# RANK-METRIC CODES OVER ARBITRARY GALOIS EXTENSIONS AND RANK ANALOGUES OF REED-MULLER CODES 

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#### Abstract

This paper extends the study of rank-metric codes in extension fields $\mathbb{L}$ equipped with an arbitrary Galois group $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})$. We propose a framework for studying these codes as subspaces of the group algebra $\mathbb{L}[G]$, and we relate this point of view with usual notions of rank-metric codes in $\mathbb{L}^{N}$ or in $\mathbb{K}^{N \times N}$, where $N=[\mathbb{L}: \mathbb{K}]$. We then adapt the notion of error-correcting pairs to this context, in order to provide a non-trivial decoding algorithm for these codes.

We then focus on the case where $G$ is abelian, which leads us to see codewords as elements of a multivariate skew polynomial ring. We prove that we can bound the dimension of the vector space of zeroes of these polynomials, depending of their degree. This result can be seen as an analogue of AlonFüredi theorem - and by means, of Schwartz-Zippel lemma - in the rank metric. Finally, we construct the counterparts of Reed-Muller codes in the rank metric, and we give their parameters. We also show the connection between these codes and classical Reed-Muller codes in the case where $\mathbb{L}$ is a Kummer extension.


## 1. Introduction

1.1. Context. Rank-metric codes were introduced independently by Delsarte in [10] and Gabidulin in [11] for combinatorial purposes. Roth rediscovered them in [28] and showed their application to crisscross error-correction. In the same year, Gabidulin, Paramonov and Tretjakov proposed the use of rank-metric codes for cryptographic purposes, designing the GPT cryptosystem [12]. More recently, Silva, Koetter and Kschischang showed how these codes can be used in network coding [33]. This series of papers raised the interest of many researchers from different areas, who investigated their mathematical properties and further applications.

Rank-metric codes have been introduced as spaces of $N \times M$ matrices over a finite field $\mathbb{F}_{q}$ by Delsarte, while Gabidulin considered them as $\mathbb{F}_{q}$-linear spaces of vectors of length $N$ over an extension field $\mathbb{F}_{q^{M}}$. The two representations are equivalent: when choosing an $\mathbb{F}_{q^{-}}$-basis of $\mathbb{F}_{q^{N}}$, one can write each element of $\mathbb{F}_{q^{N}}$ as a column of its coordinates in this basis. Thus, the rank distance on $\mathbb{F}_{q}^{N \times M}$, defined as the rank of the difference of two matrices, is equivalent to the distance on $\mathbb{F}_{q^{N}}^{M}$ defined as the rank of the difference of the matrix representations of two vectors.

In the case $M=N$ it is also possible to view matrices as endomorphisms. More precisely, one has

$$
\mathbb{F}_{q}^{N \times N} \cong \operatorname{End}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{N}}\right) \cong \mathcal{L}[x] /\left(x^{q^{N}}-x\right)
$$

where $\mathcal{L}[x]$ is the ring of $q$-polynomials with coefficients in $\mathbb{F}_{q^{N}}$ endowed with addition and composition, and $\left(x^{q^{N}}-x\right)$ denotes the two-sided ideal spanned by $x^{q^{N}}-x \in \mathcal{L}[x]$. Recall that a $q$-polynomial (or linearized polynomial) is an element $P(x) \in \mathbb{F}_{q^{N}}[x]$ such that the exponents of monomials involved in $P$ are powers of $q$. Moreover, the matrix algebra $\mathbb{F}_{q}^{N \times N}$ is also isomorphic to the skew group algebra $\mathbb{F}_{q^{N}}[G]$, where $G=\operatorname{Gal}\left(\mathbb{F}_{q^{N}} / \mathbb{F}_{q}\right)$, endowed with the usual addition and the multiplication defined by the rule

$$
\forall g_{i}, g_{j} \in G, b_{i}, b_{j} \in \mathbb{F}_{q^{N}}, \quad\left(b_{i} g_{i}\right) \circ\left(b_{j} g_{j}\right)=\left(b_{i} g_{i}\left(b_{j}\right)\right)\left(g_{i} \circ g_{j}\right)
$$

We refer to [34] for a complete presentation of these equivalent representations.

[^0]The above isomorphisms make it easier to study the algebraic structure of rank-metric codes, and have been used for designing rank-metric codes with good parameters. This is the case of the well-known family of Gabidulin codes $[10,11]$. They were first defined as the subspace of linearized polynomials of degree at most $q^{k-1}$, which corresponds to $\operatorname{Span}_{\mathbb{F}_{q^{N}}}\left\{\sigma^{i} \mid i=0, \ldots, k-1\right\} \subseteq \mathbb{F}_{q^{N}}[G]$, where $\sigma$ is the $q$-Frobenius automorphism. This family has been then generalized by Kshevetskiy and Gabidulin in [17], by taking the subspace $\operatorname{Span}_{\mathbb{F}_{q^{N}}}\left\{\theta^{i} \mid i=0, \ldots, k-1\right\} \subseteq \mathbb{F}_{q^{N}}[G]$, where $\theta$ is any generator of the Galois group $G$.

This point of view was crucial for generalizing Gabidulin codes over arbitrary cyclic Galois extensions. In a series of papers, Augot, Loidreau and Robert [4, 3, 5] investigated on the case where $G:=\langle\theta\rangle$ is the Galois group of a degree $N$ cyclic extension $\mathbb{L} / \mathbb{K}$ (see also [29, Section VI]). The same ring isomorphisms hold between $\mathbb{K}^{N \times N}, \operatorname{End}_{\mathbb{K}}(\mathbb{L})$ and the skew group algebra $\mathbb{L}[G]=\mathbb{L}[\theta]$, and hence one can define a Gabidulin code as the $\mathbb{L}$-subspace in $\mathbb{L}[G]$ generated by $\theta^{i}$ for $i=0, \ldots, k-1$.

Gabidulin codes are considered as analogues in the rank metric of Reed-Solomon codes. Indeed, ReedSolomon codes are obtained by considering the $\mathbb{F}_{q}$-subspace $\operatorname{Span}_{\mathbb{F}_{q}}\left\{x^{i} \mid i=0, \ldots, k-1\right\} \subseteq \mathbb{F}_{q}[x]$. The analogy can also be seen via their generator matrices. For Reed-Solomon codes, the evaluation of the monomials $x^{i}$ s on a subset of $\mathbb{F}_{q}$ yields a Vandermonde matrix, while for Gabidulin codes the Moore matrix is obtained by the action of the $\theta^{i}$ 's on a $\mathbb{K}$-linearly independent subset of $\mathbb{L} / \mathbb{K}$. Another analogy can be found by studying the systematic generator matrices, which produces Cauchy matrices for ReedSolomon codes, and their $q$-analogue for Gabidulin codes [25].

Central to current research trends is the idea of finding constructions in the Hamming metric that have a counterpart in the rank metric, in order to obtain analogous objects. For instance, a problem is whether one can construct Reed-Muller type codes for the rank metric. Recall that $q$-ary Reed-Muller codes in $m$ variables are obtained by considering the $\mathbb{F}_{q}$-subspace $\mathbf{S p a n}_{\mathbb{F}_{q}}\left\{x_{1}^{i_{1}} \cdots x_{m}^{i_{m}} \mid i_{1}+\cdots+i_{m} \leqslant r\right\} \subseteq$ $\mathbb{F}_{q}\left[x_{1}, \ldots, x_{m}\right]$ for a certain degree $r$, and then evaluating all the polynomials in this subspace in every point of $\mathbb{F}_{q}^{m}$. In order to obtain the same analogy as the one between Gabidulin and Reed-Solomon codes, one should construct $m$ distinct automorphisms $\theta_{1}, \ldots, \theta_{m} \in G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})$ which commute and span disjoints subgroups of $G$ of order $n$, and then define the space

$$
\mathrm{RM}_{\mathbb{L} / \mathbb{K}}(r, n, m):=\operatorname{Span}_{\mathbb{L}}\left\{\theta_{1}^{i_{1}} \circ \cdots \circ \theta_{m}^{i_{m}} \mid i_{1}+\cdots+i_{m} \leqslant r\right\}
$$

This notably requires that $G$ contains a subgroup isomorphic to $(\mathbb{Z} / n \mathbb{Z})^{m}$.
In the finite field setting, Galois groups are cyclic. This explains why up to now, no one succeeded in constructing Reed-Muller codes for the rank metric that share the parameters of classical Reed-Muller codes. Indeed, if one tries to get a subspace of $\mathbb{F}_{q^{N}}[G]$ of the form $\mathrm{RM}_{\mathbb{L} / \mathbb{K}}(r, n, m)$, then one has to choose the $\theta_{i}$ 's as powers of the same generator $\theta$, obtaining a generalized Gabidulin code or, more generally, a rank-metric code satisfying a Roos-like bound [21, 1].
1.2. Overview. Motivated by this intuition, in this paper we study the general theory of rank-metric codes over arbitrary Galois extensions. We first investigate the isomorphisms $\mathbb{K}{ }^{N \times N} \cong \operatorname{End}_{\mathbb{K}}(\mathbb{L}) \cong \mathbb{L}[G]$, showing equivalent definitions of the rank metric. This also allows us to define the counterparts of Moore matrices and Dickson matrices for general Galois extensions, which are fundamental objects in order to determine the rank of a linearized polynomial. We prove that the definitions of these matrices are consistent with the finite field case, and they have exactly the same properties. We then adapt the notion of error-correcting pairs to the context of codes in $\mathbb{L}[G]$. Error-correcting pairs were originally introduced by Pellikaan [26], and a rank-metric version was recently proposed by Martínez-Peñas and Pellikaan [22].

Once developed the general theory of codes in $\mathbb{L}[G]$ for arbitrary finite groups $G$, we restrict to the case of abelian groups, which was the main motivation of our project. In this context, elements of the group algebra can be seen as skew polynomials in $\theta_{1}, \ldots, \theta_{m}$, where $G=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle$. We prove an upper bound on the dimension of their space of zeros, depending on their degree. This result can be seen as an analogue of Alon-Füredi theorem and Schwartz-Zippel lemma in the rank metric setting.

We then naturally define $\theta$-Reed-Muller codes as mentioned before, and study their parameters. It turns out that this construction produces rank-metric codes with the same parameters as $q$-ary Reed-Muller codes. Furthemore, when restricting to Kummer extensions with Galois group $G \simeq \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}$, the $\theta$-Reed-Muller code shares the structure of an affine variety code or affine cartesian code (see [13, 19]).

Notice that in [14], Geiselmann and Ulmer also proposed a generalisation of Reed-Muller codes by using skew polynomial rings. However their work significantly differs from ours, since they use iterated rings with non-trivial derivation in order to stand out from classical Reed-Muller codes.
1.3. Organisation. The paper is structured as follows. In Section 2 we recall basic notions in algebra that are useful to define the rank metric on arbitrary Galois extension fields (Section 3). Dickson matrices are introduced in Section 4 where we also determine their algebraic properties. We are then able to define and describe the properties of rank-metric codes in $\mathbb{L}[G]$ in Section 5 and their error-correcting pairs in Section 6. Next, Section 7 is dedicated to the case of abelian groups $G$, in which the analogues of Alon-Füredi theorem and Schwartz-Zippel lemma for skew polynomials are proved. Finally, Section 8 is devoted to the construction and analysis of Reed-Muller codes in $\mathbb{L}[G]$ and their connection to the Hamming setting.

## 2. Preliminaries

2.1. Notation. Given a field $\mathbb{K}$, the elements of $\mathbb{K}^{n}$ are represented as row vectors and denoted using bold face lower case letters: $\boldsymbol{a}, \boldsymbol{b}, \ldots$. However, there might be an exception to this rule: given a finite extension $\mathbb{L}$ of $\mathbb{K}$, a vector in $\mathbb{L}^{n}$ whose entries form a $\mathbb{K}$-basis of $\mathbb{L}$ will be denoted with calligraphic letters such as $\mathcal{B}$. Matrices are denoted with capital letters: $A, B$, etc. The space of matrices with $m$ rows and $n$ columns with entries in $\mathbb{K}$ is denoted by $\mathbb{K}^{m \times n}$. The transposition of a vector $\boldsymbol{v} \in \mathbb{K}^{n}$ or a matrix $M \in \mathbb{K}^{m \times n}$ is denoted by $\boldsymbol{v}^{\top}$ and $M^{\top}$ respectively.

Given vector spaces $V_{1}, V_{2}$ over a field $\mathbb{K}$ with respective bases $\mathcal{B}_{1}, \mathcal{B}_{2}$ and a $\mathbb{K}$-linear map $f: V_{1} \rightarrow V_{2}$, we denote by $A\left(f, \mathcal{B}_{1}, \mathcal{B}_{2}\right)$ the matrix representation of $f$ in these bases. That is to say, $A\left(f, \mathcal{B}_{1}, \mathcal{B}_{2}\right)$ is the matrix whose columns are the decompositions in $\mathcal{B}_{2}$ the elements $f(\boldsymbol{b})$ when $\boldsymbol{b}$ ranges over the basis $\mathcal{B}_{1}$. Given a vector $x \in V_{1}$, we denote $\boldsymbol{x} \in \mathbb{K}^{\operatorname{dim} V_{1}}$ its representation in basis $\mathcal{B}_{1}$. Then, the vector $\boldsymbol{y} \in \mathbb{K}^{\operatorname{dim} V_{2}}$ such that

$$
\boldsymbol{y}^{\top}=A\left(f, \mathcal{B}_{1}, \mathcal{B}_{2}\right) \cdot \boldsymbol{x}^{\top}
$$

is the representation of $f(x)$ in the basis $\mathcal{B}_{2}$. Finally, when $\mathcal{B}=\mathcal{B}_{1}=\mathcal{B}_{2}$, the matrix is denoted by $A(f, \mathcal{B})$.
According to this definition, the kernel of a matrix is referred to its right kernel, i.e. given $M \in \mathbb{K}^{m \times n}$

$$
\operatorname{ker} M:=\left\{\boldsymbol{x} \in \mathbb{K}^{n} \mid M \cdot \boldsymbol{x}^{\top}=0\right\}
$$

2.2. Skew group algebras. Let $\mathbb{L} / \mathbb{K}$ be a Galois extension of finite degree $N:=[\mathbb{L}: \mathbb{K}]$, and $G:=$ $\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\{g_{1}, \ldots, g_{N}\right\}$ be its Galois group. The group algebra $\mathbb{L}[G]$ is defined as

$$
\mathbb{L}[G]:=\left\{\sum_{i=1}^{N} a_{i} g_{i} \mid a_{i} \in \mathbb{L}\right\}
$$

The set $\mathbb{L}[G]$ is naturally an $\mathbb{L}$-vector space of dimension $N$. It also has a ring structure via the multiplication $*$ defined on monomials by $\left(a_{i} g_{i}\right) *\left(a_{j} g_{j}\right)=\left(a_{i} a_{j}\right)\left(g_{i} g_{j}\right)$ and then extended by associativity and distributivity. However, in this paper we will not consider this ring structure, but the one defined by the composition $\circ$, that is given on monomials by

$$
\left(a_{i} g_{i}\right) \circ\left(a_{j} g_{j}\right)=\left(a_{i} g_{i}\left(a_{j}\right)\right)\left(g_{i} g_{j}\right)
$$

and then extended by associativity and distributivity. With this operation $\mathbb{L}[G]$ is a non-commutative ring. In addition, every element $a=\sum_{i} a_{i} g_{i} \in \mathbb{L}[G]$ can be seen as a $\mathbb{K}$-linear map

$$
\left\{\begin{array}{lll}
\mathbb{L} & \longrightarrow & \mathbb{L}  \tag{1}\\
x & \longmapsto & a(x):=\sum_{i} a_{i} g_{i}(x) .
\end{array}\right.
$$

Theorem 1. The map sending every $a \in \mathbb{L}[G]$ onto the corresponding $\mathbb{K}$-endomorphism of $\mathbb{L}$ is $a \mathbb{K}$-linear isomorphism between $\mathbb{L}[G]$ and $\operatorname{End}_{\mathbb{K}}(\mathbb{L})$.

Proof. The map is clearly $\mathbb{K}$-linear. Moreover, $G=\left\{g_{1}, \ldots, g_{N}\right\}$ is a set of distinct characters $\mathbb{L}^{\times} \rightarrow \mathbb{L}^{\times}$, defined as $x \mapsto g_{i}(x)$. Hence, by Artin's theorem of independence of characters the map is injective. The claim follows then by observing that both $\mathbb{L}[G]$ and $\operatorname{End}_{\mathbb{K}}(\mathbb{L})$ have dimension $N^{2}$ over $\mathbb{K}$.
2.3. Trace of extension fields and its duality theory. For a Galois extension $\mathbb{L} / \mathbb{K}$, the trace map is a special element in $\mathbb{L}[G]$ which gives rise to a well-known duality theory.
Definition 2. Let $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})$ be the Galois group of the extension $\mathbb{L} / \mathbb{K}$. Then, the trace map is defined as

$$
\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}:\left\{\begin{array}{rll}
\mathbb{L} & \longrightarrow & \mathbb{K} \\
x & \longmapsto & \sum_{g \in G} g(x) .
\end{array}\right.
$$

The corresponding element of $\mathbb{L}[G]$ is $\operatorname{Tr}:=\sum_{g \in G} g$.
It is well-known that for separable extensions, and hence for Galois extensions, the trace map induces a duality between $\mathbb{L}$ and $\operatorname{Hom}_{\mathbb{K}}(\mathbb{L}, \mathbb{K})$.
Theorem 3 (Duality of the trace). Let $\mathbb{L} / \mathbb{K}$ be a Galois extension. The map

$$
\langle\cdot, \cdot\rangle_{\operatorname{tr}}:\left\{\begin{array}{rll}
\mathbb{L} \times \mathbb{L} & \longrightarrow & \mathbb{K}  \tag{2}\\
(x, y) & \longmapsto & \operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(x y)
\end{array}\right.
$$

is a symmetric nondegenerate bilinear form, which induces a duality isomorphism

$$
\left\{\begin{array}{rll}
\mathbb{L} & \longrightarrow & \operatorname{Hom}_{\mathbb{K}}(\mathbb{L}, \mathbb{K}) \\
x & \longmapsto & T_{x}
\end{array}\right.
$$

where $T_{x}(y)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(x y)$ for every $y \in \mathbb{L}$.
The duality result in Theorem 3 also implies that for any ordered basis $\mathcal{B}=\left(b_{1}, \ldots, b_{N}\right)$ of $\mathbb{L} / \mathbb{K}$ there exists a dual (ordered) basis $\mathcal{B}^{*}=\left(b_{1}^{*}, \ldots, b_{N}^{*}\right)$ with respect to the bilinear form $\langle\cdot, \cdot\rangle_{\text {tr }}$. Such a dual basis satisfies

$$
\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(b_{i} b_{j}^{*}\right)= \begin{cases}1 & \text { if } i=j  \tag{3}\\ 0 & \text { if } i \neq j\end{cases}
$$

2.4. Adjunction. The trace bilinear form $\langle\cdot, \cdot\rangle_{\operatorname{tr}}$ introduced in Theorem 3 Equation (2) yields a notion of adjunction. Given $f \in \mathbb{L}[G]$, the adjoint of $f$ with respect to the trace bilinear form $\langle\cdot, \cdot\rangle_{\text {tr }}$ is denoted by $\tau(f)$. It is the unique element $\tau(f) \in \mathbb{L}[G]$ satisfying

$$
\begin{equation*}
\forall x, y \in \mathbb{L}, \quad\langle f(x), y\rangle_{\operatorname{tr}}=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(f(x) y)=\langle x, \tau(f)(y)\rangle_{\mathrm{tr}} \tag{4}
\end{equation*}
$$

Lemma 4. The adjunction map $\tau: \mathbb{L}[G] \rightarrow \mathbb{L}[G]$ is a $\mathbb{K}$-linear map satisfying
(i) $\forall a \in \mathbb{L}, \tau(a)=a$;
(ii) $\forall g \in G, \tau(g)=g^{-1}$;
(iii) $\forall u, v \in \mathbb{L}[G], \tau(u \circ v)=\tau(v) \circ \tau(u)$;
(iv) $\tau$ is an involution, i.e. $\forall u \in \mathbb{L}[G], \tau \circ \tau(u)=u$.

