EXISTENCE AND HYDRODYNAMIC LIMIT FOR A PAVERI-FONTANA TYPE KINETIC TRAFFIC MODEL

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ABSTRACT. We study a Paveri-Fontana type model, which describes the evolution of the mesoscopic distribution of vehicles through a combined effect of adjustment of the velocity with respect to nearby vehicles, and slowing down and speeding up of the vehicles arising as a result of exchange of velocity with the vehicles on the same location on the road. We first prove the global-in-time existence of weak solutions. The proof is via energy, L^p, and compact support estimates together with velocity averaging lemma. The combined effect of alignment nature of Q_r, which keeps the characteristic from spreading, and the dissipative nature of Q_i, which gives the uniform control on the size of the distribution function, is crucially used in the estimates. We also rigorously establish a hydrodynamic limit to the presureless Euler equation by employing the relative entropy combined with the Monge-Kantorovich-Rubinstein distance.

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1. INTRODUCTION

The study of the traffic flow at the kinetic level started in the seminal work [27, 28] in which Prigogine proposed a kinetic model that explains the traffic flow in a single lane road through the transport, exchange of velocity, and the relaxation to the desired velocity distribution. The relaxation part, however, was criticized by many to be unrealistic, since the desired velocity distribution is a priori prescribed. In order to remedy this drawback, Paveri-Fontana [25] proposed a model where the density function has a further state variable, desired velocity, see Section 1.1 for more details. In the case when the desired velocity is fixed to be the local vehicle velocity, which is reasonable since the drivers in the road adjust their velocities according to the vehicles around it, the Paveri-Fontana model takes the following form:

$$\partial_t f + v\partial_x f = Q(f), \tag{1.1}$$

subject to initial data

$$f(x,v,0) =: f_0(x,v).$$

The vehicle distribution function $f = f(x,v,t)$ denotes the number density function on the phase point $(x,v) \in \mathbb{R}\times\mathbb{R}_+$ at time $t \in \mathbb{R}_+$. The local vehicle density $\rho = \rho(x,t)$ and the local vehicle velocity $u = u(x,t)$ are defined by

$$\rho(x,t) := \int_0^\infty f(x,v,t)\,dv \quad \text{and} \quad u(x,t) := \frac{\int_0^\infty v f(x,v,t)\,dv}{\int_0^\infty f(x,v,t)\,dv},$$

respectively. The operator $Q = Q(f)$, referred to very often as the collision operator in the literature of collisional kinetic theory, is in charge of the interactions between vehicles and their effects on the states. The operator Q consists of relaxation operator $Q_r = Q_r(f)$ and the interaction operator $Q_i = Q_i(f)$:

$$Q(f) = Q_r(f) + Q_i(f).$$

In our case, the relaxation operator Q_r is given by

$$Q_r(f) = \partial_v((v-u)f),$$

which explains the driver's adjustment of velocity with respect to the traffic condition around it. The interaction operator Q_i is presented as the difference of the gain term $Q_i^+ = Q_i^+(f)$ and the loss term $Q_i^- = Q_i^-(f)$:

$$Q_i(f) = Q_i^+(f) - Q_i^-(f),$$

where

$$Q_i^+(f) = f(x,v,t)\int_v^\infty (v_* - v) f(x,v_*,t)\,dv_*$$

and

$$Q_i^-(f) = f(x,v,t)\int_0^v (v - v_*) f(x,v_*,t)\,dv_*.$$

The gain term Q_i^+ represents the slowing down of the cars running faster than v, while the loss term Q_i^- denotes the acceleration of the cars running slower than v. Putting together, the operator Q leads to the concentration of the velocity distribution. In particular, the interaction operator Q_i can be more clearly manifested in the following reformulation:

$$Q_i(f)(x,v,t) = f(x,v,t)\int_0^\infty (v_* - v) f(x,v_*,t)\,dv_* = \rho(x,t)\big(u(x,t) - v\big) f(x,v,t).$$

In view of this, we can understand that the relaxation term is attracting the trajectory towards the desired velocity, while the collision operator rearranges the velocity distribution of the cars to be concentrated on the desired velocity. Therefore, our model (1.1) indicates that the car distribution will eventually converge to the fluid dynamic mono-kinetic configuration.

Being one of the pioneering models in the kinetic theory of traffic flow, to the best of authors knowledge, the existence of solutions for (1.1) and appropriate scaling limit have never been studied in the literature, which is the main motivation of the current work.

1.1. **Formal derivation of** (1.1) **from the Paveri-Fontana model.** As mentioned before, Paveri-Fontana model takes into account the desired velocity as a further state variable w. More precisely, the Paveri-Fontana model [25] is given by

$$\begin{aligned}\partial_t g + v\partial_x g = -\partial_v((w-v)g/T) + (1-P)f(x,v,t)\int_v^\infty (v_*-v)g(x,v_*,w,t)\,dv_* \\ - (1-P)g(x,v,w,t)\int_0^v (v-v_*)f(x,v_*,t)\,dv_*,\end{aligned} \tag{1.2}$$

where $g = g(x,v,w,t)$ is a generalized distribution, T and P denote the relaxation time and the probability of passing, both depend on the g, respectively. Note that the density function f can be recovered by integrating g with respect to the desired velocity variable w, i.e.,

$$\int_{\mathbb{R}_+} g(x,v,w,t)\,dw = f(x,v,t).$$

This together with (1.2) yields that

$$\partial_t f + v\partial_x f = -\partial_v\left(\frac{1}{T}\int_{\mathbb{R}_+} wg\,dw - \frac{vf}{T}\right) + (1-P)\rho(u-v)f.$$

We now assume that the desired velocity is given as the local vehicle velocity, i.e.,

$$g(x,v,w,t) = f(x,v,t)\otimes\delta_{u(x,t)}(w).$$

Furthermore, we assume that the relaxation time T and the probability of non-passing $1-P$ are constants and normalized to unity. These simplifications lead to our main kinetic traffic flow model (1.1).

1.2. **Main results.** Before we define our solution concept and state the main results, we introduce several norms, function spaces, and notational conventions.

- For functions $f(x,v)$ and $g(x)$, $\|f\|_{L^p}$ and $\|g\|_{L^p}$ denote the usual $L^p(\mathbb{R}\times\mathbb{R}_+)$-norm and $L^p(\mathbb{R})$-norm, respectively.
- For any nonnegative integer s, H^s denotes the s-th order L^2 Sobolev space.
- $\mathcal{C}^s([0,T];E)$ is the set of s-times continuously differentiable functions from an interval $[0,T]\subset\mathbb{R}$ into a Banach space E, and $L^p(0,T;E)$ is the set of the L^p functions from an interval $(0,T)$ to a Banach space E.
- We denote by C a generic, not necessarily identical, positive constant. $C = C(\alpha,\beta,\dots)$ or $C = C_{\alpha,\beta,\dots}$ represents the positive constant depending on $\alpha,\beta,\dots$.

We then define the notion of weak solutions to the system (1.1).

Definition 1.1. *Let $T>0$. We say that f is a weak solution to the system* (1.1) *on the time interval $[0,T]$ if the following conditions are satisfied:*

(i) $f \in L^\infty(0,T;(L^1_+\cap L^\infty)(\mathbb{R}\times\mathbb{R}_+))$,

(ii) *for all $\phi\in\mathcal{C}^1_c(\mathbb{R}\times\mathbb{R}_+\times[0,T])$ with $\phi(\cdot,\cdot,T)=0$,*

$$\begin{aligned}-\int_{\mathbb{R}\times\mathbb{R}_+} f_0(x,v)\phi(x,v,0)\,dxdv - \int_0^T\int_{\mathbb{R}\times\mathbb{R}_+} f\left(\partial_t\phi + v\partial_x\phi + (u-v)\partial_v\phi\right)dxdvdt \\ = \int_0^T\int_{\mathbb{R}\times\mathbb{R}_+}\rho(u-v)f\phi\,dxdvdt.\end{aligned}$$

We are now ready to state our first main result on the existence theory.

Theorem 1.1. *Let $T > 0$. Suppose that the initial data $f_0 \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is compactly supported in both position and velocity. We also assume that $f_0(x, 0) = 0$. Then there exists at least one weak solution to the equation (1.1) in the sense of Definition 1.1 such that*

(i) *f is compactly supported both in position and velocity, with $f(x, 0, t) = 0$,*
(ii) $\|f\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+\times(0,T))} \le C\|f_0\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)}$,
(iii) *The total energy is non-increasing in time:*

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f\,dxdv + \int_0^t \int_{\mathbb{R}\times\mathbb{R}_+} (u-v)^2 f\,dxdvds \le \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_0\,dxdv,$$

for almost every $t \in (0, T)$.

Remark 1.1. *The statement (i) in the theorem above means that the vehicles can run only at a finite speed, and never come to a halt on the road. The former is natural in that the vehicles cannot run with an infinite speed, and the latter is reasonable when, for example, the vehicles are running on a highway.*

Remark 1.2. *The property $f(x, 0, t) = 0$ is also crucially used in the proof since otherwise the boundary terms in the integration by parts with respect to the velocity variable do not vanish.*

The proof combines the alignment effect of Q_r and the dissipative nature of Q_i. We first construct approximate solutions f_ε parametrized by a smoothing parameter ε, see Section 3.3 for details. The alignment property of Q_r prevents the spreading of the characteristics and keep the support of the distriubtion function compact, leading to the conclusion that the approximate solution f_ε is bounded for each ε globally in time. Then, the dissipative nature of Q_i with respect to vdv and v^2dv gives the coercivity that controls the boundedness of such approximate solutions uniformly in ε:

$$\|f_\varepsilon(\cdot,\cdot,t)\|_{L^\infty} \le C\|f_{0,\varepsilon}\|_{L^\infty} + C\int_0^t \int_{\mathbb{R}\times\mathbb{R}_+} \frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1 + \varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdvds.$$

These two estimates, the boundedness of f_ε and the non-spreading of trajectories, are strong enough enough to control the high nonlinearity of the collision operator in the weak limit, and enables one to derive the global-in-time weak solution, instead of having to resort to weaker notions of solutions such as the renormalized solutions.

Our next goal is to study the asymptotic analysis of (1.1). More precisely, we are interested in the limit $\varepsilon \to 0$ in the following equation:

$$\partial_t f^\varepsilon + v\partial_x f^\varepsilon = \frac{1}{\varepsilon}Q(f^\varepsilon) = \frac{1}{\varepsilon}\partial_v((v-u^\varepsilon)f^\varepsilon) + \frac{1}{\varepsilon}\rho^\varepsilon(u^\varepsilon - v)f^\varepsilon, \tag{1.3}$$

subject to the initial data

$$f^\varepsilon(x, v, 0) =: f_0^\varepsilon(x, v)$$

with

$$\rho^\varepsilon(x,t) := \int_0^\infty f^\varepsilon(x,v,t)\,dv, \quad \rho^\varepsilon(x,t)u^\varepsilon(x,t) := \int_0^\infty v f^\varepsilon(x,v,t)\,dv,$$

which is obtained from rewriting (1.1) in the typical Euler scaling:

$$x \to x/\varepsilon \quad \text{and} \quad t \to t/\varepsilon.$$

As $\varepsilon \to 0$, we expect the mono-kinetic ansatz for f^ε, i.e.,

$$f^\varepsilon(x,v,t) \to \rho(x,t) \otimes \delta_{u(x,t)}(v)$$

in the sense of distributions. Here ρ and u satisfy the following pressureless Euler equations:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2) = 0. \end{aligned} \tag{1.4}$$

The main tool we employ for the hydrodynamic limit from (1.3) to (1.4) is based on the relative entropy method for kinetic flocking model developed in [20], see also [11]. In [20], however, the diffusion term plays a significant role in that the entropy functional is strictly convex with respect to both ρ and ρu. In the absence of the diffusion term, the functional is not convex for ρ any more, see Section 5 for details, and the strategy used in [20] breaks down. Remedies to overcome this are suggested in recent works [3, 7, 12], in which a suitable estimate with respect to the second-order Wasserstein distance is augmented to provide the convergence of ρ^ε. Inspired by these works, we adopt the Monge-Kantorovich-Rubinstein (in short MKR) distance:

$$d_{MKR}(\rho_1, \rho_2) := \inf_\gamma \int_{\mathbb{R}\times\mathbb{R}} |x - y|\, d\gamma(x, y), \quad \text{for} \quad \rho_1, \rho_2 \in \mathcal{P}_1(\mathbb{R}),$$

where the infimum runs over all transference plans, i.e., all probability measures γ on $\mathbb{R}\times\mathbb{R}$ with first and second marginals ρ_1 and ρ_2, respectively. Here $\mathcal{P}_1(\mathbb{R})$ stands for the set of probability measures on $\mathbb{R}$ with finite first-order moment. Note that MKR distance is equivalent to the bounded Lipschitz distance, and it is also called first-order Wasserstein distance. We employ the idea recently developed in [7] to have the quantitative estimate for error between local densities in MKR distance under suitable assumptions on the initial data.

