MINIMUM DEGREES FOR POWERS OF PATHS AND CYCLES

ENG KEAT HNG

ABSTRACT. We study minimum degree conditions under which a graph G contains kth powers of paths and cycles of arbitrary specified lengths. We determine precise thresholds, assuming that the order of G is large. This extends a result of Allen, Böttcher and Hladký [J. Lond. Math. Soc. (2) 84(2) (2011), 269–302] concerning the containment of squares of paths and squares of cycles of arbitrary specified lengths and settles a conjecture of theirs in the affirmative.

1. INTRODUCTION

The study of conditions on vertex degrees in a host graph G for the appearance of a target graph H is a major theme in extremal graph theory. A classical result in this area is the following theorem of Dirac about the existence of a Hamiltonian cycle.

Theorem 1.1 (Dirac [5]). Every graph on $n \ge 3$ vertices with minimum degree at least $\frac{n}{2}$ has a Hamiltonian cycle.

We write C_{ℓ} (resp. P_{ℓ}) for a cycle (resp. path) of *length* ℓ , that is, a cycle (resp. path) on ℓ vertices. The kth power of a graph G, denoted by G^k , is obtained from G by joining every pair of vertices at distance at most k. In 1962, Pósa conjectured an analogue of Dirac's theorem for the containment of the square of a Hamiltonian cycle. This conjecture was extended in 1974 by Seymour to general powers of a Hamiltonian cycle.

Conjecture 1 (Pósa–Seymour Conjecture [17]). Let $k \in \mathbb{N}$. A graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{kn}{k+1}$ contains the kth power of a Hamiltonian cycle.

Fan and Kierstead made significant progress, proving an approximate version of this conjecture for the square of paths and the square of cycles in sufficiently large graphs [8] and determining the best-possible minimum degree condition for the square of a Hamiltonian path [9]. Komlós, Sárközy and Szemerédi confirmed the truth of the Pósa–Seymour Conjecture for sufficiently large graphs.

Theorem 1.2 (Komlós, Sárközy and Szemerédi [13]). For every positive integer k, there exists an integer $n_0 = n_0(k)$ such that for all integers $n \ge n_0$, any graph G on n vertices with minimum degree at least $\frac{kn}{k+1}$ contains the kth power of a Hamiltonian cycle.

In fact, their proof asserts a stronger result, guaranteeing kth powers of cycles of all lengths divisible by k + 1 between k + 1 and n, in addition to the kth power of a Hamiltonian cycle. The divisibility condition is necessary as balanced complete (k + 1)-partite graphs contain kth powers of cycles of no other length.

Theorem 1.3 (Komlós–Sárközy–Szemerédi [13]). For every positive integer k, there exists an integer $n_0 = n_0(k)$ such that for all integers $n \ge n_0$, any graph G on n vertices with minimum degree $\delta(G) \ge \frac{kn}{k+1}$ contains the kth power of a cycle $C_{(k+1)\ell}^k$ for any $1 \le \ell \le \frac{n}{k+1}$.

Recently there has been an interest in generalising the Pósa-Seymour Conjecture. Allen, Böttcher and Hladký [3] determined the exact minimum degree threshold for a large graph to contain the square of a cycle of a given length. Staden and Treglown [18] proved a degree sequence analogue for the square of a Hamiltonian cycle. Ebsen, Maesaka, Reiher, Schacht

Department of Mathematics, London School of Economics, Houghton Street, London, WC2A 2AE, United Kingdom, e.hng@lse.ac.uk, supported by an LSE PhD Studentship.



FIGURE 1. The behaviour of $pp_3(n, \delta)$

and Schülke [6] showed that inseparable graphs which are sufficiently uniformly dense contain powers of Hamiltonian cycles. Recently, Lang and Sanhueza-Matamala [15] introduced the concept of Hamilton frameworks and proved that robust aperiodic Hamilton frameworks contain powers of Hamiltonian cycles. There has also been related work in the hypergraph setting for tight cycles and tight components. Rödl, Ruciński and Szemerédi [16] established the minimum codegree threshold for a tight Hamiltonian cycle in k-uniform hypergraphs. Allen, Böttcher, Cooley and Mycroft [1] proved an asymptotically tight result on the minimum codegree threshold for a tight cycle of a given length in k-partite k-uniform hypergraphs. The problem of minimum codegree thresholds for tight components of a given size has also been studied by Georgakopoulos, Haslegrave and Montgomery [10].

In this paper we are interested in exact minimum degree thresholds for the appearance of the kth power of a path P_{ℓ}^k and the kth power of a cycle C_{ℓ}^k . One possible guess as to what minimum degree $\delta = \delta(G)$ will guarantee which length $\ell = \ell(n, \delta)$ of kth power of a path (or longest kth power of a cycle) is the following. Since the minimum degree threshold for the kth power of a Hamiltonian cycle (or path) is roughly the same as that for a spanning K_{k+1} -factor, perhaps this remains true for smaller ℓ . If this were true, it would mean that one could expect that $\ell(n, \delta)$ would be roughly $(k + 1)(k\delta - (k - 1)n)$. This is characterised by (k + 1)-partite extremal examples, which are exemplified by the k = 3 example in Figure 2a.

However, this was shown not to give the correct answer by Allen, Böttcher and Hladký [3]. For the case k = 2 (see Theorem 1.4), they determined sharp thresholds attained by a family of extremal graphs which exhibit not a linear dependence between the length of the longest square of a path and the minimum degree, but rather piecewise linear dependence with jumps at certain points. In order to state the result of [3] as well as our result, we first introduce the following functions. Given positive integers k, n, δ with $\delta \in \left(\frac{(k-1)n}{k}, n-1\right]$, we define

(1)
$$r_p(k,n,\delta) := \max\left\{r \in \mathbb{N} : \left\lfloor \frac{(k-1)\delta - (k-2)n}{r} \right\rfloor > k\delta - (k-1)n\right\},$$
$$r_c(k,n,\delta) := \max\left\{r \in \mathbb{N} : \left\lceil \frac{(k-1)\delta - (k-2)n}{r} \right\rceil > k\delta - (k-1)n\right\}.$$

Note that $r_p(k, n, \delta)$ and $r_c(k, n, \delta)$ are almost always the same, differing only for a very small number of values of δ . Setting $s_p(k, n, \delta) := \left\lceil \frac{(k-1)\delta - (k-2)n}{r_p(k, n, \delta)} \right\rceil$ and $s_c(k, n, \delta) := \left\lceil \frac{(k-1)\delta - (k-2)n}{r_c(k, n, \delta)} \right\rceil$, we define

(2)
$$pp_{k}(n,\delta) := \min\left\{ (k-1)\left(\left\lfloor \frac{s_{p}(k,n,\delta)}{2} \right\rfloor + 1\right) + s_{p}(k,n,\delta), n \right\}, \\ pc_{k}(n,\delta) := \min\left\{ (k-1)\left\lfloor \frac{s_{c}(k,n,\delta)}{2} \right\rfloor + s_{c}(k,n,\delta), n \right\}.$$



FIGURE 2. Graphs for k = 3

Note that the functions $pc_k(n, \delta)$ and $pp_k(n, \delta)$ satisfy $pc_k(n, \delta) \leq pp_k(n, \delta)$. They also behave very similarly and differ only by a constant (dependent only on k) when r_p and r_c are equal. The behaviour of $pp_3(n, \delta)$ is illustrated in Figure 1.

Before we discuss the result of Allen, Böttcher and Hladký [3] and our result, we shall define two closely related families of graphs which will serve as examples of extremal graphs. We obtain the *n*-vertex graph $G_p(k, n, \delta)$ by starting with the disjoint union of k - 1 independent sets I_1, \ldots, I_{k-1} and $r := r_p(k, n, \delta)$ cliques X_1, \ldots, X_r with $|I_1| = \cdots = |I_{k-1}| = n - \delta$ and $|X_1| \ge \cdots \ge |X_r| \ge |X_1| - 1$. Then, insert all edges between X_i and I_j for each $(i, j) \in [r] \times [k-1]$ and all edges between I_i and I_j for each $(i, j) \in {\binom{[k-1]}{2}}$. This is a natural generalisation of the construction in [3]. Figure 2b shows an example with k = 3. Construct the graph $G_c(k, n, \delta)$ in the same way as $G_p(k, n, \delta)$ but with $r := r_c(k, n, \delta)$ and with arbitrary selection of a vertex $v \in X_1$ and insertion of all edges between v and X_i for each $i \in [r]$ such that $|X_i| \neq |X_1|$.

Let us now discuss kth powers of paths and cycles in $G_p(k, n, \delta)$ and $G_c(k, n, \delta)$ respectively. We focus on the former as the discussion for the latter is analogous. Consider an arbitrary kth power of a path $P_{\ell}^k \subseteq G_p(k, n, \delta)$ with its vertices in a natural order. Any k+1 consecutive vertices form a clique, so any k+1 consecutive vertices contain vertices from at most one clique X_i . Therefore, P_{ℓ}^k contains vertices from at most one clique X_i . Since P_{ℓ}^k has independence number $\left\lceil \frac{\ell}{k+1} \right\rceil$ and I_i is independent for each $i \in [k-1]$, we have $\ell - (k-1) \left\lceil \frac{\ell}{k+1} \right\rceil \leq \left\lceil \frac{(k-1)\delta - (k-2)n}{r_p(k,n,\delta)} \right\rceil$ and thus we deduce $\ell \leq pp_k(n,\delta)$. Finally, observe that we can construct a copy of $P^k_{pp_k(n,\delta)}$ in $G_p(k, n, \delta)$ as follows. Repeatedly take an unused vertex from I_i for each $i \in [k-1]$ and two unused vertices from X_1 in turn, until all vertices of X_1 are used and skipping I_i for each $i \in [k-1]$ if they become entirely used before X_1 does.

The following is the result of Allen, Böttcher and Hladký [3] for the case k = 2. It states that $pp_2(n, \delta)$ and $pc_2(n, \delta)$ are the maximal lengths of squares of paths and squares of cycles, respectively, guaranteed in an *n*-vertex graph G with minimum degree δ . Furthermore, G also contains any shorter square of a cycle with length divisible by 3. These results are tight with $G_p(2, n, \delta)$ and $G_p(2, n, \delta)$ serving as extremal examples. In fact, both graphs contain the squares of cycles C_{ℓ}^2 for all lengths $3 \leq \ell \leq \mathrm{pc}_2(n, \delta)$ such that $\chi(C_{\ell}^2) \leq 4$. If G does not contain any one of these squares of cycles with chromatic number 4, then (ii) of Theorem 1.4 guarantees even longer squares of cycles C_{ℓ}^2 in G, where ℓ is divisible by 3.

Theorem 1.4 (Allen, Böttcher and Hladký [3]). For any $\nu > 0$ there exists an integer n_0 such that for all integers $n > n_0$ and $\delta \in [(\frac{1}{2} + \nu)n, \frac{2n-1}{3}]$ the following holds for all graphs G on n vertices with minimum degree $\delta(G) \ge \delta$.

- (i) $P^2_{\mathrm{pp}_2(n,\delta)} \subseteq G$ and $C^2_{\ell} \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [3, \mathrm{pc}_2(n,\delta)]$ such that 3 divides ℓ . (ii) Either $C^2_{\ell} \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [3, \mathrm{pc}_2(n,\delta)]$ and $\chi(C^2_{\ell}) \leq 4$, or $C^2_{\ell} \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [3, 6\delta - 3n - \nu n]$ such that 3 divides ℓ .

It was conjectured by Allen, Böttcher, and Hladký [3, Conjecture 24] that their result can be naturally generalised to higher powers. Our main result is that their conjecture is indeed true.

Note that $\chi(C_{\ell}^k) \leq k+2$ holds for all $\ell \geq k^2 + k$, so this condition excludes only a number of lengths which is a function of k.

Theorem 1.5. Given an integer $k \ge 3$ and $0 < \nu < 1$ there exists an integer n_0 such that for all integers $n \ge n_0$ and $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn}{k+1}\right)$ the following holds for all graphs G on n vertices with minimum degree $\delta(G) \ge \delta$.

- (i) $P_{\mathrm{pp}_{k}(n,\delta)}^{k} \subseteq G$ and $C_{\ell}^{k} \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, \mathrm{pc}_{k}(n,\delta)]$ such that k+1 divides ℓ .
- (ii) Either $C_{\ell}^k \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, \mathrm{pc}_k(n, \delta)]$ such that $\chi(C_{\ell}^k) \leq k+2$, or $C_{\ell}^k \subseteq G$ for every $\ell \in \mathbb{N}$ with $\ell \in [k+1, (k+1)(k\delta (k-1)n) \nu n]$ such that k+1 divides ℓ .

As with the result of Allen, Böttcher and Hladký [3], our result is also tight with $G_p(k, n, \delta)$ and $G_c(k, n, \delta)$ serving as extremal examples. Note that (ii) implies the *k*th powers of cycles case of (i), as the latter is precisely the common part of the two cases in (ii). Hence, it will be sufficient to prove (ii) and the first part of (i).

We remark that while our proof uses the same basic strategy as used in [3] for the proof of Theorem 1.4 (that is, combining the regularity method and the stability method), our proof is not merely a generalisation of the proof of Theorem 1.4. In particular, the proof of our Stability Lemma turns out to be much more complex than in [3], and the analysis requires new insights.

The remainder of the paper is organised as follows. In Section 2 we introduce our notation and tools. In Section 3 we outline our proof strategy and state the key lemmas in our proof. Then, we provide a proof of Theorem 1.5 which applies these lemmas. The main difficulty in our proof is proving a Stability Lemma (that is, Lemma 3.3). In Section 4 we provide proofs for two special cases of our Stability Lemma and introduce a family of configurations which enables analysis of the general case. In Section 5 we analyse the aforementioned family of configurations and develop greedy-type methods, which we will subsequently use in Section 6 in the proof of the general case of our Stability Lemma. Finally, we provide a proof of our Extremal Lemma in Section 7.

2. Preliminaries

In this section we introduce the notation we will use and provide various tools we will need. We will also establish some useful properties of the functions introduced in (1) and (2).

2.1. Notation. Write \mathbb{N} for the set of positive integers and \mathbb{N}_0 for the set $\mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}_0$ write [m] for the set $\{1, \ldots, m\}$ and $[m]_0$ for $[m] \cup \{0\}$. For a graph G denote its vertex set and edge set by V(G) and E(G) respectively. Set v(G) := |V(G)| and e(G) := |E(G)|. For sets $X, Y \subseteq V(G)$, set $E(X, Y) := \{xy \in E(G) : x \in X, y \in Y\}$ and e(X, Y) := |E(X, Y)|. Let G[X] denote the subgraph of G induced by X. For a vertex $v \in V(G)$ and a subset $A \subseteq V(G)$ we denote by $\Gamma_G(v; A)$ the neighbourhood in A of v in G and write $\deg_G(v; A)$ for its cardinality $|\Gamma_G(v; A)|$. Given $X \subseteq V(G)$ let $\Gamma_G(X; A) := \bigcap_{v \in X} \Gamma_G(v; A)$ denote the common neighbourhood in A of vertices from X in G and write $\deg_G(X; A)$ for its cardinality $|\Gamma_G(X; A)|$. We will omit the set brackets in $\Gamma_G(\{v_1, \ldots, v_\ell\}; A)$ and $\deg_G(\{v_1, \ldots, v_\ell\}; A)$, and write $\Gamma_G(v_1, \ldots, v_\ell; A)$ and $\deg_G(v_1, \ldots, v_\ell; A)$ respectively instead. We omit the graph G in the subscripts if it is clear from context. Furthermore, we omit the set A if we intend A = V(G). Denote the minimum degree of a graph G by $\delta(G)$. We write $v_1 \cdots v_\ell$ to denote a clique in G with vertices v_1, \ldots, v_ℓ . For an event \mathcal{A} we write $\mathbf{1}_{\mathcal{A}}$ to denote its indicator function.

2.2. Tools. We will need the following simple observations about matchings in graphs with given minimum degree.

Lemma 2.1.

(i) A graph G contains a matching with $\min\{\delta(G), \lfloor \frac{|V(G)|}{2} \rfloor\}$ edges.

(ii) Let $G = (U \cup V, E)$ be a bipartite graph with vertex classes U and V such that every vertex in U has degree at least u and every vertex in V has degree at least v. Then G contains a matching with min $\{u + v, |U|, |V|\}$ edges.

Proof. For (i), let M be a maximum matching in G. We are done unless there are two vertices $u, v \in V(G)$ not contained in M. M is maximal so all neighbours of u and v are contained in M. There cannot be an edge u'v' in M with $uv', vu' \in E(G)$ by maximality of M, since then uv'u'v would be an M-augmenting path. But this means that $\deg(u; e) + \deg(v; e) \leq 2$ for each $e \in M$, which implies that

$$\delta(G) + \delta(G) \le \deg(u) + \deg(v) = \sum_{e \in M} \deg(u; e) + \deg(v; e) \le 2|M|$$

and hence $|M| \ge \delta(G)$.

For (ii), let M be a maximum matching in G. We are done unless there are vertices $u \in U$ and $v \in V$ not contained in M. There cannot be an edge u'v' in M with $uu', vv' \in E$ by maximality of M, since then uu'v'v would be an M-augmenting path. Now M is maximal so all neighbours of u and v are contained in M. This means that $\deg(u; e) + \deg(v; e) \leq 1$ for each $e \in M$, which implies that

$$u + v \le \deg(u) + \deg(v) = \sum_{e \in M} \deg(u; e) + \deg(v; e) \le |M|$$
$$u + v.$$

and hence $|M| \ge u + v$.

It will be useful to have the following simple observations about sizes of common neighbourhoods and maximal cliques.

Lemma 2.2. Let $k \in \mathbb{N}$ be an integer, u_1, \ldots, u_k be vertices of a graph G and $U \subseteq V(G)$. Then $\deg(u_1, \ldots, u_k; U) \geq \sum_{i=1}^k \deg(u_i; U) - (k-1)|U|$. In particular, if $\delta(G) \geq \delta$ then $\deg(u_1, \ldots, u_k) \geq k\delta - (k-1)n$.

Proof. Let $X := \{u_i \mid i \in [k]\}$. Count $\rho := \sum_{i \in [k], v \in U} \mathbf{1}_{\{vu_i \in E(G)\}}$ in two ways. On the one hand, $\rho = \sum_{i \in [k]} \sum_{v \in U} \mathbf{1}_{\{vu_i \in E(G)\}} = \sum_{i \in [k]} \deg(u_i; U)$. On the other hand, noting that vertices in $U \setminus \Gamma(X)$ have at most k - 1 neighbours in X, we obtain

$$\begin{split} \rho &= \sum_{v \in U} \sum_{i \in [k]} \mathbf{1}_{\{vu_i \in E(G)\}} = \sum_{v \in U} \deg(v; X) \\ &= \sum_{v \in \Gamma(X; U)} \deg(v; X) + \sum_{v \in U \setminus \Gamma(X)} \deg(v; X) \\ &\leq k \deg(X; U) + (k-1)(|U| - \deg(X; U)) = \deg(X; U) + (k-1)|U|. \end{split}$$

It follows that $\deg(X;U) \geq \sum_{i=1}^{k} \deg(u_i;U) - (k-1)|U|$. Furthermore, if $\delta(G) \geq \delta$ then $\deg(u_i) \geq \delta$ for each $i \in [k]$, so it follows immediately that $\deg(X) \geq k\delta - (k-1)n$. \Box

Lemma 2.3. Let j, k and ℓ be integers satisfying $1 \leq j \leq \ell \leq k+1$ and let G be a graph on n vertices with minimum degree $\delta(G) > \frac{(k-1)n}{k}$. Then every copy of K_j in G can be extended to a copy of K_ℓ in G.

Proof. Fix integers k, ℓ satisfying $1 \leq \ell \leq k+1$ and proceed by backwards induction on j. The case $j = \ell$ is trivial. For $1 \leq j < \ell$, note that by Lemma 2.2 a copy of K_j has common neighbourhood of size at least $j\delta - (j-1)n > 0$. Therefore, we can extend it to a copy of K_{j+1} by adding to it a vertex in its common neighbourhood. The resultant copy of K_{j+1} can be extended to a copy of K_ℓ by the induction hypothesis.

The following is a classical result of Hajnal and Szemerédi [11].

Theorem 2.4 (Hajnal and Szemerédi [11]). For any graph G on n vertices with maximum degree $\Delta(G)$ and any integer $r \geq \Delta(G) + 1$, there is a partition of V(G) into r independent sets which are each of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

For our purposes we will need the following corollary of Theorem 2.4.

Corollary 2.5. Let $k \in \mathbb{N}$. Let G be a graph on $n \ge k(k+1)$ vertices with $\delta := \delta(G) \ge \frac{(k-1)n}{k}$. Then G contains $\min\left\{k\delta - (k-1)n, \left\lfloor \frac{n}{k+1} \right\rfloor\right\}$ vertex-disjoint copies of K_{k+1} .

Proof of Corollary 2.5. First consider $\delta \in \left(\frac{(k-1)n}{k}, \frac{kn}{k+1}\right)$. Apply Theorem 2.4 to \overline{G} with $r := \Delta(\overline{G}) + 1 = n - \delta \in \left(\frac{n}{k+1}, \frac{n}{k}\right)$. Each part in the resultant partition has size $\left\lceil \frac{n}{r} \right\rceil = k + 1$ or $\left\lfloor \frac{n}{r} \right\rfloor = k$, so there are $n - rk = k\delta - (k-1)n$ pairwise disjoint independent sets of size k + 1. These correspond to $k\delta - (k-1)n$ vertex-disjoint copies of K_{k+1} in G.

Now consider $\delta \geq \frac{kn}{k+1}$. Apply Theorem 2.4 to \overline{G} with $r := \lfloor \frac{n}{k+1} \rfloor > \Delta(\overline{G})$. Each part in the resultant partition has size $\lceil \frac{n}{r} \rceil \geq k+1$ or $\lfloor \frac{n}{r} \rfloor \geq k+1$, so there are $r = \lfloor \frac{n}{k+1} \rfloor$ pairwise disjoint independent sets of size at least k+1 in \overline{G} . These correspond to $\lfloor \frac{n}{k+1} \rfloor$ vertex-disjoint copies of K_{k+1} in G.

For our purposes the following corollary of Theorem 1.2 will be useful.

Corollary 2.6. For every integer $k \in \mathbb{N}$, there exists an integer $n_0 = n_0(k)$ such that for all integers $n \ge n_0$, any graph G on n vertices with minimum degree at least $\frac{kn-1}{k+1}$ contains the kth power of a Hamiltonian path.

Proof. Fix an integer $k \in \mathbb{N}$. Theorem 1.2 produces an integer n_0 . Let G be a graph on $n \ge n_0$ vertices with minimum degree at least $\frac{kn-1}{k+1}$. Obtain a new graph G^* by adding to G a vertex adjacent to all other vertices. Note that $\delta(G^*) \ge \frac{k(n+1)}{k+1}$, so we can appeal to Theorem 1.2 to find a copy of C_{n+1}^k in G^* . Deleting the additional vertex from this copy of C_{n+1}^k in G^* yields the desired copy of P_n^k in G.

The following theorem of Andrásfai, Erdős and Sós gives a sufficient condition for a K_k -free graph to be in fact (k-1)-partite.

Theorem 2.7 (Andrásfai, Erdős and Sós [4]). Let $k \ge 3$ be an integer. A K_k -free graph G on n vertices with minimum degree $\delta(G) > \frac{3k-7}{3k-4}n$ is (k-1)-partite.

Denote by $d_G(u, v)$ the distance between vertices $u, v \in V(G)$ in a connected graph G. A connected graph G on n vertices is *panconnected* if for each pair $u, v \in V(G)$ of vertices and each $d_G(u, v) < \ell \leq n$ there is a path in G with ℓ vertices which has u and v as endpoints. The following theorem of Williamson gives a sufficient minimum degree condition for a graph to be panconnected.

Theorem 2.8 (Williamson [20]). Every graph G on $n \ge 4$ vertices with $\delta(G) \ge \frac{n}{2} + 1$ is panconnected.

The following theorem of Erdős and Stone gives a sufficient condition for a graph to contain $K_{t,t,t}$, the complete tripartite graph on three sets of vertices of size t.

Theorem 2.9 (Erdős and Stone [7]). Given $t \in \mathbb{N}$ and $\rho > 0$, there exists $n_0 = n_0(t, \rho)$ such that every graph on $n \ge n_0$ vertices with at least $(\frac{1}{2} + \rho) \binom{n}{2}$ edges contains a copy of $K_{t,t,t}$.

2.3. Properties of some functions. In this subsection, we collect some analytical data about the functions r_p , r_c , pp_k and pc_k . Note that for fixed $k, n \in \mathbb{N}$ the functions $r_p(k, n, \cdot)$ and $r_c(k, n, \cdot)$ are monotone decreasing while the functions $pp_k(n, \cdot)$ and $pc_k(n, \cdot)$ are monotone increasing. Note that the definition of $r := r_p(k, n, \delta)$ in (1) is equivalent to $r = \left\lfloor \frac{(k-1)\delta - (k-2)n}{k\delta - (k-1)n+1} \right\rfloor$,

from which we obtain

(3)
$$\frac{n-\delta-1}{k\delta-(k-1)n+1} < r \le \frac{(k-1)\delta-(k-2)n}{k\delta-(k-1)n+1},$$

(4)
$$\frac{[(k-1)r+1]n - (r+1)}{kr+1} < \delta \le \frac{[(k-1)(r-1)+1]n - r}{k(r-1)+1}$$

The following lemma gives bounds on the functions r_p , pp_k and pc_k .

Lemma 2.10. Given an integer $k \ge 3$ and $\mu > 0$, there exists $\eta_0 > 0$ such that for every $0 < \eta < \eta_0$ there exists $n_2 \in \mathbb{N}$ such that the following hold for all $n \ge n_2$. Let $r_0 \in \mathbb{N}$ satisfy $r_p(n,\gamma) \le r_0$ for all $\gamma \ge \left(\frac{k-1}{k} + \mu\right)n$. For $\delta \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{k}{k+1} - 2\eta\right)n\right]$ we have

(5)
$$r_p(k, n, \delta + \eta n) \ge 2$$

(6)
$$\operatorname{pp}_k(n,\delta) \le \left(1 - \frac{\eta}{10r_0}\right) \operatorname{pp}_k(n,\delta + \frac{\eta}{2}n),$$

(7)
$$pp_k(n,\delta+\eta n) \le \frac{k+1}{2} \left(\frac{(k-1)(\delta+3\eta n) - (k-2)n}{r_p(k,n,\delta+\eta n)} - 2 \right),$$

(8)
$$\delta - \frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r_p(k, n, \delta + \eta n)} > \frac{3k-4}{3}(n-\delta), \quad and$$

(9)
$$pp_k(n,\delta+\eta n) \le \frac{19}{20}(k+1)(k\delta-(k-1)n) - 2 \le (k+1)(k\delta-(k-1)n) - 10k^2\eta n.$$

For
$$\delta' \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{2k-1}{2k+1} - 2\eta\right)n\right] \cup \left[\left(\frac{2k-1}{2k+1} + \eta\right)n, \left(\frac{k}{k+1} - 2\eta\right)n\right] =: A$$
 we have
(10) $\operatorname{pp}_k(n, \delta' + \eta n) \leq \frac{3}{4}(k+1)(k\delta' - (k-1)n).$

For
$$\delta'' \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{3k-2}{3k+1} - 2\eta\right)n\right] \cup \left[\left(\frac{3k-2}{3k+1} + \eta\right)n, \left(\frac{2k-1}{2k+1} - 2\eta\right)n\right] =: B \text{ we have}$$

(11) $\operatorname{pp}_k(n, \delta'' + \eta n) \leq \frac{2}{2}(k+1)(k\delta'' - (k-1)n).$

For
$$\delta''' \ge \left(\frac{k}{k+1} - 2\eta\right) n$$
 we have
(12) $r_p(k, n, \delta''' + \eta n) \le 2.$

For
$$\delta_1 \in \left[\left(\frac{k-1}{k} + \mu \right) n, \frac{kn-1}{k+1} \right)$$
 we have
(13) $\operatorname{pp}_k(n, \delta_1) \leq \min\left\{ (k+1)(k\delta_1 - (k-1)n) - 10k^2\eta n - (k+1), \frac{11n}{20} \right\}$.
For $\delta_2 \in \left[\left(\frac{k-1}{k} + \mu \right) n, \frac{kn-1}{k+1} \right]$ we have
(14) $\operatorname{pc}_k(n, \delta_2) \leq \frac{11n}{20}$.

Proof. Let $k \ge 3$ be an integer and $\mu > 0$. Pick $\eta_0 = \frac{\mu}{200k^2}$. For $0 < \eta < \eta_0$, pick $n_2 = \max\left\{\frac{10r_0}{\eta}, 100k\right\}$. Let $n \ge n_2$ be an integer.

Let $\delta \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{k}{k+1} - 2\eta\right)n\right]$. Set $\delta_+ := \delta + \eta n, r := r_p(n, \delta)$ and $r' := r_p(n, \delta_+)$. If r' = 1, then by (4) we have $\delta_+ \geq \frac{kn-1}{k+1} > \left(\frac{k}{k+1} - \eta\right)n$. This gives a contradiction, so we have τ

 $r' \geq 2$, i.e. (5). By (2) we have

$$\begin{aligned} pp_k(n,\delta) &\leq \frac{k+1}{2} \left(\frac{(k-1)\delta - (k-2)n}{r} \right) + \frac{3k-1}{2} \\ &\leq \left(1 - \frac{\eta}{10r_0} \right) \left(\frac{k+1}{2} \left(\frac{(k-1)(\delta + \frac{\eta n}{2}) - (k-2)n}{r'} \right) + \frac{k-1}{2} \right) \\ &\leq \left(1 - \frac{\eta}{10r_0} \right) pp_k \left(n, \delta + \frac{\eta n}{2} \right), \end{aligned}$$

so we have (6). By (2) we have

$$pp_k(n, \delta_+) \le \frac{k+1}{2} \left(\frac{(k-1)\delta_+ - (k-2)n}{r'} \right) + \frac{3k-1}{2} \\ \le \frac{k+1}{2} \left(\frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'} - 2 \right),$$

so we have (7). By (1), for some $q \in [r', r' + 1]$ we have

$$(k-1)\delta_{+} - (k-2)n = q(k\delta_{+} - (k-1)n + 1),$$

so we have

$$\frac{(k-1)(\delta+3\eta n) - (k-2)n}{r'} = (k\delta_+ - (k-1)n + 1)\frac{q}{r'} + \frac{(k-1)\eta n}{r'}$$

Since $n - \delta - 1 \ge (q - 1)(k\delta - (k - 1)n + 1)$ and $r' \ge 2$, we have (8). By (1) we have

(15)
$$\frac{(k-1)\delta_+ - (k-2)n}{r'+1} < k\delta_+ - (k-1)n+1.$$

Hence, by (2) we have $pp_k(n, \delta_+) \leq \frac{k+1}{2}(k\delta_+ - (k-1)n + 1)\frac{r'+1}{r'} + \frac{3k-1}{2}$. Since $r' \geq 2$, we have (9) because

$$pp_k(n, \delta_+) \le \frac{3(k+1)}{4} (k\delta_+ - (k-1)n + 1) + \frac{3k-1}{2}$$
$$\le \frac{19}{20} (k+1)(k\delta - (k-1)n) - 2$$
$$\le (k+1)(k\delta - (k-1)n) - 10k^2\eta n.$$

Let $\delta' \in A$. By (4) we have $r' \ge 2$. If $r' \ge 3$, then we have

$$pp_k(n, \delta'_+) \le \frac{2(k+1)}{3}(k\delta'_+ - (k-1)n + 1) + \frac{3k-1}{2} \le \frac{3}{4}(k+1)(k\delta' - (k-1)n),$$

which gives (10). If r' = 2, we have $\delta' \in \left[\left(\frac{2k-1}{2k+1} + \eta\right)n, \left(\frac{k}{k+1} - 2\eta\right)n\right] =: A'$. By considering (15) for $\delta_+ = \frac{(2k-1)n}{2k+1}$ and the coefficient of δ_+ on both sides of the inequality, for $\delta' \in A'$ we can strengthen the inequality to

$$\frac{(k-1)\delta'_{+} - (k-2)n}{3} < k\delta' - (k-1)n - 4.$$

From this, we obtain $pp_k(n, \delta' + \eta n) \leq \frac{3}{4}(k+1)(k\delta' - (k-1)n)$ by an argument analogous to that for (9). For $\delta'' \in B$ we obtain (11) by an argument analogous to that for $\delta' \in A$. Let $\delta''' \geq \left(\frac{k}{k+1} - 2\eta\right)n$. If $r' \geq 3$, then by (4) we have $\delta'''_{+} \leq \frac{(2k-1)n-3}{2k+1} < \left(\frac{k}{k+1} - \eta\right)n$. This gives a contradiction, so we have $r' \leq 2$.

Let $\delta_1 \in \left[\left(\frac{k-1}{k} + \mu\right)n, \frac{kn-1}{k+1}\right)$. Since $pp_k(n, \cdot)$ is monotone increasing, we have $pp_k(n, \delta_1) \leq pp_k(n, \frac{kn-2}{k+1}) \leq \frac{n}{2} + k \leq \frac{11n}{20}$. By (4) we have $r \geq 2$ and by (1) we have $\frac{(k-1)\delta_1 - (k-2)n}{r+1} < k\delta_1 - (k-1)n + 1$. Then, by (2) we have

$$pp_k(n,\delta_1) \le \frac{3(k+1)}{4} (k\delta_1 - (k-1)n + 1) + \frac{3k-1}{2} \\\le (k+1)(k\delta_1 - (k-1)n) - 10k^2\eta n - (k+1),$$

so we obtain (13). Let $\delta_2 \in \left[\left(\frac{k-1}{k} + \mu\right)n, \frac{kn-1}{k+1}\right]$. Since $\mathrm{pc}_k(n, \cdot)$ is monotone increasing, we have $\mathrm{pc}_k(n, \delta_2) \leq \mathrm{pc}_k(n, \frac{kn-1}{k+1}) \leq \frac{n}{2} + k \leq \frac{11n}{20}$, which gives (14).

3. Main Lemmas and proof of Theorem 1.5

Our proof of Theorem 1.5 uses the well-established technique that combines the regularity method, which involves the joint application of Szemerédi's regularity lemma [19] and the blowup lemma of Komlós, Sárközy and Szemerédi [12], and the stability method. However, this provides only a loose framework for proofs of this kind. In particular, the proof of our stability lemma is significantly more involved than in [3] and is the main contribution of this paper. For our application we will define the concept of a connected K_{k+1} -component of a graph, which generalises the concept of a connected triangle component of a graph introduced by Allen, Böttcher and Hladký [3].

In this section we explain how we utilise connected K_{k+1} -components, the regularity method and the stability method. We first introduce the necessary definitions and formulate our main lemmas. Then, we detail how these lemmas imply Theorem 1.5 at the end of this section.

3.1. Connected K_{k+1} -components and K_{k+1} -factors. Fix $k \in \mathbb{N}$ and let G be a graph. A K_{k+1} -walk is a sequence t_1, \ldots, t_p of copies of K_k in G such that for every $i \in [p-1]$ there is a copy c_i of K_{k+1} in G such that $t_i, t_{i+1} \subseteq c_i$. We say that t_1 and t_p are K_{k+1} -connected in G. A K_{k+1} -component of G is a maximal set C of copies of K_k in G such that every pair of copies of K_k in C is K_{k+1} -connected. Observe that this induces an equivalence relation on the copies of K_k of G whose equivalence classes are the K_{k+1} -components of G. The vertices of a K_{k+1} -component C are all vertices v of G such that v is a vertex of a copy of K_k in C. The size of a K_{k+1} -component C is the number of vertices of C, which we denote by |C|.

A K_{k+1} -factor F in G is a collection of vertex-disjoint copies of K_{k+1} in G. F is a connected K_{k+1} -factor if all copies of K_k of F are contained in the same K_{k+1} -component of G. The size of a K_{k+1} -factor F is the number of vertices covered by F. Let $\operatorname{CKF}_{k+1}(G)$ denote the maximum size of a connected K_{k+1} -factor in G. For $\ell \in [k]$ the copies of K_ℓ of a K_{k+1} -component C are all copies of K_ℓ of G which can be extended to a copy of K_k in C. For $\ell > k$ the copies of K_ℓ of a K_{k+1} -component C are all copies of K_ℓ of G to which a copy of K_k in C can be extended. In a (slight) abuse of notation, we shall write $K_\ell \subseteq C$ to mean that there is such a copy of K_ℓ .

3.2. **Regularity method.** Our proof uses a combination of Szemerédi's regularity lemma [19] and the blow-up lemma by Komlós, Sárközy and Szemerédi [12]. Generally, the regularity lemma produces a partition of a dense graph that is suitable for an application of the blow-up lemma, which enables us to embed a target graph in a large host graph. We first introduce some terminology to formulate the versions of these two lemmas we will use.

