omega$ ) and we specify the parameter $\alpha$ to be

$$
\begin{equation*}
\alpha=2\|\mathbf{U}\|_{*}, \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
\|\mathbf{U}\|_{*}=\|\mathbf{U}\|_{L^{\infty}(\mathcal{O})}+\|\operatorname{div}(\mathbf{U})\|_{L^{\infty}(\mathcal{O})}+\left\|\left.\mathbf{U}\right|_{\Omega}\right\|_{C^{2}(\bar{\Omega})} . \tag{24}
\end{equation*}
$$

Also, $\xi$ is eventually specified in (60). Since the main goal of this manuscript is to have the semigroup wellposedness in the subspace $H_{N}^{\perp}$, in what follows, for the sake of simplicity, we will use the notation

$$
\left.(\mathcal{A}+B)\right|_{H_{N}^{\perp}}=(\mathcal{A}+B) .
$$

Before beginning our wellposedness analysis, we firstly need to justify that the semigroup generator is indeed $H_{N}^{\perp}$ - invariant. This is given in the following lemma:

Lemma 3 The operator $(\mathcal{A}+B)$ is $H_{N}^{\perp}$ - invariant; that is $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_{N}^{\perp} \subset H_{N}^{\perp} \rightarrow H_{N}^{\perp}$.
Proof. Let $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in H_{N}^{\perp}, \widetilde{\varphi}=\left[\widetilde{p}_{0}, \widetilde{u}_{0}, \widetilde{w}_{1}, \widetilde{w}_{2}\right] \in H_{N}$. Recalling the adjoint operator $\mathcal{A}^{*}$ in (106) we have

$$
\begin{gathered}
(\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=\left(\varphi, \mathcal{A}^{*} \widetilde{\varphi}\right)_{\mathcal{H}}=\left(\varphi, L_{1} \widetilde{\varphi}\right)_{\mathcal{H}}+\left(\varphi, L_{2} \widetilde{\varphi}\right)_{\mathcal{H}}=0+\left(\varphi, L_{2} \widetilde{\varphi}\right)_{\mathcal{H}} \\
=\int_{\mathcal{O}} p_{0} \operatorname{div}(\mathbf{U}) 1 d \mathcal{O}+\int_{\Omega} \Delta w_{1} \Delta \AA^{-1}\left\{\operatorname{div}\left[U_{1}, U_{2}\right]\right\} 1 d \Omega \\
=\int_{\mathcal{O}} p_{0} \operatorname{div}(\mathbf{U}) 1 d \mathcal{O}+\int_{\Omega} w_{1} \operatorname{div}\left[U_{1}, U_{2}\right] 1 d \Omega \\
=\int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) p_{0} 1 d \mathcal{O}-\int_{\Omega}\left(\nabla w_{1} \cdot \mathbf{U}\right) 1 d \Omega \\
=\int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) p_{0} 1 d \mathcal{O}-\int_{\Omega} \Delta\left(\nabla w_{1} \cdot \mathbf{U}\right) \Delta \AA^{-1}(1) d \Omega \\
=\left(\left[\begin{array}{c}
\operatorname{div}(\mathbf{U}) p_{0} \\
0 \\
-\nabla w_{1} \cdot \mathbf{U} \\
0
\end{array}\right],\left[\begin{array}{c}
1 \\
0 \\
\AA^{-1}(1) \\
0
\end{array}\right]\right)_{\mathcal{H}}
\end{gathered}
$$

$$
=-(B \varphi, \widetilde{\varphi})_{\mathcal{H}}
$$

which yields that

$$
(\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=-(B \varphi, \widetilde{\varphi})_{\mathcal{H}}
$$

or

$$
((\mathcal{A}+B) \varphi, \widetilde{\varphi})_{\mathcal{H}}=0
$$

for every $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in H_{N}^{\perp}$. Hence, $(\mathcal{A}+B)$ is $H_{N}^{\perp}$-invariant.

## 3 Wellposedness

This section is devoted to showing the semigroup wellposedness of the PDE system (2)-(4). The main result of this paper is given as follows:

Theorem 4 Let Condition 2 hold. Moreover, let $\|\mathbf{U}\|_{*}$ be sufficiently small. Then the operator $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_{N}^{\perp} \rightarrow H_{N}^{\perp}$, as defined via (13) and (14), generates a strongly continuous semigroup $\left\{e^{(\mathcal{A}+B) t}\right\}_{t \geq 0}$ on $H_{N}^{\perp}$. Hence, for every initial data $\left[p_{0}, u_{0}, w_{1_{0}}, w_{2_{0}}\right] \in H_{N}^{\perp}$, the solution [ $\left.p(t), u(t), w_{1}(t), w_{2}(t)\right]$ of problem (2)-(4) is given continuously by

$$
\left[\begin{array}{c}
p(t)  \tag{25}\\
u(t) \\
w_{1}(t) \\
w_{2}(t)
\end{array}\right]=e^{(\mathcal{A}+B) t}\left[\begin{array}{c}
p_{0} \\
u_{0} \\
w_{1_{0}} \\
w_{2_{0}}
\end{array}\right] \in C\left([0, T] ; H_{N}^{\perp}\right)
$$

Moreover, this semigroup is uniformly bounded in time with respect to the standard $\mathcal{H}$-inner product. (With respect to the special norm in (18), the semigroup is in fact a contraction.)

Remark 5 In point of fact, for ambient field $\mathbf{U}$ smooth enough, the operator $(\mathcal{A}+B)$ generates a continuous semigroup in the entire phase space $\mathcal{H}$. This conclusion can be straightforwardly obtained by invoking the machinery of [9]. However, this wellposedness on all of $\mathcal{H}$ has its downsides: (i) The ambient field requires the stronger regularity $\mathbf{H}^{3}(\mathcal{O})$ (ii) the argumentation in [7, 9], which partly involves linear perturbation theory, will culminate in the semigroup of $(\mathcal{A}+B)$ not having a uniform bound; in fact the semigroup estimate on all of $\mathcal{H}$ will be of exponential order.

To prove Theorem 4, we will appeal to Lumer-Phillips Theorem that requires the analysis of the dissipativity and maximality properties of the semigroup generator $(\mathcal{A}+B)$. We start with the dissipativity for which our main tool will be the use of the inner product defined in (17):

### 3.1 Dissipativity of the Generator $(\mathcal{A}+B)$

We show the dissipativity property of the generator operator $(\mathcal{A}+B)$ in the following lemma:
Lemma 6 With reference to problem (2)- (4), the semigroup generator $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_{N}^{\perp} \subset$ $H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ is dissipative with respect to inner product $((\cdot, \cdot))_{H_{N}^{\perp}}$ for $\|\mathbf{U}\|_{*}$ (defined in (24)) small enough. In particular, for $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in D(\mathcal{A}+B) \cap H_{N}^{\perp}$,

$$
\begin{equation*}
\operatorname{Re}(([\mathcal{A}+B] \varphi, \varphi))_{H_{N}^{\perp}} \leq-\frac{\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}}{4}-\frac{\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}}{4}-\frac{\xi\left\|p_{0}\right\|_{\mathcal{O}}^{2}}{2}-\frac{\xi\left\|\Delta w_{1}\right\|_{\Omega}^{2}}{2} \tag{26}
\end{equation*}
$$

where $\xi$ is specified in (60).
Proof. Given $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in D(\mathcal{A}+B) \cap H_{N}^{\perp}$, we have

$$
\begin{gathered}
(([\mathcal{A}+B] \varphi, \varphi))_{H_{N}^{\perp}}=\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}, p_{0}\right)_{\mathcal{O}} \\
+\left(-\nabla p_{0}+\operatorname{div} \sigma\left(u_{0}\right)-\eta u_{0}-\mathbf{U} \nabla u_{0}, u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}} \\
+\left(-\nabla p_{0}+\operatorname{div} \sigma\left(u_{0}\right)-\eta u_{0}-\mathbf{U} \nabla u_{0}, \xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
-\alpha\left(D\left(g \cdot \nabla\left[w_{2}+\mathbf{U} \nabla w_{1}\right]\right) e_{3}, u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\xi\left(\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}, w_{2}+\mathbf{U} \nabla w_{1}\right), u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}} \\
+\xi^{2}\left(\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}, w_{2}+\mathbf{U} \nabla w_{1}\right), \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\left(\Delta w_{2}, \Delta w_{1}\right)_{\Omega}+\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta w_{1}\right)_{\Omega} \\
+\left(\left.p_{0}\right|_{\Omega}-\left.\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
+\left(h_{\alpha} \cdot \nabla\left[w_{2}+\mathbf{U} \nabla w_{1}\right], w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
\quad-\left(\Delta^{2} w_{1}, w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
+\xi\left(w_{2}+\mathbf{U} \nabla w_{1}, w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} .
\end{gathered}
$$

After integration by parts we then arrive at

$$
\begin{gather*}
(([\mathcal{A}+B] \varphi, \varphi))_{H_{N}^{\perp}}=-\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}-\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{U})\left[\left|u_{0}\right|^{2}-\left|p_{0}\right|^{2}\right] d \mathcal{O} \\
+2 i \operatorname{Im}\left[\left(p_{0}, \operatorname{div}\left(u_{0}\right)\right)_{\mathcal{O}}+\left(\Delta w_{2}, \Delta w_{1}\right)_{\Omega}\right]-i \operatorname{Im}\left[\left(\mathbf{U} \nabla p_{0}, p_{0}\right)_{\mathcal{O}}+\left(\mathbf{U} \nabla u_{0}, u_{0}\right)_{\mathcal{O}}\right] \\
+\sum_{j=1}^{8} I_{j} \tag{27}
\end{gather*}
$$

where above the $I_{j}$ are given by:

$$
\begin{gather*}
I_{1}=\left(\nabla p_{0}-\operatorname{div} \sigma\left(u_{0}\right)+\eta u_{0}+\mathbf{U} \nabla u_{0}, \alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}} \\
-\alpha\left(\left.p_{0}\right|_{\Omega}-\left.\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, g \cdot \nabla w_{1}\right)_{\Omega},  \tag{28}\\
I_{2}=\left(-\nabla p_{0}+\operatorname{div} \sigma\left(u_{0}\right)-\eta u_{0}-\mathbf{U} \nabla u_{0}, \xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\xi\left(\Delta^{2} w_{1}, w_{1}\right)_{\Omega} \\
+\left(\left.p_{0}\right|_{\Omega}-\left.\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, \xi w_{1}\right)_{\Omega},  \tag{29}\\
I_{3}=-\alpha\left(D\left(g \cdot \nabla\left[w_{2}+\mathbf{U} \nabla w_{1}\right]\right) e_{3}, u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}},  \tag{30}\\
I_{4}=\xi\left(\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}, w_{2}+\mathbf{U} \nabla w_{1}\right), u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}},  \tag{31}\\
I_{5}=\xi^{2}\left(\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}, w_{2}+\mathbf{U} \nabla w_{1}\right), \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}},  \tag{32}\\
I_{6}=\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta w_{1}\right)_{\Omega}-\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega},  \tag{33}\\
I_{7}=\left(h_{\alpha} \cdot \nabla\left[w_{2}+\mathbf{U} \nabla w_{1}\right], w_{2}\right)_{\Omega}, \tag{34}
\end{gather*}
$$

$$
\begin{align*}
I_{8} & =\left(h_{\alpha} \cdot \nabla\left[w_{2}+\mathbf{U} \nabla w_{1}\right], h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
& +\xi\left(w_{2}+\mathbf{U} \nabla w_{1}, w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \tag{35}
\end{align*}
$$

where we also recall the definition $h_{\alpha}=\left.\mathbf{U}\right|_{\Omega}-\alpha g$. In the course of estimating the terms (28)-(35) above, we will invoke the polynomial

$$
\begin{equation*}
r(a)=a+a^{2}+a^{3} . \tag{36}
\end{equation*}
$$

We start with $I_{1}$; integrating by parts, we have

$$
\begin{align*}
I_{1}= & -\alpha\left(p_{0}, \operatorname{div}\left[D\left(g \cdot \nabla w_{1}\right) e_{3}\right]\right)_{\mathcal{O}}+\alpha\left(\sigma\left(u_{0}\right), \epsilon\left(D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}}\right. \\
& +\alpha \eta\left(u_{0}, D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}}+\alpha\left(\mathbf{U} \nabla u_{0}, D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}} \tag{37}
\end{align*}
$$

