

Leray's backward self-similar solutions to the 3D Navier-Stokes equations in Morrey spaces

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Abstract

In this paper, it is shown that there does not exist a non-trivial Leray's backward self-similar solution to the 3D Navier-Stokes equations with profiles in Morrey spaces $\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)$ provided $3/2 < q < 6$, or in $\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)$ provided $6 \leq q < \infty$ and $2 < l \leq q$. This generalizes the corresponding results obtained by Nečas-Ráuzička-Šverák [19, Acta. Math. 176 (1996)] in $L^3(\mathbb{R}^3)$, Tsai [25, Arch. Ration. Mech. Anal. 143 (1998)] in $L^p(\mathbb{R}^3)$ with $p \geq 3$, Chae-Wolf [3, Arch. Ration. Mech. Anal. 225 (2017)] in Lorentz spaces $L^{p,\infty}(\mathbb{R}^3)$ with $p > 3/2$, and Guevara-Phuc [11, SIAM J. Math. Anal. 12 (2018)] in $\dot{\mathcal{M}}^{q,\frac{12-2q}{3}}(\mathbb{R}^3)$ with $12/5 \leq q < 3$ and in $L^{q,\infty}(\mathbb{R}^3)$ with $12/5 \leq q < 6$.

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1 Introduction

We study the following incompressible Navier-Stokes equations in three-dimensional space

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

where u stands for the flow velocity field, the scalar function π represents the pressure. The initial velocity u_0 satisfies $\operatorname{div} u_0 = 0$.

In a seminal work [16], Leray introduced the backward self-similar solutions to construct singular solutions to the 3D Navier-Stokes equations (1.1). The so-called backward self-similar solution is a weak solution (u, π) of (1.1) satisfying, for $a > 0, T \in \mathbb{R}$,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2a(T-t)}} U\left(\frac{x}{\sqrt{2a(T-t)}}\right), \\ \pi(x, t) &= \frac{1}{2a(T-t)} \Pi\left(\frac{x}{\sqrt{2a(T-t)}}\right), \end{aligned} \quad (1.2)$$

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where $U = (U_1, U_2, U_3)$ and Π are defined in \mathbb{R}^3 , and the pair $(u(x, t), \pi(x, t))$ is defined in $\mathbb{R}^3 \times (-\infty, T)$. We obtain a singular solution at $t = T$ if $U \neq 0$ and

$$-\Delta U + aU + a(y \cdot \nabla)U + U \cdot \nabla U + \nabla \Pi = 0, \quad \operatorname{div} U = 0, \quad y \in \mathbb{R}^3. \quad (1.3)$$

The first breakthrough of backward self-similar solutions was due to Nečas-Růžička-Šverák [19]. They ruled out the existence of Leray's backward self-similar solutions to the 3D Navier-Stokes system if a weak solution U of equations (1.3) is in $L^3(\mathbb{R}^3)$ (see Section 2 for the definition of weak solutions of (1.3)). Subsequently, various results involving non-existence of Leray's backward self-similar non-trivial solutions were obtained by Tsai in [25]. Precisely, he proved that Leray's backward self-similar solution is trivial under the condition that $U \in L^p(\mathbb{R}^3)$ with $3 < p < \infty$, and the solution $U \in L^\infty(\mathbb{R}^3)$ in system (1.3) is a constant. In addition, Tsai's second result is that there does not exist a non-trivial solution of the form (1.2) if u satisfy the local energy inequality

$$\sup_{t < s < T} \int_{B_{x_0}(r)} \frac{1}{2} |u(x, s)|^2 dx + \int_t^T \int_{B_{x_0}(r)} |\nabla u(x, s)|^2 dx ds < \infty, \quad (1.4)$$

for some ball $B_{x_0}(r)$ and some $t < T$. Here, $B_{x_0}(r)$ denotes the ball of center x_0 and radius r .

Very recently, Chae-Wolf [3] and Guevara-Phuc [11] independently made progress in this direction. On one hand, Chae-Wolf [3] proved that if U belong to the Lorentz spaces $L^{q, \infty}(\mathbb{R}^3)$ with $q > \frac{3}{2}$ or

$$\|U\|_{L^q(B_{y_0}(1))} + \|\nabla U\|_{L^2(B_{y_0}(1))} = o(|y_0|^{1/2}), \quad \text{as } |y_0| \rightarrow \infty, \quad (1.5)$$

then U must be identically zero. Roughly speaking, the proof in [3] relied on ε -regularity criteria without pressure at one scale, the decay at infinity in Lorentz spaces and Tsai's first result mentioned above. On the other hand, Guevara-Phuc [11] showed that there does not exist a non-trivial solution under the condition that U is in Morrey spaces $\dot{\mathcal{M}}^{q, \frac{12-2q}{3}}(\mathbb{R}^3)$ with $12/5 \leq q < 3$, or $U \in L^{q, \infty}(\mathbb{R}^3)$ with $12/5 \leq q < 6$, or $U \in L^6(\mathbb{R}^3)$. Their arguments were based on Riesz potentials in Morrey spaces, Sobolev spaces with negative indices and Tsai's second aforementioned result. The definitions of relevant function spaces can be found in Section 2. Notice that there holds the following embedding relation

$$L^q(\mathbb{R}^3) \hookrightarrow L^{q, \infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}^{q, l}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}^{q, 1}(\mathbb{R}^3), \quad 1 \leq l < q. \quad (1.6)$$

This fact can be found in [5].

The well-posedness of the 3D Navier-Stokes equations in Morrey spaces was studied in [5, 14, 15, 18, 24]. Compared with the Lebesgue spaces $L^q(\mathbb{R}^3)$ and Lorentz spaces $L^{q, \infty}(\mathbb{R}^3)$, $C_0^\infty(\mathbb{R}^3)$ is not dense in Morrey spaces. It is worth pointing out that Sawano [21] showed that $\dot{\mathcal{M}}^{q, l_2}(\mathbb{R}^n)$ is not dense in $\dot{\mathcal{M}}^{q, l_1}(\mathbb{R}^n)$ if $1 < l_1 < l_2 \leq q$. Based on this, the first objective of this paper is to generalize Guevara and Phuc's result in [11] to general Morrey spaces. Our result is reformulated as

Theorem 1.1. Let $U \in W_{\text{loc}}^{1, 2}(\mathbb{R}^3)$ be a weak solution of (1.3). If

$$U \in \dot{\mathcal{M}}^{q, l}(\mathbb{R}^3) \quad \text{with} \quad 2 < l \leq q < \infty, \quad (1.7)$$

then $U \equiv 0$.

Remark 1.1. This theorem extends all the known results involving non-existence of Leray's backward self-similar non-trivial solutions to the 3D Navier-Stokes equations in [3, 11, 19, 25].

We follow the path of [11, 19, 25] to prove Theorem 1.1. First, in contrast to [11], we construct pressure Π in the Morrey space directly under condition (1.7) without the application of Riesz potentials in Morrey spaces and dual spaces of Sobolev spaces. Second, as [11], our main target is to deduce (1.4) by deriving the energy bound (1.9) in terms of (1.7) from the local energy inequality (2.2). The key tool in [11] is to utilize the duality between $W^{-1,2}(B)$ and the Sobolev space $W_0^{1,2}(B)$ to bound energy flux in local energy inequality (2.2), namely, for $12/5 \leq q < 3$,

$$\begin{aligned} \left| \int_{T-R^2}^T \int_B (|u|^2 + 2\pi) u \cdot \nabla \phi dx dt \right| &= \left| \int_{T-R^2}^T \langle |u|^2 + 2\pi, u \cdot \nabla \phi \rangle_{W^{-1,2}(B), W_0^{1,2}(B)} dt \right| \\ &\leq C \int_{T-R^2}^T \lambda^{2-\frac{6}{q}}(t) \|U\|_{\dot{\mathcal{M}}^{q, \frac{12-2q}{3}}(\mathbb{R}^3)}^2 \|\nabla u\|_{L^2(B)} dt, \end{aligned} \quad (1.8)$$

where the Hardy-Littlewood-Sobolev inequality for Riesz potentials in Morrey spaces was used and $\lambda(t) = [2a(T-t)]^{-1/2}$. It is worth remarking that the upper bound $q < 3$ in [11] comes from the employment of Hardy-Littlewood-Sobolev inequality, but this strategy breaks down in the case (1.7). Here, we make full use of the Meyer-Gerard-Oru interpolation inequality (2.14) and the fact that Morrey spaces $\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)$ can be embedded in Besov spaces $\dot{B}_{\infty, \infty}^{-\frac{3}{q}}(\mathbb{R}^3)$ for $1 < q < \infty$. This helps us control energy flux and the first term in the left hand side of (1.8) as follows

$$\begin{aligned} &\int_{T-\frac{R^2}{4}}^T \|u\|_{L^3(B(R/2))}^3 dt \\ &\leq C \left(\sup_{T-R^2 \leq t \leq T} \|u\|_{L^2(B(R))}^2 + \int_{T-R^2}^T \|\nabla u\|_{L^2(B(R))}^2 dt \right)^{\frac{3}{2}-\frac{1}{\alpha}} \left(\int_{T-\frac{R^2}{4}}^T \lambda^{p-\frac{3p}{q}}(t) \|U\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^p dt \right)^{\frac{2}{p\alpha}}, \end{aligned}$$

where $\alpha = 2/p + 3/q < 2$. Additionally, we apply the pressure decomposition as [12] to bound the second term in the left hand side of (1.8). To sum up, we get the following energy bound

$$\begin{aligned} &\|u\|_{L^\infty(T-\frac{R^2}{4}, T; L^2(B(R/2)))}^2 + \|\nabla u\|_{L^2(T-\frac{R^2}{4}, T; L^2(B(R/2)))}^2 \\ &\leq CR^{3-2\alpha} \left(\int_{T-R^2}^T \lambda^{p-\frac{3p}{q}}(t) \|U\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^p dt \right)^{\frac{2}{p}} + CR^{\frac{6-5\alpha}{2-\alpha}} \left(\int_{T-R^2}^T \lambda^{p-\frac{3p}{q}}(t) \|U\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^p dt \right)^{\frac{4}{p(2-\alpha)}} \\ &\quad + CR^{1-\frac{6}{m}} \left(\int_{T-R^2}^T \lambda^{2-\frac{3}{m}}(t) \|\Pi\|_{\dot{\mathcal{M}}^{m,1}(\mathbb{R}^3)} dt \right)^2. \end{aligned} \quad (1.9)$$

This together with (1.4) means Theorem 1.1.