In order to state our second main result of the present work, we present the notion of strong solutions to the pressureless Euler system (1.4) in the proposition below.

Proposition 1.1. *Let $s > 2$, and suppose $(\rho_0, u_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ and $\rho_0 > 0$ on $\mathbb{R}$. Then, for any positive constants $N < M$, there exists a positive constant T_0 depending only on N and M such that if $\|(\rho_0, u_0)\|_{H^s\times H^{s+1}} \le N$, then the system (1.4) with (ρ_0, u_0) admits a unique solution $(\rho, u) \in \mathcal{C}([0, T_0]; H^s(\mathbb{R})) \times \mathcal{C}([0, T_0]; H^{s+1}(\mathbb{R}))$ satisfying*

$$\sup_{0\le t\le T_0} \|(\rho(\cdot, t), u(\cdot, t))\|_{H^s\times H^{s+1}} \le M.$$

Remark 1.3. *The weak solution for the scaled Paveri-Fontana model (1.3) can be shown to exist globally in time using the exactly same argument as was employed for Theorem 1.1. The existence and uniqueness of strong solutions for the pressureless Euler system (1.4) can be obtained by using almost the same argument as in [9].*

Theorem 1.2. *Let f^ε be a weak solution to the equation (1.3) and (ρ, u) be a strong solution to the system (1.4) on the time interval $[0, T]$. Suppose that the initial data f_0^ε and (ρ_0, u_0) satisfy the following assumptions.*

(H1) *The initial data are well-prepared:*

$$\int_{\mathbb{R}} \rho_0^\varepsilon (u_0 - u_0^\varepsilon)^2\, dx = \mathcal{O}(\varepsilon) \quad \text{and} \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}_+} v^2 f_0^\varepsilon\, dv - \rho_0 u_0^2 \right) dx = \mathcal{O}(\varepsilon).$$

(H2) *The local densities ρ_0 and ρ_0^ε satisfy*

$$d_{MKR}(\rho_0, \rho_0^\varepsilon) = \mathcal{O}(\sqrt{\varepsilon}).$$

Then we obtain

$$\sup_{0\le t\le T} \int_{\mathbb{R}} \rho^\varepsilon(x, t)(u^\varepsilon - u)^2(x, t)\, dx \le \mathcal{O}(\varepsilon) \quad \text{and} \quad \sup_{0\le t\le T} d_{MKR}(\rho^\varepsilon(\cdot, t), \rho(\cdot, t)) \le \mathcal{O}(\sqrt{\varepsilon}).$$

In particular, we have

$$\sup_{0\le t\le T} d_{MKR}(f^\varepsilon(\cdot,\cdot,t),\rho(\cdot,t)\otimes\delta_{u(\cdot,t)}(\cdot)) \le \mathcal{O}(\sqrt{\varepsilon}),$$

that is, f^ε converges to $\rho\otimes\delta_u$ weakly- as measures.*

We now briefly overview references on the traffic models relevant to our works. We confine ourselves to the kinetic traffic models, since the literature is enormous, see [2, 23, 26] for a survey on mathematical models of vehicular traffic at different scales of descriptions. Prigogine in [27, 28] suggested a Boltzmann type traffic model, which is, as mentioned above, the first kinetic model for traffic flow. The Paveri-Fontana model, which is the main concern of the current work, is introduced in [25] to remedy the controversies on the assumption used in the Prigogine model that the traffic system relaxes to a fixed velocity configuration. Klar and Wigner discussed the necessity of considering the length of the interaction, and introduced Enskog type kinetic models in [21, 22]. Vlasov-Fokker-Planck type models can be found in [15, 17]. In [29], Puppo et al introduced a Boltzmann type traffic model for which an analytic expression for the steady states can be secured. The BGK type relaxational approximate model for traffic is derived in [16]. For kinetic equations for multi-lane traffic model, we refer to [14, 17]. We also note that the studies of the closely related velocity alignment type kinetic equations have attracted a lot of attentions recently since they arise in various contexts such as a coarse grain limit of relevant flocking models [5, 6, 8, 10, 18, 20, 24].

This paper is organized as follows. In Section 2, we provide several preliminary lemmas. We introduce a regularized equation of (1.1) mollifying the local vehicle velocity u in the relaxation term Q_r and the interaction operator Q_i in Section 3. We also provide some uniform bound estimates of the approximated solutions with respect to the regularization parameters. Using these uniform bound estimates, we prove the global-in-time existence of weak solutions in Section 4. Finally, in Section 5, using the relative entropy method combined with the estimate of MKR distance between local densities, we show that the scaled Paveri-Fontana model (1.3) converges to the pressureless Euler system (1.4).

2. Preliminaries: Auxiliary Lemmas

In this section, we provide several auxiliary lemmas which will be used significantly for our results later. We first state the velocity averaging lemma whose proof can be found in [4, 13, 19].

Lemma 2.1. *For $1\le p<3/2$, let $\{g^n\}_{n\in\mathbb{N}}$ be bounded in $L^p(\mathbb{R}\times\mathbb{R}_+\times(0,T))$. Suppose that f^n is bounded in $L^\infty(0,T;(L^1\cap L^\infty)(\mathbb{R}\times\mathbb{R}_+))$ and $v^2 f^n$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}\times\mathbb{R}_+))$. If f^n and g^n satisfy the equation*

$$\partial_t f^n + v\partial_x f^n = \partial_v^k g^n, \quad f^n|_{t=0} = f_0 \in L^p(\mathbb{R}\times\mathbb{R}_+),$$

for a multi-index k. Then, for any $\psi(v)$, such that $|\psi(v)|\le Cv$ as $v\to+\infty$, the sequence

$$\left\{\int_{\mathbb{R}_+} f^n\psi(v)\,dv\right\}_{n\in\mathbb{N}}$$

is relatively compact in $L^p(\mathbb{R}\times(0,T))$.

In the lemma below, we show the dissipative estimates of the interaction operator $Q_i(f)$.

Lemma 2.2. *Suppose that f is a solution of* (1.1) *with sufficient integrability. Then we have*

$$\int_0^\infty Q_i(f)\,dv = 0, \quad \int_0^\infty vQ_i(f)\,dv = -\int_0^\infty \rho(u-v)^2 f\,dv, \quad \int_0^\infty v^2 Q_i(f)\,dv \le 0.$$

Proof. By definition of u, we easily obtain

$$\int_0^\infty Q_i(f)\,dv = \rho\int_0^\infty (u-v)f\,dv = 0.$$

Using this fact, we compute

$$\int_0^\infty vQ_i(f)\,dv = \int_0^\infty\int_{\mathbb{R}} \rho v(u-v)f\,dxdv = -\int_0^\infty\int_{\mathbb{R}} \rho(u-v)^2 f\,dxdv.$$

In order to prove the third relation, we observe that

$$\int_0^\infty v^2 f\,dv \le \left(\int_0^\infty f\,dv\right)^{1/3}\left(\int_0^\infty v^3 f\,dv\right)^{2/3}$$

and

$$u = \frac{\int_0^\infty vf\,dv}{\int_0^\infty f\,dv} \le \frac{\rho^{2/3}\left(\int_0^\infty v^3 f\,dv\right)^{1/3}}{\int_0^\infty f\,dv} = \frac{\left(\int_0^\infty v^3 f\,dv\right)^{1/3}}{\left(\int_0^\infty f\,dv\right)^{1/3}},$$

so that

$$u\int_0^\infty v^2 f\,dv \le \int_0^\infty v^3 f\,dv.$$

Therefore, we have

$$\int_0^\infty v^2 Q_i(f)\,dv = \int_0^\infty \rho v^2(u-v)f\,dv = \rho\left(u\int_0^\infty v^2 f\,dv - \int_0^\infty v^3 f\,dv\right) \le 0.$$

□

The following lemma is standard. We, however, record it in a separate lemma since we need unusual assumption that $f(x,v,t)=0$ when $v=0$, so that the boundary term in the integration by parts vanishes.

Lemma 2.3. *Suppose that f is a solution of* (1.1) *with sufficient integrability. Furthermore, we assume that f is compactly supported in x and v at $t>0$ with $f(x,0,t)=0$. Then we have*

$$\int_0^\infty \partial_v Q_r(f)\,dv = 0, \qquad \int_0^\infty v\partial_v Q_r(f)\,dv = 0,$$

and

$$\frac{1}{2}\int_0^\infty v^2 Q_r(f)\,dv = -\int_0^\infty (u-v)^2 f\,dv.$$

Proof. Since the proof is straightforward, we only consider the third relation. From the assumption, we get $f(x,0,t)=0$ and $\lim_{v\to+\infty} v^3 f(x,v,t)=0$, so that

$$\begin{aligned}
\frac{1}{2}\int_0^\infty v^2\partial_v\left((u-v)f\right)dv &= \frac{v^2}{2}(u-v)f\Big|_{v=0}^{v=\infty} - \int_0^\infty v(u-v)f\,dv \\
&= \int_0^\infty (u-v)^2 f\,dv - u\int_0^\infty (u-v)f\,dv \\
&= \int_0^\infty (u-v)^2 f\,dv.
\end{aligned}$$

In the last line, we used the following equality:

$$\int_0^\infty (u-v)f\,dv = 0.$$

□

From the above two lemmas, we have a priori energy estimates which follow directly by integrating (1.1) with respect to $(1, v, v^2)\,dv$, respectively.

Lemma 2.4. *Suppose that f is a solution of* (1.1) *with sufficient integrability. Furthermore, we assume that f is compactly supported in x and v at $t > 0$ with $f(x, 0, t) = 0$. Then we have*

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} f\,dxdv = 0, \quad \frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} vf\,dxdv + \int_{\mathbb{R}\times\mathbb{R}_+} \rho(u-v)^2 f\,dxdv = 0,$$

and

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f\,dxdv + \int_{\mathbb{R}\times\mathbb{R}_+} (u-v)^2 f\,dxdv \le 0.$$

3. Global-in-time existence for a regularized equation

In this section, we consider a regularized equation of (1.1). Inspired by [18], we regularize the local vehicle velocity u by using a mollifier $\theta_\varepsilon(x) = \varepsilon^{-1}\theta(x/\varepsilon)$ with $0 \le \theta \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfying

$$supp\,\theta \subseteq (-1, 1) \quad \text{and} \quad \int_{\mathbb{R}} \theta(x)\,dx = 1.$$

This removes the singularity in the relaxation term. More precisely, our regularized equation of (1.1) is defined as follows:

$$\partial_t f_\varepsilon + v\partial_x f_\varepsilon + \partial_v((u_\varepsilon^\varepsilon - v)f_\varepsilon) = \frac{\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1 + \varepsilon\rho_\varepsilon(1 + u_\varepsilon)}, \tag{3.1}$$

subject to regularized initial data

$$f_\varepsilon(x, v, 0) =: f_{0,\varepsilon}(x, v).$$

Here, the regularized local vehicle velocity $u_\varepsilon^\varepsilon$ is defined by

$$\begin{aligned} u_\varepsilon^\varepsilon(x,t) &:= \left(\int_{\mathbb{R}\times\mathbb{R}_+} \theta_\varepsilon(x-y) w f_\varepsilon(y,w,t)\,dydw\right) \Big/ \left(\varepsilon + \left(\int_{\mathbb{R}\times\mathbb{R}_+} \theta_\varepsilon(x-y) f_\varepsilon(y,w,t)\,dydw\right)\right) \\ &= \frac{((\rho_\varepsilon u_\varepsilon)\star\theta_\varepsilon)(x,t)}{\varepsilon + (\rho_\varepsilon\star\theta_\varepsilon)(x,t)}, \end{aligned}$$

and $f_{0,\varepsilon}$ denotes a smooth approxmiation of f_0 such that

- (i) $f_{0,\varepsilon}$ is compactly supported and it is contained in $[C_{x,0}, C_{x,1}] \times [C_{v,0}, C_{v,1}]$ with $C_{x,0} < C_{x,1}$ and $0 < C_{v,0} < C_{v,1}$.
- (ii) $f_{0,\varepsilon}$ converges strongly to f_0 in $L^\infty(\mathbb{R}\times\mathbb{R}_+)$ as $\varepsilon \to 0$.

Note that our main equation (1.1) can be formally recovered from (3.1) in the limit $\varepsilon \to 0$.