Let G = (V, E) be a graph and $d, \varepsilon \in (0, 1]$. For disjoint nonempty $U, W \subseteq V$ the density of the pair (U, W) is $d(U, W) := \frac{e(U, W)}{|U||W|}$. A pair (U, W) is ε -regular if $|d(U', W') - d(U, W)| \le \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \ge \varepsilon |U|$ and $|W'| \ge \varepsilon |W|$. An ε -regular partition of G is a partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ of V with $|V_0| \le \varepsilon |V|$, $|V_i| = |V_j|$ for all $i, j \in [\ell]$, and such that for all but at most εk^2 pairs $(i, j) \in [\ell]^2$, the pair (V_i, V_j) is ε -regular.

Given $d \in (0, 1)$, a pair of disjoint vertex sets (V_i, V_j) in a graph G is (ε, d) -regular if it is ε -regular and has density at least d. An ε -regular partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ of a graph G is an (ε, d) -regular partition if the following is true. For every $i \in [k]$ and every vertex $v \in V_i$, there

are at most $(d + \varepsilon)n$ edges incident to v which are not contained in (ε, d) -regular pairs of the partition. Given an (ε, d) -partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ of a graph G, we define a graph R, which we call the *reduced graph* of the partition of G, where R = (V(R), E(R)) has $V(R) = \{V_1, \ldots, V_\ell\}$ and $V_i V_j \in E(R)$ whenever (V_i, V_j) is an (ε, d) -regular pair. We say that G has (ε, d) -reduced graph R and call the partition classes V_i with $i \in [\ell]$ clusters of G.

The classical Szemerédi regularity lemma [19] states that every large graph has an ε -regular partition with a bounded number of parts. Here we state the so-called minimum degree form of Szemerédi's regularity lemma (see, e.g., Lemma 7 in conjunction with Proposition 9 in [14]).

Lemma 3.1 (Regularity Lemma, minimum degree form). Given $\varepsilon \in (0, 1)$ and $m_0 \in \mathbb{N}$, there is an integer $m_1 \in \mathbb{N}$ such that the following holds for all $d, \gamma \in (0, 1)$ such that $\gamma \geq 2d + 4\varepsilon$. Every graph G on $n \geq m_1$ vertices with $\delta(G) \geq \gamma n$ has an (ε, d) -reduced graph R on m vertices with $m_0 \leq m \leq m_1$ and $\delta(R) \geq (\gamma - d - 2\varepsilon)m$.

This lemma asserts the existence of a reduced graph R of G which 'inherits' the high minimum degree of G. We shall use this property to reduce our original problem of finding the kth power of a path (or cycle) in a graph on n vertices with minimum degree γn to the problem of finding an *arbitrary* connected K_{k+1} -factor of a desired size in a graph R on m vertices with minimum degree $(\gamma - d - \varepsilon)n$ (see Lemma 3.2). The new problem seeks a much less specific subgraph (connected K_{k+1} -factor) than the original problem and is therefore easier to tackle.

This kind of problem reduction is possible due to the Blow-up Lemma, which enables the embedding of a large bounded degree target graph H into a graph G with reduced graph R if there is a homomorphism from H to a subgraph T of R which does not 'overuse' any cluster of T. For our purposes we apply this lemma with $T = K_{k+1}$ and obtain for each copy of K_{k+1} in a connected K_{k+1} -factor F the kth power of a path which almost fills up the corresponding clusters of G. The K_{k+1} -connectedness of F then enables us to link up these kth power of paths and obtain kth powers of paths or cycles of the desired length (see Lemma 3.2 (i) and (ii)). In addition, the Blow-up Lemma allows for some control of the start-vertices and end-vertices of the kth power of a path constructed in this manner (see Lemma 3.2 (iii)).

The following lemma sums up what we obtain from this embedding strategy. This is an application of standard methods and we will provide its proof in Appendix A.

Lemma 3.2 (Embedding Lemma). For any integer $k \ge 2$ and any d > 0 there exists $\varepsilon_{EL} > 0$ with the following property. Given $0 < \varepsilon < \varepsilon_{EL}$, for every $m_{EL} \in \mathbb{N}$ there exists $n_{EL} \in \mathbb{N}$ such that the following hold for any graph G on $n \ge n_{EL}$ vertices with (ε, d) -reduced graph R of G on $m \le m_{EL}$ vertices.

- (i) $C_{(k+1)\ell}^k \subseteq G$ for every $\ell \in \mathbb{N}$ with $(k+1)\ell \leq (1-d)\operatorname{CKF}_{k+1}(R)\frac{n}{m}$.
- (ii) If $K_{k+2} \subseteq C$ for each K_{k+1} -component C of R, then $C_{\ell}^k \subseteq G$ for every $\ell \in \mathbb{N}$ with $k+1 \leq \ell \leq (1-d) \operatorname{CKF}_{k+1}(R) \frac{n}{m}$ and $\chi(C_{\ell}^k) \leq k+2$.

Furthermore, let \mathcal{T}' be a connected K_{k+1} -factor in a K_{k+1} -component C of R which contains a copy of K_{k+2} , let $u_{1,1} \ldots u_{1,k}$ and $u_{2,1} \ldots u_{2,k}$ be vertex-disjoint copies of K_k in G, and suppose that $X_{1,1} \ldots X_{1,k}$ and $X_{2,1} \ldots X_{2,k}$ are (not necessarily disjoint) copies of K_k in C in R such that $u_{i,j} \ldots u_{i,k}$ has at least $\frac{2dn}{m}$ common neighbours in the cluster $X_{i,j}$ for each $(i,j) \in [2] \times [k]$. Let A be a set of at most $\frac{\varepsilon n}{m}$ vertices of G disjoint from $\{u_{1,1}, \ldots, u_{1,k}, u_{2,1}, \ldots, u_{2,k}\}$. Then

(iii) $P_{\ell}^k \subseteq G$ for every $\ell \in \mathbb{N}$ with $3m^{k+1} \leq \ell \leq (1-d)|\mathcal{T}'|\frac{n}{m}$, such that P_{ℓ}^k starts in $u_{1,1} \ldots u_{1,k}$ and ends in $u_{2,k} \ldots u_{2,1}$ (in those orders), P_{ℓ}^k contains no element of A, and at most $\frac{\varepsilon n}{m}$ vertices of P_{ℓ}^k are not in $\bigcup \mathcal{T}'$.

Note that the copies of K_{k+2} required in (ii) are essential to the embedding kth powers of cycles which not (k + 1)-chromatic.

3.3. Stability method. The regularity method as just described leaves us with the task of finding a sufficiently large connected K_{k+1} -factor F in a reduced graph R of G. However, this is insufficient on its own. The Embedding Lemma (Lemma 3.2) gives the kth power of a

path which covers a proportion of G roughly the same as the proportion λ of R covered by F. Furthermore, the extremal graphs for kth powers of paths and connected K_{k+1} -factors are the same, but the relative minimum degree $\gamma_R = \delta(R)/|V(R)|$ of R is in general slightly smaller than $\gamma_G = \delta(G)/|V(G)|$. Consequently we cannot expect that λ is larger than the proportion $pp_k(v(R), \gamma_R v(R))/v(R)$ a maximum kth power of a path covers in a graph with relative minimum degree γ_R , and hence λ is smaller than the proportion $pp_k(v(G), \gamma_G v(G))/v(G)$ we would like to cover for relative minimum degree γ_G . This is where our stability approach comes into the picture.

Roughly speaking, we will be more ambitious and aim for a connected K_{k+1} -factor in R larger than guaranteed by the relative minimum degree (see Lemma 3.3 (C1) and (C2)). We prove that we will fail to find this larger connected K_{k+1} -factor only if R (and hence G) is 'very close' to the extremal graph $G_p(k, v(R), \delta(R))$, in which case we will say that R is near-extremal (see Lemma 3.3 (C3)). The following lemma, which we call Stability Lemma and prove in Section 4, does precisely this. Note that this lemma guarantees the copies of K_{k+2} required in the Embedding Lemma (Lemma 3.2). We remark that the proof of Stability Lemma is our main contribution as it is significantly more involved than in [3] and the analysis requires new insights.

Lemma 3.3 (Stability Lemma). Given an integer $k \geq 3$ and $\mu > 0$, for any sufficiently small $\eta > 0$ there exists an integer m_0 such that if $\delta \in \left[(\frac{k-1}{k} + \mu)n, \frac{kn}{k+1} \right]$ and G is a graph on $n \ge m_0$ vertices with minimum degree $\delta(G) > \delta$, then either

- (C1) $\operatorname{CKF}_{k+1}(G) \ge (k+1)(k\delta (k-1)n)$, or
- (C2) $\operatorname{CKF}_{k+1}(G) \ge \operatorname{pp}_k(n, \delta + \eta n), or$
- (C3) G has k-1 vertex-disjoint independent sets of combined size at least $(k-1)(n-\delta)-5k\eta n$ (k-1)n and for each such component X all copies of K_k in G containing at least one vertex of X are K_{k+1} -connected in G.

Moreover, in (C2) and (C3) each K_{k+1} -component of G contains a copy of K_{k+2} .

Note that (C1) gives a connected K_{k+1} -factor whose size is significantly larger than $pp_k(n, \delta)$, which is the maximum size we can guarantee in general (see Figure 1 for an illustration in the case k = 3). We also remark that since $pp_k(n, \delta)$ is a function with relatively large jumps at certain points, around these points (C2) gives a connected K_{k+1} -factor whose size is significantly larger than $pp_k(n, \delta)$. Furthermore, we clarify that the 'components' in (C3) refer to the usual connected components of a graph rather than K_{k+1} -components. It is notable that the two functions $pp_k(n,\delta)$ and $pc_k(n,\delta)$ are sufficiently similar that Stability Lemma handles both. We will only need to distinguish between the kth powers of paths and the kth powers of cycles when we consider near-extremal graphs.

It remains to handle graphs with near-extremal reduced graphs. We have a great deal of structural information about these graphs, which we use to directly find the desired kth powers of paths and cycles. The following lemma, which we call Extremal Lemma, handles the nearextremal case. We will provide a proof of this lemma in Section 7. Note that in this proof we will make use of our Embedding Lemma (Lemma 3.2). Accordingly Lemma 3.4 inherits the upper bound m_{EL} on the number of clusters from Lemma 3.2.

Lemma 3.4 (Extremal Lemma). Given an integer $k \ge 3$, $0 < \nu < 1$ and $0 < \eta, d \le \frac{\nu^4}{(k+1)^{13}10^8}$, there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every integer m_{EL} , there is an integer N such that the following holds. Let G be a graph on $n \ge N$ vertices with $\delta(G) \ge \delta \in \mathbb{R}$. $\left[\left(\frac{k-1}{k}+\nu\right)n,\frac{kn}{k+1}\right)$ and R be an (ε,d) -reduced graph of G on $m \leq m_{EL}$ vertices with a partition $\left(\bigsqcup_{i=1}^{k-1} I_i\right) \sqcup \left(\bigsqcup_{j=1}^{\ell} B_j\right)$ of V(R) with $\ell \ge 2$. Suppose that (i) each K_{k+1} -component of R contains a copy of K_{k+2} ,

(ii) I_1, \ldots, I_{k-1} are independent sets in R with $\left|\bigcup_{i=1}^{k-1} I_i\right| \ge ((k-1)(n-\delta) - 5k\eta n)\frac{m}{n}$,

(iii) for each $i \in [\ell]$ we have $0 < |B_i| \le \frac{19m}{10n}(k\delta - (k-1)n)$, all copies of K_k in R containing at least one vertex of B_i are K_{k+1} -connected in R, and for $j \in [\ell] \setminus \{i\}$ there are no edges between B_i and B_j in R.

Then G contains $P_{\mathrm{pp}_k(n,\delta)}^k$ and C_{ℓ}^k for each $\ell \in [k+1, \mathrm{pc}_k(n,\delta)]$ such that $\chi(C_{\ell}^k) \leq k+2$.

We now have all the ingredients for the proof of our main theorem. The Regularity Lemma (Lemma 3.1) provides a regular partition with reduced graph R of the host graph G and Stability Lemma (Lemma 3.3) tells us that R either contains a large connected K_{k+1} -factor or is nearextremal. We find the kth powers of long paths and cycles in G by applying Embedding Lemma (Lemma 3.2) in the first case and Extremal Lemma (Lemma 3.4) in the second case.

Proof of Theorem 1.5. We first set up the necessary constants. Let $k \ge 3$ and $0 < \nu < 1$. Set $\mu := \left(1 - \frac{2}{2k^2 + 2k + 1}\right)\nu$ and choose $\eta > 0$ to be small enough for Lemmas 2.10, 3.3 and 3.4. In particular, $\eta \le \frac{\nu^4}{(k+1)^{13}10^8}$. Given k, μ, η from above as input, Lemmas 2.10 and 3.3 produce positive integers m'_0 and m_2 respectively. Let $r_0 \in \mathbb{N}$ satisfy $r_p(n, \gamma) \le r_0$ for all $\gamma \ge \left(\frac{k-1}{k} + \mu\right)n$. Set $d := \frac{\eta}{5r_0}$ and $m_0 := \max\{m'_0, m_2, d^{-1}\}$. With k and d as input, Lemma 3.2 then produces $\varepsilon_{EL} > 0$. For ν, η and d, Lemma 3.4 produces $\varepsilon_0 > 0$. Set $\varepsilon := \frac{1}{2}\min\{\varepsilon_{EL}, \varepsilon_0, \frac{\eta}{5r_0}\}$ and choose an integer m_{EL} such that Lemma 3.1 guarantees the existence of an (ε, d) -regular partition with at least m_0 and at most m_{EL} parts. With the constants $\nu, \eta, d, \varepsilon, m_{EL}$ from above as input, Lemma 3.2 and Lemma 3.4 produce n_{EL} and N respectively. Corollary 2.6 returns n_{HP} with k as input. Finally, set $n_0 := \max\{n_{HP}, m_{EL}, n_{EL}, N\}$.

Let $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn}{k+1}\right)$ and let G be a graph on $n \ge n_0$ vertices with minimum degree $\delta(G) \ge \delta$. As remarked after Theorem 1.5, observe that it suffices to show that $P_{\text{pp}_k(n,\delta)}^k \subseteq G$ and that (ii) of Theorem 1.5 holds. Furthermore, we will need to treat the case $\delta = \frac{kn-1}{k+1}$ separately from the rest because in this case $P_{\text{pp}_k(n,\delta)}^k$ is actually Hamiltonian.

Let us consider the case when $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right)$. We first apply Lemma 3.1 to G to obtain an (ε, d) -reduced graph R on $m_0 \leq m \leq m_{EL}$ vertices with $\delta(R) \geq \delta' := \left(\frac{\delta}{n} - d - 2\varepsilon\right)m$. Note that $\delta' \in \left[\left(\frac{k-1}{k} + \mu\right)m, \frac{km}{k+1}\right]$. Then we apply Lemma 3.3 to R. According to this lemma, there are three possibilities.

Firstly, we could have $\operatorname{CKF}_{k+1}(R) \geq (k+1)(k\delta' - (k-1)m)$. Now Lemma 3.2(i) guarantees that G contains C_{ℓ}^k for each positive integer $\ell \leq (1-d)\operatorname{CKF}_{k+1}(R)\frac{n}{m}$ divisible by k+1; since by choice of d and ε we have $(1-d)(k+1)(k\delta' - (k-1)m)\frac{n}{m} \geq (k+1)(k\delta - (k-1)n) - 10k^2\eta n$, G contains C_{ℓ}^k for each positive integer $\ell \leq (k+1)(k\delta - (k-1)n) - \nu n$ divisible by k+1, i.e. the second case of Theorem 1.5(ii) holds. Since $P_{\ell}^k \subseteq C_{\ell}^k$ and we have (13), it follows that Gcontains $P_{\operatorname{pp}_k(n,\delta)}^k$.

Secondly, we could have that $\operatorname{CKF}_{k+1}(R) \ge \operatorname{pp}_k(m, \delta' + \eta m)$ and every K_{k+1} -component of R contains a copy of K_{k+2} . By Lemma 3.2(ii) we have that G contains C_{ℓ}^k for each integer $k+1 \le \ell \le (1-d)\operatorname{CKF}_{k+1}(R)\frac{n}{m}$ such that $\chi(C_{\ell}^k) \le k+2$; since by (6), $P_{\ell}^k \subseteq C_{\ell}^k$ and choice of η, d and ε we have $(1-d)\operatorname{CKF}_{k+1}(R)\frac{n}{m} \ge (1-d)\operatorname{pp}_k(n,\delta + \frac{\eta n}{2}) \ge \operatorname{pp}_k(n,\delta) \ge \operatorname{pc}_k(n,\delta)$, we conclude that G contains $P_{\operatorname{pp}_k(n,\delta)}^k$ and C_{ℓ}^k for each integer $k+1 \le \ell \le \operatorname{pc}_k(n,\delta)$ such that $\chi(C_{\ell}^k) \le k+2$, i.e. the first case of Theorem 1.5(ii) holds.

Thirdly, we could have that R is near-extremal. In this case each K_{k+1} -component of R contains a copy of K_{k+2} and R contains k-1 vertex-disjoint independent sets of combined size at least $(k-1)(m-\delta') - 5k\eta m \ge ((k-1)(n-\delta) - 5k\eta n)\frac{m}{n}$ whose removal disconnects R into components which are each of size at most $\frac{19}{10}(k\delta' - (k-1)m) \le \frac{19m}{10n}(k\delta - (k-1)n)$ and for each such component X all copies of K_k in R containing at least one vertex of X are K_{k+1} -connected in R. But now G and R satisfy the conditions of Lemma 3.4, so it follows that G contains $P^k_{pp_k(n,\delta)}$ and C^k_{ℓ} for each integer $k+1 \le \ell \le pc_k(n,\delta)$ such that $\chi(C^k_{\ell}) \le k+2$, i.e. the first case of Theorem 1.5(ii) holds.

Now it remains to deal with the special case $\delta = \frac{kn-1}{k+1}$. As with the main case, we apply Lemma 3.1 to G to obtain a reduced graph R, apply Lemma 3.3 to R with three possible outcomes and then apply Lemma 3.2(i), Lemma 3.2(ii) and Lemma 3.4 in the first, second and third cases respectively to obtain kth powers of cycles of the appropriate lengths. Finally, by Corollary 2.6 G contains a copy of $P_n^k = P_{pp_k(n,\delta)}^k$.

4. PROVING OUR STABILITY LEMMA

In this section we provide a proof of our stability lemma for connected K_{k+1} -factors, Lemma 3.3. We divide the proof of Lemma 3.3 into three lemmas, which correspond to three different cases as follows.

- (1) G has just one K_{k+1} -component (see Lemma 4.1),
- (2) G has a K_{k+1} -component C which does not contain a copy of K_{k+2} (see Lemma 4.2),
- (3) G has at least two K_{k+1} -components and each K_{k+1} -component contains a copy of K_{k+2} (see Lemma 4.3).

In the first case, the result follows from an application of a classical result of Hajnal and Szemerédi [11] in the form of Lemma 4.1. In the second case, the result follows from an inductive argument in the form of Lemma 4.2. Finally, we handle the third case in the form of Lemma 4.3. This turns out to be the main work and we will provide a sketch of its proof at the end of this section.

We now state Lemmas 4.1, 4.2 and 4.3, and provide a proof of Lemma 3.3 applying these lemmas. We will provide proofs of Lemmas 4.1 and 4.2 right after our proof of Lemma 3.3. Finally, we will introduce a family of configurations in Section 4.1 to prepare for the substantially more involved proof of Lemma 4.3. We will analyse this family of configurations and develop greedy-type methods for the construction of connected K_{k+1} -factors in Section 5. These will be applied in the proof of Lemma 4.3, which will be provided in Section 6.

Lemma 4.1. Let $k \in \mathbb{N}$ and $\delta \in \left[\frac{(k-1)n}{k}, \frac{kn}{k+1}\right]$. Let G be a graph on $n \ge k(k+1)$ vertices with minimum degree $\delta(G) \ge \delta$ and exactly one K_{k+1} -component. Then $\operatorname{CKF}_{k+1}(G) \ge (k+1)(k\delta - (k-1)n)$.

Lemma 4.2. Let $k \in \mathbb{N}$. Let G be a graph on n vertices with minimum degree $\delta(G) \geq \delta \geq \frac{(k-1)n}{k}$. Suppose that G has a K_{k+1} -component C which does not contain a copy of K_{k+2} . Then there is a set of $k\delta - (k-1)n$ vertex-disjoint copies of K_{k+1} which are all in C.

Lemma 4.3. Given an integer $k \geq 3$ and $\mu > 0$, for any sufficiently small $\eta > 0$ there exists an integer m_1 such that if $\delta \geq (\frac{k-1}{k} + \mu)n$ and G is a graph on $n \geq m_1$ vertices with minimum degree $\delta(G) \geq \delta$ such that G has at least two K_{k+1} -components and every K_{k+1} -component of G contains a copy of K_{k+2} , then either

- (D1) $\operatorname{CKF}_{k+1}(G) \ge \operatorname{pp}_k(n, \delta + \eta n), or$
- (D2) G has k-1 vertex-disjoint independent sets of combined size at least $(k-1)(n-\delta)-5k\eta n$ whose removal disconnects G into components which are each of size at most $\frac{19}{10}(k\delta - (k-1)n)$ and for each component X all copies of K_k in G containing at least one vertex of X are K_{k+1} -connected in G.

Proof of Lemma 3.3. Given an integer $k \geq 3$, $\mu > 0$ and any $\eta > 0$ sufficiently small for application of Lemma 4.3, Lemma 4.3 produces an integer m_1 . Set $m_0 := \max\{m_1, k(k+1)\}$. Let $\delta \in \left[(\frac{k-1}{k} + \mu)n, \frac{kn}{k+1}\right]$ and let G be a graph on $n \geq m_0$ vertices with minimum degree $\delta(G) \geq \delta$.

If G has only one K_{k+1} -component then Lemma 4.1 implies $\operatorname{CKF}_{k+1}(G) \ge (k+1)(k\delta - (k-1)n)$. If G has a K_{k+1} -component C which does not contain a copy of K_{k+2} then Lemma 4.2 implies $\operatorname{CKF}_{k+1}(G) \ge (k+1)(k\delta - (k-1)n)$. In both cases we are in (C1). If G has at least two K_{k+1} -components and every K_{k+1} -component of G contains a copy of K_{k+2} , then Lemma 4.3 implies that we are in (C2) or (C3).

Next, we provide proofs for Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. Fix $k \in \mathbb{N}$ and $\delta \in \left[\frac{(k-1)n}{k}, \frac{kn}{k+1}\right]$. Let G be a graph on $n \geq k(k+1)$ vertices with minimum degree $\delta(G) \geq \delta$ and exactly one K_{k+1} -component. Corollary 2.5 implies $\operatorname{CKF}_{k+1}(G) \geq (k+1)(k\delta - (k-1)n)$.

Proof of Lemma 4.2. We proceed by induction on k. For k = 1, let x be a vertex of C and define $U := \Gamma(x) \subseteq C$. Note that a K_2 -component is a connected component, so in particular vertices with a neighbour in C are also in C. C contains no triangle, so U is an independent set. Pick a set S of δ vertices from U. Choose greedily for each $u \in S$ a distinct vertex $v \in V(G)$ such that uv is an edge. Since $S \subseteq U$ is an independent set, all these vertices are not elements of S. Since $\deg(u) \geq \delta$, we can find a distinct vertex for each $u \in S$. This yields a set M of δ vertex-disjoint edges all in C.

Now suppose $k \ge 2$. Let x be a vertex of C. Define $H := G[\Gamma(x)]$ and $C_1 := \{x_1 \dots x_{k-1} : x_{1} \dots x_{k-1} x \in C\}$. Note that H is a graph on $m := |\Gamma(x)| \ge \delta$ vertices with minimum degree $\delta(H) \ge \delta - n + m \ge \frac{k-2}{k-1}m$ and C_1 is a nonempty union of some K_k -components of H. Since C does not contain a copy of K_{k+2} , any K_k -component $C' \subseteq C_1$ of H does not contain a copy of K_{k+2} , any K_k -component $C' \subseteq C_1$ of H does not contain a copy of K_{k+1} . Let C' be such a K_k -component. Applying the induction hypothesis with H and C', we obtain a set F'' of $(k-1)\delta(H) - (k-2)m \ge (k-1)(\delta - n + m) - (k-2)m \ge k\delta - (k-1)n$ vertex-disjoint copies of K_k which are all in $C' \subseteq C_1$; since these copies of K_k lie in $\Gamma(x)$, they are also in C. Let $F' \subseteq F''$ be a subset of F'' containing $k\delta - (k-1)n$ vertex-disjoint copies of K_k -11 these vertices are not neighbours of x and in particular are not vertices of elements of F'. Since $|\Gamma(f)| \ge k\delta - (k-1)n$ by Lemma 2.2, we can find a distinct vertex for each $f \in F'$. This yields a set F of $k\delta - (k-1)n$ vertex-disjoint copies of K_{k+1} which are all in C.

4.1. Configurations. To prepare for the proof of Lemma 4.3, we first introduce some definitions useful for the analysis of the graph structure. Let G be a graph with K_{k+1} -components C_1, \ldots, C_r . The K_{k+1} -interior $\operatorname{int}_k(G)$ of G is the set of vertices of G which are in more than one of the K_{k+1} -components. For a K_{k+1} -component C_i , the interior $\operatorname{int}(C_i)$ of C_i is the set of vertices of C_i which are in $\operatorname{int}_k(G)$. The exterior $\operatorname{ext}(C_i)$ of C_i is the set of vertices of C_i which are in no other K_{k+1} -component of G. To give an example, by definition the graph $G_p(k, n, \delta)$ has $r_p(k, n, \delta) K_{k+1}$ -components; its K_{k+1} -interior is the disjoint union of the k-1 independent sets I_1, \ldots, I_{k-1} (using notation from the definition of $G_p(k, n, \delta)$ on page 3 in Section 1) and its component exteriors are the cliques $X_1, \ldots, X_{r_p(k,n,\delta)}$. Note that $\operatorname{int}_k(G_p(k, n, \delta))$ induces a complete (k-1)-partite graph and in particular contains no copy of K_k .

A key part of our proof of Lemma 4.3 involves the analysis of the case in which $\operatorname{int}_k(G)$ contains a copy of K_k , which corresponds to the case in which $\operatorname{int}_2(G)$ contains an edge in [3]. However, unlike in the case in [3], in our case the graph structure is not immediately amenable to the construction of connected K_{k+1} -factors. To overcome this, we introduce a family of configurations in this subsection which gives graph structures that facilitate the construction of connected K_{k+1} -factors.

Definition 1 (Configurations). Let $j, k, \ell \in \mathbb{N}$ satisfy $1 \leq j < \ell \leq k$. We say that a graph G contains the configuration $\dagger_k(\ell, j)$ if there is a (multi)set

$$\{u_i : i \in [k]\} \sqcup \{v_i : j < i \le \ell\} \sqcup \{w_{i,h} : j < i \le \ell, h \in [\ell - 1]\}$$

of (not necessarily distinct) vertices in V(G) such that

(CG1) $u_1 \ldots u_k$ is a copy of K_k in a K_{k+1} -component C of G,

(CG2) $u_1 \ldots u_j v_{j+1} \ldots v_\ell u_{\ell+1} \ldots u_k$ is a copy of K_k of G not in C, and

(CG3) $u_{\ell+1} \dots u_k u_p w_{p,1} \dots w_{p,\ell-1}$ is a copy of K_k of G not in C for every j .

We say that G does not contain the configuration $\dagger_k(\ell, j)$ if there is no such (multi)set of vertices in V(G).



FIGURE 3. $\dagger_3(3,1), \dagger_3(3,2)$ and $\dagger_3(2,1)$

One may regard the configuration $\dagger_k(\ell, j)$ as a collection of copies of K_k which satisfies the following.

- (i) There are $k \ell$ vertices common to all the copies of K_k .
- (ii) There is a 'central' copy of K_k in some K_{k+1} -component C and all the other copies of K_k do not belong to C.
- (iii) After deleting the $k \ell$ common vertices from the copies of K_k , we obtain a collection of copies of K_{ℓ} . The 'central' copy of K_{ℓ} shares j vertices with one other copy of K_{ℓ} and each of its remaining vertices has one copy of K_{ℓ} 'dangling' off it.

Note that these configurations are by no means distinct, since 'non-central' copies of K_k and vertices not on the same copy of K_k need not be distinct. For example, a graph that contains $\dagger_3(3,2)$ also contains $\dagger_3(3,1)$ – set $v_2 := u_2, w_{2,1} := u_1, w_{2,2} := v_3$. The family of configurations for k = 3 can be found in Figure 3.

Now let us sketch the proof of Lemma 4.3. We will distinguish two cases as follows.

- (i) $\operatorname{int}_k(G)$ contains a copy of K_k ,
- (ii) $\operatorname{int}_k(G)$ does not contain a copy of K_k .

Case (i) is equivalent to G containing $\dagger_k(k, 1)$, so G contains a member of our family of configurations. By Lemma 5.1, G in fact contains a configuration of the form $\dagger_k(\ell + 1, \ell)$. Consider the configuration of this form contained in G with minimal ℓ . We will distinguish two cases: when $\ell = 1$ and when $\ell > 1$. In the first case, Lemma 5.2 will tell us that common neighbourhoods of a certain form are independent sets, which will enable us to apply Lemma 5.4 to obtain the desired large connected K_{k+1} -factor. In the second case, we know that G does not contain $\dagger_k(2, 1)$. Lemma 5.3 (i) will tell us that common neighbourhoods of a certain form are independent sets and we will be able to apply Lemma 5.5 to obtain the desired large connected K_{k+1} -factor. We remark that the argument presented above for the second case is inadequate when δ is close to $\frac{(2k-1)n}{2k+1}$. We will use an essentially similar but more tailored approach in the form of Lemmas 5.3 (ii) and 5.6.

In Case (ii), G resembles our extremal graphs and has enough structure for the application of our construction methods to obtain the desired large connected K_{k+1} -factor. This approach works for most values of δ below $\frac{(2k-1)n}{2k+1}$. For $\delta \geq \frac{(2k-1)n}{2k+1}$ however, we find that our greedy-type methods are insufficient. To overcome this, we will employ a Hall-type argument in the form of Lemma 6.8.

5. Structure and methods

In this section we develop useful techniques for our proof of Lemma 4.3. These include structural results pertaining to the family of configurations defined in Section 4.1 and procedures for constructing connected K_{k+1} -factors.

5.1. Configurations and structure. In this subsection we prove structural facts about our family of configurations which are useful for our proof of Lemma 4.3.

A key argument in our proof of Lemma 4.3 is that a graph without a sufficiently large connected K_{k+1} -factor in fact contains no member of the family of configurations defined previously in Section 4.1. The following lemma establishes an inductive-like relationship between the members of our family of configurations.

Lemma 5.1. Let $j, k, \ell \in \mathbb{N}$ satisfy $3 \leq j+2 \leq \ell \leq k$ and G be a graph on n vertices with minimum degree $\delta > \frac{(k-1)n}{k}$ and at least two K_{k+1} -components. Suppose that G does not contain $\dagger_k(\ell, j+1), \ \dagger_k(\ell, \ell-1)$ or $\dagger_k(\ell-j, 1)$. Then G does not contain $\dagger_k(\ell, j)$.

Proof. Suppose that G contains $\dagger_k(\ell, j)$. By Definition 1, there is a (multi)set

$$\{u_i : i \in [k]\} \sqcup \{v_i : j < i \le \ell\} \sqcup \{w_{i,h} : j < i \le \ell, h \in [\ell - 1]\}$$

of vertices in V(G) such that (CG1)–(CG3) hold. Observe that the vertices $u_1, \ldots, u_k, v_{j+1}, \ldots, v_{\ell}$ are all distinct: if $u_a = v_b$ for some $j < a, b \leq \ell$, the copy of K_k containing v_b would share at least $k - \ell + j + 1$ vertices with the 'central' copy of K_k , thereby yielding $\dagger_k(\ell, j + 1)$.

For each $j < i \leq \ell$ define $S_i := \Gamma(v_i, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k) \setminus \{u_i\}$ and $f_i := u_1 \ldots u_{i-1}u_{i+1} \ldots u_k$. Set $f' := u_1 \ldots u_j u_{\ell+1} \ldots u_k$. Let $j < i \leq \ell$. Observe that v_i has at least two nonneighbours in $\{u_{j+1}, \ldots, u_\ell\}$. Indeed, without loss of generality, suppose that v_i is adjacent to the vertices $u_{j+1}, \ldots, u_{\ell-1}$. Here $B := f'u_{j+1} \ldots u_{\ell-1}v_i$ is a copy of K_k sharing at least k - 1vertices with $u_1 \ldots u_k$ and at least $k - \ell + j + 1$ vertices with $f'v_{j+1} \ldots v_\ell$. If $B \in C$, then by taking B as the 'central' copy of K_k we have $\dagger_k(\ell, j + 1)$. If $B \notin C$, then with B replacing $f'v_{j+1} \ldots v_\ell$ we have $\dagger_k(\ell, \ell - 1)$.

Now applying Lemma 2.2 with $U = V(G) \setminus \{u_1, \ldots, u_k, v_i\}$, we have

$$|S_i| \ge (k-1)(\delta-k) + (\delta-k+2) - (k-1)(n-k-1) = k\delta - (k-1)n + 1 > 0.$$

Pick $s_i \in S_i$ and complete $f's_iv_i$ to a copy $Z_i := f's_iv_iz_{i,1}\ldots z_{i,\ell-j-2}$ of K_k by Lemma 2.3. Observe that $f_is_i \in C$: if not, then $u_iw_{i,1}\ldots w_{i,k-1} \notin C$, $f_iu_i \in C$ and $f_is_i \notin C$ would yield $\dagger_k(\ell, \ell-1)$ with f_iu_i as the 'central' copy of K_k . Furthermore, we have $Z_i \in C$: if not, then $u_pw_{p,1}\ldots w_{p,k-1}\notin C$ for $j , <math>f_is_i \in C$ and $Z_i\notin C$ would yield $\dagger_k(\ell-j,1)$ with f_is_i as the 'central' copy of K_k . But now $Z_i \in C$ for $j < i \leq \ell$ with $f'v_{j+1}\ldots v_\ell\notin C$ as the 'central' copy of K_k yields $\dagger_k(\ell-j,1)$, which is a contradiction.

The following lemma collects structural properties useful for the construction of connected K_{k+1} -factors in graphs which contain $\dagger_k(2, 1)$.

Lemma 5.2. Let $k \ge 2$. Let $f = u_1 \dots u_{k-1}$ be a copy of K_{k-1} in a graph G which lies in distinct K_{k+1} -components C_1 and C_2 of G. Let uv and wu_k be edges of G such that $fu, fv \in C_1$ and $fw, fu_k \in C_2$. Then $\Gamma(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k, w, u, v)$ is an independent set for each $i \in [k-1]$.

Proof. Fix $i \in [k-1]$. Let $U := u_1 \dots u_{i-1}u_{i+1} \dots u_{k-1}$. Suppose that $X := \Gamma(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_k, w, u, v)$ contains an edge u'v'. Note that $Uu_i u \in C_1$ and $Uu_i uv$ is a copy of K_{k+1} in G so $Uuv \in C_1$; also Uuvu'v' is a copy of K_{k+2} in G so $Uu'v' \in C_1$. On the other hand, $Uu_k w \in C_2$ and $Uu_k wu'v'$ is a copy of K_{k+2} in G so $Uu'v' \in C_2$. Since no copy of K_k is in more than one K_{k+1} -component, this is a contradiction. Hence, X contains no edge and is therefore an independent set.

The following lemma provides structural properties useful for the construction of connected K_{k+1} -factors in graphs containing $\dagger_k(\ell, \ell-1)$ for some $3 \le \ell \le k$ but not $\dagger_k(2, 1)$.

Lemma 5.3. Let $k \ge 2$ and $i \in [k-1]$ be integers. Let G be a graph which does not contain $\dagger_k(2,1)$ and $f = u_1 \ldots u_{k-1}$ be a copy of K_{k-1} in G which lies in distinct K_{k+1} -components C_1 and C_2 of G. Let uv be an edge of G such that $fu, fv \in C_1$ and w be a vertex of G such that $fw \in C_2$. Then

- (i) $\Gamma(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{k-1},w,u,v)$ is an independent set.
- (ii) $\Gamma(x_1, \ldots, x_{i-1}, u_{i+1}, \ldots, u_{k-1}, w, u, v)$ is an independent set for any copy $g := x_1 \ldots x_{i-1}$ of K_{i-1} in G such that we have $x_j \in \Gamma(u_{j+1}, \ldots, u_{k-1}, w, u, v)$ for each j < i.

