SIAM J. NUMER. ANAL.
Vol. 59, No. 6, pp. 3077–3097

NUMERICAL MAXIMIZATION OF THE p-LAPLACIAN ENERGY OF A TWO-PHASE MATERIAL*

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Abstract. For a diffusion problem modeled by the p-Laplacian operator, we are interested in obtaining numerically the two-phase material which maximizes the internal energy. We assume that the amount of the best material is limited. In the framework of a relaxed formulation, we present two algorithms, a feasible directions method and an alternating minimization method. We show the convergence for both of them, and we provide an estimate for the error. Since for $p > 2$ both methods are only well-defined for a finite-dimensional approximation, we also study the difference between solving the finite-dimensional and the infinite-dimensional problems. Although the error bounds for both methods are similar, numerical experiments show that the alternating minimization method works better than the feasible directions one.

Key words. optimal design, two-phase material, p-Laplacian operator, feasible directions method, alternating minimization method

AMS subject classifications. 49M05, 49J20

DOI. 10.1137/20M1353563

1. Introduction.

The aim of the present work is the numerical resolution of an optimal design problem. It corresponds to the maximization of the energy for a nonlinear diffusion process in a two-phase material modeled by the p-Laplacian operator. Namely, we are interested in the control problem

$$
\left\{
\begin{array}{c}
\displaystyle\max_{\omega} \frac{1}{p} \int_{\Omega} \left(\alpha \mathcal{X}_\omega + \beta \left(1 - \mathcal{X}_\omega\right)\right) |\nabla u|^p dx \\
-\mathrm{div}\left(\left(\alpha \mathcal{X}_\omega + \beta \left(1 - \mathcal{X}_\omega\right)\right) |\nabla u|^{p-2} \nabla u\right) = f \quad \text{in } \Omega \\
u \in W_0^{1,p}(\Omega), \quad \omega \subset \Omega \text{ measurable}, \quad |\omega| \leqslant \kappa,
\end{array}
\right. \tag{1.1}
$$

with Ω a bounded open set in $\mathbb{R}^N$, $N \geqslant 2$, $p \in (1, \infty)$, $\alpha, \beta, \kappa > 0$, $\alpha < \beta$, and $f \in W^{-1,p'}(\Omega)$. Here α and β are the diffusion constants corresponding to the two materials that we want to mix in order to maximize the corresponding functional. If we do not impose any restrictions on the amount of material α (i.e., $\kappa \geqslant |\Omega|$), then the solution is the trivial one given by $\omega = \Omega$. Thus, the interesting case corresponds to $\kappa < |\Omega|$. This problem has been extensively studied for $p = 2$ ([1], [6], [8], [12],

*Received by the editors July 17, 2020; accepted for publication (in revised form) August 23, 2021; published electronically December 21, 2021.
https://doi.org/10.1137/20M1353563
Funding: The first author has been partially supported by Project MTM2017-83583 of the Ministerio de Ciencia, Innovación y Universidades of Spain. The second author is partially supported by PFBasal-001 (CeBiB), FB210005 (CMM), and ACE210010, BASAL funds for centers of excellence from ANID-Chile and by Regional Program NEMBICA STIC190013 STIC-AmSud. The third author has been partially supported by the CONICYT PFCHA/DOCTORADO BECAS CHILE/2018-21182101.
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[14], [15], [20]). In this case, it models, for example, the optimal rearrangement of two materials in the cross section of a beam in order to minimize its torsion (in this application $f = 1$). Analogously, for $p \in (1, 2) \cup (2, \infty)$, the p-Laplacian operator models the torsional creep in the cross section of a beam [9]. Therefore, problem (1.1) corresponds to find the two-phase material which minimizes the torsion in nonlinear elasticity, assuming that the amount of the best material is limited. As usual for this type of problem ([17], [18]), it has no solution in general ([6], [7], [20]). Thus, it is necessary to work with a relaxed formulation which can be obtained from the homogenization theory ([1], [19], [21]). In the present case, it has been proved in [7] ([20] for $p = 2$) that such relaxation is given by

$$\left\{\begin{array}{c} \displaystyle\max_{u,\theta} \frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}dx \\ \displaystyle -\mathrm{div}\Big(\frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}}\nabla u\Big) = \frac{1}{\beta} f \quad \text{in } \Omega \\ \displaystyle u \in W_0^{1,p}(\Omega), \quad \theta \in L^\infty(\Omega;[0,1]), \quad \int_\Omega \theta\, dx \leqslant \kappa, \end{array}\right. \tag{1.2}$$

with $c = (\frac{\beta}{\alpha})^{\frac{1}{p-1}} - 1$. In this formulation, the materials α and β have been replaced by mixtures of them obtained by laminations. The new control variable θ represents the proportion of the best material α used in the mixture.

The problem can also be formulated in a simple way as the following calculus of variation problem:

$$\left\{\begin{array}{c} \displaystyle\min_{u,\theta} \left\{\frac{1}{p}\int_\Omega \frac{|\nabla u|^p}{(1+c\theta)^{p-1}}dx - \frac{1}{\beta}\langle f, u\rangle\right\} \\ \displaystyle \theta \in L^\infty(\Omega;[0,1]), \quad u \in W_0^{1,p}(\Omega), \quad \int_\Omega \theta dx \leqslant \kappa. \end{array}\right. \tag{1.3}$$

The numerical resolution of (1.3) for $p = 2$ has been the subject of several works. In this way, some numerical simulations have been carried out in [12] and [14] using a multigrid method. In [1] and [22], the convergence of the alternating minimization algorithm has been shown, using the optimality conditions. In [3], the convergence of a projected gradient method has been studied.

For $p \neq 2$, the use of the optimality conditions implies the resolution of the p-Laplacian equation in each iteration. This is a problem which has been considered, for example, in [11] and [13] using a steepest descent method. We also refer the reader to [10], where a reformulation of the p-Laplacian is given in order to use an augmented Lagrangian method. In these works, the order of convergence is linear in the best case.

In the present paper, we introduce two algorithms to solve (2.1). The first one is based on the Frank–Wolfe algorithm, also known as the feasible direction method. The second one is an alternating minimization method. In both of them, we choose a descent direction in $H_0^1(\Omega)$ instead of $W_0^{1,p}(\Omega)$ and we solve a linear problem instead of a p-Laplacian, which, as we said above, is very expensive from a computational point of view. For $p > 2$, this forces us to work with a discretized version of the problem because $H_0^1(\Omega)$ is not contained in $W_0^{1,p}(\Omega)$.

We prove the convergence of both methods obtaining estimates for the rate of convergence. In the best of the cases ($p \geqslant 2$), we only have a convergence of order $1/i$, with i the number of iterations. This is due to the nonstrict convexity of the problem. In this sense, we can observe that solving the minimum in θ in problem

(1.3) and using the Kuhn–Tucher theorem, we can rewrite (1.3) as (see [6], [7], [12], [14])

$$\max_{\mu\geqslant 0}\ \min_{u\in W_0^{1,p}(\Omega)}\left\{\int_\Omega F(\mu,\nabla u)\,dx-\frac{\mu^p(p-1)c}{p}\kappa-\frac{1}{\beta}\langle f,u\rangle\right\} \tag{1.4}$$

with

$$F(\mu,\xi)=\begin{cases}\dfrac{1}{p}\dfrac{|\xi|^p}{(1+c)^{p-1}}+\dfrac{\mu^p(p-1)c}{p} & \text{if } \mu\leqslant\dfrac{|\xi|}{(1+c)}\\[2ex] \mu^{p-1}|\xi|-\dfrac{\mu^p(p-1)}{p} & \text{if } \dfrac{|\xi|}{(1+c)}<\mu<|\xi|\\[2ex] \dfrac{1}{p}|\xi|^p & \text{if } \mu\geqslant|\xi|.\end{cases}$$

Observe that F is nonstrictly convex in ξ, and it is not differentiable with respect to μ.

We also prove the convergence of the solutions of the discretized problem toward the solutions of the continuous one. Moreover, taking a regular sequence of triangulations in Ω of diameter $h>0$ and discretizing $W_0^{1,p}(\Omega)$ and $L^\infty(\Omega)$ by the usual P_1 and P_0 finite elements, respectively, we show that the difference between the minimum for the continuous and the discretized problem is of order h. In order to prove this result, we assume the existence of a solution (u,θ) for (1.3) such that u is in $W^{1,\infty}(\Omega)$, ∇u belongs to $BV(\Omega)^N$, and θ belongs to $BV(\Omega)$. Some smoothness results for problem (1.3) can be found in [6] and [14] for $p=2$ and [7] for $p\in(1,\infty)$; we also refer the reader to [5] for the relaxed problem corresponding to take the minimum in (1.1) instead of the maximum one. These smoothness results imply that u is in $W^{1,\infty}(\Omega)$, the flow $\sigma=|\nabla u|^{p-2}\nabla u/(1+c\theta)^{p-1}$ is in $H^1(\Omega)^N$, and the derivatives of θ in the direction of σ are in $L^2(\Omega)$. However, this is not enough to get ∇u and θ BV-functions. Nevertheless, this assumption seems to be satisfied in the numerical experiments.

The paper is organized as follows. In section 2, we recall some known results for problem (1.3) which have been proved in [7] (see [6], [20], for $p=2$). In section 3, we state the main results of the paper. Section 4 is devoted to prove the results in section 3. Finally, in section 5, we illustrate the results of the paper with some numerical simulations. They show that the alternating minimization method converges faster than the feasible direction method.