It should be mentioned that the extra restriction that $q > 2$ and $l > 2$ in (1.7) resulted from the construction process of pressure $\Pi = \mathcal{R}_i \mathcal{R}_j (U_i U_j)$ in the proof of Theorem 1.1. Partially motivated by Chae-Wolf in [3], our next target is to utilize local suitable weak solutions (see Definition 2.3) to remove this restriction.

Theorem 1.2. Let $U \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$ be a weak solution of (1.3). If

$$U \in \dot{\mathcal{M}}^{q,1}(\mathbb{R}^3) \quad \text{with} \quad \frac{3}{2} < q < 6, \quad (1.10)$$

then $U \equiv 0$.

Remark 1.2. According to (1.6), this theorem is an improvement of corresponding results in [3, 11, 19, 25].

The proof of Theorem 1.2 is based on a combination of techniques from [3, 25, 29]. Our starting point is to set up a new Caccioppoli type inequality below

$$\begin{aligned}
& \|u\|_{L^3(T-\frac{R^2}{4}, T; L^{\frac{18}{5}}(B(\frac{R}{2})))}^2 + \|\nabla u\|_{L^2(T-\frac{R^2}{4}, T; L^2(B(\frac{R}{2})))}^2 \\
& \leq CR^{\frac{5q-12}{3q}} \|u\|_{L^{\frac{6q}{2q-3}}(T-R^2, T; \dot{\mathcal{M}}^{q,1}(B(R)))}^2 \\
& \quad + CR^{\frac{4q-15}{2q-3}} \|u\|_{L^{\frac{6q}{2q-3}}(T-R^2, T; \dot{\mathcal{M}}^{q,1}(B(R)))}^{\frac{6q}{2q-3}} + CR^{\frac{2q-6}{q}} \|u\|_{L^{\frac{6q}{2q-3}}(T-R^2, T; \dot{\mathcal{M}}^{q,1}(B(R)))}^3.
\end{aligned} \tag{1.11}$$

This inequality is derived by local suitable weak solutions to the 3D Navier-Stokes equations (1.1) and the aforementioned Meyer-Gerard-Oru interpolation inequality. Various Caccioppoli type inequalities were recently established in [3, 12, 13, 29, 30]. All the proofs rest on local suitable weak solutions originated in [29, 30] by Wolf. It is known that any usual suitable weak solution to the Navier-Stokes system enjoys the local energy inequality (2.8) (see [4, Appendix A, p.1372]). The novelty of local suitable weak solutions is that, as stated in [3, 29, 30], the relevant local energy inequality (2.8) removed the non-local effect of the pressure term. Based on this, Caccioppoli type inequality (1.11) and (1.10) allow us to derive that

$$\|\nabla U\|_{L^2(B_{y_0}(1))} = o(|y_0|^{1/2}) \quad \text{and} \quad \|U\|_{L^3(B_{y_0}(1))} = o(|y_0|^{2/3}), \quad \text{as } |y_0| \rightarrow \infty.$$

However, in light of (1.5), this is not enough to show that $U \equiv 0$. Notice that (1.11) implies $\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy < \infty$, our approach to overcome this difficulty is to construct the pressure Π via A_p weighted inequalities for singular integrals, which enables us to obtain $\pi \in L_{\text{loc}}^{3/2}$ in terms of (1.2). A similar argument has been used by Tsai in [25, 26] for the proof of his second result (1.4). Adopting this approach together with the local energy inequality (2.8), we deduce (1.4) and finally prove Theorem 1.2.

Roughly, the following four figures summarize known results about non-existence of Leray's backward self-similar non-trivial solutions in the framework of Morrey spaces $\dot{\mathcal{M}}^{q,p}(\mathbb{R}^3)$.

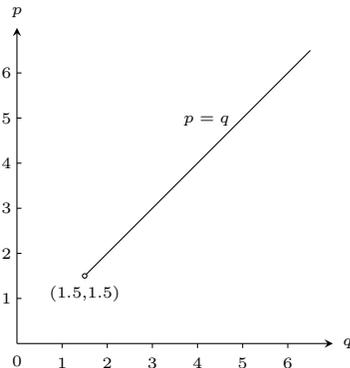


Figure 1: Region of Nečas-Růžička-Šverák, Tsai and Chae-Wolf

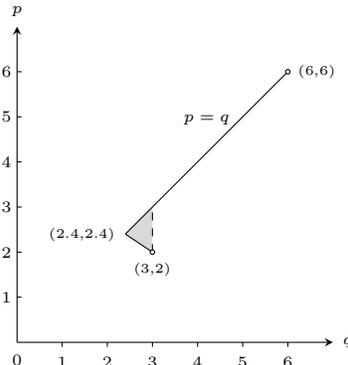


Figure 2: Region of Guevara-Phuc

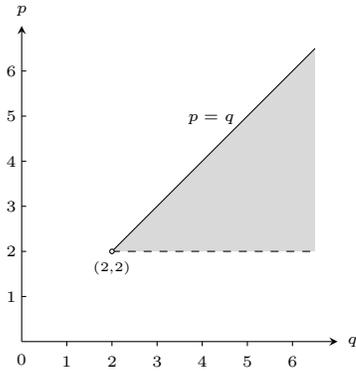


Figure 3: Region of Theorem 1.1

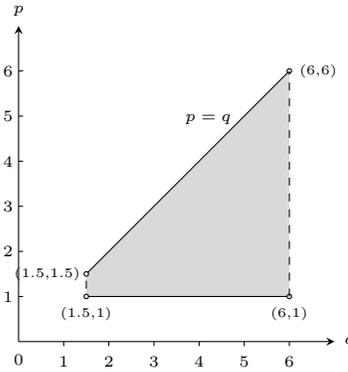


Figure 4: Region of Theorem 1.2

Eventually we would like to mention that, as a by-product of the energy bound and the Caccioppoli type inequality obtained in the proof of Theorem 1.1 and Theorem 1.2, one can establish some ε -regularity criteria at one scale in Morrey spaces. To the best knowledge of the authors, this is the first ε -regularity criterion involving Morrey spaces, which is of independent interest. For the details, see Corollary 3.3 in Section 3 and Corollary 4.2 in Section 4.

This paper is organized as follows. In the second section, we recall the definitions of various function spaces and those of suitable weak solutions including local suitable weak solutions. In addition, we present some auxiliary lemmas. Section 3 is devoted to the proof of Theorem 1.1. To this end, along the line of [11, 19, 25], we construct the pressure Π in Morrey spaces, and then localize the Meyer-Gerard-Oru interpolation inequality. This together with the local energy inequality yields the energy bound, which concludes the proof of our first theorem. In Section 4, we deal with the Caccioppoli type inequality by local suitable weak solutions and local Meyer-Gerard-Oru interpolation inequality obtained in Section 3. Then A_p weighted inequalities enable us to recover the pressure. Finally, applying the local energy inequality (2.8) again and (1.4), we complete the proof of Theorem 1.2.

2 Function spaces and some known facts

2.1 Function spaces

In this section, we begin with definitions of various function spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions on \mathbb{R}^n , endowed with the usual topology, and denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, namely, the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak *-topology. The classical Sobolev space $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, is equipped with the norm $\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}$. A function $f \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ if and only if $f \in W^{1,p}(B)$ for every ball $B \subset \mathbb{R}^n$. The space $C_0^\infty(\Omega)$ is the set of all the smooth compactly supported functions on Ω . Let $W_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$. Let $W^{-1,p}(\Omega)$ denote the dual of the Sobolev space $W_0^{1,p}(\Omega)$.

Next, we present the definitions of Lorentz spaces and Morrey spaces. For $q \in [1, \infty]$,

let

$$L^{q,\infty}(\mathbb{R}^3) = \{f : f \text{ is a measurable function on } \mathbb{R}^3 \text{ and } \|f\|_{L^{q,\infty}(\mathbb{R}^3)} < \infty\}$$

be the Lorentz space $L^{q,\infty}$ defined by means of the quasinorm

$$\|f\|_{L^{q,\infty}(\mathbb{R}^3)} = \sup_{\alpha>0} \alpha |\{x \in \mathbb{R}^3 : |f(x)| > \alpha\}|^{\frac{1}{q}},$$

where $|E|$ represents the three-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^3$. The Morrey space $\dot{\mathcal{M}}^{p,l}(\Omega)$, with $1 \leq l < \infty$, $1 \leq p \leq \infty$ and a domain $\Omega \subset \mathbb{R}^3$, is defined as the space of all measurable functions f on Ω for which the norm

$$\|f\|_{\dot{\mathcal{M}}^{p,l}(\Omega)} = \sup_{R>0} \sup_{x \in \Omega} R^{3(\frac{1}{p}-\frac{1}{l})} \left(\int_{B_x(R) \cap \Omega} |f(y)|^l dy \right)^{\frac{1}{l}} < \infty.$$

Here $B_x(R)$ represents the open ball centered at $x \in \mathbb{R}^3$ with radius $R > 0$. In particular, by using the Lebesgue differentiation theorem, one can easily prove that $\dot{\mathcal{M}}^{\infty,l}(\Omega) = L^\infty(\Omega)$.

To give the definition of Besov spaces, we denote $P_t = e^{t\Delta}$ as the heat semigroup on \mathbb{R}^n . For $\alpha < 0$, a tempered distribution f on \mathbb{R}^n belongs to the Besov space $\dot{B}_{\infty,\infty}^\alpha(\mathbb{R}^n)$ if and only if the following norm

$$\|f\|_{\dot{B}_{\infty,\infty}^\alpha(\mathbb{R}^n)} = \sup_{t>0} t^{-\frac{\alpha}{2}} \|P_t f\|_{L^\infty(\mathbb{R}^n)}$$

is finite.