In the following two subsections, we prove the proposition below on the global-in-time existence of weak solutions and some uniform bound estimates of the regularized equation (3.1).

Proposition 3.1. *Let $T > 0$. For any $\varepsilon > 0$, there exists at least one weak solution f_ε of the regularized equation* (3.1) *on the interval $[0, T]$ in the sense of Definition 1.1.*

3.1. **Approximated solutions.**

In order to obtain the existence of solutions to the regularized equation (3.1), we first construct the approximated solutions in the following way:

$$\partial_t f_\varepsilon^{n+1} + v\partial_x f_\varepsilon^{n+1} + \partial_v\left((u_\varepsilon^{\varepsilon,n} - v)f_\varepsilon^{n+1}\right) = \frac{\rho_\varepsilon^n(u_\varepsilon^n - v)f_\varepsilon^{n+1}}{1 + \varepsilon\rho_\varepsilon^n(1 + u_\varepsilon^n)}, \tag{3.2}$$

with the initial data and first iteration step:

$$f_\varepsilon^n(x, v, t)|_{t=0} = f_{0,\varepsilon}(x, v) \quad (x, v) \in \mathbb{R}\times\mathbb{R}_+ \quad \text{for all } n \ge 1$$

and

$$f_\varepsilon^0(x,v,t) = f_{0,\varepsilon}(x,v), \quad (x,v,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T).$$

Here

$$u_\varepsilon^{\varepsilon,n}(x,t) = \frac{((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(x,t)}{\varepsilon + (\rho_\varepsilon^n \star \theta_\varepsilon)(x,t)}$$

with

$$\rho_\varepsilon^n u_\varepsilon^n = \int_{\mathbb{R}_+} v f_\varepsilon^n \, dv \quad \text{and} \quad \rho_\varepsilon^n = \int_{\mathbb{R}_+} f_\varepsilon^n \, dv.$$

To get the existence of solutions to the approximated equation (3.2), we need to estimate the support of f_ε^n, and for this, we introduce the following forward characteristics:

$$Z_\varepsilon^{n+1}(s) := (X_\varepsilon^{n+1}(s), V_\varepsilon^{n+1}(s)) := (X_\varepsilon^{n+1}(s;0,x,v), V_\varepsilon^{n+1}(s;0,x,v))$$

defined by

$$\begin{aligned} &\frac{d}{ds} X_\varepsilon^{n+1}(s) = V_\varepsilon^{n+1}(s), \quad 0 \le s \le T, \\ &\frac{d}{ds} V_\varepsilon^{n+1}(s) = u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(s), s) - V_\varepsilon^{n+1}(s), \end{aligned} \tag{3.3}$$

with the initial data

$$Z_\varepsilon^{n+1}(0) = (x,v) \in \mathbb{R} \times \mathbb{R}_+.$$

Due to the regularization, the characteristics (3.3) is well-defined, thus global-in-time existence of solutions to (3.2) can be obtained by a standard existence theory. It follows from (3.3) that $V_\varepsilon^{n+1}(s)$ satisfies

$$V_\varepsilon^{n+1}(t) = ve^{-t} + e^{-t} \int_0^t u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(s), s) e^s \, ds. \tag{3.4}$$

We now define

$$R_X[f] := \max_{x \in supp_x f} x \quad \text{and} \quad R_V[f] := \max_{v \in supp_v f} v,$$

and

$$r_X[f] := \min_{x \in supp_x f} x \quad \text{and} \quad r_V[f] := \min_{v \in supp_v f} v,$$

where $supp_x f$ and $supp_v f$ represent x- and v-projections of $supp f$, respectively.

Proposition 3.2. *For any $T > 0$ and $n \in \mathbb{N}$, there exists a unique solution f_ε^n of the regularized and linearized equation* (3.2) *such that*

(i) *f_ε^n is compactly supported in both x and v satisfying*

$$C_{x,0} + C_{v,0}(1 - e^{-t}) \le r_X[f_\varepsilon^n] \le R_X[f_\varepsilon^n] \le C_{x,1} + TC_{v,1}$$

and

$$e^{-t} C_{v,0} < r_V[f_\varepsilon^n] \le R_V[f_\varepsilon^n] \le C_{v,1}.$$

Here the positive constants $C_{x,i}, C_{v,i}, i = 1,2$ are appeared in the beginning of this section.

(ii) *$f_\varepsilon^n \in L^\infty(0,T; L^\infty(\mathbb{R} \times \mathbb{R}_+))$ satisfies*

$$\sup_{0 \le t \le T} \sup_{n \in \mathbb{N}} \|f_\varepsilon^n(\cdot,\cdot,t)\|_{W^{1,\infty}} \le C_{\varepsilon,T,f_{0,\varepsilon}},$$

for $\varepsilon \in (0,1)$, where $C_{\varepsilon,T,f_{0,\varepsilon}} > 0$ depends on ε, T, and $\|f_{0,\varepsilon}\|_{W^{1,\infty}}$, but independent of n.

Remark 3.1. *Note that the size of the support does not depend on ε or n. We also note that (i) clearly implies $f_\varepsilon^n(x,0,t) = 0$ for $[0,T]$.*

Proof of Proposition 3.2. (i) In view of the observation that the solution to (3.2) is written in the characteristic formulation:

$$f_\varepsilon^{n+1}(X_\varepsilon^{n+1}(t), V_\varepsilon^{n+1}(t)) = e^{\int_0^t (A_\varepsilon^n(s)+1)\,ds} f_{0,\varepsilon}(x,v)$$

where $A_\varepsilon^n(s)$ denotes

$$A_\varepsilon^n(s) := \frac{\rho_\varepsilon^n(X_\varepsilon^{n+1}(s),s)\left(u_\varepsilon^n(X_\varepsilon^{n+1}(s),s) - V_\varepsilon^{n+1}(s)\right)}{1+\varepsilon\rho_\varepsilon^n(X_\varepsilon^{n+1}(s),s)\left(1+u_\varepsilon^n(X_\varepsilon^{n+1}(s),s)\right)}.$$

We see that it is enough to derive the compactness of the characteristics. We first estimate $R_V[f_\varepsilon^{n+1}(t)]$. Note that

$$u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(s),s) \le \frac{\int_{\mathbb{R}\times\mathbb{R}_+} v f_\varepsilon^n(X_\varepsilon^{n+1}(s)-y,v,t)\theta_\varepsilon(y)\,dydv}{\int_{\mathbb{R}\times\mathbb{R}_+} f_\varepsilon^n(X_\varepsilon^{n+1}(s)-y,v,t)\theta_\varepsilon(y)\,dydv} \le R_V[f_\varepsilon^n(t)].$$

Then it follows from (3.4) that

$$\begin{aligned} V_\varepsilon^{n+1}(t) &= ve^{-t} + e^{-t}\int_0^t u_\varepsilon^{\varepsilon,n}(X_\varepsilon^{n+1}(s),s)e^s\,ds \\ &\le R_V[f_{0,\varepsilon}]e^{-t} + e^{-t}\int_0^t R_V[f_\varepsilon^n(s)]e^s\,ds. \end{aligned}$$

This implies

$$e^t R_V[f_\varepsilon^{n+1}(t)] \le R_V[f_{0,\varepsilon}] + \int_0^t e^s R_V[f_\varepsilon^n(s)]\,ds.$$

For notational simplicity, we set $A^n(t) := e^t R_V[f_\varepsilon^n(t)]$ for $n\in\mathbb{N}$ and iterate this relation to get

$$\begin{aligned} A^n(t) &\le R_V[f_{0,\varepsilon}] + \int_0^t A^{n-1}(s)\,ds \\ &\le R_V[f_{0,\varepsilon}] + \int_0^t R_V[f_{0,\varepsilon}]\,ds + \int_0^t\int_0^s A^{n-2}(s_1)\,ds_1ds \\ &\le R_V[f_{0,\varepsilon}] + tR_V[f_{0,\varepsilon}] + \int_0^t\int_0^s R_V[f_{0,\varepsilon}]\,ds_1ds + \int_0^t\int_0^s\int_0^{s_1} A^{n-3}(s_2)\,ds_2ds_1ds \\ &\le \quad \cdots \\ &\le \left(\sum_{n=0}^\infty \frac{t^n}{n!}\right) R_V[f_{0,\varepsilon}] \\ &\le e^t R_V[f_{0,\varepsilon}]. \end{aligned}$$

Thus we obtain

$$e^t R_V[f_\varepsilon^n(t)] \le e^t R_V[f_{0,\varepsilon}], \quad \text{i.e.,} \quad R_V[f_\varepsilon^n(t)] \le R_V[f_{0,\varepsilon}] < C_{v,1} \tag{3.5}$$

for all $n\in\mathbb{N}$. Moreover, we get

$$V_\varepsilon^{n+1}(t) \ge ve^{-t} \ge e^{-t}r_V[f_{0,\varepsilon}],$$

and subsequently, this asserts

$$r_V[f_\varepsilon^n(t)] > e^{-t}r_V[f_{0,\varepsilon}] \ge e^{-t}C_{v,0},$$

due to the compact support assumption on $f_{0,\varepsilon}$. This gives the compactness of the velocity support and $f_\varepsilon^n(x,0,t)=0$. The compactness of position directly follows from this.

(ii) We now estimate $\|f_\varepsilon^{n+1}\|_{L^\infty}$ uniformly in n. For $1 \le p < \infty$, we find

$$\begin{aligned}
&\frac{d}{dt}\|f_\varepsilon^{n+1}\|_{L^p}^p \\
&\quad = p\int_{\mathbb{R}\times\mathbb{R}_+} (f_\varepsilon^{n+1})^{p-1}\left(-v\partial_x f_\varepsilon^{n+1} - \partial_v\left((u_\varepsilon^{\varepsilon,n}-v)f_\varepsilon^{n+1}\right) + \frac{\rho_\varepsilon^n(u_\varepsilon^n - v)f_\varepsilon^{n+1}}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\right)dxdv \\
&\quad = (p-1)\|f_\varepsilon^{n+1}\|_{L^p}^p + p\int_{\mathbb{R}\times\mathbb{R}_+} (f_\varepsilon^{n+1})^p \frac{\rho_\varepsilon^n(u_\varepsilon^n - v)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\,dxdv.
\end{aligned}$$

For the last term on the right side of the above equality, we use (3.5) to obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}\times\mathbb{R}_+} (f_\varepsilon^{n+1})^p \frac{\rho_\varepsilon^n(u_\varepsilon^n - v)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\,dxdv\right| &\le \int_{\mathbb{R}\times\mathbb{R}_+} (f_\varepsilon^{n+1})^p \frac{\rho_\varepsilon^n(u_\varepsilon^n + R_V[f_\varepsilon^{n+1}])}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\,dxdv \\
&\le C\int_{\mathbb{R}\times\mathbb{R}_+} (f_\varepsilon^{n+1})^p \frac{\rho_\varepsilon^n(u_\varepsilon^n + 1)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\,dxdv \\
&\le \frac{C}{\varepsilon}\|f_\varepsilon^{n+1}\|_{L^p}^p,
\end{aligned}$$

where $C > 0$ is independent of n, p and ε. Thus we get

$$\frac{d}{dt}\|f_\varepsilon^{n+1}\|_{L^p} \le C\left(1+\frac{1}{\varepsilon}\right)\|f_\varepsilon^{n+1}\|_{L^p},$$

where $C > 0$ is independent of n, p and ε. We now use Grönwall's lemma and send $p \to \infty$ to have

$$\|f_\varepsilon^{n+1}\|_{L^\infty} \le \|f_{0,\varepsilon}\|_{L^\infty} e^{C(1+\varepsilon^{-1})T},$$

where $C > 0$ is independent of n and ε.