2. Previous results.

As we mentioned in the introduction, our aim in the present paper is to numerically solve the optimal design problem (1.1). Since it has no solution in general, we work with the relaxed formulation (1.3), which, renaming f/β by f to simplify the notation, can be written as

$$\min\left\{\mathcal{F}(\theta,u):\ \theta\in L^\infty(\Omega;[0,1]),\quad u\in W_0^{1,p}(\Omega),\quad \int_\Omega\theta\,dx\leqslant\kappa\right\}, \tag{2.1}$$

with

$$\mathcal{F}(\theta,u)=\frac{1}{p}\int_\Omega\frac{|\nabla u|^p}{(1+c\,\theta)^{p-1}}\,dx-\langle f,u\rangle. \tag{2.2}$$

Here Ω is a bounded open set of $\mathbb{R}^N$, $N\geqslant 2$, $p\in(1,\infty)$, $c>0$, $\kappa\in(0,|\Omega|)$, and f is a distribution in $W^{-1,p'}(\Omega)$. Since $\mathcal{F}$ is convex in (θ,u) and coercive in u, $W_0^{1,p}(\Omega)$ is reflexive and $L^\infty(\Omega;[0,1])$ is bounded and then sequentially compact for the weak-$*$ topology in $L^\infty(\Omega)$, the existence of solution is straightforward. However, $\mathcal{F}$ is not strictly convex, and therefore the uniqueness is not clear.

The relaxed formulation (2.1) has been obtained in [7]. In this paper, we have also obtained some optimality conditions and some equivalent formulations. As a consequence, we got some uniqueness and smoothness results (see [6], [12], [14], [20] for related results in the case $p = 2$).

Thanks to the convexity of $\mathcal{F}$, Kuhn–Tucker's theorem easily provides the following system of optimality conditions ([7]).

PROPOSITION 2.1. *A pair* $(\hat\theta, \hat u)$ *is a solution of* (2.1) *if and only if there exists* $\hat\mu \geqslant 0$ *such that the following hold:*

If $\hat\mu = 0$, *then*

$$\hat\theta = 1 \text{ a.e. in } \{x \in \Omega : \ \nabla \hat u(x) \neq 0\}, \qquad \big|\{x \in \Omega : \ \nabla \hat u(x) \neq 0\}\big| \leqslant \kappa, \tag{2.3}$$

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla \hat u|^{p-2}}{(1+c)^{p-1}} \nabla \hat u \right) = f \quad \text{in } \Omega, \\ \hat u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

If $\hat\mu > 0$, *then*

$$\hat\theta = \max\left\{0, \min\left\{1, \frac{1}{c}\left(\frac{|\nabla \hat u|}{\hat\mu} - 1\right)\right\}\right\}, \qquad \int_\Omega \hat\theta \, dx = \kappa, \tag{2.5}$$

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla \hat u|^{p-2}}{(1+c\hat\theta)^{p-1}} \nabla \hat u \right) = f \quad \text{in } \Omega, \\ \hat u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Remark 2.1. The expression of $\hat\theta$ in Proposition 2.1 is obtained by solving (see [7], [20])

$$\min\left\{ \int_\Omega \frac{|\nabla \hat u|^p}{(1+c\theta)^{p-1}} dx : \ \theta \in L^\infty(\Omega; [0,1]), \ \int_\Omega \theta \, dx \leqslant \kappa \right\}. \tag{2.7}$$

The constant $\hat\mu \geqslant 0$ is a Lagrange multiplier corresponding to the constraint $\int_\Omega \theta \, dx \leqslant \kappa$.

We observe that for an arbitrary function $\hat u \in W_0^{1,p}(\Omega)$ (not necessarily a solution for (2.1)), the solutions of (2.7) can be explicitly obtained using Kuhn–Tucker's theorem, which shows that $\hat\theta$ is a solution if and only if there exists $\hat\mu \geqslant 0$ such that

$$\hat\mu \left(\int_\Omega \hat\theta dx - \kappa \right) = 0$$

and $\hat\theta$ is a solution of

$$\min\left\{ \int_\Omega \frac{|\nabla \hat u|^p}{(1+c\theta)^{p-1}} dx + \hat\mu \int_\Omega \hat\theta \, dx : \ \theta \in L^\infty(\Omega; [0,1]) \right\}. \tag{2.8}$$

This provides the following rule to solve (2.7).

If $\hat u$ is such that $|\{\nabla \hat u \neq 0\}| \leqslant \kappa$, then $\hat\theta$ is any function in $L^\infty(\Omega; [0,1])$ satisfying

$$\hat\theta = 1 \ \text{ a.e. in } \{x \in \Omega : \ \nabla \hat u(x) \neq 0\}, \qquad \int_\Omega \hat\theta \, dx \leqslant \kappa.$$

In the other case, denoting for $\mu > 0$

$$\theta_\mu := \max\left\{0, \min\left\{1, \frac{1}{c}\left(\frac{|\nabla \hat u|}{\mu} - 1\right)\right\}\right\}$$

and defining $G : (0,\infty) \to \big[0, |\Omega|\big]$ by

$$G(\mu) = \int_\Omega \theta_\mu dx \quad \forall \mu \in (0,\infty),$$

we have that the set of solutions of (2.7) is given by

$$\big\{\theta_\mu \in L^\infty(\Omega;[0,1]) : \ \mu > 0, \quad G(\mu) = \kappa\big\}. \tag{2.9}$$

Remark that the equation $G(\mu) = \kappa$ has a solution (not unique in general) due to G decreasing, continuous, and

$$\lim_{\mu\to 0} G(\mu) = \big|\{x \in \Omega : \ \nabla\hat{u}(x) \neq 0\}\big|, \qquad \lim_{\mu\to\infty} G(\mu) = 0.$$

Numerically, the equation $G(\mu) = \kappa$ can be easily solved using, for example, a dichotomy method.

In [7] (see [20] for $p = 2$), it has also been proved that introducing the flow

$$\sigma = \frac{|\nabla u|^{p-2}}{(1+c\theta)^{p-1}}\nabla u,$$

we have that (2.1) is equivalent to the min-max problem

$$\min_{\substack{-\operatorname{div}\sigma = f\\ \sigma\in L^{p'}(\Omega)^N}} \ \max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\, dx \leqslant \kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}dx = \max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\, dx \leqslant \kappa}} \ \min_{\substack{-\operatorname{div}\sigma = f\\ \sigma\in L^{p'}(\Omega)^N}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}dx. \tag{2.10}$$

Taking into account that the functional

$$\sigma \in L^2(\Omega)^N \to \max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega \theta\, dx \leqslant \kappa}} \int_\Omega (1+c\,\theta)|\sigma|^{p'}dx$$

is strictly convex, we get the uniqueness of the optimal flow. Moreover, using (1.4), we get the following smoothness results for the solutions of (2.1).

THEOREM 2.1. *For every solution $(\hat{\theta},\hat{u})$ of* (2.1), *the flow $\hat{\sigma}$ defined by*

$$\hat{\sigma} = \frac{|\nabla\hat{u}|^{p-2}}{(1+c\hat{\theta})^{p-1}}\nabla\hat{u} \tag{2.11}$$

is uniquely defined.

If f belongs to $W^{1,1}(\Omega)\cap L^q(\Omega)$, $q > N$, and Ω is a $C^{1,1}$ domain, then $\hat{\sigma}$ belongs to $H^1(\Omega)^N \cap L^\infty(\Omega)^N$. Moreover, there exists $C > 0$, which only depends on N, p, c, and Ω, such that

$$\|\hat{\sigma}\|_{H^1(\Omega)^N\cap L^\infty(\Omega)^N} \leqslant C\left(\|f\|_{W^{1,1}(\Omega)\cap L^q(\Omega)} + \hat{\mu}\right), \tag{2.12}$$

with $\hat{\mu}$ given by Proposition 2.1.

The function $\hat{\theta}$ satisfies

$$\hat{\theta}(x) = \begin{cases} 1 & \text{if } |\hat{\sigma}| > \hat{\mu},\\ 0 & \text{if } |\hat{\sigma}| < \hat{\mu},\end{cases} \tag{2.13}$$

and decomposing $\hat{\sigma} = (\hat{\sigma}_1,\dots,\hat{\sigma}_N)$, we have

$$\partial_i\hat{\theta}\hat{\sigma}_j - \partial_j\hat{\theta}\hat{\sigma}_i = (1+c\hat{\theta})(\partial_j\hat{\sigma}_i - \partial_i\hat{\sigma}_j)\chi_{\{|\hat{\sigma}|=\hat{\mu}\}} \in L^2(\Omega), \quad 1 \leqslant i,j \leqslant N. \tag{2.14}$$

Remark 2.2. Theorem 2.1 has been proved in [7], where some other regularity results depending on the smoothness of f have been obtained. The case $p = 2$ has been first proved in [6]. Observe that $\hat{\sigma}$ in $L^\infty(\Omega)^N$ implies that $\hat{u}$ belongs to $W^{1,\infty}(\Omega)$. This was previously shown in [14] for $p = 2$.

3. Algorithms and main results. In this section, we present two variants of a descent algorithm to numerically solve problem (2.1). We also show the convergence of both algorithms.

A first attempt to construct an algorithm is to use an alternate method consisting in minimizing in u, then in θ, and so on. That is, assuming an approximation (u_i, θ_i) of a solution of (2.1), we compute u_{i+1} as a solution of

$$\min_{v\in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p}\int_\Omega \frac{|\nabla v|^p}{(1+c\theta_i)^{p-1}} - \langle f, v\rangle \right\} \tag{3.1}$$

and then θ_{i+1} as a solution of

$$\min_{\substack{\theta\in L^\infty(\Omega;[0,1])\\ \int_\Omega \theta\, dx \leqslant \kappa}} \left\{ \frac{1}{p}\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta)^{p-1}} dx \right\}. \tag{3.2}$$

This method works well if $p = 2$, but for $p \neq 2$, problem (3.1) is a p-Laplacian problem, which is very expensive to solve from the computational point of view due to the nonlinearity of the corresponding Euler–Lagrange equation.