As usual, given a Schwartz function f on \mathbb{R}^n , the Fourier transform \hat{f} of f is given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx,$$

and the inverse Fourier transform f^\vee is defined as

$$f^\vee(\xi) = \widehat{f}(-\xi)$$

for all $\xi \in \mathbb{R}^n$. Furthermore, for $s \geq 0$, we define $\Lambda^s f$ by

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi),$$

where the notation Λ stands for the square root of the negative Laplacian $(-\Delta)^{1/2}$. Now introduce the homogenous Sobolev norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^n)}$ as follows

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \|\Lambda^s f\|_{L^2(\mathbb{R}^n)}.$$

Denote H^s as the standard inhomogenous Sobolev space with the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|\Lambda^s f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}.$$

For $q \in [1, \infty]$, the notation $L^q(0, T; X)$ stands for the set of all measurable functions $f(x, t)$ on the interval $(0, T)$ with values in X and $\|f(\cdot, t)\|_X$ belonging to $L^q(0, T)$. Throughout this paper, we denote

$$B(r) := \{y \in \mathbb{R}^3 \mid |y| < r\}, \quad Q(r) := B(r) \times (T - r^2, T).$$

The average integral of a function h on the ball $B(r)$ is defined by $\bar{h}_r := \frac{1}{|B(r)|} \int_{B(r)} h$. For simplicity, we write

$$\|\cdot\|_{L^p L^q(Q(r))} := \|\cdot\|_{L^p(T-r^2, T; L^q(B(r)))} \quad \text{and} \quad \|\cdot\|_{L^p(Q(r))} := \|\cdot\|_{L^p L^p(Q(r))}.$$

We will use the summation convention on repeated indices. C is an absolute constant which may be different from line to line unless otherwise stated in this paper.

2.2 Suitable weak solutions

Definition 2.1. A divergence-free vector field $U = (U_1, U_2, U_3) \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$ is called a weak solution of (1.3), if for all divergence-free vector field $\phi = (\phi_1, \phi_2, \phi_3) \in C_0^\infty(\mathbb{R}^3)$ one has

$$\int_{\mathbb{R}^3} (\nabla U \cdot \nabla \phi + [aU + a(y \cdot \nabla)U + (U \cdot \nabla)U] \cdot \phi) dy = 0. \quad (2.1)$$

Now, for the convenience of readers, we recall the classical definition of suitable weak solutions to the Navier-Stokes system (1.1).

Definition 2.2. A pair (u, π) is called a suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,

- (1) $u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; \dot{H}^1(\mathbb{R}^3))$, $\pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3))$;
- (2) (u, π) solves (1.1) in $\mathbb{R}^3 \times (-T, 0)$ in the sense of distributions;
- (3) (u, π) satisfies the following inequality, for a.e. $t \in [-T, 0]$,

$$\begin{aligned} & \int_{\mathbb{R}^3} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{-T}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \\ & \leq \int_{-T}^t \int_{\mathbb{R}^3} |u|^2 (\partial_s \phi + \Delta \phi) dx ds + \int_{-T}^t \int_{\mathbb{R}^3} u \cdot \nabla \phi (|u|^2 + 2\pi) dx ds, \end{aligned} \quad (2.2)$$

where non-negative function $\phi(x, s) \in C_0^\infty(\mathbb{R}^3 \times (-T, 0))$.

Lemma 2.1. [12] Let Φ denote the standard normalized fundamental solution of Laplace equation in \mathbb{R}^3 . For $0 < \xi < \eta$, we consider smooth cut-off function $\psi \in C_0^\infty(B(\frac{\xi+3\eta}{4}))$ such that $0 \leq \psi \leq 1$ in $B(\eta)$, $\psi \equiv 1$ in $B(\frac{3\xi+5\eta}{8})$ and $|\nabla^k \psi| \leq C/(\eta - \xi)^k$ with $k = 1, 2$ in $B(\eta)$. Then we may split pressure π in (1.1) as below

$$\pi(x) := \pi_1(x) + \pi_2(x) + \pi_3(x), \quad x \in B(\frac{\xi + \eta}{2}), \quad (2.3)$$

where

$$\begin{aligned} \pi_1(x) &= -\partial_i \partial_j \Phi * (u_j u_i \psi), \\ \pi_2(x) &= 2\partial_i \Phi * (u_j u_i \partial_j \psi) - \Phi * (u_j u_i \partial_i \partial_j \psi), \\ \pi_3(x) &= 2\partial_i \Phi * (\pi \partial_i \psi) - \Phi * (\pi \partial_i \partial_i \psi). \end{aligned}$$

Moreover, there holds

$$\|\pi_1\|_{L^{3/2}(Q(\frac{\xi+\eta}{2}))} \leq C \|u\|_{L^3(Q(\frac{\xi+3\eta}{4}))}^2; \quad (2.4)$$

$$\|\pi_2\|_{L^{3/2}(Q(\frac{\xi+\eta}{2}))} \leq \frac{C\eta^3}{(\eta - \xi)^3} \|u\|_{L^3(Q(\frac{\xi+3\eta}{4}))}^2; \quad (2.5)$$

$$\|\pi_3\|_{L^1 L^2(Q(\frac{\xi+\eta}{2}))} \leq \frac{C\eta^{3/2}}{(\eta - \xi)^3} \|\pi\|_{L^1(Q(\frac{\xi+3\eta}{4}))}. \quad (2.6)$$

2.3 Local suitable weak solutions

We begin with the Wolf's local pressure projection $\mathcal{W}_{p,\Omega} : W^{-1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ ($1 < p < \infty$). More precisely, for any $f \in W^{-1,p}(\Omega)$, we define $\mathcal{W}_{p,\Omega}(f) = \nabla\pi$, where π satisfies (2.7). Let Ω be a bounded domain with $\partial\Omega \in C^1$. According to the L^p theorem of Stokes system in [7, Theorem 2.1, p.149], there exists a unique pair $(u, \pi) \in W^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$-\Delta u + \nabla\pi = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad \int_{\Omega} \pi dx = 0. \quad (2.7)$$

Moreover, this pair is subject to the inequality

$$\|u\|_{W^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leq C\|f\|_{W^{-1,p}(\Omega)}.$$

Let $\nabla\pi = \mathcal{W}_{p,\Omega}(f)$ ($f \in L^p(\Omega)$), then $\|\pi\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$, where we used the fact that $L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$. Moreover, from $\Delta\pi = \operatorname{div} f$, we see that $\|\nabla\pi\|_{L^p(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|\pi\|_{L^p(\Omega)}) \leq C\|f\|_{L^p(\Omega)}$. Now, we present the definition of local suitable weak solutions to Navier-Stokes equations (1.1).

Definition 2.3. *A pair (u, π) is called a local suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,*

- (1) $u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; \dot{H}^1(\mathbb{R}^3))$, $\pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3))$;
- (2) (u, π) solves (1.1) in $\mathbb{R}^3 \times (-T, 0)$ in the sense of distributions;
- (3) The local energy inequality reads, for a.e. $t \in [-T, 0]$ and non-negative function $\phi(x, s) \in C_0^\infty(\mathbb{R}^3 \times (-T, 0))$,

$$\begin{aligned} & \int_{B(R)} |v|^2 \phi(x, t) dx + \int_{-T}^t \int_{B(R)} |\nabla v|^2 \phi(x, s) dx ds \\ & \leq \int_{-T}^t \int_{B(R)} |v|^2 (\Delta\phi + \partial_s \phi) dx ds + \int_{-T}^t \int_{B(R)} |v|^2 u \cdot \nabla\phi dx ds \\ & \quad + \int_{-T}^t \int_{B(R)} \phi(u \otimes v : \nabla^2 \pi_h) dx ds + \int_{-T}^t \int_{B(R)} \phi \pi_1 v \cdot \nabla\phi dx ds + \int_{-T}^t \int_{B(R)} \phi \pi_2 v \cdot \nabla\phi dx ds. \end{aligned} \quad (2.8)$$

Here, $\nabla\pi_h = -\mathcal{W}_{p,B(R)}(u)$, $\nabla\pi_1 = \mathcal{W}_{p,B(R)}(\Delta u)$, $\nabla\pi_2 = -\mathcal{W}_{p,B(R)}(u \cdot \nabla u)$, $v = u + \nabla\pi_h$. In addition, $\nabla\pi_h, \nabla\pi_1$ and $\nabla\pi_2$ meet the following facts

$$\|\nabla\pi_h\|_{L^p(B(R))} \leq C\|u\|_{L^p(B(R))}, \quad (2.9)$$

$$\|\pi_1\|_{L^2(B(R))} \leq C\|\nabla u\|_{L^2(B(R))}, \quad (2.10)$$

$$\|\pi_2\|_{L^{p/2}(B(R))} \leq C\|u\|_{L^p(B(R))}^2. \quad (2.11)$$

We list some interior estimates of the harmonic equation $\Delta h = 0$, which will be frequently utilized later. Let $1 \leq p, q \leq \infty$ and $0 < r < \rho$, then there holds

$$\|\nabla^k h\|_{L^q(B(r))} \leq \frac{Cr^{\frac{3}{q}}}{(\rho - r)^{\frac{3}{p} + k}} \|h\|_{L^p(B(\rho))}, \quad (2.12)$$

$$\|h - \bar{h}_r\|_{L^q(B(r))} \leq \frac{Cr^{\frac{3}{q}+1}}{(\rho - r)^{\frac{3}{q}+1}} \|h - \bar{h}_\rho\|_{L^q(B(\rho))}. \quad (2.13)$$

The proof of (2.12) rests on the mean value property of harmonic functions. This together with mean value theorem leads to (2.13). We leave the details to the reader.

2.4 Meyer-Gerard-Oru interpolation inequality

For the convenience of the readers, we recall Meyer-Gerard-Oru interpolation inequality in [17, 20] below

$$\|f\|_{L^m(\mathbb{R}^n)} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{2}{m}} \|f\|_{\dot{B}_{\infty,\infty}^{-\frac{2s}{m-2}}(\mathbb{R}^n)}^{1-\frac{2}{m}} \quad \text{with } s > 0 \text{ and } 2 < m < \infty. \quad (2.14)$$

In addition, we have the following embedding relation between Morrey spaces and Besov spaces (see [20, Section 3.2, Lemma 1])

$$\|f\|_{\dot{B}_{\infty,\infty}^{-\frac{3}{q}}(\mathbb{R}^3)} \leq C \|f\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)} \quad \text{with } 1 < q < \infty. \quad (2.15)$$

These two inequalities play an important role in our proof.

3 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. In Subsection 3.1, we define Π by $\mathcal{R}_i \mathcal{R}_j(U_i U_j)$ and examine that it satisfies the equations (1.3). In the second subsection we establish the energy bounds. Invoking energy bounds and (1.4), we get the desired results in the last subsection.

3.1 The construction of the pressure

Lemma 3.1. *Suppose that U is a weak solution of (1.3) and $U \in \dot{\mathcal{M}}^{2p,2q}(\mathbb{R}^3)$ with $1 < q \leq p < \infty$. Let $\tilde{\Pi} = \mathcal{R}_i \mathcal{R}_j(U_i U_j)$. Then there holds*

$$(1) \quad \|\tilde{\Pi}\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)} \leq C \|U\|_{\dot{\mathcal{M}}^{2p,2q}(\mathbb{R}^3)}^2. \quad (3.1)$$

(2) $\tilde{\Pi}$ meets

$$-\Delta \tilde{\Pi} = \partial_i \partial_j (U_i U_j),$$

in the distributional sense.