Proof. For any $a, x, y \in \mathbb{L}$, we have $\langle a x, y\rangle_{\operatorname{tr}}=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(a x y)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(x a y)=\langle x, a y\rangle_{\operatorname{tr}}$, which proves (i). Let $g \in G$ and $x, y \in \mathbb{L}$, we have $\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}(g(x) y)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(g\left(x g^{-1}(y)\right)\right)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(x g^{-1}(y)\right)$. This proves (ii). Finally (iii) is a direct consequence of (4) and (iv) is a consequence of the symmetry of $\langle\cdot, \cdot\rangle_{\mathrm{tr}}$.

As a consequence, we get an explicit definition of $\tau$ :

$$
\tau:\left\{\begin{array}{ccc}
\mathbb{L}[G] & \longrightarrow & \mathbb{L}[G]  \tag{5}\\
u=\sum_{g \in G} u_{g} g & \longmapsto & \sum_{g \in G} g\left(u_{g^{-1}}\right) g
\end{array}\right.
$$

Actually, $\tau$ can be seen as a "transpose" map in $\mathbb{L}[G]$. In particular, if there exists an orthogonal $\mathbb{K}$-basis $\mathcal{B}$ of $\mathbb{L}$ with respect to $\langle\cdot, \cdot\rangle_{\text {tr }}$, then for any $c \in \mathbb{L}[G]$ we have $A(\tau(c), \mathcal{B})=A(c, \mathcal{B})^{\top}$.

Observe that this notion is well-known and studied in the context of finite fields (see [31, 20]), which we illustrate in the following example.
Example 5. Suppose that $\mathbb{K}=\mathbb{F}_{q}$ and $\mathbb{L}=\mathbb{F}_{q^{N}}$. We have that $G=\operatorname{Gal}\left(\mathbb{F}_{q^{N}} / \mathbb{F}_{q}\right)=\langle\theta\rangle$, where $\theta$ is the $q$-Frobenius automorphism. Then all the elements of the Galois group are of the form $\theta^{i}(\alpha)=\alpha^{q^{i}}$, for $\alpha \in \mathbb{F}_{q^{N}}$. Now, fix an element $a \in \mathbb{F}_{q^{N}}[\theta]$ that we write as $a=\sum_{i=0}^{N-1} a_{i} \theta^{i}$. Hence, the adjoint of $a$ is

$$
\tau(a)=\sum_{i=0}^{N-1} \theta^{i}\left(a_{N-i}\right) \theta^{i}=\sum_{i=0}^{N-1} a_{N-i}^{q^{i}} \theta^{i}
$$

where by convention, $a_{N}:=a_{0}$. It is clear that this coincides with the usual notion given for example in [31].

## 3. Rank metric and Moore matrices over arbitrary Galois extensions

In this section we focus on the elements of $\mathbb{L}[G]$, where $G$ is the Galois group of an arbitrary Galois extension $\mathbb{L} / \mathbb{K}$. In particular, we show that we can determine the rank of any element in several equivalent ways.
Definition 6. Let $\mathbb{L} / \mathbb{K}$ be a field extension, and let $M$ be a positive integer. For a given vector $\boldsymbol{v}=$ $\left(v_{1}, \ldots, v_{M}\right) \in \mathbb{L}^{M}$, we define the $\mathbb{K}$-rank of $\boldsymbol{v}$, as the quantity

$$
\operatorname{rk}_{\mathbb{K}}(\boldsymbol{v}):=\operatorname{dim}_{\mathbb{K}} \operatorname{Span}_{\mathbb{K}}\left\{v_{1}, \ldots, v_{M}\right\}
$$

We now introduce the analogue of the Moore/Wronskian matrix, for any finite Galois group $G$.
Definition 7. Let $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\{g_{1}, \ldots, g_{N}\right\}$ and $\boldsymbol{v} \in \mathbb{L}^{N}$. We define the $G$-Moore matrix of $\boldsymbol{v}$ as

$$
M_{G}(\boldsymbol{v}):=\left(\begin{array}{cccc}
g_{1}\left(v_{1}\right) & g_{1}\left(v_{2}\right) & \cdots & g_{1}\left(v_{N}\right) \\
g_{2}\left(v_{1}\right) & g_{2}\left(v_{2}\right) & \cdots & g_{2}\left(v_{N}\right) \\
\vdots & \vdots & & \vdots \\
g_{N}\left(v_{1}\right) & g_{N}\left(v_{2}\right) & \cdots & g_{N}\left(v_{N}\right)
\end{array}\right) \in \mathbb{L}^{N \times N} .
$$

Given an ordered $\mathbb{K}$-basis $\mathcal{B}$ of $\mathbb{L}$, one can define in a very similar fashion the Moore matrix $M_{G}(\mathcal{B})$. In addition, this matrix is related to the Moore matrix of the dual basis $\mathcal{B}^{*}$ defined in Section 2.3.

Lemma 8. Let $\mathcal{B}=\left(\beta_{1}, \ldots, \beta_{N}\right)$ be an ordered basis of $\mathbb{L} / \mathbb{K}$. Then

$$
M_{G}(\mathcal{B})^{-1}=M_{G}\left(\mathcal{B}^{*}\right)^{\top}
$$

where $\mathcal{B}^{*}$ is the dual basis of $\mathcal{B}$ with respect to the bilinear form $\langle\cdot, \cdot\rangle_{\operatorname{tr}}$.
Proof. The $(i, j)$-th entry of $M_{G}\left(\mathcal{B}^{*}\right)^{\top} M_{G}(\mathcal{B})$ is equal to $\sum_{\ell} g_{\ell}\left(\beta_{i}^{*}\right) g_{\ell}\left(\beta_{j}\right)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(\beta_{i}^{*} \beta_{j}\right)$. Therefore, by (3), we get $M_{G}\left(\mathcal{B}^{*}\right)^{\top} M_{G}(\mathcal{B})=\mathrm{Id}$.

A strong interest of the Moore matrix lies in the next statement.
Proposition 9. For every $\boldsymbol{v} \in \mathbb{L}^{N}$, it holds

$$
\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)=\operatorname{rk}_{\mathbb{K}}(\boldsymbol{v})
$$

Proof. Set $r:=\operatorname{rk}_{\mathbb{K}}(\boldsymbol{v})=\operatorname{dim}_{\mathbb{K}} \operatorname{Span}_{\mathbb{K}}\left\{v_{1}, \ldots, v_{N}\right\}$. We want to prove that $\mathrm{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)=r$. By definition of $\mathrm{rk}_{\mathbb{K}}(\boldsymbol{v})$, there exist an $r$-tuple of $\mathbb{K}$-linearly independent elements $u_{1}, \ldots, u_{r} \in \mathbb{L}$ and an invertible matrix $S \in \mathbb{K}^{N \times N}$, such that $\operatorname{Span}_{\mathbb{K}}\left\{v_{1}, \ldots, v_{N}\right\}=\operatorname{Span}_{\mathbb{K}}\left\{u_{1}, \ldots, u_{r}\right\}$ and

$$
\boldsymbol{v} \cdot S=\left(u_{1}, \ldots, u_{r}, 0, \ldots, 0\right)=: \boldsymbol{u}
$$

Observe that $M_{G}(\boldsymbol{v}) \cdot S=M_{G}(\boldsymbol{v} \cdot S)=M_{G}(\boldsymbol{u})$ since $S$ is defined over $\mathbb{K}$ and hence fixed by $G$. Consequently, the last $N-r$ columns of $M_{G}(\boldsymbol{v}) \cdot S$ are zero. Therefore

$$
\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right) \cdot S=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{u})\right) \leqslant r
$$

Now, let us prove that the $r$ first columns of $M_{G}(\boldsymbol{u})=M_{G}(\boldsymbol{v}) \cdot S$ are $\mathbb{L}$-linearly independent. Suppose that there exist $\lambda_{1}, \ldots, \lambda_{r} \in \mathbb{L}$ satisfying

$$
\begin{equation*}
\forall i \in\{1, \ldots, N\}, \quad \sum_{j=1}^{r} \lambda_{j} g_{i}\left(u_{j}\right)=0 \tag{6}
\end{equation*}
$$

Without loss of generality, one can suppose that $\lambda_{1} \neq 0$. By Theorem 3, there exists $a \in \mathbb{L}$ such that $\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(a \lambda_{1}\right) \neq 0$. Thus, after possibly replacing $\lambda_{1}, \ldots, \lambda_{r}$ by $a \lambda_{1}, \ldots, a \lambda_{r}$, one can assume that there exist $\lambda_{i}$ 's $\in \mathbb{L}$ satisfying (6) and such that $\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(\lambda_{1}\right) \neq 0$. Next, (6) is equivalent to

$$
\forall i \in\{1, \ldots, N\}, \quad \sum_{j=1}^{r} g_{i}^{-1}\left(\lambda_{j}\right) u_{j}=0
$$

Summing up these $N$ equations, we get a $\mathbb{K}$-linear relation on the $u_{i}$ 's:

$$
\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(\lambda_{1}\right) u_{1}+\cdots+\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(\lambda_{r}\right) u_{r}=0
$$

and this linear relation is nontrivial since $\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(\lambda_{1}\right) \neq 0$. This yields a contradiction since the $u_{i}$ 's are $\mathbb{K}$-linearly independent. Therefore:

$$
r=\operatorname{rk}_{\mathbb{K}}(\boldsymbol{v})=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{u})\right)=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)
$$

As a consequence, we get a generalization of the well-known result over finite fields that characterizes bases of extension fields in terms of their associated Moore matrix.

Corollary 10. A vector $\boldsymbol{v} \in \mathbb{L}^{N}$ is an ordered basis of $\mathbb{L} / \mathbb{K}$ if and only if $\operatorname{det}\left(M_{G}(\boldsymbol{v})\right) \neq 0$.
The previous results give properties of the rank metric on $\mathbb{L}^{N}$ by relating the rank of an element with the rank of its Moore matrix. Let us now investigate the rank metric in $\mathbb{L}[G]$.

Definition 11. Let $\mathbb{L} / \mathbb{K}$ be a finite extension with Galois group $G$. The $\mathbb{K}$-rank of an element $a \in \mathbb{L}[G]$ is defined as the rank of the corresponding $\mathbb{K}$-endomorphism of $\mathbb{L}$ (see (1)).

Given a vector $\boldsymbol{b}=\left(b_{1}, \ldots, b_{M}\right) \in \mathbb{L}^{M}$, let us now define the evaluation map

$$
\mathrm{ev}_{\boldsymbol{b}}:\left\{\begin{array}{rll}
\mathbb{L}[G] & \longrightarrow & \mathbb{L}^{M}  \tag{7}\\
a & \longmapsto & \left(a\left(b_{1}\right), \ldots, a\left(b_{M}\right)\right) .
\end{array}\right.
$$

For $a \in \mathbb{L}[G]$, the vector $\operatorname{ev}_{\boldsymbol{b}}(a) \in \mathbb{L}^{M}$ is called the evaluation vector of $a$ at $\boldsymbol{b}$. One can easily see that $\operatorname{rk}_{\mathbb{K}}(a)=\operatorname{rk}_{\mathbb{K}}\left(\operatorname{ev}_{\mathcal{B}}(a)\right)$ for every basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$.

Definition 12. The left-annihilator of an element $a \in \mathbb{L}[G]$ is defined as

$$
\operatorname{Ann}_{\mathbb{L}[G]}(a):=\{f \in \mathbb{L}[G] \mid f \circ a=0\}
$$

Observe that $\operatorname{Ann}_{\mathbb{L}[G]}(a)$ is an $\mathbb{L}$-subspace and a left-ideal in $\mathbb{L}[G]$.
Proposition 13. For every $a \in \mathbb{L}[G]$ we have

$$
\mathrm{rk}_{\mathbb{K}}(a)=\operatorname{dim}_{\mathbb{L}}\left(\mathbb{L}[G] / \operatorname{Ann}_{\mathbb{L}[G]}(a)\right)
$$

Proof. Let us set $\boldsymbol{v}=\operatorname{ev}_{\mathcal{B}}(a)$ for some basis $\mathcal{B}=\left(\beta_{1}, \ldots, \beta_{N}\right)$ of $\mathbb{L} / \mathbb{K}$. We have

$$
\operatorname{dim}_{\mathbb{L}}\left(\mathbb{L}[G] / \operatorname{Ann}_{\mathbb{L}[G]}(a)\right)=N-\operatorname{dim}_{\mathbb{L}}\left(\operatorname{Ann}_{\mathbb{L}[G]}(a)\right)
$$

and Proposition 9 yields $\mathrm{rk}_{\mathbb{K}}(a)=\mathrm{rk}_{\mathbb{K}}(\boldsymbol{v})=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)$. To prove the result, we will prove that $\mathrm{Ann}_{\mathbb{L}[G]}(a)$ and $\operatorname{ker}_{\mathbb{L}} M_{G}(\boldsymbol{v})^{\top}$ are isomorphic. Indeed, consider the natural $\mathbb{L}$-isomorphism

$$
\varphi:\left\{\begin{array}{rll}
\mathbb{L}[G] & \longrightarrow & \mathbb{L}^{N} \\
\sum_{i} \lambda_{i} g_{i} & \longmapsto & \left(\lambda_{1}, \ldots, \lambda_{N}\right) .
\end{array}\right.
$$

One can see that $\varphi\left(\operatorname{Ann}_{\mathbb{L}[G]}(a)\right)=\operatorname{ker}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})^{\top}\right)$. Indeed, $\left(\lambda_{1}, \ldots, \lambda_{N}\right) \in \operatorname{ker}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})^{\top}\right)$ if and only if

$$
\forall j \in\{1, \ldots, N\}, \quad 0=\left(\sum_{i} \lambda_{i} g_{i}\right)\left(v_{j}\right)=\left(\sum_{i} \lambda_{i} g_{i}\right)\left(a\left(\beta_{j}\right)\right)=\left(\left(\sum \lambda_{i} g_{i}\right) \circ a\right)\left(\beta_{j}\right)
$$

Since $\mathcal{B}$ is a basis, this holds if and only if $\left(\sum \lambda_{i} g_{i}\right) \circ a=0$, which is equivalent to say that $\varphi^{-1}\left(\lambda_{1}, \ldots, \lambda_{N}\right)=$ $\left(\sum \lambda_{i} g_{i}\right) \in \operatorname{Ann}_{\mathbb{L}[G]}(a)$. This proves that $\varphi\left(\operatorname{Ann}_{\mathbb{L}[G]}(a)\right) \supseteq \operatorname{ker}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)$ and the converse inclusion can be proved in a similar fashion.

To sum up, let $a \in \mathbb{L}[G]$ and define $\mathrm{wt}_{I}(a):=\operatorname{dim}_{\mathbb{L}}\left(\mathbb{L}[G] / \operatorname{Ann}_{\mathbb{L}[G]}(a)\right)$. If we set $\boldsymbol{v}=\operatorname{ev}_{\mathcal{B}}(a)$ for some basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$, then we have proved:

$$
\operatorname{rk}_{\mathbb{K}}(a)=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)=\mathrm{wt}_{I}(a)
$$

## 4. Dickson matrices for elements in $\mathbb{L}[G]$

In this section, we study Dickson matrices in the context of arbitrary Galois groups. Before giving their definition let us introduce a notation. Consider the left action of $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})$ on itself and denote by $\sigma_{i} \in \mathfrak{S}_{N}$ the permutation associated to $g_{i} \in G$, i.e. $g_{i} g_{j}=g_{\sigma_{i}(j)}$ for all $i, j \in\{1, \ldots, N\}$.

Definition 14. Let us fix some ordering $\left(g_{1}, \ldots, g_{N}\right)$ of the group $G$ Let $a=\sum_{i} a_{i} g_{i} \in \mathbb{L}[G]$. The $G$-Dickson matrix associated to $a$ is defined as $D_{G}(a)=\left(d_{i, j}\right) \in \mathbb{L}^{N \times N}$ defined by

$$
d_{i, j}=g_{j}\left(a_{\sigma_{j}^{-1}(i)}\right), \quad \forall i, j \in\{1, \ldots, N\}
$$

Example 15. When $\mathbb{K}=\mathbb{F}_{q}$ and $\mathbb{L}=\mathbb{F}_{q^{N}}$, we have that $G=\operatorname{Gal}\left(\mathbb{F}_{q^{N}} / \mathbb{F}_{q}\right)=\langle\theta\rangle$, where $\theta$ is the $q$ Frobenius automorphism. Then choosing the ordered $\mathbb{F}_{q^{N}}$-basis $\left(\operatorname{Id}, \theta, \ldots, \theta^{N-1}\right.$ ) of $\mathbb{F}_{q^{N}}[G]$, we get that the $G$-Dickson matrix $D_{G}(a)$ of an element $a=a_{1} \mathrm{Id}+a_{2} \theta+\cdots+a_{N} \theta^{N-1} \in \mathbb{F}_{q^{N}}[G]$ is given by:

$$
D_{G}(a)=\left(\begin{array}{cccc}
a_{1} & a_{N}^{q} & \cdots & a_{2}^{q^{N-1}} \\
a_{2} & a_{1}^{q} & \cdots & a_{3}^{q N-1} \\
\vdots & & \ddots & \\
a_{N} & a_{N-1}^{q} & \cdots & a_{1}^{q^{N-1}}
\end{array}\right)
$$

This matrix is usually known as the Dickson matrix associated to $a(x)=\sum_{N=1}^{N} a_{i} x^{q^{i-1}} \in \mathcal{L}[x]$, where we recall that $\mathcal{L}[x]$ denotes the ring of linearized polynomials. Since $\mathcal{L}[x] /\left(x^{q^{N}}-x\right) \cong \mathbb{F}_{q^{N}}[G]$, this explains the relation between the $G$-Dickson matrix and the usual Dickson matrix over finite fields.

In the sequel we give two distinct interpretations of these matrices.
4.1. The right multiplication map. If $a=\sum_{i} a_{i} g_{i} \in \mathbb{L}[G]$, then for every $j \in\{1, \ldots, N\}$ we have

$$
g_{j} \circ\left(\sum_{i=1}^{N} a_{i} g_{i}\right)=\sum_{i=1}^{N} g_{j}\left(a_{i}\right) g_{j} g_{i}=\sum_{i=1}^{N} g_{j}\left(a_{i}\right) g_{\sigma_{j}(i)}
$$

Now, let us consider the $\mathbb{L}$-linear map

$$
\mu:\left\{\begin{array}{ccc}
\mathbb{L}[G] & \longrightarrow & \operatorname{Hom}_{\mathbb{L}}(\mathbb{L}[G], \mathbb{L}[G])  \tag{8}\\
a & \longmapsto & (f \mapsto f \circ a)
\end{array}\right.
$$

Proposition 16. Let $a \in \mathbb{L}[G]$. Then, the matrix representing $\mu(a)$ in the basis $\left(g_{1}, \ldots, g_{N}\right)$ is the $G$-Dickson matrix $D_{G}(a)$.