Using the fact that Dirichlet map $D \in L\left(H_{0}^{\frac{1}{2}+\epsilon}(\Omega), H^{1}(\mathcal{O})\right)$, we have

$$
\begin{equation*}
I_{1} \leq r\left(\|\mathbf{U}\|_{*}\right) C\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \tag{38}
\end{equation*}
$$

We continue with $I_{2}$; using the definition of the map $\psi(\cdot, \cdot)$ in 19) and integrating by parts we get

$$
\begin{gathered}
I_{2}=-\xi \int_{\mathcal{O}}\left|p_{0}\right|^{2} d \mathcal{O}-\xi\left(\sigma\left(u_{0}\right), \epsilon\left(\nabla \psi\left(p_{0}, w_{1}\right)\right)\right)_{\mathcal{O}} \\
+\xi\left\langle\sigma\left(u_{0}\right) n-p_{0} n,\left(\nabla \psi\left(p_{0}, w_{1}\right), n\right) n\right\rangle_{\partial \mathcal{O}}-\eta\left(u_{0}, \xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
\left(-\mathbf{U} \nabla u_{0}, \xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\Delta^{2} w_{1}, \xi w_{1}\right)_{\Omega} \\
+\left(\left.p_{0}\right|_{\Omega}-\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\left.\lambda \operatorname{div}\left(u_{0}\right)\right|_{\Omega}, \xi w_{1}\right)_{\Omega},\right.
\end{gathered}
$$

whence we obtain

$$
\begin{gather*}
I_{2} \leq-\xi\left\|p_{0}\right\|_{\mathcal{O}}^{2}-\xi\left\|\Delta w_{1}\right\|_{\Omega}^{2}+\xi r\left(\|\mathbf{U}\|_{*}\right) C\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
+\xi C\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}\left[\left\|p_{0}\right\|_{\mathcal{O}}+\left\|\Delta w_{1}\right\|_{\Omega}\right]\right\} . \tag{39}
\end{gather*}
$$

For $I_{3}$ : recalling the boundary condition

$$
\left.\left(u_{0}\right)_{3}\right|_{\Omega}=w_{2}+\mathbf{U} \nabla w_{1}
$$

making use of Lemma 6.1 of [9] and considering the assumptions made on the geometry in Condition 2. we have

$$
\begin{align*}
I_{3} & \leq \alpha C\left\|g \cdot \nabla\left(u_{0}\right)_{3}\right\|_{H^{-\frac{1}{2}}(\Omega)}\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}} \\
& \leq C\left[r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\}+\xi^{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\}\right] \tag{40}
\end{align*}
$$

where we have also implicitly used the Sobolev Embedding Theorem. To continue with $I_{4}$ :

$$
I_{4}=\xi\left(\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}(\mathbf{U}) p_{0}, 0\right), u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}}
$$

$$
\begin{gather*}
+\xi\left(\nabla \psi\left(-\operatorname{div}\left(u_{0}\right), u_{0} \cdot \mathbf{n}\right), u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right)_{\mathcal{O}} \\
=I_{4 a}+I_{4 b} \tag{41}
\end{gather*}
$$

Since $\left.\mathbf{U} \cdot \mathbf{n}\right|_{\partial \mathcal{O}}=\mathbf{0}$, we have that $\left(\mathbf{U} \nabla p_{0}+\operatorname{div}(\mathbf{U}) p_{0}\right) \in\left[H^{1}(\mathcal{O})\right]^{\prime}$ with

$$
\begin{equation*}
\left\|\mathbf{U} \nabla p_{0}+\operatorname{div}(\mathbf{U}) p_{0}\right\|_{\left[H^{1}(\mathcal{O})\right]^{\prime}} \leq C\|\mathbf{U}\|_{*}\left\|p_{0}\right\|_{\mathcal{O}} \tag{42}
\end{equation*}
$$

By Lax-Milgram Theorem, we then have

$$
\begin{gather*}
I_{4 a} \leq C \xi\left\|\nabla \psi\left(-\mathbf{U} \nabla p_{0}-\operatorname{div}(\mathbf{U}) p_{0}, 0\right)\right\|_{\mathcal{O}}\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}\right\|_{\mathcal{O}} \\
\leq C \xi r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \tag{43}
\end{gather*}
$$

and similarly

$$
\begin{equation*}
I_{4 b} \leq C \xi r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} . \tag{44}
\end{equation*}
$$

Now, applying (43)-(44) to (41) gives

$$
\begin{equation*}
I_{4} \leq C \xi r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \tag{45}
\end{equation*}
$$

Estimating $I_{5}$ : we proceed as before done for $I_{4}$ and invoke (42), Lax Milgram Theorem and the estimate (21) to have

$$
\begin{equation*}
I_{5} \leq C \xi^{2}\left[\|\mathbf{U}\|_{*}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\}+\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}\right] \tag{46}
\end{equation*}
$$

For $I_{6}$, in order to estimate the second term in (33), we follow the standard calculations used for the flux multipliers and the commutator symbol given by

$$
\begin{equation*}
[P, Q] f=P(Q f)-Q(P f) \tag{47}
\end{equation*}
$$

for the differential operators $P$ and $Q$. Hence,

$$
\begin{align*}
-\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega} & =\left(\nabla \Delta w_{1}, \nabla\left(h_{\alpha} \cdot \nabla w_{1}\right)\right)_{\Omega}  \tag{48}\\
& =-\left(\Delta w_{1}, \Delta\left(h_{\alpha} \cdot \nabla w_{1}\right)\right)_{\Omega}+\int_{\partial \Omega}\left(h_{\alpha} \cdot \nu\right)\left|\Delta w_{1}\right|^{2} d \partial \Omega \tag{49}
\end{align*}
$$

where, in the first identity we have directly invoked the clamped plate boundary conditions, and in the second we have used the fact that $w_{1}=\partial_{\nu} w_{1}=0$ on $\partial \Omega$ which yields that

$$
\frac{\partial}{\partial \nu}\left(h_{\alpha} \cdot \nabla w_{1}\right)=\left(h_{\alpha} \cdot \nu\right) \frac{\partial^{2} w_{1}}{\partial \nu}=\left(h_{\alpha} \cdot \nu\right)\left(\left.\Delta w_{1}\right|_{\partial \Omega}\right) .
$$

(See [27] or [28, p.305]). Using the commutator bracket $[\cdot, \cdot]$, we can rewrite the latter relation as $-\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega}=-\left(\Delta w_{1},\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}\right)_{\Omega}-\left(\Delta w_{1}, h_{\alpha} \cdot \nabla\left(\Delta w_{1}\right)\right)_{\Omega}+\int_{\partial \Omega}\left(h_{\alpha} \cdot \nu\right)\left|\Delta w_{1}\right|^{2} d \partial \Omega$.

With Green's relations once more:

$$
\begin{align*}
-\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega}= & -\left(\Delta w_{1},\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}\right)_{\Omega}-\frac{1}{2} \int_{\partial \Omega}\left(h_{\alpha} \cdot \nu\right)\left|\Delta w_{1}\right|^{2} d \partial \Omega \\
& +\frac{1}{2} \int_{\Omega}\left[\operatorname{div}\left(h_{\alpha}\right)\right]\left|\Delta w_{1}\right|^{2} d \Omega-i \operatorname{Im}\left(\Delta w_{1}, h_{\alpha} \cdot \nabla\left(\Delta w_{1}\right)\right)_{\Omega} \\
& +\int_{\partial \Omega}\left(h_{\alpha} \cdot \nu\right)\left|\Delta w_{1}\right|^{2} d \partial \Omega \tag{50}
\end{align*}
$$

Thus,

$$
\begin{align*}
-\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega}= & -\left(\Delta w_{1},\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}\right)_{\Omega}+\frac{1}{2} \int_{\partial \Omega}\left(h_{\alpha} \cdot \nu\right)\left|\Delta w_{1}\right|^{2} d \partial \Omega \\
& +\frac{1}{2} \int_{\Omega}\left[\operatorname{div}\left(h_{\alpha}\right)\right]\left|\Delta w_{1}\right|^{2} d \Omega-i \operatorname{Im}\left(\Delta w_{1}, h_{\alpha} \cdot \nabla\left(\Delta w_{1}\right)\right) \tag{51}
\end{align*}
$$

Since $h_{\alpha}=\left.\mathbf{U}\right|_{\Omega}-\alpha g$, where $g$ is an extension of $\nu(\mathbf{x})$, we will have then
$-\operatorname{Re}\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega}=\frac{1}{2} \int_{\partial \Omega}(\mathbf{U} \cdot \nu-\alpha)\left|\Delta w_{1}\right|^{2} d \partial \Omega+\frac{1}{2} \int_{\Omega} \operatorname{div}\left(h_{\alpha}\right)\left|\Delta w_{1}\right|^{2} d \Omega-\operatorname{Re}\left(\Delta w_{1},\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}\right)_{\Omega}$
Since we can explicitly compute the commutator

$$
\begin{aligned}
{\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}=} & \left(\Delta h_{1}\right)\left(\partial_{x_{1}} w_{1}\right)+2\left(\partial_{x_{1}} h_{1}\right)\left(\partial_{x_{1}}^{2} w_{1}\right)+2\left(\partial_{x_{2}} h_{2}\right)\left(\partial_{x_{2}}^{2} w_{1}\right)+\left(\Delta h_{2}\right)\left(\partial_{x_{2}} w_{1}\right) \\
& +2 \operatorname{div}\left(h_{\alpha}\right)\left(\partial_{x_{1}} \partial_{x_{2}} w_{1}\right),
\end{aligned}
$$

and

$$
\begin{equation*}
\left\|\left[\Delta, h_{\alpha} \cdot \nabla\right] w_{1}\right\|_{L^{2}(\Omega)} \leq C r\left(\|\mathbf{U}\|_{*}\right)\left\|\Delta w_{1}\right\|_{L^{2}(\Omega)} \tag{53}
\end{equation*}
$$

combining (52)-(53) we eventually get

$$
\begin{equation*}
-\operatorname{Re}\left(\Delta^{2} w_{1}, h_{\alpha} \cdot \nabla w_{1}\right)_{\Omega} \leq \frac{1}{2} \int_{\partial \Omega}[\mathbf{U} \cdot \nu-\alpha]\left|\Delta w_{1}\right|^{2} d \partial \Omega+\operatorname{Cr}\left(\|\mathbf{U}\|_{*}\right)\left\|\Delta w_{1}\right\|_{\Omega}^{2} \tag{54}
\end{equation*}
$$