(3) $(U, \tilde{\Pi})$ smoothly solves

$$-\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla \tilde{\Pi} = 0 \quad \text{in } \mathbb{R}^3. \quad (3.2)$$

Here the solution Π is unique up to a constant.

Proof. (1) The boundedness of singular integral operators $R_i R_j$ on Morrey spaces $\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)$ with $1 < q \leq p < \infty$ means (3.1).

(2) To show the second assertion, assume for a while, we have proved that

$$\int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) \Delta \varphi dx = - \int_{\mathbb{R}^3} \chi_{B(R)} U_i U_j \partial_i \partial_j \varphi dx, \quad \varphi \in C_0^\infty(\mathbb{R}^3), \quad (3.3)$$

where $\chi_{B(R)}$ is the characteristic function of $B(R)$. The problem reduces to that of passing to the limit as R tends to infinity. Indeed, note that, for any $f \in \dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)$ and $b > 3 - \frac{3q}{p}$,

$$\int_{\mathbb{R}^3} |f|^q (1 + |x|^2)^{-\frac{b}{2}} dx \leq C \|f\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}^q.$$

This fact can be found in [14]. To make our paper more self-contained and more readable, we present the proof of the above fact as follows

$$\begin{aligned} \int_{\mathbb{R}^3} |f|^q (1 + |x|^2)^{-\frac{b}{2}} dx &\leq \int_{|x| \leq 1} |f|^q (1 + |x|^2)^{-\frac{b}{2}} dx + \int_{|x| > 1} |f|^q (1 + |x|^2)^{-\frac{b}{2}} dx \\ &\leq \int_{|x| \leq 1} |f|^q dx + \sum_{k=0}^{\infty} \int_{2^k < |x| \leq 2^{k+1}} |f|^q (1 + |x|^2)^{-\frac{b}{2}} dx \\ &\leq \|f\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}^q + C \sum_{k=0}^{\infty} (2^{k+1})^{-b-3(\frac{q}{p}-1)} \|f\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}^q \\ &\leq C \|f\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}^q. \end{aligned}$$

In addition, note that $w(x) = (1 + |x|^2)^{-\frac{b}{2}}$ satisfies the Muckenhoupt A_q -condition if $0 \leq b < 3$. Therefore, for $p < \infty$, we choose b such that $3 - \frac{3q}{p} < b < 3$. Then, the Hölder inequality and the classical Calderón-Zygmund Theorem with A_q weights yield that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(\mathcal{R}_i \mathcal{R}_j (U_i U_j) - \mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) \right) \Delta \varphi dx \right| \\ &\leq \left(\int_{\mathbb{R}^3} |\mathcal{R}_i \mathcal{R}_j (U_i U_j - \chi_{B(R)} U_i U_j)|^q w(x) dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} |\Delta \varphi|^{\frac{q}{q-1}} w(x)^{\frac{1}{1-q}} dx \right)^{1-\frac{1}{q}} \\ &\leq C \left(\int_{\mathbb{R}^3} |U_i U_j - \chi_{B(R)} U_i U_j|^q w(x) dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^3} |\Delta \varphi|^{\frac{q}{q-1}} w(x)^{\frac{1}{1-q}} dx \right)^{1-\frac{1}{q}} \\ &\leq C \|U_i U_j\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}, \end{aligned}$$

Likewise,

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(U_i U_j - \chi_{B(R)} U_i U_j \right) \partial_i \partial_j \varphi dx \right| \\ &\leq C \|U_i U_j\|_{\dot{\mathcal{M}}^{p,q}(\mathbb{R}^3)}. \end{aligned}$$

Thus we can apply the Lebesgue's dominated convergence theorem to obtain

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) \Delta \varphi dx = \int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (U_i U_j) \Delta \varphi dx,$$

and

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} \chi_{B(R)} U_i U_j \partial_i \partial_j \varphi dx = \int_{\mathbb{R}^3} U_i U_j \partial_i \partial_j \varphi dx. \quad (3.4)$$

Combining this and (3.3), we know that

$$\int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (U_i U_j) \Delta \varphi dx = - \int_{\mathbb{R}^3} U_i U_j \partial_i \partial_j \varphi dx, \quad (3.5)$$

which implies the second assertion. Subsequently, we need to prove (3.3) we have assumed. Since the Fourier transform is a topological isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself, we conclude that $(\Delta \varphi)^\vee \in \mathcal{S}(\mathbb{R}^3)$ from $\Delta \varphi \in C_0^\infty(\mathbb{R}^3) \subset \mathcal{S}(\mathbb{R}^3)$. Moreover, there holds $\Delta \varphi = ((\Delta \varphi)^\vee)^\wedge$. Let $s = \min\{q, 2\}$. According to the definition of the Fourier transform of tempered distributions and $\mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) \in L^s(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3)$, we discover

$$\int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) ((\Delta \varphi)^\vee)^\wedge dx = \int_{\mathbb{R}^3} (\mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j))^\wedge (\Delta \varphi)^\vee dx. \quad (3.6)$$

We recall the following fact (see [6, p.76]) that for any $f \in L^m(\mathbb{R}^3)$ with $1 < m \leq 2$,

$$(\mathcal{R}_i \mathcal{R}_j f)^\wedge(\xi) = - \frac{\xi_i \xi_j}{|\xi|^2} \hat{f}(\xi), \quad \xi \in \mathbb{R}^3. \quad (3.7)$$

From (3.6) and (3.7), we observe that

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j) \Delta \varphi dx &= \int_{\mathbb{R}^3} (\mathcal{R}_i \mathcal{R}_j (\chi_{B(R)} U_i U_j))^\wedge (\Delta \varphi)^\vee dx \\ &= - \int_{\mathbb{R}^3} \frac{\xi_i \xi_j}{|\xi|^2} (\chi_{B(R)} U_i U_j)^\wedge (|\xi|^2 \varphi^\vee) dx \\ &= - \int_{\mathbb{R}^3} \chi_{B(R)} U_i U_j (\xi_i \xi_j \varphi^\vee)^\wedge dx \\ &= - \int_{\mathbb{R}^3} \chi_{B(R)} U_i U_j \partial_i \partial_j \varphi dx. \end{aligned} \quad (3.8)$$

This confirms (3.3).

Next, we turn our attention to demonstrating (3.2). Before going further, we write

$$F := -\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla \tilde{\Pi}$$

and show that $F \equiv 0$.

It follows from $F = \nabla \tilde{\Pi} - \nabla \Pi$, $\operatorname{div} F = 0$ and $\operatorname{curl} F = 0$ that $\Delta F = 0$. Since harmonic functions are analytic, to get $F \equiv 0$, it suffices to show that $D^\alpha F(0) = 0$ for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| \geq 0$. Let $\theta \in C_0^\infty(\mathbb{R}^3)$ be any radial function which is supported in $\{x \in \mathbb{R}^3 \mid |x| < 1\}$ and has integral 1. Note that $D^\alpha F$ is also harmonic, there holds

$$D^\alpha F(0) = \varepsilon^3 \int_{\mathbb{R}^3} D^\alpha F(y) \theta(\varepsilon y) dy \quad (3.9)$$

for any $\varepsilon > 0$. (See [22, p.275] for the details.)

Integrating by parts, one computes

$$D^\alpha F(0) = \varepsilon^3 \int_{\mathbb{R}^3} D^\alpha F(y) \theta(\varepsilon y) dy = (-1)^{|\alpha|} \varepsilon^3 \int_{\mathbb{R}^3} F(y) \varepsilon^{|\alpha|} (D^\alpha \theta)(\varepsilon y) dy.$$

According to this, to show that $D^\alpha F(0) = 0$, it suffices to prove that for any $\varphi \in C_0^\infty(B(1))$,

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^3 \int_{\mathbb{R}^3} F(y) \varphi(\varepsilon y) dy = 0.$$

Integration by parts twice, the Hölder inequality and the definition of Morrey spaces guarantee that

$$\begin{aligned} \left| \varepsilon^3 \int_{\mathbb{R}^3} \Delta U \varphi(\varepsilon y) dy \right| &= \left| \varepsilon^5 \int_{\mathbb{R}^3} U \Delta \varphi(\varepsilon y) dy \right| \\ &\leq \varepsilon^5 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^{2q} dy \right)^{\frac{1}{2q}} \left(\int_{|y| < \frac{1}{\varepsilon}} |\Delta \varphi(\varepsilon y)|^{\frac{2q}{2q-1}} dy \right)^{1 - \frac{1}{2q}} \\ &\leq C \varepsilon^{2 + \frac{3}{2p}} \|U\|_{\mathcal{M}^{2p, 2q}(\mathbb{R}^3)}. \end{aligned}$$

A slight modification of the latter argument yields

$$\begin{aligned} \left| \varepsilon^3 \int_{\mathbb{R}^3} U \varphi(\varepsilon y) dy \right| &\leq \varepsilon^3 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^{2q} dy \right)^{\frac{1}{2q}} \left(\int_{|y| < \frac{1}{\varepsilon}} |\varphi(\varepsilon y)|^{\frac{2q}{2q-1}} dy \right)^{1 - \frac{1}{2q}} \\ &\leq C \varepsilon^{\frac{3}{2p}} \|U\|_{\mathcal{M}^{2p, 2q}(\mathbb{R}^3)}. \end{aligned}$$

In virtue of integration by parts once again, we see that

$$\varepsilon^3 \int_{\mathbb{R}^3} (y \cdot \nabla) U(y) \varphi(\varepsilon y) dy = -\varepsilon^3 \int_{\mathbb{R}^3} 3U(y) \varphi(\varepsilon y) dy - \varepsilon^3 \int_{\mathbb{R}^3} U(y) \{(\varepsilon y) \cdot \nabla \varphi(\varepsilon y)\} dy. \quad (3.10)$$

In the same manner as above, we can also bound $|\varepsilon^3 \int_{\mathbb{R}^3} (y \cdot \nabla) U(y) \varphi(\varepsilon y) dy|$.