Remark 17. The $G$-Dickson matrix is the matrix associated to the $\mathbb{L}$-linear map $\mu(a)$, given by the right composition by $a$. One can also consider the map $\mathbb{L}[G] \rightarrow \mathbb{L}[G]$ given by the left composition $f \mapsto a \circ f$. However this map is only semilinear.
4.2. The element of the group algebra after a base field extension. Given $a \in \mathbb{L}[G]$, the element $a$ induces a $\mathbb{K}$-endomorphism $a: \mathbb{L} \rightarrow \mathbb{L}$. We claim that the transposition of its $G$-Dickson matrix represents this endomorphism after a base field extension. To understand this fact, we introduce the map

$$
\nu:\left\{\begin{array}{ccc}
\mathbb{L}[G] & \longrightarrow & \operatorname{Hom}_{\mathbb{L}}\left(\mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}, \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}\right) \\
a & \longmapsto & \operatorname{Id} \otimes a
\end{array}\right.
$$

and will study in depth the maps of the form $\operatorname{Id} \otimes a: \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L} \rightarrow \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}$. Let $\alpha$ be a primitive element of $\mathbb{L} / \mathbb{K}$, and consider the $\mathbb{K}$-linear map given by the multiplication by $\alpha$

$$
m_{\alpha}:\left\{\begin{array}{llc}
\mathbb{L} & \longrightarrow & \mathbb{L} \\
x & \longmapsto & \alpha x
\end{array}\right.
$$

In the $\mathbb{K}$-basis $\left(1, \alpha, \alpha^{2}, \ldots, \alpha^{N-1}\right)$ this map is represented by the companion matrix of the minimal polynomial of $\alpha$ over $\mathbb{K}$. Therefore, its eigenvalues are nothing but the $g(\alpha)$ for $g \in G$. For a suitable
choice of eigenvectors basis in $\mathbb{L} \otimes_{\mathbb{K}} \mathbb{L} \simeq \mathbb{L}^{N}$ the map $\operatorname{Id} \otimes m_{\alpha}$ has a diagonal matrix representation

$$
\left(\begin{array}{cccc}
\alpha & & & (0)  \tag{9}\\
& g_{2}(\alpha) & & \\
& & \ddots & \\
(0) & & & g_{N}(\alpha)
\end{array}\right)
$$

where we ordered the elements of $G$ so that $g_{1}=\mathrm{Id}$. This matrix representation is associated to a basis of eigenvectors of $\mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}$. Let us make a particular choice of normalisation for them. Choose $v \in \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}$ an eigenvector of $\operatorname{Id} \otimes m_{\alpha}$ with respect to the eigenvalue $\alpha$. That is to say $\left(\operatorname{Id} \otimes m_{\alpha}\right)(v)=\alpha \cdot v$. For $g \in G$ we define $v_{g}:=\left(\operatorname{Id} \otimes g^{-1}\right)(v)$.
Proposition 18. Let $g \in G$. Then $v_{g}$ is an eigenvector of $I d \otimes m_{\alpha}$ with respect to the eigenvalue $g(\alpha)$.
Proof. First, note that $g \circ m_{\alpha}=m_{g(\alpha)} \circ g$. Therefore, we have

$$
\begin{aligned}
\left(\operatorname{Id} \otimes m_{\alpha}\right) \circ\left(\operatorname{Id} \otimes g^{-1}\right)(v) & =\operatorname{Id} \otimes\left(m_{\alpha} \circ g^{-1}\right)(v) \\
& =\operatorname{Id} \otimes\left(g^{-1} \circ m_{g(\alpha)}\right)(v) \\
& =\left(\operatorname{Id} \otimes g^{-1}\right) \circ\left(\operatorname{Id} \otimes m_{g(\alpha)}\right)(v)
\end{aligned}
$$

Since $\alpha$ is a primitive element of $\mathbb{L} / \mathbb{K}$, there exists a polynomial $P \in \mathbb{K}[X]$ such that $g(\alpha)=P(\alpha)$ and hence $P\left(m_{\alpha}\right)=m_{g(\alpha)}$. Moreover, since $v$ is an eigenvector of $\mathrm{Id} \otimes m_{\alpha}$ with respect to the eigenvalue $\alpha$, then it is an eigenvector of $P\left(\operatorname{Id} \otimes m_{\alpha}\right)=\mathrm{Id} \otimes P\left(m_{\alpha}\right)$ with respect to the eigenvalue $P(\alpha)$. Therefore,

$$
\begin{aligned}
\left(\operatorname{Id} \otimes m_{\alpha}\right) \circ\left(\operatorname{Id} \otimes g^{-1}\right)(v) & =\left(\operatorname{Id} \otimes g^{-1}\right) \circ\left(\operatorname{Id} \otimes m_{g(\alpha)}\right)(v) \\
& =\left(\operatorname{Id} \otimes g^{-1}\right)(P(\alpha)(v)) \\
& =P(\alpha) \cdot\left(\operatorname{Id} \otimes g^{-1}\right)(v) \\
& =g(\alpha) \cdot\left(\operatorname{Id} \otimes g^{-1}\right)(v)
\end{aligned}
$$

In summary $v_{g}:=\left(\operatorname{Id} \otimes g^{-1}\right)(v)$ is an eigenvector of $\operatorname{Id} \otimes m_{\alpha}$ with respect to $g(\alpha)$.
Therefore, in the basis $\left(v_{g}\right)_{g \in G}$ the multiplication by an element $\alpha \in \mathbb{L}$ is represented by a diagonal matrix. Next, the action of elements of $G$ will be represented by permutation matrices as suggests the next statement.

Proposition 19. Let $g, h \in G$, then $(I d \otimes g)\left(v_{h}\right)=v_{h g^{-1}}$.
Proof. $(\operatorname{Id} \otimes g) \circ\left(\operatorname{Id} \otimes h^{-1}\right)(v)=\left(\operatorname{Id} \otimes\left(h g^{-1}\right)^{-1}\right)(v)=v_{h g^{-1}}$.
As a conclusion, $G$ acts by permutation on eigenvectors $\left(v_{g}\right)_{g \in G}$.
4.3. Relating these two approaches. Now, let us try to relate matrix representations of $\mu(a)$ and $\nu(a)$.

Theorem 20. Let

$$
\Lambda:\left\{\begin{array}{ccc}
\mathbb{L}[G] & \longrightarrow & \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L} \\
\sum_{g} a_{g} g & \longmapsto & \sum_{g} a_{g} v_{g}
\end{array}\right.
$$

then for any $a \in \mathbb{L}[G]$, we have

$$
\mu(\tau(a))=\Lambda^{-1} \circ \nu(a) \circ \Lambda,
$$

where $\tau$ is the adjunction map introduced in Section 2.4. From the matrix point of view:

$$
D_{G}(a)=A\left(\mu(a),\left(g_{1}, \ldots, g_{N}\right)\right)=A\left(\nu(a),\left(v_{g_{1}}, \ldots, v_{g_{N}}\right)\right)^{\top}
$$

Proof. Note first that the map $\nu$ is a ring homomorphism, while $\mu$ is a ring anti-homomorphism: for any $a, b \in \mathbb{L}[G]$, we have $\mu(a \circ b)=\mu(b) \circ \mu(a)$. Thus, we introduce the map $\mu^{\prime}: a \mapsto \mu(\tau(a))$ which is a ring homomorphism and we will show that $\mu^{\prime}$ and $\nu$ have conjugated images under $\Lambda$. Since we have ring homomorphisms it is sufficient to prove that the property is satisfied by generators, i.e. elements of $\mathbb{L}$ and elements of $G$.

First consider the case of an element $a \in \mathbb{L}$. We already proved that $\nu(a)$ has a diagonal representation in the basis $\left(v_{g}\right)_{g \in G}$. On the other hand for any $g \in G$, we have

$$
\mu^{\prime}(a)(g)=g \circ \tau(a)=g \circ a=g(a) \cdot g .
$$

Hence, $g$ is an eigenvector of $\mu^{\prime}(a)$ with respect to the eigenvalue $g(a)$ and hence has the very same matrix representation. Formally, $\mu^{\prime}(a)=\Lambda^{-1} \circ \nu(a) \circ \Lambda$.

Next, consider an element $g \in G$. By Proposition 19 , for any $h \in G$ we have $\nu(g)\left(v_{h}\right)=v_{h g^{-1}}$. On the other hand,

$$
\mu^{\prime}(g)(h)=h \circ \tau(g)=h g^{-1}
$$

Hence, here again, we deduce that $\mu^{\prime}(g)=\Lambda^{-1} \circ \nu(g) \circ \Lambda$. This concludes the proof.
Corollary 21. Let $\mathcal{B}$ be a $\mathbb{K}$-basis of $\mathbb{L}$ and $\mathcal{B}_{\mathbb{L}}:=(1 \otimes b)_{b \in \mathcal{B}}$ the corresponding $\mathbb{L}$-basis of $\mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}$. Let $\boldsymbol{v}$ be the representation of the eigenvector $v \in \mathbb{L} \otimes_{\mathbb{K}} \mathbb{L}$ in the basis $\mathcal{B}_{\mathbb{L}}$. Then, for any $a \in \mathbb{L}[G]$

$$
D_{G}(a)^{\top}=M_{G}(\boldsymbol{v})^{\top} A(a, \mathcal{B})\left(M_{G}(\boldsymbol{v})^{\top}\right)^{-1}
$$

Proof. The matrix $D_{G}(a)^{\top}$ represents $\nu(a)$ in the basis $\left(v_{g}\right)_{g \in G}$. On the other hand $M_{G}(\boldsymbol{v})^{\top}$ can be interpreted as the change of basis matrix from $\mathcal{B}_{\mathbb{L}}$ to $\left(v_{g}\right)_{g \in G}$.
4.4. Properties of Dickson matrices. The previous observations permit first to assert the following statement.

Lemma 22. For any $a \in \mathbb{L}[G]$, we have

$$
D_{G}(a)^{\top}=D_{G}(\tau(a)) .
$$

Next, if we define the following algebra, $\mathcal{D}(\mathbb{L} / \mathbb{K}):=\left\{D_{G}(a)^{\top} \mid a \in \mathbb{L}[G]\right\} \subseteq \mathbb{L}^{N \times N}$, then we get a new ring isomorphism:

$$
\mathcal{D}(\mathbb{L} / \mathbb{K}) \cong \mathbb{L}[G] \cong \operatorname{End}_{\mathbb{K}}(\mathbb{L}) \cong \mathbb{K}^{N \times N}
$$

In addition, the rank of an element of $\mathbb{L}[G]$ can obviously be interpreted in terms of the rank of its $G$-Dickson matrix, as it holds for the finite field case (see e.g. [23, 34, 9]).
Theorem 23. Let $a \in \mathbb{L}[G]$ and $\boldsymbol{v}:=\left(a\left(\beta_{1}\right), \ldots, a\left(\beta_{N}\right)\right)$ for some basis $\mathcal{B}=\left(\beta_{1}, \ldots, \beta_{N}\right)$ of $\mathbb{L} / \mathbb{K}$. Then,

$$
\operatorname{rk}_{\mathbb{K}}(a)=\operatorname{wt}_{I}(a)=\operatorname{rk}_{\mathbb{L}}\left(M_{G}(\boldsymbol{v})\right)=\operatorname{rk}_{\mathbb{L}}\left(D_{G}(a)\right)
$$

## 5. RANK-METRIC CODES

The theory of rank-metric codes has been essentially always studied in the context of extension fields with cyclic Galois groups. For the special case of finite fields, the reader is referred to [32]. In this section, we consider the case of general Galois extensions $\mathbb{L} / \mathbb{K}$ of finite degree $N=[\mathbb{L}: \mathbb{K}]=|\operatorname{Gal}(\mathbb{L} / \mathbb{K})|$.
5.1. Equivalent representations of codes. According to Sections 3 and 4, we can define the rank metric in several equivalent ways. In $\mathbb{L}[G]$, the rank distance is defined as

$$
\mathrm{d}(a, b):=\operatorname{rk}_{\mathbb{K}}(a-b), \quad \text { for any } a, b \in \mathbb{L}[G]
$$

Definition 24. An $\mathbb{L}$-linear rank-metric code $\mathcal{C}$ is an $\mathbb{L}$-subspace of $\mathbb{L}[G]$, equipped with the rank distance. The dimension of $\mathcal{C}$ is its dimension as an $\mathbb{L}$-vector space, and its minimum rank-distance is the integer

$$
\mathrm{d}(\mathcal{C}):=\min \{\mathrm{d}(a, b) \mid a, b \in \mathcal{C}, a \neq b\}
$$

An $\mathbb{L}$-linear rank-metric code $\mathcal{C} \subseteq \mathbb{L}[G]$ of $\mathbb{L}$-dimension $k$ and minimum rank distance $d$ will be also called an $[N, k, d]_{\mathbb{L}[G]}$ code, where $N:=|G|$, or simply $[N, k]_{\mathbb{L}[G]}$ code, if the minimum rank distance is not known/relevant.

Rank-metric codes have been previously studied in other ambient spaces. First, in spaces of matrices, the rank distance is defined as

$$
\mathrm{d}:\left\{\begin{array}{ccc}
\mathbb{K}^{N \times M} \times \mathbb{K}^{N \times M} & \longrightarrow & \mathbb{N} \\
(A, B) & \longmapsto & \mathrm{rk}_{\mathbb{K}}(A-B) .
\end{array}\right.
$$

Codes in this setting are usually called matrix rank-metric codes. Linear codes are $K$-dimensional $\mathbb{K}$ subspaces of $\mathbb{K}^{N \times M}$ and they are denoted by $[N \times M, K]_{\mathbb{K}}$ codes (or $[N \times M, K, d]_{\mathbb{K}}$ codes if the minimum distance is known).

As in classical literature, we can also define the rank distance on vectors over $\mathbb{L}$ as

$$
\mathrm{d}:\left\{\begin{array}{ccc}
\mathbb{L}^{M} \times \mathbb{L}^{M} & \longrightarrow & \mathbb{N} \\
(\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \mathrm{rk}_{\mathbb{K}}(\boldsymbol{u}-\boldsymbol{v})
\end{array}\right.
$$

Here, codes are called vector rank-metric codes. Linear codes in this framework are $k$-dimensional $\mathbb{L}$ subspaces of $\mathbb{L}^{M}$ and they are denoted by $[M, k]_{\mathbb{L} / \mathbb{K}}$ codes (or $[M, k, d]_{\mathbb{L} / \mathbb{K}}$ codes if the minimum distance is known).
5.1.1. From vector codes to matrix codes. In the theory of rank-metric codes there is a procedure for going from an $[M, k, d]_{\mathbb{L} / \mathbb{K}}$ code to an $[N \times M, N k, d]_{\mathbb{K}}$ code. Fix an ordered basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$, and write every element of $\mathbb{L}$ in coordinates with respect to $\mathcal{B}$, resulting in a column vector in $\mathbb{K}^{N}$. In the same way, we can transform a vector $\boldsymbol{v} \in \mathbb{L}^{M}$ to a matrix in $\mathbb{K}^{N \times M}$, which we denote by $\operatorname{Ext}_{\mathcal{B}}(\boldsymbol{v})$. Hence, for an $[M, k, d]_{\mathbb{L} / \mathbb{K}}$ code $\mathcal{C}$ and a fixed ordered basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$ we define

$$
\operatorname{Ext}_{\mathcal{B}}(\mathcal{C}):=\left\{\operatorname{Ext}_{\mathcal{B}}(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathcal{C}\right\} \subseteq \mathbb{K}^{N \times M}
$$

which is an $[N \times M, N k, d]_{\mathbb{K}}$ code.
5.1.2. From $\mathbb{L}[G]$-codes to vector codes. Now, we briefly explain the relation between rank-metric codes in $\mathbb{L}[G]$ and vector rank-metric codes in $\mathbb{L}^{N}$. Let $\mathcal{C} \subseteq \mathbb{L}[G]$ be an $[N, k, d]_{\mathbb{L}[G]}$ code and fix an ordered basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$. Then, we define the code

$$
\mathcal{C}(\mathcal{B}):=\left\{\operatorname{ev}_{\mathcal{B}}(c) \mid c \in \mathcal{C}\right\}
$$

By Theorem 23 the map $\mathcal{C} \mapsto \mathcal{C}(\mathcal{B})$ is an isometry between spaces $(\mathbb{L}[G], \mathrm{d})$ and $\left(\mathbb{L}^{N}, \mathrm{~d}\right)$, and hence the code $\mathcal{C}(\mathcal{B})$ is an $[N, k, d]_{\mathbb{L} / \mathbb{K}}$ vector rank-metric code. Moreover, if we fix two ordered bases $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ of $\mathbb{L} / \mathbb{K}$, and let $X \in \mathbb{K}^{N \times N}$ be the change-of-basis matrix such that $\mathcal{B}_{1}=\mathcal{B}_{2} X$, then we have

$$
\begin{equation*}
\mathcal{C}\left(\mathcal{B}_{1}\right)=\mathcal{C}\left(\mathcal{B}_{2}\right) \cdot X=\left\{\boldsymbol{v} X \mid \boldsymbol{v} \in \mathcal{C}\left(\mathcal{B}_{2}\right)\right\} \tag{10}
\end{equation*}
$$

One may note that the two codes $\mathcal{C}\left(\mathcal{B}_{1}\right)$ and $\mathcal{C}\left(\mathcal{B}_{2}\right)$ are equivalent in the sense of vector rank-metric codes (see [24] for the finite field case.). In particular, they are isometric with respect to the rank metric.
5.1.3. From $\mathbb{L}[G]$-codes to matrix codes. Finally, if we fix two ordered bases $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$ of $\mathbb{L} / \mathbb{K}$, we can transform the $[N, k, d]_{\mathbb{L}[G]}$ code $\mathcal{C}$ in the vector $\operatorname{code} \mathcal{C}\left(\mathcal{B}_{1}\right)$ and then to the matrix code $\operatorname{Ext}_{\mathcal{B}_{2}}\left(\mathcal{C}\left(\mathcal{B}_{1}\right)\right)$. This last matrix code satisfies,

$$
\operatorname{Ext}_{\mathcal{B}_{2}}\left(\mathcal{C}\left(\mathcal{B}_{1}\right)\right)=\left\{A\left(c, \mathcal{B}_{1}, \mathcal{B}_{2}\right) \mid c \in \mathcal{C}\right\}
$$

In the special case in which $\mathcal{B}_{1}=\mathcal{B}_{2}=: \mathcal{B}$, we get the code

$$
\operatorname{Ext}_{\mathcal{B}}(\mathcal{C}(\mathcal{B})):=\{A(c, \mathcal{B}) \mid c \in \mathcal{C}\}
$$

Example 25. Let us fix $\mathbb{K}=\mathbb{Q}$, and $\mathbb{L}$ to be the splitting field of the polynomial $x^{3}-p$, where $p$ is a prime number. This means that $\mathbb{L}=\mathbb{Q}(\zeta, \sqrt[3]{p})$, where $\zeta$ is a primitive 3 rd root of unity satisfying $\zeta^{2}+\zeta+1=0$. The Galois group $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})$ is isomorphic to the symmetric group $\mathfrak{S}_{3}$ and it is generated by the automorphisms $\sigma_{1}$ and $\sigma_{2}$, defined as

$$
\sigma_{1}:\left\{\begin{array}{ll}
\zeta & \mapsto \zeta^{2} \\
\sqrt[3]{p} & \mapsto \sqrt[3]{p}
\end{array} \quad \text { and } \quad \sigma_{2}: \begin{cases}\zeta & \mapsto \zeta \\
\sqrt[3]{p} & \mapsto \zeta \sqrt[3]{p}\end{cases}\right.
$$