Proof. Note that $u_1 \ldots u_{i-1}$ is a copy of K_{i-1} such that for each j < i we have $u_j \in \Gamma(u_{j+1}, \ldots, u_{k-1}, w, u, v)$, so (i) follows from (ii). Hence, it remains to prove (ii).

Fix a copy $g := x_1 \dots x_{i-1}$ of K_{i-1} such that for each j < i we have $x_j \in \Gamma(u_{j+1}, \dots, u_{k-1}, w, u, v)$. For each $j \in [i]$ set $U'_j := u_1 \dots u_{j-1}, g_j := x_1 \dots x_{j-1}, U_j := u_{j+1} \dots u_{k-1}$ and $f_j := U'_j U_j$. Set $U_0 := f$. We shall prove by induction that $g_j U_j uv \in C_1$ for each $j \in [i]$. For j = 1, note that $fu \in C_1$ and fuv is a copy of K_{k+1} in G so $f_1 uv = g_1 U_1 uv \in C_1$. For $j \ge 2$, note that $g_{j-1}U_{j-1}uv \in C_1$ by the induction hypothesis and $g_j U_{j-1}uv$ is a copy of K_{k+1} in G so $g_i U_j uv \in C_1$, completing the proof by induction. In particular, we have $gU_i uv \in C_1$.

Set $X := \Gamma(x_1, \ldots, x_{i-1}, u_{i+1}, \ldots, u_{k-1}, w, u, v)$ and suppose that there is an edge u'v' with $u', v' \in X$. Now by the definitions of g and X we have that $gU_iuvu'v'$ is a copy of K_{k+2} in G, so we have $gU_iu'v' \in C_1$. Furthermore, since $g_jU_juv \in C_1$ and $g_jU_{j-1}uv$ is a copy of K_{k+1} in G for each $j \in [i]$, we have $g_jU_{j-1}u \in C_1$ for each $j \in [i]$.

Now we shall prove that $g_j U_{j-1} w \in C_2$ for each $j \in [i]$. For j = 1, we have $fw = g_1 U_0 w \in C_2$. For $j \geq 2$, observe that the induction hypothesis implies that $g_j U_{j-1} w \in C_2$: if not, then $g_{j-1}U_{j-2}u \in C_1$ from before, $g_{j-1}U_{j-2}w \in C_2$ by the induction hypothesis and $g_j U_{j-1}w \notin C_2$ would yield $\dagger_k(2,1)$ with $g_{j-1}U_{j-2}w \in C_2$ as the 'central' copy of K_k and $g_{j-1}U_{j-1}$ as the common vertices. This completes the proof by induction. In particular, we have $gU_{i-1}w \in C_2$. Now observe that $gU_iwu' \in C_2$: if not, then $gU_{i-1}u \in C_1$ from before, $gU_{i-1}w \in C_2$ and $gU_iwu' \notin C_2$ would yield $\dagger_k(2,1)$ with $gU_{i-1}w \in C_2$ as the 'central' copy of K_k and gU_i as the common vertices. Finally, $gU_iwu'v'$ is a copy of K_{k+1} in G so $gU_iu'v' \in C_2$, which contradicts our earlier deduction that $gU_iu'v' \in C_1$. Hence, X contains no edge and is therefore an independent set.

5.2. Constructing connected K_{k+1} -factors. In this subsection we develop greedy-type procedures for constructing connected K_{k+1} -factors which exploit certain structures in graphs of interest, including those proved in Section 5.1. Lemmas 5.4, 5.5, and 5.6 serve to formalise the achievable outcomes of these procedures.

Lemma 5.4 represents a greedy-type procedure for constructing connected K_{k+1} -factors in a graph using two parallel processes following two closely related partitions of the vertex set. The purpose of this lemma is to obtain sufficiently large connected K_{k+1} -factors in graphs containing $\dagger_k(2, 1)$. The sets A and A' in Lemma 5.4 contain the vertices avoided by the two parallel processes. Note that the larger A and A' are, the smaller s_1 and t_1 are. Since the sizes of s_1 and t_1 will often be the key determinants of the attainable size of a connected K_{k+1} -factor, we will think of A and A' as 'bad' sets. We remark that while we formally allow the quantities s_1, s_2, t_1 and t_2 to be negative to reduce the overall proof complexity, in practice they will always be non-negative.

Lemma 5.4. Let $2 \leq b \leq c \leq k$ be integers. Let G be a graph on n vertices with minimum degree $\delta = \delta(G) > \frac{(k-1)n}{k}$. Suppose there are two partitions of V(G), one with vertex classes $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and another with vertex classes $V_1, V_2, X_1, \ldots, X_{k-2}, X', A'$, such that

- (a) $U_1 \cap V_1 = \emptyset$,
- (b) there are no edges between U_1 and U_2 and between V_1 and V_2 ,
- (c) all copies of K_k in G with at least two vertices from U_1 and all other vertices from $\bigcup_{i=1}^{k-2} X_i$, or at least two vertices from V_1 and all other vertices from $\bigcup_{i=1}^{k-2} X_i$, are K_{k+1} connected,
- (d) $|X_i| \leq n \delta$ for $i \in [k 1]$ and $|X'| \leq n \delta$, and
- (e) $X_i \cap \Gamma(g)$ is an independent set for each (i, g) where $i \in [k-2]$ and g is a clique of order at least i with at least two vertices from U_1 and all other vertices from $\bigcup_{j=1}^{i-1} X_j$, or at least two vertices from V_1 and all other vertices from $\bigcup_{j=1}^{i-1} X_j$.

Let F^{U} be a collection of vertex-disjoint copies of K_{b} in U_{1} and F^{V} be a collection of vertexdisjoint copies of K_{c} in V_{1} . Set $s_{1} := \frac{k\delta - (k-1)n + (b-1)|U_{2}| - |A|}{2b-1}$ and $s_{2} := \frac{k\delta - (k-1)n + (b-1)|U_{2}| - |V_{1}|}{2b-1}$. Set $t_{1} := \frac{k\delta - (k-1)n + (c-1)|V_{2}| - |A'| - |U_{1}|(c-1)/b}{2c-1}$ and $t_{2} := \frac{k\delta - (k-1)n + (c-1)|V_{2}| - |U_{1}| - |U_{1}|(c-1)/b}{2c-1}$. Let $d_1, d_2 \geq 0$ satisfy $|V_2| \geq 2d_2 + d_1$. Then G contains a connected K_{k+1} -factor of size at least

$$(k+1)\min\left\{|F^{U}|, \left\lfloor\frac{|U_{2}|}{2}\right\rfloor, d_{1}, s_{1}, s_{2}\right\} + (k+1)\min\left\{|F^{V}|, d_{2}, t_{1}, t_{2}\right\}.$$

Moreover, if F^V is empty then G contains a connected K_{k+1} -factor of size at least

$$(k+1)\min\left\{|F^U|, \left\lfloor \frac{|U_2|}{2} \right\rfloor, d_1, s_1, s_2\right\}.$$

The proof of this lemma proceeds as follows. We first describe the greedy-type procedure used to construct a connected K_{k+1} -factor. Then, we prove that our procedure does indeed produce a connected K_{k+1} -factor of the desired size. This will turn out to be an inductive argument where we will need to justify that we can make 'good' choices at each step and the quantities s_1, s_2, t_1, t_2 are chosen to ensure success. For example, a copy of K_k extending a copy of K_b in F^U has at least $k\delta - (k-1)n + (b-1)|U_2| - |A|$ common neighbours not in some 'bad' set A; on the other hand, each copy of K_b in F^U may render up to 2b - 1 of these common neighbours unavailable, so $|F^U| \leq s_1$ ensures that there is still an available common neighbour.

Proof. Let $F_b^U \subseteq F^U$ and $F_c^V \subseteq F^V$ satisfy

$$|F_b^U| = \max\left\{0, \min\left\{|F^U|, \left\lfloor \frac{|U_2|}{2} \right\rfloor, d_1, s_1, s_2\right\}\right\} \text{ and } |F_c^V| = \max\left\{0, \min\left\{|F^V|, d_2, t_1, t_2\right\}\right\}.$$

In what follows, we use vertices in $U_1, X_1, \ldots, X_{k-2}$ to extend each clique in F_b^U to a copy of K_k and vertices in $U_1, X_1, \ldots, X_{k-2}$ to extend each clique in F_c^V to a copy of K_k . These copies of K_k will then be extended to copies of K_{k+1} using vertices outside of $U_1, V_1, X_1, \ldots, X_{k-2}$. Note that the resultant copies of K_{k+1} will be K_{k+1} -connected by (c).

We build up our desired connected K_{k+1} -factor by running two parallel processes, one starting from F_b^U in U_1 and the other starting from F_c^V in V_1 . Each process is a two-stage step-by-step process performing steps in tandem with the other process. Set $\overline{F}_{b-1}^U, \overline{F}_{c-1}^V := \emptyset$. Stage one has steps $j = 1, \ldots, k - b + 1$. In step $j \in [c - b]$ of stage one, we extend copies of K_{b+j-1} in F_{b+j-1}^U to vertex-disjoint copies of K_{b+j} where possible. For each copy of K_{b+j-1} in F_{b+j-1}^U in turn we pick greedily, where possible, a common neighbour in U_1 which is not covered by $\overline{F}_{b-1}^U, \ldots, \overline{F}_{b+j-2}^U, F_{b+j-1}^U$ or previously chosen common neighbours. Since the vertices selected lie in U_1 , F_c^V is contained in V_1 and $U_1 \cap V_1 = \emptyset$ by (a), no vertex of F_c^V is selected. Let \overline{F}_{b+j-1}^U be the collection of copies of K_{b+j-1} in F_{b+j-1}^U which could not be extended and let F_{b+i}^U be the collection of vertex-disjoint copies of K_{b+j} which result from extending copies of K_{b+j-1} in F_{b+j-1}^U . In step $j \in [k-b+1] \setminus [c-b]$ of stage one, we extend copies of K_{b+j-1} in F_{b+j-1}^U to vertex-disjoint copies of K_{b+j} and copies of K_{c+j-1} in F_{c+j-1}^V to vertex-disjoint copies of K_{c+j} where possible. For each copy of K_{b+j-1} in F_{b+j-1}^U in turn we pick greedily, where possible, a common neighbour in U_1 which is not covered by $\overline{F}_{b-1}^U, \ldots, \overline{F}_{b+j-2}^U, F_{b+j-1}^U$ or previously chosen common neighbours. Let \overline{F}_{b+j-1}^U be the collection of copies of K_{b+j-1} in F_{b+j-1}^U which could not be extended and let F_{b+j}^U be the collection of vertex-disjoint copies of K_{b+j} which result from extending copies of K_{b+j-1} in F_{b+j-1}^U . We do the same with F_{c+j-1}^V within V_1 . We end stage one after step k - b + 1. Set $\overline{F}_{k+1}^U := F_{k+1}^U$ and $\overline{F}_{k+1}^V := F_{k+1}^V$. At this point, we have collections \overline{F}_i^U and \overline{F}_i^V of vertex-disjoint copies of K_i and K_j respectively, for each $i = b, \ldots, k + 1$ and $j = c, \ldots, k + 1$, some of which may be empty. Let $\overline{F}^U = \bigcup_{i=b}^{k+1} \overline{F}_i^U$ and $\overline{F}^V = \bigcup_{i=c}^{k+1} \overline{F}_i^V$. Note that $|\overline{F}^U| = |F_b^U|$ and $|\overline{F}^V| = |F_c^V|$. Order the elements of $\overline{F}^U \cup \overline{F}^V$ with those in \overline{F}^U coming before those in \overline{F}^V , those in each of \overline{F}^U and \overline{F}^V in increasing size order, and those in each of \overline{F}^U and \overline{F}^V of the same size in some arbitrary order. We will use this ordering when attempting to extend cliques in stage two.

We begin stage two with $\widetilde{F}_0^U := \overline{F}^U$ and $\widetilde{F}_0^V := \overline{F}^V$. Stage two has steps $j = 1, \ldots, k-1$. In step $j \in [k-2]$ we attempt to extend each clique in \widetilde{F}_{j-1}^U and \widetilde{F}_{j-1}^V of order at most k by one vertex using X_j . We will extend cliques in the order mentioned previously. For each clique of order at most k in \widetilde{F}_{j-1}^U and \widetilde{F}_{j-1}^V in turn we pick greedily, where possible, a common neighbour in X_j which is outside the previously chosen common neighbours. Let \widetilde{F}_j^U and \widetilde{F}_j^V be the collections of both cliques in \widetilde{F}_{j-1}^U and \widetilde{F}_{j-1}^V respectively of order k+1 and cliques resulting from the attempts to extend each clique of order at most k in \widetilde{F}_{j-1}^U and \widetilde{F}_{k-2}^V by one vertex, no matter whether they were successful or not. In step k-1 we attempt to extend each clique of order at most k in \widetilde{F}_{k-2}^U by one vertex using vertices of G outside of $U_1 \cup V_1 \cup \left(\bigcup_{i=1}^{k-2} X_i\right)$ in a manner similar to that in earlier steps of stage two. We end stage two after step k-1 with collections \widetilde{F}_{k-1}^U and \widetilde{F}_{k-1}^V of $|F_b^U|$ and $|F_c^V|$ vertex-disjoint cliques in G respectively.

We shall prove that \tilde{F}_{k-1}^U and \tilde{F}_{k-1}^V are collections of $|F_b^U|$ and $|F_c^V|$ vertex-disjoint copies of K_{k+1} respectively. In fact, we shall prove that \tilde{F}_j^U and \tilde{F}_j^V are collections of $|F_b^U|$ and $|F_c^V|$ vertex-disjoint cliques of order at least j+2 respectively for each $j = b-2, \ldots, k-1$. We shall first consider \tilde{F}_j^U . We proceed by induction on j. The j = b-2 case is trivial. Consider \tilde{F}_j^U for $j \ge b-1$. By the induction hypothesis, \tilde{F}_{j-1}^U is a collection of $|F_b^U|$ vertex-disjoint cliques of order at least j + 1. Hence, it suffices to show that the copies of K_{j+1} in \tilde{F}_{j-1}^U are all extended to copies of K_{j+2} in step j to prove our claim. Observe that this holds trivially when $|F_b^U| = 0$, so in what follows it is enough to consider when $|F_b^U| = \min\left\{|F^U|, \left|\frac{|U_2|}{2}\right|, d_1, s_1, s_2\right\}$.

Let f be a copy of K_{j+1} in \widetilde{F}_{j-1}^U with $\ell \ge b$ vertices in U_1 and \overline{f} be its corresponding clique in \overline{F}^U . In particular, \overline{f} has ℓ vertices. Note that f has vertices from only $X_1, \ldots, X_{j-1}, U_1$ and at most one vertex from each X_i . Define $I := \{i : |f \cap X_i| = 1\}$. Let \overline{v}_i be the vertex of f in X_i for each $i \in I$.

First consider the case $j \leq k-2$. Every vertex v of U_1 has at least $\delta - |A| - \deg(v; U_1) - \sum_{h \neq j} \deg(v; X_h)$ neighbours in X_j and for each $i \in I$ the vertex \overline{v}_i has at least $\delta - |A| - |U_2| - \deg(\overline{v}_i; U_1) - \sum_{h \neq j} \deg(\overline{v}_i; X_h)$ neighbours in X_j . By application of Lemma 2.2 and noting that $|X_j| = n - |A| - |U_2| - |U_1| - \sum_{h \neq j} |X_h|$, the number of common neighbours of f in X_j is at least

$$a_{j} := \sum_{v \in \overline{f}} \left(\delta - |A| - |U_{2}| - \deg(v; U_{1}) - \sum_{h \neq j} \deg(v; X_{h}) \right) \\ + \sum_{i \in I} \left(\delta - |A| - |U_{2}| - \deg(\overline{v}_{i}; U_{1}) - \sum_{h \neq j} \deg(\overline{v}_{i}; X_{h}) \right) \\ - j \left(n - |A| - |U_{2}| - |U_{1}| - \sum_{h \neq j} |X_{h}| \right) \\ = (j+1)\delta - jn + (\ell - 1)|U_{2}| - \left(\sum_{v \in f} \deg(v; U_{1}) - j|U_{1}| \right) \\ - \sum_{h \neq j} \left(\sum_{v \in f} \deg(v; X_{h}) - j|X_{h}| \right) - |A|.$$

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Now we seek appropriate estimations of the terms in our expression. Since \overline{f} could not be extended in step $\ell-b+1$ of stage one, $\Gamma(\overline{f}; U_1)$ contains only vertices from elements of $\overline{F}_b^U, \ldots, \overline{F}_\ell^U$ and $F_{\ell+1}^U$. For each $b \leq h \leq \ell$ the elements of \overline{F}_h^U contain h vertices each while the elements of $F_{\ell+1}^U$ contain $\ell + 1$ vertices each. Furthermore, we have $|F_b^U| = |\overline{F}_b^U| + \cdots + |\overline{F}_\ell^U| + |F_{\ell+1}^U|$ by the definitions of $\overline{F}_b^U, \ldots, \overline{F}_\ell^U$. Hence, by applying Lemma 2.2 to U_1 and f, we obtain

(17)
$$\sum_{v \in f} \deg(v; U_1) - j|U_1| \le \deg(f; U_1) \le \deg(\overline{f}; U_1) \le \ell |F_b^U| + |F_{\ell+1}^U|.$$

For $h \in [k-1]$, by applying Lemma 2.2 to X_h and f, we get

(18)
$$\sum_{v \in f} \deg(v; X_h) - j|X_h| \le \deg(f; X_h).$$

For $0 \leq i < j$ let $f_i \in \widetilde{F}_i^U$ be the clique corresponding to f right before step i + 1 of stage two, so $f_i = \overline{f} \cup \{\overline{v}_h : h \in I \cap [i]\}$. Let $h \in I$. By the induction hypothesis f_{h-1} is a clique of order at least h + 1 with at least two vertices from U_1 and all other vertices from $\bigcup_{j=1}^{h-1} X_j$, so \overline{v}_h has no neighbour in $\Gamma(f_{h-1}; X_h)$ by (e) applied with $(i, g) = (h, f_{h-1})$. Hence, we have deg $(f_h; X_h) = 0$ for all $h \in I$. Together with (18) and the fact that deg $(f; X_h) \leq \text{deg}(f_h; X_h)$ for all $h \in I$, we obtain

(19)
$$\sum_{h \in I} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le \sum_{h \in I} \deg(f_h; X_h) = 0.$$

Given $i \notin I, i < j$, the clique f_i was not extended in step i of stage two. It follows that its common neighbourhood in X_i contains only vertices used to extend cliques that came before it in the size ordering, of which there were fewer than $m := |F_b^U| - |F_{\ell+1}^U|$. Noting further that $[j-1]\setminus I$ contains $\ell-2$ elements, by (18) and that $\deg(f; X_h) \leq \deg(f_h; X_h)$ for all $h \in [j-1]\setminus I$, we get

(20)
$$\sum_{h\in[j-1]\setminus I} \left(\sum_{v\in f} \deg(v; X_h) - j |X_h| \right) \leq \sum_{h\in[j-1]\setminus I} \deg(f_h; X_h) \leq (\ell-2)(m-1).$$

By (d) $|X_h| \leq n - \delta$ for $h \in [k - 1]$, so by (18) we have

(21)
$$\sum_{h=j+1}^{k-1} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le \sum_{h=j+1}^{k-1} |X_h| \le (k-j-1)(n-\delta).$$

By (16), (17), (19), (20), (21) and that $m = |F_b^U| - |F_{\ell+1}^U|$, we obtain

$$\begin{aligned} a_j &\geq (j+1)\delta - jn - |A| + (\ell-1)|U_2| - \ell |F_b^U| - |F_{\ell+1}^U| - (\ell-2)m \\ &- (k-j-1)(n-\delta) \\ &\geq k\delta - (k-1)n - |A| + (\ell-1)|U_2| - (\ell+1)|F_b^U| - (\ell-3)m. \end{aligned}$$

Since $\ell \ge b$ and by the definition of F_b^U and m we have $|U_2| \ge 2|F_b^U| \ge |F_b^U| + m$ and $|F_b^U| \ge m$, we obtain

$$\begin{aligned} a_j &\geq k\delta - (k-1)n - |A| + (\ell-1)|U_2| - (\ell+1)|F_b^U| - (\ell-3)m \\ &= k\delta - (k-1)n - |A| + (\ell-2)(|U_2| - |F_b^U| - m) + |U_2| - 3|F_b^U| + m \\ &\geq k\delta - (k-1)n - |A| + (b-2)(|U_2| - |F_b^U| - m) + |U_2| - 3|F_b^U| + m \\ &\geq k\delta - (k-1)n - |A| + (b-1)|U_2| - (2b-1)|F_b^U| + m. \end{aligned}$$

Now by the definition of s_1 we have

$$a_j \ge k\delta - (k-1)n - |A| + (b-1)|U_2| - (2b-1)|F_b^U| + m$$

 $\ge (2b-1)(s_1 - |F_b^U|) + m \ge m$

so we are indeed able to pick a vertex in X_j to extend f.

For the case j = k - 1, an analysis analogous to (16) implies that the number of common neighbours of f outside of $U_1 \cup U_2 \cup V_1 \cup \left(\bigcup_{i=1}^{k-2} X_i\right)$ is at least

$$\begin{split} a_{k-1} &:= k\delta - (k-1)n + (\ell-1)|U_2| - \left(\sum_{v \in f} \deg(v; U_1) - (k-1)|U_1|\right) \\ &- \sum_{h=1}^{k-2} \left(\sum_{v \in f} \deg(v; X_h) - (k-1)|X_h|\right) - |V_1|. \end{split}$$

By (17), (19), (20) and that $m = |F_b^U| - |F_{\ell+1}^U|$, we obtain

$$a_{k-1} \ge k\delta - (k-1)n - |V_1| + (\ell - 1)|U_2| - (\ell + 1)|F_b^U| - (\ell - 3)m.$$

Since $\ell \ge b$ and by the definition of F_b^U and m we have $|U_2| \ge 2|F_b^U| \ge |F_b^U| + m$ and $|F_b^U| \ge m$, we obtain

$$\begin{aligned} a_{k-1} &\geq k\delta - (k-1)n - |V_1| + (\ell-1)|U_2| - (\ell+1)|F_b^U| - (\ell-3)m \\ &= k\delta - (k-1)n - |V_1| + (\ell-2)(|U_2| - |F_b^U| - m) + |U_2| - 3|F_b^U| + m \\ &\geq k\delta - (k-1)n - |V_1| + (b-2)(|U_2| - |F_b^U| - m) + |U_2| - 3|F_b^U| + m \\ &\geq k\delta - (k-1)n - |V_1| + (b-1)|U_2| - (2b-1)|F_b^U| + m. \end{aligned}$$

Now by the definition of s_2 we have

$$a_{k-1} \ge k\delta - (k-1)n - |V_1| + (b-1)|U_2| - (2b-1)|F_b^U| + m$$

$$\ge (2b-1)(s_2 - |F_b^U|) + m \ge m$$

so we are indeed able to pick a vertex outside of $U_1 \cup U_2 \cup V_1 \cup \left(\bigcup_{i=1}^{k-2} X_i\right)$ to extend f. This proves that copies of K_{j+1} in \widetilde{F}_{j-1}^U are all extended to copies of K_{j+2} in step j and so by induction \widetilde{F}_j^U is a collection of $|F_b^U|$ vertex-disjoint cliques of order at least j+2 for each $j=b-2,\ldots,k-1$. In particular, \widetilde{F}_{k-1}^U is a collection of $|F_b^U|$ vertex-disjoint copies of K_{k+1} .

The proof for the \widetilde{F}_j^V case is very similar to that for the \widetilde{F}_j^U case. We also proceed by induction on j and here the j = c - 2 case is trivial. As in the \widetilde{F}_j^U case, the desired outcome holds trivially when $|F_c^V| = 0$, so in what follows it is enough to consider when $|F_c^V| = \min\{|F^V|, d_2, t_1, t_2\}$. Let f be a copy of K_{j+1} in \widetilde{F}_{j-1}^V with $\ell \ge c$ vertices in V_1 and \overline{f} be its corresponding clique in \overline{F}^V . In particular, \overline{f} has ℓ vertices. Define $I := \{i : |f \cap X_i| = 1\}$ and let \overline{v}_i be the vertex of f in X_i for each $i \in I$. Note that f has vertices from only $X_1, \ldots, X_{j-1}, U_1$ and at most one vertex from each X_i .

Consider $c-1 \leq j \leq k-1$. Let $m' := |F_c^V| - |F_{\ell+1}^V|$. An analysis analogous to (16) implies that the number of common neighbours of f in X_j is at least

(22)
$$b_{j} := (j+1)\delta - jn + (\ell-1)|V_{2}| - \left(\sum_{v \in f} \deg(v; V_{1}) - j|V_{1}|\right) - \sum_{h \neq j} \left(\sum_{v \in f} \deg(v; X_{h}) - j|X_{h}|\right) - |A'|.$$

By analyses similar to those for (17), (19), (20) and (21), we also have

(23)
$$\sum_{v \in f} \deg(v; V_1) - j |V_1| \le \ell |F_c^V| + |F_{\ell+1}^V|,$$

(24)
$$\sum_{h \in I} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le 0,$$

(25)
$$\sum_{h \in [j-1] \setminus I} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le (\ell - 2)(m' + |F_b^U| - 1),$$

(26)
$$\sum_{h=j+1}^{k-1} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le (k-j-1)(n-\delta),$$

respectively. The 'additional' term of $|F_b^U|$ in (25) (cf. (20)) arises because in each step of stage two we extend the cliques corresponding to F_b^U before those corresponding to F_c^V . Now by (22), (23), (24), (25) and (26) and that $m' = |F_c^V| - |F_{\ell+1}^V|$, we obtain

$$b_{j} \geq (j+1)\delta - jn - |A'| + (\ell - 1)|V_{2}| - \ell |F_{c}^{V}| - |F_{\ell+1}^{V}| - (\ell - 2)(m' + |F_{b}^{U}|) - (k - j - 1)(n - \delta) \geq k\delta - (k - 1)n - |A'| + (\ell - 1)|V_{2}| - (\ell + 1)|F_{c}^{V}| - (\ell - 2)|F_{b}^{U}| - (\ell - 3)m'.$$

Since $\ell \geq c$ and by the definition of F_b^U , F_c^V , d_1 , d_2 and m we have $|F_c^V| \geq m'$ and $|V_2| \geq 2d_2 + d_1 \geq 2|F_c^V| + |F_b^U| \geq |F_c^V| + |F_b^U| + m'$, we obtain

$$\begin{split} b_j &\geq k\delta - (k-1)n - |A'| + (\ell-1)|V_2| - (\ell+1)|F_c^V| - (\ell-2)|F_b^U| \\ &- (\ell-3)m' \\ &= k\delta - (k-1)n - |A'| + (\ell-2)(|V_2| - |F_c^V| - |F_b^U| - m') \\ &+ |V_2| - 3|F_c^V| + m' \\ &\geq k\delta - (k-1)n - |A'| + (c-2)(|V_2| - |F_c^V| - |F_b^U| - m') \\ &+ |V_2| - 3|F_c^V| + m' \\ &\geq k\delta - (k-1)n - |A'| + (c-1)|V_2| - (2c-1)|F_c^V| - (c-2)|F_b^U| + m' \end{split}$$

Now by the definition of t_1 we have

$$b_j \ge k\delta - (k-1)n - |A'| + (c-1)|V_2| - (2c-1)|F_c^V| - (c-2)|F_b^U| + m'$$

$$\ge (2c-1)(t_1 - |F_c^V|) + m' + |F_b^U| \ge m' + |F_b^U|$$

so we are indeed able to pick a vertex in X_j to extend f.

For the case j = k - 1, an analogous analysis implies that the number of common neighbours of f outside of $U_1 \cup V_1 \cup V_2 \cup \left(\bigcup_{i=1}^{k-2} X_i\right)$ is at least

$$b_{k-1} := k\delta - (k-1)n + (\ell-1)|V_2| - \left(\sum_{v \in f} \deg(v; V_1) - (k-1)|V_1|\right)$$
$$-\sum_{h=1}^{k-2} \left(\sum_{v \in f} \deg(v; X_h) - (k-1)|X_h|\right) - |U_1|.$$

By (23), (24) and (25) and that $m' = |F_c^V| - |F_{\ell+1}^V|$, we obtain

$$b_{k-1} \ge k\delta - (k-1)n - |U_1| + (\ell-1)|V_2| - \ell |F_c^V| - |F_{\ell+1}^V| - (\ell-2)(m' + |F_b^U|) \ge k\delta - (k-1)n - |U_1| + (\ell-1)|V_2| - (\ell+1)|F_c^V| - (\ell-2)|F_b^U| - (\ell-3)m'.$$

Since $\ell \ge c$ and by the definition of F_b^U , F_c^V , d_1 , d_2 and m we have $|F_c^V| \ge m'$ and $|V_2| \ge 2d_2 + d_1 \ge 2|F_c^V| + |F_b^U| \ge |F_c^V| + |F_b^U| + m'$, we obtain

$$\begin{split} b_{k-1} &\geq k\delta - (k-1)n - |U_1| + (\ell-1)|V_2| - (\ell+1)|F_c^V| - (\ell-2)|F_b^U| \\ &- (\ell-3)m' \\ &= k\delta - (k-1)n - |U_1| + (\ell-2)(|V_2| - |F_c^V| - |F_b^U| - m') \\ &+ |V_2| - 3|F_c^V| + m' \\ &\geq k\delta - (k-1)n - |U_1| + (c-2)(|V_2| - |F_c^V| - |F_b^U| - m') \\ &+ |V_2| - 3|F_c^V| + m' \\ &\geq k\delta - (k-1)n - |U_1| + (c-1)|V_2| - (2c-1)|F_c^V| - (c-2)|F_b^U| \\ &+ m'. \end{split}$$

Now by the definition of t_2 we have

$$b_{k-1} \ge k\delta - (k-1)n - |U_1| + (c-1)|V_2| - (2c-1)|F_c^V| - (c-2)|F_b^U| + m'$$

$$\ge (2c-1)(t_2 - |F_c^V|) + m' + |F_b^U| \ge m' + |F_b^U|$$

so we are indeed able to pick a vertex outside of $U_1 \cup V_1 \cup V_2 \cup \left(\bigcup_{i=1}^{k-2} X_i\right)$ to extend f. This proves that copies of K_{j+1} in \widetilde{F}_{j-1}^V are all extended to copies of K_{j+2} in step j and so by induction \widetilde{F}_j^V is a collection of $|F_c^V|$ vertex-disjoint cliques of order at least j+2 for each $j = c-2, \ldots, k-1$. In particular, \widetilde{F}_{k-1}^{V} is a collection of $|F_c^{V}|$ vertex-disjoint copies of K_{k+1} .

It remains to show that $\widetilde{F}_{k-1}^U \cup \widetilde{F}_{k-1}^V$ is a connected K_{k+1} -factor. Now $\widetilde{F}_{k-1}^U \cup \widetilde{F}_{k-1}^V$ consists of copies of K_k in G with either at least two vertices from U_1 and all other vertices from $\bigcup_{i=1}^{k-1} X_i$, or at least two vertices from V_1 and all other vertices from $\bigcup_{i=1}^{k-2} X_i$, so by (c) the copies of K_k in $\widetilde{F}_{k-1}^U \cup \widetilde{F}_{k-1}^V$ are pairwise K_{k+1} -connected. Hence, $\widetilde{F}_{k-1}^U \cup \widetilde{F}_{k-1}^V$ is a connected K_{k+1} -factor of size at least $(k+1)(|F_b^U|+|F_c^V|)$. If F^V is empty then we have $|F_c^V|=0$, so \widetilde{F}_{k-1}^U is a connected K_{k+1} -factor of size at least $(k+1)|F_{h}^{U}|$. \square

Lemma 5.5 is both the single partition analogue of and a straightforward consequence of Lemma 5.4. We will use it to find large connected K_{k+1} -factors when $int_k(G)$ contains a copy of K_k , specifically in Lemmas 6.4 and 6.5.

Lemma 5.5. Let $2 \le b \le k$ be integers. Let G be a graph on n vertices with minimum degree $\delta =$ $\delta(G) > \frac{(k-1)n}{k}$. Suppose there is a partition of V(G) into vertex classes $U_1, U_2, X_1, \ldots, X_{k-1}, A_k$ such that

- (a) there are no edges between U_1 and U_2 ,
- (b) all copies of K_k in G with at least two vertices from U_1 and all other vertices from $\bigcup_{i=1}^{k-2} X_i \text{ are } K_{k+1}\text{-connected},$ (c) $|X_i| \le n - \delta \text{ for } i \in [k-1], \text{ and}$
- (d) $X_i \cap \Gamma(g)$ is an independent set for each (i, g) where $i \in [k-2]$ and g is a clique of order at least i with at least two vertices from U_1 and all other vertices from $\bigcup_{j=1}^{i-1} X_j$.

Set $s_1 := \frac{k\delta - (k-1)n + (b-1)|U_2| - |A|}{2b-1}$. Let F be a collection of vertex-disjoint copies of K_b in U_1 . Then G contains a connected K_{k+1} -factor of size at least

$$(k+1)\min\left\{|F|, \left\lfloor\frac{|U_2|}{2}\right\rfloor, s_1\right\}.$$

Proof. Fix integers $2 \le b \le k$ and set c := b. Fix a graph G and a partition of V(G) with vertex classes $U_1, U_2, X_1, \ldots, X_{k-1}, A$ satisfying the lemma hypothesis. Define $V_1 = X' = A' = F^V :=$ $\emptyset, F^U := F$ and $V_2 := V(G) \setminus \left(\bigcup_{i=1}^{k-2} X_i \right)$. Set $d_1 := |V_2|, d_2 := 0$. Then the result follows by application of Lemma 5.4, noting that $|V_2| \ge |U_2|$ and $|V_1| = 0$.

We will find that Lemma 5.5 is sometimes inadequate, especially when $int_k(G)$ does not contains a copy of K_k . This is partly due to the strength of conditions (b) and (d) forcing a large 'bad' set A. The conditions are necessary when b > 2, but we can weaken these conditions and sometimes do better when b = 2. We present this as Lemma 5.6. In this case, we require a smaller set of copies of K_k in G to be K_{k+1} -connected and $X_i \cap \Gamma(g)$ to be an independent set for a smaller set of copies g of K_{i+1} with $g \subseteq U_1 \cup \left(\bigcup_{j=1}^{i-1} X_j\right)$.

 $\delta(G) > \frac{(k-1)n}{k}$. Suppose there is a partition of V(G) into vertex classes $U_1, U_2, X_1, \ldots, X_{k-1}, A$ such that

- (a) there are no edges between U_1 and U_2 ,
- (b) all copies of K_k in G comprising an edge of $G[U_1]$ and a vertex from each of X_1, \ldots, X_{k-2} are K_{k+1} -connected,
- (c) $|X_i| \leq n \delta$ for $i \in [k-1]$, and
- (d) $X_i \cap \Gamma(g)$ is an independent set for each (i,g) where $i \in [k-2]$ and g is a copy of K_{i+1} comprising an edge of $G[U_1]$ and a vertex from each of X_1, \ldots, X_{i-1} .

Let F be a matching in U_1 . Set $q := k\delta - (k-1)n + |U_2| - |U_1| - |A|$. Then G contains a connected K_{k+1} -factor of size at least $(k+1) \min \{|F|, q\}$.

The proof approach is similar to that of Lemma 5.4; however, in this case we skip stage one and it turns out that we never fail to extend in stage two. Note that a copy of K_k extending an edge from F has at least q common neighbours outside of both U_1 (which contains F) and a 'bad' set A.

Proof. Let $\overline{F} \subseteq F$ satisfy $|\overline{F}| = \max\{0, \min\{|F|, q\}\}$. We will eventually extend each edge of \overline{F} to a copy of K_{k+1} using vertices in X_1, \ldots, X_{k-1} . Note that the resultant copies of K_{k+1} will be K_{k+1} -connected by (b).