On the other hand, it follows from the divergence-free condition that

$$\begin{aligned} &\varepsilon^3 \int_{\mathbb{R}^3} (U \cdot \nabla) U(y) \varphi(\varepsilon y) dy \\ &= \varepsilon^3 \int_{\mathbb{R}^3} \operatorname{div} U \otimes U(y) \varphi(\varepsilon y) dy \\ &= -\varepsilon^4 \int_{\mathbb{R}^3} U \otimes U(y) \cdot \nabla \varphi(\varepsilon y) dy, \end{aligned} \quad (3.11)$$

which in turn implies that

$$\begin{aligned} &\left| \varepsilon^3 \int_{\mathbb{R}^3} (U \cdot \nabla) U(y) \varphi(\varepsilon y) dy \right| \\ &\leq \varepsilon^4 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^{2q} dy \right)^{\frac{1}{q}} \left(\int_{|y| < \frac{1}{\varepsilon}} |\nabla \varphi(\varepsilon y)|^{\frac{q}{q-1}} dy \right)^{1 - \frac{1}{q}} \\ &\leq C \varepsilon^{1 + \frac{3}{p}} \|U\|_{\mathcal{M}^{2p, 2q}(\mathbb{R}^3)}^2. \end{aligned} \quad (3.12)$$

Eventually, we need to bound the last term involving $\nabla \tilde{\Pi}$. Integration by parts gives

$$\varepsilon^3 \int_{\mathbb{R}^3} \nabla \tilde{\Pi}(y) \varphi(\varepsilon y) dy = -\varepsilon^4 \int_{\mathbb{R}^3} \tilde{\Pi}(y) \nabla \varphi(\varepsilon y) dy. \quad (3.13)$$

Furthermore, a variant of (3.12) provides the estimate

$$\left| \varepsilon^4 \int_{\mathbb{R}^3} \tilde{\Pi}(y) \nabla \varphi(\varepsilon y) dy \right| \leq C \varepsilon^{1 + \frac{3}{p}} \|\tilde{\Pi}\|_{\mathcal{M}^{p, q}(\mathbb{R}^3)} \leq C \varepsilon^{1 + \frac{3}{p}} \|U\|_{\mathcal{M}^{2p, 2q}(\mathbb{R}^3)}^2.$$

This verifies $D^\alpha F(0) = 0$ and completes the proof of Lemma 3.1. \square

3.2 Energy bounds

Proposition 3.2. *Assume that (u, π) is a suitable weak solution to the 3D Navier-Stokes system (1.1). Let $\alpha = 2/p + 3/q$ and the pair (p, q) satisfy*

$$1 \leq 2/p + 3/q < 2, \quad \text{with } 1 < p < \infty. \quad (3.14)$$

Then there holds, for any $m \geq 1$,

$$\begin{aligned} & \|u\|_{L^\infty(T - \frac{R^2}{4}, T; L^2(B(R/2)))}^2 + \|\nabla u\|_{L^2(T - \frac{R^2}{4}, T; L^2(B(R/2)))}^2 \\ & \leq CR^{3-2\alpha} \|u\|_{L^p(T - R^2, T; \dot{\mathcal{M}}^{q,1}(B(R)))}^2 \\ & \quad + CR^{\frac{6-5\alpha}{2-\alpha}} \|u\|_{L^p(T - R^2, T; \dot{\mathcal{M}}^{q,1}(B(R)))}^{\frac{4}{2-\alpha}} + CR^{1-\frac{6}{m}} \|\pi\|_{L^1(T - R^2, T; \dot{\mathcal{M}}^{m,1}(B(R)))}^2. \end{aligned} \quad (3.15)$$

Remark 3.1. We refer the readers to [10] and [12] for recent progress on energy bounds of suitable weak solutions.

In the spirit of [10, 12], we conclude ε -regularity criteria at one scale in Morrey spaces from the above energy bounds. Previous related results in Lorentz spaces can be found in [1, 28]. In addition, a summary of ε -regularity criteria at one scale is given in [12].

Corollary 3.3. *Let the pair (u, π) be a suitable weak solution to the 3D Navier-Stokes system (1.1) in $Q(1)$. There exists an absolute positive constant ε_1 such that if the pair (u, π) satisfy*

$$\|u\|_{L^p(-1,0; \dot{\mathcal{M}}^{q,1}(B(1)))} + \|\pi\|_{L^1(-1,0; \dot{\mathcal{M}}^{m,1}(B(1)))} \leq \varepsilon_1, \quad (3.16)$$

where

$$1 \leq 2/p + 3/q < 2, 1 < p < \infty \text{ and } m \geq 1,$$

then $u \in L^\infty(Q(1/2))$.

Remark 3.2. This result extends regularity criteria via Lebesgue spaces in [12] to Morrey spaces. We refer the readers to [9] for various ε -regularity criteria at all scales in Lebesgue spaces.

For abbreviation, we set

$$E(r) = \sup_{T-r^2 \leq t \leq T} \|u(\cdot, t)\|_{L^2(B(r))}^2 + \int_{T-r^2}^T \|\nabla u\|_{L^2(B(r))}^2 dt.$$

Lemma 3.4. *For $0 < \xi < \eta$, let $r = \frac{\xi+3\eta}{4}$ and $\alpha = 2/p + 3/q$ with the pair (p, q) satisfying (3.14). Then there exists an absolute constant C independent of ξ and η , such that*

$$\int_{T-r^2}^T \|u\|_{L^3(B(r))}^3 dt \leq C \eta^{\frac{3(2-\alpha)}{2\alpha}} \left(1 + \frac{\eta^2}{(\eta - \xi)^2}\right)^{\frac{3}{2} - \frac{1}{\alpha}} E^{\frac{3}{2} - \frac{1}{\alpha}}(\eta) \left(\int_{T-r^2}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\eta))}^p dt\right)^{\frac{2}{p\alpha}}. \quad (3.17)$$

Proof. For any pair (p, q) satisfying (3.14), we can select $p' < p$ such that the value of $\frac{2}{p'} + \frac{3}{q}$ is very close to 2 and smaller than 2. Due to the Hölder inequality only in time direction, it is enough to consider the case that $\frac{2}{p} + \frac{3}{q}$ is close to 2. To proceed further, we set

$$m = \frac{6\alpha}{3\alpha - 2}.$$

Taking $\alpha \rightarrow 2^-$, we conclude from some elementary computations that

$$3 < m \leq \frac{2q+6}{3}. \quad (3.18)$$

Invoking interpolation inequality (2.14), we see that

$$\begin{aligned} \|u\|_{L^m(\mathbb{R}^3)} &\leq C \|u\|_{\dot{H}^{\frac{3(m-2)}{2q}}(\mathbb{R}^3)}^{\frac{2}{m}} \|u\|_{\dot{B}_{\infty,\infty}^{-\frac{3}{q}}(\mathbb{R}^3)}^{1-\frac{2}{m}} \\ &\leq C \|u\|_{\dot{H}^{\frac{3(m-2)}{2q}}(\mathbb{R}^3)}^{\frac{2}{m}} \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{1-\frac{2}{m}}, \end{aligned} \quad (3.19)$$

where we also used (2.15).

Using the Hölder inequality and (3.19), we arrive at

$$\begin{aligned} \|u\|_{L^3(B(r))}^3 &\leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \|u\|_{L^m(B(r))}^3 \\ &\leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \|u\|_{L^m(\mathbb{R}^3)}^3 \\ &\leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \|u\|_{\dot{H}^{\frac{3(m-2)}{2q}}(\mathbb{R}^3)}^{\frac{6}{m}} \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{3-\frac{6}{m}}. \end{aligned} \quad (3.20)$$

From the Gagliardo-Nirenberg inequality and (3.18), we get

$$\|u\|_{\dot{H}^{\frac{3(m-2)}{2q}}(\mathbb{R}^3)} \leq C \|u\|_{L^2(\mathbb{R}^3)}^{\frac{2q-3m+6}{2q}} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3m-6}{2q}}.$$

Substituting this into (3.20), we deduce that

$$\|u\|_{L^3(B(r))}^3 \leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{3(2q-3m+6)}{mq}} \|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3(3m-6)}{mq}} \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{3-\frac{6}{m}}.$$

Integrating this inequality in time and applying the Hölder inequality, we know that

$$\begin{aligned} &\int_{T-r^2}^T \|u\|_{L^3(B(r))}^3 dt \\ &\leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \sup_{T-r^2 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{3(2q-3m+6)}{mq}} \int_{T-r^2}^T \|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3(3m-6)}{mq}} \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{3-\frac{6}{m}} dt \\ &\leq Cr^{9(\frac{1}{3}-\frac{1}{m})} \sup_{T-r^2 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{3(2q-3m+6)}{mq}} \left(\int_{T-r^2}^T \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 dt \right)^{\frac{3(3m-6)}{2mq}} \\ &\quad \times \left(\int_{T-r^2}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{\frac{2q(3m-6)}{2qm-3(3m-6)}} dt \right)^{\frac{2qm-3(3m-6)}{2qm}}, \end{aligned}$$

namely,

$$\begin{aligned} &\int_{T-r^2}^T \|u\|_{L^3(B(r))}^3 dt \\ &\leq Cr^{\frac{3(2-\alpha)}{2\alpha}} \left(\sup_{T-r^2 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^3)}^2 + \int_{T-r^2}^T \|\nabla u\|_{L^2(\mathbb{R}^3)}^2 dt \right)^{\frac{3}{2}-\frac{1}{\alpha}} \left(\int_{T-r^2}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^p dt \right)^{\frac{2}{p\alpha}}. \end{aligned} \quad (3.21)$$

Choose a cut-off function $\psi \in C_0^\infty(B(\eta))$ satisfying $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B(\frac{\xi+3\eta}{4})$ and $|\nabla\psi| \leq \frac{C}{|\eta-\xi|}$. Replacing u by ψu in (3.21) and taking $r = \frac{\xi+3\eta}{4}$, we get

$$\begin{aligned} & \int_{T-r^2}^T \|u\|_{L^3(B(r))}^3 dt \\ & \leq C\eta^{\frac{3(2-\alpha)}{2\alpha}} \left(1 + \frac{\eta^2}{(\eta-\xi)^2}\right)^{\frac{3}{2}-\frac{1}{\alpha}} E^{\frac{3}{2}-\frac{1}{\alpha}}(\eta) \left(\int_{T-r^2}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\eta))}^p dt\right)^{\frac{2}{p\alpha}}. \end{aligned}$$

Here we used the fact below

$$\|\psi u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)} \leq C\|u\|_{\dot{\mathcal{M}}^{q,1}(B(\eta))},$$

which can be derived from the definition of Morrey spaces. This completes the proof. \square

Proof of Proposition 3.2. Indeed, consider $0 < R/2 \leq \xi < \frac{3\xi+\eta}{4} < \frac{\xi+\eta}{2} < \frac{\xi+3\eta}{4} < \eta \leq R$. Let $\phi(x, t)$ be a non-negative smooth function supported in $Q(\frac{\xi+\eta}{2})$ such that $\phi(x, t) \equiv 1$ on $Q(\frac{3\xi+\eta}{4})$, $|\nabla\phi| \leq C/(\eta-\xi)$ and $|\nabla^2\phi| + |\partial_t\phi| \leq C/(\eta-\xi)^2$.