Consider the $[6,3]_{\mathbb{L}[G]}$ rank-metric code given by

$$
\mathcal{C}:=\left\{a \cdot \operatorname{Id}+b \cdot \sigma_{1}+c \cdot \sigma_{2} \mid a, b, c \in \mathbb{L}\right\}
$$

We fix the following ordered basis $\mathcal{B}=\left(1, \zeta, \sqrt[3]{p}, \zeta \sqrt[3]{p}, \sqrt[3]{p}^{2}, \zeta \sqrt[3]{p}{ }^{2}\right)$ of $\mathbb{L} / \mathbb{K}$. Then, the $[6,3]_{\mathbb{L} / \mathbb{K}}$ code $\mathcal{C}(\mathcal{B})$ is generated by the matrix

$$
\left(\begin{array}{cccccc}
1 & \zeta & \sqrt[3]{p} & \zeta \sqrt[3]{p} & \sqrt[3]{p} & \zeta \sqrt[3]{p} \\
1 & -(\zeta+1) & \sqrt[3]{p} & -(\zeta+1) \sqrt[3]{p} & \sqrt[3]{p}^{2} & -(\zeta+1) \sqrt[3]{p} \bar{p}^{2} \\
1 & \zeta & \zeta \sqrt[3]{p} & -(\zeta+1) \sqrt[3]{p} & -(\zeta+1) \sqrt[3]{p} & \sqrt[3]{p}^{2}
\end{array}\right)
$$

Moreover, we can also determine the $[6 \times 6,18]_{\mathbb{K}}$ matrix $\operatorname{code}^{\operatorname{Ext}}{ }_{\mathcal{B}}(\mathcal{C}(\mathcal{B}))$. The matrices that represent the scalar multiplication by the six elements of the basis are of the form $A^{i} B^{j}$ for $i \in\{0,1\}$ and $j \in\{0,1,2\}$, where

$$
A=\left(\begin{array}{cccccc}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{array}\right) \quad \text { and } \quad B=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & p & 0 \\
0 & 0 & 0 & 0 & 0 & p \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)
$$

This is due to the fact that $A$ and $B$ represent the multiplication by $\zeta$ and $\sqrt[3]{p}$ respectively. Hence, by writing the three row vectors of the generator matrix of $\mathcal{C}(\mathcal{B})$ with respect to the basis $\mathcal{B}$, we see that the code $\operatorname{Ext}_{\mathcal{B}}(\mathcal{C}(\mathcal{B}))$ is the $\mathbb{Q}$-span of the set

$$
\left\{A^{i} B^{j}, A^{i} B^{j} X, A^{i} B^{j} Y \mid 0 \leqslant i \leqslant 1,0 \leqslant j \leqslant 2\right\}
$$

where

$$
X=\left(\begin{array}{cccccc}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1
\end{array}\right) \quad \text { and } \quad Y=\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{array}\right)
$$

are the matrices representing $\sigma_{1}$ and $\sigma_{2}$ in the basis $\mathcal{B}$. In other words, $X$ and $Y$ are the Ext $\mathcal{B}_{\mathcal{B}}$ of the vectors

$$
\left(1,-(\zeta+1), \sqrt[3]{p},-(\zeta+1) \sqrt[3]{p}, \sqrt[3]{p}^{2},-(\zeta+1) \sqrt[3]{p}\right) \quad \text { and } \quad\left(1, \zeta, \zeta \sqrt[3]{p},-(\zeta+1) \sqrt[3]{p},-(\zeta+1) \sqrt[3]{p}, \sqrt[3]{p}^{2}\right)
$$

respectively.
5.2. Duality for rank-metric codes. Here, we study the different notions of duality for rank-metric codes, according to the three representations mentioned above and how they are related.
5.2.1. Matrix codes. First, on the space of matrices we consider the standard bilinear form for matrices, given by

$$
\left\{\begin{array}{ccc}
\mathbb{K}^{N \times M} \times \mathbb{K}^{N \times M} & \longrightarrow & \mathbb{K} \\
(A, B) & \longmapsto & \operatorname{Tr}\left(A B^{\top}\right),
\end{array}\right.
$$

where $\operatorname{Tr}$ denotes the matrix trace. The dual code of an $[N \times M, K]_{\mathbb{K}}$ code $\mathcal{C}$ is then

$$
\mathcal{C}^{\perp}:=\left\{A \in \mathbb{K}^{N \times M} \mid \operatorname{Tr}\left(A B^{\top}\right)=0 \text { for all } B \in \mathcal{C}\right\}
$$

Since the standard bilinear form is nondegenerate, then $\mathcal{C}^{\perp}$ is an $[N \times M, N M-K]$ code.
5.2.2. Vector codes. For vector rank-metric codes, the duality is always taken with respect to the standard inner product. Hence, for an $[M, k]_{\mathbb{L} / \mathbb{K}}$ code, its dual code is the $[M, M-k]_{\mathbb{L} / \mathbb{K}}$ code given by

$$
\mathcal{C}^{\perp}:=\left\{\boldsymbol{u} \in \mathbb{L}^{M} \mid \boldsymbol{u} \cdot \boldsymbol{v}^{\top}=0 \text { for all } \boldsymbol{v} \in \mathcal{C}\right\}
$$

5.2.3. $\mathbb{L}[G]$-codes. Finally, we introduce the following bilinear form on $\mathbb{L}[G]$ - which is also called standard bilinear form over finite fields - defined as

$$
\langle\cdot, \cdot\rangle_{\mathbb{L}[G]}:\left\{\begin{array}{ccc}
\mathbb{L}[G] \times \mathbb{L}[G] & \longrightarrow & \mathbb{L} \\
\left(a=\sum_{g \in G} a_{g} g, b=\sum_{g \in G} b_{g} g\right) & \longmapsto & \sum_{g \in G} a_{g} b_{g}
\end{array}\right.
$$

This bilinear form is also nondegenerate, and given an $[N, k]_{\mathbb{L}[G]}$ rank-metric code, we define its dual code as the $[N, N-k]_{\mathbb{L}[G]}$ code

$$
\mathcal{C}^{\perp}:=\left\{a \in \mathbb{L}[G] \mid\langle a, b\rangle_{\mathbb{L}[G]}=0 \text { for all } b \in \mathcal{C}\right\}
$$

In Section 5.1, we have seen how codes in these three points of view are related. This can be extended to duality. For instance, the relation between the duality of matrix and vector rank-metric codes over finite fields has been already studied in $[16,27]$. With the same proof, it is easy to see that for any $[M, k]_{\mathbb{L} / \mathbb{K}}$ code $\mathcal{C} \subseteq \mathbb{L}^{M}$ and any ordered basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$ with dual basis $\mathcal{B}^{*}$, it holds

$$
\operatorname{Ext}_{\mathcal{B}}(\mathcal{C})^{\perp}=\operatorname{Ext}_{\mathcal{B}^{*}}\left(\mathcal{C}^{\perp}\right)
$$

Now, let us show how rank-metric codes in $\mathbb{L}[G]$ are related to vector rank-metric codes in $\mathbb{L}^{N}$. Let $\alpha \in \mathbb{L}$ be a normal element of $\mathbb{L} / \mathbb{K}$, i.e. the set $\{g(\alpha) \mid g \in G\}$ is a basis of $\mathbb{L} / \mathbb{K}$. If we fix some ordering for the elements of $G$, say $g_{1}, \ldots, g_{N}$, then we get an ordered normal basis $\boldsymbol{a}=\left(g_{1}(\alpha), \ldots, g_{N}(\alpha)\right) \in \mathbb{L}^{N}$. One can prove that the dual basis of an ordered normal basis is normal with respect to the same ordering of elements of $G$.

Theorem 26. Let $\boldsymbol{a}=\left(g_{1}(\alpha), \ldots, g_{N}(\alpha)\right) \in \mathbb{L}^{N}$ be an ordered normal basis, where $\alpha \in \mathbb{L}$. Then, there exists $\beta \in \mathbb{L}$ such that $\boldsymbol{b}=\left(g_{1}(\beta), \ldots, g_{N}(\beta)\right) \in \mathbb{L}^{N}$ is the dual basis of $\boldsymbol{a}$.

Proof. Without loss of generality, assume that $g_{1}$ is the identity element. Let $\boldsymbol{b}=\left(b_{1}, \ldots, b_{N}\right) \in \mathbb{L}^{N}$ be the unique dual basis of $\boldsymbol{a}$ and define $\beta:=b_{1}$. Then, by $G$-invariance of the trace,

$$
\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(g_{i}(\beta) g_{j}(\alpha)\right)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(b_{1} g_{i}^{-1} g_{j}(\alpha)\right)
$$

is 0 if $i \neq j$, and 1 otherwise. Hence, $\left(g_{1}(\beta), \ldots, g_{N}(\beta)\right)$ is dual to $\boldsymbol{a}$, and by uniqueness, $b_{j}=g_{j}(\beta)$ for every $j$.

From this result, we can relate the notions of duality of rank-metric codes, when $G$ is abelian.
Theorem 27. Let $\mathcal{C}$ be an $[N, k]_{\mathbb{L}[G]}$ code and let $\mathcal{B}$ be an ordered basis of $\mathbb{L} / \mathbb{K}$. Moreover, assume that $G$ is abelian. Then

$$
\operatorname{Ext}_{\mathcal{B}}\left(\mathcal{C}^{\perp}\right)=\operatorname{Ext}_{\mathcal{B}^{*}}(\mathcal{C})^{\perp}
$$

Proof. First, we fix a normal basis $\boldsymbol{a}=\left(g_{1}(\alpha), \ldots, g_{N}(\alpha)\right)$, which always exists thanks to the normal basis theorem. By Theorem 26 there exists $\beta \in \mathbb{L}$, such that $\boldsymbol{b}:=\left(g_{1}(\beta), \ldots, g_{N}(\beta)\right)=\boldsymbol{a}^{*}$. Moreover, we have

$$
\begin{aligned}
\operatorname{ev}_{\boldsymbol{a}}\left(g_{i}\right) \mathrm{ev}_{\boldsymbol{b}}\left(g_{j}\right)^{\top} & =\sum_{\ell=1}^{N} g_{i}\left(a_{\ell}\right) g_{j}\left(b_{\ell}\right)=\sum_{\ell=1}^{N} g_{i}\left(g_{\ell}(\alpha)\right) g_{j}\left(g_{\ell}(\beta)\right) \\
& =\sum_{\ell=1}^{N} g_{\ell}\left(g_{i}(\alpha) g_{j}(\beta)\right)=\operatorname{Tr}_{\mathbb{L} / \mathbb{K}}\left(g_{i}(\alpha) g_{j}(\beta)\right) \\
& =\delta_{i, j}=\left\langle g_{i}, g_{j}\right\rangle_{\mathbb{L}[G]} .
\end{aligned}
$$

This shows that in this case $\mathcal{C}^{\perp}(\boldsymbol{a})=(\mathcal{C}(\boldsymbol{b}))^{\perp}$.
Now, suppose that $\mathcal{B}=\left(b_{1}, \ldots, b_{N}\right)$ is a generic ordered basis. There exists an invertible matrix $X \in \mathbb{K}^{N \times N}$ such that $\mathcal{B}=\boldsymbol{a} X$. Moreover, we also have that $\mathcal{B}^{*}=\boldsymbol{b}\left(X^{-1}\right)^{\top}$. Hence, we get

$$
\begin{aligned}
\operatorname{ev}_{\mathcal{B}}\left(g_{i}\right) \operatorname{ev}_{\mathcal{B}^{*}}\left(g_{j}\right)^{\top} & =\operatorname{ev}_{\boldsymbol{a} X}\left(g_{i}\right) \operatorname{ev}_{\boldsymbol{b}\left(X^{-1}\right)^{\top}}\left(g_{j}\right)^{\top} \\
& =\operatorname{ev}_{\boldsymbol{a}}\left(g_{i}\right) X\left(\operatorname{ev}_{\boldsymbol{b}}\left(g_{j}\right)\left(X^{-1}\right)^{\top}\right)^{\top} \\
& =\operatorname{ev}_{\boldsymbol{a}}\left(g_{i}\right) X X^{-1} \operatorname{ev}_{\boldsymbol{b}}\left(g_{j}\right)^{\top} \\
& =\delta_{i, j}=\left\langle g_{i}, g_{j}\right\rangle_{\mathbb{L}[G]} .
\end{aligned}
$$

## 6. Error-correcting pairs in $\mathbb{L}[G]$

In this section, we make a first step towards decoding codes seen as $\mathbb{L}$-subspaces of $\mathbb{L}[G]$. We adapt the notion of rank error-correcting pairs (rank-ECP) introduced by Martínez-Peñas and Pellikaan [22], which themselves were counterparts of Hamming metric error-correcting pairs [26].
Note. From now on and for convenience sake, we always suppose that the group $G$ is equipped with some total ordering and we allow ourselves to index rows and columns of matrices with elements of $G$. Given $A \in \mathbb{L}^{|G| \times|G|}$ and $g, h \in G$, we denote by $A_{g, h}$ the entry of $A$ at row $i$ and column $j$, where $g$ (resp. $h$ ) is the $i$-th (resp. $j$-th) element of $G$ with respect to this ordering. As a consequence, from Definition 14, $G$-Dickson matrices are defined as

$$
D_{G}(a)=\left(h\left(a_{h^{-1} g}\right)\right)_{g, h \in G}
$$

The following statement is useful in the sequel.
Proposition 28. For any $a, b, c \in \mathbb{L}[G]$, we have

$$
\langle a \circ \tau(b), c\rangle_{\mathbb{L}[G]}=\langle a, c \circ b\rangle_{\mathbb{L}[G]} .
$$

Proof. According to the description of $G$-Dickson matrices in Section 4.1, the maps

$$
\left\{\begin{array} { c l c } 
{ \mathbb { L } [ G ] } & { \longrightarrow } & { \mathbb { L } [ G ] } \\
{ x } & { \longmapsto } & { x \circ b }
\end{array} \quad \text { and } \quad \left\{\begin{array}{clc}
\mathbb{L}[G] & \longrightarrow & \mathbb{L}[G] \\
x & \longmapsto & x \circ \tau(b)
\end{array}\right.\right.
$$

are represented in the canonical basis of $\mathbb{L}[G]$ by the $G$-Dickson matrices $D_{G}(b)$ and $D_{G}(\tau(b))$. From Lemma 22, these matrices are transpose to each other. Thus, since the elements of $G$ form an orthonormal basis with respect to $\langle\cdot, \cdot\rangle_{\mathbb{L}[G]}$, the corresponding maps are adjoint to each other.
6.1. Support. First recall a very classical fact in adjunction which is that, given $a \in \mathbb{L}[G]$, then we have $\operatorname{ker}(a)^{\perp}=\operatorname{Im}(\tau(a))$, where the dual is taken with respect to $\langle\cdot, \cdot\rangle_{\mathrm{tr}}$. Now, let us introduce the notion of support of an element of $\mathbb{L}[G]$.
Definition 29. The support of an element $a \in \mathbb{L}[G]$ is defined as the orthogonal of $\operatorname{ker}(a)$ with respect to $\langle\cdot, \cdot\rangle_{\text {tr }}$. Namely,

$$
\operatorname{Supp}(a):=\operatorname{ker}(a)^{\perp}=\operatorname{Im}(\tau(a)) \subseteq \mathbb{L}
$$

This definition can appear to be slightly different from the usual one as given for instance in [15, § 2] for matrix codes, where the support of a matrix is its column space. However, our definition can be understood as a row space. Indeed, the support $\operatorname{Im}(\tau(a))$ of $a$ can be interpreted as the column space of a matrix representing $\tau(a)$ and hence as the row space of a matrix representing $a$. In particular, we have that $\operatorname{dim}_{\mathbb{K}}(\operatorname{Supp}(a))=\operatorname{rk}(a)$.

Finally, let us recall the notion of shortening which is for instance introduced in [7, Definition 3.2] (see also [32, Definition 14]).
Definition 30. Let $\mathcal{C} \subseteq \mathbb{L}[G]$ be a code and $I$ be a $\mathbb{K}$-subspace of $\mathbb{L}$. The shortening of $\mathcal{C}$ at $I$ is defined as

$$
\operatorname{Short}_{I}(\mathcal{C}):=\{c \in \mathcal{C} \mid I \subseteq \operatorname{ker}(c)\}
$$

6.2. Error correcting pairs. The product of two codes $\mathcal{A}, \mathcal{B} \subseteq \mathbb{L}[G]$ is defined as:

$$
\mathcal{B} \circ \mathcal{A}:=\operatorname{Span}_{\mathbb{L}}\{b \circ a \mid a \in \mathcal{A}, b \in \mathcal{B}\}
$$

Notice that, generally, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$. Next, given two codes $\mathcal{A}, \mathcal{B} \subseteq \mathbb{L}[G]$ and some $e \in \mathbb{L}[G]$, we define

$$
\mathcal{K}(e):=\left\{a \in \mathcal{A} \mid\langle b \circ a, e\rangle_{\mathbb{L}[G]}=0, \forall b \in \mathcal{B}\right\} \subseteq \mathbb{L}[G]
$$

Then we have the following result.
Proposition 31. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathbb{L}[G]$ be codes such that $\mathcal{B} \circ \mathcal{A} \subseteq \mathcal{C}^{\perp}$. Let $r=c+e \in \mathbb{L}[G]$, where $c \in \mathcal{C}$ and $e \in \mathbb{L}[G]$. Denote $I=\operatorname{Supp}(e)$. Then,
(1) $\mathcal{K}(r)=\mathcal{K}(e)$,
(2) $\operatorname{Short}_{I}(\mathcal{A}) \subseteq \mathcal{K}(e)$,
(3) if $\operatorname{rk}(e)<\mathrm{d}\left(\mathcal{B}^{\perp}\right)$, then $\operatorname{Short}_{I}(\mathcal{A})=\mathcal{K}(e)$.

Proof.
(1) This holds since $\langle b \circ a, r\rangle_{\mathbb{L}[G]}=\langle b \circ a, c\rangle_{\mathbb{L}[G]}+\langle b \circ a, e\rangle_{\mathbb{L}[G]}$ and, from $b \circ a \in \mathcal{B} \circ \mathcal{A} \subseteq \mathcal{C}^{\perp}$ we have $\langle b \circ a, c\rangle_{\mathbb{L}[G]}=0$.
(2) Let $a \in \operatorname{Short}_{I}(\mathcal{A})$. Then $I \subseteq \operatorname{ker}(a)$ and hence $\operatorname{ker}(a)^{\perp}=\operatorname{Im}(\tau(a)) \subseteq I^{\perp}=\operatorname{ker}(e)$, and hence $e \circ \tau(a)=0$. Thus, from Corollary 28, we get $\langle b \circ a, e\rangle_{\mathbb{L}[G]}=\langle b, e \circ \tau(a)\rangle_{\mathbb{L}[G]}=0$ for any $b \in \mathcal{B}$.
(3) Assume $\operatorname{rk}(e) \leqslant \mathrm{d}\left(\mathcal{B}^{\perp}\right)$, and let us prove that $\mathcal{K}(e) \subseteq \operatorname{Short}_{I}(\mathcal{A})$. If $a \in \mathcal{K}(e)$, then we have $e \circ \tau(a) \in \mathcal{B}^{\perp}$ by definition of $\mathcal{K}(e)$. Since $\operatorname{rk}(e)<\mathrm{d}\left(\mathcal{B}^{\perp}\right)$, necessarily $e \circ \tau(a)=0$ which yields $\operatorname{ker}(a)^{\perp}=\operatorname{Im} \tau(a) \subseteq \operatorname{ker}(e)=I^{\perp}$, or equivalently, $I \subseteq \operatorname{ker}(a)$.