Moreover, for the first term of (33), we have

$$
\begin{gathered}
\left.\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta w_{1}\right)_{\Omega}=\left(\mathbf{U} \nabla w_{1}\right), \Delta w_{1}\right)_{\Omega}-\left([\mathbf{U} \cdot \nabla, \Delta] w_{1}, \Delta w_{1}\right)_{\Omega} \\
\quad=\int_{\partial \Omega}(\mathbf{U} \cdot \nu)\left|\Delta w_{1}\right|^{2} d \partial \Omega-\int_{\partial \Omega} \operatorname{div}(\mathbf{U})\left|\Delta w_{1}\right|^{2} d \partial \Omega \\
\quad-\left([\mathbf{U} \cdot \nabla, \Delta] w_{1}, \Delta w_{1}\right)_{\Omega}-\int_{\Omega} \Delta w_{1} \mathbf{U} \cdot \nabla\left(\Delta w_{1}\right) d \Omega
\end{gathered}
$$

where we also use the commutator expression in (47). This gives us

$$
\begin{equation*}
\operatorname{Re}\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta w_{1}\right)_{\Omega} \leq \frac{1}{2} \int_{\partial \Omega}(\mathbf{U} \cdot \nu)\left|\Delta w_{1}\right|^{2} d \partial \Omega+C r\left(\|\mathbf{U}\|_{*}\right)\left\|\Delta w_{1}\right\|_{\Omega}^{2} \tag{55}
\end{equation*}
$$

Now applying (54)-(55) to (33), we obtain

$$
\begin{equation*}
\operatorname{Re} I_{6} \leq \int_{\partial \Omega}\left[\mathbf{U} \cdot \nu-\frac{\alpha}{2}\right]\left|\Delta w_{1}\right|^{2} d \partial \Omega+C r\left(\|\mathbf{U}\|_{*}\right)\left\|\Delta w_{1}\right\|_{\Omega}^{2} \tag{F}
\end{equation*}
$$

To estimate $I_{7}$ : since $w_{2} \in H_{0}^{1}(\Omega)$, we have

$$
\begin{gathered}
\operatorname{Re}\left(h_{\alpha} \cdot \nabla w_{2}, w_{2}\right)_{\Omega}=-\frac{1}{2} \int_{\Omega} \operatorname{div}\left(h_{\alpha}\right)\left|w_{2}\right|^{2} d \Omega \\
=-\frac{1}{2} \int_{\Omega} \operatorname{div}\left(h_{\alpha}\right)\left|\left(u_{0}\right)_{3}-\mathbf{U} \nabla w_{1}\right|^{2} d \Omega
\end{gathered}
$$

after using the boundary condition in (A.v). Applying the last relation to RHS of (34) and recalling that $h_{\alpha}=\left.\mathbf{U}\right|_{\Omega}-\alpha g$, we get

$$
\begin{gather*}
\operatorname{Re} I_{7}=\operatorname{Re}\left(h_{\alpha} \cdot \nabla w_{2}, w_{2}\right)_{\Omega}+\operatorname{Re}\left(h_{\alpha} \cdot \nabla\left(\mathbf{U} \nabla w_{1}\right),\left(u_{0}\right)_{3}-\mathbf{U} \nabla w_{1}\right)_{\mathcal{O}} \\
\leq C r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \tag{56}
\end{gather*}
$$

where we also implicitly use Sobolev Trace Theorem. Lastly, for the term $I_{8}$, we proceed in a manner similar to that adopted for $I_{7}$ and we have

$$
\begin{gather*}
I_{8}=\left(h_{\alpha} \cdot \nabla\left(u_{0}\right)_{3}, h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
+\xi\left(\left(u_{0}\right)_{3},\left(u_{0}\right)_{3}-\mathbf{U} \cdot \nabla w_{1}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right)_{\Omega} \\
\leq C\left[r\left(\|\mathbf{U}\|_{*}\right)+\xi^{2}\right]\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
+C \xi\left[\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+r\left(\|\mathbf{U}\|_{*}\right)\left\{\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\}\right] \tag{57}
\end{gather*}
$$

Now, if we apply (38)-(57) to RHS of (27), we obtain

$$
\begin{gather*}
\operatorname{Re}(([\mathcal{A}+B] \varphi, \varphi))_{H_{N}^{\perp}} \leq-\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}-\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}-\xi\left\|p_{0}\right\|_{\mathcal{O}}^{2}-\xi\left\|\Delta w_{1}\right\|_{\Omega}^{2} \\
\quad+\int_{\partial \Omega}\left[\mathbf{U} \cdot \nu-\frac{\alpha}{2}\right]\left|\Delta w_{1}\right|^{2} d \partial \Omega \\
+C\left[r_{\mathbf{U}}+\xi r_{\mathbf{U}}+\xi^{2}+\xi\right]\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2} \\
+C\left[r_{\mathbf{U}}+\xi r_{\mathbf{U}}+\xi^{2}+\xi^{2} r_{\mathbf{U}}\right]\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
+C \xi\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}+\left\|\Delta w_{1}\right\|_{\Omega}\right\} \tag{58}
\end{gather*}
$$

where, for the simplicity, we have set $r_{\mathbf{U}}=r\left(\|\mathbf{U}\|_{*}\right)$. We recall now the value of $\alpha=2\|\mathbf{U}\|_{*}$ to get

$$
\operatorname{Re}(([\mathcal{A}+B] \varphi, \varphi))_{H_{\perp}^{\perp}} \leq-\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}-\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}-\xi\left\|p_{0}\right\|_{\mathcal{O}}^{2}-\xi\left\|\Delta w_{1}\right\|_{\Omega}^{2}
$$

$$
\begin{align*}
&+\left[\left(C_{1}+C_{2} r_{\mathbf{U}}\right) \xi^{2}+C_{2} r_{\mathbf{U}} \xi+C_{2} r_{\mathbf{U}}\right]\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
&+\frac{1}{2}\left\{\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}\right\} \\
&+ C_{3}\left[r_{\mathbf{U}}+\xi r_{\mathbf{U}}+\xi^{2}+\xi\right]\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2} \tag{59}
\end{align*}
$$

where the positive constants $C_{1}, C_{2}$ and $C_{3}$ are obtained with the application of Holder-Young and Korn's inequalities and $C_{2}$ depends on the constant in Korn's inequality. We now specify $\xi$ be a zero of the equation

$$
\left(C_{1}+C_{2} r_{\mathbf{U}}\right) \xi^{2}+\left(C_{2} r_{\mathbf{U}}-\frac{1}{2}\right) \xi+C_{2} r_{\mathbf{U}}=0
$$

Namely,

$$
\begin{equation*}
\xi=\frac{\frac{1}{2}-C_{2} r_{\mathbf{U}}}{2\left(C_{1}+C_{2} r_{\mathbf{U}}\right)}-\frac{\sqrt{\left(\frac{1}{2}-C_{2} r_{\mathbf{U}}\right)^{2}-4 C_{2}\left(C_{1}+C_{2} r_{\mathbf{U}}\right) r_{\mathbf{U}}}}{2\left(C_{1}+C_{2} r_{\mathbf{U}}\right)} \tag{60}
\end{equation*}
$$

where the radicand is nonnegative for $\|\mathbf{U}\|_{*}$ sufficiently small. Then 59 becomes

$$
\begin{aligned}
& \operatorname{Re}(([\mathcal{A}+B] \varphi, \varphi))_{H_{\perp}} \leq-\frac{\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}}{4}-\eta \frac{\left\|u_{0}\right\|_{\mathcal{O}}^{2}}{4}-\frac{\xi}{2}\left\|p_{0}\right\|_{\mathcal{O}}^{2}-\frac{\xi}{2}\left\|\Delta w_{1}\right\|_{\Omega}^{2} \\
&-\frac{\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}}{4}-\eta \frac{\left\|u_{0}\right\|_{\mathcal{O}}^{2}}{4} \\
&+C_{K}\left[r_{\mathbf{U}}+\xi r_{\mathbf{U}}+\xi^{2}+\xi\right]\left\{\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}\right\}
\end{aligned}
$$

With $\xi$ as prescribed in (60), we now have the dissipativity estimate (26), for $\|\mathbf{U}\|_{*}$ small enough. (Here we also implicitly re-use Korn's inequality and $C_{K}$ is the constant there). This concludes the proof of Lemma 6.

### 3.2 Maximality of the Generator $(\mathcal{A}+B)$

In order to complete the proof of Theorem 4, we also need to show that the semigroup generator $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_{N}^{\perp} \subset H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ is maximal dissipative. This is given in the following lemma:

Lemma 7 With reference to problem (2)-(4), the semigroup generator $(\mathcal{A}+B): D(\mathcal{A}+B) \cap H_{N}^{\perp} \subset$ $H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ is maximal dissipative. In other words, the following range condition holds:

$$
\begin{equation*}
\operatorname{Range}[\lambda I-(\mathcal{A}+B)]=H_{N}^{\perp} \tag{61}
\end{equation*}
$$

for some $\lambda>0$.

## Proof of Lemma 7

Proof of relation (61) is based on showing that $[\lambda I-(\mathcal{A}+B)]^{-1} \in \mathcal{L}\left(H_{N}^{\perp}\right)$. For this, we appeal to linear operator theory and exploit Lemma 12 in Appendix as our main tool. So, with respect to

Lemma 12 the requirements to be shown are:
$(\mathbf{M}-\mathbf{I}) \operatorname{Range}[\lambda I-(\mathcal{A}+B)]$ is dense in $H_{N}^{\perp}$,
$(\mathbf{M}-\mathbf{I I})[\lambda I-(\mathcal{A}+B)]$ is a closed operator.
( $\mathbf{M}-\mathbf{I I I}$ ) There is an $m>0$ such that

$$
\||[\lambda I-(\mathcal{A}+B)] \varphi|\|_{H_{N}^{\perp}} \geq m\||\varphi|\|_{H_{N}^{\perp}}
$$

for all $\varphi \in D([\lambda I-(\mathcal{A}+B)]) \cap H_{N}^{\perp}=D(\mathcal{A}+B) \cap H_{N}^{\perp}$.