We next differentiate (3.2) with respect to x to find

$$\partial_t\partial_x f_\varepsilon^{n+1} + v\partial_{xx} f_\varepsilon^{n+1} + \partial_{xv}\left((u_\varepsilon^{\varepsilon,n}-v)f_\varepsilon^{n+1}\right) = \partial_x\left(\frac{\rho_\varepsilon^n(u_\varepsilon^n - v)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}f_\varepsilon^{n+1}\right).$$

Then we estimate

$$\begin{aligned}
\frac{d}{dt}\|\partial_x f_\varepsilon^{n+1}\|_{L^p}^p &= p\int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2}\partial_x f_\varepsilon^{n+1}\left(-v\partial_{xx} f_\varepsilon^{n+1} - \partial_{xv}\left((u_\varepsilon^{\varepsilon,n}-v)f_\varepsilon^{n+1}\right)\right)dxdv \\
&\quad + p\int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2}\partial_x f_\varepsilon^{n+1}\partial_x\left(\frac{\rho_\varepsilon^n(u_\varepsilon^n - v)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}f_\varepsilon^{n+1}\right)dxdv \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, we use the integration by parts to obtain

$$I_1 = -\int_{\mathbb{R}\times\mathbb{R}_+} \partial_x(|\partial_x f_\varepsilon^{n+1}|^p)v\,dxdv = 0.$$

For I_2, we get

$$\begin{aligned}
I_2 &= -p\int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2}\partial_x f_\varepsilon^{n+1}\left(-\partial_x f_\varepsilon^{n+1} + \partial_x u_\varepsilon^{\varepsilon,n}\partial_v f_\varepsilon^{n+1} + (u_\varepsilon^{\varepsilon,n}-v)\partial_{xv} f_\varepsilon^{n+1}\right)dxdv \\
&\le p\|\partial_x f_\varepsilon^{n+1}\|_{L^p}^p + C_\varepsilon p\|f_\varepsilon^n\|_{L^\infty}\|\partial_x f_\varepsilon^{n+1}\|_{L^p}^{p-1}\|\partial_v f_\varepsilon^{n+1}\|_{L^p} + \|\partial_x f_\varepsilon^{n+1}\|_{L^p}^p.
\end{aligned}$$

Here we used

$$
\begin{aligned}
|\partial_x u_\varepsilon^{\varepsilon,n}| &= \left| \frac{1}{(\varepsilon + \rho_\varepsilon^n \star \theta_\varepsilon)^2} \left(\partial_x((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(\rho_\varepsilon^n \star \theta_\varepsilon) - ((\rho_\varepsilon^n u_\varepsilon^n) \star \theta_\varepsilon)(\partial_x \rho_\varepsilon^n \star \theta_\varepsilon)\right)\right| \\
&\le \left| \frac{(\rho_\varepsilon^n u_\varepsilon^n) \star \partial_x \theta_\varepsilon}{\varepsilon + \rho_\varepsilon^n \star \theta_\varepsilon} \right| + C \left| \frac{\rho_\varepsilon^n \star \partial_x \theta_\varepsilon}{\varepsilon + \rho_\varepsilon^n \star \theta_\varepsilon} \right| \\
&\le C_\varepsilon \|f_\varepsilon^n\|_{L^\infty}
\end{aligned}
$$

and

$$
\begin{aligned}
&- p \int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2} \partial_x f_\varepsilon^{n+1} (u_\varepsilon^{\varepsilon,n} - v) \partial_{xv} f_\varepsilon^{n+1} dxdv \\
&\quad = \int_{\mathbb{R}\times\mathbb{R}_+} \partial_v (|\partial_x f_\varepsilon^{n+1}|^p)(u_\varepsilon^{\varepsilon,n} - v)\, dxdv \\
&\quad = \|\partial_x f_\varepsilon^{n+1}\|_{L^p}^p.
\end{aligned}
$$

For the estimate of I_3, we notice from (3.5) that

$$
|\partial_x(\rho_\varepsilon^n u_\varepsilon^n)| = \left| \int_{\mathbb{R}_+} v \partial_x f_\varepsilon^n \, dv \right| \le C \|\partial_x f_\varepsilon^n\|_{L^p}
$$

and

$$
|\partial_x \rho_\varepsilon^n| = \left| \int_{\mathbb{R}_+} \partial_x f_\varepsilon^n \, dv \right| \le C \|\partial_x f_\varepsilon^n\|_{L^p}.
$$

This gives

$$
\begin{aligned}
\left| \partial_x \left(\frac{\rho_\varepsilon^n (u_\varepsilon^n - v)}{1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n)} \right) \right| &= \left| \frac{\partial_x(\rho_\varepsilon^n u_\varepsilon^n) - v \partial_x \rho_\varepsilon^n}{(1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n))} - \frac{\varepsilon \rho_\varepsilon^n (u_\varepsilon^n - v)((\partial_x \rho_\varepsilon^n + \partial_x(\rho_\varepsilon^n u_\varepsilon^n)))}{(1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n))^2} \right| \\
&\le C(1 + |v|) \|\partial_x f_\varepsilon^n\|_{L^p},
\end{aligned}
$$

for $\varepsilon \in (0,1)$, and subsequently this asserts

$$
\begin{aligned}
I_3 &= p \int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2} \partial_x f_\varepsilon^{n+1} \partial_x \left(\frac{\rho_\varepsilon^n (u_\varepsilon^n - v)}{1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n)} \right) f_\varepsilon^{n+1} \, dxdv \\
&\quad + p \int_{\mathbb{R}\times\mathbb{R}_+} |\partial_x f_\varepsilon^{n+1}|^{p-2} \partial_x f_\varepsilon^{n+1} \frac{\rho_\varepsilon^n (u_\varepsilon^n - v)}{1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n)} \partial_x f_\varepsilon^{n+1} \, dxdv \\
&\le Cp \|\partial_x f_\varepsilon^{n+1}\|_{L^p}^{p-1} \|f_\varepsilon^{n+1}\|_{L^p} + C_\varepsilon p \|\partial_x f_\varepsilon^{n+1}\|_{L^p}^p.
\end{aligned}
$$

Thus we obtain

$$
\frac{d}{dt} \|\partial_x f_\varepsilon^{n+1}\|_{L^p} \le C_{\varepsilon,T,f_{0,\varepsilon}} \left(1 + \frac{1}{p}\right) \|f_\varepsilon^{n+1}\|_{W^{1,p}},
$$

where $C_{\varepsilon,T,f_{0,\varepsilon}} > 0$ depends on ε, T, and $\|f_{0,\varepsilon}\|_{L^\infty}$. Similarly, we also find

$$
\begin{aligned}
\frac{d}{dt} \|\partial_v f_\varepsilon^{n+1}\|_{L^p}^p &= p \int_{\mathbb{R}\times\mathbb{R}_+} |\partial_v f_\varepsilon^{n+1}|^{p-2} \partial_v f_\varepsilon^{n+1} \left(-\partial_x f_\varepsilon^{n+1} - \partial_{vv}((u_\varepsilon^{\varepsilon,n} - v) f_\varepsilon^{n+1}) \right) dxdv \\
&\quad + p \int_{\mathbb{R}\times\mathbb{R}_+} |\partial_v f_\varepsilon^{n+1}|^{p-2} \partial_v f_\varepsilon^{n+1} \frac{\rho_\varepsilon^n}{1 + \varepsilon \rho_\varepsilon^n (1 + u_\varepsilon^n)} \partial_v ((u_\varepsilon^n - v) f_\varepsilon^{n+1}) \, dxdv \\
&=: J_1 + J_2 + J_3.
\end{aligned}
$$

Here $J_i, i=1,2,3$ can be estimated as follows.

$$
\begin{aligned}
J_1 &\le p\|\partial_v f_\varepsilon^{n+1}\|_{L^p}^{p-1}\|\partial_x f_\varepsilon^{n+1}\|_{L^p},\\
J_2 &= -p\int_{\mathbb{R}\times\mathbb{R}_+}|\partial_v f_\varepsilon^{n+1}|^{p-2}\partial_v f_\varepsilon^{n+1}\partial_v(-f_\varepsilon^{n+1}+(u_\varepsilon^{\varepsilon,n}-v)\partial_v f_\varepsilon^{n+1})\,dxdv\\
&= 2p\|\partial_v f_\varepsilon^{n+1}\|_{L^p}^p - \|\partial_v f_\varepsilon^{n+1}\|_{L^p}^p,\\
J_3 &= p\int_{\mathbb{R}\times\mathbb{R}_+}|\partial_v f_\varepsilon^{n+1}|^{p-2}\partial_v f_\varepsilon^{n+1}\frac{\rho_\varepsilon^n}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}(-f_\varepsilon^{n+1}+(u_\varepsilon^n-v)\partial_v f_\varepsilon^{n+1})\,dxdv\\
&\le C_\varepsilon p\|\partial_v f_\varepsilon^{n+1}\|_{L^p}^{p-1}\|f_\varepsilon^{n+1}\|_{L^p}+C_\varepsilon p\|\partial_v f_\varepsilon^{n+1}\|_{L^p}^p.
\end{aligned}
$$

Hence we have

$$
\frac{d}{dt}\|\partial_v f_\varepsilon^{n+1}\|_{L^p}\le C_\varepsilon\|f_\varepsilon^{n+1}\|_{W^{1,p}}.
$$

We now combine all of the above estimates to arrive at

$$
\frac{d}{dt}\|f_\varepsilon^{n+1}\|_{W^{1,p}}\le C_{\varepsilon,T,f_{0,\varepsilon}}\left(1+\frac{1}{p}\right)\|f_\varepsilon^{n+1}\|_{W^{1,p}}.
$$

Then, applying Grönwall's lemma and letting $p\to\infty$, we get

$$
\|f_\varepsilon^{n+1}(\cdot,\cdot,t)\|_{W^{1,\infty}}\le\|f_{0,\varepsilon}\|_{W^{1,\infty}}e^{C_{\varepsilon,T,f_{0,\varepsilon}}},
$$

for $t\in[0,T]$, where $C_{\varepsilon,T,f_{0,\varepsilon}}>0$ is independent of n. □

3.2. **Proof of Proposition 3.1: Existence of weak solutions of** (3.1)**.** In this subsection, we establish that the global-in-time existence of weak solutions to the regularized equation (3.1). For this, we first show that the approximation sequence $\{f^n\}_{n\in\mathbb{N}}$ is Cauchy. It follows from (3.2) that

$$
\begin{aligned}
&\partial_t(f_\varepsilon^{n+1}-f_\varepsilon^n)+v\partial_x(f_\varepsilon^{n+1}-f_\varepsilon^n)+\partial_v((u_\varepsilon^{\varepsilon,n}-v)(f_\varepsilon^{n+1}-f_\varepsilon^n))+(u_\varepsilon^{\varepsilon,n}-u_\varepsilon^{\varepsilon,n-1})\partial_v f_\varepsilon^n\\
&\quad=\frac{\rho_\varepsilon^n(u_\varepsilon^n-v)}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}(f_\varepsilon^{n+1}-f_\varepsilon^n)+\frac{f_\varepsilon^n}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\left(\rho_\varepsilon^n u_\varepsilon^n-\rho_\varepsilon^{n-1}u_\varepsilon^{n-1}-v(\rho_\varepsilon^n-\rho_\varepsilon^{n-1})\right)\\
&\qquad+f_\varepsilon^n\rho_\varepsilon^{n-1}(u_\varepsilon^{n-1}-v)\left(\frac{1}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}-\frac{1}{1+\varepsilon\rho_\varepsilon^{n-1}(1+u_\varepsilon^{n-1})}\right).
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^p\\
&\quad= -p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)\partial_v((u_\varepsilon^{\varepsilon,n}-v)(f_\varepsilon^{n+1}-f_\varepsilon^n))\,dxdv\\
&\qquad - p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)(u_\varepsilon^{\varepsilon,n}-u_\varepsilon^{\varepsilon,n-1})\partial_v f_\varepsilon^n dxdv\\
&\qquad + p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)\rho_\varepsilon^n(u_\varepsilon^n-v)(f_\varepsilon^{n+1}-f_\varepsilon^n)\,dxdv\\
&\qquad + p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)f_\varepsilon^n(\rho_\varepsilon^n u_\varepsilon^n-\rho_\varepsilon^{n-1}u_\varepsilon^{n-1})\,dxdv\\
&\qquad - p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)v(\rho_\varepsilon^n-\rho_\varepsilon^{n-1}))\,dxdv\\
&\qquad + p\int_{\mathbb{R}\times\mathbb{R}_+}|f_\varepsilon^{n+1}-f_\varepsilon^n|^{p-2}(f_\varepsilon^{n+1}-f_\varepsilon^n)\\
&\qquad\qquad\qquad \times\frac{\varepsilon f_\varepsilon^n\rho_\varepsilon^{n-1}(u_\varepsilon^{n-1}-v)(\rho_\varepsilon^{n-1}-\rho_\varepsilon^n+\rho_\varepsilon^{n-1}u_\varepsilon^{n-1}-\rho_\varepsilon^n u_\varepsilon^n)}{(1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n))(1+\varepsilon\rho_\varepsilon^{n-1}(1+u_\varepsilon^{n-1}))}\,dxdv\\
&\quad=: \sum_{i=1}^6 K_i,
\end{aligned}
$$

where $K_i, i=1,\dots,6$ can be estimated as follows.