We are now able to introduce error-correcting pairs in the context of codes in $\mathbb{L}[G]$. The definition is identical to the one given by Martínez-Peñas and Pellikaan [22] in the context of rank-metric codes over finite fields.

Definition 32. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathbb{L}[G]$ be three codes. The pair $(\mathcal{A}, \mathcal{B})$ is a $t$-error-correcting pair for $\mathcal{C}$ if the following holds:
(1) $\mathcal{B} \circ \mathcal{A} \subseteq \mathcal{C}^{\perp}$,
(2) $\operatorname{dim}_{\mathbb{L}}(\overline{\mathcal{A}})>t$,
(3) $\mathrm{d}\left(\mathcal{B}^{\perp}\right)>t$,
(4) $\mathrm{d}(\mathcal{A})+\mathrm{d}(\mathcal{C})>|G|$.

Before showing how a $t$-error-correcting pair for a code $\mathcal{C} \subseteq \mathbb{L}[G]$ enables to decode errors of rank up to $t$, we need a couple of technical lemmas.

Lemma 33. Let $\left(a_{i}\right)_{i},\left(b_{i}\right)_{i} \in \mathbb{L}[G]^{M}$ and $\left(c_{i}\right)_{i} \in \mathbb{L}^{M}$. The system of $\mathbb{K}$-linear equations

$$
\left\langle a_{i} \circ x, b_{i}\right\rangle_{\mathbb{L}[G]}=c_{i}, \quad i=1, \ldots, M
$$

with unknown $x \in \mathbb{L}[G]$, can be solved in $O\left(\min (M, N) M N^{4}\right)$ operations over $\mathbb{K}$, where $N=[\mathbb{L}: \mathbb{K}]$.
Proof. Let us fix a basis $\left(\beta_{1}, \ldots, \beta_{N}\right)$ of $\mathbb{L} / \mathbb{K}$. One writes $x=\sum_{g \in G} \sum_{j=1}^{N} x_{g}^{(j)} \beta_{j} g \in \mathbb{L}[G]$, where $x_{g}^{(j)} \in \mathbb{K}$. Then, we have

$$
\left\langle a_{i} \circ x, b_{i}\right\rangle_{\mathbb{L}[G]}=\sum_{g, h \in G} a_{i, g} g\left(x_{h}\right) b_{i, g h}=\sum_{g, h \in G} \sum_{j=1}^{N} a_{i, g} b_{i, g h} g\left(\beta_{j}\right) x_{h}^{(j)} .
$$

If we set $u_{i, h}^{(j)}=\sum_{g \in G} a_{i, g} b_{i, g h} g\left(\beta_{j}\right)$, then we end up with the system of $\mathbb{K}$-linear equations

$$
\sum_{j=1}^{N} \sum_{h \in G} u_{i, h}^{(j)} x_{h}^{(j)}=c_{i}, \quad i=1, \ldots, M
$$

where $u_{i, h}^{(j)} \in \mathbb{L}, c_{i} \in \mathbb{L}$ and $x_{h}^{(j)} \in \mathbb{K}$. Using any basis of $\mathbb{L} / \mathbb{K}$, these $M$ equations can be written as $M N$ equations over $\mathbb{K}$, with $N^{2}$ unknowns $\left\{x_{h}^{(j)}\right\}$. Classical linear algebra algorithms solve this problem in $O\left(\min \left(M N, N^{2}\right) M N^{3}\right)=O\left(\min (M, N) M N^{4}\right)$ operations over $\mathbb{K}$.

Lemma 34. Let $c \in \mathcal{C}$ and $r=c+e \in \mathbb{L}[G]$, where $\operatorname{Supp}(e) \subseteq J$ for some $\mathbb{K}$-vector space $J \subseteq \mathbb{L}$ such that $\operatorname{dim}_{\mathbb{K}}(J)<\mathrm{d}(\mathcal{C})$. Then, $c$ is the unique element in $\mathcal{C}$ such that $\operatorname{Supp}(r-c) \subseteq J$. Moreover, the codeword $c$ can be found by solving a system of linear equations over $\mathbb{K}$, with $O\left(N^{3}\right)$ equations and $O\left(N^{2}\right)$ unknowns in $\mathbb{K}$, where $N=[\mathbb{L}: \mathbb{K}]$.

Proof. Assume $c, c^{\prime} \in \mathcal{C}$ satisfy $\operatorname{Supp}(r-c) \subseteq J$ and $\operatorname{Supp}\left(r-c^{\prime}\right) \subseteq J$. Then,

$$
\operatorname{Supp}\left(c-c^{\prime}\right)=\operatorname{Im}\left(\tau\left(c-c^{\prime}\right)\right) \subseteq \operatorname{Im}\left(\tau\left(r-c^{\prime}\right)\right)+\operatorname{Im}(\tau(r-c)) \subseteq J
$$

If $c \neq c^{\prime}$, then $\operatorname{dim}_{\mathbb{K}} \operatorname{Supp}\left(c-c^{\prime}\right)=\operatorname{rk}\left(c-c^{\prime}\right) \geqslant \mathrm{d}(\mathcal{C})$ and we obtain a contradiction. Thus, $c=c^{\prime}$.

In order to compute $c$, it suffices to solve the system of $\mathbb{K}$-linear equations

$$
\left\{\begin{array}{l}
\left\langle c, u_{i}\right\rangle_{\mathbb{L}[G]}=0 \\
(r-c)\left(w_{k}\right)=0
\end{array}\right.
$$

where $\left\{u_{i}\right\}_{i}$ is an $\mathbb{L}$-basis of $\mathcal{C}^{\perp}$ and $\left\{w_{k}\right\}_{k}$ is a $\mathbb{K}$-basis of $J^{\perp}$. One gets a system of $O\left(N^{2}\right)$ equations of the form given in Lemma 33 which yields the result.

Theorem 35. Assume that $(\mathcal{A}, \mathcal{B})$ is a t-error-correcting pair for $\mathcal{C} \subseteq \mathbb{L}[G]$, where $2 t+1 \leqslant \mathrm{~d}(\mathcal{C})$. Then, there exists a deterministic algorithm Dec which runs in $O\left(N^{7}\right)$ operations over $\mathbb{K}$ given as input $r=c+e$ where $c \in \mathcal{C}$ and $e \in \mathbb{L}[G]$ satisfies $\operatorname{rk}_{\mathbb{L}}(e) \leqslant t$, outputs the codeword $c$.
Proof. Given $r=c+e$, the algorithm first computes $\mathcal{K}(r)$; it consists of solving the system of equations

$$
\left\langle b_{i} \circ x, r\right\rangle_{\mathbb{L}[G]}=\left\langle x, a_{j}\right\rangle_{\mathbb{L}[G]}=0
$$

where the unknown is $x \in \mathbb{L}[G]$ and where $\left\{b_{i}\right\}$ is an $\mathbb{L}$-basis of $\mathcal{B}$ and $\left\{a_{j}\right\}$ is an $\mathbb{L}$-basis of $\mathcal{A}^{\perp}$. This can be done in $O\left(N^{6}\right)$ operations over $\mathbb{K}$ by Lemma 33.

Denote $I=\operatorname{Supp}(e)$. Since $\mathcal{K}(r)=\mathcal{K}(e)=\operatorname{Short}_{I}(\mathcal{A})$ by Proposition 31, one can now take an arbitrary nonzero element $a \in \mathcal{K}(r)$. Define $J=\operatorname{ker}(a)=\operatorname{Supp}(a)^{\perp}$ and notice that $J$ contains $I$. Using the last condition in the definition of error-correcting pairs, we get

$$
\operatorname{dim}_{\mathbb{K}}(J)=|G|-\operatorname{rk}(a) \leqslant|G|-\mathrm{d}(\mathcal{A})<\mathrm{d}(\mathcal{C}) .
$$

Thus, from Lemma 34 one can find $c$ by solving another system of linear equations, requiring $O\left(N^{7}\right)$ operations over $\mathbb{K}$.

## 7. The abelian case: $\theta$-polynomials

In this section, we assume that

$$
G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \cong \mathbb{Z} / n_{1} \mathbb{Z} \times \mathbb{Z} / n_{2} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}
$$

From now, we will also write elements of $\mathbb{L}[G]$ with uppercase characters, e.g. $P \in \mathbb{L}[G]$, since they will be viewed as polynomials.
7.1. Definition. Multivariate linearized polynomials can be defined as follows. Let $\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{m}\right)$ be a vector of generators of $G$. For a given $\boldsymbol{i}=\left(i_{1}, \ldots, i_{m}\right) \in \mathbb{N}^{m}$, we denote by $\boldsymbol{\theta}^{\boldsymbol{i}}$ the element $\theta_{1}^{i_{1}} \circ \cdots \circ \theta_{m}^{i_{m}} \in G$ and we write $|\boldsymbol{i}|:=i_{1}+\cdots+i_{m}$. Since $\theta_{i}^{n_{i}}=\theta_{i}^{0}=\mathrm{Id}$, we can actually consider only tuples $\boldsymbol{i}$ belonging to $\Delta(\boldsymbol{n}):=\Delta\left(n_{1}\right) \times \cdots \times \Delta\left(n_{m}\right)$, where $\Delta(t):=\{0,1, \ldots, t-1\}$ and $\boldsymbol{n}:=\left(n_{1}, \ldots, n_{m}\right)$. In this way, we have that $G=\left\{\boldsymbol{\theta}^{\boldsymbol{i}} \mid \boldsymbol{i} \in \Delta(\boldsymbol{n})\right\}$ and hence, every $P \in \mathbb{L}[G]$ has a unique representation as

$$
P=\sum_{i \in \Delta(\boldsymbol{n})} b_{i} \boldsymbol{\theta}^{i}
$$

We also define $1:=(1, \ldots, 1) \in \mathbb{N}^{m}$. This will be used in Sections 7 and 8 .
Definition 36. A $\boldsymbol{\theta}$-polynomial is an element $P=\sum_{\boldsymbol{i} \in \Delta(\boldsymbol{n})} b_{\boldsymbol{i}} \boldsymbol{\theta}^{\boldsymbol{i}}$ belonging to the skew group algebra $\mathbb{L}[G]=\mathbb{L}\left[\theta_{1}, \ldots, \theta_{m}\right]$. If $P$ is non-zero, then the $\boldsymbol{\theta}$-degree of $P$ is the quantity

$$
\operatorname{deg}_{\boldsymbol{\theta}}(P):=\max \left\{|\boldsymbol{i}| \mid \boldsymbol{i} \in \Delta(\boldsymbol{n}), b_{\boldsymbol{i}} \neq 0\right\}
$$

Observe that $\boldsymbol{\theta}$-polynomials are just elements of $\mathbb{L}[G]$, endowed with a notion of degree. This notion will be useful for defining $\boldsymbol{\theta}$-Reed-Muller codes and bounding their minimum distance.
7.2. Alon-Füredi Theorem and Schwartz-Zippel Lemma for $\theta$-polynomials. In this section we show that we have an analogue of the celebrated Alon-Füredi Theorem [2, Theorem 5] and Schwartz-Zippel Lemma [30, Corollary 1].

Let $m$ and $N$ be positive integers with $m \leqslant N$ and let $\boldsymbol{a}=\left(a_{1}, \ldots, a_{m}\right) \in \mathbb{N}^{m}$ be a vector of positive integers. Define the integer $f(\boldsymbol{a}, N)$ as

$$
f(\boldsymbol{a}, N):=\min \left\{\prod_{i=1}^{m} b_{i} \mid \boldsymbol{b}-\mathbf{1} \in \Delta(\boldsymbol{a}) \text { and }|\boldsymbol{b}|=N\right\} .
$$

Lemma 37. [8, Lemma 2.2] Suppose $a_{1} \geqslant a_{2} \geqslant \ldots \geqslant a_{m}$. Let $N \in \mathbb{N}$ be such that $N-m=\sum_{i=1}^{s}\left(a_{i}-1\right)+\ell$ for some $s \in\{0, \ldots, m\}$ and $\ell$ such that $0 \leqslant \ell<a_{s+1}$. Then

$$
\begin{equation*}
f(\boldsymbol{a}, N)=(\ell+1) \prod_{i=1}^{s} a_{s} \tag{11}
\end{equation*}
$$

We recall now the classical versions of Alon-Füredi Theorem and Schwartz-Zippel Lemma. For this purpose, we introduce the following notation. Let $\mathbb{F}$ be a field and let $S \subseteq \mathbb{F}^{m}$ be a fixed set. Moreover, let $p \in \mathbb{F}\left[x_{1}, \ldots, x_{m}\right]$ be a multivariate polynomial. We denote by $U_{S}(p)$ and $V_{S}(p)$ the set of non-zeros and of zeros, respectively, of $p$ in $S$, that is

$$
U_{S}(p):=\{\boldsymbol{u} \in S \mid p(\boldsymbol{u}) \neq 0\}, \quad V_{S}(p):=\{\boldsymbol{v} \in S \mid p(\boldsymbol{v})=0\}
$$

Theorem 38 (Alon-Füredi Theorem). [2, Theorem 5] Let $S=S_{1} \times \cdots \times S_{m} \subseteq \mathbb{F}^{m}$ be a finite grid with $S_{i} \subseteq \mathbb{F}$ and $\left|S_{i}\right|=n_{i}$, where $n_{1} \geqslant n_{2} \geqslant \cdots \geqslant n_{m} \geqslant 1$. Let $p \in \mathbb{F}\left[x_{1}, \ldots, x_{m}\right]$ be a polynomial that is not identically 0 on $S$, and let $\bar{p}$ be the polynomial $p$ modulo the ideal $\left(p_{1}\left(x_{1}\right), \ldots, p_{m}\left(x_{m}\right)\right)$, where $p_{i}\left(x_{i}\right)=\prod_{s \in S_{i}}\left(x_{i}-s\right)$. Then

$$
\left|U_{S}(p)\right| \geqslant\left(n_{s}-\ell\right) \prod_{i=1}^{s-1} n_{i}
$$

where $\ell$ and $s$ are the unique integers satisfying $\operatorname{deg} \bar{p}=\sum_{i=s+1}^{k}\left(n_{i}-1\right)+\ell$, with $1 \leqslant s \leqslant k$ and $1 \leqslant \ell<n_{s}$.
Lemma 39 (Schwartz-Zippel Lemma). [30, Corollary 1]. Let $S=S_{1} \times \cdots \times S_{m} \subseteq \mathbb{F}^{m}$ be a finite grid with $S_{i} \subseteq \mathbb{F}$ and $\left|S_{i}\right| \geqslant 1$ for each $i \in\{1, \ldots, m\}$. Let $p \in \mathbb{F}\left[x_{1}, \ldots, x_{k}\right]$ be a nonzero polynomial. Then,

$$
\left|V_{S}(p)\right| \leqslant \frac{\operatorname{deg}(p)}{\min \left\{\left|S_{1}\right|, \ldots,\left|S_{m}\right|\right\}}|S|
$$

At this point, we are ready to state the Alon-Füredi Theorem for $\boldsymbol{\theta}$-polynomials, which is the central result of this section.

Theorem 40 (Alon-Füredi Theorem for $\boldsymbol{\theta}$-polynomials). Let $\boldsymbol{n}=\left(n_{1}, \ldots, n_{m}\right)$ be an m-tuple of nonnegative integers such that $n_{1} \geqslant n_{2} \geqslant \cdots \geqslant n_{m} \geqslant 2$ and let $G=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \simeq \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}$ be the Galois group of a field extension $\mathbb{L} / \mathbb{K}$. Moreover, let $P \in \mathbb{L}[G]$ be nonzero. Then

$$
\operatorname{rk}(P) \geqslant\left(n_{s}-\ell\right) \prod_{i=1}^{s-1} n_{i}
$$

where $\ell$ and $s$ are the unique integers satisfying $\operatorname{deg}_{\boldsymbol{\theta}}(P)=\sum_{i=s+1}^{m}\left(n_{i}-1\right)+\ell$, with $0 \leqslant \ell<n_{s}$.
Proof. Let $P=\sum_{\boldsymbol{i} \in \Delta(\boldsymbol{n})} b_{\boldsymbol{i}} \boldsymbol{\theta}^{\boldsymbol{i}}$ be a $\boldsymbol{\theta}$-polynomial with $\operatorname{deg}_{\boldsymbol{\theta}}(P)=\sum_{i=s+1}^{m}\left(n_{i}-1\right)+\ell$. Our goal is to find an ordering of $G$ for which $\operatorname{rk}(P)=\operatorname{rk}_{\mathbb{L}}\left(D_{G}(P)\right)$ can be easily bounded. Let us fix a monomial order $\prec$ on $\mathbb{N}^{m}$, which is a refinement of the total degree, that is, for each finite set $\mathcal{S} \subset \mathbb{N}^{m}$, the maximal element in $\mathcal{S}$ with respect to $\prec$ has also maximal total degree among the elements of $\mathcal{S}$. We write the group

$$
G=\left\{\theta^{i^{(1)}}, \ldots, \theta^{i^{(N)}}\right\}
$$

according to the order $\prec$ restricted to $\Delta(\boldsymbol{n})=\left\{\boldsymbol{i}^{(1)}, \ldots, \boldsymbol{i}^{(N)}\right\}$. We also denote $\mathrm{lt}_{\prec}(P)=\boldsymbol{\theta}^{\boldsymbol{i}^{(s)}}$ the leading term of $P$, and $\mathrm{lc}_{\prec}(P)=b_{\boldsymbol{i}^{(s)}}$ its leading coefficient, for some $\boldsymbol{i}^{(s)}=\left(u_{1}, \ldots, u_{m}\right) \in \Delta(\boldsymbol{n})$.