STEP (M-I): Firstly, to prove that Range $[\lambda I-(\mathcal{A}+B)]$ is dense in $H_{N}^{\perp}$, we use the fact that

$$
\text { Range }[\lambda I-(\mathcal{A}+B)]=\operatorname{Null}\left([\lambda I-(\mathcal{A}+B)]^{*}\right)^{\perp}
$$

which is given in the following lemma:
Lemma 8 Let parameter $\lambda>0$ be given. Then for $\|\mathbf{U}\|_{*}$ sufficiently small,

$$
\operatorname{Null}\left[\lambda I-(\mathcal{A}+B)^{*}\right]=\{0\}
$$

Proof. Suppose that $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in D\left((\mathcal{A}+B)^{*}\right) \cap H_{N}^{\perp}$ satisfies

$$
\begin{equation*}
\left[\lambda I-(\mathcal{A}+B)^{*}\right] \varphi=0 . \tag{62}
\end{equation*}
$$

In PDE terms, this is

$$
\left\{\begin{array}{c}
\lambda p_{0}-\mathbf{U} \nabla p_{0}-\operatorname{div}\left(u_{0}\right)=0 \quad \text { in } \mathcal{O}  \tag{63}\\
\lambda u_{0}-\nabla p_{0}-\operatorname{div} \sigma\left(u_{0}\right)+\eta u_{0}-\mathbf{U} \nabla u_{0}+\operatorname{div}(\mathbf{U}) u_{0}=0 \quad \text { in } \mathcal{O} \\
u_{0} \cdot n=0 \quad \text { on } S \\
u_{0} \cdot n=w_{2} \quad \text { on } \Omega \\
\lambda w_{1}+w_{2}-\AA^{-1}\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)-\Delta^{2} w_{1}\right]_{\Omega} \\
-\mathbf{U} \cdot \nabla w_{1}-\Delta \AA^{-1} \nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)=0 \quad \text { in } \Omega\right. \\
\lambda w_{2}+\left.\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}-\Delta^{2} w_{1}=0 \quad \text { in } \Omega \\
\left.w_{1}\right|_{\partial \Omega}=\left.\frac{\partial w_{1}}{\partial \nu}\right|_{\partial \Omega}=0
\end{array}\right.
$$

Since we have from (62)

$$
\begin{equation*}
0=\lambda\|\varphi\|_{\mathcal{H}}^{2}-\left((\mathcal{A}+B)^{*} \varphi, \varphi\right)_{\mathcal{H}} \tag{64}
\end{equation*}
$$

integrating by parts as usual, we get

$$
\begin{gathered}
\lambda\|\varphi\|_{\mathcal{H}}^{2}+\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2} \\
=-\frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{U})\left[\left|p_{0}\right|^{2}+3\left|u_{0}\right|^{2}\right] d \mathcal{O} \\
+\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)-\Delta^{2} w_{1}\right]_{\Omega}, w_{1}\right)_{\Omega}
\end{gathered}
$$

$$
\begin{equation*}
+\left(\Delta\left[\mathbf{U} \cdot \nabla w_{1}\right], \Delta w_{1}\right)_{\Omega}+\left(\nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)\right), \Delta w_{1}\right)_{\Omega} \tag{65}
\end{equation*}
$$

To handle the terms on RHS of (65), we firstly invoke the map given in (19) and apply the multiplier $\nabla \psi\left(p_{0}, w_{1}\right)$ to the fluid equation 63$)_{2}$. This gives

$$
\begin{gather*}
\lambda\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\nabla p_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\operatorname{div} \sigma\left(u_{0}\right), \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\eta\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\mathbf{U} \nabla u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}+\left(\operatorname{div}(\mathbf{U}) u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}=0 \tag{66}
\end{gather*}
$$

Let us look at the terms of (66):

$$
\begin{align*}
& -\left(\nabla p_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}=\int_{\partial \mathcal{O}}\left(p_{0} \cdot n\right) \nabla \psi\left(p_{0}, w_{1}\right) d \partial \mathcal{O} \\
& \quad+\int_{\mathcal{O}} p_{0} \operatorname{div}\left(\nabla \psi\left(p_{0}, w_{1}\right)\right) d \mathcal{O} \\
& \quad=-\int_{\mathcal{O}}\left|p_{0}\right|^{2} d \mathcal{O}-\int_{\Omega} p_{0} w_{1} d \Omega \tag{67}
\end{align*}
$$

Also,

$$
\begin{gather*}
-\left(\operatorname{div} \sigma\left(u_{0}\right), \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
=\left(\sigma\left(u_{0}\right), \epsilon\left(\nabla \psi\left(p_{0}, w_{1}\right)\right)\right)_{\mathcal{O}}-\left\langle\sigma\left(u_{0}\right) \cdot n, \nabla \psi\left(p_{0}, w_{1}\right)\right\rangle_{\partial \mathcal{O}} \\
+\eta\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \tag{68}
\end{gather*}
$$

Applying (67)-(68) to (66), we then have

$$
\begin{gather*}
\int_{\mathcal{O}}\left|p_{0}\right|^{2} d \mathcal{O}=\lambda\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\mathbf{U} \nabla u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\left(\operatorname{div}(\mathbf{U}) u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}-\left(\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, w_{1}\right)_{\Omega} \\
+\left(\sigma\left(u_{0}\right), \epsilon\left(\nabla \psi\left(p_{0}, w_{1}\right)\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \tag{69}
\end{gather*}
$$

Subsequently, we apply the multiplier $w_{1}$ to the structural equation in $(63)_{7}$, and use (69) to get

$$
\begin{gather*}
\int_{\mathcal{O}}\left|p_{0}\right|^{2} d \mathcal{O}+\left(\Delta^{2} w_{1}, w_{1}\right)_{\Omega}=\lambda\left(w_{2}, w_{1}\right)_{\Omega}+\lambda\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\left(\sigma\left(u_{0}\right), \epsilon\left(\nabla \psi\left(p_{0}, w_{1}\right)\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
-\left(\mathbf{U} \nabla u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}}+\left(\operatorname{div}(\mathbf{U}) u_{0}, \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \tag{70}
\end{gather*}
$$

To estimate the terms on RHS of 70, we appeal to the elliptic regularity results for solutions of second order BVPs on corner domains [21. At this point, using the geometrical assumptions in Condition 2 and the higher regularity estimate

$$
\|\psi(p, w)\|_{H^{2}(\mathcal{O})} \leq C\left[\|p\|_{\mathcal{O}}+\left\|w_{e x t}\right\|_{H^{\frac{1}{2}+\varepsilon}(\partial \mathcal{O})}\right]
$$

$$
\begin{equation*}
\leq C\left[\|p\|_{\mathcal{O}}+\|w\|_{H_{0}^{2}(\Omega)}\right] \tag{71}
\end{equation*}
$$

where

$$
w_{e x t}(x)=\left\{\begin{array}{c}
0, \quad x \in S \\
w(x), \quad x \in \Omega
\end{array}\right.
$$

we obtain

$$
\begin{equation*}
\left.\int_{\mathcal{O}}\left|p_{0}\right|^{2} d \mathcal{O}+\int_{\Omega}\left|\Delta w_{1}\right|^{2} d \Omega \leq C_{\epsilon} r\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{72}
\end{equation*}
$$

Here, we also used Holder-Young Inequalities and $r(\cdot)$ and $\|\mathbf{U}\|_{*}$ are given as in (36) and (24), respectively. Now, to proceed with the second term on RHS of (65):

$$
\begin{gather*}
\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)-\Delta^{2} w_{1}\right]_{\Omega}, w_{1}\right)_{\Omega} \\
=\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, w_{1}\right)_{\Omega} \\
-\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\} \Delta^{2} w_{1}, w_{1}\right)_{\Omega} \\
=K_{1}+K_{2} \tag{73}
\end{gather*}
$$

For $K_{1}$ :

$$
\begin{gather*}
K_{1}=\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, w_{1}\right)_{\Omega} \\
=-\left(\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, \mathbf{U} \cdot \nabla w_{1}\right)_{\Omega} \tag{74}
\end{gather*}
$$

To handle the term on RHS of 74 : Let $D_{\Omega}: H_{0}^{\frac{1}{2}+\epsilon}(\Omega) \rightarrow H^{1}(\mathcal{O})$ be defined by

$$
D_{\Omega} g=f \Leftrightarrow\left\{\begin{array}{r}
-\Delta f=0 \quad \text { in } \mathcal{O}  \tag{75}\\
\left.f\right|_{S}=0 \quad \text { on } S \\
\left.f\right|_{\Omega}=g \quad \text { on } \Omega
\end{array}\right.
$$

Therewith,

$$
\begin{align*}
& \left(\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, \mathbf{U} \cdot \nabla w_{1}\right)_{\Omega}=\left(\sigma\left(u_{0}\right), \epsilon\left(D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}} \\
+ & \left(\nabla p_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}}+\left(\operatorname{div} \sigma\left(u_{0}\right), D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}} \\
= & \left(\sigma\left(u_{0}\right), \epsilon\left(D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}} \\
+ & \lambda\left(u_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}-\left(\mathbf{U} \cdot \nabla u_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}+\left(\operatorname{div}(\mathbf{U}) u_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}} \tag{76}
\end{align*}
$$

Now, applying (76) to RHS of (74), and invoking (72) we then have

$$
\begin{align*}
\left|K_{1}\right|= & \left|\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}, w_{1}\right)_{\Omega}\right| \\
& \left.\leq \operatorname{Cr}\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{77}
\end{align*}
$$

where again $r(\cdot)$ and $\|\mathbf{U}\|_{*}$ are given as in (36) and (24), respectively. Let us now continue with $K_{2}$ :

$$
K_{2}=-\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\} \Delta^{2} w_{1}, w_{1}\right)_{\Omega}
$$

$$
\begin{equation*}
=\left(\Delta^{2} w_{1}, \mathbf{U} \cdot \nabla w_{1}\right)_{\Omega} \tag{78}
\end{equation*}
$$

If we argue as in the estimates (50)-(51) by replacing $h_{\alpha}$ with $\mathbf{U}$, we then have

$$
\begin{gather*}
\left(\Delta^{2} w_{1}, \mathbf{U} \cdot \nabla w_{1}\right)_{\Omega}=\left(\Delta w_{1},[\Delta, \mathbf{U} \cdot \nabla] w_{1}\right)_{\Omega} \\
-\frac{1}{2} \int_{\partial \Omega}(\mathbf{U} \cdot \nu)\left|\Delta w_{1}\right|^{2} d \partial \Omega-\frac{1}{2} \int_{\Omega} \operatorname{div}(\mathbf{U})\left|\Delta w_{1}\right|^{2} d \Omega \tag{79}
\end{gather*}
$$

For the second term on RHS of $\sqrt[79]{79}$, let $\gamma(x)$ be a $C^{2}$-extension of the normal vector $\nu(\mathbf{x})$ to the boundary of $\Omega$. Applying the multiplier $\gamma \cdot \nabla w_{1}$ to the structral equation $\left.{ }^{63}\right)_{7}$, we get

$$
\begin{equation*}
\left(\Delta^{2} w_{1}, \gamma \cdot \nabla w_{1}\right)_{\Omega}=\left(\left.\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, \gamma \cdot \nabla w_{1}\right)_{\Omega}+\lambda\left(w_{2}, \gamma \cdot \nabla w_{1}\right)_{\Omega} \tag{80}
\end{equation*}
$$

Revoking the elliptic map (75), we have

$$
\begin{gather*}
\left(\left.\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, \gamma \cdot \nabla w_{1}\right)_{\Omega} \\
=\left(\sigma\left(u_{0}\right), \epsilon\left(D_{\Omega}\left(\gamma \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, D_{\Omega}\left(\gamma \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(D_{\Omega}\left(\gamma \cdot \nabla w_{1}\right)\right)\right)_{\mathcal{O}} \\
+\lambda\left(u_{0}, D_{\Omega}\left(\gamma \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}-\left(\mathbf{U} \cdot \nabla u_{0}, D_{\Omega}\left(\gamma \cdot \nabla w_{1}\right)\right)_{\mathcal{O}}+\left(\operatorname{div}(\mathbf{U}) u_{0}, D_{\Omega}\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)_{\mathcal{O}} \tag{81}
\end{gather*}
$$