The local energy inequality (2.2), the decomposition of pressure in Lemma 2.1 and the Hölder inequality ensure that

$$\begin{aligned} & \int_{B(\frac{\eta+\xi}{2})} |u(x, t)|^2 \phi(x, t) dx + 2 \iint_{Q(\frac{\eta+\xi}{2})} |\nabla u|^2 \phi dx ds \\ & \leq \frac{C\eta^{5/3}}{(\eta-\xi)^2} \left(\iint_{Q(\frac{\xi+3\eta}{4})} |u|^3 dx ds\right)^{2/3} + \frac{C}{(\eta-\xi)} \iint_{Q(\frac{\xi+3\eta}{4})} |u|^3 dx ds \\ & \quad + \frac{C\eta^3}{(\eta-\xi)^4} \|u\|_{L^3(Q(\frac{\xi+3\eta}{4}))}^3 + \frac{C\eta^{3/2}}{(\eta-\xi)^4} \|\pi\|_{L^1(Q(\eta))} \|u\|_{L^\infty L^2(Q(\eta))} \\ & =: I + II + III + IV. \end{aligned} \tag{3.22}$$

Combining (3.17) and the Young inequality, we obtain

$$\begin{aligned} I & \leq \frac{C\eta^{3+\alpha}}{(\eta-\xi)^{3\alpha}} \left(1 + \frac{\eta^2}{(\eta-\xi)^2}\right)^{\frac{3\alpha-2}{2}} \|u\|_{L^p(T-\eta^2, T; \dot{\mathcal{M}}^{q,1}(B(\eta)))}^2 + \frac{1}{6}E(\eta), \\ II & \leq \frac{C\eta^3}{(\eta-\xi)^{\frac{2\alpha}{2-\alpha}}} \left(1 + \frac{\eta^2}{(\eta-\xi)^2}\right)^{\frac{3\alpha-2}{2-\alpha}} \|u\|_{L^p(T-\eta^2, T; \dot{\mathcal{M}}^{q,1}(B(\eta)))}^{\frac{4}{2-\alpha}} + \frac{1}{6}E(\eta), \\ III & \leq \frac{C\eta^{\frac{3(\alpha+2)}{2-\alpha}}}{(\eta-\xi)^{\frac{8\alpha}{(2-\alpha)}}} \left(1 + \frac{\eta^2}{(\eta-\xi)^2}\right)^{\frac{3\alpha-2}{2-\alpha}} \|u\|_{L^p(T-\eta^2, T; \dot{\mathcal{M}}^{q,1}(B(\eta)))}^{\frac{4}{2-\alpha}} + \frac{1}{6}E(\eta). \end{aligned} \tag{3.23}$$

In light of the definition of Morrey spaces, we see that

$$\|\pi\|_{L^1(B(\eta))} \leq C\eta^{3-\frac{3}{m}} \|\pi\|_{\dot{\mathcal{M}}^{m,1}(B(\eta))}.$$

Using the Young inequality again, we conclude that

$$\begin{aligned} IV & \leq \frac{C\eta^3}{(\eta-\xi)^8} \|\pi\|_{L^1(Q(\eta))}^2 + \frac{1}{6}E(\eta) \\ & \leq \frac{C\eta^{9-\frac{6}{m}}}{(\eta-\xi)^8} \|\pi\|_{L^1(T-\eta^2, T; \dot{\mathcal{M}}^{m,1}(B(\eta)))}^2 + \frac{1}{6}E(\eta). \end{aligned} \tag{3.24}$$

After plugging (3.23)-(3.24) into (3.22), we apply the classical Iteration Lemma [8, Lemma V.3.1, p.161] to finish the proof. \square

Proof of Corollary 3.3. Recall the ε -regularity criteria below shown in [12]: $u \in L^\infty(Q(1/2))$ provided that

$$\|u\|_{L^p(-1,0;L^q(B(1)))} + \|\pi\|_{L^1(Q(1))} \leq \varepsilon_3, \quad (3.25)$$

where

$$1 \leq 2/p + 3/q < 2 \quad \text{and} \quad 1 < p < \infty.$$

Note that for any $m \geq 1$,

$$\|\pi\|_{L^1(B(r))} \leq Cr^{3-\frac{3}{m}} \|\pi\|_{\dot{\mathcal{M}}^{m,1}(B(r))}.$$

This together with the energy bound (3.15) and (3.25) means (3.16). \square

3.3 Proof of Theorem 1.1

Proof. Let $\lambda = [2a(T-t)]^{-1/2}$. By means of an elementary change of variables and the definition of Morrey spaces, we have

$$\begin{aligned} R^{3(\frac{1}{q}-\frac{1}{l})} \left(\int_{B_x(R) \cap B_{x_0}(r)} |u(y,t)|^l dy \right)^{\frac{1}{l}} &\leq R^{3(\frac{1}{q}-\frac{1}{l})} \left(\int_{B_x(R)} |u|^l dy \right)^{\frac{1}{l}} \\ &= R^{3(\frac{1}{q}-\frac{1}{l})} \left(\int_{B_x(R)} |\lambda U(\lambda y)|^l dy \right)^{\frac{1}{l}} \\ &\leq \lambda^{1-\frac{3}{q}} \|U\|_{\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)}. \end{aligned}$$

Hence, it follows from (3.1) that for any $r > 0$,

$$\begin{aligned} \|u\|_{\dot{\mathcal{M}}^{q,l}(B_{x_0}(r))} &\leq \lambda^{1-\frac{3}{q}} \|U\|_{\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)}, \\ \|\pi\|_{\dot{\mathcal{M}}^{q/2,l/2}(B_{x_0}(r))} &\leq \lambda^{2-\frac{6}{q}} \|\tilde{\Pi}\|_{\dot{\mathcal{M}}^{q/2,l/2}(\mathbb{R}^3)} \leq C \lambda^{2-\frac{6}{q}} \|U\|_{\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)}^2. \end{aligned} \quad (3.26)$$

Recall that $\lambda(t) = (2a(T-t))^{-1/2}$ and choose $p > 1$ such that $2-p < p - \frac{3p}{q} < 2$, then there hold

$$\begin{aligned} \int_{T-r^2}^T \|u\|_{\dot{\mathcal{M}}^{q,l}(B_{x_0}(r))}^p dt &\leq \int_{T-r^2}^T \lambda^{p-\frac{3p}{q}} \|U\|_{\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)}^p dt < \infty, \\ \int_{T-r^2}^T \|\pi\|_{\dot{\mathcal{M}}^{q/2,l/2}(B_{x_0}(r))} dt &\leq C \int_{T-r^2}^T \lambda^{2-\frac{6}{q}} \|U\|_{\dot{\mathcal{M}}^{q,l}(\mathbb{R}^3)}^2 dt < \infty. \end{aligned} \quad (3.27)$$

At this stage, the proof of Theorem 1.1 follows at once from Proposition 3.2 and (1.4). \square

4 Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into three steps. In Step 1, utilizing the local suitable weak solutions and local Meyer-Gerard-Oru interpolation inequality (3.17), we establish a new Caccioppoli type inequality. Step 2 is devoted to constructing the pressure Π via A_p weighted inequalities. In the last step, an application of local energy inequalities leads to the proof of Theorem 1.2.

In what follows, we set

$$\|\cdot\|_{L^p \dot{\mathcal{M}}^{q,l}(Q(r))} := \|\cdot\|_{L^p(T-r^2, T; \dot{\mathcal{M}}^{q,l}(B(r)))}.$$

4.1 Caccioppoli type inequality

Proposition 4.1. *Assume that u is a local suitable weak solution to the Navier-Stokes equations (1.1). Then there holds for any $q > 3/2$,*

$$\begin{aligned} & \|u\|_{L^3 L^{\frac{18}{5}}(Q(\frac{R}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{R}{2}))}^2 \\ & \leq CR^{\frac{5q-12}{3q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{M}^{q,1}(Q(R))}^2 + CR^{\frac{4q-15}{2q-3}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{M}^{q,1}(Q(R))}^{\frac{6q}{2q-3}} + CR^{\frac{2q-6}{q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{M}^{q,1}(Q(R))}^3. \end{aligned}$$

As a straightforward consequence of the above proposition, we establish a regularity criterion at one scale without pressure in Morrey spaces for local suitable weak solutions to (1.1). The Hölder inequality together with any result of [3, 13, 27, 29, 30] and this proposition yields the desired.

Corollary 4.2. *Let the pair (u, π) be a local suitable weak solution to the 3D Navier-Stokes system (1.1) in $Q(1)$. There exists an absolute positive constant ε_2 such that if the pair (u, π) satisfy*

$$\int_{-1}^0 \|u\|_{\dot{M}^{q,1}(B(1))}^{\frac{6q}{2q-3}} dt \leq \varepsilon_2 \quad \text{with } q > 3/2, \quad (4.1)$$

then $u \in L^\infty(Q(1/2))$.

Proof of Proposition 4.1. Consider $0 < R/2 \leq r < \frac{3r+\rho}{4} < \frac{r+\rho}{2} < \rho \leq R$. Let $\phi(x, t)$ be a non-negative smooth function supported in $Q(\frac{r+\rho}{2})$ such that $\phi(x, t) \equiv 1$ on $Q(\frac{3r+\rho}{4})$, $|\nabla \phi| \leq C/(\rho - r)$ and $|\nabla^2 \phi| + |\partial_t \phi| \leq C/(\rho - r)^2$.