Instead of using the above alternate method, we can also try to use a gradient method, i.e., an iterative method where the iterations are defined through $u_{i+1} = u_i + t_i v_{i+1}$, $\theta_{i+1} = \theta_i + s_i(\vartheta_{i+1} - \theta_i)$ for some $t_i, s_i \in (0, 1)$, with $(v_{i+1}, \vartheta_{i+1})$ a solution of

$$\begin{cases} \displaystyle\min_{\|v\|_{W_0^{1,p}(\Omega)}\leqslant 1} \left\{ \frac{1}{p}\int_\Omega \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}} \nabla u_i\cdot\nabla v\, dx - \langle f, v\rangle \right\}, \\ \displaystyle\max_{\substack{\vartheta\in L^\infty(\Omega;[0,1])\\ \int_\Omega \vartheta\, dx\leqslant\kappa}} \int_\Omega \frac{|\nabla u_i|^p}{(1+c\theta_i)^p}\,\vartheta\, dx, \end{cases} \tag{3.3}$$

but the minimization in v also implies the resolution of a p-Laplacian problem. To avoid this difficulty, we can replace the constraint $\|v\|_{W_0^{1,p}(\Omega)} \leqslant 1$ by $\|v\|_{H_0^1(\Omega)} \leqslant 1$. This is a feasible direction method. In each iteration, we look for the direction of maximum descent of $\mathcal{F}$ in the convex set:

$$\left\{ (v,\vartheta) \in H_0^1(\Omega)\times L^\infty(\Omega;[0,1]) \;:\; \|v\|_{H_0^1(\Omega)} \leqslant 1,\; \int_\Omega \vartheta dx \leqslant \kappa \right\}.$$

The maximum direction with respect to ϑ is simple to calculate. Namely, reasoning as in Remark 2.1, we have the following.

If $|\{|\nabla u_i| > 0\}| \leqslant \kappa$, then ϑ_i is any function in $L^\infty(\Omega; [0, 1])$ such that

$$\chi_{\{|\nabla u_i|>0\}} \leqslant \vartheta_i, \qquad \int_\Omega \vartheta_i\, dx \leqslant \kappa. \tag{3.4}$$

In another case, we introduce $H : (0,\infty) \to [0, |\Omega|]$ by

$$H(\mu) = \big|\{x\in\Omega :\; |\nabla u_i(x)| > (1+c\theta_i)\mu\}\quad \forall\, \mu \geqslant 0.$$

Then H is a decreasing function, continuous on the right and satisfying

$$\lim_{\mu\to 0^+} H(\mu) = \big|\{x\in\Omega :\ |\nabla u_i(x)|>0\}\big|, \qquad \lim_{\mu\to\infty} H(\mu)=0.$$

This ensures the existence of $\mu_i>0$ (not unique in general) such that

$$H(\mu_i)\leqslant \kappa \leqslant \lim_{\mu\to\mu_i^-} H(\mu),$$

which can be easily numerically obtained by a dichotomy rule. For such μ_i, the maximum direction in θ in (3.3), ϑ_i, is given by any function in $L^\infty(\Omega;[0,1])$ such that

$$\chi_{\{|\nabla u_i|>(1+c\theta_i)\mu_i\}}\leqslant \vartheta_i, \qquad \int_\Omega \vartheta_i\,dx=\kappa. \tag{3.5}$$

A similar result holds if we use a finite-dimensional approximation consisting in choosing θ taking constant values in the elements of a given mesh. On the other hand, the maximum descent direction with respect to v is unique, and it is the solution of a linear equation. However, we observe that for $p>2$, the sequence of functions $\{u_i\}$ generated by the method is not in $W_0^{1,p}(\Omega)$. Thus, the algorithm has only a sense of using a finite-dimensional space instead of $L^\infty(\Omega)\times W_0^{1,p}(\Omega)$. In such a case, all the norms are equivalent. However it would be necessary to prove the convergence of the solutions of the discretized problem to the continuous one.

With these considerations, we are going to be interested in the problem

$$\min\left\{\mathcal{F}(\theta,u):\quad \theta\in\Theta,\quad u\in V,\quad \int_\Omega \theta\,dx\leqslant\kappa,\quad \theta\in[0,1] \text{ a.e. in } \Omega\right\}, \tag{3.6}$$

with Θ and V finite-dimensional subspaces of $L^\infty(\Omega)$ and $H_0^1(\Omega)\cap W_0^{1,p}(\Omega)$, respectively. As in the continuous problem, it is not clear that (3.6) has a unique solution, but for every solution (θ^*,u^*), the flow

$$\sigma^*=\frac{|\nabla u^*|^{p-2}}{(1+c\theta^*)^{p-1}}\nabla u^* \tag{3.7}$$

is unique because it is a solution of (see (2.10))

$$\min\left\{\max_{\substack{\theta\in L^\infty(\Omega:[0,1])\\ \int_\Omega\theta\,dx\leqslant\kappa}}\int_\Omega(1+c\,\theta)|\sigma|^{p'}dx:\ \sigma\in L^{p'}(\Omega)^N,\quad \int_\Omega\sigma\cdot\nabla v\,dx=\langle f,v\rangle,\ \forall\, v\in V\right\},$$

where the function to minimize is strictly convex.

As an example of practical interest, we can consider a regular triangular mesh $\mathcal{T}_h$ of $\overline{\Omega}$ with maximum diameter $h>0$ and the Lagrange finite element spaces

$$\Theta_h=\Big\{v=\sum_{\tau\in\mathcal{T}_h}\alpha_\tau\mathcal{X}_\tau\ :\alpha_\tau\in\mathbb{R}\quad\forall\tau\in\mathcal{T}_h\Big\}, \tag{3.8}$$

$$V_h=\big\{v\in C_0^0(\Omega)\ :\ v|_\tau\in\mathbb{P}_1(\tau)\quad\forall\tau\in\mathcal{T}_h\big\}, \tag{3.9}$$

with $\mathbb{P}_1(\tau)$ the space of affine functions in τ.

Since the minimization of $\mathcal{F}$ in θ for u fixed is simple to carry out in practice (see (2.1) for the infinite-dimensional case; the finite-dimensional one is analogous), we can also consider a variant of the previous algorithm consisting in directly computing the minimum in θ in each iteration.

With these considerations, we present the following two algorithms.

Algorithm 1.

Initialization: $i=1$, $\theta_0\in\Theta$, $u_0\in V$, $a,b\in(0,1)$.

1: Set v_i a solution of

$$\int_\Omega \nabla v_i\cdot\nabla\phi\,dx = \langle f,\phi\rangle - \int_\Omega \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}}\nabla u_i\cdot\nabla\phi\,dx \quad \forall\phi\in V. \tag{3.10}$$

2: Choose the step length by $t_i=b^j$ (Armijo's rule), with j the smallest nonnegative integer such that

$$\mathcal{F}(\theta_i,u_i+t_iv_i)\leqslant\mathcal{F}(\theta_i,u_i)-at_i\int_\Omega|\nabla v_i|^2dx, \tag{3.11}$$

and set $u_{i+1}=u_i+t_iv_i$.

3: Set ϑ_i a solution of

$$\max\left\{\int_\Omega\frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}\vartheta\,dx:\quad \vartheta\in\Theta,\ 0\leqslant\vartheta\leqslant1 \text{ a.e. in } \Omega,\ \int_\Omega\vartheta dx\leqslant\kappa\right\}. \tag{3.12}$$

4: Choose $s_i=b^k$, with k the smallest nonnegative integer such that

$$\mathcal{F}(\theta_i+s_i(\vartheta_i-\theta_i),u_{i+1})\leqslant\mathcal{F}(\theta_i,u_{i+1})-as_i\frac{c(p-1)}{p}\int_\Omega\frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\vartheta_i-\theta_i)\,dx, \tag{3.13}$$

and set $\theta_{i+1}=\theta_i+s_i(\vartheta_i-\theta_i)$.

Algorithm 2.

Initialization: $i=0$, $u_0\in V$, $a,b\in(0,1)$.

1: Set $v_i\in V$ the solution of (3.10).
2: Choose the step length by $t_i=b^j$ with j the smallest nonnegative integer such that (3.11) is satisfied, and set $u_{i+1}=u_i+t_iv_i$.
3: Set θ_{i+1} a solution of

$$\min\left\{\int_\Omega\frac{|\nabla u_i|^p}{(1+c\vartheta)^{p-1}}dx:\quad \vartheta\in\Theta,\ 0\leqslant\vartheta\leqslant1 \text{ a.e. in } \Omega,\ \int_\Omega\vartheta\,dx\leqslant\kappa\right\}. \tag{3.14}$$

Remark 3.1. Since by definition (3.10) of v_i we have

$$\lim_{t\to0}\frac{\mathcal{F}(\theta_i,u_i+tv_i)-\mathcal{F}(\theta_i,u_i)}{t}=-\int_\Omega|\nabla v_i|^2dx$$

and $a<1$, we get that

$$\mathcal{F}(\theta_i,u_i+tv_i)-\mathcal{F}(\theta_i,u_i)\leqslant -at\int_\Omega|\nabla v_i|^2dx,$$

for $0<t$ small enough. This proves the existence of t_i satisfying (3.11). A similar argument shows the existence of s_i in (3.13).

Remark 3.2. If $p=2$, then $t_i=1$ for both algorithms and every $i\geqslant0$. The second method agrees in this case with the one given in [1] Theorem 5.1.5, and [22].