Consider the $G$-Dickson matrix $D_{G}(P)$ with respect to this order on $G$. In the first column of $D_{G}(P)$, the $(s, 1)$-entry is $b_{\boldsymbol{i}^{(s)}} \neq 0$, and the $(j, 1)$-entry is 0 for every $s<j \leqslant N$. Let us define

$$
\mathcal{T}:=\left\{\boldsymbol{\theta}^{\boldsymbol{v}} \mid v_{i}<n_{i}-u_{i}\right\} \subseteq G \quad \text { and } \quad t:=|\mathcal{T}|=\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)
$$

We also order $\mathcal{T}=\left\{\boldsymbol{\theta}^{\boldsymbol{j}^{(1)}}, \ldots, \boldsymbol{\theta}^{\boldsymbol{j}^{(t)}}\right\}$ according to $\prec$, i.e. $\boldsymbol{\theta}^{\boldsymbol{j}^{(1)}} \prec \cdots \prec \boldsymbol{\theta}^{\boldsymbol{j}^{(t)}}$.
Let us now fix $i \in\{1, \ldots, t\}$. By definition, the column of $D_{G}(P)$ corresponding to $\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}} \in \mathcal{T}$ is given by the coordinates of the $\boldsymbol{\theta}$-polynomial $\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}} \circ P$ in the basis $\left\{\boldsymbol{\theta}^{\boldsymbol{i}^{(1)}}, \ldots, \boldsymbol{\theta}^{\boldsymbol{i}^{(N)}}\right\}$. We have that $\operatorname{lt}_{\prec}\left(\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}} \circ P\right)=\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}+\boldsymbol{i}^{(s)}}=\boldsymbol{\theta}^{\boldsymbol{i}^{\left(s_{i}\right)}}$, for a suitable positive integer $s_{i} \leqslant N$. Moreover, by definition of
a monomial order, we have $s=s_{1}<s_{2}<\cdots<s_{t} \leqslant N$, and in the column corresponding to $\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}}$, all the elements with row index $j$ for $s_{i}<j \leqslant N$ are equal to 0 . Furthermore, the element with row index $s_{i}$ equals $\boldsymbol{\theta}^{\boldsymbol{j}^{(i)}}\left(b_{\boldsymbol{i}^{(s)}}\right) \neq 0$. Therefore, the submatrix $D_{\mathcal{T}}$ of $D_{G}(P)$ obtained by taking the columns corresponding to $\mathcal{T}$ and the rows $s_{1}, \ldots, s_{t}$, is an upper triangular $t \times t$ matrix of the form

$$
D_{\mathcal{T}}=\left(\begin{array}{ccccc}
\boldsymbol{\theta}^{\boldsymbol{j}^{(1)}}\left(b_{\boldsymbol{i}^{(s)}}\right) & & & & \\
& \boldsymbol{\theta}^{\boldsymbol{j}^{(2)}}\left(b_{\boldsymbol{i}^{(s)}}\right) & & (*) & \\
& & \ddots & & \\
& (0) & & \boldsymbol{\theta}^{\boldsymbol{j}^{(t-1)}}\left(b_{\boldsymbol{i}^{(s)}}\right) & \\
& & & & \boldsymbol{\theta}^{\boldsymbol{j}^{(t)}\left(b_{\left.\boldsymbol{i}^{(s)}\right)}\right.}
\end{array}\right)
$$

with nonzero elements on the diagonal. Hence, by Theorem 23, we have

$$
\operatorname{rk}(P)=\operatorname{rk}_{\mathbb{L}}\left(D_{G}(P)\right) \geqslant \operatorname{rk}_{\mathbb{L}}\left(D_{\mathcal{T}}\right)=|\mathcal{T}|=\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)
$$

We conclude the proof by observing that

$$
\begin{aligned}
f\left(\boldsymbol{n},\left(\sum_{i=1}^{m} n_{i}\right)-\operatorname{deg}_{\boldsymbol{\theta}}(P)\right) & =\min \left\{\prod_{i=1}^{m} v_{i}\left|\boldsymbol{v}-\mathbf{1} \in \Delta(\boldsymbol{n}),|\boldsymbol{v}|=\left(\sum_{i} n_{i}\right)-\operatorname{deg}_{\boldsymbol{\theta}}(P)\right\}\right. \\
& =\min \left\{\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)\left|\boldsymbol{u} \in \Delta(\boldsymbol{n}),|\boldsymbol{u}|=\operatorname{deg}_{\boldsymbol{\theta}}(P)\right\} .\right.
\end{aligned}
$$

and from Lemma 37 we get the desired result.

Remark 41. From the proof of Theorem 40 one can easily see that the result can be refined if we make further assumptions on the element $P \in \mathbb{L}[G]$. Indeed, if there exists a monomial order $\prec^{\prime}$ on $\mathbb{N}^{m}$ such that $\operatorname{lc}_{\prec^{\prime}}(P)=\boldsymbol{\theta}^{\boldsymbol{u}}$ with $|\boldsymbol{u}|<\operatorname{deg}_{\boldsymbol{\theta}}(P)$, using the same proof with the monomial order $\prec^{\prime}$, one gets that

$$
\operatorname{rk}(P) \geqslant f\left(\boldsymbol{n},\left(\sum_{i=1}^{m} n_{i}\right)-|\boldsymbol{u}|\right)
$$

In the Hamming metric, the effects of the choice of monomial orders for designing codes with better minimum distance have been intensively studied by Geil and Thomsen in [13].

Actually, the Alon-Füredi Theorem for $\boldsymbol{\theta}$-polynomials allows to prove an analogue in the rank metric of the well-known Schwartz-Zippel lemma. This can be stated as follows.

Corollary 42 (Schwartz-Zippel Lemma for $\boldsymbol{\theta}$-polynomials). Let $\boldsymbol{n}=\left(n_{1}, \ldots, n_{m}\right)$ be an m-tuple of nonnegative integers, let $G=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \simeq \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}$ be the Galois group of a field extension $\mathbb{L} / \mathbb{K}$, and let $P \in \mathbb{L}[G]$. Then, we have:

$$
\operatorname{dim}_{\mathbb{K}} \operatorname{ker}(P) \leqslant \frac{\operatorname{deg}_{\boldsymbol{\theta}}(P)}{\min \left\{n_{1}, \ldots, n_{m}\right\}} \cdot \prod_{i=1}^{m} n_{i}
$$

Proof. Without loss of generality, we can assume $m_{1} \geqslant n_{2} \geqslant \cdots \geqslant n_{m} \geqslant 2$, so that $\min \left\{n_{1}, \ldots, n_{m}\right\}=n_{m}$. If $\operatorname{deg}_{\boldsymbol{\theta}}(P) \geqslant n_{m}$ there is nothing to prove. Hence, suppose $\operatorname{deg}_{\theta}<n_{m}$. Using Theorem 40 we obtain

$$
\operatorname{dim}_{\mathbb{K}} \operatorname{ker}(P)=\prod_{i=1}^{m} n_{i}-\operatorname{rk}(P) \geqslant \prod_{i=1}^{m} n_{i}-\left(n_{m}-\operatorname{deg}_{\boldsymbol{\theta}}(P)\right) \prod_{i=1}^{m-1} n_{i}=\operatorname{deg}_{\boldsymbol{\theta}}(P) \prod_{i=1}^{m-1} n_{i}
$$

## 8. $\theta$-REED-Muller codes

In this section, we introduce and develop the theory of $\boldsymbol{\theta}$-Reed-Muller codes. They can be seen either as the counterparts of Reed-Muller codes in the rank metric, or as the multivariate version of Gabidulin codes.
8.1. Definition. We assume to work in the setting described in Section 7.

Definition 43. Let $\mathbb{L} / \mathbb{K}$ be a Galois extension such that $G:=\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \cong \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times$ $\mathbb{Z} / n_{m} \mathbb{Z}$ and let $r \in \mathbb{N}$ such that $r \leqslant \sum_{i}\left(n_{i}-1\right)$. The $\boldsymbol{\theta}$-Reed-Muller code of order $r$ and type $\boldsymbol{n}$ is

$$
\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n}):=\left\{P \in \mathbb{L}[G] \mid \operatorname{deg}_{\boldsymbol{\theta}}(P) \leqslant r\right\} \subseteq \mathbb{L}[G]
$$

Remark 44. The definition of $\boldsymbol{\theta}$-Reed-Muller codes depends on the choice of generators $\boldsymbol{\theta}$ of the Galois group $G$. This is somehow similar to the case of (generalized) Gabidulin codes.

Remark 45. Given a basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$, the vectorial version of $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is then

$$
\operatorname{RM}_{\boldsymbol{\theta}, \mathcal{B}}(r, \boldsymbol{n}):=\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})(\mathcal{B})=\left\{\operatorname{ev}_{\mathcal{B}}(P) \mid P \in \mathbb{L}[G], \operatorname{deg}_{\boldsymbol{\theta}}(P) \leqslant r\right\} \subseteq \mathbb{L}^{N}
$$

where $\operatorname{ev}_{\mathcal{B}}(P)$ is the evaluation vector as defined in (7).
Example 46. Let $\mathbb{K}=\mathbb{Q}(\zeta)$ where $\zeta^{2}+\zeta+1=0$. Consider $\mathbb{L} / \mathbb{K}$ a Galois extension of degree 6 given by $\mathbb{L}=\mathbb{K}(\sqrt{p}, \sqrt[3]{q})$, where $p$ and $q$ are two distinct primes. Then

$$
\mathcal{B}=\left(1, \sqrt{p}, \sqrt[3]{q}, \sqrt{p} \sqrt[3]{q}, \sqrt[3]{q^{2}}, \sqrt{p} \sqrt[3]{q^{2}}\right) \in \mathbb{L}^{6}
$$

is an ordered $\mathbb{K}$-basis of $\mathbb{L}$. Moreover we have $G=\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\langle\theta_{1}, \theta_{2}\right\rangle$ where

$$
\theta_{1}:\left\{\begin{aligned}
\sqrt{p} & \mapsto(-1) \cdot \sqrt{p} \\
\sqrt[3]{q} & \mapsto 1 \cdot \sqrt[3]{q}
\end{aligned} \quad \text { and } \quad \theta_{2}:\left\{\begin{array}{rl}
\sqrt{p} & \mapsto 1 \cdot \sqrt{p} \\
\sqrt[3]{q} & \mapsto \zeta \cdot \sqrt[3]{q}
\end{array} .\right.\right.
$$

We observe that $\theta_{1}^{2}=\theta_{2}^{3}=\operatorname{Id}$, hence $\boldsymbol{n}=\left(n_{1}, n_{2}\right)=(2,3)$ and $N=|\operatorname{Gal}(\mathbb{L} / \mathbb{K})|=n_{1} n_{2}=6$.
Let now $r=1$. The $\left(\theta_{1}, \theta_{2}\right)$-Reed-Muller code of order $r$ is

$$
\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})=\left\{a \cdot \operatorname{Id}+b \cdot \theta_{1}+c \cdot \theta_{2} \mid a, b, c \in \mathbb{L}\right\} \subseteq \mathbb{L}[G]
$$

Its vectorial version with respect to the basis $\mathcal{B}=\left(b_{1}, \ldots, b_{6}\right)$ defined above, has the following generator matrix:

$$
\left(\begin{array}{cccccc}
b_{1} & b_{2} & b_{3} & b_{4} & b_{5} & b_{6} \\
b_{1} & -b_{2} & b_{3} & -b_{4} & b_{5} & -b_{6} \\
b_{1} & b_{2} & \zeta b_{3} & \zeta b_{4} & \zeta^{2} b_{5} & \zeta^{2} b_{6}
\end{array}\right)
$$

8.2. Parameters of $\theta$-Reed-Muller codes. We now compute the dimension and the minimum rank distance of $\boldsymbol{\theta}$-Reed-Muller codes.
Proposition 47. The dimension of $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is equal to the cardinality of the set $\{\boldsymbol{i} \in \Delta(\boldsymbol{n})||\boldsymbol{i}| \leqslant r\}$, that in turn is equal to

$$
k(r, \boldsymbol{n})=\sum_{\ell=0}^{r} c(\ell, \boldsymbol{n})=\sum_{\ell=0}^{r}\left[z^{\ell}\right] \prod_{j=1}^{m}\left(\frac{1-z^{n_{j}}}{1-z}\right)
$$

where $c(\ell, \boldsymbol{n})$ of the integer $\ell$ in at most $m$ parts in which the $j$-th part is at most $n_{j}-1$ and $\left[z^{\ell}\right] p(z)$ denotes the coefficient of $z^{\ell}$ in the polynomial $p(z)$.
Proof. By definition a set of generators for the $\boldsymbol{\theta}$-Reed-Muller code is given by the set $\left\{\boldsymbol{\theta}^{\boldsymbol{i}}|\boldsymbol{i} \in \Delta(\boldsymbol{n}),|\boldsymbol{i}| \leqslant\right.$ $r\}$. Moreover these $\boldsymbol{\theta}$-monomials are linearly independent over $\mathbb{L}$, by Artin's theorem. Therefore the dimension of the code is equal to the cardinality $k(r, \boldsymbol{n})$ of the set $\{\boldsymbol{i} \in \Delta(\boldsymbol{n})||\boldsymbol{i}| \leqslant r\}$. Let $c(\ell, \boldsymbol{n})$ denote the number of weak compositions of the integer $\ell$ in at most $m$ parts in which the $j$-th part is at most $n_{j}-1$. Then,

$$
k(r, \boldsymbol{n})=\sum_{\ell=0}^{r} c(\ell, \boldsymbol{n})
$$

Since it is well-known that $c(\ell, \boldsymbol{n})=\left[z^{\ell}\right] \prod_{j=1}^{m}\left(\frac{1-z^{n_{j}}}{1-z}\right)$, we can conclude.
For every $i \in\{1, \ldots, m\}$ we also consider the subgroup $G_{i}=\left\langle\theta_{j} \mid j \in\{1, \ldots, m\} \backslash\{i\}\right\rangle$, and the corresponding fixed field

$$
\mathbb{L}_{i}:=\mathbb{L}^{G_{i}}=\left\{a \in \mathbb{L} \mid \sigma(a)=a, \text { for every } \sigma \in G_{i}\right\}
$$

Before determining the minimum distance of $\boldsymbol{\theta}$-Reed-Muller codes, we define an object of particular interest in the case of cyclic extensions.

Proposition 48. [4, Theorem 2] Let $\mathbb{L} / \mathbb{K}$ be a cyclic Galois extension of degree $n$, with Galois group $G=\langle\theta\rangle$. Let $V:=\operatorname{Span}_{\mathbb{K}}\left\{v_{1}, \ldots, v_{r}\right\} \subseteq \mathbb{L}$ be a $\mathbb{K}$-subspace of dimension $r \geqslant 0$. Then, there exists $a$ unique monic $\theta$-polynomial $P_{V} \in \mathbb{L}[\theta]$ of $\bar{\theta}$-degree $r$ such that $P_{V}(V)=\{0\}$. Moreover, the polynomial $P_{V}$ is defined by induction as:

$$
P_{V}= \begin{cases}I d & \text { if } r=0 \\ \left(\theta-\frac{\theta\left(P_{V_{1}}\left(v_{r}\right)\right)}{P_{V_{1}}\left(v_{r}\right)}\right) \circ P_{V_{1}} & \text { if } r \geqslant 1\end{cases}
$$

where $V_{1}:=\operatorname{Span}_{\mathbb{K}}\left\{v_{1}, \ldots, v_{r-1}\right\}$.
Proof. The existence and uniqueness follows from the fact that $\mathbb{L}[\theta]$ is a left Euclidean domain. In particular, the left ideal $I:=\{P \in \mathbb{L}[\theta] \mid P(v)=0$ for every $v \in V\}$ is principal. In addition, $I$ contains $P_{V}$. Moreover, it is well-known that the dimension of the kernel of a $\theta$-polynomial is bounded from above by its $\theta$-degree. This can be deduced, for instance, from Corollary 42. Therefore, $P_{V}$ is a monic element of $I$ of the least possible degree. Hence it is a generator of $I$. Moreover, the polynomial defined by the recursive formula has $\theta$-degree $r$, is monic and it annihilates the subspace $V$.

The polynomial $P_{V}$ defined by Proposition 48 is called the annihilator polynomial of the subspace $V$. In the finite field case, this coincides with the notion of annihilator or subspace polynomial, which is a linearized polynomial of degree $q^{r}$ whose roots are exactly the elements of an $r$-dimensional $\mathbb{F}_{q}$-subspace of $\mathbb{F}_{q^{n}}$.

Theorem 49. Let $r$ be a positive integer and $\boldsymbol{n}=\left(n_{1}, \ldots, n_{m}\right) \in \mathbb{N}^{m}$ be a vector such that $n_{1} \geqslant n_{2} \geqslant$ $\cdots \geqslant n_{m} \geqslant 2$. Then the minimum rank distance of the code $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is equal to

$$
d(r, \boldsymbol{n})=\min \left\{\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)\left|\boldsymbol{u}=\left(u_{1}, \ldots, u_{m}\right) \in \Delta(\boldsymbol{n}),|\boldsymbol{u}| \leqslant r\right\}\right.
$$

In particular, $d(r, \boldsymbol{n})=1$ if $r \geqslant \sum_{i=1}^{m}\left(n_{i}-1\right)$, and otherwise

$$
d(r, \boldsymbol{n})=\left(n_{s}-\ell\right) \prod_{i=1}^{s-1} n_{i}
$$

where $\ell$ and $s$ are the unique integers satisfying $r=\sum_{i=s+1}^{m}\left(n_{i}-1\right)+\ell$, with $0 \leqslant \ell<n_{s}$.
Proof. Lower bound. First, it is easy to observe that the minimum is met for an element $\boldsymbol{u}$ such that $|\boldsymbol{u}|=r$. At this point, the lower bound directly follows from Theorem 40, since the minimum distance is the minimum rank of $P$ among all the nonzero $P \in \mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ of $\boldsymbol{\theta}$-degree equal to $r$.
Upper bound. Let now $r \geqslant 1$, and $\ell$, $s$ the unique integers satisfying $r=\sum_{i=s+1}^{m}\left(n_{i}-1\right)+\ell$, with $1 \leqslant \ell<n_{s}$. For every $i \in\{s+1, \ldots, m\}$, choose a $\mathbb{K}$-subspace $V_{i}$ of $\mathbb{L}_{i}$ with dimension $n_{i}-1$ that does not contain $\mathbb{K}$, that is, $V_{i} \cap \mathbb{K}=\{0\}$. Moreover, choose $V_{s}$ to be any $\mathbb{K}$-subspace of $\mathbb{L}_{s}$ of dimension $\ell$ that does not contain $\mathbb{K}$. For each $i \in\{s, \ldots, m\}$, let $P_{i} \in \mathbb{L}_{i}\left[\theta_{i}\right]$ to be the annihilator $\theta_{i}$-polynomial of $V_{i}$. Observe that if $j \neq i$, then for every $x \in \mathbb{L}_{j}$ we have $P_{i}(x)=P_{i}(1) x$. Thus, define $\tilde{P}_{i}:=P_{i}(1)^{-1} P_{i}$, and consider the $\boldsymbol{\theta}$-polynomial $\tilde{P}:=\tilde{P}_{s} \circ \tilde{P}_{s+1} \circ \cdots \circ \tilde{P}_{m}$. We then have $\tilde{P}\left(V_{i}\right)=0$ for every $i \geqslant s$.