Moreover, proceeding as in 79, we get

$$
\begin{gather*}
\left(\Delta^{2} w_{1}, \gamma \cdot \nabla w_{1}\right)_{\Omega}=\left(\Delta w_{1},[\Delta, \gamma \cdot \nabla] w_{1}\right)_{\Omega} \\
-\frac{1}{2} \int_{\partial \Omega}\left|\Delta w_{1}\right|^{2} d \partial \Omega-\frac{1}{2} \int_{\Omega} \operatorname{div}(\gamma)\left|\Delta w_{1}\right|^{2} d \Omega \tag{82}
\end{gather*}
$$

Now, applying (81), (82) to (80), using (53) (replacing $h_{\alpha}$ with $\gamma$ ) and subsequently re-invoking (72), we obtain

$$
\begin{equation*}
\left.\int_{\partial \Omega}\left|\Delta w_{1}\right|^{2} d \partial \Omega \leq C r\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{83}
\end{equation*}
$$

Combining now (78), (79), (83) and (72), we have

$$
\begin{gather*}
\left|K_{2}\right|=\left|\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\} \Delta^{2} w_{1}, w_{1}\right)_{\Omega}\right| \\
\left.\leq C r\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{84}
\end{gather*}
$$

Hence, the second term of (65) can be handled by

$$
\begin{gather*}
\left|\left(\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)-\Delta^{2} w_{1}\right]_{\Omega}, w_{1}\right)_{\Omega}\right| \\
\leq\left|K_{1}\right|+\left|K_{2}\right| \\
\left.\leq C r\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{85}
\end{gather*}
$$

Also, for the third and fourth terms of (65):

$$
\left(\Delta\left[\mathbf{U} \cdot \nabla w_{1}\right], \Delta w_{1}\right)_{\Omega}+\left(\nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)\right), \Delta w_{1}\right)_{\Omega}
$$

$$
\begin{gathered}
=\left(\mathbf{U} \cdot \nabla\left(\Delta w_{1}\right), \Delta w_{1}\right)_{\Omega}+\left([\Delta, \mathbf{U} \cdot \nabla] w_{1}, \Delta w_{1}\right)_{\Omega}+\left(\nabla\left[\mathbf{U} \cdot \nabla w_{1}\right], \nabla\left(\Delta w_{1}\right)\right)_{\Omega} \\
=\frac{1}{2} \int_{\partial \Omega}(\mathbf{U} \cdot \nu)\left|\Delta w_{1}\right|^{2} d \partial \Omega-\frac{1}{2} \int_{\Omega} \operatorname{div}(\mathbf{U})\left|\Delta w_{1}\right|^{2} d \Omega \\
+\left([\Delta, \mathbf{U} \cdot \nabla] w_{1}, \Delta w_{1}\right)_{\Omega}-\left(\mathbf{U} \cdot \nabla w_{1}, \Delta^{2} w_{1}\right)_{\Omega}
\end{gathered}
$$

Proceeding as done above, we then have

$$
\begin{align*}
& \left|\left(\Delta\left[\mathbf{U} \cdot \nabla w_{1}\right], \Delta w_{1}\right)_{\Omega}+\left(\nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)\right), \Delta w_{1}\right)_{\Omega}\right| \\
& \left.\leq C r\left(\|\mathbf{U}\|_{*}\right)\left\{\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\lambda\|\varphi\|_{\mathcal{H}}^{2}\right\} \tag{86}
\end{align*}
$$

Finally, if we apply the estimates (72), (85) and (86) to RHS of (65), we arrive at

$$
\begin{gathered}
\left.\lambda\|\varphi\|_{\mathcal{H}}^{2}+\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2} \\
\leq C\|\mathbf{U}\|_{*}\left\{\lambda\|\varphi\|_{\mathcal{H}}^{2}+\left(\sigma\left(u_{0}\right), \epsilon\left(u_{0}\right)\right)_{\mathcal{O}}+\eta\left\|u_{0}\right\|_{\mathcal{O}}^{2}\right\}
\end{gathered}
$$

For $\|\mathbf{U}\|_{*}$ small enough-independent of $\lambda>0$ - we infer that the solution $\varphi$ of 62 is zero which concludes the proof of Lemma 8 .

STEP (M-II): We continue with showing that $[\lambda I-(\mathcal{A}+B)]$ is a closed operator. For this, it will be enough to prove the following lemma:

Lemma 9 The operator $\mathcal{A}+B: D(\mathcal{A}+B) \cap H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ is closed.
Proof. Let $\left\{\varphi_{n}\right\}=\left\{\left[p_{0 n}, u_{0 n}, w_{1 n}, w_{2 n}\right]\right\} \subseteq D(\mathcal{A}+B) \cap H_{N}^{\perp}$ satisfy

$$
\begin{aligned}
\varphi_{n} & \rightarrow \varphi \text { in } H_{N}^{\perp} \\
(\mathcal{A}+B) \varphi_{n} & \rightarrow \varphi^{*} \text { in } H_{N}^{\perp}
\end{aligned}
$$

We must show that $\varphi \in D(\mathcal{A}+B) \cap H_{N}^{\perp}$, and $(\mathcal{A}+B) \varphi=\varphi^{*}$. To start, via the relation 26) in Lemma 6, we have

$$
\frac{\left(\sigma\left(u_{0 m}-u_{0 n}\right), \epsilon\left(u_{0 m}-u_{0 n}\right)\right)_{\mathcal{O}}}{4} \leq-\operatorname{Re}\left(\left([\mathcal{A}+B]\left(\varphi_{m}-\varphi_{n}, \varphi_{m}-\varphi_{n}\right)\right)_{H_{N}^{\perp}}\right.
$$

from which we infer that

$$
\begin{equation*}
u_{0 n} \rightarrow u \quad \text { in } \quad H^{1}(\mathcal{O}) \tag{87}
\end{equation*}
$$

Assume that for $\varphi_{n}^{*}=\left\{\left[p_{0 n}^{*}, u_{0 n}^{*}, w_{1 n}^{*}, w_{2 n}^{*}\right]\right\} \subseteq H_{N}^{\perp}$

$$
\begin{equation*}
(\mathcal{A}+B) \varphi_{n}=\varphi_{n}^{*} \tag{88}
\end{equation*}
$$

In PDE terms this gives

$$
\left\{\begin{array}{c}
-\mathbf{U} \nabla p_{0 n}-\operatorname{div}\left(u_{0 n}\right)-\operatorname{div}(\mathbf{U}) p_{0 n}=p_{0 n}^{*} \quad \text { in } \mathcal{O}  \tag{89}\\
-\nabla p_{0 n}+\operatorname{div} \sigma\left(u_{0 n}\right)-\eta u_{0 n}-\mathbf{U} \nabla u_{0 n}=u_{0 n}^{*} \quad \text { in } \mathcal{O} \\
w_{2 n}+\mathbf{U} \nabla w_{1 n}=w_{1 n}^{*} \text { in } \Omega \\
p_{0 n}-\left.\left[2 \nu \partial_{x_{3}}\left(u_{0 n}\right)_{3}+\lambda \operatorname{div}\left(u_{0 n}\right)\right\}\right|_{\Omega}-\Delta^{2} w_{1 n}=w_{2 n}^{*} \quad \text { in } \Omega
\end{array}\right.
$$

If we read off the first equation in (89) to have

$$
\mathbf{U} \nabla p_{0 n}=-\operatorname{div}\left(u_{0 n}\right)-\operatorname{div}(\mathbf{U}) p_{0 n}-p_{0 n}^{*}
$$

and take upon the limit when $n \rightarrow \infty$ we get

$$
\begin{equation*}
\mathbf{U} \nabla p_{0}=\left[-\operatorname{div}\left(u_{0}\right)-\operatorname{div}(\mathbf{U}) p_{0}-p_{0}^{*}\right] \in L^{2}(\mathcal{O}) \tag{90}
\end{equation*}
$$

Moreover, using the third equation in (89), we have

$$
\begin{equation*}
w_{2}=\lim _{n \rightarrow \infty} w_{2 n}=\lim _{n \rightarrow \infty}\left[w_{1 n}^{*}-\mathbf{U} \cdot \nabla w_{1 n}\right]=\left[w_{1}^{*}-\mathbf{U} \cdot \nabla w_{1}\right] \in H_{0}^{1}(\Omega) \tag{91}
\end{equation*}
$$

In addition, from the domain criteria for $(\mathcal{A}+B)$, we have $u_{0 n}=\mu_{0 n}+\widetilde{\mu}_{0 n}$, where $\mu_{0 n} \in \mathbf{V}_{0}$ and $\widetilde{\mu}_{0 n} \in H^{1}(\mathcal{O})$ satisfies

$$
\tilde{\mu}_{0 n}= \begin{cases}0 & \text { on } S \\ \left(w_{2 n}+\mathbf{U} \cdot \nabla w_{1 n}\right) \mathbf{n} & \text { on } \Omega\end{cases}
$$

Since $\mathbf{V}_{0}$ is closed, then by (87), (91) and the Sobolev Trace Theorem, we have

$$
\begin{equation*}
u_{0}=\mu_{0}+\widetilde{\mu}_{0}, \tag{92}
\end{equation*}
$$

where $\mu_{0} \in \mathbf{V}_{0}$ and $\widetilde{\mu}_{0} \in H^{1}(\mathcal{O})$ satisfies

$$
\widetilde{\mu}_{0}= \begin{cases}0 & \text { on } S \\ \left(w_{2}+\mathbf{U} \cdot \nabla w_{1}\right) \mathbf{n} & \text { on } \Omega\end{cases}
$$

Furthermore, we recall the form of the adjoint $(\mathcal{A}+B)^{*}: D(\mathcal{A}+B)^{*} \cap H_{N}^{\perp} \subset H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ in 106) and given arbitrary $\Phi \in \mathcal{D}(\mathcal{O})$ we will have then $[0, \Phi, 0,0] \in D(\mathcal{A}+B)^{*} \cap H_{N}^{\perp}$. Therewith, we have

$$
\begin{aligned}
& \left(\varphi,(\mathcal{A}+B)^{*}\left[\begin{array}{l}
0 \\
\Phi \\
0 \\
0
\end{array}\right]\right)_{\mathcal{H}}=\lim _{n \rightarrow \infty}\left(\varphi_{n},(\mathcal{A}+B)^{*}\left[\begin{array}{l}
0 \\
\Phi \\
0 \\
0
\end{array}\right]\right)_{\mathcal{H}} \\
& =\lim _{n \rightarrow \infty}\left((\mathcal{A}+B) \varphi_{n},\left[\begin{array}{c}
0 \\
\Phi \\
0 \\
0
\end{array}\right]\right)_{\mathcal{H}}=\left(\left(\varphi^{*},\left[\begin{array}{l}
0 \\
\Phi \\
0 \\
0
\end{array}\right]\right)_{\mathcal{H}},\right.
\end{aligned}
$$

or

$$
\left(p_{0}, \operatorname{div}(\Phi)\right)_{\mathcal{O}}+\left(u_{0}, \operatorname{div} \sigma(\Phi)-\eta \Phi+\mathbf{U} \cdot \nabla \Phi+\operatorname{div}(\mathbf{U}) \Phi\right)_{\mathcal{O}}=\left(u_{0}^{*}, \Phi\right)_{\mathcal{O}}
$$