Our main result is given by theorem 3.1 below which provides the convergence for both algorithms. Before stating it, we need the following definition.

DEFINITION 3.3. *For $p > 1$, we define $\gamma_p > 0$ by*

$$\begin{cases} \|v\|_{H_0^1(\Omega)} \leqslant \gamma_p \|v\|_{W_0^{1,p}(\Omega)} & \text{if } 1 < p < 2 \\ \|v\|_{W_0^{1,p}(\Omega)} \leqslant \gamma_p \|v\|_{H_0^1(\Omega)} & \text{if } p \geqslant 2 \end{cases} \qquad \forall\, v \in V. \tag{3.15}$$

Remark 3.4. Clearly $\gamma_2 = 1$, while for $p \neq 2$ and V replaced by a sequence of finite-dimensional spaces V_h such that

$$\lim_{h\to 0} \min_{v\in V_h} \|v - v_h\|_{W_0^{1,p}(\Omega)} = 0 \quad \forall\, v \in W_0^{1,p}(\Omega),$$

we have that $\gamma_p = \gamma_{p,h}$ tends to infinity when h goes to zero. For example, in the case where the spaces V_h are given by (3.9), with $\mathcal{T}_h$ a sequence of regular meshes of diameter h, we have

$$\gamma_{p,h} = O\left(\frac{1}{h^{N|\frac{1}{2}-\frac{1}{p}|}}\right). \tag{3.16}$$

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $p \in (1,\infty)$, $f \in W^{-1,p'}(\Omega)$, and Θ, V finite-dimensional subspaces of $L^\infty(\Omega)$ and $W_0^{1,p}(\Omega)$, respectively. Taking $(\theta_i, u_i) \in \Theta \times V$, the sequence defined by Algorithm 1 or 2, and denoting by $\mathcal{F}^*$ the minimum value of (3.6) and by e_i the sequence of errors*

$$e_i = \mathcal{F}(\theta_i, u_i) - \mathcal{F}^* \geqslant 0, \quad i \geqslant 0, \tag{3.17}$$

we have that e_i is a decreasing sequence and that there exists $C > 0$ depending on $a, b, u_0, \theta_0, f, c, N$, and p such that

$$e_i \leqslant \begin{cases} C\gamma_p^p\, i^{-\frac{1}{p-1}} & \text{if } 1 < p < 2 \\ C\gamma_p^4\, i^{-1} & \text{if } p \geqslant 2 \end{cases} \qquad \forall i \geqslant 1. \tag{3.18}$$

Moreover, the sequence

$$\sigma_i = \frac{|\nabla u_i|^{p-2}}{(1 + c\theta_i)^{p-1}} \nabla u_i \tag{3.19}$$

converges strongly to σ^* defined by (3.7) in $L^{p'}(\Omega)^N$. Namely, there exists $C > 0$ as above such that

$$\int_\Omega \big(|\sigma^*|+|\sigma_i|\big)^{p-2}|\sigma^*-\sigma|^2 dx \leqslant \begin{cases} C\big(1 + \gamma_p \|u_0\|_{W_0^{1,p}(\Omega)}\big)(e_i - e_{i+1})^{\frac{1}{p'}} & \text{if } 1 < p < 2, \\ C\big(|1 + \gamma_p^2 \|u_0\|^{p-1}_{W_0^{1,p}(\Omega)}\big)(e_i - e_{i+1})^{\frac{1}{2}} & \text{if } p \geqslant 2. \end{cases} \tag{3.20}$$

Remark 3.5. In the continuous case $V = W_0^{1,p}(\Omega)$ and $1 < p < 2$, the classical regularity results for the Poisson equation show that the solution v_i of (3.10) satisfies the estimate

$$\|v_i\|_{W_0^{1,p'}(\Omega)} \leqslant C\big(\|u_i\|_{W_0^{1,p}(\Omega)} + \|f\|_{W^{-1,p'}(\Omega)}\big), \tag{3.21}$$

with $C > 0$ depending only on p and Ω. Thanks to this result, we can deduce that in this case (3.18) holds true with γ_p replaced by one. Similar results to (3.21) also hold for special choices of spaces V; see, e.g., [4, Theorem 8.5.3]. With these choices, we can eliminate the dependence in γ_p of estimate (3.18) for $1 < p < 2$.

Remark 3.6. In the case of the p-Laplacian problem, i.e.,

$$\min_{u\in W_0^{1,p}(\Omega)}\left\{\frac{1}{p}\int_\Omega |\nabla u|^p dx-\langle f,u\rangle\right\},$$

we can consider the following algorithm, similar to Algorithms 1 and 2.

Initialization: $i=0$, $u_0\in V$, $a,b\in(0,1)$.

1: Set $v_i\in V$ the solution of

$$\int_\Omega \nabla v_i\cdot\nabla\phi\,dx=\langle f,\phi\rangle-\int_\Omega |\nabla u_i|^{p-2}\nabla u_i\cdot\nabla\phi\,dx\quad \forall\phi\in V. \tag{3.22}$$

2: Choose the step length by $t_i=b^j$ with j the smallest nonnegative integer such that

$$\begin{aligned}&\frac{1}{p}\int_\Omega |\nabla(u_i+t_iv_i)|^p dx-\langle f,u_i+t_iv_i\rangle\\ &\leqslant \frac{1}{p}\int_\Omega |\nabla u_i|^p dx-\langle f,u_i\rangle-at_i\int_\Omega |\nabla v_i|^2dx,\end{aligned}$$

and set $u_{i+1}=u_i+t_iv_i$.

Then a similar reasoning to the one used below to prove Theorem 3.1 shows the estimates

$$\mathcal{F}(u_i)-\mathcal{F}^*\leqslant\begin{cases}\dfrac{C\gamma_p^{\frac{2p}{2-p}}}{i^{\frac{2(p-1)}{2-p}}} & \text{if } p<2,\\ C^i & \text{if } p=2,\\ \dfrac{C\gamma_p^{\frac{2p}{p-2}}}{i^{\frac{p}{p-2}}} & \text{if } p>2,\end{cases} \tag{3.23}$$

with $C<1$ for $p=2$. Similarly to Remark 3.5, the dependence of the estimate on γ_p can be suppressed for $1<p<2$ in the continuous case or V finite-dimensional but satisfying further assumptions. Observe that estimates (3.23) are better than the ones obtained in Theorem 3.1. This is due to the strict convexity of the p-Laplacian operator, which does not hold in our case.

We finish this section studying the convergence of the solutions of the discrete problem to the solutions of the continuous one. The next result is an immediate consequence of the convexity of $\mathcal{F}$ and therefore is given without proof.

PROPOSITION 3.1. *Assume two sequences of spaces $\Theta_h\subset L^\infty(\Omega)$ and $V_h\subset W_0^{1,p}(\Omega)$ such that*

- *for every $\theta\in L^\infty(\Omega)$, with $\theta\geqslant 0$, there exists a sequence $\theta_h\in\Theta_h$ such that*

$$0\leqslant\theta_h\leqslant\|\theta\|_{L^\infty(\Omega)},\qquad \int_\Omega\theta_h dx\leqslant\int_\Omega\theta dx,\qquad \theta_h\to\theta\ \text{ in } L^1(\Omega); \tag{3.24}$$

- *for every $u\in W_0^{1,p}(\Omega)$, there exists a sequence $u_h\in V_h$ such that*

$$u_h\to u\ \text{ in } W_0^{1,p}(\Omega). \tag{3.25}$$

Then, defining $\mathcal{F}_h^*$ as the value of the minimum in (3.6) *with Θ and V replaced by Θ_h and V_h, respectively, and $\hat{\mathcal{F}}$ as the value of the minimum in* (3.1)*, we have*

$$\lim_{h\to 0}\mathcal{F}_h^*=\hat{\mathcal{F}}. \tag{3.26}$$

Moreover, defining σ_h^* by (3.7)*, with (θ^*, u^*) any solution of* (3.6) *for $\Theta = \Theta_h$, $V = V_h$, we have*

$$\sigma_h^* \to \hat{\sigma} \quad in\ L^{p'}(\Omega)^N, \tag{3.27}$$

with $\hat{\sigma}$ defined by (2.11)*.*

An example of spaces satisfying properties (3.24) and (3.25) is given by (3.8) and (3.9). In this case, we have the following improvement

THEOREM 3.2. *Assume Ω a polygonal open set, $f \in W^{-1,\infty}(\Omega) \cap L^1(\Omega)$, and that there exists a solution $(\hat{\theta}, \hat{u})$ of* (2.1) *such that*

$$\hat{\theta} \in BV(\Omega), \quad \hat{u} \in W^{1,\infty}(\Omega), \quad \nabla\hat{u} \in BV(\Omega)^N. \tag{3.28}$$

We also consider a regular sequence $\mathcal{T}_h$ of triangulations in Ω by N-simplexes and define the spaces Θ_h and V_h by (3.8) *and* (3.9)*, respectively. Then there exists $C > 0$, depending on Ω, p, and the functions $\hat{\theta}$, $\hat{u}$, such that denoting by $\hat{\mathcal{F}}$ and $\mathcal{F}_h^*$ the minimum values of* (2.1) *and* (3.6)*, respectively, with $\Theta = \Theta_h$ and $V = V_h$, we have*

$$\hat{\mathcal{F}} \leqslant \mathcal{F}_h^* \leqslant \hat{\mathcal{F}} + Ch \quad \forall\, h > 0. \tag{3.29}$$

Moreover, the functions σ_h^* and σ^* defined as in Proposition 3.1 *satisfy*

$$\int_\Omega (|\sigma^*| + |\sigma_h^*|)^{p-2} |\sigma^* - \sigma_h^*|^2 dx \leqslant Ch. \tag{3.30}$$

Remark 3.7. In Theorem 2.1, we recalled some smoothness results for problem (2.1). Contrary to Theorem 3.2, they assumed that Ω is $C^{1,1}$ instead of a polygonal set. Indeed, assuming Ω a smooth convex set, Theorem 3.2 could still be applied, taking a sequence of regular meshes for polygonal subsets of Ω which fulfill Ω as the limit. Even with this assumption, we do not know that θ and ∇u are in $BV(\Omega)$ and $BV(\Omega)^N$, but numerical simulations usually provide solutions which seem to satisfy these assumptions.