Given $j \in\{s, \ldots, m\}$ let us define

$$
\mathcal{U}_{j}:=\mathbb{L}_{1} \cdots \mathbb{L}_{j-1} V_{j}=\mathbb{L}_{(j-1)} V_{j} \subseteq \mathbb{L}
$$

where for two $\mathbb{K}$-subspaces $W, W^{\prime}$ of $\mathbb{L}$, we define $W W^{\prime}:=\operatorname{Span}_{\mathbb{K}}\left\{w w^{\prime} \mid w \in W, w^{\prime} \in W^{\prime}\right\}$, and $\mathbb{L}_{(j-1)}$ denotes the compositum of $\mathbb{L}_{1}, \ldots, \mathbb{L}_{j-1}$. Then, we see that for every $j \geqslant s$ we have $\operatorname{ker}(\tilde{P}) \supseteq \mathcal{U}_{j}$ and $\mathcal{U}_{j} \cap\left(\mathcal{U}_{j+1}+\cdots+\mathcal{U}_{m}\right)=\{0\}$. Therefore,

$$
\operatorname{rk}(\tilde{P})=\prod_{i=1}^{m} n_{i}-\operatorname{dim}(\operatorname{ker}(\tilde{P})) \leqslant \prod_{i=1}^{m} n_{i}-\sum_{j=s}^{m} \operatorname{dim}\left(\mathcal{U}_{j}\right)
$$

Since $\operatorname{dim}\left(\mathcal{U}_{s}\right)=\ell \prod_{i=1}^{s-1} n_{i}$ and $\operatorname{dim}\left(\mathcal{U}_{j}\right)=\left(n_{j}-1\right) \prod_{i=1}^{j-1} n_{i}$ for $j \geqslant s+1$, this yields

$$
\operatorname{rk}(\tilde{P}) \leqslant \prod_{i=1}^{m} n_{i}-\sum_{j=s+1}^{m}\left(n_{j}-1\right) \prod_{i=1}^{j-1} n_{i}-\ell \prod_{i=1}^{s-1} n_{i}=\left(n_{s}-\ell\right) \prod_{j=1}^{s-1} n_{j}
$$

from which we get the desired upper bound.
8.3. Duality. In this section we study the duality properties of $\boldsymbol{\theta}$-Reed-Muller codes, showing that such a family is essentially closed under duality (see Proposition 50). For this purpose, let us denote $\boldsymbol{\theta}_{\text {inv }}=$ $\left(\theta_{1}^{-1}, \ldots, \theta_{m}^{-1}\right)$. It is clear that $\boldsymbol{\theta}_{\text {inv }}$ is also a system of generators for the Galois group $G$. Let us also denote $\boldsymbol{\theta}^{-1}=\theta_{1}^{-1} \circ \cdots \circ \theta_{m}^{-1}$.
Proposition 50. Let $p=\sum_{j=1}^{m}\left(n_{j}-1\right)$. Then we have:

$$
\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})^{\perp}=\mathrm{RM}_{\boldsymbol{\theta}_{\mathrm{inv}}}(p-r-1, \boldsymbol{n}) \circ \boldsymbol{\theta}^{-\mathbf{1}}=\boldsymbol{\theta}^{-\mathbf{1}} \circ \mathrm{RM}_{\boldsymbol{\theta}_{\mathrm{inv}}}(p-r-1, \boldsymbol{n}) .
$$

Proof. It is clear that the dual of $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is the $\mathbb{L}$-span of the set

$$
\left\{\boldsymbol{\theta}^{i} \mid \boldsymbol{i} \in \Delta(\boldsymbol{n}), i_{1}+\cdots+i_{m}>r\right\} .
$$

Observe that we can write $\boldsymbol{\theta}^{\boldsymbol{i}}=\left(\boldsymbol{\theta}^{-\mathbf{1}}\right)^{\boldsymbol{n - 1 - \boldsymbol { i }}} \circ \boldsymbol{\theta}^{\mathbf{- 1}}=\boldsymbol{\theta}^{-\mathbf{1}} \circ\left(\boldsymbol{\theta}^{-\mathbf{1}}\right)^{\boldsymbol{n - 1 - i}}$. Moreover, $\boldsymbol{i} \in \Delta(\boldsymbol{n})$ with $\sum_{j=1}^{m} i_{j}>r$ if and only if $\boldsymbol{n}-\mathbf{1}-\boldsymbol{i} \in \Delta(\boldsymbol{n})$ with $\sum_{j=1}^{m} n_{j}-1-i_{j} \leqslant p-r-1$. This concludes the proof.

Proposition 50 can be translated in the vector setting as follows.
Corollary 51. Let $\mathcal{B}$ be a basis of $\mathbb{L} / \mathbb{K}$ and $p=\sum_{j=1}^{m}\left(n_{j}-1\right)$. Then we have:

$$
\left(\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})(\mathcal{B})\right)^{\perp}=\mathrm{RM}_{\boldsymbol{\theta}_{\text {inv }}}(p-r-1, \boldsymbol{n})\left(\boldsymbol{\theta}^{-\mathbf{1}}\left(\mathcal{B}^{*}\right)\right) .
$$

Proof. Combining Theorem 27 and Proposition 50, we get that

$$
\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})(\mathcal{B})^{\perp}=\left(\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})^{\perp}\right)\left(\mathcal{B}^{*}\right)=\left(\mathrm{RM}_{\boldsymbol{\theta}_{\text {inv }}}(p-r-1, \boldsymbol{n}) \circ \boldsymbol{\theta}^{-\mathbf{1}}\right)\left(\mathcal{B}^{*}\right)
$$

At this point one can observe that for every $P \in \mathbb{L}[G]$, it holds $\operatorname{ev}_{\mathcal{B}^{*}}\left(P \circ \boldsymbol{\theta}^{-\mathbf{1}}\right)=\operatorname{ev}_{\boldsymbol{\theta}^{-1}\left(\mathcal{B}^{*}\right)}(P)$, giving

$$
\left(\operatorname{RM}_{\boldsymbol{\theta}_{\text {inv }}}(p-r-1, \boldsymbol{n}) \circ \boldsymbol{\theta}^{-\mathbf{1}}\right)\left(\mathcal{B}^{*}\right)=\operatorname{RM}_{\boldsymbol{\theta}_{\text {inv }}}(p-r-1, \boldsymbol{n})\left(\boldsymbol{\theta}^{-\mathbf{1}}\left(\mathcal{B}^{*}\right)\right) .
$$

8.4. Decoding $\theta$-Reed-Muller codes. In this section, we shortly explain how error-correcting pairs allow to decode $\boldsymbol{\theta}$-Reed-Muller codes up to some error weight. The decoding capability is however nonoptimal, and we leave open the question of the decoding $\boldsymbol{\theta}$-Reed-Muller codes up to half their minimum distance.

The key point is to notice the following.
Lemma 52. Let $r, r^{\prime} \geqslant 0$ such that $r+r^{\prime} \leqslant p:=\sum_{i=1}^{m}\left(n_{i}-1\right)$. Then we have

$$
\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n}) \circ \mathrm{RM}_{\boldsymbol{\theta}}\left(r^{\prime}, \boldsymbol{n}\right)=\mathrm{RM}_{\boldsymbol{\theta}}\left(r+r^{\prime}, \boldsymbol{n}\right) .
$$

Proof. This is clear since $\operatorname{deg}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^{\boldsymbol{i}} \boldsymbol{\theta}^{\boldsymbol{j}}\right) \leqslant r+r^{\prime}$ whenever $\operatorname{deg}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^{\boldsymbol{i}}\right) \leqslant r$ and $\operatorname{deg}_{\boldsymbol{\theta}}\left(\boldsymbol{\theta}^{\boldsymbol{j}}\right) \leqslant r^{\prime}$.
We recall that $d(r, \boldsymbol{n})$ and $k(r, \boldsymbol{n})$ respectively represent the minimum distance and the dimension of $\boldsymbol{\theta}$-Reed-Muller. Their definition are given in Theorem 49 and Proposition 47.
Proposition 53. Let $r, t \geqslant 0$ and assume that $2 t+1 \leqslant d(r, \boldsymbol{n})$ Set $N=\prod_{i=1}^{m} n_{i}$ and $p:=\sum_{i=1}^{m}\left(n_{i}-1\right)$. Let $\mathcal{A}=\mathrm{RM}_{\boldsymbol{\theta}_{\text {inv }}}(a, \boldsymbol{n})$ and $\mathcal{B}=\mathrm{RM}_{\boldsymbol{\theta}_{\text {inv }}}(b, \boldsymbol{n})$ be such that
(1) $a+b \leqslant p-r-1$,
(2) $k(a, \boldsymbol{n})>t$,
(3) $d(p-1-b, \boldsymbol{n})>t$,
(4) $d(a, \boldsymbol{n})+d(r, \boldsymbol{n})>N$.

Then, $(\mathcal{A}, \mathcal{B})$ is an error-correcting pair for $\mathcal{C}=\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n}) \circ \boldsymbol{\theta}^{-\mathbf{1}}$.
Proof. It follows from the definition of error-correcting pairs and the duality results from Proposition 51.

A natural question is to compute the maximum decoding radius $t$ one can get with a $t$-error correcting pair for a given code $\mathcal{C}_{r}=\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$. In the following example, we initiate this study by considering the simplest non-trivial case $\boldsymbol{n}=(n, n), n \geqslant 2$.

Example 54. Let us fix $\boldsymbol{n}=(n, n)$ and $r \leqslant 2 n-3$. For clarity let us also use the simpler notation $d(x):=d(x, \boldsymbol{n})$ and $k(x):=k(x, \boldsymbol{n})$. The goal is to find the maximum $t$ for which there exists a pair $(a, b)$ such that $\left(\operatorname{RM}_{\boldsymbol{\theta}_{\text {inv }}}(a, \boldsymbol{n}), \mathrm{RM}_{\boldsymbol{\theta}_{\text {inv }}}(b, \boldsymbol{n})\right)$ is a $t$-error-correcting pair for $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$. In other words, we look for

$$
t_{\max }=\max \left\{\min \{k(a), d(2 n-3-b)\}-1 \mid d(a)+d(r) \geqslant n^{2}+1 \text { and } a+b \leqslant 2 n-3-r\right\}
$$

In this context, we have

$$
d(x)= \begin{cases}n^{2}-n x & \text { if } 0 \leqslant x \leqslant n-1 \\ 2 n-1-x & \text { if } n \leqslant x \leqslant 2 n-2\end{cases}
$$

and

$$
k(x)= \begin{cases}\frac{(x+1)(x+2)}{2} & \text { if } 0 \leqslant x \leqslant n-1 \\ n^{2}-\frac{(2 n-1-x)(2 n-2-x)}{2} & \text { if } n \leqslant x \leqslant 2 n-2\end{cases}
$$

Maps $d$ and $k$ are illustrated in Figure 1.
If $r \geqslant n-1$, then $d(r)=2 n-1-r$ and one needs to set $a=0$ to fulfill the condition $d(a)+d(r) \geqslant n^{2}+1$. Thus $t_{\text {max }}=0$, which means that $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ admits no non-trivial error-correcting pair of the desired form.

Therefore, let us consider the more interesting case $r \leqslant n-2$. Define $u=2 n-3-a-b$. Since $d(r)=n^{2}-n r$, we have

$$
t_{\max }=\max \{\min \{k(a), d(a+u)\}-1 \mid d(a) \geqslant n r+1 \text { and } u \geqslant r\}
$$

For any fixed $a$, the map $u \mapsto \min \{k(a), d(a+u)\}$ is decreasing, therefore $t_{\text {max }}$ is reached for $u=r$. Moreover, the condition $d(a) \geqslant n r+1$ is equivalent to $a \leqslant n-r-1$. We also see that $d(\cdot)$ is decreasing and $k(\cdot)$ is increasing, thus $t_{\text {max }}=\min \{k(\lfloor\alpha\rfloor), d(\lceil\alpha\rceil+r)\}-1$ where $\alpha \in[0, n-1-r]$ is the only real number satisfying $k(\alpha)=d(\alpha+r)$. A simple computation shows that $\alpha=-n-\frac{3}{2}+\sqrt{3 n^{2}+(3-2 r) n+\frac{1}{4}}$.

Asymptotically, let us set $\rho=\lim _{n \rightarrow \infty} \frac{r}{n}$. Then we see that $\alpha=(\sqrt{3-2 \gamma}-1) n+O(\sqrt{n})$, hence $t_{\max }=(2-\gamma-\sqrt{3-2 \gamma}) n^{2}+O\left(n^{3 / 2}\right)$. It means that the corresponding error-correcting pair can correct approximately $(2-\gamma-\sqrt{3-2 \gamma}) n^{2}$ errors, while the unique decoding radius of $\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is $\left\lfloor\frac{d(r)-1}{2}\right\rfloor \simeq$ $\frac{1-\gamma}{2} n^{2}$. See Figure 1 for a comparison.


Figure 1. On the left, representation of the minimum distance $d(r)$ and the dimension $k(r)$ of $\mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ depending on $r$, for $\boldsymbol{n}=(n, n)$. On the right, representation of relative decoding radii of $\mathrm{RM}_{\boldsymbol{\theta}}(\gamma n, \boldsymbol{n})$ with $n \gg 1$, depending on $\gamma$.
8.5. Connection with classical Reed-Muller codes. In this section we prove a relation between $\boldsymbol{\theta}$-Reed-Muller codes and affine cartesian codes in the specific setting where the base field $\mathbb{K}$ contains all the $n_{i}$-th roots of unity. For convenience, we restrict our study to $\boldsymbol{\theta}$-Reed-Muller codes of type $\boldsymbol{n}=(n, \ldots, n) \in \mathbb{N}^{m}$, for which affine cartesian codes are classical $q$-ary Reed-Muller codes. See [13, 19] for more details on affine cartesians codes.

We therefore consider a Galois extension $\mathbb{L} / \mathbb{K}$ of degree $N=n^{m}$, such that $\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \cong$ $(\mathbb{Z} / n \mathbb{Z})^{m}$. Furthermore, we assume that $\mathbb{L} / \mathbb{K}$ is a Kummer extension, hence $x^{n}-1$ completely splits in linear factors in $\mathbb{K}$. Equivalently, $\mathbb{K}$ contains all the $n$-th roots of unity.

We give some additional notation now. Fix $i \in\{1, \ldots, m\}$. The subgroup $G_{i}:=\left\langle\theta_{j}\right| j \in\{1, \ldots, m\} \backslash$ $\{i\}\rangle$ yields a fixed field $\mathbb{L}_{i}:=\mathbb{L}^{G_{i}}$, for $i \in\{1, \ldots, m\}$. Let us also define $\mathbb{E}_{i}:=\mathbb{L}^{\theta_{i}}$. We see that $\mathbb{L}=\mathbb{L}_{i} \mathbb{E}_{i}$, $\mathbb{L}_{i} \cap \mathbb{E}_{i}=\mathbb{K}$ and $\mathbb{L}=\mathbb{L}_{1} \mathbb{L}_{2} \cdots \mathbb{L}_{m}$. Moreover, since $\mathbb{L} / \mathbb{K}$ is a Kummer extension and $\left[\mathbb{L}_{i}: \mathbb{K}\right]=n$, the extension $\mathbb{L}_{i} / \mathbb{K}$ is also a Kummer extension with Galois group $\operatorname{Gal}\left(\mathbb{L}_{i} / \mathbb{K}\right)=\left\langle\theta_{i}\right\rangle \cong \mathbb{Z} / n \mathbb{Z}$. Additionally, for this kind of extensions we have the following theorem, which is a consequence of the more general abelian Kummer theory (see [18, Ch. VI, Sec. 8]).
Theorem 55. Let $\mathbb{L} / \mathbb{K}$ be an abelian extension and $\mathbb{K}$ contains the $n$-th roots of unity. If $\operatorname{Gal}(\mathbb{L} / \mathbb{K})$ has exponent ${ }^{1} n$, then $\mathbb{L}=\mathbb{K}\left(\sqrt[n]{a_{1}}, \ldots, \sqrt[n]{a_{m}}\right)$ for some $a_{1}, \ldots, a_{m} \in \mathbb{K}^{*}$. Conversely, every extension $\mathbb{K}\left(\sqrt[n]{a_{1}}, \ldots, \sqrt[n]{a_{m}}\right)$ is abelian of exponent $n$.

As a consequence of Theorem 55, there exist $a_{i} \in \mathbb{K}$ and $\alpha_{i} \in \mathbb{L}_{i}$ such that $\alpha_{i}^{n}=a_{i}$ and $\mathbb{L}_{i}=\mathbb{K}\left(\alpha_{i}\right)$. This implies that the set $A_{i}:=\left\{\alpha_{i}^{j} \mid j=0,1, \ldots, n-1\right\}$ is a $\mathbb{K}$-basis of $\mathbb{L}_{i} / \mathbb{K}$ and

$$
A_{1} \cdot A_{2} \cdots A_{m}:=\left\{\prod_{i=1}^{m} \alpha_{i}^{j_{i}} \mid j_{1}, \ldots, j_{m} \in\{0, \ldots, n-1\}\right\}
$$

is a $\mathbb{K}$-basis of $\mathbb{L} / \mathbb{K}$. Furthermore, $\mathbb{L}=\mathbb{K}\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ and we have

$$
\theta_{i}^{s}\left(\alpha_{j}^{r}\right)= \begin{cases}\alpha_{j}^{r} & \text { if } i \neq j  \tag{12}\\ \zeta_{n}^{r s} \alpha_{j}^{r} & \text { if } i=j\end{cases}
$$

where $\zeta_{n} \in \mathbb{K}$ is a primitive $n$-th root of unity. Consider now for $i \in\{0, \ldots, m\}$ the set $\mathcal{B}_{i}:=A_{1} \cdots A_{i}$, where $\mathcal{U} \cdot \mathcal{V}=\{u v, u \in \mathcal{U}, v \in \mathcal{V}\}$. By convention, $\mathcal{B}_{0}:=\{1\}$. Moreover, for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$ and $\boldsymbol{i}=\left(i_{1}, \ldots, i_{m}\right) \in \Delta(n)^{m}$, we write $\boldsymbol{\alpha}^{i}:=\prod_{j=1}^{m} \alpha_{j}^{i_{j}}$. We consider the reverse lexicographic order $\prec$ on $\mathbb{N}^{m}$, from which we reorder the set $\Delta(n)^{m}=\left\{\boldsymbol{i}_{1}, \ldots \boldsymbol{i}_{N}\right\}$. With this notation $\mathcal{B}_{m}=\left\{\boldsymbol{\alpha}^{i_{1}}, \ldots, \boldsymbol{\alpha}^{i_{N}}\right\}$, and for every $t \in\{1, \ldots, m\}$ we have $\mathcal{B}_{t}=\left\{\boldsymbol{\alpha}^{i_{1}}, \ldots, \boldsymbol{\alpha}^{i_{n} t}\right\}$. In particular, it holds that

$$
\begin{equation*}
\mathcal{B}_{m}=\bigcup_{j=1}^{n} \alpha_{m}^{j} \cdot \mathcal{B}_{m-1} \tag{13}
\end{equation*}
$$

Different bases of $\mathbb{L} / \mathbb{K}$ produce equivalent vector codes (in the rank-metric sense). For this reason, we can restrict our study to $\mathrm{RM}_{\boldsymbol{\theta}, \mathcal{B}}(r, \boldsymbol{n}) \subseteq \mathbb{L}^{N}$ for the specific basis $\mathcal{B}=\mathcal{B}_{m}$ defined above. We already know that a basis for the space $\operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ is given by the set $T_{r, \boldsymbol{n}}=\left\{\boldsymbol{\theta}^{\boldsymbol{i}}|\boldsymbol{i} \in \Delta(\boldsymbol{n}),|\boldsymbol{i}| \leqslant r\}\right.$. We define $\overline{\boldsymbol{\theta}}:=\left(\theta_{1}, \ldots, \theta_{m-1}\right)$ and $\overline{\boldsymbol{n}}:=(n, \ldots, n) \in \mathbb{N}^{m-1}$ and we write

$$
\begin{equation*}
T_{r, \boldsymbol{n}}=\bigcup_{j=0}^{r}\left\{\overline{\boldsymbol{\theta}}^{i} \theta_{m}^{j}|\boldsymbol{i} \in \Delta(\overline{\boldsymbol{n}}), 0 \leqslant j<n,|\boldsymbol{i}| \leqslant r-j\}=\bigcup_{j=0}^{r} \theta_{m}^{j} T_{r-j, \overline{\boldsymbol{n}}}\right. \tag{14}
\end{equation*}
$$

where $T_{r, \bar{n}}=\varnothing$ whenever $r<0$. Furthermore, for a given $s \in\{0, \ldots, m\}$, we denote by $\operatorname{Diag}\left(\mathcal{B}_{s}\right)$ the $n^{s} \times n^{s}$ diagonal matrix whose entries are given by $\boldsymbol{\alpha}^{i}$, ordered in the reverse lexicographic order $\prec$.