$$
\begin{aligned}
&K_1=(p+1)\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^p,\\
&K_2\le C_{\varepsilon,T,f_{0,\varepsilon}}p\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^{p-1}\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty},\\
&K_3\le C_{\varepsilon,T,f_{0,\varepsilon}}p\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^p,\\
&K_4\le C_{\varepsilon,T,f_{0,\varepsilon}}p\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^{p-1}\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty},\\
&K_5\le Cp\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^{p-1}\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty},\\
&K_6\le C_{\varepsilon,T,f_{0,\varepsilon}}p\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}^{p-1}\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty}.
\end{aligned}
$$

Here we used

$$
\begin{aligned}
|\rho_\varepsilon^n-\rho_\varepsilon^{n-1}| &= \left|\int_{\mathbb{R}_+}(f_\varepsilon^n-f_\varepsilon^{n-1})\,dv\right|\le C\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty},\\
|\rho_\varepsilon^n u_\varepsilon^n-\rho_\varepsilon^{n-1}u_\varepsilon^{n-1}| &= \left|\int_{\mathbb{R}_+}v(f_\varepsilon^n-f_\varepsilon^{n-1})\,dv\right|\le C\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty}, \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&|u_\varepsilon^{\varepsilon,n}-u_\varepsilon^{\varepsilon,n-1}|\\
&\quad=\left|\frac{(\rho_\varepsilon^n u_\varepsilon^n-\rho_\varepsilon^{n-1}u_\varepsilon^{n-1})\star\theta_\varepsilon}{\varepsilon+\rho_\varepsilon^n}+\frac{((\rho_\varepsilon^{n-1}u_\varepsilon^{n-1})\star\theta_\varepsilon)(\rho^n-\rho^{n-1})\star\theta_\varepsilon}{(\varepsilon+\rho_\varepsilon^n)(\varepsilon+\rho_\varepsilon^{n-1})}\right|\\
&\quad\le C_{\varepsilon,T,f_{0,\varepsilon}}\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty}. \qquad (3.7)
\end{aligned}
$$

Thus we have

$$
\frac{d}{dt}\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}\le C_{\varepsilon,T,f_{0,\varepsilon}}\left(\|f_\varepsilon^{n+1}-f_\varepsilon^n\|_{L^p}+\|f_\varepsilon^n-f_\varepsilon^{n-1}\|_{L^\infty}\right),
$$

where $C_{\varepsilon,T,f_{0,\varepsilon}} > 0$ is independent of n. This, together with applying Grönwall's lemma and passing $p \to \infty$, yields

$$\|(f_\varepsilon^{n+1} - f_\varepsilon^n)(\cdot,\cdot,t)\|_{L^\infty} \le C_{\varepsilon,T,f_{0,\varepsilon}} \int_0^t \|(f_\varepsilon^n - f_\varepsilon^{n-1})(\cdot,\cdot,s)\|_{L^\infty}\, ds.$$

This concludes that f^n is Cauchy in $L^\infty(0,T; L^\infty(\mathbb{R}\times\mathbb{R}_+))$ from which, for a fixed $\varepsilon > 0$, there exists a limiting function f_ε such that

$$\sup_{0\le t\le T} \|(f_\varepsilon^n - f_\varepsilon)(\cdot,\cdot,t)\|_{L^\infty} \to 0,$$

as $n \to \infty$. Due to (3.6) and (3.7), we can easily show that the liming function f_ε solves the regularized equation (3.1).

3.3. **Proof of Proposition 3.1: Uniform-in-ε bound estimates.** In this part, we establish several uniform-in-ε estimates for f_ε.

3.3.1. *Support estimates.* We recall from Proposition 3.2 that

$$f_\varepsilon^n(x,v) = 0 \quad \text{if} \quad (x,v) \in (\mathbb{R}\times\mathbb{R}_+) \setminus \big([C_{x,0}, C_{x,1} + TC_{v,1}] \times [e^{-T}C_{v,0}, C_{v,1}]\big)$$

for $0 \le t \le T$. Since $C_{x,i}$ and $C_{v,i}$ $(i = 0,1)$ do not depend on ε and f_ε^n converges uniformly to f_ε, this implies that the support of f_ε is also contained in the same area:

$$[C_{x,0}, C_{x,1} + TC_{v,1}] \times [e^{-T}C_{v,0}, C_{v,1}].$$

That is,

$$R_V[f_\varepsilon(t)] \le C_{v,1}, \quad R_X[f_\varepsilon(t)] \le C_{x,1} + TC_{v,1}$$

and

$$r_X[f_\varepsilon(t)] \ge C_{x,0}, \quad r_V[f_\varepsilon(t)] \ge e^{-T}C_{v,0}.$$

Note that this automatically implies $f_\varepsilon(x,0,t) = 0$.

3.3.2. *Uniform bounds of the moment and energy estimates.* We first provide the uniform energy estimate. It follows from [18, Lemma 2.5] that

$$\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \theta_\varepsilon(x-y) \frac{\rho_\varepsilon(x)}{\theta_\varepsilon \star \rho_\varepsilon(x)}\, dx \le C, \tag{3.8}$$

where $C > 0$ is independent of ε. On the other hand, a straightforward computation yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon\, dxdv &= \int_{\mathbb{R}\times\mathbb{R}_+} v\cdot(u_\varepsilon^\varepsilon - v) f_\varepsilon\, dxdv + \int_{\mathbb{R}\times\mathbb{R}_+} \frac{v^2\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\, dxdv\\ &\le \frac{1}{2}\int_{\mathbb{R}} |u_\varepsilon^\varepsilon|^2 \rho_\varepsilon\, dx - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon\, dxdv\\ &\le \frac{1}{2}\int_{\mathbb{R}} |u_\varepsilon^\varepsilon|^2\rho_\varepsilon\, dx,\end{aligned}$$

where we used Lemma 2.4:

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}_+} \frac{v^2\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\, dxdv &= \int_{\mathbb{R}} \frac{\rho_\varepsilon}{1+\varepsilon\rho_\varepsilon^n(1+u_\varepsilon^n)}\left(u_\varepsilon\int_{\mathbb{R}_+} v^2 f_\varepsilon\, dv - \int_{\mathbb{R}_+} v^3 f_\varepsilon\, dv\right) dx\\ &\le 0,\end{aligned}$$

and

$$v\cdot(u_\varepsilon^\varepsilon - v) \le \frac{(u_\varepsilon^\varepsilon)^2 + v^2}{2} - v^2 = \frac{(u_\varepsilon^\varepsilon)^2 - v^2}{2}.$$

Note that

$$|u_\varepsilon^\varepsilon(x,t)|^2 \le \left|\frac{\int_{\mathbb{R}\times\mathbb{R}_+} \theta_\varepsilon(x-y) w f_\varepsilon(y,w,t)\,dydw}{\theta_\varepsilon \star \rho_\varepsilon(x,t)}\right|^2 \le \frac{\int_{\mathbb{R}\times\mathbb{R}_+} \theta_\varepsilon(x-y) w^2 f_\varepsilon(y,w,t)\,dydw}{\theta_\varepsilon \star \rho_\varepsilon(x,t)},$$

and this together with (3.8) gives

$$\begin{aligned}\int_{\mathbb{R}} \rho_\varepsilon |u_\varepsilon^\varepsilon|^2\,dx &\le \int_{\mathbb{R}\times\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \theta_\varepsilon(x-y)\frac{\rho_\varepsilon(x)}{\theta_\varepsilon \star \rho_\varepsilon(x)}\,dx\right) w^2 f_\varepsilon(y,w)\,dydw \\ &\le C\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon(x,v)\,dxdv,\end{aligned} \tag{3.9}$$

where $C>0$ is independent of ε. Hence we have

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon\,dxdv \le C\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon\,dxdv,$$

i.e.,

$$\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_\varepsilon\,dxdv \le C\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_{0,\varepsilon}\,dxdv,$$

where $C>0$ is independent of ε.

We next show the moment estimate. By multiplying the regularized equation (3.1) by v and integrating the resulting equation over $\mathbb{R}\times\mathbb{R}_+$, we find

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} v f_\varepsilon\,dxdv = -\int_{\mathbb{R}\times\mathbb{R}_+} v\partial_v((u_\varepsilon^\varepsilon - v)f_\varepsilon)\,dxdv + \int_{\mathbb{R}\times\mathbb{R}_+} \frac{v\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv.$$

Here the last term on the right hand side of the above equation can be estimated as

$$\int_{\mathbb{R}\times\mathbb{R}_+} \frac{v\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv = -\int_{\mathbb{R}\times\mathbb{R}_+} \frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv,$$

due to

$$\int_{\mathbb{R}_+} (u_\varepsilon - v)f_\varepsilon\,dv = 0.$$

For the estimate of the first term on the right hand side, we use the uniform estimate above, (3.9), and the fact that

$$\int_{\mathbb{R}} \rho_\varepsilon u_\varepsilon\,dx = \int_{\mathbb{R}\times\mathbb{R}_+} v f_\varepsilon\,dxdv \ge 0$$

to get

$$\begin{aligned}
-\int_{\mathbb{R}\times\mathbb{R}_+} v\partial_v((u^\varepsilon_\varepsilon - v)f_\varepsilon)\,dxdv &= \int_{\mathbb{R}\times\mathbb{R}_+}(u^\varepsilon_\varepsilon - v)f_\varepsilon\,dxdv \\
&= \int_{\mathbb{R}}(\rho_\varepsilon u^\varepsilon_\varepsilon - \rho_\varepsilon u_\varepsilon)\,dx \\
&\le \int_{\mathbb{R}}\rho_\varepsilon u^\varepsilon_\varepsilon\,dx \\
&\le \frac{1}{2}\int_{\mathbb{R}}\rho_\varepsilon\,dx + \frac{1}{2}\int_{\mathbb{R}}\rho_\varepsilon|u^\varepsilon_\varepsilon|^2\,dx \\
&\le C\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_\varepsilon\,dxdv \\
&\le C\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_{0,\varepsilon}\,dxdv,
\end{aligned}$$

where $C > 0$ is independent of ε. Thus, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}_+} vf_\varepsilon\,dxdv \le C\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_{0,\varepsilon}\,dxdv - \int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv.$$

By integrating the above differential inequality with respect to time, we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}_+} vf_\varepsilon\,dxdv + \int_0^t\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdvds \\
&\quad\le \int_{\mathbb{R}\times\mathbb{R}_+} vf_{0,\varepsilon}\,dxdv + C_T\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_{0,\varepsilon}\,dxdv \qquad (3.10)\\
&\quad\le C_T\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_{0,\varepsilon}\,dxdv,
\end{aligned}$$

where $C > 0$ is independent of ε.

3.3.3. *Uniform L^∞-bound of f_ε.* For $1 \le p < \infty$, we find

$$\begin{aligned}
\frac{d}{dt}\|f_\varepsilon\|^p_{L^p} &= p\int_{\mathbb{R}\times\mathbb{R}_+}(f_\varepsilon)^{p-1}\left(-\partial_v\left((u^\varepsilon_\varepsilon - v)f_\varepsilon\right) + \frac{\rho_\varepsilon(u_\varepsilon - v)f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\right)dxdv \\
&=: L_1 + L_2.
\end{aligned}$$

Here L_1 can be easily estimated as

$$L_1 = (p-1)\|f_\varepsilon\|^p_{L^p}.$$

For the estimate of L_2, we obtain

$$\begin{aligned}
L_2 &= p\int_{\mathbb{R}\times\mathbb{R}_+}(f_\varepsilon)^p\frac{\rho_\varepsilon(u_\varepsilon - v)}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv \\
&\le p\|f_\varepsilon\|^{p-1/2}_{L^\infty}\int_{\mathbb{R}\times\mathbb{R}_+}(f_\varepsilon)^{1/2}\frac{\rho_\varepsilon|u_\varepsilon - v|}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv \\
&\le Cp\|f_\varepsilon\|^{p-1/2}_{L^\infty}\left(\int_{\mathbb{R}}\frac{\rho_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dx\right)^{1/2}\left(\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv\right)^{1/2} \\
&\le Cp\|f_\varepsilon\|^{p-1/2}_{L^\infty}\left(\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv\right)^{1/2},
\end{aligned}$$

where we used the uniform bound estimates of supports of f_ε. Thus we have

$$\begin{aligned}\frac{d}{dt}\|f_\varepsilon\|_{L^p} &\le C\|f_\varepsilon\|_{L^p} + C\|f_\varepsilon\|_{L^p}^{1/2}\left(\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv\right)^{1/2}\\ &\le C\|f_\varepsilon\|_{L^p} + \int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdv.\end{aligned}$$

We now use Grönwall's lemma and pass to the limit $p \to \infty$ to conclude

$$\begin{aligned}\|f_\varepsilon(\cdot,\cdot,t)\|_{L^\infty} &\le C\|f_{0,\varepsilon}\|_{L^\infty} + C\int_0^t\int_{\mathbb{R}\times\mathbb{R}_+}\frac{\rho_\varepsilon(u_\varepsilon - v)^2 f_\varepsilon}{1+\varepsilon\rho_\varepsilon(1+u_\varepsilon)}\,dxdvds\\ &\le C\|f_{0,\varepsilon}\|_{L^\infty} + C_T\int_{\mathbb{R}\times\mathbb{R}_+}(1+v^2)f_{0,\varepsilon}\,dxdv,\end{aligned}$$

where $C > 0$ is independent of ε and we used the uniform moment estimate (3.10).