From (3.16), (3.18), and (3.29), we get the following.

COROLLARY 3.1. *In the assumptions of Theorem* 3.2*, we have the estimates*

$$0 \leqslant \mathcal{F}(\theta_{i,h}, u_{i,h}) - \hat{\mathcal{F}} \leqslant \begin{cases} C\Big(\dfrac{1}{h^{N(1-\frac{p}{2})} i^{p-1}} + h\Big) & \text{if } 1 < p < 2, \\ C\Big(\dfrac{1}{h^{2N(1-\frac{2}{p})} i} + h\Big) & \text{if } p \geqslant 2. \end{cases} \tag{3.31}$$

Here $\hat{\mathcal{F}}$ denotes the minimum value of (2.1)*, $(\theta_{i,h}, u_{i,h})$ is the ith pair obtained by any of the algorithms, and Θ_h, V_h are defined by* (3.8) *and* (3.9)*, respectively.*

4. Convergence proof. We dedicate this section to prove the results stated in the previous one. In order to simplify the proof of Theorem 3.1, we start with the following lemma.

LEMMA 4.1. *Assume $p \in (1, \infty)$. Then we have the following:*

1. *There exists $C > 0$, depending only on p, such that for every $\xi, \eta \in \mathbb{R}^N$, we get*

$$\big||\eta|^p - |\xi|^p - p|\xi|^{p-2}\xi\cdot(\eta - \xi)\big| \leqslant \begin{cases} C|\xi - \eta|^p & \text{if } p < 2, \\ C\big(|\xi| + |\eta|\big)^{p-2} |\xi - \eta|^2 & \text{if } p \geqslant 2. \end{cases} \tag{4.1}$$

2. *There exists $C > 0$, depending only on p and c, such that for every $q, r \in [0,1]$, we get*

$$\left| \frac{1}{(1+cr)^{p-1}} - \frac{1}{(1+cq)^{p-1}} + \frac{(p-1)c(r-q)}{(1+cq)^p} \right| \leqslant C|r-q|^2. \tag{4.2}$$

Proof. In order to show (4.1), we first recall the following property of the function $\xi \in \mathbb{R}^N \mapsto |\xi|^{p-2}\xi \in \mathbb{R}^N$: There exists $c_p > 0$ such that for every $\xi, \eta \in \mathbb{R}^N$, we have

$$\big||\eta|^{p-2}\eta - |\xi|^{p-2}\xi\big| \leqslant \begin{cases} c_p|\xi-\eta|^{p-1} & \text{if } p<2, \\ c_p\big(|\xi|+|\eta|\big)^{p-2}|\eta-\xi| & \text{if } p \geqslant 2. \end{cases} \tag{4.3}$$

By the mean value theorem, for every $\xi, \eta \in \mathbb{R}^N$, there exists $\lambda \in (0,1)$ such that

$$|\eta|^p - |\xi|^p = p\big|\lambda\xi + (1-\lambda)\eta\big|^{p-2}\big(\lambda\xi + (1-\lambda)\eta\big) \cdot (\eta - \xi), \tag{4.4}$$

where, thanks to (4.3), we have

$$\begin{aligned} &\Big|\big|\lambda\xi + (1-\lambda)\eta\big|^{p-2}\big(\lambda\xi + (1-\lambda)\eta\big) - |\xi|^{p-2}\xi\Big| \\ &\leqslant \begin{cases} c_p|\xi-\eta|^{p-1} & \text{if } p<2, \\ 2^{p-2}c_p\big(|\xi|+|\eta|\big)^{p-2}|\eta-\xi| & \text{if } p \geqslant 2. \end{cases} \end{aligned} \tag{4.5}$$

This proves (4.1). Let us now show (4.2). As above, for every $q, r \in [0,1]$, the mean value theorem provides the existence of $\lambda \in (0,1)$ such that

$$\frac{1}{(1+cr)^{p-1}} - \frac{1}{(1+cq)^{p-1}} = -\frac{(p-1)c(r-q)}{\big(1+c(\lambda q + (1-\lambda)r)\big)^p}, \tag{4.6}$$

where

$$\left| \frac{1}{\big(1+c(\lambda q + (1-\lambda)r)\big)^p} - \frac{1}{(1+cq)^p} \right| = \frac{\Big|(1+cq)^p - \big(1+c(\lambda q + (1-\lambda)r)\big)^p\Big|}{\big(1+c(\lambda q + (1-\lambda)r)\big)^p(1+cq)^p}.$$

Using here the mean value theorem in the numerator, that the denominator is bigger or equal than 1, and that $q, r \in [0,1]$, we get

$$\begin{aligned} &\left| \frac{1}{\big(1+c(\lambda q + (1-\lambda)r)\big)^p} - \frac{1}{(1+cq)^p} \right| \leqslant pc^p(2+c(q+r))^{p-1}|q-r| \\ &\leqslant p2^{p-1}c^p(1+c)^{p-1}|q-r|. \end{aligned} \tag{4.7}$$

Inequalities (4.6) and (4.7) show (4.2). ∎

The proof of Theorem 3.1 also uses the following lemma, which has been obtained in [13, Lemma 1].

LEMMA 4.2. *Assume $\nu > 0$, $\gamma > 1$ and a sequence of positive numbers λ_n such that*

$$\lambda_n - \lambda_{n+1} \geqslant \nu\lambda_n^\gamma \quad \forall\, n \geqslant 0.$$

Then, for $r = 1/(\gamma - 1)$, we have

$$\lambda_n \leqslant \frac{1}{n^r} \max\left\{\lambda_0, \Big(\frac{2^r-1}{\nu}\Big)^r\right\}. \tag{4.8}$$

Proof of Theorem 3.1. Let us first prove estimate (3.18) for Algorithm 1.

For every $i \geqslant 0$, estimate (4.1), Hölder's inequality and definition (3.10) of v_i imply the existence of $C > 0$ depending only on p such that the following hold:

If $1 < p < 2$,

$$\begin{aligned}
&\mathcal{F}(\theta_i, u_i + tv_i) - \mathcal{F}(\theta_i, u_i) \\
&\leqslant t\left(\int_\Omega \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}} \nabla u_i \cdot \nabla v_i \, dx - \langle f, v_i\rangle\right) + Ct^p \int_\Omega |\nabla v_i|^p dx \\
&= -t\|v_i\|^2_{H^1_0(\Omega)} + Ct^p \|v_i\|^p_{W^{1,p}_0(\Omega)}.
\end{aligned} \tag{4.9}$$

If $p \geqslant 2$,

$$\begin{aligned}
\mathcal{F}(\theta_i, u_i + tv_i) - \mathcal{F}(\theta_i, u_i) &\leqslant t\left(\int_\Omega \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}} \nabla u_i \cdot \nabla v_i \, dx - \langle f, v_i\rangle\right) \\
&\quad + Ct^2\Big(\|u_i\|_{W^{1,p}_0(\Omega)} + \|u_i + tv_i\|_{W^{1,p}_0(\Omega)}\Big)^{p-2} \|v_i\|^2_{W^{1,p}_0(\Omega)} \\
&= -t\|v_i\|^2_{H^1_0(\Omega)} + Ct^2\Big(\|u_i\|_{W^{1,p}_0(\Omega)} + \|u_i + tv_i\|_{W^{1,p}_0(\Omega)}\Big)^{p-2} \|v_i\|^2_{W^{1,p}_0(\Omega)}.
\end{aligned} \tag{4.10}$$

Now we observe that if $t_i < 1$, then, by definition of t_i, we have

$$\mathcal{F}(\theta_i, u_i + bt_iv_i) - \mathcal{F}(\theta_i, u_i) > -abt_i\|v_i\|^2_{H^1_0(\Omega)}.$$

Combined with (4.9) or (4.10), this proves the existence of $\tau > 0$, which only depends on a, b, and p such that

$$t_i \geqslant \begin{cases} \min\left\{1, \tau\dfrac{\|v_i\|^{\frac{2}{p-1}}_{H^1_0(\Omega)}}{\|v_i\|^{p'}_{W^{1,p}_0(\Omega)}}\right\} & \text{if } 1 < p < 2, \\[2ex] \min\left\{1, \tau\dfrac{\|v_i\|^2_{H^1_0(\Omega)}}{\Big(\|u_i\|_{W^{1,p}_0(\Omega)} + \|u_{i+1}\|_{W^{1,p}_0(\Omega)}\Big)^{p-2}\|v_i\|^2_{W^{1,p}_0(\Omega)}}\right\} & \text{if } p \geqslant 2. \end{cases} \tag{4.11}$$

On the other hand, inequality (4.2) implies the existence of another constant $C > 0$ depending only on p and c such that

$$\begin{aligned}
&\mathcal{F}(\theta_i + s(\vartheta_i - \theta_i), u_{i+1}) - \mathcal{F}(\theta_i, u_{i+1}) \\
&\leqslant -s\frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\vartheta_i - \theta_i)\, dx + Cs^2\int_\Omega |\nabla u_{i+1}|^p|\vartheta_i - \theta_i|^2 dx,
\end{aligned}$$

which, reasoning as above, implies the existence of $\lambda > 0$ depending only on a, b, c, and p such that

$$s_i \geqslant \min\left\{1, \lambda\frac{\displaystyle\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\vartheta_i - \theta_i)dx}{\displaystyle\int_\Omega |\nabla u_{i+1}|^p|\vartheta_i - \theta_i|^2 dx}\right\}. \tag{4.12}$$