With this notation, we can now study the generator matrix of the $k$-dimensional code $\mathrm{RM}_{\boldsymbol{\theta}, \mathcal{B}_{m}}(r, \boldsymbol{n})$.
Proposition 56. Let $G_{r, m} \in \mathbb{L}^{k \times N}$ be the generator matrix of $\mathrm{RM}_{\boldsymbol{\theta}, \mathcal{B}_{m}}(r, \boldsymbol{n})$ obtained by evaluating the $\boldsymbol{\theta}$-monomials in $T_{r, \boldsymbol{n}}$. Then $G_{r, m}=Y_{r, m} \operatorname{Diag}\left(\mathcal{B}_{m}\right)$, where
(1) If $r=0$, then $Y_{0, m}=(1,1, \ldots, 1)$.
(2) If $m=1$, then

$$
Y_{r, 1}=\left(\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1 \\
1 & \zeta_{n} & \zeta_{n}^{2} & \ldots & \zeta_{n}^{n-1} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \zeta_{n}^{r} & \zeta_{n}^{2 r} & \ldots & \zeta_{n}^{(n-1) r}
\end{array}\right)
$$

[^1]
## If $r \geqslant 1$ and $m \geqslant 2$, then

$$
Y_{r, m}=\left(\begin{array}{ccccc}
Y_{r, m-1} & Y_{r, m-1} & Y_{r, m-1} & \cdots & Y_{r, m-1} \\
Y_{r-1, m-1} & \zeta_{n} Y_{r-1, m-1} & \zeta_{n}^{2} Y_{r-1, m-1} & \cdots & \zeta_{n}^{n-1} Y_{r-1, m-1} \\
Y_{r-2, m-1} & \zeta_{n}^{2} Y_{r-2, m-1} & \zeta_{n}^{4} Y_{r-2, m-1} & \ldots & \zeta_{n}^{2(n-1)} Y_{r-2, m-1} \\
\vdots & \vdots & \vdots & & \vdots \\
Y_{0, m-1} & \zeta_{n}^{r} Y_{0, m-1} & \zeta_{n}^{2 r} Y_{0, m-1} & \cdots & \zeta_{n}^{r(n-1)} Y_{0, m-1}
\end{array}\right)
$$

Proof. (1) If $r=0$, then $\operatorname{RM}_{\boldsymbol{\theta}}(0, \boldsymbol{n})=\operatorname{Span}_{\mathbb{L}}\{\operatorname{Id}\}$, and hence for every ordered basis $\mathcal{B}$ of $\mathbb{L} / \mathbb{K}$, we have $G_{0, m}=(1, \ldots, 1) \operatorname{Diag}(\mathcal{B})$. In particular, it holds for $\mathcal{B}_{m}$.
(2) If $m=1$, then we are in the case of a cyclic Galois group $G=\langle\theta\rangle$. It is easy to see by (12), that the action of $\theta$ leads to $Y_{r, 1}$ being a Vandermonde matrix.
(3) We order the elements in $T_{r, \boldsymbol{n}}$ according to the reverse lexicographic order $\theta_{1} \prec \ldots \prec \theta_{m}$, and evaluate them in increasing order. This leads to a block division of $G_{r, m}$, in which the first block of rows corresponds to the evaluation of $T_{r, \overline{\boldsymbol{n}}}$, the second block of rows to the evaluation of $\theta_{m} T_{r-1, \bar{n}}$, and so on as explained in (14). Moreover, we have also ordered the elements of the basis $\mathcal{B}_{m}$ according to the reverse lexicographic order, which leads to a columns division of $G_{r, m}$ in blocks as explained in (13). The first block of columns correspond to $\mathcal{B}_{m-1}$, the second block of columns to $\alpha_{m} \cdot \mathcal{B}_{m-1}$ and so on. To sum up, this produces a block structure of $G_{r, m}$ in which the $(i, j)$-block corresponds to the evaluation of $\theta_{m}^{i-1} T_{r-i+1, \bar{n}}$ in $\alpha_{m}^{j-1} \cdot \mathcal{B}_{m-1}$.

Now, by (12) we have $\sigma\left(\alpha_{m}\right)=\alpha_{m}$ for every $\sigma \in T_{r-i+1, \bar{n}}$. Moreover, it holds that $\theta_{m}\left(\mathcal{B}_{m-1}\right)=$ $\mathcal{B}_{m-1}$ and $\theta_{m}^{i-1}\left(\alpha_{m}^{j-1}\right)=\zeta_{m}^{(i-1)(j-1)} \alpha_{m}$. By definition, the matrix associated to $T_{r-i+1, \bar{n}}\left(\mathcal{B}_{m-1}\right)$ is $Y_{r-i+1, m-1} \operatorname{Diag}\left(\alpha_{m}^{j-1} \cdot \mathcal{B}_{m-1}\right)$. Hence, the $(i, j)$-block of $G_{r, m}$ is equal to

$$
\zeta_{m}^{(i-1)(j-1)} Y_{r-i+1, m-1} \operatorname{Diag}\left(\alpha_{m}^{j-1} \cdot \mathcal{B}_{m-1}\right)
$$

which gives the desired result.

As a byproduct we now show that we get a characterization of the generator matrix $G_{r, m}$ which relates $\boldsymbol{\theta}$-Reed-Muller codes with classical Reed-Muller codes (or affine variety codes or affine cartesian codes). Consider the set

$$
P_{r, m}:=\left\{p \in \mathbb{K}\left[x_{1}, \ldots, x_{m}\right] \mid \operatorname{deg} p \leqslant r\right\}
$$

For a finite subset $U \subset \mathbb{K}$ with cardinality $n$, we consider the set $X:=U \times \cdots \times U=U^{m}$, and a total order on it, such that we can write $X=\left\{u_{1}, \ldots, u_{n^{m}}\right\}$. Then the classical Reed-Muller code (or affine variety code, or affine cartesian code) on $X$ is

$$
\operatorname{HRM}_{X}(r, m)=\left\{\left(p\left(u_{1}\right), \ldots, p\left(u_{n^{m}}\right)\right) \mid p \in P_{r, m}\right\} \subseteq \mathbb{K}^{N}
$$

Theorem 57. [13, Proposition 5][19, Theorem 3.8] If $r \geqslant 1$ and $U$ has cardinality $n \geqslant 2$, then the code $\operatorname{HRM}_{X}(r, m)$ is an $[N, k, d]_{\mathbb{K}}$ code in the Hamming metric, with $N=n^{m}$ and $d=(n-\ell) n^{m-s-1}$, where $\ell$ and $s$ are the unique non-negative integers such that $r=s(n-1)+\ell$ and $0 \leqslant \ell<n-1$.

We now consider the special case when $U=U_{n}$ is the set of $n$-th roots of unity. Every element in $\left(U_{n}\right)^{m}$ is of the form $\left(\zeta_{n}^{j_{1}}, \zeta_{n}^{j_{2}}, \ldots, \zeta_{n}^{j_{m}}\right)=: \zeta^{j}$, where $\boldsymbol{j}=\left(j_{1}, \ldots, j_{m}\right) \in \Delta(n)^{m}$. We order the elements $\boldsymbol{\zeta}^{\boldsymbol{j}}$ 's of $X:=U_{n}^{m}$ according to the reverse lexicographic order on $\Delta(n)^{m}$, and we obtain the following result.

Theorem 58. The $\boldsymbol{\theta}$-Reed-Muller code $\operatorname{RM}_{\boldsymbol{\theta}, \mathcal{B}_{m}}(r, \boldsymbol{n})$ has a generator matrix of the form $G_{r, \boldsymbol{n}}:=Y_{r, m} \operatorname{Diag}\left(\mathcal{B}_{m}\right)$, where $Y_{r, m} \in \mathbb{K}^{k \times N}$ is the generator matrix of the classical Reed-Muller codes $\operatorname{HRM}_{X}(r, m)$ obtained by evaluating the monomials on the points of $X:=\left(U_{n}\right)^{m}$.
Proof. The generator matrix for a classical Reed-Muller codes $\operatorname{HRM}_{X}(r, m)$ follows the same recursive relations described in Proposition 56 part 3, with the same initial conditions given in 1 and 2.

In the general case of $\operatorname{Gal}(\mathbb{L} / \mathbb{K}) \cong \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}$, i.e. for a $\boldsymbol{\theta}$-Reed-Muller code of type $\boldsymbol{n}=\left(n_{1}, \ldots, n_{m}\right)$, a similar result can be shown. More specifically, let $V_{i}:=\left\{x \in \mathbb{K} \mid x^{n_{i}}=1\right\}$ and $X:=V_{1} \times \cdots \times V_{m}$. Then the code $\operatorname{RM}_{\boldsymbol{\theta}, \mathcal{B}_{m}}(r, \boldsymbol{n})$ has a generator matrix which is equal to the generator matrix of the code $\operatorname{HRM}_{X}(r, m)$ multiplied on the right by $\operatorname{Diag}(\mathcal{B})$, where $\mathcal{B}$ is the ordered $\mathbb{K}$-basis
of $\mathbb{L} / \mathbb{K}$ with respect to the reverse lexicographic order, which is constructed as explained for the case $n_{1}=\cdots=n_{m}=n$.

## 9. Conclusion and open problems

In this paper was presented a general description of codes seen as subspaces of the group algebra $\mathbb{L}[G]$ with arbitrary Galois extensions $\mathbb{L} / \mathbb{K}$. Analogues of Reed-Muller codes were constructed as an application, but there remains some way to go towards practicality of these codes.

First, one can wonder whether $\boldsymbol{\theta}$-Reed-Muller codes can be decoded up to hald their minimum distance. Such decoding algorithms are known for Hamming-metric Reed-Muller codes over finite fields. However they require to embed the code in a Reed-Solomon code over the extension field $\mathbb{L}$, and to use the decoder attached to this code. To our opinion, this technique seems difficult to adapt in our context, given the fact that there is no way to embed a $\boldsymbol{\theta}$-Reed-Muller code into a Gabidulin code (since $G$ is not cyclic).

Second, the lack of practicality of our codes relies on the fact that, if $\mathbb{L} / \mathbb{K}$ is not cyclic, then $\mathbb{L}$ cannot be a finite field. This raises the two following issues: (i) find Galois extensions $\mathbb{L} / \mathbb{K}$ in which computations are efficiently doable (so-called effective fields), and (ii) find maps $\pi: \mathbb{L} \rightarrow \mathbb{F}$, where $\mathbb{F}$ is an effective field, such that $\pi$ sends a code $\mathcal{C} \subseteq \mathbb{L}[G]$ to a "good" code $\pi(\mathcal{C}) \subseteq \mathbb{F}^{n}$ whose properties can be derived from those of $\mathcal{C}$.

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## Appendix A. A second proof for the minimum distance lower bound

The algebra $\mathbb{L}[G]$ can also be represented as a skew polynomial ring modulo a particular two-sided ideal. Let us recall that the skew polynomial ring $\mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}]=\mathbb{L}\left[x_{1}, \ldots, x_{m} ; \theta_{1}, \ldots, \theta_{m}\right]$ is the ring of polynomials $Q(\boldsymbol{x})=Q\left(x_{1}, \ldots, x_{n}\right)$ where the addition is defined as in the usual polynomial ring, and the multiplication follows the following rules

$$
\begin{aligned}
x_{i} x_{j} & =x_{j} x_{i} & & \text { for any } i, j \in\{1, \ldots, m\} \\
x_{i} a & =\theta_{i}(a) x_{i} & & \text { for any } a \in \mathbb{L}
\end{aligned}
$$

and is extended by associativity. It is known that the center of this ring is $\mathbb{K}\left[\boldsymbol{x}^{\boldsymbol{n}}\right]$, and the ideal generated by $\left(x_{1}^{n_{1}}-1, x_{2}^{n_{2}}-1, \ldots, x_{m}^{n_{m}}-1\right)$ is two-sided. We will indicate such ideal by $I_{\boldsymbol{n}}$.

The ring $\mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}]$ is a very particular case of left Poincaré-Birkhoff-Witt ring, for which the theory of Gröbner basis is well-defined and it works practically in the same way as for commutative rings. For a deeper understanding on the topic, we refer the interested reader to [6].
Theorem 59. Let $G:=\operatorname{Gal}(\mathbb{L} / \mathbb{K})=\left\langle\theta_{1}, \ldots, \theta_{m}\right\rangle \cong \mathbb{Z} / n_{1} \mathbb{Z} \times \cdots \times \mathbb{Z} / n_{m} \mathbb{Z}$. Then the map

$$
\Phi:\left\{\begin{array}{cll}
\mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] & \longrightarrow & \mathbb{L}[G] \\
\sum_{\boldsymbol{i} \in \mathbb{N}^{m}} b_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} & \longmapsto & \sum_{\boldsymbol{i} \in \mathbb{N}^{m}} b_{\boldsymbol{i}} \boldsymbol{\theta}^{\boldsymbol{i}}
\end{array}\right.
$$

is a surjective ring homomorphism with $\operatorname{ker} \Phi=I_{\boldsymbol{n}}$. In particular, it induces an isomorphism $\bar{\Phi}$ : $\mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] / I_{\boldsymbol{n}} \rightarrow \mathbb{L}[G]$.

With this framework in mind, we propose a second proof of the lower bound on the rank of a nonzero $\boldsymbol{\theta}$-polynomial $P$ given in Theorem 40. Precisely, we will prove the following: if $P \in \operatorname{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$, then

$$
\mathrm{rk}_{\mathbb{K}}(P) \geqslant \min \left\{\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)\left|\boldsymbol{u}=\left(u_{1}, \ldots, u_{m}\right) \in \Delta(\boldsymbol{n}),|\boldsymbol{u}| \leqslant r\right\}\right.
$$

Second proof: Let $P=\sum_{\boldsymbol{i} \in \Delta(\boldsymbol{n})} b_{\boldsymbol{i}} \boldsymbol{\theta}^{\boldsymbol{i}} \in \mathrm{RM}_{\boldsymbol{\theta}}(r, \boldsymbol{n})$ be a $\boldsymbol{\theta}$-polynomial. Observe again that the minimum is attained for a $\boldsymbol{\theta}$-polynomial of $\boldsymbol{\theta}$-degree equal to $r$, and we set

$$
\delta:=\min \left\{\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)\left|\boldsymbol{u}=\left(u_{1}, \ldots, u_{m}\right) \in \Delta(\boldsymbol{n}),|\boldsymbol{u}|=r\right\} .\right.
$$

Therefore, we need to prove that $\operatorname{wt}_{I}(P) \geqslant \delta$, where $\operatorname{wt}_{I}(P)=\operatorname{dim}_{\mathbb{L}}\left(\mathbb{L}[G] / \operatorname{Ann}_{\mathbb{L}[G]}(P)\right)$. Equivalently, we have to show that that there exists an $\mathbb{L}$-subspace $T$ of $\mathbb{L}[G]$ of dimension at least $\delta$ such that $T \cap$ $\operatorname{Ann}_{\mathbb{L}[G]}(P)=\{0\}$. Consider the isomorphism $\bar{\Phi}: \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] / I_{\boldsymbol{n}} \rightarrow \mathbb{L}[G]$ introduced in Theorem 59. Using this isomorphism, our goal is equivalent to finding an $\mathbb{L}$-subspace $V$ of $\mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] / I_{\boldsymbol{n}}$ of dimension at least $\delta$, such that $g(\boldsymbol{x}) \bar{\Phi}^{-1}(P)(\boldsymbol{x}) \neq 0 \bmod I_{n}$ for every $g(\boldsymbol{x}) \in V$.

First, we observe that the set $\left\{x_{1}^{n_{1}}-1, \ldots, x_{m}^{n_{m}}-1\right\}$ is a universal Gröbner basis for the ideal $I_{n}$. We choose the representative $\bar{P}(\boldsymbol{x}) \in \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}]$ of $\bar{\Phi}^{-1}(P)(\boldsymbol{x})$ reduced modulo the Gröbner basis $\left\{x_{1}^{n_{1}}-\right.$ $\left.1, \ldots, x_{m}^{n_{m}}-1\right\}$, that is $\bar{P}(\boldsymbol{x})=\sum_{\boldsymbol{i} \in \Delta(\boldsymbol{n})} b_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \in \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}]$. Moreover, we fix a monomial order $\prec$, and we consider the leading term of $\bar{P}(\boldsymbol{x})$ with respect to $\prec$, that is $l^{\ell}(\bar{P}(\boldsymbol{x}))=\boldsymbol{x}^{\boldsymbol{u}}$, for $\boldsymbol{u}=\left(u_{1}, \ldots, u_{m}\right)$, and we consider the set

$$
Z=\left\{f(\boldsymbol{x}) \in \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] \mid \operatorname{deg}_{x_{i}}(f)<n_{i}-u_{i}, i=1, \ldots, m\right\}
$$

Note that $Z \cap I_{\boldsymbol{n}}=\{0\}$. This is due to the fact that the set $\left\{x_{1}^{n_{1}}-1, \ldots, x_{m}^{n_{m}}-1\right\}$ is a universal Gröbner basis for the ideal $I_{\boldsymbol{n}}$ and none of the monomials in $Z$ belongs to monomial ideal spanned by the leading terms of the generators of $I_{\boldsymbol{n}}$, namely $\mathrm{lt}_{\prec}\left(I_{\boldsymbol{n}}\right)=\left(x_{1}^{n_{1}}, \ldots, x_{m}^{n_{m}}\right)$. Therefore, the canonical projection $\pi: \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] \rightarrow \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] / I_{\boldsymbol{n}}$ is injective when restricted to $Z$.

At this point let us take an arbitrary skew polynomial $f(\boldsymbol{x}) \in Z$ and consider its leading term $\mathrm{lt}_{\prec}(f(\boldsymbol{x}))=\boldsymbol{x}^{\boldsymbol{v}}$, where, by definition of the space $Z$, we have $\boldsymbol{v}=\left(v_{1}, \ldots, v_{m}\right)$ and $v_{i}<n_{i}-u_{i}$ for all $i=1, \ldots, m$. Then,

$$
\operatorname{lt}_{\prec}(f(\boldsymbol{x}) \bar{P}(\boldsymbol{x}))=\operatorname{lt}_{\prec}(f(\boldsymbol{x})) \operatorname{lt}_{\prec}(\bar{P}(\boldsymbol{x}))=\boldsymbol{x}^{\boldsymbol{v}} \boldsymbol{x}^{\boldsymbol{u}}=\boldsymbol{x}^{\boldsymbol{u}+\boldsymbol{v}}
$$

Since $u_{i}+v_{i}<n_{i}$ for every $i$, we have that $\operatorname{lt}_{\prec}(f(\boldsymbol{x}) \bar{P}(\boldsymbol{x})) \notin\left(x_{1}^{n_{1}}, \ldots, x_{m}^{n_{m}}\right)=\operatorname{lt}_{\prec}\left(I_{\boldsymbol{n}}\right)$. Therefore, $f(\boldsymbol{x}) \bar{P}(\boldsymbol{x}) \notin I_{\boldsymbol{n}}$. Denote by $\pi: \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] \rightarrow \mathbb{L}[\boldsymbol{x} ; \boldsymbol{\theta}] / I_{\boldsymbol{n}}$ the canonical projection modulo the ideal $I_{\boldsymbol{n}}$. Hence, $\pi(f(\boldsymbol{x})) \pi(\bar{P}(\boldsymbol{x}))=\pi(f(\boldsymbol{x})) \bar{\Phi}^{-1}(P) \neq 0$. Thus, the space $V:=\pi(Z)$ is such that $g(\boldsymbol{x}) \bar{\Phi}^{-1}(P)(\boldsymbol{x}) \neq 0$ $\bmod I_{\boldsymbol{n}}$ for every $g(\boldsymbol{x}) \in V$. Moreover,

$$
\operatorname{dim}_{\mathbb{L}}(\pi(Z))=\operatorname{dim}_{\mathbb{L}}(Z)=\prod_{i=1}^{m}\left(n_{i}-u_{i}\right)
$$

which concludes the proof.


[^0]:    Inria \& LIX, CNRS UMR 7161, École Polytechnique, 91120 Palaiseau Cedex, France
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[^1]:    ${ }^{1} \mathrm{~A}$ group $G$ is said to have exponent $n$ if every element $g \in G$ satisfies $g^{n}=\mathrm{id}$