4. Global existence of weak solutions: Proof of Theorem 1.1

In this section, we provide the details of proof of Theorem 1.1.

4.1. **Strong compactness of ρ_ε and u_ε.** From the argument in the previous section, we see that there exists $f \in L^1(\mathbb{R}\times\mathbb{R}_+\times(0,T))$ such that f_ε, $f_\varepsilon v$ converge to f, fv weakly in $L^1(\mathbb{R}\times\mathbb{R}_+\times(0,T))$ respectively, which also implies

$$\rho_\varepsilon = \int_{\mathbb{R}_+} f_\varepsilon\,dv \rightharpoonup \int_{\mathbb{R}_+} f\,dv = \rho \quad \text{and} \quad \rho_\varepsilon u_\varepsilon = \int_{\mathbb{R}_+} v f_\varepsilon\,dv \rightharpoonup \int_{\mathbb{R}_+} vf\,dv = \rho u,$$

in $L^1(\mathbb{R}\times(0,T))$, respectively. Thanks to Lemma 2.1, the above convergences actually are strong, which also give the almost everywhere convergences of the macroscopic fields ρ_ε and $\rho_\varepsilon u_\varepsilon$:

$$\rho_\varepsilon \to \rho \quad \text{a.e on } \mathbb{R}\times[0,T] \quad \text{and} \quad \rho_\varepsilon u_\varepsilon \to \rho u \quad \text{a.e on } \mathbb{R}\times[0,T]. \tag{4.1}$$

4.2. **$f_\varepsilon u_\varepsilon^\varepsilon$ converges to fu in $L^\infty(0,T;L^p(\mathbb{R}\times\mathbb{R}_+))$.** In this part, we show that

$$f_\varepsilon u_\varepsilon^\varepsilon \rightharpoonup fu \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{R}\times\mathbb{R}_+)) \quad \text{for} \quad p\in(1,3/2)\,.$$

Even though the proof is almost the same with [18], we briefly present it for the completeness of our work. It follows from (4.1) together with [18, Lemma 6] that

$$(\rho_\varepsilon u_\varepsilon)\star\theta_\varepsilon \to \rho u, \quad \rho_\varepsilon\star\theta_\varepsilon \to \rho \quad \text{a.e.} \quad \text{and} \quad L^p(\mathbb{R}\times(0,T))\text{-strong},$$

up to a subsequence, for all $p \in (1,3/2)$. Let

$$\rho_\varepsilon^\varphi = \int_{\mathbb{R}_+} f_\varepsilon\varphi(v)\,dv,$$

for a given test function $\varphi(v)$. Consider a test function $\psi(x,v,t) := \phi(x,t)\varphi(v)$ with $\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,T))$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. Then we find

$$\int_0^T\int_{\mathbb{R}\times\mathbb{R}_+} f_\varepsilon u_\varepsilon^\varepsilon\psi\,dxdvdt = \int_0^T\int_{\mathbb{R}} u_\varepsilon^\varepsilon\rho_\varepsilon^\varphi\phi\,dxdt.$$

Note that if $p\in(1,3/2)$, then $p/(2-p)\in(1,3)$, and this gives the following uniform boundedness in ε:

$$\|u_\varepsilon^\varepsilon\rho_\varepsilon^\varphi\|_{L^p} \le \|\varphi\|_{L^\infty}\|\rho_\varepsilon\|_{L^{p/(2-p)}}^{1/2}\|\sqrt{\rho_\varepsilon}u_\varepsilon^\varepsilon\|_{L^2} < \infty.$$

This implies that there exists a function $m \in L^\infty(0,T;L^p(\mathbb{R}))$ such that

$$u_\varepsilon^\varepsilon\rho_\varepsilon^\varphi \rightharpoonup m \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{R})) \quad \text{for all} \quad p\in(1,3/2)$$

up to a subsequence. We now prove

$$m = u\rho^{\varphi}, \quad \text{where} \quad \rho^{\varphi} = \int_{\mathbb{R}_+} f\varphi\, dv \text{ and } \rho u = \int_{\mathbb{R}_+} vf\, dv.$$

By using the set $E_R^0 := \{(x,t) \in (-R,R) \times (0,T) \,:\, \rho(x,t) = 0\}$, we estimate

$$\|u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi\|_{L^p(E_R^0)} \le C\|\rho_\varepsilon\|_{L^{p/(2-p)}(E_R^0)}^{1/2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus it suffices to check

$$m = u\rho^{\varphi} \quad \text{whenever} \quad \rho > 0.$$

For this, we introduce a set

$$E_R^\delta := \{(x,t) \in (-R,R) \times (0,T) \,:\, \rho(x,t) > \delta\}.$$

Due to the compactness of ρ_ε and $\rho_\varepsilon \star \theta_\varepsilon$, by Egorov's theorem, for any $\eta > 0$, there exists a set $C_\eta \subset E_R^\delta$ with $|E_R^\delta \setminus C_\eta| < \eta$ on which ρ_ε and $\rho_\varepsilon \star \theta_\varepsilon$ uniformly converge to ρ. This asserts $\rho_\varepsilon \star \theta_\varepsilon > \delta/2$ in C_η for $\varepsilon > 0$ small enough. Thus we obtain

$$u_\varepsilon^\varepsilon \rho_\varepsilon^\varphi = \frac{(\rho_\varepsilon u_\varepsilon) \star \theta_\varepsilon}{\varepsilon + \rho_\varepsilon \star \theta_\varepsilon} \rho_\varepsilon^\varphi \to m = u\rho^\varphi \quad \text{in} \quad C_\eta.$$

This further yields

$$m = u\rho^\varphi \quad \text{on} \quad \{\rho > 0\},$$

since $\eta > 0$, $R > 0$, and $\delta > 0$ were arbitrary. Consequently, we have

$$\int_0^T \int_{\mathbb{R}\times\mathbb{R}_+} f_\varepsilon u_\varepsilon^\varepsilon \psi\, dxdvdt \to \int_0^T \int_{\mathbb{R}} u\rho^\varphi \phi dxdt = \int_0^T \int_{\mathbb{R}\times\mathbb{R}_+} fu\psi\, dxdvdt$$

for all test functions of the form $\psi(x,v,t) = \phi(x,t)\varphi(v)$.

4.3. $(\rho_\varepsilon u_\varepsilon f_\varepsilon)/(1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon))$ **converges to** $\rho u f$ **in** $L^\infty(0,T;L^p(\mathbb{R}\times\mathbb{R}_+))$**.** Employing the same notations with that in the previous section, we get

$$\int_0^T \int_{\mathbb{R}\times\mathbb{R}_+} \frac{\rho_\varepsilon u_\varepsilon f_\varepsilon}{1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon)} \psi\, dxdvdt = \int_0^T \int_{\mathbb{R}} \frac{\rho_\varepsilon u_\varepsilon \rho_\varepsilon^\varphi}{1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon)} \phi\, dxdt.$$

Note that if $p \in (1, 3/2)$, then $3p/(2-p) \in (3,9)$, and this gives the following uniform boundedness in ε:

$$\left\| \frac{\rho_\varepsilon u_\varepsilon \rho_\varepsilon^\varphi}{1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon)} \right\|_{L^p} \le \|\rho_\varepsilon u_\varepsilon \rho_\varepsilon^\varphi\|_{L^p} \le \|\varphi\|_{L^\infty} \|\rho_\varepsilon\|_{L^{3p/(2-p)}}^{3/2} \|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^2} < \infty.$$

This again gives the existence of a function $m \in L^\infty(0,T;L^p(\mathbb{R}))$ such that

$$\frac{\rho_\varepsilon u_\varepsilon \rho_\varepsilon^\varphi}{1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon)} \rightharpoonup m \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{R})) \quad \text{for all} \quad p \in (1,3/2)$$

up to a subsequence. We then show that

$$m = \rho u \rho^\varphi, \quad \text{where} \quad \rho^\varphi = \int_{\mathbb{R}_+} f\varphi\, dv \text{ and } \rho u = \int_{\mathbb{R}_+} vf\, dv.$$

By using the set $E_R^0 = \{(x,t) \in (-R,R) \times (0,T) \,:\, \rho(x,t) = 0\}$ again, we obtain

$$\|u_\varepsilon \rho_\varepsilon^\varphi\|_{L^p(E_R^0)} \le C\|\rho_\varepsilon\|_{L^{3p/(2-p)}(E_R^0)}^{3/2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Again by considering the set $E_{\mathbb{R}}^\delta$ defined as before and employing almost the same argument as in the previous section, we can show that

$$m = \rho u \rho^\varphi \quad \text{whenever} \quad \rho > 0.$$

Hence we have

$$\int_0^T \int_{\mathbb{R}\times\mathbb{R}_+} \frac{\rho_\varepsilon u_\varepsilon f_\varepsilon}{1+\varepsilon(\rho_\varepsilon + \rho_\varepsilon u_\varepsilon)} \psi \, dxdvdt \to \int_0^T \int_{\mathbb{R}} u\rho^\varphi \phi dxdt = \int_0^T \int_{\mathbb{R}\times\mathbb{R}_+} f u \psi \, dxdvdt$$

for all test functions of the form $\psi(x,v,t) = \phi(x,t)\varphi(v)$.

Proof of Theorem 1.1. Equipped with the previous convergence estimates, there is no problem with passing to the limit $\varepsilon \to 0$ in (3.1) to conclude that the limiting function f is a weak solution to (1.1) in the sense of Definition 1.1. □

5. Hydrodynamic limit: Proof of Theorem 1.2

In this section, we present the details of proof of Theorem 1.2. As mentioned before, our proof relies on the relative entropy argument.

5.1. **Relative entropy estimate.** We first rewrite the equations (1.4) as a conservative form:

$$U_t + \nabla \cdot A(U) = 0,$$

where

$$m = \rho u, \quad U := \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad A(U) := \begin{pmatrix} m \\ m^2/\rho \end{pmatrix}.$$

Then the above system have the macro entropy form $E(U) := m^2/(2\rho)$. Note that the entropy defined above is not strictly convex with respect to ρ. We now define the relative entropy functional $\mathcal{H}$ as follows.

$$\mathcal{H}(\bar{U}|U) := E(\bar{U}) - E(U) - DE(U)(\bar{U} - U) \quad \text{with} \quad \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \end{pmatrix}, \tag{5.1}$$

where $DE(U)$ denotes the derivation of E with respect to ρ, m, i.e.,

$$DE(U) = \begin{pmatrix} -m^2/(2\rho^2) \\ m/\rho \end{pmatrix}.$$

This asserts

$$\mathcal{H}(\bar{U}|U) = \frac{\bar{\rho}\bar{u}^2}{2} - \frac{\rho u^2}{2} - \frac{u^2}{2}(\rho - \bar{\rho}) - u \cdot (\bar{\rho}\bar{u} - \rho u) = \frac{\bar{\rho}}{2}(u - \bar{u})^2.$$

As mentioned in Introduction, the relative entropy defined above does not give any information about the discrepancy between ρ and $\bar{\rho}$.