Let $\nabla \pi_h = \mathcal{W}_{3, B(\frac{r+3\rho}{4})}(u)$, then, from (2.9)-(2.11), we have

$$\|\nabla \pi_h\|_{L^3(Q(\frac{r+3\rho}{4}))} \leq C \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}, \quad (4.2)$$

$$\|\pi_1\|_{L^2(Q(\frac{r+3\rho}{4}))} \leq C \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}, \quad (4.3)$$

$$\|\pi_2\|_{L^{\frac{3}{2}}(Q(\frac{r+3\rho}{4}))} \leq C \| |u|^2 \|_{L^{\frac{3}{2}}(Q(\frac{r+3\rho}{4}))}. \quad (4.4)$$

Since $v = u + \nabla \pi_h$, the Hölder inequality and (4.2) allow us to write

$$\begin{aligned} \iint_{Q(\rho)} |v|^2 |\Delta \phi^4 + \partial_t \phi^4| & \leq \frac{C}{(\rho - r)^2} \iint_{Q(\frac{r+\rho}{2})} (|u|^2 + |\nabla \pi_h|^2) \\ & \leq \frac{C\rho^{5/3}}{(\rho - r)^2} \left(\iint_{Q(\frac{r+\rho}{2})} |u|^3 + |\nabla \pi_h|^3 \right)^{\frac{2}{3}} \\ & \leq \frac{C\rho^{5/3}}{(\rho - r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2. \end{aligned} \quad (4.5)$$

The Hölder inequality, $v = u + \nabla \pi_h$ and (4.2) ensure

$$\left| \iint_{Q(\rho)} |v|^2 \phi^3 u \cdot \nabla \phi \right| \leq \frac{C}{(\rho - r)} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3. \quad (4.6)$$

It follows from the interior estimate of harmonic functions (2.12) and (4.2) that

$$\begin{aligned}\|\nabla^2 \pi_h\|_{L^{20/7}(Q(\frac{r+\rho}{2}))} &\leq \frac{C(r+\rho)}{(\rho-r)^2} \|\nabla \pi_h\|_{L^3(Q(\frac{r+3\rho}{4}))} \\ &\leq \frac{C\rho}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))},\end{aligned}$$

which in turn implies that

$$\begin{aligned}&\left| \iint_{Q(\rho)} \phi^4(u \otimes v : \nabla^2 \pi_h) \right| \\ &\leq \|v\phi^2\|_{L^3(Q(\frac{r+\rho}{2}))} \|u\|_{L^3(Q(\frac{r+\rho}{2}))} \|\nabla^2 \pi_h\|_{L^3(Q(\frac{r+\rho}{2}))} \\ &\leq \frac{C\rho}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3.\end{aligned}\tag{4.7}$$

The Hölder inequality, (4.3) and Young's inequality yield that

$$\begin{aligned}\left| \iint_{Q(\rho)} \phi^3 \pi_1 v \cdot \nabla \phi \right| &\leq \frac{C}{(\rho-r)} \|v\|_{L^2(Q(\frac{r+\rho}{2}))} \|\pi_1\|_{L^2(Q(\frac{r+\rho}{2}))} \\ &\leq \frac{C}{(\rho-r)^2} \|v\|_{L^2(Q(\frac{r+\rho}{2}))}^2 + \frac{1}{16} \|\pi_1\|_{L^2(Q(\frac{r+3\rho}{4}))}^2 \\ &\leq \frac{C\rho^{5/3}}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}^2.\end{aligned}\tag{4.8}$$

We deduce by the Hölder inequality and (4.4) that

$$\left| \iint_{Q(\rho)} \phi^3 \pi_2 v \cdot \nabla \phi \right| \leq \frac{C}{(\rho-r)} \|v\phi^2\|_{L^3(Q(\frac{r+\rho}{2}))} \|\pi_2\|_{L^{\frac{3}{2}}(Q(\frac{r+\rho}{2}))} \leq \frac{C}{(\rho-r)} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3.\tag{4.9}$$

Inserting (4.5)-(4.9) into the local energy inequality (2.8), we arrive at

$$\begin{aligned}\sup_{T-\rho^2 \leq t \leq T} \int_{B(\rho)} |v\phi^2|^2 + \iint_{Q(\rho)} |\nabla(v\phi^2)|^2 &\leq \frac{C\rho^{5/3}}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 + \frac{C\rho}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 \\ &\quad + \frac{C}{(\rho-r)} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}^2.\end{aligned}\tag{4.10}$$

The interior estimate of harmonic functions (2.12) and (4.2) provide the bound

$$\|\nabla \pi_h\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 \leq \frac{Cr^{\frac{5}{3}}}{(\rho-r)^2} \|\nabla \pi_h\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 \leq \frac{Cr^{\frac{5}{3}}}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2.\tag{4.11}$$

Combining the triangle inequality, (4.10) and (4.11), we discover that

$$\begin{aligned}\|u\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 &\leq \|v\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 + \|\nabla \pi_h\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 \\ &\leq C \left\{ \|v\|_{L^\infty L^2(Q(r))}^2 + \|\nabla v\|_{L^2(Q(r))}^2 \right\} + \frac{Cr^{\frac{5}{3}}}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 \\ &\leq \frac{C\rho^{5/3}}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 + \frac{C\rho}{(\rho-r)^2} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 \\ &\quad + \frac{C}{(\rho-r)} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}^2.\end{aligned}\tag{4.12}$$

According to (2.12) and (4.2) again, we ascertain

$$\|\nabla^2 \pi_h\|_{L^2(Q(r))}^2 \leq \frac{Cr^3}{(\rho-r)^5} \|\nabla \pi_h\|_{L^2(Q(\frac{r+\rho}{2}))}^2 \leq \frac{Cr^3 \rho^{5/3}}{(\rho-r)^5} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2.$$

Owing to the triangle inequality and (4.10), we have

$$\begin{aligned} \|\nabla u\|_{L^2(Q(r))}^2 &\leq \|\nabla v\|_{L^2(Q(r))}^2 + \|\nabla^2 \pi_h\|_{L^2(Q(r))}^2 \\ &\leq \left\{ \frac{C\rho^{5/3}}{(\rho-r)^2} + \frac{Cr^3 \rho^{5/3}}{(\rho-r)^5} \right\} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 \\ &\quad + \left\{ \frac{C\rho}{(\rho-r)^2} + \frac{C}{(\rho-r)} \right\} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 + \frac{1}{16} \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}^2. \end{aligned} \quad (4.13)$$

Adding (4.12) to (4.13), we derive that

$$\|u\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 + \|\nabla u\|_{L^2(Q(r))}^2 \quad (4.14)$$

$$\begin{aligned} &\leq \left\{ \frac{C\rho^{5/3}}{(\rho-r)^2} + \frac{Cr^3 \rho^{5/3}}{(\rho-r)^5} \right\} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^2 \\ &\quad + \left\{ \frac{C\rho}{(\rho-r)^2} + \frac{C}{(\rho-r)} \right\} \|u\|_{L^3(Q(\frac{r+3\rho}{4}))}^3 + \frac{1}{8} \|\nabla u\|_{L^2(Q(\frac{r+3\rho}{4}))}^2. \end{aligned} \quad (4.15)$$

Thus, the key ingredient is to control $\|u\|_{L^3}^3$ on the right hand side of the above inequality. To this end, invoking the Hölder inequality and (2.14), we see that

$$\begin{aligned} \|u\|_{L^3(B(\frac{r+3\rho}{4}))}^3 &\leq C\rho^{9(\frac{1}{3}-\frac{3}{2q+6})} \|u\|_{L^{\frac{2q+6}{3}}(B(\frac{r+3\rho}{4}))}^3 \\ &\leq C\rho^{\frac{3(2q-3)}{2q+6}} \|u\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{18}{2q+6}} \|u\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{3-\frac{18}{2q+6}}. \end{aligned} \quad (4.16)$$

Choose a cut-off function $\psi \in C_0^\infty(B(\rho))$ satisfying $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $B(\frac{r+3\rho}{4})$ and $|\nabla \psi| \leq \frac{C}{\rho-r}$. In view of triangle inequality and the classical Poincaré inequality, we deduce that

$$\begin{aligned} \|\nabla[(u - \bar{u}_{B(\rho)})\phi]\|_{L^2(B(\rho))} &\leq \|\phi \nabla u\|_{L^2(B(\rho))} + \|\nabla \phi(u - \bar{u}_{B(\rho)})\|_{L^2(B(\rho))} \\ &\leq (C + \frac{C\rho}{\rho-r}) \|\nabla u\|_{L^2(B(\rho))} \\ &\leq \frac{C\rho}{\rho-r} \|\nabla u\|_{L^2(B(\rho))}. \end{aligned} \quad (4.17)$$

We calculate

$$\begin{aligned} \|(u - \bar{u}_{B(\rho)})\phi\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)} &\leq \|u\phi\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))} + \|\phi \bar{u}_{B(\rho)}\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))} \\ &\leq C \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}. \end{aligned} \quad (4.18)$$

Thanks to the triangle inequality again and (4.16)-(4.18), we infer that

$$\begin{aligned} \|u\|_{L^3(B(\frac{r+3\rho}{4}))}^3 &\leq C\rho^{\frac{3(2q-3)}{2q+6}} \|u - \bar{u}_{B(\rho)}\|_{L^{\frac{2q+6}{3}}(B(\frac{r+3\rho}{4}))}^3 + \|\bar{u}_{B(\rho)}\|_{L^3(B(\frac{r+3\rho}{4}))}^3 \\ &\leq C\rho^{\frac{3(2q-3)}{2q+6}} \|\phi(u - \bar{u}_{B(\rho)})\|_{L^{\frac{2q+6}{3}}(B(\rho))}^3 + \|\bar{u}_{B(\rho)}\|_{L^3(B(\frac{r+3\rho}{4}))}^3 \\ &\leq C\rho^{\frac{3(2q-3)}{2q+6}} \|(u - \bar{u}_{B(\rho)})\phi\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{18}{2q+6}} \|(u - \bar{u}_{B(\rho)})\phi\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{3-\frac{18}{2q+6}} + C\rho^{3-\frac{9}{q}} \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^3 \\ &\leq \frac{C\rho^{\frac{8q-3}{2q+6}}}{(\rho-r)} \|\nabla u\|_{L^2(B(\rho))}^{\frac{18}{2q+6}} \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^{3-\frac{18}{2q+6}} + C\rho^{3-\frac{9}{q}} \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^3. \end{aligned}$$

From the Hölder inequality, we know that

$$\begin{aligned}
& \int_{T-\frac{(r+3\rho)^2}{16}}^T \|u\|_{L^3(B(\frac{r+3\rho}{4}))}^3 dt \\
& \leq \frac{C\rho^{\frac{8q-3}{2q+6}}}{(\rho-r)} \int_{T-\frac{(r+3\rho)^2}{16}}^T \|\nabla u\|_{L^2(B(\rho))}^{\frac{18}{2q+6}} \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^{\frac{6q}{2q+6}} dt + C\rho^{3-\frac{6}{q}} \int_{T-\frac{(r+3\rho)^2}{16}}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^3 dt \\
& \leq \frac{C\rho^{\frac{8q-3}{2q+6}}}{(\rho-r)} \left(\int_{T-\frac{(r+3\rho)^2}{16}}^T \|\nabla u\|_{L^2(B(\rho))}^2 dt \right)^{\frac{9}{2q+6}} \left(\int_{T-\frac{(r+3\rho)^2}{16}}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^{\frac{6q}{2q-3}} dt \right)^{\frac{2q-3}{2q+6}} \\
& \quad + C\rho^{3-\frac{6}{q}} \left(\int_{T-\frac{(r+3\rho)^2}{16}}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(\rho))}^{\frac{6q}{2q-3}} dt \right)^{\frac{2q-3}{2q}},
\end{aligned}$$

where we have used the fact that $\frac{18}{2q+6} < 2$ and $3 \leq \frac{6q}{2q-3}$.