We build up our desired connected K_{k+1} -factor step-by-step, starting with the aforementioned matching $\widetilde{F}_0 := \overline{F}$ in U_1 . We have steps $j = 1, \ldots, k-1$. In step j we extend each copy of K_{j+1} in \widetilde{F}_{j-1} to a copy of K_{j+2} using X_j . For each copy of K_{j+1} in \widetilde{F}_{j-1} in turn we pick greedily a common neighbour in X_j which is outside the previously chosen common neighbours to obtain a collection F_j of |F| vertex-disjoint copies of K_{j+2} . We claim that this is always possible for all $j \in [k-1]$. Observe that this holds trivially when $|\overline{F}| = 0$, so in what follows it is enough to consider when $|\overline{F}| = \min\{|F|, q\}.$

Let f be a copy of K_{j+1} in \widetilde{F}_{j-1} . Note that f has exactly one vertex in each X_i for i < j, exactly two vertices in U_1 and none elsewhere. Let v_1 and v_2 be the vertices of f in U_1 , and let \overline{v}_i be the vertex of f in X_i for each i < j. Every vertex v of U_1 has at least δ – $|A| - |U_1| - \sum_{h \neq j} \deg(v; X_h)$ neighbours in X_j , and for each i < j the vertex \overline{v}_i has at least $\delta - |A| - |U_2| - |U_1| - \sum_{h \neq j} \deg(\overline{v}_i; X_h) \text{ neighbours in } X_j. \text{ By application of Lemma 2.2 and}$ noting that $|X_j| = n - |U_2| - |U_1| - |A| - \sum_{h \neq j} |X_h|$, the number of common neighbours of f24 in X_j is at least

$$a_{j} := \sum_{i=1}^{2} \left(\delta - |U_{1}| - |A| - \sum_{h \neq j} \deg(v_{i}; X_{h}) \right) \\ + \sum_{i=1}^{j-1} \left(\delta - |U_{2}| - |U_{1}| - |A| - \sum_{h \neq j} \deg(\overline{v}_{i}; X_{h}) \right) \\ - j \left(n - |U_{2}| - |U_{1}| - |A| - \sum_{h \neq j} |X_{h}| \right).$$

Grouping terms together, we obtain

(27)
$$a_{j} = (j+1)\delta - jn + |U_{2}| - |U_{1}| - \sum_{h=1}^{j-1} \left(\sum_{v \in f} \deg(v; X_{h}) - j |X_{h}| \right) - \sum_{h=j+1}^{k-1} \left(\sum_{v \in f} \deg(v; X_{h}) - j |X_{h}| \right) - |A|.$$

For $h \in [k-1]$, by applying Lemma 2.2 to X_h and f, we get

(28)
$$\sum_{v \in f} \deg(v; X_h) - j|X_h| \le \deg(f; X_h).$$

For $0 \leq i < j$ let $f_i \in \widetilde{F}_i$ be the clique corresponding to f right before step i + 1, so $f_i = \{v_1, v_2\} \cup \{\overline{v}_h : h \in [i]\}$. Let $1 \leq h < j$. Now f_{h-1} is a clique of order h + 1 comprising two vertices from U_1 and a vertex from each of X_1, \ldots, X_{h-1} , so \overline{v}_h has no neighbour in $\Gamma(f_{h-1}; X_h)$ by (d) applied with $(i, g) = (h, f_{h-1})$. Hence, we have $\deg(f_h; X_h) = 0$ for all $h \in I$. Together with (28) and the fact that $\deg(f; X_h) \leq \deg(f_h; X_h)$ for all $h \in [j-1]$, we obtain

(29)
$$\sum_{h=1}^{j-1} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le \sum_{h=1}^{j-1} \deg(f_h; X_h) = 0.$$

By (c) $|X_h| \le n - \delta$ for $h \in [k - 1]$, so by (28) we have

(30)
$$\sum_{h=j+1}^{k-1} \left(\sum_{v \in f} \deg(v; X_h) - j |X_h| \right) \le \sum_{h=j+1}^{k-1} |X_h| \le (k-j-1)(n-\delta).$$

Putting together (27), (29) and (30), we get $a_j \ge q \ge |\overline{F}|$, so we are indeed able to pick a vertex in X_j to extend f. This proves that copies of K_{j+1} in \widetilde{F}_{j-1} are all extended to copies of K_{j+2} in step j. Therefore, we terminate after step k-1 with a collection \widetilde{F}_{k-1} of $|\overline{F}|$ vertex-disjoint copies of K_{k+1} in G. All copies of K_k in G comprising an edge of $G[U_1]$ and a vertex from each of X_1, \ldots, X_{k-2} are K_{k+1} -connected by (b), so \widetilde{F}_{k-1} is in fact a connected K_{k+1} -factor in G of size at least $(k+1) \min\{|F|, q\}$.

6. The proof of Lemma 4.3

In this section we provide a proof of Lemma 4.3, our stability result for graphs with at least two K_{k+1} -components where each K_{k+1} -component contains a copy of K_{k+2} . We start with a couple of preparatory lemmas which collect some observations about K_{k+1} -components.

The first lemma states that K_{k+1} -components cannot be too small, that there are no edges between the exteriors of different components and that certain spots in a K_{k+1} -component induce a graph with minimum degree $k\delta - (k-1)n$. **Lemma 6.1.** Let $k \in \mathbb{N}$ and let G be a graph on n vertices with minimum degree $\delta(G) > \frac{(k-1)n}{k}$. Then

- (i) each K_{k+1} -component C satisfies $|C| > \delta$,
- (ii) for distinct K_{k+1} -components C and C' there are no edges between ext(C) and ext(C'),
- (iii) for each K_{k+1} -component C, each copy $u_1 \dots u_{k-1}$ of K_{k-1} of C, and $U = \{v : u_1 \dots u_{k-1}v \in C\}$, we have $\delta(G[U]) \ge k\delta (k-1)n$ and $|U| \ge k\delta (k-1)n + 1$.

Proof. For (i) let M be a maximal clique in C. Note that $|M| \ge k + 1$. Count $\rho := \sum_{m \in M, u \in V(G)} \mathbf{1}_{\{mu \in E(G)\}}$ in two ways. On the one hand,

$$\rho = \sum_{m \in M} \sum_{u \in V(G)} \mathbf{1}_{\{mu \in E(G)\}} = \sum_{m \in M} \deg(m) \ge |M| \delta.$$

On the other hand, noting that each vertex of G which is not a vertex of C is adjacent to at most k-1 vertices of M, while each vertex of C is adjacent to at most |M|-1 vertices of M, we obtain

$$\begin{split} \rho &= \sum_{u \in V(G)} \sum_{m \in M} \mathbf{1}_{\{mu \in E(G)\}} = \sum_{u \in V(G)} \deg(u; M) \\ &\leq \sum_{u \in C} |M| - 1 + \sum_{u \notin C} k - 1 = |C|(|M| - k) + (k - 1)n \end{split}$$

and so $|M|\delta - (k-1)n \leq |C|(|M|-k)$. Since $(k-1)n < k\delta$ we conclude that $|C| > \delta$.

For (ii) suppose that u is a vertex in ext(C), v is a vertex in ext(C') and uv is an edge in G. Apply Lemma 2.3 to complete uv to a copy of K_k in G. Since this copy of K_k contains a vertex from each of ext(C) and ext(C'), it is in both C and C', which is a contradiction.

For (iii) note that U is non-empty as $u_1 \ldots u_{k-1}$ is a copy of K_{k-1} of C. Let $u_k \in U$, so by definition $u_1 \ldots u_k \in C$. Since $\Gamma(u_1, \ldots, u_k) \subseteq U$, by Lemma 2.2 we have $\deg(u_k; U) = \deg(u_1, \ldots, u_k) \ge k\delta - (k-1)n$. Now $\{u_k\} \cup \Gamma(u_k; U) \subseteq U$ so $|U| \ge k\delta - (k-1)n + 1$. \Box

The next lemma says that graphs with more than one K_{k+1} -component have a non-empty K_{k+1} -interior and gives a lower bound on the size of said K_{k+1} -interior. This is an easy consequence of Lemma 6.1(i).

Lemma 6.2. Let $k \ge 2$ be an integer. Let G be a graph on n vertices with minimum degree $\delta(G) = \delta > \frac{(k-1)n}{k}$ and more than one K_{k+1} -component. Then

- (i) $|\operatorname{int}_k(G)| \ge 2\delta n + 2 > 0$, and
- (ii) for each K_{k+1} -component C of G we have $|\operatorname{ext}(C)| \leq n \delta 1$.

Proof. For (i), let C and C' be distinct K_{k+1} -components of G. Lemma 6.1(i) tells us that $|C|, |C'| > \delta$. $\operatorname{int}_k(G)$ contains all vertices which are vertices of both C and C' so $|\operatorname{int}_k(G)| \ge |C_1| + |C_2| - n \ge 2\delta - n + 2 > 0$.

For (ii), let C' be a K_{k+1} -component of G distinct from C. Now ext(C) contains no vertex of C' and by Lemma 6.1(i) we have $|C'| > \delta$, so it follows that $|ext(C)| \le n - \delta - 1$. \Box

Central to our proof of Lemma 4.3 is the construction of sufficiently large connected K_{k+1} -factors. Lemma 6.1(iii) enables us to find spots in a K_{k+1} -component which induce a graph with minimum degree $k\delta - (k-1)n$. In our proof of Lemma 4.3, we will often use this to find a large matching in such spots (this is possible due to Lemma 2.1(i)). The family of configurations introduced in Section 4.1, the structural analysis in Section 5.1 and our construction procedures in Section 5.2 will then enable us to extend such a matching to a connected K_{k+1} -factor.

As mentioned in Section 4.1, our proof of Lemma 4.3 considers two cases – when $\operatorname{int}_k(G)$ contains a copy of K_k and when $\operatorname{int}_k(G)$ does not contains a copy of K_k . In the first case, we prove that if $\operatorname{int}_k(G)$ contains a copy of K_k then $\operatorname{CKF}_{k+1}(G) \ge \operatorname{pp}_k(n, \delta + \eta n)$. In fact, we prove the contrapositive statement in Lemma 6.3, which involves proving that if $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$, then G does not contain the configurations $\dagger_k(\ell, j)$ for all $1 \le j < \ell \le k$: it follows immediately from the definition of $\operatorname{int}_k(G)$ that any copy of K_k in $\operatorname{int}_k(G)$ acts as the 'central'

copy of K_k in an instance of the configuration $\dagger_k(k, 1)$. We will use structural properties of these configurations proved in Section 5.1 and clique factor construction procedures from Section 5.2 to do so.

Lemma 6.3. Let $k \ge 3$ be an integer and $\mu > 0$. Let $\eta > 0$ and $n \in \mathbb{N}$ satisfy $\frac{1}{n} \ll \eta \ll \mu, \frac{1}{k}$. Let G be a graph on n vertices with minimum degree $\delta(G) \ge \delta \ge \left(\frac{k-1}{k} + \mu\right)n$ and at least two K_{k+1} -components. Suppose $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$. Then G does not contain the configuration $\dagger_k(\ell, j)$ for all j, ℓ such that $1 \le j < \ell \le k$. In particular, $\operatorname{int}_k(G)$ is K_k -free.

We prove Lemma 6.3 by induction on some function $f(j, \ell)$. While the specific choices of proof method and induction function are motivated by technical considerations, let us discuss the underlying ideas of our proof. We work in the context where $\operatorname{int}_k(G)$ contains a copy of K_k , which is equivalent to G containing $\dagger_k(k, 1)$ by definition, and we want to construct a sufficiently large connected K_{k+1} -factor. To this end, we seek increasingly structured graph configurations; this is achieved as a consequence of Lemma 5.1. Roughly speaking, the larger the interfaces between copies of K_k in different K_{k+1} -components, the more highly structured the configuration. Eventually, we arrive at a configuration of the form $\dagger_k(\ell, \ell - 1)$, which represents the 'pinnacle of evolution' with copies of K_k in different K_{k+1} -components that share a copy of K_{k-1} . These possess sufficient structure for the construction of a sufficiently large connected K_{k+1} -factor; we handle them in Lemmas 6.4 and 6.5. For technical reasons, we need treat $\dagger_k(2, 1)$ separately. We first consider the $j + 1 = \ell = 2$ case.

Lemma 6.4. Let $k \ge 3$ be an integer and $\mu > 0$. Let $\eta > 0$ and $n \in \mathbb{N}$ satisfy $\frac{1}{n} \ll \eta \ll \mu, \frac{1}{k}$. Let G be a graph on n vertices with minimum degree $\delta(G) \ge \delta \ge \left(\frac{k-1}{k} + \mu\right)n$ and at least two K_{k+1} -components. Suppose $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$. Then G does not contain the configuration $\dagger_k(2, 1)$.

Proof. Let $0 < \eta < \min\{\frac{1}{1000k^2}, \eta_0(k, \mu)\}$ and $n_1 := \max\{n_2(k, \mu, \eta), \frac{2}{\eta}\}$ with $\eta_0(k, \mu)$ and $n_2(k, \mu, \eta)$ given by Lemma 2.10. Suppose that G contains the configuration $\dagger_k(2, 1)$, so by Definition 1 there are vertices $u_1, \ldots, u_k, v_2, w_{2,1}$ in V(G) such that (CG1)–(CG3) hold. Say $f := u_2 \ldots u_k$ lies in distinct K_{k+1} -components C_1, \ldots, C_p and $f' := u_1 u_3 \ldots u_k$ lies in distinct K_{k+1} -components C_1, \ldots, C_p and $f' := u_1 u_3 \ldots u_k$ lies in distinct K_{k+1} -components C_1', \ldots, C_q' with $p, q \ge 2$ and $f_0 := u_1 \ldots u_k \in C_1 = C_1'$. Define

$$U_i = \{y : fy \in C_i\}$$
 for $i \in [p]$ and $V_j = \{y : f'y \in C'_i\}$ for $j \in [q]$,

so $\{U_i\}_{i\in[p]}$ and $\{V_j\}_{j\in[q]}$ partition $\Gamma(f)$ and $\Gamma(f')$ respectively. Note that

(31)
$$U_i \cap V_j = \emptyset \quad \text{for all } (i,j) \in ([p] \times [q]) \setminus \{(1,1)\}.$$

By Lemma 6.1(iii) we have

$$|U_i|, |V_j| \ge k\delta - (k-1)n + 1$$

for all $i \in [p], j \in [q]$. Since we have $\deg(u_1, \ldots, u_k) \ge k\delta - (k-1)n > 0$ by Lemma 2.2, we can pick a vertex $w \in \Gamma(f_0) \subseteq U_1 \cap V_1$. Now w has no neighbours in $\left(\bigcup_{1 \le j \le p} U_i\right) \cup \left(\bigcup_{1 \le j \le q} V_j\right)$, so by (32) we have

(33)
$$\delta \le \deg(w) < n - \sum_{1 \le i \le p} |U_i| - \sum_{1 \le j \le q} |V_j| \le n - 2(k\delta - (k-1)n + 1)$$

and we obtain $\delta \leq \frac{(2k-1)n-3}{2k+1} < \left(\frac{k}{k+1} - 2\eta\right)n$. By Lemma 2.2 we have

(34)
$$|\Gamma(f)| = \sum_{i \in [p]} |U_i| \ge (k-1)\delta - (k-2)n \text{ and}$$
$$|\Gamma(f')| = \sum_{j \in [q]} |V_j| \ge (k-1)\delta - (k-2)n,$$

so we obtain

$$|U_1| = |\Gamma(f)| - \sum_{1 < i \le p} |U_i| \stackrel{(33)}{\ge} |\Gamma(f)| - (n - \delta - 1) + \sum_{1 < j \le q} |V_j|$$

$$\stackrel{(34)}{\ge} k\delta - (k - 1)n + 1 + \sum_{1 < j \le q} |V_j| \stackrel{(32)}{\ge} 2(k\delta - (k - 1)n + 1).$$

By symmetry we have $|V_1| \ge 2(k\delta - (k-1)n + 1)$. We have $p, q \ge 2$, so $U_2, V_2 \ne \emptyset$ and we can pick $u \in U_2$ and $v \in V_2$. Now u and v have no neighbours in U_1 and V_1 respectively, so we conclude that

(36)

(35)

$$|U_1|, |V_1| < n - \delta.$$

We now define

$$\begin{split} X_{k-1}, Y_{k-1} &:= V(G) \setminus \Gamma(u_k), \\ X_i, Y_i &:= \Gamma(u_{i+2}, \dots, u_k) \setminus \Gamma(u_{i+1}) \text{ for } i \in [k-2] \setminus \{1\}, \\ X_1, X_1' &:= \Gamma(u_3, \dots, u_k) \setminus \Gamma(u_2), \quad Y_1, Y_1' &:= \Gamma(u_3, \dots, u_k) \setminus \Gamma(u_1), \\ X_i' &:= \Gamma(u_2, \dots, u_i, u_{i+2}, \dots, u_k) \setminus \Gamma(u_{i+1}) \text{ for } i \in [k-1] \setminus \{1\}, \\ Y_i' &:= \Gamma(u_1, u_3, \dots, u_i, u_{i+2}, \dots, u_k) \setminus \Gamma(u_{i+1}) \text{ for } i \in [k-1] \setminus \{1\}, \\ Z_i' &:= X_{i+1}' \cap Y_{i+1}'; Z_i'' &:= Z_i' \cap \Gamma(w) \text{ for } i \in [k-2]; \\ A &:= \bigcup_{i=1}^{k-1} (X_i \setminus X_i') = \bigcup_{i=2}^{k-1} (X_i \setminus X_i'), A' &:= \bigcup_{i=1}^{k-1} (Y_i \setminus Y_i') = \bigcup_{i=2}^{k-1} (Y_i \setminus Y_i'), \\ A'' &:= (\bigcup_{i=1}^{k-2} Z_i') \setminus \Gamma(w); \quad B &:= A \cup A' \cup A''. \end{split}$$

Note that A is the set of vertices in G with at least two non-neighbours in f. Count $\rho := \sum_{v \in V(G), u \in f} \mathbf{1}_{\{vu \notin E(G)\}}$ in two ways. On the one hand,

$$\rho = \sum_{u \in f} \left(\sum_{v \in V(G)} \mathbf{1}_{\{vu \notin E(G)\}} \right) = \sum_{u \in f} |V(G) \setminus \Gamma(u)| \le (k-1)(n-\delta)$$

On the other hand, we have

$$\rho = \sum_{v \in V(G)} \left(\sum_{u \in f} \mathbf{1}_{\{vu \notin E(G)\}} \right) = \sum_{v \in V(G)} |f \setminus \Gamma(v)| \ge n - |\Gamma(f)| + |A|.$$

Hence, by (34) we obtain $|A| \leq \sum_{i \in [p]} |U_i| - n + (k-1)(n-\delta)$. Similarly, A' is the set of vertices in G with at least two non-neighbours in f'. Hence, $|A'| \leq \sum_{j \in [q]} |V_j| - n + (k-1)(n-\delta)$. No vertex in $\left(\bigcup_{1 < i \leq p} U_i\right) \cup \left(\bigcup_{1 < j \leq q} V_j\right) \cup A''$ is adjacent to w, so $|A''| \leq n - \delta - 1 - \sum_{1 < i \leq p} |U_i| - \sum_{1 < j \leq q} |V_j|$. Therefore, we conclude that

(37)
$$|B| \le |U_1| + |V_1| - 2[k\delta - (k-1)n + 1] - (n-\delta - 1).$$

Let $1 < h \le p$. Lemma 6.1(iii) tells us that $\delta(G[U_h]) \ge k\delta - (k-1)n$, so we have a matching M in U_h with $|M| = \min\{k\delta - (k-1)n, \lfloor \frac{|U_h|}{2} \rfloor\}$ by Lemma 2.1(i). We check the conditions to apply Lemma 5.5 for b = 2 with $U_h, \bigcup_{i \ne h} U_i, Z''_1, \ldots, Z''_{k-2}, X_1, B$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively. By definition U_h and $\bigcup_{i \ne h} U_i$ partition $\Gamma(f)$. For each $i \in [k-2]$ the set Z''_i consists of the neighbours of w whose only non-neighbour in f_0 is u_{i+2} . The set X_1 consists of the non-neighbours of w whose only non-neighbour in f is u_2 . The set B consists of the non-neighbours of w whose only non-neighbours in $f \in [k-2]$ and the vertices with at least two non-neighbours in f'. Hence, $U_h, \bigcup_{i \ne h} U_i, Z''_1, \ldots, Z''_{k-2}, X_1, B$

form a partition of V(G) such that there are no edges between U_h and $\bigcup_{i \neq h} U_i$. Note that $Z''_i \subseteq V(G) \setminus \Gamma(u_{i+2})$ for $i \in [k-2]$ and $X_1 \subseteq V(G) \setminus \Gamma(u_2)$, so $|Z''_i| \leq n - \delta$ for each $i \in [k-2]$ and $|X_1| \leq n - \delta$. For each $(e, i) \in E(G[U_h]) \times [k-2]$, by applying Lemma 5.2 for i+1 with u_1, \ldots, u_k, w as themselves, C_h as C_1 , C_1 as C_2 and e as uv, we have that $Z''_i \cap \Gamma(e)$ is an independent set. Furthermore, all copies of K_k in G with at least two vertices from U_h and all other vertices from $\left(\bigcup_{i=1}^{k-2} Z''_i\right)$ are K_{k+1} -connected: we can construct a K_{k+1} -walk from such a copy g of K_k to f_0 by a step-by-step vertex replacement of the vertices of g with the vertices of f_0 .

Since the requisite conditions are satisfied, we apply Lemma 5.5 for b = 2 with $U_h, \bigcup_{i \neq h} U_i, Z''_1, \ldots, Z''_{k-2}, X_1, B$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively; since $\sum_{i \neq h} |U_i| \ge |U_1|$ and by noting (35), (36) and (37), we obtain that $\operatorname{CKF}_{k+1}(G)$ is at least

$$\begin{aligned} (k+1)\min\left\{k\delta - (k-1)n, \left\lfloor \frac{|U_h|}{2} \right\rfloor, \left\lfloor \frac{\sum_{i \neq h} |U_i|}{2} \right\rfloor, \frac{k\delta - (k-1)n + \sum_{i \neq h} |U_i| - |B|}{3} \right\} \\ \geq (k+1)\min\left\{k\delta - (k-1)n, \left\lfloor \frac{|U_h|}{2} \right\rfloor\right\}. \end{aligned}$$

Since (7) and (9) hold and $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$, we deduce that

(38)
$$|U_h| < 2(k\delta - (k-1)n) \text{ and } |U_h| < \frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'}$$

where $r' := r_p(n, \delta + \eta n)$. If furthermore $\delta \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{2k-1}{2k+1} - 2\eta\right)n\right]$, by (10) we also deduce that

(39)
$$|U_h| < \frac{3}{2}(k\delta - (k-1)n) + 1.$$

By symmetry we also have that for all $1 < j \leq q$,

(40)
$$|V_j| < 2(k\delta - (k-1)n) \text{ and } |V_j| < \frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'}$$

and if furthermore $\delta \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{2k-1}{2k+1} - 2\eta\right)n\right]$ then (41) $|V_i| < \frac{3}{2}(k\delta - (k-1)n) + 1.$

Let $1 < i \le p$ and $1 < j \le q$. By Lemma 2.1(i) and Lemma 6.1(iii) and noting (38) and (40), there are matchings M_u and M_v in U_i and V_j respectively with $|M_u| = \min\{\left\lfloor \frac{|U_i|}{2}\right\rfloor, k\delta - (k - 1)n\} = \left\lfloor \frac{|U_i|}{2}\right\rfloor$ and $|M_v| = \min\{\left\lfloor \frac{|V_j|}{2}\right\rfloor, k\delta - (k - 1)n\} = \left\lfloor \frac{|V_j|}{2}\right\rfloor$. Without loss of generality, suppose $|V_j| \ge |U_i|$. Let uv be an edge in U_i . Note that u and v each has at most $n - \delta - 1 - \sum_{h \ne i} |U_h| \le |U_i| - (k\delta - (k - 1)n + 1)$ non-neighbours outside of $\left(\bigcup_{h \ne i} U_h\right) \cup \{u, v\}$. Hence, $\Gamma(u, v)$ has at most $2[|U_i| - (k\delta - (k - 1)n + 1)]$ non-neighbours outside of $\left(\bigcup_{h \ne i} U_h\right) \cup \{u, v\}$. Since we have $|V_j| \ge |U_i|$ and by (31) V_j is disjoint from $\left(\bigcup_{h \ne i} U_h\right) \cup \{u, v\}$, by (38) we obtain (42) $|\Gamma(u, v; V_j)| \ge |V_j| - 2[|U_i| - (k\delta - (k - 1)n + 1)] > 0.$

Hence, we may pick $x \in \Gamma(u, v; V_j)$. Suppose $\Gamma(u, v; V_j)$ is an independent set. Now x has no neighbour in $\left(\bigcup_{h\neq j} V_h\right) \cup \Gamma(u, v; V_j)$, so we have $n - \delta \ge \sum_{h\neq j} |V_h| + |\Gamma(u, v; V_j)| \ge |\Gamma(f')| - 2[|U_i| - (k\delta - (k-1)n + 1)]$ by (34) and (42). Hence, we obtain

(43)
$$|V_j| \ge |U_i| \ge k\delta - (k-1)n + 1 + \frac{|\Gamma(f')| - (n-\delta)}{2} \stackrel{(34)}{\ge} \frac{3}{2}(k\delta - (k-1)n) + 1.$$

Note that $U_i \cap V_j = \emptyset$ and w has no neighbours in $U_i \cup V_j$, so $n-\delta > |U_i \cup V_j| \ge 3[k\delta - (k-1)n+1]$, which implies $\delta \le \frac{(3k-2)n-4}{3k+1} < \left(\frac{2k-1}{2k+1} - 2\eta\right)n$. However, this means that (43) contradicts (39). Therefore, there is an edge u'v' in $\Gamma(u, v; V_j)$. Then $uvu'v'u_3 \dots u_k$ is a copy of K_{k+2} with $uvu_3 \dots u_k \in C_i$ and $u'v'u_3 \dots u_k \in C'_j$ so $C_i = C'_j$. Noting that $i \in [p] \setminus \{1\}$ and $j \in [q] \setminus \{1\}$ are arbitrary, we deduce that in fact p = q = 2.

In what follows, we check the conditions to apply Lemma 5.4 for b = c = 2 with U_2, U_1, V_2, V_1 , $Z''_1, \ldots, Z''_{k-2}, X_1, Y_1, B, B, M_u$ and M_v as the required inputs $U_1, U_2, V_1, V_2, X_1, \ldots, X_{k-1}, X', A$, A', F^U and F^V respectively. We know from earlier that $U_2, U_1, Z''_1, \ldots, Z''_{k-2}, X_1, B$ form a partition of V(G) such that there are no edges between U_2 and U_1 , that $|Z''_i| \leq n - \delta$ for each $i \in [k-2]$ and $|X_1| \leq n-\delta$, that $Z''_i \cap \Gamma(e)$ is an independent set for each $(e,i) \in E(G[U_2]) \times [k-2]$ and that all copies of K_k in G with at least two vertices from U_2 and all other vertices from $\left(\bigcup_{i=1}^{k-2} Z_i''\right)$ are K_{k+1} -connected. By swapping the roles of u_1 and u_2 , we also have that $V_2, V_1, Z''_1, \ldots, Z''_{k-2}, Y_1, B$ form a second partition of V(G) such that there are no edges between V_1 and V_2 , that $|Y_1| \leq n - \delta$, that $Z''_i \cap \Gamma(e)$ is an independent set for each $(e,i) \in E(G[V_2]) \times [k-2]$ and that all copies of K_k in G with at least two vertices from V_2 and all other vertices from $\left(\bigcup_{i=1}^{k-2} Z_i''\right)$ are K_{k+1} -connected. Since the conditions are satisfied, we apply Lemma 5.4 with the given inputs and $d_1 = d_2 =$

 $\left|\frac{|V_1|}{3}\right|$ to obtain that $\operatorname{CKF}_{k+1}(G)$ is at least

$$(k+1) \min\left\{ \left\lfloor \frac{|U_2|}{2} \right\rfloor, \left\lfloor \frac{|U_1|}{2} \right\rfloor, \left\lfloor \frac{|V_1|}{3} \right\rfloor, \frac{k\delta - (k-1)n - |B| + |U_1|}{3}, \frac{k\delta - (k-1)n + |U_1| - |V_2|}{3} \right\} + (k+1) \min\left\{ \left\lfloor \frac{|V_2|}{2} \right\rfloor, \left\lfloor \frac{|V_1|}{3} \right\rfloor, \frac{k\delta - (k-1)n - |B| + |V_1| - |U_2|/2}{3}, \frac{k\delta - (k-1)n + |V_1| - 3|U_2|/2}{3} \right\}.$$

By (33), (34), (35), (36), (37) and (38), this is at least

$$(k+1)\left(\min\left\{\left\lfloor\frac{|U_2|}{2}\right\rfloor,\frac{2(k\delta-(k-1)n)}{3}\right\}+\min\left\{\left\lfloor\frac{|V_2|}{2}\right\rfloor,\frac{2(k\delta-(k-1)n)}{3}-\frac{|U_2|}{6}\right\}\right)$$

Now by Lemma 6.1(iii) we have $\left\lfloor \frac{|U_2|}{2} \right\rfloor$, $\left\lfloor \frac{|V_2|}{2} \right\rfloor \ge \frac{(k\delta - (k-1)n}{2}$ and we have (38), so in fact it is at least $(k+1)(k\delta - (k-1)n) \ge pp_k(n, \delta + \eta n)$ by (9). However, this is a contradiction so G does not contain $\dagger_k(2,1)$.

Next, we consider the $3 \le j + 1 = \ell \le k$ case.

Lemma 6.5. Let k, ℓ be integers satisfying $3 \le \ell \le k$ and let $\mu > 0$. Let $\eta > 0$ and $n \in \mathbb{N}$ satisfy $\frac{1}{n} \ll \eta \ll \mu, \frac{1}{k}$. Let G be a graph on n vertices with minimum degree $\delta(G) \ge \delta \ge \left(\frac{k-1}{k} + \mu\right) n$ and at least two K_{k+1} -components. Suppose $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$ and G does not contain $\dagger_k(q,p)$ for all $p < q < \ell$. Then G does not contain the configuration $\dagger_k(\ell, \ell-1)$.

Proof. Let $0 < \eta < \min\{\frac{1}{1000k^2}, \eta_0(k, \mu)\}$ and $n_1 := \max\{n_2(k, \mu, \eta), \frac{2}{\eta}\}$ with $\eta_0(k, \mu)$ and $n_2(k,\mu,\eta)$ given by Lemma 2.10. Suppose that G contains the configuration $\dagger_k(\ell,\ell-1)$, so by Definition 1 there are vertices $u_1, \ldots, u_k, v_\ell, w_{\ell,1}, \ldots, w_{\ell,\ell-1}$ in V(G) such that (CG1)-(CG3) hold. Set $f_i := u_1 \dots u_{i-1} u_{i+1} \dots u_{\ell-1} u_{\ell+1} \dots u_k$ for each $i \in [\ell - 1]$. Let f := $u_1 \ldots u_{\ell-1} u_{\ell+1} \ldots u_k$. Observe that u_ℓ and v_ℓ are distinct vertices: if not, $fu_\ell = fv_\ell$ would be a copy of K_k in two different K_{k+1} -components, giving a contradiction. Furthermore, $v_\ell u_\ell$ is not an edge: if not, $fu_{\ell}v_{\ell}$ would be a copy of K_{k+1} in G where fu_{ℓ} and fv_{ℓ} would belong to different K_{k+1} -components, giving a contradiction. Hence, $u_1, \ldots, u_k, v_\ell, w_{\ell,1}, \ldots, w_{\ell,\ell-1}$ are all distinct vertices. Set $X_i := \Gamma(f_i) \setminus \{u_i\}$ for $i \in [\ell-1], X := \bigcup_{i=1}^{\ell-1} X_i$ and $Y_j := \Gamma(u_\ell, v_\ell, w_{\ell,j}; X)$ for $j \in [\ell - 1]$.

We claim that $Y_j = \emptyset$ for some $j \in [\ell - 1]$. Indeed, suppose that $Y_j \neq \emptyset$ for all $j \in [\ell - 1]$. Pick $y_j \in Y_j$ for each $j \in [\ell-1]$. Fix a function $\phi: [\ell-1] \to [\ell-1]$ such that $y_j \in X_{\phi(j)}$. Observe that $y_j f_{\phi(j)} u_\ell \in C$ for each $j \in [\ell - 1]$: if not, then $f u_\ell \in C$, $f v_\ell \notin C$ and $y_j f_{\phi(j)} u_\ell \notin C$ would yield $\dagger_k(2,1)$ with fu_ℓ as the 'central' copy of K_k and $f_{\phi(j)}$ as the common vertices. Similarly, we have $y_j f_{\phi(j)} v_\ell \notin C$ for each $j \in [\ell-1]$. Now for each $j \in [\ell-1]$ apply Lemma 2.3 to complete $u_{\ell} \dots u_k w_{\ell,j} y_j$ to a copy $D_j := u_{\ell} \dots u_k w_{\ell,j} y_j y_{j,1} \dots y_{j,\ell-3}$ of K_k . Observe that $D_j \in C$ for each $j \in [\ell-1]$: if not, then $y_j f_{\phi(j)} v_\ell \notin C$, $y_j f_{\phi(j)} u_\ell \in C$ and $D_j \notin C$ would yield $\dagger_k (\ell-1, \ell-2)$ with $y_j f_{\phi(j)} u_\ell$ as the 'central' copy of $K_k, y_j u_{\ell+1} \dots u_k$ as the common vertices and D_j 'dangling off' u_{ℓ} . But now $D_j \in C$ for $j \in [\ell - 1]$ with $u_{\ell} \dots u_k w_{\ell,1} \dots w_{\ell,\ell-1} \notin C$ as the 'central' copy of K_k

yields $\dagger_k(\ell - 1, 1)$ with $u_\ell \dots u_k$ as the common vertices, giving a contradiction. Hence, Y_j is empty for some $j \in [\ell - 1]$.