Lemma 5.1. *The relative entropy $\mathcal{H}$ defined in* (5.1) *satisfies the following equality.*

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \mathcal{H}(\bar{U}|U)\, dx \\ &\quad = \int_{\mathbb{R}} \partial_t E(\bar{U})\, dx - \int_{\mathbb{R}} \nabla(DE(U)) : A(\bar{U}|U)\, dx - \int_{\mathbb{R}} DE(U)\left(\partial_t \bar{U} + \nabla \cdot A(\bar{U})\right) dx, \end{aligned}$$

where $A(\bar{U}|U)$ is the relative flux functional given by

$$A(\bar{U}|U) := A(\bar{U}) - A(U) - DA(U)(\bar{U} - U).$$

Proof. It follows from (5.1) that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\mathcal{H}(\bar{U}|U)\,dx &= \int_{\mathbb{R}}\partial_t E(\bar{U})\,dx - \int_{\mathbb{R}} DE(U)(\partial_t\bar{U} + \nabla\cdot A(\bar{U}))\,dx \\ &\quad + \int_{\mathbb{R}} D^2E(U)\nabla\cdot A(U)(\bar{U}-U) + DE(U)\nabla\cdot A(\bar{U})\,dx \\ &=: \sum_{i=1}^{3} I_i,\end{aligned}$$

where I_3 can be easily estimated as

$$\begin{aligned}I_3 &= \int_{\mathbb{R}}(\nabla DE(U)) : \big(DA(U)(\bar{U}-U) - A(\bar{U})\big)\,dx \\ &= -\int_{\mathbb{R}}(\nabla DE(U)) : \big(A(\bar{U}|U) + A(U)\big)\,dx \\ &= -\int_{\mathbb{R}}(\nabla DE(U)) : A(\bar{U}|U)\,dx.\end{aligned}$$

Here we used the fact from [20] that

$$\int_{\mathbb{R}}(\nabla DE(U)) : A(U)\,dx = 0.$$

□

We now set

$$m^\varepsilon := \rho^\varepsilon u^\varepsilon \quad \text{and} \quad U^\varepsilon := \begin{pmatrix}\rho^\varepsilon \\ m^\varepsilon\end{pmatrix} \quad \text{with} \quad \rho^\varepsilon := \int_{\mathbb{R}_+} f^\varepsilon\,dv \quad \text{and} \quad m^\varepsilon := \int_{\mathbb{R}_+} vf^\varepsilon\,dv,$$

where f^ε is a weak solution to the equation (1.3) in the sense of Definition 1.1.

Proposition 5.1. *Let f^ε be a weak solution to the equation* (1.3) *and (ρ, u) be a strong solution to the system* (1.4) *on the time interval $[0,T]$. Suppose that the assumptions* **(H1)**–**(H2)** *hold. Then we have*

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\mathcal{H}(U^\varepsilon(t)|U(t))\,dx \le \mathcal{O}(\varepsilon). \tag{5.2}$$

Proof. By the relative entropy estimate in Lemma 5.1, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\mathcal{H}(U^\varepsilon|U)\,dx \\ &\quad = \int_{\mathbb{R}}\mathcal{H}(U_0^\varepsilon|U_0)\,dx + \int_{\mathbb{R}}(E(U^\varepsilon) - E(U_0^\varepsilon))\,dx - \int_0^t\int_{\mathbb{R}}\nabla(DE(U)) : A(U^\varepsilon|U)\,dxds \\ &\qquad - \int_0^t\int_{\mathbb{R}} DE(U)\left(\partial_s U^\varepsilon + \nabla\cdot A(U^\varepsilon)\right)\,dxds \\ &\quad =: \sum_{i=1}^{4} J_i^\varepsilon.\end{aligned}$$

The assumption **(H1)** gives $J_1^\varepsilon = \mathcal{O}(\varepsilon)$. For the estimate of J_2^ε, we first notice that

$$(u^\varepsilon)^2 = \left(\frac{\int_{\mathbb{R}_+} vf^\varepsilon\,dv}{\int_{\mathbb{R}_+} f^\varepsilon\,dv}\right)^2 \le \frac{\int_{\mathbb{R}_+} v^2 f^\varepsilon\,dv}{\rho^\varepsilon}, \quad \text{i.e.,} \quad \rho^\varepsilon(u^\varepsilon)^2 \le \int_{\mathbb{R}_+} v^2 f^\varepsilon\,dv,$$

and this implies

$$E(U^\varepsilon) = \frac{1}{2}\int_{\mathbb{R}} \rho^\varepsilon (u^\varepsilon)^2\,dx \le \frac{1}{2}\int_{\mathbb{R}_+} v^2 f^\varepsilon\,dxdv. \tag{5.3}$$

Thus, by adding and subtracting, we find

$$\begin{aligned} J_2^\varepsilon &= \int_{\mathbb{R}} E(U^\varepsilon)\,dx - \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f^\varepsilon\,dxdv \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f^\varepsilon\,dxdv - \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_0^\varepsilon\,dxdv \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_0^\varepsilon\,dxdv - \int_{\mathbb{R}} E(U_0)\,dx \\ &\le 0 + 0 + \mathcal{O}(\varepsilon). \end{aligned} \tag{5.4}$$

Here we also used Lemma 2.4 and **(H1)**. We next use [3, Proposition 2.2] to estimate

$$J_3^\varepsilon \le C\int_0^t \int_{\mathbb{R}} \mathcal{H}(U^\varepsilon|U)\,dxds,$$

by using the following identity:

$$\int_{\mathbb{R}} |A(U^\varepsilon|U)|\,dx = \int_{\mathbb{R}} \rho^\varepsilon (u^\varepsilon - u)^2\,dx.$$

Moreover, by Theorem 1.1, see also Lemma 2.4, we get that f^ε satisfies

$$\int_0^t \int_{\mathbb{R}\times\mathbb{R}_+} f^\varepsilon (u^\varepsilon - v)^2\,dxdvds \le \frac{\varepsilon}{2}\int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_0^\varepsilon\,dxdv, \tag{5.5}$$

and this together with the assumption (**H1**) asserts

$$\begin{aligned} J_4^\varepsilon &\le \|\partial_x u\|_{L^\infty}\int_0^t \int_{\mathbb{R}} \left|\int_{\mathbb{R}_+} ((u^\varepsilon)^2 - v^2) f^\varepsilon\,dv\right| dxds \\ &\le \|\partial_x u\|_{L^\infty}\int_0^t \int_{\mathbb{R}\times\mathbb{R}_+} f^\varepsilon (u^\varepsilon - v)^2\,dxdvds \\ &\le C\varepsilon, \end{aligned}$$

where $C = C(\|\partial_x u\|_{L^\infty}, \int_{\mathbb{R}\times\mathbb{R}_+} f_0^\varepsilon v^2\,dxdv) > 0$. We finally combine all of the above estimates to conclude the proof. □

5.2. **Proof of Theorem 1.2.** We now show that the MKR distance can be bounded by the relative entropy, which directly gives the quantitative error estimate between ρ and ρ^ε.

Note that the local densities ρ and ρ^ε satisfy

$$\partial_t \rho + \partial_x(\rho u) = 0 \quad \text{and} \quad \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = 0,$$

respectively. Let us define forward characteristics $X(t) := X(t;0,x)$ and $X^\varepsilon(t) := X^\varepsilon(t;0,x)$, $t \in [0,T]$ as solutions to

$$\partial_t X(t) = u(X(t),t) \quad \text{and} \quad \partial_t X^\varepsilon(t) = u^\varepsilon(X^\varepsilon(t),t) \tag{5.6}$$

with $X(0) = X^\varepsilon(0) = x \in \mathbb{R}$, respectively. Since u is bounded and Lipschitz continuous on the time interval $[0,T]$, we find the solution ρ uniquely exists and it can be determined as the push-forward of the its initial densities through the flow maps X, i.e., $\rho(t) = X(t;0,\cdot)\#\rho_0$.

Here $\cdot \# \cdot$ stands for the push-forward of a probability measure by a measurable map, more precisely, $\nu = \mathcal{T}\#\mu$ for probability measure μ and measurable map $\mathcal{T}$ implies

$$\int_{\mathbb{R}} \varphi(y)\, d\nu(y) = \int_{\mathbb{R}} \varphi(\mathcal{T}(x))\, d\mu(x),$$

for all $\varphi \in \mathcal{C}_b(\mathbb{R})$. On the other hand, due to the lack of regularity of u^ε, it is not clear to have the existence of solutions X^ε. To handle this issue, we recall the following proposition from [1, Theorem 8.2.1], see also [12, Proposition 3.3].

Proposition 5.2. *Let $T > 0$ and $\rho : [0,T] \to \mathcal{P}(\mathbb{R})$ be a narrowly continuous solution of* (5.6)*, that is, ρ is continuous in the duality with continuous bounded functions, for a Borel vector field u satisfying*

$$\int_0^T \int_{\mathbb{R}} |u(x,t)|^p \rho(x,t)\, dxdt < \infty, \tag{5.7}$$

for some $p > 1$. Let $\Gamma_T : [0,T] \to \mathbb{R}$ denote the space of continuous curves. Then there exists a probability measure η on $\Gamma_T \times \mathbb{R}$ satisfying the following properties:

(i) *η is concentrated on the set of pairs (γ, x) such that γ is an absolutely continuous curve satisfying*
$$\dot\gamma(t) = u(\gamma(t), t),$$
for almost everywhere $t \in (0,T)$ with $\gamma(0) = x \in \mathbb{R}$.

(ii) *ρ satisfies*
$$\int_{\mathbb{R}} \varphi(x)\rho\, dx = \int_{\Gamma_T \times \mathbb{R}} \varphi(\gamma(t))\, d\eta(\gamma, x),$$
for all $\varphi \in \mathcal{C}_b(\mathbb{R})$, $t \in [0,T]$.

Note that it follows from (5.3) and (5.4) that

$$\int_{\mathbb{R}} (u^\varepsilon)^2 \rho^\varepsilon\, dx \le \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f^\varepsilon\, dxdv \le \int_{\mathbb{R}\times\mathbb{R}_+} v^2 f_0^\varepsilon\, dxdv < \infty,$$

i.e., (5.7) holds for $p = 2$, and thus by Proposition 5.2, we have the existence of a probability measure η^ε in $\Gamma_T \times \mathbb{R}$, which is concentrated on the set of pairs (γ, x) such that γ is a solution of

$$\dot\gamma(t) = u^\varepsilon(\gamma(t), t), \tag{5.8}$$

with $\gamma(0) = x$. Furthermore, it holds

$$\int_{\mathbb{R}} \varphi(x)\rho^\varepsilon(x,t)\, dx = \int_{\Gamma_T \times \mathbb{R}} \varphi(\gamma(t))\, d\eta^\varepsilon(\gamma, x), \tag{5.9}$$

for all $\varphi \in \mathcal{C}_b(\mathbb{R})$, $t \in [0,T]$.

We now consider the push-forward of the ρ_0^ε through the flow map X and denote it by $\bar\rho^\varepsilon$, i.e., $\bar\rho^\varepsilon = X\#\rho_0^\varepsilon$. Then for bounded Lipschitz function ϕ, we find

$$\left| \int_{\mathbb{R}} \phi(x)(\rho(x) - \bar\rho^\varepsilon(x))\, dx \right| = \left| \int_{\mathbb{R}} \phi(X(t))(\rho_0(x) - \rho_0^\varepsilon(x))\, dx \right| \le C d_{MKR}(\rho_0, \rho_0^\varepsilon), \tag{5.10}$$

where $C > 0$ is independent of ε, and we used the fact that the bounded Lipschitz distance is equivalent to MKR distance and $\phi(X)$ is bounded and Lipschitz. Indeed, we get

$$\begin{aligned} |X(t;0,x) - X(t;0,y)| &\le |x-y| + \int_0^t |u(X(s;0,x)) - |u(X(s;0,y))|\, ds \\ &\le |x-y| + \|\partial_x u\|_{L^\infty} \int_0^t |X(s;0,x) - X(s;0,y)|\, ds \end{aligned}$$

and apply Grönwall's lemma to derive the Lipschitz continuity of the characteristic flow $X(t;0,x)$ in x, and subsequently, this asserts

$$|\phi(X(t;0,x)) - \phi(X(t;0,y))| \le \|\phi\|_{Lip}|X(t;0,x) - X(t;0,y)| \le \|\phi\|_{Lip}\|X\|_{Lip}|x-y|,$$

where $\|\cdot\|_{Lip}$ denotes the Lipschitz constant given by

$$\|\phi\|_{Lip} := \sup_{x\neq y\in\mathbb{R}} \frac{|\phi(x)-\phi(y)|}{|x-y|}.$$

Thus we obtain from (5.10) that

$$d_{MKR}(\rho(t), \bar{\rho}^\varepsilon(t)) \le C d_{MKR}(\rho_0, \rho_0^\varepsilon), \tag{5.11}$$

for $t \in [0,T]$, where $C>0$ is independent of $\varepsilon > 0$. We next estimate the error between $\bar{\rho}^\varepsilon$ and ρ^ε. For this, we note that by the disintegration theorem of measures, see [1], we can write

$$d\eta^\varepsilon(\gamma, x) = \eta_x^\varepsilon(d\gamma) \otimes \rho_0^\varepsilon(x)\,dx,$$

where $\{\eta_x^\varepsilon\}_{x\in\mathbb{R}}$ is a family of probability measures on Γ_T concentrated on solutions of (5.8). We then introduce a measure ν^ε on $\Gamma_T \times \Gamma_T \times \mathbb{R}$ defined by

$$d\nu^\varepsilon(\gamma, x, \sigma) = \eta_x^\varepsilon(d\gamma) \otimes \delta_{X(\cdot;0,x)}(d\sigma) \otimes \rho_0^\varepsilon(x)\,dx.$$