Plugging this inequality into (4.15) and using the Young inequality, we obtain

$$\begin{aligned}
& \|u\|_{L^3 L^{\frac{18}{5}}(Q(r))}^2 + \|\nabla u\|_{L^2(Q(r))}^2 \\
& \leq \left\{ \frac{C\rho^{5/3}}{(\rho-r)^2} + \frac{Cr^3\rho^{5/3}}{(\rho-r)^5} \right\}^{\frac{q+3}{q}} \frac{\rho^{\frac{8q-3}{3q}}}{(\rho-r)^{\frac{2q+6}{3q}}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(\rho))}^2 \\
& \quad + \left\{ \frac{C\rho^{5/3}}{(\rho-r)^2} + \frac{Cr^3\rho^{5/3}}{(\rho-r)^5} \right\} \rho^{\frac{2q-4}{q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(\rho))}^2 \\
& \quad + \left\{ \frac{C\rho}{(\rho-r)^2} + \frac{C}{(\rho-r)} \right\}^{\frac{2q+6}{2q-3}} \frac{\rho^{\frac{8q-3}{2q-3}}}{(\rho-r)^{\frac{2q+6}{2q-3}}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(\rho))}^{\frac{6q}{2q-3}} \\
& \quad + \left\{ \frac{C\rho}{(\rho-r)^2} + \frac{C}{(\rho-r)} \right\} \rho^{\frac{3q-6}{q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(\rho))}^3 + \frac{3}{16} \|\nabla u\|_{L^2(Q(\rho))}^2.
\end{aligned}$$

Now, we are in a position to apply the classical Iteration Lemma [8, Lemma V.3.1, p.161] to find that

$$\begin{aligned}
& \|u\|_{L^3 L^{\frac{18}{5}}(Q(\frac{R}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{R}{2}))}^2 \\
& \leq CR^{\frac{5q-12}{3q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(R))}^2 + CR^{\frac{4q-15}{2q-3}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(R))}^{\frac{6q}{2q-3}} + CR^{\frac{2q-6}{q}} \|u\|_{L^{\frac{6q}{2q-3}} \dot{\mathcal{M}}^{q,1}(Q(R))}^3.
\end{aligned}$$

This achieves the proof of this proposition. \square

4.2 Proof of Theorem 1.2

Proof. It follows from (3.26) and $\frac{3}{2} < q < 6$ that for any $r > 0$,

$$\int_{T-R^2}^T \|u\|_{\dot{\mathcal{M}}^{q,1}(B(r))}^{\frac{6q}{2q-3}} dt \leq \int_{T-R^2}^T \lambda^{\frac{6q}{2q-3}(1-\frac{3}{q})} \|U\|_{\dot{\mathcal{M}}^{q,1}(\mathbb{R}^3)}^{\frac{6q}{2q-3}} dt < \infty.$$

Thus, this together with Proposition 4.1 implies

$$\|u\|_{L^3(Q(\frac{R}{2}))}^2 + \|\nabla u\|_{L^2(Q(\frac{R}{2}))}^2 < \infty. \tag{4.19}$$

Let $\lambda_0 = (2aT)^{-1/2}$. Changing the order of integration, we apply (4.19) to deduce that

$$\begin{aligned} \iint_{Q(1)} |u|^3 dx dt &= \int_{\mathbb{R}^3} |U|^3 \lambda_0^2 \min\{|y|^{-2}, \lambda_0^{-2}\} dy, \\ \iint_{Q(1)} |\nabla u|^2 dx dt &= \int_{\mathbb{R}^3} |\nabla U|^2 2\lambda_0^2 \min\{|y|^{-1}, \lambda_0^{-1}\} dy, \end{aligned} \quad (4.20)$$

which in turn implies that

$$\|\nabla U\|_{L^2(B_{y_0}(1))} = o(|y_0|^{1/2}) \quad \text{and} \quad \|U\|_{L^3(B_{y_0}(1))} = o(|y_0|^{2/3}), \quad \text{as } |y_0| \rightarrow \infty.$$

Next, to employ the local energy inequality (2.2) and (1.4), we take into account recovering pressure Π via (4.20). Observe that (4.20) yields

$$\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy < \infty, \quad (4.21)$$

we can define

$$\tilde{\Pi} = \mathcal{R}_i \mathcal{R}_j (U_i U_j),$$

where $U = (U_1, U_2, U_3)$ is determined by (4.21). Due to the classical Calderón-Zygmund Theorem with $A_{3/2}$ weights, there holds

$$\|\tilde{\Pi}|y|^{-\frac{4}{3}}\|_{L^{3/2}(\mathbb{R}^3)} \leq C \|U_i U_j |y|^{-\frac{4}{3}}\|_{L^{3/2}(\mathbb{R}^3)}. \quad (4.22)$$

This means the local integrability of $\tilde{\Pi}$. Thus, Weyl's lemma guarantees that $\tilde{\Pi}$ is smooth.

To proceed further, the fact that $U \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} |U|^3 (1 + |y|)^{-2} dy < \infty$ allows us to revise the proof of Lemma 3.1 to show that $\tilde{\Pi}$ satisfies $-\Delta \Pi = \partial_i \partial_j (U_i U_j)$ in the distributional sense. Here, we just prove that $\nabla \Pi = \nabla \tilde{\Pi}$.

We use the same notations given in the proof of Lemma 3.1. By integration by parts, we have

$$\begin{aligned} \varepsilon^3 \left| \int_{\mathbb{R}^3} \Delta U \varphi(\varepsilon y) dy \right| &= \varepsilon^5 \left| \int_{\mathbb{R}^3} U |y|^{-\frac{2}{3}} |y|^{\frac{2}{3}} \Delta \varphi(\varepsilon y) dy \right| \\ &\leq \varepsilon^5 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^3 |y|^{-2} dy \right)^{\frac{1}{3}} \left(\int_{|y| < \frac{1}{\varepsilon}} |y| |\Delta \varphi(\varepsilon y)|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} \\ &\leq C \varepsilon^{\frac{7}{3}} \left(\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy \right)^{\frac{1}{3}}. \end{aligned}$$

Exactly as the above, we get

$$\begin{aligned} \varepsilon^3 \left| \int_{\mathbb{R}^3} U \varphi(\varepsilon y) dy \right| &\leq \varepsilon^3 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^3 |y|^{-2} dy \right)^{\frac{1}{3}} \left(\int_{|y| < \frac{1}{\varepsilon}} |y| |\varphi(\varepsilon y)|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} \\ &\leq C \varepsilon^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy \right)^{\frac{1}{3}}. \end{aligned}$$

In view of integrating by parts once again, we see that

$$\varepsilon^3 \int_{\mathbb{R}^3} (y \cdot \nabla) U(y) \varphi(\varepsilon y) dy = -\varepsilon^3 \int_{\mathbb{R}^3} 3U(y) \varphi(\varepsilon y) dy - \varepsilon^3 \int_{\mathbb{R}^3} U(y) \{(\varepsilon y) \cdot \nabla \varphi(\varepsilon y)\} dy. \quad (4.23)$$

Hence, proceeding as the above, we can also control $|\varepsilon^3 \int_{\mathbb{R}^3} (y \cdot \nabla) U(y) \varphi(\varepsilon y) dy|$.

Thanks to the divergence-free condition, we arrive at

$$\begin{aligned} & \varepsilon^3 \int_{\mathbb{R}^3} (U \cdot \nabla) U(y) \varphi(\varepsilon y) dy \\ &= -\varepsilon^4 \int_{\mathbb{R}^3} U \otimes U(y) \cdot \nabla \varphi(\varepsilon y) dy, \end{aligned} \tag{4.24}$$

which in turn implies that

$$\begin{aligned} & \varepsilon^3 \left| \int_{\mathbb{R}^3} (U \cdot \nabla) U(y) \varphi(\varepsilon y) dy \right| \\ & \leq \varepsilon^4 \left(\int_{|y| < \frac{1}{\varepsilon}} |U|^3 |y|^{-2} dy \right)^{\frac{2}{3}} \left(\int_{|y| < \frac{1}{\varepsilon}} |y|^4 |\nabla \varphi(\varepsilon y)|^3 dy \right)^{\frac{1}{3}} \\ & \leq C \varepsilon^{\frac{5}{3}} \left(\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy \right)^{\frac{2}{3}}. \end{aligned} \tag{4.25}$$

It remains to bound the last term involving pressure $\tilde{\Pi}$. Integration by parts gives

$$\varepsilon^3 \int_{\mathbb{R}^3} \nabla \tilde{\Pi}(y) \varphi(\varepsilon y) dy = -\varepsilon^4 \int_{\mathbb{R}^3} \tilde{\Pi}(y) \nabla \varphi(\varepsilon y) dy. \tag{4.26}$$

Furthermore, due to (4.22), a variant of (4.25) provides the estimate

$$\varepsilon^4 \left| \int_{\mathbb{R}^3} \tilde{\Pi}(y) \nabla \varphi(\varepsilon y) dy \right| \leq C \varepsilon^{\frac{5}{3}} \left(\int_{\mathbb{R}^3} |U|^3 |y|^{-2} dy \right)^{\frac{2}{3}}.$$

Therefore, we get $\nabla \Pi = \nabla \tilde{\Pi}$. Then we can define π by $\tilde{\Pi}$ via (1.2). Note that

$$\iint_{Q(1)} |\pi|^{\frac{3}{2}} dx dt = \int_{\mathbb{R}^3} |\tilde{\Pi}|^{\frac{3}{2}} \lambda_0^2 \min\{|y|^{-2}, \lambda_0^{-2}\} dy.$$

From this and (4.22), we know that $\pi \in L^{3/2}(Q(1))$. This together with $u \in L^3(Q(1))$ implies (1.4) via local energy inequality (2.2), which concludes the proof of Theorem 1.2. \square

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