Upon an integration by parts this relation now becomes

$$
-\left(\nabla p_{0}, \Phi\right)_{\mathcal{O}}+\left(\operatorname{div} \sigma\left(u_{0}\right), \Phi\right)_{\mathcal{O}}-\eta\left(u_{0}, \Phi\right)_{\mathcal{O}}-\left(\mathbf{U} \cdot \nabla u_{0}, \Phi\right)_{\mathcal{O}}=\left(u_{0}^{*}, \Phi\right)_{\mathcal{O}}, \quad \forall \Phi \in \mathcal{D}(\mathcal{O})
$$

Applying a density argument to the above relation gives

$$
\begin{equation*}
-\nabla p_{0}+\operatorname{div} \sigma\left(u_{0}\right)-\eta u_{0}-\mathbf{U} \cdot \nabla u_{0}=u_{0}^{*} \in L^{2}(\mathcal{O}) \tag{93}
\end{equation*}
$$

A further integration by parts assigns a meaning to the trace $\left[\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right]_{\partial \mathcal{O}}$ in $H^{-\frac{1}{2}}-$ sense. What is more: If $\gamma_{0}^{+}(\cdot) \in L\left(H^{\frac{1}{2}}(\partial \mathcal{O}), H^{1}(\mathcal{O})\right)$ is the right inverse of Sobolev Trace Map $\gamma_{0}(\cdot)=\left.(\cdot)\right|_{\partial \mathcal{O}}$, then for every $g \in H^{\frac{1}{2}}(\partial \mathcal{O})$, we have

$$
\begin{gathered}
\left\langle\left[\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right]_{\partial \mathcal{O}}, g\right\rangle_{\partial \mathcal{O}}=\left(\sigma\left(u_{0}\right), \epsilon\left(\gamma_{0}^{+}(g)\right)\right)_{\mathcal{O}}+\left(\operatorname{div} \sigma\left(u_{0}\right), \gamma_{0}^{+}(g)\right)_{\mathcal{O}} \\
\quad-\left(p_{0}, \operatorname{div} \gamma_{0}^{+}(g)\right)_{\mathcal{O}}-\left(\nabla p_{0}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}} \\
=\left(\sigma\left(u_{0}\right), \epsilon\left(\gamma_{0}^{+}(g)\right)\right)_{\mathcal{O}}+\eta\left(u_{0}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}}+\left(\mathbf{U} \cdot \nabla u_{0}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}} \\
\\
+\left(u_{0}^{*}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}}-\left(p_{0}, \operatorname{div} \gamma_{0}^{+}(g)\right)_{\mathcal{O}} \\
=\lim _{n \rightarrow \infty}\left[\left(\sigma\left(u_{0 n}\right), \epsilon\left(\gamma_{0}^{+}(g)\right)\right)_{\mathcal{O}}+\eta\left(u_{0 n}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}}+\left(\mathbf{U} \cdot \nabla u_{0 n}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}}\right. \\
+ \\
\left.+\left(u_{0 n}^{*}, \gamma_{0}^{+}(g)\right)_{\mathcal{O}}-\left(p_{0 n}, \operatorname{div} \gamma_{0}^{+}(g)\right)_{\mathcal{O}}\right] \\
=\lim _{n \rightarrow \infty}\left\langle\left[\sigma\left(u_{0 n}\right) \mathbf{n}-p_{0 n} \mathbf{n}\right]_{\partial \mathcal{O}}, g\right\rangle_{\partial \mathcal{O}}
\end{gathered}
$$

That is

$$
\begin{equation*}
\left[\sigma\left(u_{0 n}\right) \mathbf{n}-p_{0 n} \mathbf{n}\right]_{\partial \mathcal{O}} \rightarrow\left[\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right]_{\partial \mathcal{O}} \quad \text { in } \quad H^{\frac{1}{2}}(\partial \mathcal{O}) \tag{94}
\end{equation*}
$$

The last relation in turn allows us to pass to limit in (89) ${ }_{4}$, and we get

$$
\begin{equation*}
\left.\left[p_{0}-\left(2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right)\right]\right|_{\Omega}-\Delta^{2} w_{1}=w_{2}^{*} \in L^{2}(\Omega) \tag{95}
\end{equation*}
$$

Lastly, from (92) and (93) and the Lax-Milgram Theorem, the flow component $u_{0}=\mu_{0}+\widetilde{\mu}_{0}$ can be characterized via the solution $\mu_{0} \in \mathbf{V}_{0}$ of the following variational problem for all $\chi \in \mathbf{V}_{0}$ :

$$
\begin{gathered}
\left(\sigma\left(\mu_{0}\right), \epsilon(\chi)\right)_{\mathcal{O}}+\eta\left(\mu_{0}, \chi\right)_{\mathcal{O}}=-\left(\sigma\left(\widetilde{\mu}_{0}\right), \epsilon(\chi)\right)_{\mathcal{O}}-\eta\left(\widetilde{\mu}_{0}, \chi\right)_{\mathcal{O}} \\
+\left(p_{0}, \operatorname{div}(\chi)\right)_{\mathcal{O}}-\left(\mathbf{U} \cdot \nabla u_{0}, \chi\right)_{\mathcal{O}}-\left(u_{0}^{*}, \chi\right)_{\mathcal{O}}
\end{gathered}
$$

An integration by parts with respect to this relation now gives for all $\chi \in V_{0}$,

$$
\begin{gathered}
-\left(\operatorname{div} \sigma\left(u_{0}\right), \chi\right)_{\mathcal{O}}+\eta\left(u_{0}, \chi\right)_{\mathcal{O}}+\left\langle\sigma\left(u_{0}\right) \mathbf{n}, \chi\right\rangle_{\partial \mathcal{O}} \\
=-\left(\nabla p_{0}, \chi\right)_{\mathcal{O}}+\left\langle p_{0} \mathbf{n}, \chi\right\rangle_{\partial \mathcal{O}}-\left(\mathbf{U} \cdot \nabla u_{0}, \chi\right)_{\mathcal{O}}-\left(u_{0}^{*}, \chi\right)_{\mathcal{O}}
\end{gathered}
$$

or after using (93)

$$
\left\langle\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}, \chi\right\rangle_{\partial \mathcal{O}}=0, \text { for every } \chi \in V_{0}
$$

which gives in the sense of distributions

$$
\begin{equation*}
\left[\sigma\left(u_{0}\right) \mathbf{n}-p_{0} \mathbf{n}\right] \cdot \tau=0, \quad \forall \tau \in T H^{\frac{1}{2}}(\partial \mathcal{O}) \tag{96}
\end{equation*}
$$

Hence, the estimates (87)-(96) now give the desired conclusion and completes the proof of Lemma 9.

STEP (M-III): Lastly, we prove the following fact:

Lemma 10 For given $\lambda>0$, we have the existence of a constant $\varrho>0$ such that for all $\varphi \in$ $D(\mathcal{A}+B) \cap H_{N}^{\perp}$

$$
\begin{equation*}
\left\|\left\|[\lambda I-(\mathcal{A}+B)] \varphi\left|\left\|_{H_{N}^{\perp}} \geq \varrho\right\|\right| \varphi \mid\right\|_{H_{N}^{\perp}}\right. \tag{97}
\end{equation*}
$$

where the norm $\||\cdot|\|_{H_{N}^{\perp}}$ is defined in 18$)$.
Proof. Using the estimate (26) in Lemma 6, we have for given $\lambda>0$,

$$
\begin{gather*}
(([\lambda I-(\mathcal{A}+B)] \varphi, \varphi))_{H_{N}}^{\perp} \\
\geq \lambda\||\varphi|\|_{H_{\frac{\perp}{N}}}^{2}+C_{1}\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\frac{\epsilon}{2}\left[\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right] \\
\geq \lambda\||\varphi|\|_{H_{N}^{\perp}}^{2}+\left(C_{1}-\frac{\epsilon}{2}\right)\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2}+\frac{\epsilon}{2}\left[\left\|p_{0}\right\|_{\mathcal{O}}^{2}+\left\|u_{0}\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right] \tag{98}
\end{gather*}
$$

With respect to the RHS: we firstly add and subtract, so as to have

$$
\begin{gather*}
\left\|u_{0}\right\|_{\mathcal{O}}^{2}=\left\|\left[u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right]+\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}-\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2} \\
=\left\|\left[u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right]\right\|_{\mathcal{O}}^{2} \\
+2 \operatorname{Re}\left(u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right), \alpha D\left(g \cdot \nabla w_{1}\right) e_{3}-\xi \nabla \psi\left(p_{0}, w_{1}\right)\right)_{\mathcal{O}} \\
+\left\|\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}-\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2} \tag{99}
\end{gather*}
$$

By using Holder-Young Inequalities we get

$$
\begin{gather*}
\left\|u_{0}\right\|_{\mathcal{O}}^{2} \geq(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2} \\
+\left(1-C_{\delta}\right)\left\|\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}-\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2} \tag{100}
\end{gather*}
$$

Using the boundedness of the maps $D(\cdot)$ and $\psi(\cdot, \cdot)$ defined in 22) and 21, respectively we then have

$$
\begin{gather*}
\left\|u_{0}\right\|_{\mathcal{O}}^{2} \geq(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2} \\
+C_{2}\left(1-C_{\delta}\right)\left[\|\mathbf{U}\|_{*}^{2}+\xi^{2}\right]\left\|\Delta w_{1}\right\|_{\Omega}^{2} \tag{101}
\end{gather*}
$$

Now, applying (101) to the RHS of (98), we get

$$
\begin{gather*}
(([\lambda I-(\mathcal{A}+B)] \varphi, \varphi))_{H_{N}^{\perp}} \geq \lambda\||\varphi|\|_{H_{N}^{\perp}}^{2}+\left(C_{1}-\frac{\epsilon}{2}\right)\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2} \\
+\frac{\epsilon}{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2}\right. \\
\left.+\left[1+C_{2}\left(1-C_{\delta}\right)\left[\|\mathbf{U}\|_{*}^{2}+\xi^{2}\right]\right]\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \tag{102}
\end{gather*}
$$

If we take now $\|\mathbf{U}\|_{*}$ so small such that

$$
\|\mathbf{U}\|_{*}^{2}+\xi^{2}<\frac{1}{2 C_{2}\left(C_{\delta}-1\right)},
$$

we then have

$$
\begin{gather*}
(([\lambda I-(\mathcal{A}+B)] \varphi, \varphi))_{H_{\frac{\perp}{N}}} \geq \lambda\||\varphi|\|_{H_{\frac{1}{N}}^{2}}^{2}+\left(C_{1}-\frac{\epsilon}{2}\right)\left\|u_{0}\right\|_{H^{1}(\mathcal{O})}^{2} \\
+\frac{\epsilon}{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\frac{1}{2}\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
\geq \frac{\epsilon}{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\frac{1}{2}\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
+\lambda\left\|w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\mathcal{O}}^{2} \tag{103}
\end{gather*}
$$

Using Cauchy-Schwarz now we obtain

$$
\begin{gather*}
\||[\lambda I-(\mathcal{A}+B)] \varphi|\|_{H_{\frac{1}{N}}}\||\varphi|\|_{H_{\bar{N}}^{\perp}} \\
\geq \frac{\epsilon}{2}\left\{\left\|p_{0}\right\|_{\mathcal{O}}^{2}+(1-\delta)\left\|u_{0}-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p_{0}, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\frac{1}{2}\left\|\Delta w_{1}\right\|_{\Omega}^{2}\right\} \\
+\lambda\left\|w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\mathcal{O}}^{2} \tag{104}
\end{gather*}
$$

which gives the desired estimate (97), with therein

$$
\varrho=\min \left\{\frac{\epsilon}{4}, \lambda\right\}
$$

and finishes the proof of Lemma 10, ■ Now, combining Lemma 8, Lemma 9 and Lemma 10 gives that the map $[\lambda I-(\mathcal{A}+B)]$ satisfies the requirements of Lemma 12 in Appendix which, in turn, yields that

$$
[\lambda I-(\mathcal{A}+B)]^{-1} \in \mathcal{L}\left(H_{N}^{\perp}\right)
$$

and the range condition (61) holds. This finishes the proof of Lemma 7 .
By Lemma 6 and Lemma 7, we have the desired contraction semigroup generation with respect to the special inner product $((\cdot, \cdot))_{H_{\frac{\perp}{N}}}$. Hence we have the asserted wellposedness statement of Theorem 4.