Pick $j \in [\ell - 1]$ such that $Y_j = \emptyset$, which exists by the claim above. Apply Lemma 2.2 with $U = V(G) \setminus \{u_1, \ldots, u_{\ell-1}, u_{\ell+1}, \ldots, u_k\}$ to obtain

(44)
$$|X_i| \ge (k-2)(\delta-k+2) - (k-3)(n-k+1) = (k-2)\delta - (k-3)n - 1$$

for each $i \in [\ell - 1]$. Since $X_h \cap X_i = \Gamma(f)$ for all $\{h, i\} \in {[\ell - 1] \choose 2}$, we have

(45)
$$|X| = \sum_{i=1}^{\ell-1} |X_i| - (\ell-2)|\Gamma(f)|.$$

We claim that $w_{\ell,j} \notin X$. Indeed, suppose that $w_{\ell,j} \in X$. Without loss of generality, $w_{\ell,j} \in X_1$. Observe that $w_{\ell,j}f_1u_\ell \in C$: if not, then $fu_\ell \in C$, $fv_\ell \notin C$ and $w_{\ell,j}f_1u_\ell \notin C$ would yield $\dagger_k(2,1)$ with fu_ℓ as the 'central' copy of K_k and f_1 as the common vertices. Similarly, we have $w_{\ell,j}f_1v_\ell \notin C$. But now $w_{\ell,j}f_1v_\ell \notin C$, $w_{\ell,j}f_1u_\ell \in C$ and $u_\ell \dots u_k w_{\ell,1} \dots w_{\ell,\ell-1} \notin C$ yields $\dagger_k(\ell-1,\ell-2)$ with $w_{\ell,j}f_1u_\ell$ as the 'central' copy of K_k , $u_{\ell+1} \dots u_k w_{\ell,j}$ as the common vertices and $u_\ell \dots u_k w_{\ell,1} \dots w_{\ell,\ell-1}$ 'dangling off' u_ℓ , giving a contradiction. Now apply Lemma 2.2 with $U = X \setminus \{u_\ell, v_\ell\}$ to obtain

(46)
$$|Y_j| \ge 2(\delta - n + |X|) + (\delta - n + |X| - 1) - 2(|X| - 2) \\ = |X| - 3(n - \delta - 1).$$

Denote by C' the K_{k+1} -component of G which contains fv_{ℓ} . Define

$$W_1 := \{ u \in \Gamma(f) : uf \in C \}, \quad W_2 := \{ u \in \Gamma(f) : uf \in C' \}, \\ W_3 := \{ u \in \Gamma(f) : uf \notin C, C' \}.$$

Since $u_{\ell} \in W_1$ and $v_{\ell} \in W_2$, we have $W_1, W_2 \neq \emptyset$. Moreover, we have $\Gamma(u_1, \ldots, u_k) \subseteq W_1$ and $\Gamma(u_1, \ldots, u_{\ell-1}, u_{\ell+1}, \ldots, u_k, v_{\ell}) \subseteq W_2$. Let $w_1 \in W_1$ and $w_2 \in W_2$. Note that w_1 has no neighbour in $W_2 \cup W_3$ and w_2 has no neighbour in $W_1 \cup W_3$, so

(47)
$$|W_1 \cup W_3|, |W_2 \cup W_3| \le n - \delta - 1.$$

Since Y_j is empty and (44), (45) and (46) hold, we obtain

$$0 = |Y_j| \ge (\ell - 1)((k - 2)\delta - (k - 3)n - 1) - (\ell - 2)|\Gamma(f)| - 3(n - \delta - 1).$$

By rearrangement, we obtain

(48)
$$|\Gamma(f)| = \sum_{i \in [3]} |W_i| \ge (k-2)\delta - (k-3)n - 1 + \frac{(k+1)\delta - kn + 2}{\ell - 2}.$$

By (47) and (48), we have

$$(k-1)\delta - (k-2)n + \frac{(k+1)\delta - kn + 2}{\ell - 2} \le |W_1|, |W_2| \le n - \delta - 1$$

Hence, $\delta \leq \frac{[(\ell-1)(k-1)+1]n-\ell}{(\ell-1)k+1} \leq \frac{(2k-1)n-3}{2k+1} < \left(\frac{k}{k+1} - 2\eta\right)n$. By multiplying both sides of the first upper bound on δ by $\ell - 3$ and rearranging, we obtain

$$n - \delta - 1 + \frac{(k+1)\delta - kn + 2}{\ell - 2} \ge (\ell - 2)(k\delta - (k - 1)n + 1).$$

Recalling (48) and $\ell \geq 3$, we obtain

(49)
$$|\Gamma(f)| \ge (k-1)\delta - (k-2)n + (\ell-2)(k\delta - (k-1)n + 1) \\ \ge (k-1)\delta - (k-2)n + k\delta - (k-1)n + 1$$

and

(50)
$$|W_1|, |W_2| \ge (\ell - 1)[k\delta - (k - 1)n + 1] \ge 2[k\delta - (k - 1)n + 1].$$

Now pick a vertex $w \in W_2$ and define

$$\begin{aligned} Z_{i} &:= \Gamma(u_{i+2}, \dots, u_{k}) \setminus \Gamma(u_{i+1}) \text{ for } \ell \leq i \leq k-1, \\ Z_{i} &:= \Gamma(u_{i+1}, \dots, u_{\ell-1}, u_{\ell+1} \dots, u_{k}) \setminus \Gamma(u_{i}) \text{ for } i \in [\ell-1]; \\ Z'_{i} &:= \Gamma(u_{1}, \dots, u_{i-1}, u_{i+1}, \dots, u_{\ell-1}, u_{\ell+1}, \dots, u_{k}) \setminus \Gamma(u_{i}) \text{ for } i \in [\ell-1], \\ Z'_{i} &:= \Gamma(u_{1}, \dots, u_{\ell-1}, u_{\ell+1} \dots, u_{i}, u_{i+2}, \dots, u_{k}) \setminus \Gamma(u_{i+1}) \text{ for } \ell \leq i \leq k-1; \\ Z''_{i} &:= Z'_{i} \cap \Gamma(w) \text{ for } i \in [k-1]; \\ A_{1} &:= \bigcup_{i=1}^{k-1} (Z_{i} \setminus Z'_{i}), A_{2} &:= \left(\bigcup_{i=1}^{k-1} Z'_{i}\right) \setminus \Gamma(w), A := A_{1} \cup A_{2}. \end{aligned}$$

Note that $|A_1|$ is the number of vertices in G with at least two non-neighbours in f. Count $\rho := \sum_{v \in V(G), u \in f} \mathbf{1}_{\{vu \notin E(G)\}}$ in two ways. On the one hand,

$$\rho = \sum_{u \in f} \left(\sum_{v \in V(G)} \mathbf{1}_{\{vu \notin E(G)\}} \right) = \sum_{u \in f} |V(G) \setminus \Gamma(u)| \le (k-1)(n-\delta).$$

On the other hand,

$$\rho = \sum_{v \in V(G)} \left(\sum_{u \in f} \mathbf{1}_{\{vu \notin E(G)\}} \right) = \sum_{v \in V(G)} |f \setminus \Gamma(v)| \ge n - |\Gamma(f)| + |A_1|.$$

Hence, we have $|A_1| \leq |\Gamma(f)| - n + (k-1)(n-\delta)$. No vertex in $W_1 \cup W_3 \cup A_2$ is adjacent to w, so $|A_2| \leq n - \delta - 1 - |W_1| - |W_3|$. Therefore, noting that $|\Gamma(f)| = \sum_{i \in [3]} |W_i|$, we obtain

(51)
$$|A| \le |W_2| - [k\delta - (k-1)n + 1].$$

Lemma 6.1(iii) tells us that $\delta(G[W_1]) \geq k\delta - (k-1)n$, so by Lemma 2.1(i) and (50) we have a matching M of size $|M| = k\delta - (k-1)n$ in W_1 . We shall check the conditions to apply Lemma 5.5 for b = 2 with $W_1, W_2 \cup W_3, Z''_1, \ldots, Z''_{k-1}, A$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively. By definition W_1 and $W_2 \cup W_3$ partition $\Gamma(f)$. For each $i \in [k-1]$ the set Z''_i consists of the neighbours of w whose only non-neighbour in f is u_i if $i < \ell$ and u_{i+1} if $i \ge \ell$. The set A consists of the non-neighbours of w with exactly one non-neighbour in f and the vertices with at least two non-neighbours in f. Hence, $W_1, W_2 \cup W_3, Z''_1, \ldots, Z''_{k-1}, A$ form a partition of V(G) such that there are no edges between W_1 , W_2 and W_3 . Given $i \in [k-1]$ there exists $j \in [k]$ such that $Z''_i \subseteq V(G) \setminus \Gamma(u_j)$ so $|Z''_i| \leq n - \delta$. For each $(e, i) \in E(G[W_1]) \times [k-1]$, by applying Lemma 5.3(i) for i with $u_1, \ldots, u_{\ell-1}, w$ as themselves, u_{a+1} as u_a for $\ell \leq a < k$, C as C_1 , C' as C_2 and e as uv, we have that $Z''_i \cap \Gamma(e)$ is an independent set. Furthermore, all copies of K_k in G with at least two vertices from W_1 and all other vertices from $\left(\bigcup_{i=1}^{k-2} Z_i''\right)$ are K_{k+1} -connected: we can construct a K_{k+1} -walk from such a copy g of K_k to fu_ℓ by a step-by-step vertex replacement of the vertices of g with the vertices of fu_{ℓ} .

Since the requisite conditions are satisfied, we apply Lemma 5.5 for b = 2 with $W_1, W_2 \cup$ $W_3, Z_1'', \ldots, Z_{k-1}'', A$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively; noting that (50) and (51) hold, we obtain that $CKF_{k+1}(G)$ is at least

$$(k+1) \min\left\{k\delta - (k-1)n, \left\lfloor \frac{|W_2 \cup W_3|}{2} \right\rfloor, \frac{2[k\delta - (k-1)n] + |W_3| + 1}{3}\right\} \\ \ge (k+1) \min\left\{k\delta - (k-1)n, \frac{2[k\delta - (k-1)n] + |W_3| + 1}{3}\right\}.$$

First suppose there is a vertex $u \in W_3$. Since $\Gamma(u, f) \subseteq W_3$, by Lemma 2.2 we have $|W_3| \geq$ $|\Gamma(u, f)| \ge k\delta - (k-1)n$. This implies $pp_k(n, \delta + \eta n) > CKF_{k+1}(G) \ge (k+1)(k\delta - (k-1)n)$, which contradicts (9). Hence, we have $W_3 = \emptyset$. We distinguish three cases.

Case 1: $\delta \in \left[\left(\frac{k-1}{k} + \mu\right)n, \left(\frac{3k-2}{3k+1} - 2\eta\right)n\right] \cup \left[\left(\frac{3k-2}{3k+1} + \eta\right)n, \left(\frac{2k-1}{2k+1} - 2\eta\right)n\right]$. In this case, we have $\operatorname{pp}_k(n, \delta + \eta n) > \operatorname{CKF}_{k+1}(G) \geq \frac{2(k+1)(k\delta - (k-1)n)}{3}$, which contradicts (11).

Case 2: $\delta \in \left[\left(\frac{3k-2}{3k+1}-2\eta\right)n, \left(\frac{3k-2}{3k+1}+\eta\right)n\right]$. Without loss of generality, we have $|W_1| \ge |W_2|$. By the upper bound on δ , we have $n-\delta-1 \ge 3(k\delta-(k-1)n+1)-(3k+1)\eta n-2$. Now together with (49) we obtain $|W_1| \ge \frac{|\Gamma(f)|}{2} \ge \frac{9}{4}(k\delta-(k-1)n)+3$ so $\left\lfloor\frac{|W_1|}{3}\right\rfloor \ge \frac{3}{4}(k\delta-(k-1)n)$. Note that $\delta(G[W_1]) \ge |W_1| - (n-\delta-|W_2|)$. By Corollary 2.5 applied to $G[W_1]$ with k = 2, (49) and (50), the number of vertex-disjoint triangles in $G[W_1]$ is at least

$$\min\left\{ |\Gamma(f)| + |W_2| - 2(n-\delta), \left\lfloor \frac{|W_1|}{3} \right\rfloor \right\}$$

$$\geq \min\left\{ 4(k\delta - (k-1)n) - (n-\delta), \left\lfloor \frac{|W_1|}{3} \right\rfloor \right\} \geq \frac{3}{4}(k\delta - (k-1)n).$$

Let T be a collection of $\frac{3}{4}(k\delta - (k-1)n)$ vertex-disjoint triangles in $G[W_1]$. We apply Lemma 5.5 for b = 3 with $W_1, W_2, Z''_1, \ldots, Z''_{k-1}, A$ and T as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively; the requisite conditions have already been shown to be satisfied at the preceding application of Lemma 5.5. Noting (50), we obtain that $\operatorname{CKF}_{k+1}(G)$ is at least

$$\begin{aligned} &(k+1)\min\left\{\frac{3}{4}(k\delta-(k-1)n), \left\lfloor\frac{|W_2|}{2}\right\rfloor, \frac{2[k\delta-(k-1)n]+|W_2|+1}{5}\right\} \\ &\geq \frac{3}{4}(k+1)(k\delta-(k-1)n), \end{aligned}$$

so $pp_k(n, \delta + \eta n) > \frac{3}{4}(k+1)(k\delta - (k-1)n)$, which contradicts (10).

Case 3:
$$\delta \in \left[\left(\frac{2k-1}{2k+1} - 2\eta\right)n, \frac{(2k-1)n-3}{2k+1}\right]$$
. Define

$$\widetilde{Z}_i = Z_i \cap \Gamma(w) \text{ for } i \in [k-1] \text{ and } \widetilde{A} := \left(\bigcup_{i=1}^{k-1} Z_i\right) \setminus \Gamma(w).$$

No vertex in $W_1 \cup \widetilde{A}$ is adjacent to w, so

(52)
$$|A| \le n - \delta - 1 - |W_1|.$$

By Lemma 6.1(iii) we have $\delta(G[W_1]) \geq k\delta - (k-1)n$, so there is a matching M of size $|M| = k\delta - (k-1)n$ in W_1 by Lemma 2.1(i) and (50). We shall check the conditions to apply Lemma 5.6 with $W_1, W_2, \tilde{Z}_1, \ldots, \tilde{Z}_{k-1}, \tilde{A}$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively. W_1 and W_2 partition $\Gamma(f)$ by definition. For each $i \in [k-1]$ the set \tilde{Z}_i consists of the neighbours v of w such that max $\{j \in [k] \setminus \{\ell\} : vu_j \notin E(G)\}$ is well-defined and equal to i if $i < \ell$ and to i + 1 if $i \geq \ell$. The set \tilde{A} consists of the non-neighbours of w with at least one non-neighbour in f. Hence, $W_1, W_2, \tilde{Z}_1, \ldots, \tilde{Z}_{k-1}, \tilde{A}$ form a partition of V(G) such that there are no edges between W_1 and W_2 . Given $i \in [k-1]$ there is $j \in [k]$ such that $\tilde{Z}_i \subseteq V(G) \setminus \Gamma(u_j)$ so $|\tilde{Z}_i| \leq n - \delta$. Let $i \in [k-1]$ and let g be a copy of K_{i+1} comprising an edge e of $G[W_1]$ and a copy g' of K_{i-1} with a vertex from each of $\tilde{Z}_1, \ldots, \tilde{Z}_{i-1}$. By applying Lemma 5.3 (ii) for i with $u_1, \ldots, u_{\ell-1}, w$ as themselves, u_{a+1} as u_a for $\ell \leq a < k$, C as C_1 , C' as C_2 , e as uv and g' as g, we have that $\tilde{Z}_i \cap \Gamma(g)$ is an independent set. Furthermore, all copies of K_k in G comprising an edge of $G[W_1]$ and a vertex from each of $\tilde{Z}_1, \ldots, \tilde{Z}_{k-2}$ are K_{k+1} -connected: we can construct a K_{k+1} -walk from such a copy g of K_k to fu_ℓ by a step-by-step vertex replacement of the vertices of g with the vertices of fu_{ℓ} .

Since the requisite conditions are satisfied, we apply Lemma 5.6 with the objects $W_1, W_2, Z_1, \ldots, \widetilde{Z}_{k-1}, \widetilde{A}$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively; noting that in this case we have $\delta \geq \left(\frac{2k-1}{2k+1} - 2\eta\right) n$, (50) and (52), we obtain that $\operatorname{CKF}_{k+1}(G)$ is at least

$$\begin{aligned} &(k+1)\min\left\{k\delta - (k-1)n, k\delta - (k-1)n + |W_2| - |W_1| - |\widetilde{A}|\right\}\\ &\geq (k+1)\min\left\{k\delta - (k-1)n, k\delta - (k-1)n + |W_2| - (n-\delta-1)\right\}\\ &\geq (k+1)\min\left\{k\delta - (k-1)n, k\delta - (k-1)n + (2k+1)\delta - (2k-1)n + 3\right\}\\ &\geq (k+1)(k\delta - (k-1)n - 2(2k+1)\eta n),\end{aligned}$$

so $pp_k(n, \delta + \eta n) > (k+1)(k\delta - (k-1)n - 2(2k+1)\eta n)$, contradicting (9).

Now we prove Lemma 6.3.

Proof of Lemma 6.3. Let $S = \{(j,\ell) \in \mathbb{Z}^2 \mid 1 \le j < \ell \le k\}$. Note that $f: S \to \left[\frac{k(k-1)}{2}\right]$ given by $f(j,l) = \frac{\ell(\ell-1)}{2} - j + 1$ is bijective and $f(j,\ell) < f(j',\ell') \iff \ell < \ell'$ or $(\ell = \ell', j' < j)$. We proceed by induction on $f(j,\ell)$. The base case $f(j,\ell) = 1$ corresponds to $(j,\ell) = (1,2)$. By Lemma 6.4, G does not contain $\dagger_k(2,1)$. For $f(j,\ell) > 1$, there are two cases to consider: $j+1 = \ell \le k$ and $j+1 < \ell \le k$.

Consider the first case $j+1 = \ell \leq k$. By the inductive hypothesis, G does not contain $\dagger_k(q, p)$ for all p, q such that $p < q < \ell$. Hence, by Lemma 6.5 G does not contain $\dagger_k(\ell, \ell - 1)$. Consider the second case $j+1 < \ell \leq k$. By the inductive hypothesis, G does not contain $\dagger_k(q, p)$ for all pairs p, q such that $p < q < \ell$ or j . Hence, by Lemma 5.1 <math>G does not contain $\dagger_k(q, p)$ for $\dagger_k(\ell, j)$. This completes the proof by induction.

Finally, G does not contain $\dagger_k(k, 1)$ so $\operatorname{int}_k(G)$ is K_k -free.

It remains to handle the case where $int_k(G)$ contains no copy of K_k . The following lemma represents an application of Lemma 5.6 for this case.

Lemma 6.6. Let $k \geq 3$ be an integer. Let G be a graph on n vertices with minimum degree $\delta(G) \geq \delta > \frac{(k-1)n}{k}$, at least two K_{k+1} -components and $\operatorname{int}_k(G)$ K_k -free. Let C_1, \ldots, C_p be the K_{k+1} -components of G. Set $q' := k\delta - (k-1)n + \sum_{j \neq 1} |\operatorname{ext}(C_j)| - |\operatorname{ext}(C_1)|$. Then

$$\operatorname{CKF}_{k+1}(G) \ge (k+1)\min\left\{k\delta - (k-1)n, \left\lfloor \frac{|\operatorname{ext}(C_1)|}{2} \right\rfloor, q'\right\}.$$

Proof. By Lemma 6.2(i) we have $|\operatorname{int}_k(G)| \ge 2\delta - n + 2 > 0$. Pick $u_{k-1} \in \operatorname{int}_k(G)$ and recursively pick $u_i \in \Gamma(u_{k-1}, \ldots, u_{i+1}; \operatorname{int}_k(G))$ for $i \in [k-2]$. By Lemma 2.2 we have

(53)
$$|\Gamma(u_{k-1}, \dots, u_{i+1}; \operatorname{int}_k(G))| \ge |\operatorname{int}_k(G)| - (k-i-1)(n-\delta) \ge (k-i+1)\delta - (k-i)n + 2 > 0$$

for each $i \in [k-1]$ so this is well-defined. For $i \in [k-1]$ define

$$L_i = \Gamma(u_{k-1}, \dots, u_{i+1}; \operatorname{int}_k(G)) \setminus \Gamma(u_i).$$

We want to apply Lemma 5.6 with $\operatorname{ext}(C_1), \bigcup_{j\neq 1} \operatorname{ext}(C_j), L_1, \ldots, L_{k-1}$ and \varnothing as $U_1, U_2, X_1, \ldots, X_{k-1}$ and A respectively. We claim that L_1, \ldots, L_{k-1} give a partition of $\operatorname{int}_k(G)$. Indeed, for each $v \in \operatorname{int}_k(G)$ we have $v \in L_h$ if and only if $h = \max\{a \in [k-1] : v \notin \Gamma(u_a)\}$; this quantity is well-defined because $\operatorname{int}_k(G)$ is K_k -free. Furthermore, each set L_i is nonempty by (53) and the fact that each u_i has at most $n - \delta$ non-neighbours. Hence, $L_1, \ldots, L_{k-1}, \operatorname{ext}(C_1), \bigcup_{j\neq 1} \operatorname{ext}(C_j)$ gives a partition of V(G). No vertex of L_i is adjacent to u_i so $|L_i| \leq n - \delta$ for each $i \in [k-1]$ and $|\operatorname{int}_k(G)| \leq (k-1)(n-\delta)$. By Lemma 6.1(ii) there are no edges between $\bigcup_{j\neq 1} \operatorname{ext}(C_j)$ and $\operatorname{ext}(C_1)$. This means that vertices in $\operatorname{ext}(C_1)$ have neighbours in only $\operatorname{ext}(C_1)$ and $\operatorname{int}_k(G)$, so $\delta(\operatorname{ext}(C_1)) \geq \delta - |\operatorname{int}_k(G)| \geq k\delta - (k-1)n$. Hence, we have a matching M in $\operatorname{ext}(C_1)$ with $|M| = \min\left\{k\delta - (k-1)n, \left\lfloor \frac{|\operatorname{ext}(C_1)|}{2} \right\rfloor\right\}$ by Lemma 2.1(i). All copies of K_k in G containing an edge of $G[\operatorname{ext}(C_1)]$ belong to C_1 , so they are all K_{k+1} -connected. Let $i \in [k-2]$ and let f be a copy of K_{i+1} comprising an edge of $G[\operatorname{ext}(C_1)]$ and a vertex from each of L_1, \ldots, L_{i-1} . Since we have $L_1 \cup \cdots \cup L_i \subseteq \Gamma(u_{k-1}, \ldots, u_{i+1}; \operatorname{int}_k(G))$, an edge in $L_i \cap \Gamma(f)$ would form a copy of K_k in $\operatorname{int}_k(G)$ is K_k -free, so $L_i \cap \Gamma(f)$ is an independent set.

Since the requisite conditions are satisfied, we apply Lemma 5.6 with the objects $ext(C_1)$, $\bigcup_{j\neq 1} ext(C_j), L_1, \ldots, L_{k-1}, \emptyset$ and M as $U_1, U_2, X_1, \ldots, X_{k-1}, A$ and F respectively to obtain that $CKF_{k+1}(G)$ is at least

$$(k+1)\min\left\{k\delta-(k-1)n,\left\lfloor\frac{|\operatorname{ext}(C_1)|}{2}
ight
ceil,q'
ight\}$$

as required.

Now we aim to prove that if $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$ and $\operatorname{int}_k(G)$ contains no copy of K_k , then $\operatorname{int}_k(G)$ is in fact (k-1)-partite and its copies of K_{k-1} lie in at least $r_p(n, \delta + \eta n)$ K_{k+1} -components.

Lemma 6.7. Let $k \geq 3$ be an integer and let $\mu > 0$. Let $\eta > 0$ and $n \in \mathbb{N}$ satisfy $\frac{1}{n} \ll \eta \ll \mu$, $\frac{1}{k}$. Let G be a graph on n vertices with at least two K_{k+1} -components and minimum degree $\delta(G) \geq \delta \geq (\frac{k-1}{k} + \mu)n$. Suppose $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$ and $\operatorname{int}_k(G)$ is K_k -free. Then $\operatorname{int}_k(G)$ is (k-1)-partite and all copies of K_{k-1} in $\operatorname{int}_k(G)$ are contained in at least $r_p(n, \delta + \eta n)$ K_{k+1} -components of G.

Proof. Let $0 < \eta < \min\{\frac{1}{1000k^2}, \eta_0(k, \mu)\}$ and $n_1 := \max\{n_2(k, \mu, \eta), \frac{2}{\eta}\}$ with $\eta_0(k, \mu)$ and $n_2(k, \mu, \eta)$ given by Lemma 2.10. Set $r' := r_p(n, \delta + \eta n)$. Let $f := u_1 \dots u_{k-1}$ be a copy of K_{k-1} in $\operatorname{int}_k(G)$ and let C_1, \dots, C_p be the K_{k+1} -components of G.

We claim that f is a copy of K_{k-1} of every K_{k+1} -component of G. Indeed, suppose f is not a copy of K_{k-1} of C_i for some $i \in [p]$. Since $|\Gamma(f)| \ge (k-1)\delta - (k-2)n$ by Lemma 2.2 and $|C_i| > \delta$ by Lemma 6.1(i), there is a vertex $w \in \Gamma(f)$ which is also a vertex of C_i . Now since $fw \notin C_i$, we have $fw \in C_j$ for some $j \neq i$ and hence w is a vertex of C_j . Since w is a vertex of both C_i and C_j , we have $w \in \operatorname{int}_k(G)$, which in turn implies that fw is a copy of K_k in $\operatorname{int}_k(G)$, contradicting our lemma hypothesis.

For $\delta \geq \left(\frac{k}{k+1} - 2\eta\right)n$, note that by Lemma 6.2(i) we have $|\operatorname{int}_k(G)| \geq 2\delta - n + 2 > \frac{3k-4}{3}(n-\delta)$, so $\delta(G[\operatorname{int}_k(G)]) \geq \delta - n + |\operatorname{int}_k(G)| > \frac{3k-7}{3k-4} |\operatorname{int}_k(G)|$. Then, Theorem 2.7 implies that $\operatorname{int}_k(G)$ is (k-1)-partite. Furthermore, by (12) and since G has at least two K_{k+1} -components, we have that all copies of K_{k-1} in $\operatorname{int}_k(G)$ are contained in at least $r' \leq 2 K_{k+1}$ -components. Therefore, it remains to consider the case $\delta < \left(\frac{k}{k+1} - 2\eta\right)n$; by (5) we have $r' \geq 2$. For each $i \in [p]$, let U_i be the set of common neighbours v of f such that $fv \in C_i$. Since $\operatorname{int}_k(G)$ is K_k -free, we have $U_i \subseteq \operatorname{ext}(C_i)$ for each $i \in [p]$. Without loss of generality, let $\operatorname{ext}(C_1)$ be a largest K_{k+1} -component exterior of G.

Let $i \neq 1$. Applying Lemma 6.6 and noting that $|\operatorname{ext}(C_1)| \geq |\operatorname{ext}(C_i)|$, we have that $\operatorname{CKF}_{k+1}(G)$ is at least

$$(k+1)\min\left\{\left\lfloor\frac{|\operatorname{ext}(C_i)|}{2}\right\rfloor,k\delta-(k-1)n\right\}.$$

Since $(k+1)(k\delta - (k-1)n) \ge pp_k(n, \delta + \eta n)$ by (9), we find that $(k+1)\left\lfloor \frac{|\operatorname{ext}(C_i)|}{2} \right\rfloor < pp_k(n, \delta + \eta n) \le \frac{k+1}{2}\left(\frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'} - 2\right)$ by (7). Hence, we have

(54)
$$|U_i| \le |\operatorname{ext}(C_i)| < \frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'}$$

By Lemma 6.1(i) we have $|\operatorname{ext}(C_i)| + |\operatorname{int}_k(G)| \ge |C_i| > \delta$, so by (8) we have

(55)
$$|\operatorname{int}_k(G)| > \delta - \frac{(k-1)(\delta + 3\eta n) - (k-2)n}{r'} > \frac{3k-4}{3}(n-\delta)$$

It follows that $\delta(G[\operatorname{int}_k(G)]) \geq \delta - n + |\operatorname{int}_k(G)| > \frac{3k-7}{3k-4} |\operatorname{int}_k(G)|$, so Theorem 2.7 implies that $\operatorname{int}_k(G)$ is (k-1)-partite. Let I_1, \ldots, I_{k-1} be the parts of $\operatorname{int}_k(G)$. For each $j \in [k-1]$ we have that I_j is an independent set, so $|I_j| \leq n - \delta$. Hence, we have $|I_j| = |\operatorname{int}_k(G)| - \sum_{h \neq j} |I_h| > (k-1)\delta - (k-2)n - \frac{(k-1)(\delta+3\eta n) - (k-2)n}{r'}$. Furthermore, each vertex in I_j is adjacent to all but at most $n - \delta - |I_j|$ vertices outside I_j .

It remains to show $p \ge r'$, so suppose p < r'. In particular, this implies $r' \ge 3$. Since (54) and $\sum_{i \in [p]} |\operatorname{ext}(C_i)| \ge \sum_{i \in [p]} |U_i| = |\Gamma(f)| \ge (k-1)\delta - (k-2)n$ hold, we obtain $|\operatorname{ext}(C_1)| \ge |\Gamma(f)| - \sum_{i \ne 1} |\operatorname{ext}(C_i)| > \frac{2[(k-1)\delta - (k-2)n] - 3(k-1)(r'-2)\eta n}{r'}$. By Lemma 6.1(ii), there are no edges between $\operatorname{ext}(C_1)$ and $\bigcup_{i \ne 1} \operatorname{ext}(C_i)$, so every vertex in $\operatorname{ext}(C_1)$ has neighbours in $\operatorname{ext}(C_1)$ and $\operatorname{int}_k(G)$ only. Hence, we have $\delta(\operatorname{ext}(C_1)) \geq \delta - |\operatorname{int}_k(G)|$. By Lemma 2.1(i), there is a matching F_0 in $ext(C_1)$ with

$$|F_0| = \min\left\{\delta - |\operatorname{int}_k(G)|, \frac{[(k-1)\delta - (k-2)n] - 3(k-1)(r'-1)\eta n}{r'}\right\}.$$

We now build up our desired connected K_{k+1} -factor step-by-step, starting from the aforementioned matching F_0 in $ext(C_1)$. We have steps $j = 1, \ldots, k - 1$. In step j, we extend the K_{j+1} -factor F_{j-1} to a K_{j+2} -factor F_j using I_j . We greedily match vertices of I_j with distinct copies of K_{j+1} of F_{j-1} to form copies of K_{j+2} . We find that $|I_j| > |F_0| \ge |F_{j-1}|$, so we stop only when we encounter a vertex $x \in I_j$ which is not a common neighbour of any remaining copy of K_{j+1} of F_{j-1} . Since at most $n - \delta - |I_j|$ copies of K_{j+1} in F_{j-1} do not have x as a common neighbour, we obtain a K_{j+2} -factor F_j with at least $|F_{j-1}| - (n-\delta) + |I_j|$ copies of K_{i+2} .

We terminate after step k-1 with a collection F_{k-1} of at least $|F_0| - (k-1)(n-\delta) + |\operatorname{int}_k(G)|$ vertex-disjoint copies of K_{k+1} in G. Since each copy of K_{k+1} in F_{k-1} uses an edge of $F_0 \subseteq$ $G[ext(C_1)]$ and (55) holds, we deduce that F_{k-1} is in fact a connected K_{k+1} -factor of size at least $(k+1)(k\delta - (k-1)n - 3(k-1)\eta n)$. By (9), this means that $\operatorname{CKF}_{k+1}(G) \ge \operatorname{pp}_k(n, \delta + \eta n)$, which is a contradiction. This completes the proof.

We prove in the following lemma that a graph which has very high minimum degree and is not near-extremal in fact contains a large connected K_{k+1} -factor. We handle this case separately as it turns out that our greedy-type methods in Section 5.2 are insufficient. To overcome this, we employ a Hall-type argument (see Lemma 2.1(ii)) for the purposes of extending our large matchings to sufficiently large connected K_{k+1} -factors.

Lemma 6.8. Let $k \ge 2$ be an integer and let $\mu > 0$. Let $\eta > 0$ and $n \in \mathbb{N}$ satisfy $\frac{1}{n} \ll \eta \ll \mu, \frac{1}{k}$. Let G be a graph on n vertices with minimum degree $\delta = \delta(G) \ge \left(\frac{2k-1}{2k+1} - 2\eta\right)n$, exactly two K_{k+1} -components, $\operatorname{int}_k(G)$ (k-1)-partite and either $|\operatorname{int}_k(G)| < (k-1)(n-\delta) - 5k\eta n$ or the larger K_{k+1} -component exterior X satisfies $|X| > \frac{19}{10}(k\delta - (k-1)n)$. Then $\operatorname{CKF}_{k+1}(G) \geq 10^{-10}$ $pp_k(n, \delta + \eta n).$

Proof. Let $0 < \eta < \min\{\frac{1}{1000k^2}, \eta_0(k, \mu), \frac{k\mu^2}{k+1}\}$ and $n_1 := \max\{n_2(k, \mu, \eta), \frac{2}{\eta}\}$ with $\eta_0(k, \mu)$ and $n_2(k,\mu,\eta)$ given by Lemma 2.10. Let C_1 and C_2 be the two K_{k+1} -components of G. There is a partition of V(G) into the vertex classes $int_k(G), ext(C_1)$ and $ext(C_2); int_k(G)$ is further partitioned into k-1 independent sets I_1, \ldots, I_{k-2} and I_{k-1} . Without loss of generality, suppose $|\operatorname{ext}(C_1)| \geq |\operatorname{ext}(C_2)|$. Since I_i is an independent set, we have

 $|I_i| \le n - \delta$ for each $i \in [k - 1]$. (56)

If $\delta \ge \left(\frac{k}{k+1} - 2\eta\right)n$, then by Lemma 6.2(i) we have $|\operatorname{int}_k(G)| \ge 2\delta - n + 2 \ge (k-1)(n-\delta) - 2(k+1)\eta n$ and by Lemma 6.2(ii) we have $|\operatorname{ext}(C_1)| \le n - \delta - 1 \le k\delta - (k-1)n + 2(k+1)\eta n \le k\delta$ $\frac{19}{10}(k\delta - (k-1)n),$ which contradicts the lemma hypothesis. Therefore, we have $\delta < \left(\frac{k}{k+1} - 2\eta\right)n.$

In particular, this means that $r' := r_p^{(k)}(n, \delta + \eta n) \ge 2$. By (56) we have $|\operatorname{int}_k(G)| \le (k-1)(n-\delta)$. By Lemma 6.1(i) we have $|C_1| > \delta$, so $|\operatorname{ext}(C_1)| > \delta$. $\delta - (k-1)(n-\delta) = k\delta - (k-1)n \ge 0$. By Lemma 6.1(ii), there are no edges between $ext(C_1)$ and $ext(C_2)$, so every vertex in $ext(C_1)$ has neighbours in $ext(C_1)$ and $int_k(G)$ only. Hence, we have $\delta(\text{ext}(C_1)) \geq \delta - |\inf_k(G)| \geq \delta - (k-1)(n-\delta) = k\delta - (k-1)n$. Therefore, we can conclude by Lemma 2.1(i) that there is matching F_0 in $ext(C_1)$ of size $|F_0| = min\{k\delta - (k-1)n, \left|\frac{|ext(C_1)|}{2}\right|\}$.

We build up the desired connected K_{k+1} -factor step-by-step, starting from the aforementioned matching F_0 . We have steps j = 1, ..., k - 1. In step j we extend the K_{j+1} -factor F_{j-1} to a K_{j+2} -factor F_j using I_j . By Lemma 6.1(ii), there are no edges between $ext(C_1)$ and $ext(C_2)$, so every vertex in $ext(C_1)$ has at least $\delta - |ext(C_1)| - \sum_{h \neq j} |I_h|$ neighbours in I_j . For each

 $i \in [j-1]$, since I_i is an independent set, every vertex of I_i has at least $\delta - (n - |I_j| - |I_i|)$ neighbours in I_j . Therefore, by Lemma 2.2 every copy of K_{j+1} in F_{j-1} has at least

$$a_j := 2\left(\delta - |\operatorname{ext}(C_1)| - \sum_{h \neq j} |I_h|\right) + \sum_{i=1}^{j-1} (\delta - n + |I_j| + |I_i|) - j|I_j|$$
$$= (j+1)\delta - (j-1)n - 2|\operatorname{ext}(C_1)| - \sum_{i=1}^{k-1} |I_i| - \sum_{i=j+1}^{k-1} |I_i|$$

common neighbours in I_j . At the same time, since I_j is an independent set, every vertex of I_j has at least $\delta - (n - |\operatorname{ext}(C_1)| - |I_1| - \cdots - |I_j|)$ neighbours in $\operatorname{ext}(C_1) \cup I_1 \cup \cdots \cup I_j$, of which all but at most $|\operatorname{ext}(C_1)| + |I_1| + \cdots + |I_j| - (j+1)|F_{j-1}|$ are in F_{j-1} . Hence, every vertex in I_j has at least

$$b_j := \delta - (n - |\operatorname{ext}(C_1)| + |I_1| + \dots + |I_j|) - (|\operatorname{ext}(C_1)| + |I_1| + \dots + |I_{j-1}| - (j+1)|F_{j-1}|) - j|F_{j-1}| = \delta - n + |I_j| + |F_{j-1}|$$

copies of K_{j+1} of F_{j-1} in its neighbourhood. Form an auxiliary bipartite graph with vertex set $F_{j-1} \cup I_j$, where $f \in F_{j-1}$ is adjacent to $u \in I_j$ if and only if fu is a copy of K_{j+2} in G. By Lemma 2.1(ii), there is a matching in the auxiliary bipartite graph with at least $\min\{a_j + b_j, |F_{j-1}|, |I_j|\}$ edges, which corresponds to a collection F_j of

(57)
$$|F_j| = \min\{a_j + b_j, |F_{j-1}|, |I_j|\}$$

vertex-disjoint copies of K_{j+2} in G. Lemma 6.2(i) tells us $|\operatorname{int}_k(G)| \ge 2\delta - n + 2$, so by (56) we have

(58)
$$|I_j| = |\operatorname{int}_k(G)| - \sum_{h \neq j} |I_h| > k\delta - (k-1)n \ge |F_0| \ge |F_{j-1}|.$$

Observe that by (56) we have

(59)
$$a_{j} + b_{j} = (j+2)\delta - jn - 2|\exp(C_{1})| - \sum_{i=j+1}^{k-1} |I_{i}| - \sum_{i\neq j} |I_{i}| + |F_{j-1}|$$
$$\geq (2k-1)\delta - (2k-3)n - 2|\exp(C_{1})| + |F_{j-1}|.$$

Since by Lemma 6.2(ii) we have $|\exp(C_1)| \leq n - \delta - 1$ and recalling our assumption that $\delta \geq \left(\frac{2k-1}{2k+1} - 2\eta\right)n$, by (59) we have

(60)
$$a_j + b_j \ge |F_{j-1}| - 2(2k+1)\eta n$$

Furthermore, by (59) and $\delta \ge \left(\frac{2k-1}{2k+1} - 2\eta\right)n$ we obtain

(61) if
$$|\operatorname{ext}(C_1)| \le \left(\frac{2}{2k+1} - (2k-1)\eta\right)n$$
, then $a_j + b_j \ge |F_{j-1}|$ for all j .

All copies of K_k in G containing an edge of $G[ext(C_1)]$ belong to C_1 , so they are all K_{k+1} connected. Therefore, F_{k-1} is a connected K_{k+1} -factor.

It remains to check that $(k+1)|F_{k-1}| \ge pp_k(n, \delta + \eta n)$. We first consider when $|F_0| = k\delta - (k-1)n$. In this case, noting (57), (58) and (60) we have that $|F_j| \ge |F_{j-1}| - 2(2k+1)\eta n$ for each $j \in [k-1]$, so F_{k-1} is a connected K_{k+1} -factor in G of size at least $(k+1)(k\delta - (k-1)n - 2(k-1)(2k+1)\eta n) \ge pp_k(n, \delta + \eta n)$ by (9). Now consider when $|F_0| = \lfloor \frac{|\operatorname{ext}(C_1)|}{2} \rfloor$. We distinguish two cases.