Using that

$$\begin{aligned} e_i - e_{i+1} &= \mathcal{F}(\theta_i, u_i) - \mathcal{F}^* - \big(\mathcal{F}(\theta_{i+1}, u_{i+1}) - \mathcal{F}^*\big) \\ &= \mathcal{F}(\theta_i, u_{i+1}) - \mathcal{F}(\theta_{i+1}, u_{i+1}) + \mathcal{F}(\theta_i, u_i) - \mathcal{F}(\theta_i, u_{i+1}); \end{aligned}$$

inequalities (3.11), (3.13), (4.11), and (4.12); and

$$\|v_i\|_{W_0^{1,p}(\Omega)} \leqslant \begin{cases} |\Omega|^{\frac{1}{p}-\frac{1}{2}} \|v_i\|_{H_0^1(\Omega)} & \text{if } 1 < p < 2, \\ \gamma_p \|v_i\|_{H_0^1(\Omega)} & \text{if } p \geqslant 2, \end{cases}$$

we deduce the existence of $C > 0$ depending only on a, b, c, p, and $|\Omega|$ such that the following hold:If $1 < p < 2$,

$$e_i - e_{i+1} \geqslant C \min\left\{1, \frac{\Big(\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\vartheta_i - \theta_i)dx\Big)^2}{\int_\Omega |\nabla u_{i+1}|^p |\vartheta_i - \theta_i|^2 dx} + \|v_i\|_{H_0^1(\Omega)}^{p'}\right\}. \tag{4.13}$$

If $p \geqslant 2$,

$$\begin{aligned} &e_i - e_{i+1} \\ &\geqslant C \min\left\{1, \frac{\Big(\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^2}(\vartheta_i - \theta_i)dx\Big)^2}{\int_\Omega |\nabla u_{i+1}|^2 |\vartheta_i - \theta_i|^2 dx} + \frac{\gamma_p^{-4}\|v_i\|^2_{W_0^{1,p}(\Omega)}}{\big(\|u_i\|_{W_0^{1,p}(\Omega)} + \|u_{i+1}\|_{W_0^{1,p}(\Omega)}\big)^{p-2}}\right\}. \end{aligned} \tag{4.14}$$

In particular, e_i is a nonnegative and nonincreasing sequence and therefore a converging sequence. In particular $e_i - e_{i+1}$ tends to zero. Moreover, e_i nonincreasing implies that $\mathcal{F}(\theta_i, u_i)$ and then $\|u_i\|_{W_0^{1,p}(\Omega)}$ are bounded.

On the other hand, thanks to the convexity of $\mathcal{F}$; $u_{i+1} = u_i + t_i v_i$, with $0 \leqslant t_i \leqslant 1$; and definitions (3.10) and (3.12) of v_i and ϑ_i, respectively, we have

$$\begin{aligned} e_i = \mathcal{F}(\theta_i, u_i) - \mathcal{F}(\theta^*, u^*) &\leqslant \int_\Omega \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}} \nabla u_i \cdot \nabla(u_i - u^*)\,dx + \langle f, u_i - u^*\rangle \\ &\quad - \frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\theta_i - \theta^*)dx + Ct_i \int_\Omega \big(|\nabla u_i| + |\nabla u_{i+1}|\big)^{p-1}|\nabla v_i|dx \\ &\leqslant \int_\Omega \nabla v_i \cdot \nabla(u_i - u^*)\,dx - \frac{c(p-1)}{p}\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\theta_i - \vartheta_i)dx \\ &\quad + C\int_\Omega \big(|\nabla u_i| + |\nabla u_{i+1}|\big)^{p-1}|\nabla v_i|dx, \end{aligned} \tag{4.15}$$

where C only depends on p and c. Combined with Hölder's inequality, (4.13), (4.14), and

$$\min\Big\{\|u^*\|_{W_0^{1,p}(\Omega)}, \min_{i\geqslant 0}\|u_i\|_{W_0^{1,p}(\Omega)}\Big\} \leqslant C\|u_0\|_{W_0^{1,p}(\Omega)},$$

which is a consequence of e_i nonincreasing and the definition of u^*, we conclude

$$e_{i-1} \leqslant \begin{cases} C\gamma_p\big(\|u_0\|_{W_0^{1,p}(\Omega)} + 1\big)(e_{i-1} - e_{i+1})^{\frac{p-1}{p}} & \text{if } 1 < p < 2, \\ C\gamma_p^2\big(\|u_0\|^p_{W_0^{1,p}(\Omega)} + 1\big)(e_{i-1} - e_{i+1})^{\frac{1}{2}} & \text{if } p \geqslant 2. \end{cases} \tag{4.16}$$

This inequality allows us to use Lemma 4.2 to get (3.18) for the first algorithm.

For Algorithm 2, using again (4.9) or (4.10), we get that (4.11) still holds true. Combined with

$$\mathcal{F}(\theta_{i+1}, u_{i+1}) \leqslant \mathcal{F}(\theta_i, u_{i+1}),$$

we have analogously to (4.13) and (4.14)

$$e_i - e_{i+1} \geqslant \begin{cases} C \min\left\{1, \|v_i\|^{p'}_{H^1_0(\Omega)}\right\} & \text{if } 1 < p < 2, \\ C \min\left\{1, \dfrac{\gamma_p^{-4}\|v_i\|^2_{W_0^{1,p}(\Omega)}}{\left(\|u_i\|_{W_0^{1,p}(\Omega)} + \|u_{i+1}\|_{W_0^{1,p}(\Omega)}\right)^{p-2}}\right\} & \text{if } p \geqslant 2. \end{cases} \tag{4.17}$$

Using then that by convexity, θ_{i+1} solution of (3.14) is equivalent to θ_{i+1} solution of

$$\max\left\{\int_\Omega \frac{|\nabla u_i|^p}{(1+c\theta_{i+1})^p}\,\vartheta\,dx : \quad \vartheta \in \Theta,\ 0 \leqslant \vartheta \leqslant 1 \text{ a.e. in } \Omega,\ \int_\Omega \vartheta dx \leqslant \kappa\right\},$$

we have similarly to (4.15)

$$e_i \leqslant \int_\Omega \nabla v_i \cdot \nabla(u_i - u^*)\,dx + C\int_\Omega \big(|\nabla u_{i-1}| + |\nabla u_i|\big)^{p-1}|\nabla v_{i-1}|dx.$$

Using here

$$e_{i-1} = e_i + e_i - e_{i-1}, \qquad e_{i-1} - e_i, e_i - e_{i+1} \leqslant e_{i-1} - e_{i+1}$$

and taking into account (4.17), we conclude similarly to (4.16)

$$e_{i-1} \leqslant \begin{cases} C\gamma_p\big(\|u_0\|_{W_0^{1,p}(\Omega)} + 1\big)(e_{i-1} - e_{i+1})^{\frac{p-1}{p}} & \text{if } 1 < p < 2, \\ C\gamma_p^2\big(\|u_0\|^p_{W_0^{1,p}(\Omega)} + 1\big)(e_{i-1} - e_{i+1})^{\frac{1}{2}} & \text{if } p \geqslant 2, \end{cases} \tag{4.18}$$

which, by Lemma 4.2, proves that (3.18) also holds true for the second algorithm.

Let us now estimate the difference between σ_i and σ^*. To simplify the exposition, we just prove the result for Algorithm 1; the proof for Algorithm 2 is completely similar.

We consider a solution (θ^*, u^*) of (3.6). Then (θ^*, σ^*) is a solution of the discrete version of (2.10). Combined with the strict convexity properties of the function $\xi \in \mathbb{R}^N \mapsto |\xi|^p \in \mathbb{R}$, we get

$$\begin{aligned} &\int_\Omega (1+c\theta^*)|\sigma^*|^{p'}dx \geqslant \int_\Omega (1+c\theta_i)|\sigma^*|^{p'}dx \\ &\geqslant \int_\Omega (1+c\theta_i)|\sigma_i|^{p'}dx + p'\int_\Omega (1+c\theta_i)|\sigma_i|^{p'-2}\sigma_i\cdot(\sigma^* - \sigma_i)\,dx \\ &\quad + \rho\int_\Omega \big(|\sigma^*| + |\sigma_i|\big)^{p-2}|\sigma^* - \sigma_i|^2dx \\ &= \int_\Omega (1+c\vartheta_i)|\sigma_i|^{p'}dx + p'\int_\Omega (1+c\theta_i)|\sigma_i|^{p'-2}\sigma_i\cdot(\sigma^* - \sigma_i)\,dx \\ &\quad + \rho\int_\Omega \big(|\sigma^*| + |\sigma_i|\big)^{p-2}|\sigma^* - \sigma_i|^2dx + c\int_\Omega (\theta_i - \vartheta_i)|\sigma_i|^{p'}dx, \end{aligned} \tag{4.19}$$

with ρ positive constants which only depend on p. Similarly, using ϑ_i as a solution of (3.12), we have

$$\begin{aligned}\int_\Omega (1+c\vartheta_i)|\sigma_i|^{p'}dx \geqslant \int_\Omega (1+c\theta^*)|\sigma^*|^{p'}dx\\ + p'\int_\Omega (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\cdot(\sigma_i-\sigma^*)\,dx + \rho\int_\Omega \big(|\sigma^*|+|\sigma_i|\big)^{p-2}|\sigma^*-\sigma_i|^2dx.\end{aligned} \tag{4.20}$$