Moreover, form the values of the parameters $\alpha$ and $\xi$ in (23) and 60), respectively, as well as the definition of $((\cdot, \cdot))_{H_{N}^{\perp}}$ in 17 , we infer that $e^{(\mathcal{A}+B) t}$ is uniformly bounded in time, in the standard $\mathcal{H}$-norm. In fact, given $\phi^{*}=\left[p^{*}, u^{*}, w_{1}^{*}, w_{2}^{*}\right] \in H_{N}^{\perp}$, set

$$
\phi(t)=\left[\begin{array}{c}
p(t)  \tag{105}\\
u(t) \\
w_{1}(t) \\
w_{2}(t)
\end{array}\right]=e^{(\mathcal{A}+B) t}\left[\begin{array}{c}
p^{*} \\
u^{*} \\
w_{1}^{*} \\
w_{2}^{*}
\end{array}\right]
$$

Then,

$$
\begin{gathered}
\|\phi(t)\|_{\mathcal{H}}^{2}=\|p\|_{\mathcal{O}}^{2}+\|u\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}+\left\|w_{2}\right\|_{\Omega}^{2} \\
\leq C\left[\|p\|_{\mathcal{O}}^{2}+\left\|u-\alpha D\left(g \cdot \nabla w_{1}\right) e_{3}+\xi \nabla \psi\left(p, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\alpha^{2}\left\|D\left(g \cdot \nabla w_{1}\right) e_{3}\right\|_{\mathcal{O}}^{2}\right. \\
\left.+\xi^{2}\left\|\nabla \psi\left(p, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\left\|\Delta w_{1}\right\|_{\Omega}^{2}+\left\|w_{2}+h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\Omega}^{2}+\left\|h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\Omega}^{2}\right]
\end{gathered}
$$

$$
\leq C\left[\left\|\left|e^{(\mathcal{A}+B) t} \phi^{*}\right|\right\|_{H_{N}^{\perp}}^{2}+\alpha^{2}\left\|D\left(g \cdot \nabla w_{1}\right) e_{3}\right\|_{\mathcal{O}}^{2}+\xi^{2}\left\|\nabla \psi\left(p, w_{1}\right)\right\|_{\mathcal{O}}^{2}+\left\|h_{\alpha} \cdot \nabla w_{1}+\xi w_{1}\right\|_{\Omega}^{2}\right] .
$$

Using the fact that $e^{(\mathcal{A}+B) t}$ is a contraction semigroup on $H_{N}^{\perp}$ with respect to the norm $\||\cdot|\|_{H_{\frac{\perp}{N}}}$, then combining this fact with (18), we have

$$
\|\phi(t)\|_{\mathcal{H}}^{2} \leq C\left[\|\mathbf{U}\|_{*}^{2}+\xi^{2}\right]\|\phi(t)\|_{\mathcal{H}}^{2}+C_{1}\left\|\phi^{*}\right\|_{\mathcal{H}}^{2}
$$

For $\|\mathbf{U}\|_{*}$ small enough, we then have

$$
\|\phi(t)\|_{\mathcal{H}} \leq C^{*}\left\|\phi^{*}\right\|_{\mathcal{H}}, \quad \text { for all } t>0 .
$$

This concludes the proof of Theorem 4

## 4 Appendix

In this section we will provide some useful lemmas that are critically used in this manuscript. In reference to problem (2)-(4), we start with defining the adjoint operator $(\mathcal{A}+B)^{*}: D\left((\mathcal{A}+B)^{*}\right) \cap$ $H_{N}^{\perp} \subset H_{N}^{\perp} \rightarrow H_{N}^{\perp}$ of the semigroup generator $\mathcal{A}+B$ in the following lemma:

Lemma 11 The adjoint operator of the generator $(\mathcal{A}+B)$ (given via (13)-(14)) is defined as

$$
\begin{align*}
& (\mathcal{A}+B)^{*}=\mathcal{A}^{*}+B^{*} \\
& =\left[\begin{array}{cccc}
\mathbf{U} \cdot \nabla(\cdot) & \operatorname{div}(\cdot) & 0 & 0 \\
\nabla(\cdot) & \operatorname{div\sigma }(\cdot)-\eta I+\mathbf{U} \cdot \nabla(\cdot) & 0 & 0 \\
0 & 0 & 0 & -I \\
-[\cdot]_{\Omega} & \left.-\left[2 \nu \partial_{x_{3}}(\cdot)\right)_{3}+\lambda \operatorname{div}(\cdot)\right]_{\Omega} & \Delta^{2} & 0
\end{array}\right] \\
& +\left[\begin{array}{cccc}
\operatorname{div}(\mathbf{U})(\cdot) & 0 & 0 & 0 \\
0 & \operatorname{div}(\mathbf{U})(\cdot) & 0 & 0 \\
\left.\AA^{-1}\left\{\operatorname{div}\left(\left[U_{1}, U_{2}\right]\right)+\mathbf{U} \cdot \nabla\right)\right\}\left.(\cdot)\right|_{\Omega} & \left.\AA^{-1}\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right\}\left[2 \nu \partial_{x_{3}}(\cdot)\right)_{3}+\lambda \operatorname{div}(\cdot)\right]_{\Omega} & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \\
& +\left[\begin{array}{cccc}
-\operatorname{div}(\mathbf{U})(\cdot) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\AA^{-1}\left\{\left(\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right) \Delta^{2}(\cdot)\right\}+\mathbf{U} \cdot \nabla(\cdot)+\Delta \AA^{-1} \nabla^{*}(\nabla \cdot(\mathbf{U} \cdot \nabla(\cdot))) & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \\
& =L_{1}+L_{2}+B^{*} \tag{106}
\end{align*}
$$

Here, $\nabla^{*} \in \mathcal{L}\left(L^{2}(\Omega),\left[H^{1}(\Omega)\right]^{\prime}\right)$ is the adjoint of the gradient operator $\nabla \in \mathcal{L}\left(H^{1}(\Omega), L^{2}(\Omega)\right)$ and the domain of $\left.(\mathcal{A}+B)^{*}\right|_{H_{N}^{\perp}}$ is given as
$D\left((\mathcal{A}+B)^{*}\right) \cap H_{N}^{\perp}=\left\{\left(p_{0}, u_{0}, w_{1}, w_{2}\right) \in L^{2}(\mathcal{O}) \times \mathbf{H}^{1}(\mathcal{O}) \times H_{0}^{2}(\Omega) \times L^{2}(\Omega):\right.$ properties $\left(\mathbf{A}^{*} . \mathbf{i}\right)-\left(\mathbf{A}^{*}\right.$. vii $)$ hold $\}$, where

1. $\left(\mathbf{A}^{*} . \mathbf{i}\right) \mathbf{U} \cdot \nabla p_{0} \in L^{2}(\mathcal{O})$
2. $\left(\mathbf{A}^{*} . \mathbf{i i}\right) \operatorname{div} \sigma\left(u_{0}\right)+\nabla p_{0} \in \mathbf{L}^{2}(\mathcal{O})\left(\right.$ So, $\left.\left[\sigma\left(u_{0}\right) \mathbf{n}+p_{0} \mathbf{n}\right]_{\partial \mathcal{O}} \in \mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O})\right)$
3. $\left(\mathbf{A}^{*} . \mathbf{i i i}\right) \Delta^{2} w_{1}-\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}-\left.p_{0}\right|_{\Omega} \in L^{2}(\Omega)$
4. ( $\mathbf{A}^{*}$.iv) $\left(\sigma\left(u_{0}\right) \mathbf{n}+p_{0} \mathbf{n}\right) \perp T H^{1 / 2}(\partial \mathcal{O})$. That is,

$$
\left\langle\sigma\left(u_{0}\right) \mathbf{n}+p_{0} \mathbf{n}, \tau\right\rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial \mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial \mathcal{O})}=0 \text { in } \mathcal{D}^{\prime}(\mathcal{O}) \text { for every } \tau \in T H^{1 / 2}(\partial \mathcal{O})
$$

5. (A*.v) The flow velocity component $u_{0}=\mathbf{f}_{0}+\widetilde{\mathbf{f}}_{0}$, where $\mathbf{f}_{0} \in \mathbf{V}_{0}$ and $\widetilde{\mathbf{f}}_{0} \in \mathbf{H}^{1}(\mathcal{O})$ satisfies

$$
\widetilde{\mathbf{f}}_{0}= \begin{cases}0 & \text { on } S \\ w_{2} \mathbf{n} & \text { on } \Omega\end{cases}
$$

(and so $\left.\mathbf{f}_{0}\right|_{\partial \mathcal{O}} \in T H^{1 / 2}(\partial \mathcal{O})$ )
6. $\left(\mathbf{A}^{*} . \mathbf{v i}\right)\left[-w_{2}+\mathbf{U} \cdot \nabla w_{1}+\Delta \AA^{-1} \nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right] \in H_{0}^{2}(\Omega)$, (and so $\left.w_{2} \in H_{0}^{1}(\Omega)\right)$
7. ( $\mathbf{A}^{*}$.vii) $\int_{\mathcal{O}}\left[\mathbf{U} \cdot \nabla p_{0}+\operatorname{div}\left(u_{0}\right)\right] d \mathcal{O}$

$$
\begin{aligned}
& +\int_{\Omega} \AA^{-1}\left\{\left(\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right)\left(\left[p_{0}+2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]_{\Omega}\right)\right\} d \Omega \\
& -\int_{\Omega} \AA^{-1}\left\{\left(\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \cdot \nabla\right) \Delta^{2} w_{1}\right\} d \Omega \\
& +\int_{\Omega}\left[\mathbf{U} \cdot \nabla w_{1}+\Delta \AA^{-1} \nabla^{*}\left(\nabla \cdot\left(\mathbf{U} \cdot \nabla w_{1}\right)\right)\right] d \Omega \\
& =0 .
\end{aligned}
$$