Case 1: $a_j + b_j \ge |F_{j-1}|$ for each $j \in [k-1]$. In this case, F_{k-1} is a connected K_{k+1} -factor in G of size $(k+1)|F_0| = (k+1) \left\lfloor \frac{|\operatorname{ext}(C_1)|}{2} \right\rfloor$. Suppose that this is less than $\operatorname{pp}_k(n, \delta + \eta n)$. By (7) and $\delta \ge \left(\frac{2k-1}{2k+1} - 2\eta\right)n$, we have $|\operatorname{ext}(C_1)| < \frac{(k-1)(\delta+3\eta n)-(k-2)n}{2} \le \frac{19}{10}(k\delta - (k-1)n)$. Furthermore, $|\operatorname{int}_k(G)| \ge n-2|\operatorname{ext}(C_1)| > (k-1)(n-\delta) - 3(k-1)\eta n$. This contradicts the lemma hypothesis.

Case 2: $a_j + b_j < |F_{j-1}|$ for some $j \in [k-1]$. By (61), this means that $|\exp(C_1)| > \left(\frac{2}{2k+1} - (2k-1)\eta\right) n \ge 2(k\delta - (k-1)n) - (2k-1)\eta n$. By (57), (58) and (60) we have $|F_j| \ge |F_{j-1}| - 2(2k+1)\eta n$ for each $j \in [k-1]$, so F_{k-1} is a connected K_{k+1} -factor in G of size at least

$$(k+1)(|F_0| - 2(k-1)(2k+1)\eta n) \ge (k+1)(k\delta - (k-1)n - 6k^2\eta n).$$

By (9) this is at least $pp_k(n, \delta + \eta n)$.

Finally, we prove Lemma 4.3.

Proof of Lemma 4.3. Given $k \geq 3$, $\mu > 0$ and $0 < \eta < \min\{\frac{1}{1000k^2}, \eta_0(k,\mu), \frac{k\mu^2}{k+1}\}$, let $m_1 := \max\{n_2(k,\mu,\eta), 2/\eta, k(k+1)\}$ with the quantities $\eta_0(k,\mu)$ and $n_2(k,\mu,\eta)$ given by Lemma 2.10. Let $\delta \geq (\frac{k-1}{k} + \mu)n$. Let G be a graph on $n \geq m_1$ vertices with minimum degree $\delta(G) \geq \delta$ and at least two K_{k+1} -components, each of which contains a copy of K_{k+2} . Let C_1, \ldots, C_ℓ be the K_{k+1} -components of G. Set $\alpha := |\operatorname{int}_k(G)|$.

Lemma 6.2(i) tells us $\operatorname{int}_k(G) \neq \emptyset$ and $|\operatorname{int}_k(G)| > 2\delta - n > (k-2)(n-\delta)$, so $\delta(G[\operatorname{int}_k(G)]) \ge \delta - n + |\operatorname{int}_k(G)| > \frac{k-3}{k-2} |\operatorname{int}_k(G)|$. Hence, any vertex in $\operatorname{int}_k(G)$ can be extended to a copy of K_{k-1} in $\operatorname{int}_k(G)$ by Lemma 2.3. In particular, $\operatorname{int}_k(G)$ contains a copy of K_{k-1} .

Suppose that (D1) does not hold. Lemma 6.3 tells us that $G[\operatorname{int}_k(G)]$ is K_k -free, so Lemma 6.7 implies that $\operatorname{int}_k(G)$ is (k-1)-partite and all copies of K_{k-1} in $G[\operatorname{int}_k(G)]$ (of which there is at least one) are contained in at least $r' := r_p(n, \delta + \eta n) K_{k+1}$ -components. Hence, G has at least $r' K_{k+1}$ -components. Since $\operatorname{int}_k(G)$ is (k-1)-partite, we have $\alpha \leq (k-1)(n-\delta)$. Lemma 6.1(i) tells us that $|C_i| > \delta$, so

(62)
$$|\operatorname{ext}(C_i)| \ge \delta - \alpha + 1 \ge k\delta - (k-1)n + 1$$

for each $i \in [\ell]$. In particular, every K_{k+1} -component has a non-empty exterior. Pick $x \in ext(C_2)$. It has at least δ neighbours, none of which are in $ext(C_1) \cup ext(C_3) \cup \cdots \cup ext(C_\ell)$ by Lemma 6.1(ii). Observe that

(63)
$$n = |\operatorname{int}_k(G)| + |\operatorname{ext}(C_1)| + \dots + |\operatorname{ext}(C_\ell)|$$
 and

(64)
$$n \ge 1 + \delta + |\exp(C_1)| + |\exp(C_3)| + \dots + |\exp(C_\ell)|.$$

Without loss of generality, suppose $ext(C_1)$ is a largest K_{k+1} -component exterior. By Lemma 6.1(ii), there are no edges between any pair of K_{k+1} -component exteriors. Note that for any K_{k+1} -component C, all copies of K_k in G containing at least one vertex of ext(C) are in C and are therefore K_{k+1} -connected in G. Hence, it is enough to prove that

$$\alpha \ge (k-1)(n-\delta) - 5k\eta n \text{ and } |\operatorname{ext}(C_1)| \le \frac{19}{10}(k\delta - (k-1)n),$$

as this would imply that (D2) holds. Suppose this is not the case.

Claim 6.9. G has exactly $r' K_{k+1}$ -components.

Proof. Suppose that $\ell \ge r' + 1$. By (63) we have $(r' + 1)(\delta - \alpha) + \alpha < n$. We consider two cases. Case 1: $\alpha < (k - 1)(n - \delta) - 5k\eta n$. Then we have

$$(r'+1)\delta < n + r'\alpha < n + r'((k-1)(n-\delta) - 5k\eta n)$$

= [(k-1)r'+1]n - (k-1)r'\delta - (kr'+1)\eta n - (4kr'-1)\eta n,

which we rearrange to obtain

$$\delta + \eta n < \frac{[(k-1)r'+1]n - (4kr'-1)\eta n}{kr'+1}$$

Comparing this with (4) applied to $r' := r_p(n, \delta + \eta n)$, we deduce $r' > (4kr' - 1)\eta n \ge 4kr' - 1 \ge r'$, which is a contradiction.

Case 2: $|\exp(C_1)| > \frac{19}{10}(k\delta - (k-1)n)$. By (62) and (64), we have

$$1 + \delta + \frac{19}{10}(k\delta - (k-1)n) + (r'-1)[k\delta - (k-1)n + 1] \le n,$$

which we simplify to

$$\frac{9}{10}(k\delta - (k-1)n) + r'[k\delta - (k-1)n] < n - \delta.$$

Since by (3) we have $r' \ge \frac{n-\delta-\eta n}{k\delta-(k-1)n+k\eta n+1}$, we deduce that

$$\frac{9}{10}(k\delta - (k-1)n) + \frac{n-\delta - \eta n}{k\delta - (k-1)n + k\eta n + 1}[k\delta - (k-1)n] < n-\delta.$$

Since $\eta < \frac{k\mu^2}{k+1}$ and $k\delta - (k-1)n \ge k\mu n$, we have

$$\begin{aligned} &(k\delta - (k-1)n + k\eta n + 1)(1 - \mu) \\ &< k\delta - (k-1)n + (k+1)\eta n - \mu(k\delta - (k-1)n) \\ &\le k\delta - (k-1)n + (k+1)\eta n - k\mu^2 n \\ &< k\delta - (k-1)n, \end{aligned}$$

so applying this to the previous inequality, we obtain

$$\frac{9}{10}k\mu n + (n - \delta - \eta n)(1 - \mu) < n - \delta.$$

However, since $\eta < \mu$ and $n - \delta < \frac{n}{k}$, this is a contradiction. Therefore, G has exactly $r' K_{k+1}$ -components.

In particular, this means that $r' \ge 2$. For r' = 2, Lemma 6.8 gives a contradiction, so it remains to consider the case $r' \ge 3$. First suppose $|\exp(C_1)| \le \sum_{h \ne 1} |\exp(C_h)|$. By Lemma 6.6, we have

$$\operatorname{CKF}_{k+1}(G) \ge (k+1) \min\left\{ \left\lfloor \frac{|\operatorname{ext}(C_1)|}{2} \right\rfloor, k\delta - (k-1)n \right\}$$

We have $\operatorname{CKF}_{k+1}(G) < \operatorname{pp}_k(n, \delta + \eta n)$ by assumption, so by (9) we have

$$(k+1)\left\lfloor \frac{|\operatorname{ext}(C_1)|}{2} \right\rfloor < \operatorname{pp}_k(n,\delta+\eta n).$$

Hence, by (7) and (9) we obtain $|\operatorname{ext}(C_1)| < \frac{(k-1)(\delta+3\eta n)-(k-2)n}{r'}$ and $|\operatorname{ext}(C_1)| \leq \frac{19}{10}(k\delta - (k-1)n)$. Then, by (63) we have $|\operatorname{int}_k(G)| = n - \sum_{i \in [r']} |\operatorname{ext}(C_i)| > (k-1)(n-\delta) - 3(k-1)\eta n$. This is contradicts our earlier supposition, so we have that

(65)
$$|\operatorname{ext}(C_1)| > \sum_{h \neq 1} |\operatorname{ext}(C_h)|$$

Set $r := r_p(n, \delta)$. By (62) and (64), we have that

$$1 + \delta + (r' - 1)(k\delta - (k - 1)n + 1) + (r' - 2)(k\delta - (k - 1)n + 1) \le n.$$

Rearranging and applying (3), we obtain

$$2r' - 3 \le \frac{n - \delta - 1}{k\delta - (k - 1)n + 1} < r \le r' + 1,$$

which gives r = r' + 1 = 4. In particular, by (4) we have $\delta \ge \left(\frac{3k-2}{3k+1} - 2\eta\right)n$. By (62) and (65), we have $|\exp(C_1)| > |\exp(C_2)| + |\exp(C_3)| \ge 2(k\delta - (k-1)n + 1)$. By (64) and the fact that $\delta \ge \left(\frac{3k-2}{3k+1} - 2\eta\right)n$, we have $|\exp(C_1)| < n - \delta - |\exp(C_3)| \le 2[k\delta - (k-1)n] + 2(3k+1)\eta n$. Finally, by Lemma 6.6 and (9), we have

$$\operatorname{CKF}_{k+1}(G) \ge (k+1) (k\delta - (k-1)n - 2(3k+1)\eta n) \ge \operatorname{pp}_k(n, \delta + \eta n),$$

which is a contradiction. This completes the proof of Lemma 4.3.

7. Near-extremal graphs

In this section we provide our proof of Lemma 3.4. To this end, we start with two useful lemmas. The first lemma will be used to construct kth powers of paths and cycles from simple paths and cycles through repeated application.

Lemma 7.1. Given $h \in \mathbb{N}$, $h' \in [h+1]$ and a graph G, let $T = t_1 \dots t_{(h+1)\ell+h'-1}$ be the hth power of a path in G and let W be a set of vertices disjoint from T. Let $Q_1 := t_1 \dots t_{h+1}$, $Q_i := t_{(h+1)(i-2)+1} \dots t_{(h+1)i}$ for each $1 < i \leq \ell$, and $Q_{\ell+1} := t_{(h+1)\ell-h} \dots t_{(h+1)\ell+h'-1}$. If there exists a permutation σ of $[\ell+1]$ such that for each $i \in [\ell+1]$ the vertices of $Q_{\sigma(i)}$ have at least i common neighbours in W, then there is the (h+1)st power of a path

$$(q_1t_1\dots t_{h+1})\dots (q_\ell t_{(h+1)\ell-h}\dots t_{(h+1)\ell})(q_{\ell+1}t_{(h+1)\ell+1}\dots t_{(h+1)\ell+h'-1})$$

in G, with $q_i \in W$ for each $i \in [\ell + 1]$, using every vertex of T. If T is a cycle on $(h + 1)\ell$ vertices we let instead $Q_1 := t_{(h+1)\ell-h} \dots t_{(h+1)\ell} t_1 \dots t_{h+1}$, $Q_i := t_{(h+1)(i-2)+1} \dots t_{(h+1)i}$ for each $1 < i \leq \ell$ and σ be a permutation on $[\ell]$. Then, under the same conditions, we have the (h+1)st power of a cycle $C_{(h+2)\ell}^{h+1}$.

Proof. Choose for each *i* in succession $q_{\sigma(i)}$ to be any so far unused common neighbour of $Q_{\sigma(i)}$. The lemma hypothesis ensures that this is always possible.

The second lemma allows us to construct paths and cycles of desired lengths which keep certain 'bad' vertices far apart. We apply Theorem 2.8 in its proof.

Lemma 7.2. Let H be a graph on $h \ge 10$ vertices and $B \subseteq V(H)$ be of size at most $\frac{h}{12}$. Suppose that every vertex in B has at least 3|B| + 1 neighbours in H, and every vertex outside B has at least $\frac{h}{2} + 2|B| + 2$ neighbours in H. Then for any $3 \le \ell \le h$ we can find a cycle C_{ℓ} of length ℓ in H on which no four consecutive vertices contain more than one vertex of B. Furthermore, if x and y are any two vertices not in B and $5 \le \ell \le h$, we can find an ℓ -vertex path P_{ℓ} whose end-vertices are x and y and on which no four consecutive vertices contain more than one vertex of $B \cup \{x, y\}$.

Proof. If we seek a path in H from x to y and $xy \notin E(H)$, add xy as a 'dummy' edge. If we seek a cycle, let xy be any edge of H such that $x, y \notin B$. Hence, it suffices to show for each $3 \leq \ell \leq h$ and each edge $xy \in V(H)$ with $x, y \notin B$ that we can find a cycle C_{ℓ} of length ℓ with xy as an edge, on which no four consecutive vertices contain more than one vertex of B and on which any four consecutive vertices including a vertex of B contain neither x nor y.

Let $H_1 := H[V(H) \setminus B]$. Since H_1 is a graph on $h - |B| \ge 4$ vertices with minimum degree $\delta(H_1) \ge \frac{h}{2} + |B| + 2 \ge \frac{h-|B|}{2} + 1$, by Theorem 2.8 H_1 is panconnected. Hence, H_1 has paths between x and y of every number of vertices from 3 to h - |B|. By adding the edge xy to these paths, we obtain cycles of every length from 3 to h - |B| with the desired properties.

To find the required cycles of length greater than h - |B|, we first construct a path P in H covering B with x as an end-vertex and xy as an edge. Let $B = \{b_1, \ldots, b_{|B|}\}$ and set $B' := B \cup \{x, y\}$. For each $i \in [|B|]$ choose distinct vertices $u_{i+1}, v_i \in V(H) \setminus B'$ adjacent to b_i . Every vertex in B has at least 3|B| + 1 neighbours in H, so we may pick these vertices to be distinct for all $i \in [|B|]$. Choose a different vertex $u_1 \in V(H) \setminus B'$ adjacent to y. We can do so as y has at least $\frac{h}{2} + 2|B| + 2$ neighbours in H and $h \geq 12|B|$. Let $i \in [|B|]$. Both u_i and v_i have $\frac{h}{2} + 2|B| + 2$ neighbours in H, so they have at least 4|B| + 4 common neighbours. At most 3|B| + 3 of these are in $B \cup \{x, y, u_1, \ldots, u_{|B|+1}, v_1, \ldots, v_{|B|}\}$, so we can find a thus far unused vertex w_i adjacent to u_i and v_i . We may pick the vertices $w_1, \ldots, w_{|B|}$ greedily as we require only |B| vertices. Hence, we obtain a path

$$P = xyu_1w_1v_1b_1u_2w_2v_2b_2\dots v_{|B|}b_{|B|}u_{|B|+1}$$

on 4|B| + 3 vertices. Notably, any cycle containing P of length at least 4|B| + 5 has the desired properties.

Let $H_2 := H[V(H) \setminus (V(P) \setminus \{x, u_{|B|+1}\})]$. Since H_2 is a graph on $h - 4|B| - 1 \ge 4$ vertices with minimum degree $\delta(H_2) \ge \frac{h}{2} - 2|B| + 1 \ge \frac{h-4|B|-1}{2} + 1$, by Theorem 2.8 H_2 is panconnected. Hence, H_2 has paths between x and $u_{|B|+1}$ of every number of vertices from 4 to h - 4|B| - 1. By adding the path P to these paths, we obtain cycles of every length from 4|B| + 5 to h with the desired properties.

Before providing the proof of Lemma 3.4 we first give an outline of our method. Recall that the Lemma is given a Szemerédi partition with a 'near-extremal' structure. We shall show that the underlying graph either also has a 'near-extremal' structure, or possesses features which lead to longer kth powers of paths and cycles than required for the conclusion of the Lemma. The complication we encounter is the insensitivity of the Szemerédi partition to misassignment of sublinearly many vertices and to editing of subquadratically many edges.

Recall that the sets I_i are subsets of V(R) and the elements of each set I_i correspond to clusters in V(G). We shall denote by $\bigcup I_i$ the union of the elements of the set I_i as clusters in V(G). We begin by separating those vertices with 'few' neighbours in $\bigcup I_i$, which we collect in a set W_i , from those with 'many' for each $i \in [k-1]$. We then show that if there are two vertex-disjoint edges in W_i , then the sets $\bigcup B_1$ and $\bigcup B_2$ 'belong' to the same K_{k+1} -component of G. We shall show that this enables us to construct very long kth powers of paths and cycles by applying Lemma 3.2.

Hence we may assume that W_i does not contain two vertex-disjoint edges, so W_i is almost independent with 'near-extremal' size. $W = \bigcup_{i=1}^{k-1} W_i$ now resembles a 'near-extremal' interior and the minimum degree condition on G will guarantee that almost every edge from W to $V(G) \setminus W$ is present. At this point, we would like to say that we can find a long path outside Wwith sufficiently nice properties (which we need because the bipartite graph $G[W, V(G) \setminus W]$ is unfortunately not actually complete) so that we can repeatedly apply Lemma 7.1 to extend it to the *k*th power of a path (and similarly for powers of cycles) using vertices from W. The purpose of Lemma 7.2 is precisely to provide paths and cycles with those nice properties. The rest of the proof then focuses on establishing the right conditions for the application of Lemma 7.2 and working out the details of the various applications of Lemma 7.1.

Proof of Lemma 3.4. Given an integer $k \geq 3$ and $0 < \nu < 1$ let $\eta > 0$ and d > 0 satisfy

(66)
$$\eta \le \frac{\nu^4}{(k+1)^{13}10^8} \text{ and } d \le \frac{\nu^4}{(k+1)^{13}10^8}$$

Given $k \geq 3$ and d > 0, Lemma 3.2 returns a constant $\varepsilon_{EL} > 0$. Set

(67)
$$\varepsilon_0 := \min\left\{\varepsilon_{EL}, \frac{\nu^4}{(k+1)^{13}10^8}\right\}.$$

Given m_{EL} and $0 < \varepsilon < \varepsilon_0$, Lemma 3.2 returns a constant n_{EL} . Given t = k and $\rho = \varepsilon^{1/2}$, Theorem 2.9 returns a constant $n_{ES} \in \mathbb{N}$. Set

(68)
$$N := \max\left\{n_{EL}, \nu^{-1}n_{ES}, 100m_{EL}^{k+2}, 100(k+1)\eta^{-1}\nu^{-1}\right\}.$$

Let $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn}{k+1}\right)$. Let G, R, and the partition $V(R) = \left(\bigcup_{i=1}^{k-1} I_i\right) \cup \left(\bigcup_{j=1}^{\ell} B_j\right)$ satisfy conditions (i)–(iii) of the lemma.

Note that the specific case of finding P_n^k in G when $\delta \geq \frac{kn-1}{k+1}$ is settled by Corollary 2.6. Therefore, by (13) and (14) it would be sufficient to find

11...

(69)
$$k$$
th powers of cycles and paths of all lengths up to $\frac{11n}{20}$.

R is an (ε, d) -reduced graph of G, so

(70)
$$\delta(R) \ge \delta' := \left(\frac{\delta}{n} - d - \varepsilon\right) m.$$

Moreover, by (ii) for each $i \in [k-1]$ clusters in I_i have δ' neighbours outside I_i in R, so

(71)
$$|I_i| \le m - \delta' = \left(1 - \frac{\delta}{n} + d + \varepsilon\right) m$$

Set $I_J := \bigcup_{i \in J} I_i$ for each $J \subseteq [k-1]$. By (iii) each cluster $C \in B_j$ has neighbours only in $B_j \cup I_{[k-1]}$ in R, so by (71) we have $\delta' \leq \deg(C) = \deg(C, B_j \cup I_{[k-1]}) \leq \deg(C, B_j) + |I_{[k-1]}| \leq \deg(C, B_j) + (k-1)(m-\delta')$. Then, by (70) we have

$$|B_j| > \deg(C, B_j) \ge k\delta' - (k-1)m$$
$$\ge \frac{m}{n}(k\delta - (k-1)n - k(d+\varepsilon)n).$$

Since $\delta \ge \left(\frac{k-1}{k} + \nu\right) n$, we have $k\delta - (k-1)n \ge k\nu n$; hence, by (66) and (67) we obtain

(72)
$$|B_j| \ge \frac{38(k\delta - (k-1)n)m}{39n} \ge \frac{38k\nu m}{39}.$$

Set $\xi := \sqrt[4]{d + \varepsilon + 6k\eta}$. By (66) and (67), we have

(73)
$$\xi \le \frac{\nu}{50(k+1)^3}.$$

For each $i \in [k-1]$ define W_i to be the set of vertices of G with no more than ξn neighbours in $\bigcup I_i$. Since $\xi > d + \varepsilon$, the independence of I_i and the definition of an (ε, d) -regular partition implies that $\bigcup I_i \subseteq W_i$. Set $W_J := \bigcup_{i \in J} W_i$ and $I_J^* := \bigcup_{i \in J} (\bigcup I_i)$ for each $J \subseteq [k-1]$. Note that by (71) and (ii) we have

(74)
$$|I_J| \ge |I_{[k-1]}| - (k-1-|J|)(m-\delta') \\\ge \frac{m}{n} |J|(n-\delta) - 5k\eta m - (k-1-|J|)(d+\varepsilon)m$$

for each $J \subseteq [k-1]$. Hence, we have

(75)
$$|I_J^*| \ge \frac{(1-\varepsilon)n}{m} |I_J|$$
$$\ge |J|(n-\delta) - 5k\eta n - (k-|J|-1)(d+\varepsilon)n - \varepsilon n$$

for each $J \subseteq [k-1]$.

The claim below states that if there are two vertex-disjoint edges in some W_i , then we have two vertex-disjoint copies of K_k on $W_{[k-1]}$.

Claim 7.3. Suppose that for some $i \in [k-1]$ there are two vertex-disjoint edges in W_i . Then, there are two vertex-disjoint copies of K_k in $W_{[k-1]}$ each comprising two vertices of W_i and a vertex of W_h for each $h \in [k-1] \setminus \{i\}$.

Proof. We consider the i = 1 case and note that an analogous argument applies for each $i \neq 1$. We prove the following statement for all $2 \leq j \leq k$ by backwards induction on j. If there are two vertex-disjoint copies of K_j on $W_{[j-1]}$ each comprising two vertices of W_1 and a vertex of W_h for each 1 < h < j, then there are two vertex-disjoint copies of K_k on $W_{[k-1]}$ each comprising two vertices of W_1 and a vertex of W_h for each 1 < h < k. Setting j = 2 then gives our desired statement for the i = 1 case.

The statement is trivially true for j = k. Consider $2 \leq j < k$. Let $u_1 \ldots u_j$ and $u'_1 \ldots u'_j$ be two vertex-disjoint copies of K_j on $W_{[j-1]}$ with $u_1, u'_1 \in W_1$ and $u_{i+1}, u'_{i+1} \in W_i$ for each $i \in [j-1]$. By definition, u_1 and u'_1 each have at most ξn neighbours in $I^*_{\{1\}}$ and u_{i+1} and u'_{i+1} each have at most ξn neighbours in $I^*_{\{i\}}$ for each $i \in [j-1]$. Then, by (66), (67), (73) and (75) we have

$$\begin{aligned} &\deg(u_1, \dots, u_j; W_j) \\ &\geq \sum_{i \in [j-1]} \left(\delta - n + |W_j| + \left| I_{\{i\}}^* \right| - \xi n \right) + \left(\delta - n + |W_j| + \left| I_{\{1\}}^* \right| - \xi n \right) \\ &- (j-1)|W_j| \\ &\geq -j(n-\delta) + |W_j| + \left| I_{[j-1]}^* \right| + \left| I_{\{1\}}^* \right| - j\xi n \\ &\geq -j(n-\delta) + \left| I_{[j]}^* \right| + \left| I_{\{1\}}^* \right| - j\xi n \\ &\geq n - \delta - 10k\eta n - (j+1)(k-1)(d+\varepsilon)n - j\xi n > 1. \end{aligned}$$

An analogous argument gives

$$\deg(u'_1, \dots, u'_j; W_j)$$

$$\geq n - \delta - 10k\eta n - (j+1)(k-1)(d+\varepsilon)n - j\xi n > 1.$$

Hence, we have two vertices $u_{j+1} \in \Gamma(u_1, \ldots, u_j; W_j)$ and $u'_{j+1} \in \Gamma(u'_1, \ldots, u'_j; W_j)$ which are distinct. Notice that $u_1 \ldots u_{j+1}$ and $u'_1 \ldots u'_{j+1}$ are two vertex-disjoint copies of K_{j+1} on $W_{[j]}$ each comprising two vertices of W_1 and a vertex of W_h for each $1 < h \leq j$, so by the inductive hypothesis there are two vertex-disjoint copies of K_k on $W_{[k-1]}$ each comprising two vertices of W_1 and a vertex of W_h for each $1 < h \leq j$.

Now suppose that for some $i \in [k-1]$ we have a copy $u_1 \ldots u_k$ of K_k on $W_{[k-1]}$ with two vertices of W_i and a vertex of W_h for each $h \in [k-1] \setminus \{i\}$. We shall consider the i = 1 case and note that for each $i \neq 1$ an analogous version of the following argument applies. Without loss of generality, let $u_1 \in W_1$ and $u_{i+1} \in W_i$ for $i \in [k-1]$. We shall count the common neighbours of $u_1 \ldots u_k$ outside $I^*_{[k-1]}$. By definition u_1 has at most ξn neighbours in $I^*_{\{1\}}$ and u_{i+1} has at most ξn neighbours in $I^*_{\{i\}}$ for each $i \in [k-1]$. Then, (66), (67), (73), (75) and the fact that $k\delta - (k-1)n \geq k\nu n$ imply that $u_1 \ldots u_k$ has at least

(76)
$$\sum_{i \in [k-1]} \left(\delta - \left| I_{[k-1]}^* \right| + \left| I_{\{i\}}^* \right| - \xi n \right) + \left(\delta - \left| I_{[k-1]}^* \right| + \left| I_{\{1\}}^* \right| - \xi n \right) \\ - (k-1) \left(n - \left| I_{[k-1]}^* \right| \right) \\ \ge (k-1)\delta - (k-2)n - \frac{k\delta - (k-1)n}{48}.$$

common neighbours outside $I_{[k-1]}^*$. Now the following claim tells us that we are done if we can find two vertex-disjoint copies of K_k which satisfy (76).

Claim 7.4. Suppose that $u_1 \ldots u_k$ and $u'_1 \ldots u'_k$ are vertex-disjoint copies of K_k in G such that each of them has at least $(k-1)\delta - (k-2)n - \frac{k\delta - (k-1)n}{48}$ common neighbours outside $I^*_{[k-1]}$. Then G contains $P^k_{pp_k(n,\delta)}$ and C^k_ℓ for each $\ell \in [k+1, pc_k(n,\delta)]$ such that $\chi(C^k_\ell) \leq k+2$.

Proof. Let D' be the set of clusters $C \in V(R) \setminus I_{[k-1]}$ such that $u_1 \ldots u_k$ has at most $\frac{2dn}{m}$ common neighbours in C. By the hypothesis, $u_1 \ldots u_k$ has at least $(k-1)\delta - (k-2)n - \frac{k\delta - (k-1)n}{48}$ common neighbours outside $I_{[k-1]}^*$. Of these, at most εn are in the exceptional set V_0 of the regular partition, and at most $\frac{2dn|D'|}{m}$ are in $\bigcup D'$. The remaining common neighbours all lie in $\bigcup (V(R) \setminus (I_{[k-1]} \cup D'))$, so by (ii) we have the inequality

$$(k-1)\delta - (k-2)n - \frac{k\delta - (k-1)n}{48} - \varepsilon n - \frac{2dn|D'|}{m}$$

$$\leq (m - |I_{[k-1]}| - |D'|)\frac{n}{m} \leq n - (k-1)(n-\delta) + 5k\eta n - |D'|\frac{n}{m}$$

Simplifying this, we obtain

$$(1-2d)\frac{n}{m}|D'| \le \varepsilon n + 5k\eta n + \frac{k\delta - (k-1)n}{48},$$

and so by (66) and (67) we have $|D'| \leq \frac{(k\delta - (k-1)n)m}{40n}$. Now let D be the set of clusters $C \in V(R) \setminus I_{[k-1]}$ such that either $u_1 \dots u_k$ or $u'_1 \dots u'_k$ has at most $\frac{2dn}{m}$ common neighbours in C. Since the same analysis holds for $u'_1 \ldots u'_k$, we obtain

(77)
$$|D| \le \frac{(k\delta - (k-1)n)m}{20n}.$$

We now show that there is a copy $X_1 \ldots X_{k-2}$ of K_{k-2} in R such that $X_j \in I_j \setminus D$ for each $j \in [k-2]$. In fact, we prove the following statement for all $i \in [k-2]$ by backwards induction on i: there is a copy $X_1 \dots X_i$ of K_i in R such that $X_j \in I_j \setminus D$ for each $j \in [i]$. Setting i = k-2then gives the desired statement.

Consider i = 1. From (74) and (77) we conclude that

$$|I_1 \setminus D| \ge \frac{m}{n} \left(n - \delta - 5k\eta n - (k-2)(d+\varepsilon)n - \frac{k\delta - (k-1)n}{20} \right)$$
$$\ge \frac{m}{n} \left(n - \delta - \frac{k\delta - (k-1)n}{10} \right) > 0,$$

so we may choose $X_1 \in I_1 \setminus D$. Now consider $1 < i \leq k - 2$. By the induction hypothesis, there is a copy $X_1 \dots X_{i-1}$ of K_{i-1} such that $X_j \in I_j \setminus D$ for each $j \in [i-1]$. By (ii) I_j is an independent set for each $j \in [i-1]$, so $\Gamma(X_1, \ldots, X_{i-1}) \cap I_{[i-1]} = \emptyset$. Then applying Lemma 2.2, (66), (67), (70), (74) and (77), we obtain

$$\deg(X_1, \dots, X_{i-1}; I_i) \ge \deg(X_1, \dots, X_{i-1}) - m + |I_{[i]}|$$

$$\ge |I_{[i]}| - (i-1)(m-\delta')$$

$$\ge \frac{m}{n}((k-1)(n-\delta) - 5k\eta n) - (k-2)(m-\delta')$$

$$= \frac{m}{n}(n-\delta - (k-2)(d+\varepsilon)n - 5k\eta n)$$

$$\ge \frac{(k\delta - (k-1)n)m}{2n} > |D|,$$

so we may pick $X_i \in \Gamma(X_1, \ldots, X_{i-1}) \cap (I_i \setminus D)$. Then, $X_1 \ldots X_i$ is a copy of K_i such that $X_j \in I_j \setminus D$ for each $j \in [i]$, concluding our inductive proof.

Hence, there is a copy $X_1 \ldots X_{k-2}$ of K_{k-2} such that $X_j \in I_j \setminus D$ for each $j \in [k-2]$. By (ii) I_j is an independent set for each $j \in [k-1]$, so $\Gamma(X_1, \ldots, X_{k-2}) \cap I_{[k-2]} = \emptyset$. Now by Lemma 2.2, (66), (67), (70), (72), (74) and (77), we have

$$\begin{aligned} &\deg(X_1, \dots, X_{k-2}; B_1) \\ &\ge \deg(X_1, \dots, X_{k-2}) - m + |B_1| + |I_{[k-2]}| \\ &\ge |B_1| + |I_{[k-2]}| - (k-2)(m-\delta') \\ &\ge |B_1| + \frac{m}{n}((k-1)(n-\delta) - 5k\eta n) - (k-1)(m-\delta') \\ &= |B_1| - (5k\eta + (k-1)(d+\varepsilon))m \\ &\ge \frac{(k\delta - (k-1)n)m}{2n} > |D|, \end{aligned}$$

so we may pick $X \in \Gamma(X_1, \ldots, X_{k-2}) \cap (B_1 \setminus D)$. By Lemma 2.2, (iii), (66), (67), (70), (71) and (77) we have

$$\deg(X_1, \dots, X_{k-2}, X; B_1) \ge \deg(X_1, \dots, X_{k-2}) - |I_{k-1}|$$
$$\ge k\delta' - (k-1)m$$
$$\ge \frac{(k\delta - (k-1)n)m}{2n} > |D|,$$

so we may pick $Y \in \Gamma(X_1, \ldots, X_{k-2}, X) \cap (B_1 \setminus D)$. An analogous argument tells us that we may pick $X' \in \Gamma(X_1, \ldots, X_{k-2}) \cap (B_2 \setminus D)$ and $Y' \in \Gamma(X_1, \ldots, X_{k-2}, X') \cap (B_2 \setminus D)$. Therefore, we have copies $X_1 \ldots X_{k-2}XY$ and $X_1 \ldots X_{k-2}X'Y'$ of K_k such that $X_j \in I_j \setminus D$ for each $j \in [k-2]$, $X, Y \in B_1 \setminus D$ and $X', Y' \in B_2 \setminus D$.

Since $\delta_R(B_1), \delta_R(B_2) \geq \delta' - |I|$ and $|B_i| > \delta_R(B_i)$ for all $i \in [2]$, by Lemma 2.1(i) we can find a matching $F_2 := M$ in $R[B_1 \cup B_2]$ with $\delta' - |I|$ edges. Using a step-by-step process with steps $1, \ldots, k - 1$, we will extend the edges of F_2 to copies of K_{k+1} each consisting of an edge of F_2 and exactly one vertex from each I_i . The final collection of copies of K_{k+1} will have size at least $k\delta' - (k-1)m - 5k\eta m$. Let $i \in [k-1]$ and let F_{i+1} be the set of at least $i\delta' - (i-1)m - \sum_{h=i}^{k-1} |I_h| - 5k\eta m$ vertex-disjoint copies of K_{i+1} which we have immediately before step i. Every cluster in I_i has at most $m - |I_i| - \delta'$ non-neighbours outside I_i . Hence, every cluster in $|I_i|$ forms a copy of K_{i+2} with at least $|F_{i+1}| - (m - |I_i| - \delta') \geq (i+1)\delta' - im - \sum_{h=i+1}^{k-1} |I_h| - 5k\eta m$ copies of K_{i+1} of F_{i+1} . Therefore, we may choose greedily clusters in I_i to obtain a set F_{i+2} of at least

$$\min\left\{ (i+1)\delta' - im - \sum_{h=i+1}^{k-1} |I_h| - 5k\eta m, |I_i| \right\}$$

$$\geq (i+1)\delta' - im - \sum_{h=i+1}^{k-1} |I_h| - 5k\eta m$$

vertex-disjoint copies of K_{i+2} formed from copies of K_{i+1} of F_{i+1} and clusters of I_i . After step k-1, we have a set $T := F_{k+1}$ of at least $k\delta' - (k-1)m - 5k\eta m$ vertex-disjoint copies of K_{k+1} each comprising an edge of M and a vertex from I_i for each $i \in [k-1]$. Let T_1 be the collection of the copies of K_{k+1} of T contained in $B_1 \cup I_{[k-1]}$ and T_2 the collection of those contained in $B_2 \cup I_{[k-1]}$. By (iii), all the copies of K_{k+1} in T_1 are in the same K_{k+1} -component as $X_1 \ldots X_{k-2}XY$ and all the copies of K_{k+1} in T_2 are in the same K_{k+1} -component as $X_1 \ldots X_{k-2}X'Y'$.