From (4.19) and (4.20), we deduce

$$\begin{aligned}0 \geqslant p'\int_\Omega \Big((1+c\theta_i)|\sigma_i|^{p'-2}\sigma_i - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_i)\,dx\\ + 2\rho\int_\Omega \big(|\sigma^*|+|\sigma_i|\big)^{p-2}|\sigma^*-\sigma_i|^2dx + c\int_\Omega (\theta_i-\vartheta_i)|\sigma_i|^{p'}dx.\end{aligned} \tag{4.21}$$

Now we use that

$$\begin{aligned}&\Big((1+c\theta_i)|\sigma_i|^{p'-2}\sigma_i - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_i)\\ &= \Big(\frac{|\nabla u^*|^{p-2}}{(1+c\theta^*)^{p-1}}\nabla u^* - \frac{|\nabla u_i|^{p-2}}{(1+c\theta_i)^{p-1}}\nabla u_i\Big)\cdot\nabla(u_i-u^*),\end{aligned}$$

which, taking into account (3.10) and that (θ^*, σ^*) satisfies the discrete version of (2.7), prove

$$\int_\Omega \Big((1+c\theta_i)|\sigma_i|^{p'-2}\sigma_i - (1+c\theta^*)|\sigma^*|^{p'-2}\sigma^*\Big)\cdot(\sigma^*-\sigma_i)\,dx = \int_\Omega \nabla(u_i-u^*)\cdot\nabla v_i\,dx. \tag{4.22}$$

Replacing this equality in (4.21) and recalling $u_{i+1} = u_i + t_i v_i$, with $0 \leqslant t_i \leqslant 1$, we get

$$\begin{aligned}2\rho\int_\Omega \big(|\sigma^*|+|\sigma_i|\big)^{p'-2}|\sigma^*-\sigma_i|^2dx \leqslant c\int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+c\theta_i)^p}(\vartheta_i-\theta_i)dx\\ + C\int_\Omega \big(|\nabla u_i|+|\nabla u^*|\big)^{p-1}|\nabla v_i|dx + p'\int_\Omega \big(|\nabla u_i|+|\nabla u^*|\big)|\nabla v_i|dx\end{aligned} \tag{4.23}$$

with C depending only on p and c. By (4.13) and (4.14), we then conclude (3.20). ☐

Proof of Theorem 3.2. For $(\hat\theta, \hat u)$ the solution of (2.1) which satisfies (3.28), $\hat\sigma$ defined by (2.11), and $h > 0$, we introduce $\hat\theta_h \in \Theta_h$, $\hat\sigma_h \in \Theta_h^N$, and $\hat u_h \in V_h$ by

$$\hat\theta_h = \frac{1}{|\tau|}\int_\tau \hat\theta\,dx, \quad \hat\sigma_h = \frac{1}{|\tau|}\int_\tau \hat\sigma\,dx \quad \forall\,\tau \in \mathcal{T}_h, \tag{4.24}$$

$$\hat u_h(x_i) = \hat u(x_i) \quad \forall\,x_i \text{ vertex of } \mathcal{T}_h. \tag{4.25}$$

Thanks to (3.28) and the regularity of $\mathcal{T}_h$, there exists $C > 0$ such that

$$h\|\hat{u}_h\|_{W^{1,\infty}(\Omega)} + \|\hat{u}_h - \hat{u}\|_{L^\infty(\Omega)} + \|\hat{u}_h - \hat{u}\|_{W_0^{1,1}(\Omega)} + \|\hat{\theta}_h - \hat{\theta}\|_{L^1(\Omega)} \leqslant Ch. \tag{4.26}$$

The definition of $\mathcal{F}$, the mean value theorem, and these estimates imply

$$\begin{aligned}\Big|\mathcal{F}(\hat{\theta},\hat{u}) - \mathcal{F}(\hat{\theta}_h,\hat{u}_h)\Big| &\leqslant C\Big(\|\nabla\hat{u}\|_{L^\infty(\Omega)^N} + \|\nabla\hat{u}_h\|_{L^\infty(\Omega)^N}\Big)^{p-1}\|\nabla(u-u_h)\|_{L^1(\Omega)^N}\\ &+ \|\nabla\hat{u}\|^p_{L^\infty(\Omega)^N}\|\hat{\theta}-\hat{\theta}_h\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}\|\hat{u}-\hat{u}_h\|_{L^\infty(\Omega)} \leqslant Ch.\end{aligned}$$

Then, since the definitions of $\hat{\mathcal{F}}$ and $\mathcal{F}_h^*$ imply

$$\mathcal{F}(\hat{\theta},\hat{u}) = \hat{\mathcal{F}} \leqslant \mathcal{F}_h^* \leqslant \mathcal{F}(\hat{\theta}_h,\hat{u}_h),$$

we conclude (3.29). On the other hand, we consider (θ_h^*, u_h^*) a solution of (3.6) with Θ and V replaced by Θ_h and V_h. We define $\hat{\sigma}$ by (2.11) and σ_h^* by

$$\sigma_h^* = \frac{|\nabla u_h^*|^{p-2}}{(1+c\theta_h^*)^{p-1}}\nabla u_h^*, \tag{4.27}$$

and we recall that, thanks to (2.10), we have

$$\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'}dx = \max\left\{\int_\Omega (1+c\theta)|\hat{\sigma}|^{p'}dx:\ \theta\in L^\infty(\Omega;[0,1]),\ \int_\Omega \theta\,dx \leqslant \kappa\right\}.$$

Thus, we deduce

$$\begin{aligned}&\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'}dx \geqslant \int_\Omega (1+c\theta_h^*)|\hat{\sigma}|^{p'}dx \geqslant \int_\Omega (1+c\theta_h^*)|\sigma_h^*|^{p'}dx\\ &+ p'\int_\Omega (1+c\theta_h^*)|\hat{\sigma}_h^*|^{p'-2}\hat{\sigma}_h^*\cdot(\hat{\sigma}-\hat{\sigma}_h^*)dx + \rho\int_\Omega (|\hat{\sigma}|+\hat{\sigma}_h^*|)^{p-2}|\hat{\sigma}-\hat{\sigma}_h^*|^2dx\end{aligned} \tag{4.28}$$

for some $\rho > 0$, which only depends on p. Using the definitions of $\hat{\sigma}$ and $\hat{\sigma}^*$ and that $(\hat{\theta},\hat{u})$, (θ^*, u_h^*) are solutions of (2.1) and (3.6), we have

$$\int_\Omega (1+c\hat{\theta})|\hat{\sigma}|^{p'}dx = \int_\Omega \frac{|\nabla\hat{u}|^p}{(1+c\hat{\theta})^{p-1}}dx = -p'\hat{\mathcal{F}},$$

$$\int_\Omega (1+c\theta_h^*)|\sigma_h^*|^{p'}dx = \int_\Omega \frac{|\nabla u_h^*|^p}{(1+c\theta_h^*)^{p-1}}dx = -p'\mathcal{F}_h^*,$$

$$p'\int_\Omega (1+c\theta_h^*)|\hat{\sigma}_h^*|^{p'-2}\hat{\sigma}_h^*\cdot(\hat{\sigma}-\hat{\sigma}_h^*)dx = p'\int_\Omega (\hat{\sigma}-\hat{\sigma}_h^*)\cdot\nabla u_h^*\,dx = 0.$$

Replacing these equalities in (4.28) and taking into account (3.29), we get (3.30). ∎

5. Numerical experiments. In this section, we present some simulations for the numerical resolution of (2.1) using the two algorithms presented in section 3. The implementation has been carried out in Python using the finite element solver FeniCs [2].

In our numerical experiments, we have taken $N = 2$, Ω the unit disc, $c = 1$, $f = 1$, and $\kappa = 1$. In this case, the solution of (2.1) is explicitly given by

$$\hat{\theta}(x) = \begin{cases} 1 & \text{if } |x| < \pi^{-\frac{1}{2}}, \\ 0 & \text{if } |x| > \pi^{-\frac{1}{2}}, \end{cases} \qquad \hat{u}(x) = \begin{cases} \dfrac{1}{2^{\frac{1}{p-1}} p'} \left(1 + \pi^{-\frac{p'}{2}} - 2|x|^{p'}\right) & \text{if } |x| < \pi^{-\frac{1}{2}}, \\ \dfrac{1}{2^{\frac{1}{p-1}} p'} \left(1 - |x|^{p'}\right) & \text{if } |x| > \pi^{-\frac{1}{2}}, \end{cases}$$

and thus

$$\hat{\mathcal{F}} = \mathcal{F}(\hat{\theta}, \hat{u}) = -\frac{\pi}{p'(2+p')2^{\frac{1}{p-1}}} \left(1 - \frac{1}{2\pi^{1+\frac{p'}{2}}}\right).$$

We solve the problem for meshes of different diameter h and $p = 1.2, 2, 100$.

The stop criterion for the first algorithm is

$$\int_\Omega |\nabla v_i|^2 dx + \frac{c(p-1)}{p} \int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+\theta_i)^p} (\vartheta_i - \theta_i)\, dx \leqslant 10^{-7} \quad \text{or } i \geqslant 2000, \tag{5.1}$$

while for the second one, it is given by

$$\int_\Omega |\nabla v_i|^2 dx + \frac{c(p-1)}{p} \int_\Omega \frac{|\nabla u_{i+1}|^p}{(1+\theta_{i+1})^p} (\theta_{i+1} - \theta_i) dx \leqslant 10^{-7} \quad \text{or } i \geqslant 2000. \tag{5.2}$$

Observe that in both cases, replacing 10^{-7} by 0 would mean that (θ_i, u_i) satisfies the optimality conditions for (2.1) and then, by the convexity of $\mathcal{F}$, that (θ_i, u_i) is a solution for (3.6).