Proof. Let $\varphi=\left[p_{0}, u_{0}, w_{1}, w_{2}\right] \in D(\mathcal{A}+B) \cap H_{N}^{\perp}, \widetilde{\varphi}=\left[\widetilde{p}_{0}, \widetilde{u}_{0}, \widetilde{w}_{1}, \widetilde{w}_{2}\right] \in D(\mathcal{A}+B)^{*} \cap H_{N}^{\perp}$. Then, we have

$$
\begin{aligned}
& (\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=-\left(\mathbf{U} \nabla p_{0}, \widetilde{p}_{0}\right)_{\mathcal{O}}-\left(\operatorname{div}\left(u_{0}\right), \widetilde{p}_{0}\right)_{\mathcal{O}}-\left(\nabla p_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}} \\
& +\left(\operatorname{div} \sigma\left(u_{0}\right), \widetilde{u}_{0}\right)_{\mathcal{O}}-\eta\left(u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}}-\left(\mathbf{U} \nabla u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}} \\
& +\left(\Delta w_{2}, \Delta \widetilde{w}_{1}\right)_{\Omega}+\left(\left.p_{0}\right|_{\Omega}-\left.\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}, \widetilde{w}_{2}\right)_{\Omega}-\left(\Delta^{2} w_{1}, \widetilde{w}_{2}\right)_{\Omega} \\
& =\left(p_{0}, \operatorname{div}(\mathbf{U}) \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \mathbf{U} \nabla \widetilde{p}_{0}\right)_{\mathcal{O}}-\left\langle u_{0} \cdot \mathbf{n}, \widetilde{p}_{0}\right\rangle_{\partial \mathcal{O}}+\left(u_{0}, \nabla \widetilde{p}_{0}\right)_{\mathcal{O}} \\
& +\left(p_{0}, \operatorname{div}\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}}-\left\langle p_{0}, \widetilde{u}_{0} \cdot \mathbf{n}\right\rangle_{\partial \mathcal{O}}-\left(\sigma\left(u_{0}\right), \epsilon\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}} \\
& +\left\langle\sigma\left(u_{0}\right) \cdot \mathbf{n}, \widetilde{u}_{0}\right\rangle_{\partial \mathcal{O}}-\eta\left(u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}} \\
& +\left(u_{0}, \operatorname{div}(\mathbf{U}) \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \mathbf{U} \nabla \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(\Delta w_{2}, \Delta \widetilde{w}_{1}\right)_{\Omega} \\
& -\left(\left.\left[2 \nu \partial_{x_{3}}\left(u_{0}\right)_{3}+\lambda \operatorname{div}\left(u_{0}\right)\right]\right|_{\Omega}-\left.p_{0}\right|_{\Omega}, \widetilde{w}_{2}\right)_{\Omega}-\left(\Delta w_{1}, \Delta \widetilde{w}_{2}\right)_{\Omega} .
\end{aligned}
$$

Using the domain criterion (A.vi), we then have from the above equality

$$
(\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=\left(p_{0}, \operatorname{div}(\mathbf{U}) \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \mathbf{U} \nabla \widetilde{p}_{0}\right)_{\mathcal{O}}
$$

$$
\begin{gathered}
-\left(w_{2}+\mathbf{U} \nabla w_{1}, \widetilde{p}_{0}\right)_{\Omega}+\left(u_{0}, \nabla \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}} \\
-\left(\sigma\left(u_{0}\right), \epsilon\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}}-\eta\left(u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \operatorname{div}(\mathbf{U}) \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \mathbf{U} \nabla \widetilde{u}_{0}\right)_{\mathcal{O}} \\
+\left(w_{2}, \Delta^{2} \widetilde{w}_{1}\right)_{\Omega}-\left(\Delta w_{1}, \Delta \widetilde{w}_{2}\right)_{\Omega} .
\end{gathered}
$$

Subsequently, integrating by parts in the third line of the last relation, we get

$$
\begin{gathered}
(\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=\left(p_{0}, \operatorname{div}(\mathbf{U}) \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \mathbf{U} \nabla \widetilde{p}_{0}\right)_{\mathcal{O}} \\
-\left(w_{2}+\mathbf{U} \nabla w_{1}, \widetilde{p}_{0}\right)_{\Omega}+\left(u_{0}, \nabla \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(\widetilde{u}_{0}\right)_{\mathcal{O}}\right. \\
+\left(u_{0}, \operatorname{div} \sigma\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}}-\left\langle u_{0}, \sigma\left(\widetilde{u}_{0}\right) \cdot \mathbf{n}\right\rangle_{\partial \mathcal{O}}-\eta\left(u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}} \\
+\left(u_{0}, \operatorname{div}(\mathbf{U}) \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \mathbf{U} \nabla \widetilde{u}_{0}\right)_{\mathcal{O}} \\
+\left(w_{2}, \Delta^{2} \widetilde{w}_{1}\right)_{\Omega}-\left(\Delta w_{1}, \Delta \widetilde{w}_{2}\right)_{\Omega} .
\end{gathered}
$$

Now, integrating by parts in the second line, and using again domain criterion (A.vi), we have

$$
\begin{gather*}
(\mathcal{A} \varphi, \widetilde{\varphi})_{\mathcal{H}}=\left(p_{0}, \operatorname{div}(\mathbf{U}) \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \mathbf{U} \nabla \widetilde{p}_{0}\right)_{\mathcal{O}} \\
-\left(w_{2},\left.\left[\widetilde{p}_{0}+2 \nu \partial_{x_{3}}\left(\widetilde{u}_{0}\right)_{3}+\lambda \operatorname{div}\left(\widetilde{u}_{0}\right)\right]\right|_{\Omega}\right)_{\Omega} \\
+\left(w_{1},\left.\left(\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \nabla\right)\left[\widetilde{p}_{0}+2 \nu \partial_{x_{3}}\left(\widetilde{u}_{0}\right)_{3}+\lambda \operatorname{div}\left(\widetilde{u}_{0}\right)\right]\right|_{\Omega}\right)_{\Omega} \\
+\left(u_{0}, \nabla \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(p_{0}, \operatorname{div}\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}}+\left(u_{0}, \operatorname{div} \sigma\left(\widetilde{u}_{0}\right)\right)_{\mathcal{O}} \\
-\eta\left(u_{0}, \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \operatorname{div}(\mathbf{U}) \widetilde{u}_{0}\right)_{\mathcal{O}}+\left(u_{0}, \mathbf{U} \nabla \widetilde{u}_{0}\right)_{\mathcal{O}} \\
+\left(w_{2}, \Delta^{2} \widetilde{w}_{1}\right)_{\Omega}-\left(\Delta w_{1}, \Delta \widetilde{w}_{2}\right)_{\Omega} . \tag{107}
\end{gather*}
$$

Also we have

$$
\begin{equation*}
(B \varphi, \widetilde{\varphi})_{\mathcal{H}}=-\left(\operatorname{div}(\mathbf{U}) p_{0}, \widetilde{p}_{0}\right)_{\mathcal{O}}+\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta \widetilde{w}_{1}\right)_{\Omega} \tag{108}
\end{equation*}
$$

For the second term of the RHS of the above equality: for any $w_{1}, \widetilde{w}_{1} \in H^{3}(\Omega)$

$$
\begin{gathered}
\left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta \widetilde{w}_{1}\right)_{\Omega}=\left\langle\frac{\partial}{\partial \nu}\left(\mathbf{U} \nabla w_{1}\right), \Delta \widetilde{w}_{1}\right\rangle_{\partial \Omega} \\
-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega} \\
=\left\langle(\mathbf{U} \cdot \nu) \Delta w_{1}, \Delta \widetilde{w}_{1}\right\rangle_{\partial \Omega}-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega}
\end{gathered}
$$

where we have used the fact that $w_{1}=\frac{\partial w_{1}}{\partial \nu}=0$ and this yields

$$
\frac{\partial}{\partial \nu}\left(\mathbf{U} \nabla w_{1}\right)=(\mathbf{U} \cdot \nu) \frac{\partial^{2} w_{1}}{\partial \nu}=(\mathbf{U} \cdot \nu)\left(\left.\Delta w_{1}\right|_{\partial \Omega}\right) .
$$

Then

$$
\begin{align*}
& \left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta \widetilde{w}_{1}\right)_{\Omega}=\left\langle\Delta w_{1}, \frac{\partial}{\partial \nu}\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right\rangle_{\partial \Omega}-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega} \\
= & \left(\Delta w_{1}, \Delta\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right)_{\Omega}+\left(\nabla \Delta w_{1}, \nabla\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right)_{\Omega}-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega} \\
= & \left(\Delta w_{1}, \Delta\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right)_{\Omega}+\left(\Delta w_{1}, \nabla^{*}\left[\nabla\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right]_{\Omega}-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega}\right. \tag{109}
\end{align*}
$$

where $\nabla^{*} \in \mathcal{L}\left(L^{2}(\Omega),\left[H^{1}(\Omega)\right]^{\prime}\right)$ is the adjoint of the gradient operator $\nabla \in \mathcal{L}\left(H^{1}(\Omega),\left[L^{2}(\Omega)\right]\right)$. To continue with the third term on RHS of (109):

$$
\begin{gather*}
-\left(\nabla\left(\mathbf{U} \nabla w_{1}\right), \nabla \Delta \widetilde{w}_{1}\right)_{\Omega}=\left(\mathbf{U} \nabla w_{1}, \Delta^{2} \widetilde{w}_{1}\right)_{\Omega} \\
=-\left(w_{1},\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \nabla\right\} \Delta^{2} \widetilde{w}_{1}\right)_{\Omega} \\
=-\left(\Delta w_{1}, \Delta \AA^{-1}\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \nabla\right\} \Delta^{2} \widetilde{w}_{1}\right)_{\Omega} \tag{110}
\end{gather*}
$$

If we take into account (110) in 109 and invoke the biharmonic operator with clamped homogeneous boundary conditions we take

$$
\begin{align*}
& \left(\Delta\left(\mathbf{U} \nabla w_{1}\right), \Delta \widetilde{w}_{1}\right)_{\Omega}=-\left(\Delta w_{1}, \Delta \AA^{-1}\left\{\operatorname{div}\left[U_{1}, U_{2}\right]+\mathbf{U} \nabla\right\} \Delta^{2} \widetilde{w}_{1}\right)_{\Omega} \\
& \quad+\left(\Delta w_{1}, \Delta\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right)_{\Omega}+\left(\Delta w_{1}, \Delta\left[\Delta \AA^{-1} \nabla^{*}\left[\nabla\left(\mathbf{U} \nabla \widetilde{w}_{1}\right)\right]\right]\right)_{\Omega} . \tag{111}
\end{align*}
$$

Now, considering (111) in (108) and combining the result with (107) gives the adjoint operator given in 106) and completes the proof of Lemma 11.

In order to establish the wellposedness result, one of the key tools that we use in our proof is the invertibility criterion of a linear, closed operator which we recall in the following lemma [32, pg.102, Lemma 3.8.18]:
Lemma 12 Let $L$ be a linear and closed operator from the Hilbert space $H$ into $H$. Then $L^{-1} \in$ $\mathcal{L}(H)$ if and only if $R(L)$ is dense in $H$ and there is an $m>0$ such that

$$
\|L f\| \geq m\|f\| \quad \text { for all } f \in D(L)
$$

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[^1]:    ${ }^{1}$ The existence of an $\mathbf{H}^{1}(\mathcal{O})$-function $\widetilde{\mathbf{f}}_{0}$ with such a boundary trace on Lipschitz domain $\mathcal{O}$ is assured; see e.g., Theorem 3.33 of 29 .