Apply Lemma 3.2 with $X_{i,j} = X_j$ for $(i,j) \in [2] \times [k-2]$ and $X_{i,k-1} = X, X_{i,k} = Y$ for $i \in [2]$ to find the kth power of a path starting with $u_1 \dots, u_k$ and ending with $u'_1 \dots u'_k$ using the copies of K_{k+1} in T_1 . Similarly, apply Lemma 3.2 with $X_{i,j} = X_j$ for $(i,j) \in [2] \times [k-2]$, $X_{i,k-1} = X', X_{i,k} = Y'$ for $i \in [2]$ and A as the set of vertices of the kth power of a path we have above which are not in $\bigcup T_1$, to find the kth power of a path starting with $u'_1 \dots, u'_k$ and ending with $u_1 \dots u_k$ using the copies of K_{k+1} in T_2 , intersecting the first only at $u_1, \dots, u_k, u'_1, \dots, u'_k$. Choosing appropriate lengths for these kth power of paths and concatenating them yields the kth power of a cycle C_ℓ^k for any $6(k+1)m^{k+1} \leq \ell \leq (k+1)(1-d) (k\delta' - (k-1)m - 5k\eta m) \frac{n}{m}$. Applying Lemma 3.2 to a copy of K_{k+2} in a K_{k+1} -component directly yields C_ℓ^k for each $k+1 \leq \ell \leq (k+1)(1-d) \frac{n}{m}$ such that $\chi(C_\ell^k) \leq k+2$. By (66) and (68) we have $(k+1)(1-d) \frac{n}{m} \geq 6(k+1)m^{k+1}$, and by (66), (67) we have $(k+1)(1-d) (k\delta' - (k-1)m - 5k\eta m) \frac{n}{m} \geq p_k(n,\delta)$. It follows that G contains C_ℓ^k for each $\ell \in [k+1, pc_k(n,\delta)]$ such that $\chi(C_\ell^k) \leq k+2$. For the case $\delta \in \left[\left(\frac{k-1}{k} + \nu \right) n, \frac{kn-1}{k+1} \right]$, note that by (66), (67) we have $(k+1)(1-d) (k\delta' - (k-1)m - 5k\eta m) \frac{n}{m} \geq p_k(n,\delta)$.

By Claim 7.3 and (76), if we can find two vertex-disjoint edges in some W_i , then we are done by Claim 7.4. Hence, we assume in the following that W_i does not contain two vertexdisjoint edges for each $i \in [k-1]$. This means that for each $i \in [k-1]$ there are two vertices in W_i which meet every edge in W_i . For each $i \in [k-1]$ let W'_i be W_i without these two vertices. Since neither of these two vertices has more than ξn neighbours in $I^*_{\{i\}} \subseteq W_i$, while $|I_i| \geq \frac{m}{k+1} - 5k\eta m - (k-2)(d+\varepsilon)m$ by (74) and because $\delta < \frac{kn}{k+1}$, there is a vertex in W_i adjacent to no vertex of W_i . By (75) we conclude that

(78)
$$\begin{aligned} |J|(n-\delta) - 5k\eta n - (k-1-|J|)(d+\varepsilon)n - \varepsilon n\\ \leq |I_J^*| \leq |W_J| \leq |J|(n-\delta) \end{aligned}$$

for each $J \subseteq [k-1]$. Set $W := W_{[k-1]}$. For each $i \in [k-1]$ the total number of non-edges between W_i and $V(G) \setminus W_i$ is at most

$$|W_i||V(G) \setminus W_i| - |W_i|(\delta - 2) = |W_i|(n - \delta + 2 - |W_i|)$$

$$\leq |W_i|((k - 1)(n - \delta) + 2 - |W|).$$

Hence, by (78) the total number of non-edges between W and $V(G) \setminus W$ is at most

$$|W|((k-1)(n-\delta)+2-|W|) \leq |W|(5k\eta n+\varepsilon n+2)$$

$$\leq 5k\eta n^2+\varepsilon n^2+2n.$$

In particular, by the definition of ξ and (68), we have

(79)
$$\left|\left\{v \in V(G) \setminus W : \deg(v; W) < |W| - \xi^2 n\right\}\right| \le \xi^2 n.$$

Recall that the sets B_i are subsets of V(R) and the elements of each set B_i correspond to clusters in V(G). We shall denote by $\bigcup B_i$ the union of the elements of the set B_i as clusters in V(G). By (iii) we have $|B_i| \leq \frac{19m(k\delta - (k-1)n)}{10n}$, which together with $\delta < \frac{kn}{k+1}$, (66), (67) and (73) implies

(80)
$$\left| \bigcup B_i \right| \le \frac{19}{10} (k\delta - (k-1)n) \le \frac{19}{20} ((k-1)\delta - (k-2)n) < (k-1)\delta - (k-2)n - \xi n - (d+\varepsilon)n.$$

By (iii) and the definition of an (ε, d) -regular partition, vertices in $\bigcup B_i$ have at most $(d + \varepsilon)n$ neighbours outside of $(\bigcup B_i) \cup I^*_{[k-1]}$; hence, by $\delta(G) \ge \delta$, (78) and (80) they have more than ξn neighbours in $\bigcup I_h$ for all $h \in [k-1]$. Now the definition of W_h implies $\bigcup B_i \cap W_h = \emptyset$ for all $(i, h) \in [\ell] \times [k-1]$, so in fact

(81)
$$\bigcup B_i \cap W = \emptyset \text{ for all } i \in [\ell].$$

Furthermore, (66), (67), (73), (78) and (80) imply that $v \in \bigcup B_i$ has at least

(82)
$$\delta - |W| - (d+\varepsilon)n \ge k\delta - (k-1)n - (d+\varepsilon)n > \left|\bigcup B_i\right|/2 + 50\xi^2 m$$

neighbours in $\bigcup B_i$.

Now for each $i \in [\ell]$ let A_i be the set of vertices in $\bigcup B_i$ which are adjacent to at least $|W| - \xi^2 n$ vertices of W. By (79) we have

(83)
$$\left| \bigcup_{i \in [\ell]} \left(\bigcup B_i \right) \setminus A_i \right| \le \xi^2 n$$

Vertices which are neither in W nor in any of the sets A_i must either be in the exceptional set V_0 or in $(\bigcup B_i) \setminus A_i$ for some *i*, so we have

(84)
$$\left| V_0 \cup \bigcup_{i \in [\ell]} \left(\bigcup B_i \right) \setminus A_i \right| \le \varepsilon n + \xi^2 n < 2\xi^2 n$$

As such, (82) implies that

(85)
$$\delta(G[A_i]) \ge |A_i|/2 + 48\xi^2 n_{46}$$

and since $|B_i| > \delta' - |I_{[k-1]}| \ge k\delta' - (k-1)m$, we have

(86)
$$|A_i| \ge \left| \bigcup B_i \right| - \xi^2 n \ge (1 - \varepsilon) \frac{n}{m} |B_i| - \xi^2 n \ge k\delta - (k - 1)n - 2\xi^2 n.$$

for each $i \in [\ell]$, where we have used (66), (67), (70) and the definition of ξ .

The following claim uses A_1 to obtain powers of cycles of all lengths up to near-extremal.

Claim 7.5. $C_{\ell}^k \subseteq G$ for each $\ell \in [k+1, \frac{k+1}{2}|A_1|]$ such that $\chi(C_{\ell}^k) \leq k+2$.

Proof. By Lemma 7.2 (with $B = \emptyset$) we find in A_1 a copy of $C_{2h'}$ for all $2h' \in [4, \min\{|A_1|, \frac{2n}{k+2}\}]$. We shall construct a copy of $C_{(k+1)h'}^k$ from this cycle by repeated application of Lemma 7.1. We have steps $j = 1, \ldots, k-1$. In step j we start with a copy of $C_{(j+1)h'}^{\mathcal{I}}$

$$T_j = q_{1,j-1} \dots q_{1,1} t_1 t_2 \dots q_{h',j-1} \dots q_{h',1} t_{2h'-1} t_{2h'}$$

in G, with $t_i \in A_1$ for $i \in [2h']$, $q_{f,g} \in W_g$ for each $f \in [h']$, $g \in [j-1]$, such that each vertex is adjacent to the immediately preceding j vertices in cyclic order.

Any 2(j + 1)-tuple of consecutive vertices on T_j comprises four vertices from A_1 and two vertices from W_i for each $i \in [j-1]$. Each vertex in A_1 has at least $|W_j| - \xi^2 n$ neighbours in W_j , while for each $i \in [j-1]$ a vertex in W_i has at least $|W_j| - (n-\delta) + \left|I_{\{i\}}^*\right| - \xi n$ neighbours in W_j . Applying Lemma 2.2 and (75), we find that every 2(j+1)-tuple of consecutive vertices on T_j has at least

$$|W_j| - 4\xi^2 n - 2(j-1)\xi n - 2(j-1)(n-\delta) + 2\left|I_{[j-1]}^*\right|$$

$$\geq |W_j| - 4\xi^2 n - 2(j-1)\xi n - 10k\eta n - 2(k-j)(d+\varepsilon)n - 2\varepsilon n$$

common neighbours in W_j . Since $\delta < \frac{kn}{k+1}$ and by (66), (67), (73) and (78), we have

$$|W_j| - 4\xi^2 n - 2(j-1)\xi n - 10k\eta n - 2(k-j)(d+\varepsilon)n - 2\varepsilon n \ge \frac{n}{k+2}$$

This means that we can apply Lemma 7.1 with G and W_j to obtain a copy of $C_{(j+2)h'}^{j+1}$

$$T_{j+1} = q_{1,j} \dots q_{1,1} t_1 t_2 \dots q_{h',j} \dots q_{h',1} t_{2h'-1} t_{2h'}$$

in G, with $t_i \in A_1$ for $i \in [2h']$, $q_{f,g} \in W_g$ for each $f \in [h'], g \in [j]$, such that each vertex is adjacent to the preceding j vertices in cyclic order. Terminating after step k-1 gives us a copy of $C_{(k+1)h'}^k$. Hence, we are able to find copies of C_h^k for $h \in [k+1, \frac{k+1}{2} \min\{|A_1|, \frac{2n}{k+2}\}]$ such that h is divisible by k+1.

To obtain a copy of C_h^k for h not divisible by k+1, we perform a procedure which we will call parity correction. Fix $g \in [k]$. We seek a copy of $C^k_{(k+1)h'+g}$ with $h' \ge g$. Let h'' := h' - g. Pick (by Theorem 2.9) vertices $a_{i,j}$ for $(i,j) \in [g] \times [3]$ in A_1 such that $a_{i,1}a_{i,2}a_{i,3}$ is a triangle for each $i \in [g]$ and $a_{i,3}a_{i+1,1}$ is an edge for $i \in [g-1]$. Let $A = \{a_{i,j} \mid (i,j) \in [g] \times [3]\}$. Apply Lemma 7.2 to find a path $P'_1 = a_{1,1}p_{2h''} \dots p_1 a_{g,3}$ in $(A_1 \setminus A) \cup \{a_{1,1}, a_{g,3}\}$ on 2(h'' + 1)vertices whose end-vertices are $a_{1,1}$ and $a_{g,3}$. For each $a \in A$, insert a dummy vertex a' into G with the same adjacencies as a. Define $P_1^{(i)} := a_{i+1,2}a_{i+1,1}a_{i,3}a'_{i,2}a'_{i,1}$ for $i \in [g-1]$ and $P_1 := a_{1,2} P_1' a_{g,2}' a_{g,1}'.$

We shall construct a copy of $C_{(k+1)h'+g}^k$ from these paths by repeatedly applying Lemma 7.1 and suitably truncating and concatenating the resultant kth powers of paths. We have steps j = 1, ..., k - 1. In step 1 we start with the paths $P_1, P_1^{(1)}, ..., P_1^{(g-1)}$. For P_1 set $Q_i = p_{2(h''-i+3)}p_{2(h''-i+3)-1}p_{2(h''-i+2)}p_{2(h''-i+2)-1}$ for $3 \le i \le h'' + 1$ as well as $Q_1 = a_{1,2}a_{1,1}, Q_2 = a_{1,2}a_{1,1}$ $a_{1,2}a_{1,1}p_{2h''}p_{2h''-1}$ and $Q_{h''+2} = p_2p_1a_{g,3}a'_{q,2}a'_{q,1}$. Then, apply Lemma 7.1 with W_1 to obtain the squared path

$$qa_{1,2}a_{1,1}q_{h'',1}p_{2h''}p_{2h''-1}\dots q_{1,1}p_2p_1q_1^{(g)}a_{g,3}a'_{g,2}a'_{g,1}$$

with $q_1^{(g)}, q_{x,1} \in W_1$ for each $x \in [h'']$, such that each vertex is adjacent to the preceding 2 vertices in cyclic order and $q_1^{(g)}$ adjacent to $a'_{g,1}$. Let P_2 be the result of replacing q in the above squared path with $a'_{1,3}$. For $P_1^{(i)}$ with $i \in [g-1]$, take $Q_1 = a_{i+1,2}a_{i+1,1}, Q_2 = a_{i+1,2}a_{i+1,1}a_{i,3}a'_{i,2}a'_{i,1}$, and apply Lemma 7.1 with W_1 to obtain the squared path

$$qa_{i+1,2}a_{i+1,1}q_1^{(i)}a_{i,3}a_{i,2}'a_{i,1}'$$

such that $q_1^{(i)} \in W_1$ adjacent to $a'_{g,1}$ and each vertex is adjacent to the preceding 2 vertices in cyclic order. Let $P_2^{(i)}$ be the result of replacing q in the above squared path with $a'_{i+1,3}$.

In step $j \ge 2$ we start with *j*th powers of paths

$$P_{j} = (q_{j-2}^{(1)})' \dots (q_{1}^{(1)})' a_{1,3}' a_{1,2} a_{1,1} q_{h'',j-1} \dots q_{h'',1} p_{2h''} p_{2h''-1} \dots q_{1,j-1} \dots q_{1,1} p_{2} p_{1} q_{j-1}^{(g)} \dots q_{1}^{(g)} a_{g,3} a_{g,2}' a_{g,1}', P_{j}^{(i)} = (q_{j-2}^{(i+1)})' \dots (q_{1}^{(i+1)})' a_{i+1,3}' a_{i+1,2} a_{i+1,1} q_{j-1}^{(i)} \dots q_{1}^{(i)} a_{i,3} a_{i,2}' a_{i,1}'$$

for each $i \in [g-1]$. We seek to apply Lemma 7.1 with W_j to each of them. For P_j take $Q_1 = (q_{j-2}^{(1)})' \dots (q_1^{(1)})' a_{1,3}' a_{1,2} a_{1,1}$,

$$Q_{2} = (q_{j-2}^{(1)})' \dots (q_{1}^{(1)})' a_{1,3}' a_{1,2} a_{1,1} q_{h'',j-1} \dots q_{h'',1} p_{2h''} p_{2h''-1},$$

$$Q_{i} = q_{h''-i+3,j-1} \dots q_{h''-i+3,1} p_{2(h''-i+3)} p_{2(h''-i+3)-1}$$

$$q_{h''-i+2,j-1} \dots q_{h''-i+2,1} p_{2(h''-i+2)} p_{2(h''-i+2)-1}$$

for each $3 \le i \le h'' + 1$, and

$$Q_{h''+2} = q_{1,j-1} \dots q_{1,1} p_2 p_1 q_{j-1}^{(g)} \dots q_1^{(g)} a_{g,3} a_{g,2}' a_{g,1}'$$

Applying Lemma 7.1 with W_j yields the (j + 1)st power of a path

$$q(q_{j-2}^{(1)})' \dots (q_1^{(1)})' a_{1,3}' a_{1,2} a_{1,1} q_{h'',j} \dots q_{h'',1} p_{2h''} p_{2h''-1} \dots q_{1,j} \dots q_{1,1} p_2 p_1 q_j^{(g)} \dots q_1^{(g)} a_{g,3} a_{g,2}' a_{g,1}'$$

with $q_j^{(g)}, q_{x,j} \in W_1$ for each $x \in [h'']$, such that each vertex is adjacent to the preceding j + 1 vertices in cyclic order and $q_j^{(g)}$ adjacent to $a'_{g,1}$. Insert a dummy vertex $(q_{j-1}^{(1)})'$ into G with the same adjacencies as $q_{j-1}^{(1)}$. Define P_{j+1} to be the above (j+1)st power of a path with q replaced by $(q_{j-1}^{(1)})'$. For $P_j^{(i)}$ with $i \in [g-1]$, take

$$Q_1 = (q_{j-2}^{(i+1)})' \dots (q_1^{(i+1)})' a_{i+1,3}' a_{i+1,2} a_{i+1,1} \text{ and}$$
$$Q_2 = (q_{j-2}^{(i+1)})' \dots (q_1^{(i+1)})' a_{i+1,3}' a_{i+1,2}' a_{i+1,1} q_{j-1}^{(i)} \dots q_1^{(i)} a_{i,3}' a_{i,2}' a_{i,1}'.$$

Applying Lemma 7.1 with W_j yields the (j + 1)st power of a path

$$q(q_{j-2}^{(i+1)})'\dots(q_1^{(i+1)})'a_{i+1,3}'a_{i+1,2}a_{i+1,1}q_j^{(i)}\dots q_1^{(i)}a_{i,3}a_{i,2}'a_{i,3}'a_{$$

such that $q_j^{(i)} \in W_j$ adjacent to $a'_{g,1}$ and each vertex is adjacent to the preceding 2 vertices in cyclic order. Insert a dummy vertex $(q_{j-1}^{(i+1)})'$ into G with the same adjacencies as $q_{j-1}^{(i+1)}$. Define $P_{j+1}^{(i)}$ to be the above (j+1)st power of a path with q replaced by $(q_{j-1}^{(i+1)})'$.

After step k - 1, we have kth powers of paths $P_{k-1}, P_{k-1}^{(1)}, \ldots, P_{k-1}^{(g-1)}$. We delete the cloned vertices from each of them and concatenate the resultant kth powers of paths to obtain the kth power of a cycle on (k + 1)h' + g vertices. Therefore, we can obtain $C_{\ell'}^k$ for every $\ell' \in [k+1, \frac{k+1}{2} \min\{|A_1|, \frac{2n}{k+2}\}]$ such that $\chi(C_{\ell'}^k) \leq k+2$. Since $pc_k(n, \delta) \leq \frac{(k+1)n}{k+2}$ by (69), we obtain the desired result.

It remains to show that we have $C_{\ell'}^k \subseteq G$ for every $\frac{k+1}{2}|A_1| \leq \ell' \leq \mathrm{pc}_k(n,\delta)$ and that in the case $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right)$ we have $P_{\mathrm{pp}_k(n,\delta)}^k \subseteq G$. To do so, we need to incorporate vertices which are not 'nice' enough to be included in the sets A_i . Define X_i as A_i together with all vertices in $V(G)\setminus W$ with at least $30\xi^2 n$ neighbours in A_i . Every vertex of $V(G)\setminus W$ has at least $\delta - |W|$ neighbours outside W, so by (78) every vertex of $V(G)\setminus W$ is in X_i for at least one i. Let $i, j \in [\ell]$ satisfy $i \neq j$. Since $A_h \subseteq \bigcup B_h$, we have $A_i \cap A_j = \emptyset$. By the definition of an (ε, d) -regular partition and (iii), vertices in A_i have at most $(d + \varepsilon)n$ neighbours outside of $(\bigcup B_i) \cup I_{[k-1]}^*$; by (81) vertices in A_i have at most $(d + \varepsilon)n < 30\xi^2 n$ neighbours in A_j . Hence, we have

$$A_i \cap X_j = \emptyset$$

Then, it follows from (84) that

(88)

$$|X_i| < |A_i| + 2\xi^2 n.$$

now show the desired outcome by considering three cases based of

We shall now show the desired outcome by considering three cases based on the values of $|X_i \cap X_j|$. The following claim deals with the case when $|X_i \cap X_j| \ge 2$ for some $i \ne j$.

Claim 7.6. Suppose that $|X_i \cap X_j| \ge 2$ for some $i \ne j$. Then we have $C_{\ell}^k \subseteq G$ for every $\frac{k+1}{2}|A_1| \le \ell \le \mathrm{pc}_k(n,\delta)$ and if further $\delta \in \left[\left(\frac{k-1}{k}+\nu\right)n, \frac{kn-1}{k+1}\right)$ we also have $P_{\mathrm{pp}_k(n,\delta)}^k \subseteq G$.

Proof. Let $i \neq j$ such that $|X_i \cap X_j| \geq 2$. Let u_1 and u_2 be distinct vertices of $X_i \cap X_j$. Let v_1 and v_2 be distinct neighbours in A_i of u_1 and u_2 respectively, and similarly w_1 and w_2 in A_j . Applying Lemma 7.2 in A_i , we can find a path from v_1 to v_2 of length α for any $4 \leq \alpha \leq |A_i| - 1$. We can find a similar path in A_j from w_1 to w_2 . Concatenating these paths with u_1 and u_2 , we can find a cycle $S_{2h'}$ of length 2h' in $X_i \cup X_j$ for any $12 \leq 2h' \leq \min\{|A_i| + |A_j| + 2, \frac{2n}{k+2}\}$. We shall construct the desired copy of $C_{(k+1)h'}^k$ from this cycle by repeated application of Lemma 7.1.

We have steps j = 1, ..., k - 1. In step j we start with a copy of $C_{(j+1)h'}^{j}$

$$T_j = q_{1,j-1} \dots q_{1,1} t_1 t_2 \dots q_{h',j-1} \dots q_{h',1} t_{2h'-1} t_{2h'}$$

in G, with $t_p \in A_i \cup A_j \cup \{u_1, u_2\}$ for $p \in [2h']$, $q_{f,g} \in W'_g$ for each $f \in [h'], g \in [j-1]$, such that each vertex is adjacent to the immediately preceding j vertices in cyclic order and no 2(j+1)-tuple of consecutive vertices on T_j uses both u_1 and u_2 .

Any 2(j+1)-tuple of consecutive vertices on T_j comprises four vertices from $A_i \cup A_j \cup \{u_1, u_2\}$ and two vertices from W'_h for each $h \in [j-1]$. Each vertex in $A_i \cup A_j$ has at least $|W'_j| - \xi^2 n$ neighbours in W'_j , u_1 and u_2 each have at least $\xi n - 2$ neighbours in W'_j , and for each $i \in [j-1]$ a vertex in W'_i has at least $|W'_j| - (n - \delta) + |I^*_{\{i\}}| - 2$ neighbours in W'_j . Hence, the four 2(j+1)-tuples which use either u_1 or u_2 each have at least

$$\begin{split} \xi n &- 2 - 3\xi^2 n - 2(j-1)(n-\delta) + 2 \left| I^*_{[j-1]} \right| \\ &\geq \xi n - 2 - 3\xi^2 n - 10k\eta n - 2(k-j+1)(d+\varepsilon)n > 100\ell \end{split}$$

common neighbours in W'_j , with the first inequality following from (78) and the second inequality following from (68), (73) and from

(89) $\ell \le \nu^{-1}.$

Every other 2(j+1)-tuple of consecutive vertices on T_j has at least

$$\begin{aligned} |W'_j| &-4\xi^2 n - 2(j-1)(n-\delta) + 2 \left| I^*_{[j-1]} \right| \\ &\geq |W'_j| - 4\xi^2 n - 10k\eta n - 2(k-j+1)(d+\varepsilon)n \end{aligned}$$

common neighbours in W'_i . By the definition of ξ , (66), (67) and (78), we have

$$|W'_{j}| - 4\xi^{2}n - 10k\eta n - 2(k - j + 1)(d + \varepsilon)n \ge \frac{n}{k + 2}.$$

This means that we can apply Lemma 7.1, with G, W'_j , and an ordering σ of the relevant 2(j+1)-tuples which has all the 2(j+1)-tuples containing u_1 or u_2 coming first, to obtain a copy of $C^{j+1}_{(j+2)h'}$

$$T_{j+1} = q_{1,j} \dots q_{1,1} t_1 t_2 \dots q_{h',j} \dots q_{h',1} t_{2h'-1} t_{2h'}$$

in G, with $t_p \in A_i \cup A_j \cup \{u_1, u_2\}$ for $p \in [2h']$, $q_{f,g} \in W'_g$ for each $f \in [h']$, $g \in [j]$, such that each vertex is adjacent to the immediately preceding j vertices in cyclic order and no 2(j+2)-tuple of consecutive vertices on T_{j+1} uses both u_1 and u_2 . Terminating after step k-1 gives us a copy of $C^k_{(k+1)h'}$. Hence, we are able to find copies of C^k_h for $h \in [k+1, \frac{k+1}{2} \min\{|A_i| + |A_j| + 2, \frac{2n}{k+2}\}]$ such that h is divisible by k + 1.

To obtain a copy of C_h^k for h not divisible by k+1, we perform a parity correction procedure. Fix $g \in [k]$. We seek a copy of $C_{(k+1)h'+g}^k$ with $h' \ge g+7$. Let $h'' := h' - g \ge 7$. Let $u_1, u_2, v_1, v_2, w_1, w_2$ be the vertices previously picked. For the purpose of parity correction, pick (by Theorem 2.9) vertices $a_{x,y}$ for $(x, y) \in [g] \times [3]$ in A_i such that $a_{x,1}a_{x,2}a_{x,3}$ is a triangle for each $x \in [g]$ and $a_{x,3}a_{x+1,1}$ is an edge for $x \in [g-1]$. Let $A' = \{a_{x,y} \mid (x,y) \in [g] \times [3]\}$. Pick a common neighbour v of v_1 and $a_{1,1}$ in A_i which is not in $A' \cup \{v_2\}$. Applying Lemma 7.1 suitably, we can find a path in $A_i \setminus (A \cup \{v, v_1\})$ from $a_{g,3}$ to v_2 of length h for any $4 \le h \le |A_i| - 3g - 2$ and a path in A_j from w_1 to w_2 of length h for any $4 \le h \le |A_i| - 1$. Concatenating these paths with $u_1, u_2, v_1, v, a_{1,1}, a_{g,3}$, we can find a path of length 2h'' + 1 in $A_i \cup A_j \cup \{u_1, u_2\}$ for any $15 \le 2h'' + 1 \le \min\{|A_i| + |A_j| - 3g + 3, \frac{2n}{k+2}\}$. This allows us to construct a copy of $C_{(k+1)h'+g}^k$ whenever $h' \ge g + 7$ by applying the method used previously. Therefore, we can obtain $C_{\ell'}^k \subseteq G$ for every $\ell' \in [k^2 + 9k + 7, \frac{k+1}{2} \min\{|A_i| + |A_j| - 3k, \frac{2n}{k+2}\}$ such that $\chi(C_{\ell'}^k) \le k+2$. By (86), (13) and (14) we have $\operatorname{pc}_k(n, \delta) \le \frac{k+1}{2} \min\{|A_i| + |A_j| - 3k, \frac{2n}{k+2}\}$, so G contains C_{ℓ}^k for every $\frac{k+1}{2}|A_1| \le \ell' \le \operatorname{pc}_k(n, \delta)$. For the case $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right\right)$ we note that $P_\ell^k \subseteq C_\ell^k$ and by (86), (13) and (69) we have $\operatorname{pp}_k(n, \delta) \le \frac{k+1}{2} \min\{|A_i| + |A_j| - 3k, \frac{2n}{k+2}\}$, so G contains $P_{\operatorname{pp}_k(n,\delta)}^k$. This completes the proof.

The following claim deals with the case when there exists $i \in [\ell]$ such that every vertex of A_i is adjacent to some vertex outside $X_i \cup W$.

Claim 7.7. Suppose that there exists $i \in [\ell]$ such that every vertex of A_i is adjacent to some vertex outside $X_i \cup W$. Then we have $C_{\ell}^k \subseteq G$ for every $\frac{k+1}{2}|A_1| \leq \ell \leq pc_k(n,\delta)$ and if further $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right)$ we also have $P_{pp_k(n,\delta)}^k \subseteq G$.

Proof. Since we have

$$A_{i} \stackrel{(86)}{\geq} \left| \bigcup B_{i} \right| - \xi^{2} n \stackrel{(72)}{\geq} \frac{38}{39} \nu (1 - \varepsilon) n - \xi^{2} n \stackrel{(73)}{\geq} 25\xi n \stackrel{(73),(89)}{>} 50\ell\xi^{2} n$$

there exists $j \neq i$ such that there are $50\xi^2 n$ vertices in A_i all adjacent to vertices of $X_j \setminus X_i$. No vertex of $X_j \setminus X_i$ is adjacent to $30\xi^2 n$ vertices of A_i (by definition of X_i), so there are two disjoint edges u_1v_1 and u_2v_2 from $u_1, u_2 \in A_i$ to $v_1, v_2 \in X_j$. Then, choosing distinct neighbours w_1 of v_1 and w_2 of v_2 in A_j and applying the same reasoning as in Claim 7.6 completes the proof.

Now we deal with the remainder case. The following claim deals with finding the kth power of a path of the desired length in this case.

Claim 7.8. Suppose that $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right)$, for each $i \neq j$ we have $|X_i \cap X_j| \leq 1$ and for each *i* there is a vertex of A_i adjacent only to vertices in $X_i \cup W$. Then we have $P_{pp_k(n,\delta)}^k \subseteq G$.

Proof. In this case we have $|X_i| \ge \delta - |W| + 1$ for each $i \in [\ell]$. We first focus on finding the kth power of a path on $pp_k(n, \delta)$ vertices when $\delta \in \left[\left(\frac{k-1}{k} + \nu\right)n, \frac{kn-1}{k+1}\right)$. Note that if $|X_i \cap X_j| = 1$ for some $i \ne j$, then we obtain the kth power of a path of the desired length as in Claim 7.6.

We required two vertices in $|X_i \cap X_j|$ previously for a cycle to cross from X_i to X_j and back to X_i , whereas here we only need one vertex for a path to cross from X_i to X_j .

Hence, assume that the sets X_i are all disjoint. This implies that $\ell \leq \frac{n-|W|}{\delta-|W|+1}$. Note that $|W| \leq (k-1)(n-\delta)$ by (78), so we have

$$\ell \le \frac{n - (k - 1)(n - \delta)}{\delta - (k - 1)(n - \delta) + 1} = \frac{(k - 1)\delta - (k - 2)n}{k\delta - (k - 1)n + 1}.$$

Now if $\ell \geq r_p(n,\delta) + 1$, we would have $r_p(n,\delta) + 1 \leq \ell \leq \frac{(k-1)\delta - (k-2)n}{k\delta - (k-1)n+1}$, and so $r_p(n,\delta) \leq \frac{n-\delta}{k\delta - (k-1)n+1}$, but by (3) we have $r_p(n,\delta) \geq \frac{n-\delta}{k\delta - (k-1)n+1}$, so we have $\ell \leq r_p(n,\delta)$. Therefore, the largest of the sets X_i , say X_1 , has at least

(90)
$$|X_1| \ge \frac{n - |W|}{\ell} \stackrel{(78)}{\ge} \frac{(k-1)\delta - (k-2)n}{\ell} \ge \frac{(k-1)\delta - (k-2)n}{r_p(n,\delta)}$$

vertices.

We wish to apply Lemma 7.2 with $H = G[X_1]$ and 'bad' vertices $B = X_1 \setminus A_1$. Note that by (88) B contains at most $2\xi^2 n$ vertices, so we have

$$|B| \stackrel{(88)}{\leq} 2\xi^2 n \stackrel{(73)}{\leq} \frac{\nu[(k-1)\delta - (k-2)n]}{100} \stackrel{(89)}{\leq} \frac{(k-1)\delta - (k-2)n}{100\ell} \stackrel{(90)}{\leq} \frac{|H|}{100}.$$

Moreover, we have $\delta(H) \ge \delta(G[X_1]) \ge 30\xi^2 n$ by definition of X_1 , so every vertex of B has at least $30\xi^2 n \ge 9 \cdot 2\xi^2 n \ge 9|B|$ neighbours in H. For $v \in X_1 \setminus B = A_1$, we have

$$\deg(v; X_1) \stackrel{(85)}{\geq} \frac{|A_1|}{2} + 48\xi^2 n \stackrel{(88)}{>} \frac{|X_1|}{2} + 47\xi^2 n$$
$$= \frac{|H|}{2} + 47\xi^2 n \stackrel{(68)}{\geq} \frac{|H|}{2} + 9|B| + 10.$$

Hence, we may apply Lemma 7.2 to obtain a path P in X_1 with $\alpha := \min\left\{|X_1|, \frac{2n}{k+2}\right\}$ vertices, on which no four consecutive vertices contain more than one vertex of B. Define $h' := \lfloor \frac{\alpha}{2} \rfloor$ and $\beta := \alpha - 2h' \in \{0, 1\}$. We shall construct the desired copy of $P_{\text{pp}_k(n,\delta)}^k$ from P by repeated application of Lemma 7.1. We have steps $j = 1, \ldots, k - 1$. In step j we start with a copy of $P_{(j+1)h'+j-1+\beta}^j$

$$T_{j} = q_{1,j-1} \dots q_{1,1} t_{1} t_{2} \dots q_{h',j-1} \dots q_{h',1} t_{2h'-1} t_{2h'}$$
$$q_{h'+1,j-1} \dots q_{h'+1,1} t_{2h'+1} \dots t_{2h'+\beta}$$

in G, with $t_p \in X_1$ for $p \in [\alpha]$, $q_{f,g} \in W'_g$ for each $f \in [h'+1]$, $g \in [j-1]$, such that each vertex is adjacent to the preceding j vertices and no 2(j+1)-tuple of consecutive vertices on T_j contains more than one vertex of B.

There are at most $2|B| \leq 4\xi^2 n \ 2(j+1)$ -tuples containing vertices of B. Any 2(j+1)-tuples of consecutive vertices on T_j comprises four vertices from X_1 and two vertices from W'_i for each $i \in [j-1]$. Each vertex in A_1 has at least $|W'_j| - \xi^2 n$ neighbours in W'_j , each vertex in B has at least $\xi n - 2$ neighbours in W'_j , and for each $i \in [j-1]$ a vertex in W'_i has at least $|W'_j| - (n-\delta) + |I^*_{\{i\}}| - 2$ neighbours in W'_j . Hence, the 2(j+1)-tuples which contain a vertex of B each have at least

$$\begin{split} \xi n &- 2 - 3\xi^2 n - 2(j-1)(n-\delta) + 2 \left| I^*_{[j-1]} \right| \\ &\geq \xi n - 2 - 3\xi^2 n - 10k\eta n - 2(k-j+1)(d+\varepsilon)n > 100\ell \end{split}$$

common neighbours in W'_j , with the first inequality following from (78) and the second inequality following from (68), (73) and (89). Every other 2(j+1)-tuple of consecutive vertices on T_j has

at least

$$\begin{aligned} |W'_j| &-4\xi^2 n - 2(j-1)(n-\delta) + 2 \left| I^*_{[j-1]} \right| \\ &\geq |W'_j| - 4\xi^2 n - 10k\eta n - 2(k-j+1)(d+\varepsilon)n \end{aligned}$$

common neighbours in W'_i . By the definition of ξ , (66), (67) and (78), we have

$$|W'_j| - 4\xi^2 n - 10k\eta n - 2(k - j + 1)(d + \varepsilon)n \ge \frac{n}{k + 2}.$$

This means that we can apply Lemma 7.1, with an ordering σ of the relevant 2(j+1)-tuples which has all the 2(j + 1)-tuples containing vertices of B coming first, to obtain a copy of $P^{j+1}_{(j+2)h'+j+\beta}$

$$T_{j+1} = q_{1,j} \dots q_{1,1} t_1 t_2 \dots q_{h',j} \dots q_{h',1} t_{2h'-1} t_{2h'}$$
$$q_{h'+1,j} \dots q_{h'+1,1} t_{2h'+1} \dots t_{2h'+\beta}$$

in G, with $t_p \in X_1$ for $p \in [\alpha]$, $q_{f,g} \in W'_g$ for each $f \in [h'+1], g \in [j]$, such that each vertex is adjacent to the preceding j vertices and no 2(j+2)-tuple of consecutive vertices on T_j contains more than one vertex of B. Terminating after step k-1 gives the kth power of a path on at least $(k+1)h' + k - 1 + \beta$ vertices. We consider two cases. First consider when $\alpha = \frac{2n}{k+2}$. In this case, we have the kth power of a path on at least

$$(k+1)\left(\frac{n}{k+2} - \frac{k+1}{k+2}\right) + k - 1 \ge \frac{(k+1)n}{k+2} - 2 \ge pp_k(n,\delta)$$

vertices, with the inequality following from (69). Otherwise, we have $\alpha = |X_1|$. Define h'' := $\left\lfloor \frac{|X_1|}{2} \right\rfloor$ and $\beta' := |X_1| - 2h'' \in \{0, 1\}$. In this case, we have the kth power of a path on at least

$$(k+1)h'' + k - 1 + \beta' = (k-1)(h''+1) + |X_1| \ge pp_k(n,\delta)$$

vertices, with the inequality following from (90) and the definition of $pp_k(n, \delta)$.

Finally, the following claim deals with finding kth powers of cycles of the desired lengths in the remainder case.

Claim 7.9. Suppose that for each $i \neq j$ we have $|X_i \cap X_j| \leq 1$ and for each *i* there is a vertex of A_i adjacent only to vertices in $X_i \cup W$. Then we have $C_\ell^k \subseteq G$ for every $\frac{k+1}{2}|A_1| \leq \ell \leq pc_k(n,\delta)$.