Depending on p, h, and the choice of the algorithm, we present in Figure 1 the convergence history of the objective function, the Lagrange multiplier μ, and the stop criterion ($\|DF\|$ denotes the left-hand sides in (5.1) and (5.2), respectively). Observe that for $p = 1.2$ and $p = 2$, the rate of convergence for both algorithms is similar. However, for $p = 100$, Algorithm 2 converges faster than Algorithm 1. Although our estimates depend on h, we do not observe this dependence in the numerical experiments for $p = 2$. This is because $\gamma_2 = 1$, and therefore, according to Remark 3.2, the step length is constant, and all the bounds in Theorem 3.1 do not depend on the mesh size.

In Figure 2, we represent the solutions (θ_i, u_i) depending on p but only for the finest mesh. Observe that the solutions obtained are very similar for both algorithms.

In Figure 3, we show the time spent in the resolution of the numerical experiments. We observe that the iterations are calculated faster for Algorithm 2 than for Algorithm 1. When the diameter of the mesh decreases, the time increases for both algorithms in the same way. Moreover, for p large, Algorithm 2 needs fewer iterations than Algorithm 1, while for p small, both algorithms use more or less the same number of iterations.

In Figure 4, we present the value of the objective function for the final interaction for each mesh size and Algorithm 1 and 2. For $p = 1.2$ and $p = 2$, both algorithms have the same behaviours. On the other hand, when $p = 100$ Algorithm 2 achieves a lower value of the objective function in the final iteration for all of the mesh sizes. This agrees with the fact that Algorithm 2 is faster than Algorithm 1, as illustrated in Figure 2.

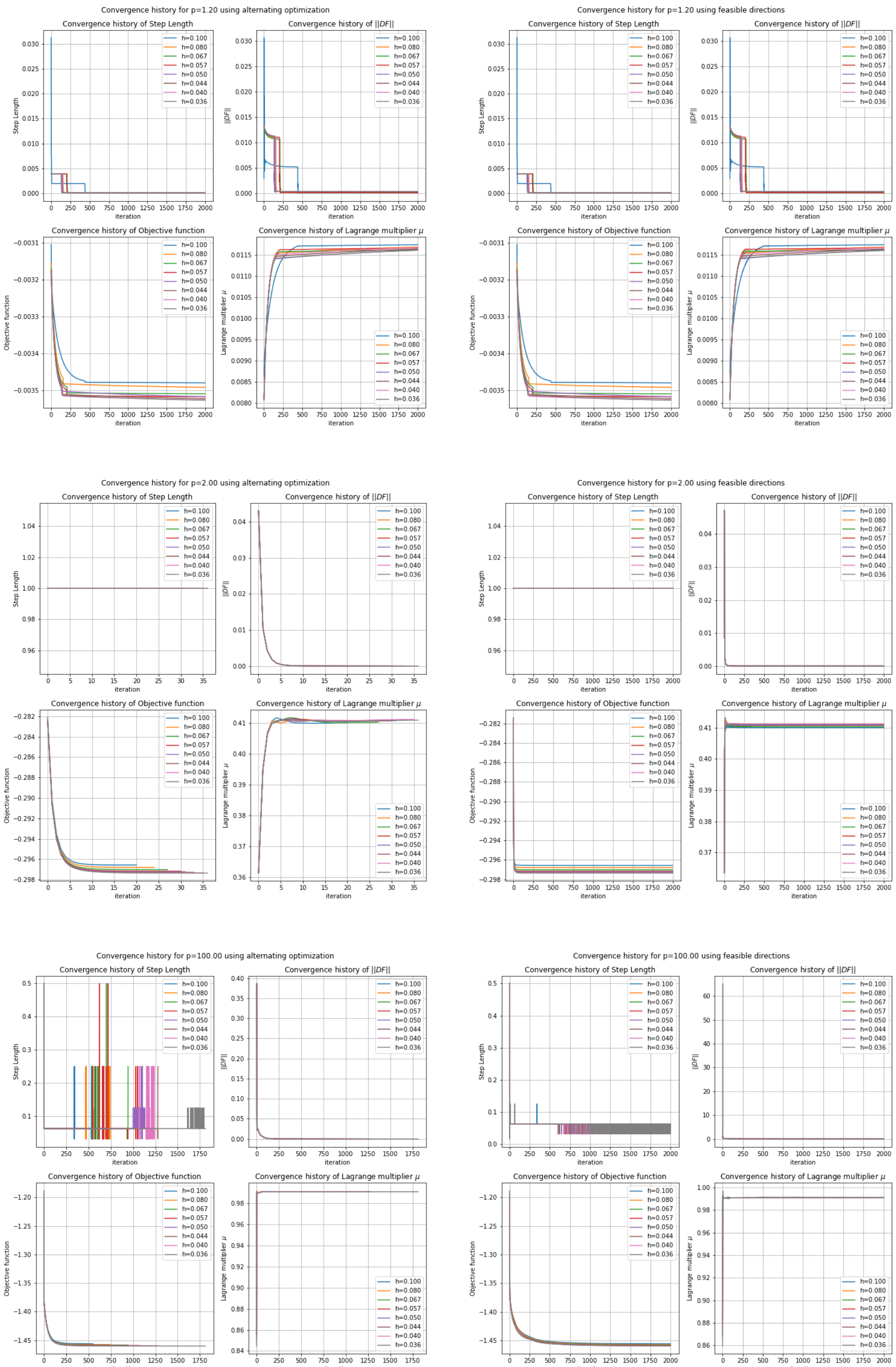

FIG. 1. *Convergence history for each p and mesh.*

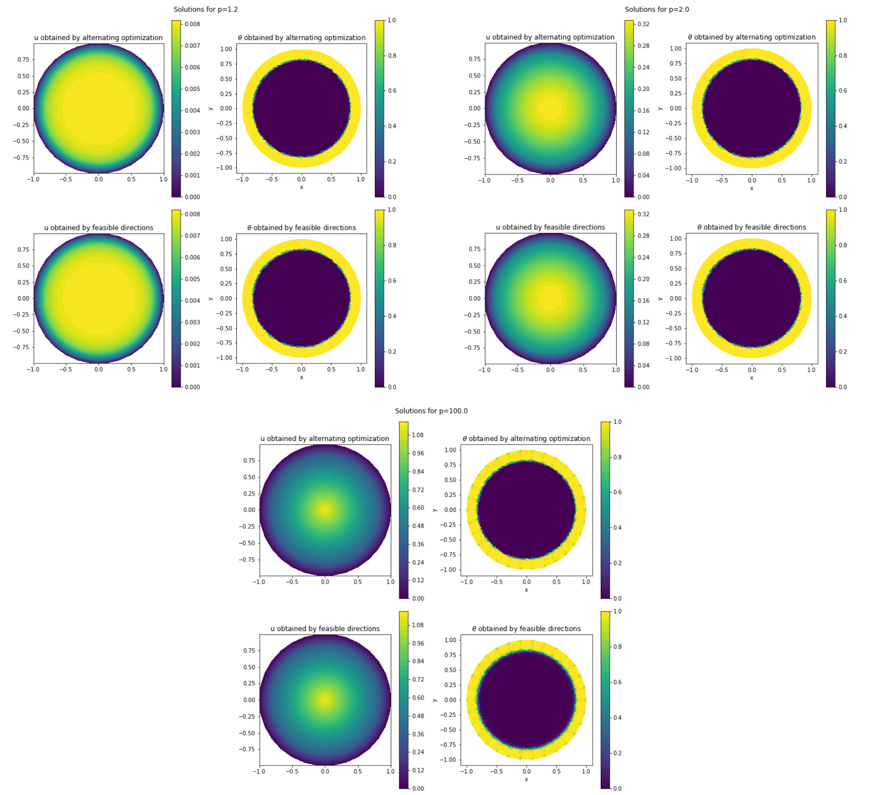

Fig. 2. *Solutions for the finest mesh.*

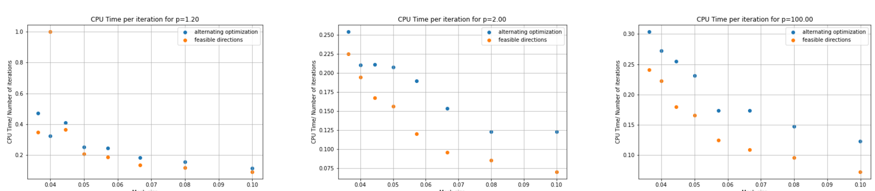

Fig. 3. *Mean CPU time by iteration in seconds for each p.*

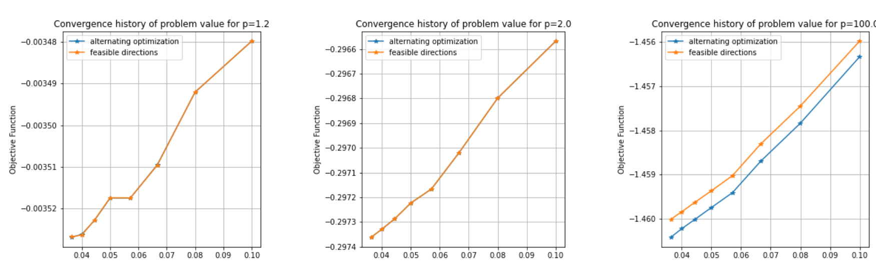

Fig. 4. *Convergence rate of minimum value as function of the mesh size for each p.*

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