We further consider an evaluation map $E_t : \Gamma_T \times \Gamma_T \times \mathbb{R} \to \mathbb{R}\times\mathbb{R}$ defined as $E_t(\gamma,\sigma,x) = (\gamma(t), \sigma(t))$. Then we readily find that measure $\pi_t^\varepsilon := (E_t)\#\nu^\varepsilon$ on $\mathbb{R}\times\mathbb{R}$ has marginals $\rho^\varepsilon(x,t)\,dx$ and $\bar{\rho}^\varepsilon(y,t)\,dy$ for $t\in[0,T]$, see (5.9). This yields

$$\begin{aligned} d_{MKR}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) &\le \int_{\mathbb{R}\times\mathbb{R}} |x-y|\,d\pi_t^\varepsilon(x,y) \\ &= \int_{\Gamma_T\times\Gamma_T\times\mathbb{R}} |\sigma(t)-\gamma(t)|\,d\nu^\varepsilon(\gamma,\sigma,x) \\ &= \int_{\Gamma_T\times\mathbb{R}} |X(t;0,x)-\gamma(t)|\,d\eta^\varepsilon(\gamma,x). \end{aligned} \tag{5.12}$$

On the other hand, it follows from (5.6) and (5.8) that

$$\begin{aligned} &|X(t;0,x)-\gamma(t)| \\ &\quad = \left| \int_0^t u(X(s;0,x)) - u^\varepsilon(\gamma(s),s)\,ds \right| \\ &\quad \le \int_0^t |u(X(s;0,x)) - u(\gamma(s),s)|\,ds + \int_0^t |u(\gamma(s),s) - u^\varepsilon(\gamma(s),s)|\,ds \\ &\quad \le \|\partial_x u\|_{L^\infty} \int_0^t |X(s;0,x)-\gamma(s)|\,ds + \int_0^t |u(\gamma(s),s) - u^\varepsilon(\gamma(s),s)|\,ds. \end{aligned}$$

Applying Grönwall's lemma to the above asserts

$$|X(t;0,x)-\gamma(t)| \le C\int_0^t |u(\gamma(s),s) - u^\varepsilon(\gamma(s),s)|\,ds,$$

where $C > 0$ is independent of $\varepsilon > 0$. Combining this with (5.12), we have

$$
\begin{aligned}
d_{MKR}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) &\le C \int_0^t \int_{\Gamma_T \times \mathbb{R}} |u(\gamma(s), s) - u^\varepsilon(\gamma(s), s)| \, d\eta^\varepsilon(\gamma, x) \, ds \\
&\le C \int_0^t \int_{\mathbb{R}} |u(x, s) - u^\varepsilon(x, s)| \rho^\varepsilon(x, s) \, dxds \\
&\le C\sqrt{T} \left(\int_0^t \int_{\mathbb{R}} (u^\varepsilon(x, s) - u(x, s))^2 \rho^\varepsilon(x, s) \, dxds \right)^{1/2} \\
&= C \left(\int_0^t \int_{\mathbb{R}} \mathcal{H}(U^\varepsilon | U) \, dxds \right)^{1/2},
\end{aligned} \tag{5.13}
$$

where $C > 0$ is independent of $\varepsilon > 0$, and we used (5.9). We then combine (5.11) and (5.13) to conclude

$$
\begin{aligned}
d_{MKR}(\rho(t), \rho^\varepsilon(t)) &\le d_{MKR}(\rho(t), \bar{\rho}^\varepsilon(t)) + d_{MKR}(\rho^\varepsilon(t), \bar{\rho}^\varepsilon(t)) \\
&\le C d_{MKR}(\rho_0, \rho_0^\varepsilon) + C \left(\int_0^t \int_{\mathbb{R}} \mathcal{H}(U^\varepsilon | U) \, dxds \right)^{1/2},
\end{aligned}
$$

where $C > 0$ is independent of $\varepsilon > 0$. We finally use the estimate (5.2) to conclude that

$$
d_{MKR}(\rho(t), \rho^\varepsilon(t)) \le C d_{MKR}(\rho_0, \rho_0^\varepsilon) + \mathcal{O}(\sqrt{\varepsilon}).
$$

We next estimate the MKR distance between f^ε and $\rho \otimes \delta_u$. For $\phi \in Lip(\mathbb{R} \times \mathbb{R}_+)$, we estimate

$$
\begin{aligned}
&\left| \int_{\mathbb{R} \times \mathbb{R}_+} \phi(x, v) f^\varepsilon(x, v) \, dxdv - \int_{\mathbb{R} \times \mathbb{R}_+} \phi(x, v) \rho(x) \, dx \otimes \delta_{u(x)}(dv) \right| \\
&\quad = \left| \int_{\mathbb{R} \times \mathbb{R}_+} \phi(x, v) f^\varepsilon(x, v) \, dxdv - \int_{\mathbb{R}} \phi(x, u(x)) \rho(x) \, dx \right| \\
&\quad \le \left| \int_{\mathbb{R} \times \mathbb{R}_+} \phi(x, v) f^\varepsilon(x, v) \, dxdv - \int_{\mathbb{R}} \phi(x, u(x)) \rho^\varepsilon(x) \, dx \right| \\
&\qquad + \left| \int_{\mathbb{R}} \phi(x, u(x)) \rho^\varepsilon(x) \, dx - \int_{\mathbb{R}} \phi(x, u(x)) \rho(x) \, dx \right|.
\end{aligned}
$$

Since both ϕ and u are Lipschitz, the second term on the right hand side of the above inequality can be bounded by

$$
C d_{MKR}(\rho^\varepsilon, \rho) \le \mathcal{O}(\sqrt{\varepsilon}),
$$

where $C > 0$ is independent of ε. For the first term, we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times\mathbb{R}_+} \phi(x,v) f^\varepsilon(x,v)\,dxdv - \int_{\mathbb{R}} \phi(x,u(x))\rho^\varepsilon(x)\,dx\right| \\
&\quad \le \|\phi\|_{Lip} \int_{\mathbb{R}\times\mathbb{R}_+} |v-u| f^\varepsilon(x,v)\,dxdv \\
&\quad \le \|\phi\|_{Lip} \left(\int_{\mathbb{R}\times\mathbb{R}_+} |v-u^\varepsilon| f^\varepsilon(x,v)\,dxdv + \int_{\mathbb{R}} |u^\varepsilon - u|\rho^\varepsilon(x)\,dx\right) \\
&\quad \le C\left(\int_{\mathbb{R}\times\mathbb{R}_+} (v-u^\varepsilon)^2 f^\varepsilon(x,v)\,dxdv + \int_{\mathbb{R}} (u^\varepsilon - u)^2\rho^\varepsilon(x)\,dx\right)^{1/2} \\
&\quad \le C\sqrt{\varepsilon},
\end{aligned}$$

due to (5.2) and (5.5), where $C > 0$ is independent of ε. Combining all of the above estimates, we have

$$\left|\int_{\mathbb{R}\times\mathbb{R}_+} \phi(x,v) f^\varepsilon(x,v)\,dxdv - \int_{\mathbb{R}\times\mathbb{R}_+} \phi(x,v)\rho(x)\,dx \otimes \delta_{u(x)}(dv)\right| \le \mathcal{O}(\sqrt{\varepsilon})$$

for any $\phi \in Lip(\mathbb{R}\times\mathbb{R}_+)$. This completes the proof.

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References

[1] L. Ambrosio, N. Gigli, and G, Savaré, Gradient flows: in metric spaces and in the space of probability measures, Springer Science & Business Media, 2008.

[2] Bellomo, N., Dogbe, C.: On the modeling of traffic and crowds: a survey of models, speculations, and perspectives. SIAM Rev. **53** (2011), 409–463.

[3] Carrillo, J. A. and Choi, Y.-P.: Quantitative error estimates for the large friction limit of Vlasov equation with nonlocal forces, preprint.

[4] Carrillo, J. A., Choi, Y.-P., and Karper, T. K.: On the analysis of a coupled kinetic-fluid model with local alignment forces, Ann. Inst. Henri Poincaré Anal. Non Linéaire, **33** (2016), 273–307.

[5] Carrillo, J. A., Choi, Y.-P., and Pérez, S.: A review on attractive-repulsive hydrodynamics for consensus in collective behavior. Active particles. Vol. 1. Advances in theory, models, and applications, 259–298, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2017.

[6] Choi, Y.-P.: Global classical solutions of the Vlasov-Fokker-Planck equation with local alignment forces, Nonlinearity, **29** (2016), 1887–1916.

[7] Choi, Y.-P.: A rigorous derivation of the hydrodynamic model for synchronization phenomena from inertial kinetic Kuramoto model, preprint.

[8] Choi, Y.-P., Ha, S.-Y., and Li, Z.: Emergent dynamics of the Cucker-Smale flocking model and its variants. Active particles. Vol. 1. Advances in theory, models, and applications, 299–331, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2017.

[9] Choi, Y.-P. and Kwon, B.: The Cauchy problem for the pressureless Euler/isentropic Navier-Stokes equations, J. Differential Equations **261** (2016), 654–711.

[10] Cucker, F. and Smale, S.: Emergent behavior in flocks. IEEE Trans. Automat. Control, **52** (2007), 852–862.

[11] Dafermos, C. M.: The second law of thermodynamics and stability, Arch. Ration. Mech. Anal., **70** (1979), 167–179.

[12] Figalli, A. and Kang, M.-J.: A rigorous derivation from the kinetic Cucker-Smale model to the pressureless Euler system with nonlocal alignment, Anal. PDE, **12** (2019), 843–866.

[13] Golse, F., Lions, P.-L., Perthame, B., and Sentis, R.: Regularity of the moments of the solution of a transport equation. J. Funct. Anal., **76** (1988), 110–125.
[14] Helbing, D. and Greiner, A.: Modeling and simulation of multilane traffic flow. Phys. Rev. E, **55**, pages 5498–5508, 1997.
[15] Herty, M. and Pareschi, L: Fokker-Planck Asymptotics for Traffic Flow Models. Kinet. Relat. Models, **1** (2010), 165–179.
[16] Herty, H., Puppo, G., Roncoroni, S., and Visconti, G.: The BGK approximation of kinetic models for traffic. Preprint.
[17] Illner, R., Klar, A., and Materne, T.: Vlasov-Fokker-Planck models for multilane traffic flow. Commun. Math. Sci., **1** (2003), 1–12.
[18] Karper, T. K. Mellet, A., and Trivisa, K.: On strong local alignment in the kinetic Cucker-Smale model, Hyperbolic conservation laws and related analysis with applications, Springer (2013), 227–242.
[19] Karper, T. K. Mellet, A., and Trivisa, K.: Existence of weak solutions to kinetic flocking models, SIAM J. Math. Anal., **45** (2013), 215–243.
[20] Karper, T. K. Mellet, A., and Trivisa, K.: Hydrodynamic limit of the kinetic Cucker-Smale flocking model, Math. Models Methods Appl. Sci., **25** (2015), 131–163.
[21] Klar, A. and Wegener, R: A kinetic model for vehicular traffic derived from a stochastic microscopic model. Transport. Theor. Stat., **25** (1996), 785–798.
[22] Klar, A. and Wegener, R.: Enskog-like kinetic models for vehicular traffic. J. Stat. Phys., **87** (1997), 91–114.
[23] Klar, A. and Wegener, R.: Traffic flow: models and numerics. In Modeling and computational methods for kinetic equations, Model. Simul. Sci. Eng. Technol., pages 219–258. Birkhäuser Boston, Boston, MA, 2004.
[24] Motsch S. and Tadmor, E.: A new model for self-organized dynamics and its flocking behavior. J. Stat. Phys., **144** (2011), 923–947.
[25] Paveri-Fontana, S.-L.: On Boltzmann-like treatments for traffic flow: a critical review of the basic model and an alternative proposal for dilute traffic analysis. Transport. Res., **9** (1975), 225–235.
[26] Piccoli, B. and Tosin, A.: Vehicular Traffic: Areview of continuum mathematical models, Mathematics of complexity anddynamical systems, Vols. 1–3, 1748–1770, Springer, New York, 2012.
[27] Prigogine, I.: A Boltzmann-like approach to the statistical theory of traffic flow. In R. Herman, editor, Theory of traffic flow, pages 158–164, Amsterdam, 1961. Elsevier.
[28] Prigogine, I. and Herman, R.: Kinetic theory of vehicular traffic. American Elsevier Publishing Co., New York, 1971.
[29] Puppo, G., Semplice, M., Tosin, A., and Visconti, G.: Kinetic models for traffic flow resulting in a reduced space of microscopic velocities. Kinet. Relat. Models, **10** (2017), 823–854.

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