Proof. First consider when there is a cycle of sets (relabelling the indices if necessary) X_1, \ldots, X_s for some $3 \le s \le \ell$ such that $X_i \cap X_{i+1} = \{u_i\}$ for each *i* and the u_i are all distinct. In this case for each i we may choose neighbours $v_i \in A_i$ and $w_i \in A_{i+1}$ of u_i , and we may insist that these 3s vertices are distinct. Similarly as before, we may apply Lemma 7.2 to each $G[A_i]$ in turn and concatenate the resulting paths, in order to find a cycle $T_{2h'}$ for every $6s \le 2h' \le \min\{|A_i| +$ $|A_j|, \frac{2n}{k+2}$ on which there are no quadruples using more than one vertex outside $\bigcup_{i \in [s]} A_i$. Arguing in a manner similar to Claim 7.6, we may repeatedly apply Lemma 7.1 to obtain a copy of $C_{(k+1)h'}^k$. Hence, we are able to find copies of C_h^k for $h \in [3s(k+1), \frac{k+1}{2} \min\{|A_i| + |A_j|, \frac{2n}{k+2}\}]$ such that h is divisible by k + 1. To obtain a copy of C_h^k for h not divisible by k + 1, we use a parity correction procedure analogous to that in Claim 7.6. Therefore, we can find copies of C_h^k for $h \in [k^2 + 3(s+1)k + (3s+1), \frac{k+1}{2}\min\{|A_i| + |A_j| - 3k, \frac{2n}{k+2}\}]$. Hence, we have $C_{\ell'}^k \subseteq G$ for every $\ell' \in [k+1, \frac{k+1}{2}\min\{|A_i| + |A_j| - 3k, \frac{2n}{k+2}\}]$ such that $\chi(C_{\ell'}^k) \leq k+2$.

Otherwise, no such cycle of sets exists. In this case, we have $\sum_{i=1}^{\ell} |X_i| \le n - |W| + \ell - 1$. Note that $|X_i| \ge \delta - |W| + 1$ for each $i \in [\ell]$, so this implies that $\ell \le \frac{n - |W| - 1}{\delta - |W|}$. Note that $|W| \leq (k-1)(n-\delta)$ by (78), so we have

$$\ell \le \frac{n - (k - 1)(n - \delta) - 1}{\delta - (k - 1)(n - \delta)} = \frac{(k - 1)\delta - (k - 2)n - 1}{k\delta - (k - 1)n}$$

Now if $\ell \geq r_c(n,\delta) + 1$, we would have $r_c(n,\delta) + 1 \leq \ell \leq \frac{(k-1)\delta - (k-2)n}{k\delta - (k-1)n}$, and so $r_c(n,\delta) \leq \frac{n-\delta}{k\delta - (k-1)n}$, but we have $r_c(n,\delta) \geq \frac{n-\delta}{k\delta - (k-1)n}$, so we have $\ell \leq r_c(n,\delta)$. Therefore, the largest of the sets X_i , say X_1 , has at least

$$|X_1| \ge \frac{n - |W|}{\ell} \ge \frac{(k - 1)\delta - (k - 2)n}{\ell} \ge \frac{(k - 1)\delta - (k - 2)n}{r_c(n, \delta)}$$

vertices.

As before, by Lemma 7.2 for each $2h' \in [4, \min\{|X_1|, \frac{2n}{k+2}\}]$ we find in X_1 a copy of $C_{2h'}$, on which no four consecutive vertices contain more than one vertex of B, and by repeated application of Lemma 7.1 we obtain the kth power of a cycle $C_{(k+1)h'}^k$ for each $(k+1)h' \in$ $[2(k+1), pc_k(n, \delta)]$. As before, we may apply a parity correction procedure for copies of C_h^k where h is not divisible by k + 1. Therefore, we have copies of C_h^k for $h \in [k+1, pc_k(n, \delta)]$ such that $\chi(C_{\ell'}^k) \leq k+2$.

Claims 7.6, 7.7, 7.8 and 7.9 collectively yield the desired outcome, completing the proof. \Box

8. Concluding Remarks

Extremal graphs and minimum degree. Our proofs provide a template for checking that $G_p(k, n, \delta)$ and $G_c(k, n, \delta)$ are the only extremal graphs up to some trivial modifications. We believe that the graph $G_p(k, n, \delta)$ remains extremal for kth powers of paths for all $\delta > \frac{(k-1)n}{k}$. However, the same is generally not true for $G_c(k, n, \delta)$ and kth powers of cycles: Allen, Böttcher and Hladký [3] sketched a construction, for infinitely many values of n, of graphs G on n vertices with $\delta(G) \geq \frac{n}{2} + \frac{\sqrt{n}}{5}$ which do not contain a copy of C_6^2 . Their construction can be generalised to one for general powers of cycles.

Long kth powers of cycles. Theorem 1.5(ii) states that if G does not contain any of various kth powers of cycles of lengths not divisible by k + 1, then G must contain kth powers of cycles of every length divisible by k + 1 up to $(k + 1)(k\delta - (k - 1)n) - \nu n$. We believe that the error term of νn can be removed, but it would involve significantly more technical work. This includes a new version of the stability lemma with more extremal cases and new extremal results corresponding to these additional extremal cases.

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APPENDIX A. EMBEDDING LEMMA

In this appendix we will provide our proof of Lemma 3.2. To do so, we shall apply a version of a graph Blow-up Lemma by Allen, Böttcher, Hàn, Kohayakawa and Person [2]. We remark that the Blow-up Lemma of Komlós, Sárközy and Szemerédi [12] is perfectly adequate for this proof; our choice of Blow-up Lemma is not driven by necessity, but rather a desire to reduce the technical complexity of our proof.

We will use the definition of (ε, d) -regular as given in Section 3.2; this involves both an upper bound and a lower bound on densities. We note that the corresponding graph Blow-up Lemma in [2] applies to a more general class of graphs; in particular, the regularity condition in [2] is weaker and involves only a lower bound.

We first introduce some terminology in order to formulate this version of the Blow-up Lemma. Let $\kappa \geq 1$. Let G and H be two graphs, on the same number of vertices, with partitions $\mathcal{V} = \{V_i\}_{i \in [r]}$ and $\mathcal{X} = \{X_i\}_{i \in [r]}$ of their respective vertex sets. We say that \mathcal{V} and \mathcal{X} are size-compatible if $|V_i| = |X_i|$ for all $i \in [r]$. Moreover, we say that (G, \mathcal{V}) is κ -balanced if there exists $m \in \mathbb{N}$ such that $m \leq |V_i| \leq \kappa m$ for all $i \in [r]$.

Let R be a graph on r vertices.

- (i) (H, X) is an *R*-partition if each part of X is nonempty, and whenever there are edges of H between X_i and X_j, the pair ij is an edge of R,
- (ii) (G, \mathcal{V}) is an (ε, d) -regular *R*-partition if for each edge $ij \in E(R)$ the pair (V_i, V_j) is (ε, d) -regular.

In this case we say that R is an (ε, d) -full-reduced graph of the partition \mathcal{V} . We remark that the notion of an (ε, d) -regular R-partition is distinct from that of an (ε, d) -reduced partition defined in Section 3: in an (ε, d) -regular R-partition, the partition \mathcal{V} does not have an exceptional set and we allow each vertex of G to be incident to possibly many edges which are not in (ε, d) -regular pairs. The notion of an (ε, d) -full-reduced graph is correspondingly distinct from that of an (ε, d) -reduced graph.

Suppose R is a graph on r vertices, (H, \mathcal{X}) is an R-partition and (G, \mathcal{V}) is a size-compatible (ε, d) -regular R-partition. Let $\alpha > 0$. A family $\bar{\mathcal{X}} = {\bar{X}_i}_{i \in [r]}$ of subsets $\bar{X}_i \subseteq X_i$ is an α -buffer for H if

- (i) the elements of \bar{X}_i are isolated vertices in H,
- (ii) $|\bar{X}_i| \ge \alpha |X_i|$ for all $i \in [r]$.

Note that this corresponds to the notion of an (α, R') -buffer for H in [2] with R' as the empty spanning subgraph of R.

Let R be a graph on r vertices, (H, \mathcal{X}) be an R-partition and (G, \mathcal{V}) be a size-compatible (ε, d) -regular R-partition, with $G \subseteq K_n$. Let $\mathcal{I} = \{I_x\}_{x \in V(H)}$ be a collection of subsets of V(G), called *image restrictions*, and $\mathcal{J} = \{J_x\}_{x \in V(H)}$ be a collection of subsets of $V(K_n) \setminus V(G)$, called *restricting vertices*. \mathcal{I} and \mathcal{J} are a $(\rho, \zeta, \Delta, \Delta_J)$ -restriction pair if the following properties hold for each $i \in [r]$ and $x \in X_i$.

(a) The set $X_i^* \subseteq X_i$ of *image restricted* vertices in X_i , that is, vertices such that $I_x \neq V_i$, has size $|X_i^*| \leq \rho |X_i|$.

- (b) If $x \in X_i^*$, then $I_x \subseteq V_i$ is of size at least $\zeta d^{|J_x|} |V_i|$.
- (c) If $x \in X_i^*$, then $|J_x| + |\Gamma_H(x)| \le \Delta$, and if $x \notin X_i^*$, then $J_x = \emptyset$.
- (d) Each vertex of K_n appears in at most Δ_J of the sets of \mathcal{J} .
- (e) If $x \in X_i^*$, then for each $xy \in E(H)$ with $y \in X_i$ the pair (V_i, V_j) is (ε, d) -regular in G.

Lemma A.1 (Allen, Böttcher, Hàn, Kohayakawa and Person [2]). For all $\Delta \geq 2, \Delta_J, \alpha, \zeta, d > 0$, $\kappa > 1$ there exists $\varepsilon, \rho > 0$ such that for all r_1 there exists $n_{BL} \in \mathbb{N}$ such that for all $n \geq n_{BL}$ the following holds. Let R be a graph on $r \leq r_1$ vertices. Let H and G be n-vertex graphs with κ -balanced size-compatible vertex partitions $\mathcal{X} = \{X_i\}_{i \in [r]}$ and $\mathcal{V} = \{V_i\}_{i \in [r]}$, respectively, which have parts of size at least $m \geq n/(\kappa r_1)$. Let $\overline{\mathcal{X}} = \{\overline{X}_i\}_{i \in [r]}$ be a family of subsets of $V(H), \mathcal{I} = \{I_x\}_{x \in V(H)}$ be a family of image restrictions, and $\mathcal{J} = \{J_x\}_{x \in V(H)}$ be a family of restricting vertices. Suppose that

- (i) $\Delta(H) \leq \Delta$, (H, \mathcal{X}) is an *R*-partition, and $\overline{\mathcal{X}}$ is an α -buffer for *H*,
- (ii) (G, \mathcal{V}) is an (ε, d) -regular R-partition,
- (iii) \mathcal{I} and \mathcal{J} form a $(\rho, \zeta, \Delta, \Delta_J)$ -restriction pair.

Then there is an embedding $\psi: V(H) \to V(G)$ such that $\psi(x) \in I_x$ for each $x \in V(H)$.

Proof of Lemma 3.2. We proceed by checking the conditions for a suitable application of Lemma A.1 to embed a relevant graph H into G. We first prove (i),(ii). Fix $k \ge 2$, d > 0 and set $\Delta = 2k, \Delta_J = k, \kappa = 2, \alpha = \frac{d}{2}, \zeta = 1$. Now Lemma A.1 outputs $\varepsilon_0, \rho_0 > 0$. We choose

$$\varepsilon_{EL} = \min\left\{\frac{\varepsilon_0}{k+1}, \frac{d^2}{8(k+1)}\right\}.$$

Given $0 < \varepsilon < \varepsilon_{EL}, r_{EL} \in \mathbb{N}$, Lemma A.1 outputs $n_{BL} \in \mathbb{N}$. We choose

$$n_{EL} = \max\left\{n_{BL}, \frac{6r_{EL}^{k+2}}{\varepsilon}, \frac{4r_{EL}}{\rho_0}\right\}.$$

Let $n \ge n_{EL}$, let G be a graph on n vertices and let R be an (ε, d) -reduced graph of G on $r \le r_{EL}$ vertices. Let V_0, V_1, \ldots, V_r be the vertex classes of the (ε, d) -regular partition of G which gives rise to R. Fix a connected K_{k+1} -factor \mathcal{F} in R which contains $c := \frac{\operatorname{CKF}_{k+1}(R)}{k+1}$ copies of K_{k+1} . Let T_1, \ldots, T_c be the copies of K_{k+1} in \mathcal{F} . Let $\mathcal{V} := \{V_1, \ldots, V_r\}$. Let R' be the empty spanning subgraph of R. Let G^* be the subgraph of G induced on \mathcal{V} and set $n^* := |V(G^*)|$. Note that (G^*, \mathcal{V}) is an (ε, d) -regular R-partition. Note that $|V_i| \ge (1 - \varepsilon)\frac{n}{r} \ge \frac{n}{2r_{EL}}$ for all $i \in [r]$, so \mathcal{V} is 2-balanced.

Let H be a copy of C_{ℓ}^k together with additional isolated vertices so that it has n^* vertices. Let v_1, \ldots, v_{n^*} be its vertices, with v_1, \ldots, v_{ℓ} being the vertices of the copy of C_{ℓ}^k in an arbitrary cyclic order. Let $C := \{v_i : i \in [\ell]\}$. Suppose that we have a vertex partition $\mathcal{X} := \{X_i\}_{i \in [r]}$ of H and a family $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$ of subsets of V(H) such that \mathcal{X} is size-compatible with $\mathcal{V}, (H, \mathcal{X})$ is an R-partition, and $\bar{\mathcal{X}}$ is an α -buffer for H. Define $\mathcal{I} := \{I_x\}_{x \in V(H)}$ and $\mathcal{J} := \{J_x\}_{x \in V(H)}$ by $I_x = V_i$ for $x \in X_i$ and $J_x = \emptyset$ for $x \in V(H)$. Note that \mathcal{I} and \mathcal{J} form a $(\rho_0, \zeta, \Delta, \Delta_J)$ -restriction pair. Then, by Lemma A.1 we will have an embedding $\phi : V(H) \to V(G^*)$, which will then complete our proof of (i) and (ii). Therefore, for suitable values of ℓ it remains to find a vertex partition \mathcal{X} of H and a family $\bar{\mathcal{X}}$ of subsets of V(H) such that \mathcal{X} is size-compatible with $\mathcal{V}, (H, \mathcal{X})$ is an R-partition and $\bar{\mathcal{X}}$ is an α -buffer for H.

We start with (i) and we will consider $\ell \leq \frac{(1-d)(k+1)cn}{r}$ divisible by k+1. We first consider the case $\ell \leq \frac{(k+1)(1-d)n}{r}$ divisible by k+1, that is, when c = 1. Let Y_1, \ldots, Y_{k+1} be the vertices of T_1 . Define $\phi: V(H) \to V(R)$ as follows. For $i \leq \ell$, set $\phi(v_i) = Y_j$ with $j \equiv i \mod k+1$. For $i > \ell$, given $\phi(v_1), \ldots, \phi(v_{i-1})$, set $\phi(v_i) = V_j$ with $j = \min\{h : |\{b < i : \phi(v_b) = V_h\}| < |V_h|\}$. Set $X_i := \phi^{-1}(V_i), \bar{X}_i := \phi^{-1}(V_i) \setminus C$ for $i \in [r]$. Define $\mathcal{X} := \{X_i\}_{i \in [r]}$ and $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$. Since all edges in H have pairs of vertices in C at most k apart in the cyclic order as endpoints and any k+1 consecutive vertices in the cyclic order are mapped to a copy of K_{k+1} in R, it follows that (H, \mathcal{X}) is an R-partition. Furthermore, for each $i \in [r]$ at most $\frac{\ell}{k+1} \leq (1-d)\frac{n}{r} \leq |V_i|$ vertices in C are mapped to V_i , so \mathcal{X} is a vertex partition of H which is size-compatible with \mathcal{V} . Finally, \bar{X}_i is a set of isolated vertices in H by definition and

$$|\bar{X}_i| = |X_i| - |C \cap X_i| \ge \left(1 - \frac{1 - d}{1 - \varepsilon}\right) |X_i| \ge \alpha |X_i|$$

for each $i \in [r]$, so $\overline{\mathcal{X}}$ is an α -buffer for H. This completes the proof in this case. We are done if c = 1, so we can assume $c \ge 2$ for the remainder of (i).

Next, we consider the case $\ell \in \left(\frac{(1-d)(k+1)n}{r}, \frac{(1-d)(k+1)cn}{r}\right]$ divisible by k+1. For each $i \in [c-1]$, fix a K_{k+1} -walk W_i whose first copy of K_k is in T_i and whose last is in T_{i+1} , which is of minimal length. We have $|W_i| \leq \binom{r}{k}$ for each $i \in [c-1]$. Let W' be the K_{k+1} -walk obtained by concatenating W_1, \ldots, W_{c-1} .

We shall now describe how to construct the sequence $Q(W, \overrightarrow{U_{11} \dots U_{1k}})$ for any K_{k+1} -walk $W = (E_1, E_2, \dots)$ in R and any orientation $\overrightarrow{U_{11} \dots U_{1k}}$ of E_1 , its first copy of K_k . We construct $Q(W, \overrightarrow{U_{11} \dots U_{1k}})$ iteratively as follows. Let $Q_1 = (U_{11}, \dots, U_{1k})$. Now for $2 \le i \le |W|$ successively, we define Q_i as follows. The last k vertices $U_{(i-1)1}, \dots, U_{(i-1)k}$ of Q_{i-1} are an orientation of E_{i-1} . For some $j \in [k]$ we have $E_i = U_{(i-1)1} \dots U_{(i-1)(j-1)} U_{(i-1)(j+1) \dots U_{(i-1)k}} U_{ik}$. Append $(U_{ik}, U_{(i-1)1}, \dots, U_{(i-1)(j-1)})$ to Q_{i-1} to create Q_i . At each step the last k vertices of Q_i are an orientation of E_i and every vertex of Q_i is adjacent in R to the k vertices preceding it in Q_i . Finally we let $Q(W, \overrightarrow{U_{11} \dots U_{1k}}) := Q_{|W|}$.

It is easy to check by induction that for any K_{k+1} -walk W whose first edge is $U_{11} \dots U_{1k}$, we have

(91)
$$|Q(W, \overrightarrow{U_{11} \dots U_{1k}})| + |Q(W, \overrightarrow{U_{1k} \dots U_{11}})| \equiv -2 \mod k+1.$$

Now consider the concatenation W' of the walks W_i . Let $U_{11} \ldots U_{1k}$ be the first copy of K_k of W_1 . If we construct $Q(W', \overrightarrow{U_{11} \ldots U_{1k}})$ then the first copy of $K_k \ U_{i1} \ldots U_{ik}$ and the last copy of $K_k \ U'_{i1} \ldots U'_{ik}$ of each W_i obtain orientations, say $\overrightarrow{U_{i1} \ldots U_{ik}}$ and $\overrightarrow{U'_{i1} \ldots U'_{ik}}$. Clearly, there are sequences \overline{Q}_i of vertices in T_i for 1 < i < c, such that $Q(W', \overrightarrow{U_{11} \ldots U_{ik}})$ is the concatenation of

$$Q(W_1, \overrightarrow{U_{11} \dots U_{1k}}), \overline{Q}_2, Q(W_2, \overrightarrow{U_{21} \dots U_{2k}}), \dots, \overline{Q}_{c-1}, Q(W_{c-1}, \overrightarrow{U_{(c-1)1} \dots U_{(c-1)k}}).$$

Let $\bar{Q}_1 := T_1 - U_{11} \dots U_{1k}$ and $\bar{Q}_c := T_c - U'_{c1} \dots U'_{ck}$. Define $f_i \equiv |\bar{Q}_i| \mod k + 1$ for $i \in [c]$. Together with (91), we obtain

$$|Q(W', \overrightarrow{U_{1k} \dots U_{11}})| + |Q(W_1, \overrightarrow{U_{11} \dots U_{1k}})| + \sum_{1 \le i \le c} (|Q(W_i, \overrightarrow{U_{i1} \dots U_{ik}})| + f_i) \equiv -2 \mod k + 1$$

and hence

(92)
$$|Q(W', \overline{U_{1k} \dots U_{11}})| + \sum_{i \in [c-1]} (|Q(W_i, \overline{U_{i1} \dots U_{ik}})| + f_i) + f_c \equiv 0 \mod k+1.$$

Let Q' denote $Q(W', \overrightarrow{U_{1k} \dots U_{11}})$ and let Q_i^* denote $Q(W_i, \overrightarrow{U_{i1} \dots U_{ik}})$ for each $i \in [c-1]$. Define q' := |Q'| and $q_i := |Q_i^*|$ for each $i \in [c-1]$. For a sequence Q of vertices of R, let $(Q)_h$ denote the *h*th term of Q.

Let $U_{11} \ldots U_{1k}$ be the first copy of K_k in W_1 . Orient it as $\overrightarrow{U_{11} \ldots U_{1k}}$. Construct Q_i^* for $i \in [c-1]$ and Q' as described before, and define q_i, f_i for $i \in [c-1]$ and q', f_c as before. Let $T_i = Y_{i1} \ldots Y_{i(k+1)}$ for $i \in [c]$ be such that $\overrightarrow{Y_{i2} \ldots Y_{i(k+1)}}$ is the oriented last copy of K_k of W_{i-1} in Q_{i-1}^* for $2 \leq i \leq c$ and $\overrightarrow{Y_{1(k+1)} \ldots Y_{12}}$ is the oriented first copy of K_k of W' in Q'. Define the

following. Let $\alpha := \sum_{i=1}^{c-1} (q_i + f_i) + f_c + q'$.

$$p_{0} := \max \left\{ p \in \mathbb{Z} \mid \ell \geq p(1-d)(k+1)\frac{n}{r} + \alpha \right\};$$

$$t_{i} = \left\{ \begin{array}{ll} (1-d)\frac{n}{r} & \text{if } i \in [p_{0}] \\ \frac{\ell-\alpha}{k+1} - p_{0}(1-d)\frac{n}{r} & \text{if } i = p_{0} + 1 \\ 0 & \text{if } i > p_{0} + 1; \end{array} \right.$$

$$L_{0} = 0, L_{j} = \sum_{i=1}^{j} [t_{i}(k+1) + q_{i} + f_{i}] \text{ for } j \in [c-1];$$

$$M_{j} = L_{j-1} + t_{j}(k+1) + f_{j} \text{ for } j \in [c].$$

Define $\phi: V(H) \to V(R)$ as follows. For $i \leq \ell$, set

$$\phi(v_i) = \begin{cases} Y_{jh} & \text{if } L_{j-1} < i \le M_j, \text{ with } h \equiv i - L_{j-1} \mod k+1 \\ (Q_j^*)_{i-M_j} & \text{if } M_j < i \le L_j \\ (Q')_{M_c+q'+1-i} & \text{if } M_c < i \le \ell. \end{cases}$$

For $i > \ell$, given $\phi(v_1), \ldots, \phi(v_{i-1})$, set $\phi(v_i) = V_j$ with $j = \min\{h : |\{b < i : \phi(v_b) = V_h\}| < \ell$ $|V_h|$.

Set $X_i := \phi^{-1}(V_i), \ \bar{X}_i := \phi^{-1}(V_i) \setminus C$ for $i \in [r]$. Define $\mathcal{X} := \{X_i\}_{i \in [r]}$ and $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$. Since all edges in H have pairs of vertices in C at most k apart in the cyclic order as endpoints and any k + 1 consecutive vertices in the cyclic order are mapped to a copy of K_{k+1} in R, it follows that (H, \mathcal{X}) is an *R*-partition. Furthermore, for each $i \in [r]$ at most $(1 - d)\frac{n}{r} + \frac{2r\binom{r}{k}}{k+1} \leq \frac{1}{r}$ $(1-d+\varepsilon)\frac{n}{r} \leq (1-\varepsilon)\frac{n}{r} \leq |V_i|$ vertices in C are mapped to V_i , so \mathcal{X} is a vertex partition of Hwhich is size-compatible with \mathcal{V} . Finally, \overline{X}_i is a set of isolated vertices in H by definition and

$$|\bar{X}_i| = |X_i| - |C \cap X_i| \ge \left(1 - \frac{1 - d + \varepsilon}{1 - \varepsilon}\right) |X_i| \ge \alpha |X_i|$$

for each $i \in [r]$, so $\bar{\mathcal{X}}$ is an α -buffer for H. This completes the proof in this case and for (i). We continue with (ii) and we will consider $\ell \leq \frac{(1-d)(k+1)cn}{r}$ satisfying $\chi(C_{\ell}^k) \leq k+2$. Pick $y \in [k] \cup \{0\}$ such that $\ell \equiv y \mod k+1$. In particular, we have $\ell \geq y(k+2)$. Let S be a copy of K_{k+2} in the same K_{k+1} -component as the copies of K_{k+1} in \mathcal{F} and let Z_1, \ldots, Z_{k+2} be the vertices of S. We first consider the case $\ell \leq \frac{(k+1)(1-d)n}{r}$ satisfying $\chi(C_{\ell}^k) \leq k+2$. Define $\phi: V(H) \to V(R)$ as follows. For $i \leq \ell$, set

$$\phi(v_i) = \begin{cases} Z_j & \text{if } i \le \ell - y(k+2), \text{ with } j \equiv i \mod k+1, \\ Z_j & \text{if } \ell - y(k+2) < i \le \ell, \text{ with } j \equiv i \mod k+2. \end{cases}$$

For $i > \ell$, given $\phi(v_1), \ldots, \phi(v_{i-1})$, set $\phi(v_i) = V_j$ with $j = \min\{h : |\{b < i : \phi(v_b) = V_h\}| < \ell$ $|V_h|$. Set $X_i := \phi^{-1}(V_i), \ \bar{X}_i := \phi^{-1}(V_i) \setminus C$ for $i \in [r]$ and define $\mathcal{X} := \{X_i\}_{i \in [r]}$ and $\bar{\mathcal{X}} := \{X_i\}_{i \in [r]}$ $\{\bar{X}_i\}_{i\in[r]}$. Since all edges in H have pairs of vertices in C at most k apart in the cyclic order as endpoints and any k + 1 consecutive vertices in the cyclic order are mapped to a copy of K_{k+1} in R, it follows that (H, \mathcal{X}) is an R-partition. Furthermore, for each $i \in [r]$ at most $\frac{\ell}{k+1} \leq (1-d)\frac{n}{r} \leq |V_i|$ vertices in C are mapped to V_i , so \mathcal{X} is a vertex partition of H which is size-compatible with \mathcal{V} . Finally, \bar{X}_i is a set of isolated vertices in H by definition and

$$|\bar{X}_i| = |X_i| - |C \cap X_i| \ge \left(1 - \frac{1 - d}{1 - \varepsilon}\right) |X_i| \ge \alpha |X_i|$$

for each $i \in [r]$, so $\bar{\mathcal{X}}$ is an α -buffer for H. This completes the proof in this case. We are done

if c = 1, so we can assume $c \ge 2$ for the remainder of (ii). Next, we consider $\ell \in \left(\frac{(1-d)(k+1)n}{r}, \frac{(1-d)(k+1)cn}{r}\right]$ satisfying $\chi(C_{\ell}^{k}) \le k+2$. For each $i \in [c-1]$, fix a K_{k+1} -walk W_{i} whose first copy of K_{k} is in T_{i} and whose last is in T_{i+1} , which is of minimal length. We have $|W_i| \leq \binom{r}{k}$ for each $i \in [c-1]$. Let W' be the K_{k+1} -walk obtained by

concatenating W_1, \ldots, W_{c-1} . Fix a K_{k+1} -walk W'' whose first copy of K_k is that of W_1 , whose last is that of W_{c-1} , which includes a copy of K_k from S and is one of minimal length satisfying these conditions. We have $|W''| \leq 2\binom{r}{k}$.

We construct the sequence $Q(W, \overline{U_{11}} \dots \overline{U_{1k}})$ for any K_{k+1} -walk $W = (E_1, E_2, \dots)$ in R and any orientation $\overline{U_{11}} \dots \overline{U_{1k}}$ of E_1 , its first copy of K_k , identically to that in (i). Let $U_{11} \dots U_{1k}$ be the first copy of K_k in W_1 and orient it as $\overline{U_{11}} \dots \overline{U_{1k}}$. Construct $Q(W', \overline{U_{11}} \dots \overline{U_{1k}})$. Then, the first copy of $K_k \underbrace{U_{i1}} \dots \underbrace{U_{ik}}_{ik}$ and the last copy of $K_k \underbrace{U'_{i1}} \dots \underbrace{U'_{ik}}_{ik}$ of each W_i obtain orientations, say $\overline{U_{i1}} \dots \overline{U'_{ik}}$ and $\overline{U'_{i1}} \dots \underbrace{U'_{ik}}_{ik}$. Construct $Q(W_i, \overline{U_{i1}} \dots \overline{U'_{ik}})$ for $i \in [c]$. Clearly, there are sequences $\overline{Q_i}$ of vertices in T_i for 1 < i < c, such that $Q(W', \overline{U_{11}} \dots \overline{U'_{1k}})$ is the concatenation of

$$Q(W_1, \overrightarrow{U_{11} \dots U_{1k}}), \overline{Q}_2, Q(W_2, \overrightarrow{U_{21} \dots U_{2k}}), \dots, \overline{Q}_{c-1}, Q(W_{c-1}, \overrightarrow{U_{(c-1)1} \dots U_{(c-1)k}}).$$

Let $\overline{Q}_1 := T_1 - U_{11} \dots U_{1k}$ and $\overline{Q}_c := T_c - U_{c1} \dots U_{ck}$. Define $f_i \equiv |\overline{Q}_i| \mod k + 1$ for $i \in [c]$. Let Q_i^* denote $Q(W_i, \overline{U_{i1} \dots U_{ik}})$ for $i \in [c-1]$ and let Q'' denote $Q(W'', \overline{U_{1k} \dots U_{11}})$. Define $q_i := |Q_i^*|$ for $i \in [c-1]$ and q'' := |Q''|. For a sequence Q of vertices of R, let $(Q)_h$ denote the hth term of Q. Let $T_i = Y_{i1} \dots Y_{i(k+1)}$ for all $i \in [c]$ be such that $\overline{Y_{i2} \dots Y_{i(k+1)}}$ is the oriented last copy of K_k of W_{i-1} in Q_{i-1} for $2 \leq i \leq c$ and $\overline{Y_{1(k+1)} \dots Y_{12}}$ is the oriented first copy of K_k of W'' in Q''.

Let $\alpha := \sum_{i=1}^{c-1} (q_i + f_i) + f_c + q''$. Pick $x \in [k] \cup \{0\}$ such that $\ell - \alpha \equiv x \mod k + 1$. Let Z_{k+2}, \ldots, Z_3 be the last k consecutive terms of Q'' which correspond to a copy of K_k in S. Define Q''' as the result of inserting x copies of Z_{k+2}, \ldots, Z_1 into Q'' right before the last occurrence of Z_{k+2}, \ldots, Z_3 in Q''. Let q''' := |Q'''|. Define $\alpha_x := \sum_{i=1}^{c-1} (q_i + f_i) + f_c + q''' = \alpha + x(k+2)$. Define the following.

$$p_{0} := \max\left\{p \in \mathbb{Z} \mid \ell \geq p(1-d)(k+1)\frac{n}{r} + \alpha_{x}\right\};$$

$$t_{i} = \begin{cases} (1-d)\frac{n}{r} & \text{if } i \in [p_{0}] \\ \frac{\ell-\alpha_{x}}{k+1} - p_{0}(1-d)\frac{n}{r} & \text{if } i = p_{0} + 1 \\ 0 & \text{if } i > p_{0} + 1; \end{cases}$$

$$L_{0} = 0, L_{j} = \sum_{i=1}^{j} [t_{i}(k+1) + q_{i} + f_{i}] \text{ for } j \in [c-1];$$

$$M_{j} = L_{j-1} + t_{j}(k+1) + f_{j} \text{ for } j \in [c].$$

Define $\phi: V(H) \to V(R)$ as follows. For $i \leq \ell$, set

$$\phi(v_i) = \begin{cases} Y_{jh} & \text{if } L_{j-1} < i \le M_j, \text{ with } h \equiv i - L_{j-1} \mod k+1\\ (Q_j^*)_{i-M_j} & \text{if } M_j < i \le L_j\\ (Q''')_{M_c+q'''+1-i} & \text{if } M_c < i \le \ell. \end{cases}$$

For $i > \ell$, given $\phi(v_1), \ldots, \phi(v_{i-1})$, set $\phi(v_i) = V_j$ with $j = \min\{h : |\{b < i : \phi(v_b) = V_h\}| < |V_h|\}.$

Set $X_i := \phi^{-1}(V_i)$, $\bar{X}_i := \phi^{-1}(V_i) \setminus C$ for $i \in [r]$. Define $\mathcal{X} := \{X_i\}_{i \in [r]}$ and $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$. Since all edges in H have pairs of vertices in C at most k apart in the cyclic order as endpoints and any k+1 consecutive vertices in the cyclic order are mapped to a copy of K_{k+1} in R, it follows that (H, \mathcal{X}) is an R-partition. Furthermore, for each $i \in [r]$ at most $(1-d)\frac{n}{r} + \frac{3r\binom{n}{k} + k(k+2)}{k+1} \leq (1-d+\varepsilon)\frac{n}{r} \leq (1-\varepsilon)\frac{n}{r} \leq |V_i|$ vertices in C are mapped to V_i , so \mathcal{X} is a vertex partition of H which is size-compatible with \mathcal{V} . Finally, \bar{X}_i is a set of isolated vertices in H by definition and

$$\bar{X}_i| = |X_i| - |C \cap X_i| \ge \left(1 - \frac{1 - d + \varepsilon}{1 - \varepsilon}\right) |X_i| \ge \alpha |X_i|$$

for each $i \in [r]$, so $\bar{\mathcal{X}}$ is an α -buffer for H. This completes the proof in this case and for (ii).

Now we prove (iii). Fix $k \ge 3$, d > 0 and let $\Delta = 2k$, $\Delta_J = k$, $\kappa = 2$, $\alpha = \frac{d}{2}$, $\zeta = 1$. Now Lemma A.1 outputs ε_0 , $\rho_0 > 0$. We choose

$$\varepsilon_{EL} = \min\left\{\frac{\varepsilon_0}{k+3}, \frac{d^2}{8(k+1)}\right\}.$$

Given $0 < \varepsilon < \varepsilon_{EL}, r_{EL} \in \mathbb{N}$, Lemma A.1 outputs $n_{BL} \in \mathbb{N}$. We choose

$$n_{EL} = \max\left\{n_{BL}, \frac{6r_{EL}^{k+2}}{\varepsilon}, \frac{4r_{EL}}{\rho_0}\right\}.$$

Let $n \geq n_{EL}$, let G be a graph on n vertices and let R^* be an (ε, d) -reduced graph of G on $r \leq r_{EL}$ vertices. Let V'_0, V'_1, \ldots, V'_r be the vertex classes of the (ε, d) -regular partition of G which gives rise to R^* . Let \mathcal{T}' be the given connected K_{k+1} -factor in R^* with $t := |\mathcal{T}'|$ copies of K_{k+1} . Let T'_1, \ldots, T'_t be the copies of K_{k+1} of \mathcal{T}' . Let $A' := \{u_{i,j} \mid (i,j) \in [2] \times [k]\}$.

Consider $T'_i = X'_{i,1} \dots X'_{i,(k+1)}$ for $i \in [t]$. Let $j \in [k+1]$. Remove the vertices of $A \cup A'$ from $X'_{i,j}$ to obtain $X_{i,j}$. We have $|X_{i,j}| \ge \varepsilon |X'_{i,j}|$ and $|X_{i,h}| \ge \varepsilon |X'_{i,h}|$, so the (ε, d) -regularity of $(X'_{i,j}, X'_{i,h})$ implies that $(X_{i,j}, X_{i,h})$ is $(2\varepsilon, d - \varepsilon)$ -regular.

Let $\{V_0, \ldots, V_r\}$ be the new vertex partition obtained by replacing each $X'_{i,j}$ with $X_{i,j}$ and let $\mathcal{V} := \{V_1, \ldots, V_r\}$. Let R be the $(2\varepsilon, d - \varepsilon)$ -full-reduced graph of the partition \mathcal{V} . Every edge of R^* carries over to R, and let V_i be the vertex of R corresponding to V'_i in R^* . Let \mathcal{T} be the connected K_{k+1} -factor in R corresponding to \mathcal{T}' . Let T_1, \ldots, T_t be the copies of K_{k+1} in \mathcal{T} , with T_i corresponding to T'_i for all $i \in [t]$. Let G^* be the subgraph of G induced on \mathcal{V} . Let $n^* := |V(G^*)|$. Here (G^*, \mathcal{V}) is a $(2\varepsilon, d-\varepsilon)$ -regular R-partition. Note that $|V_i| \ge (1-3\varepsilon)\frac{n}{r} \ge \frac{n}{2r_{EL}}$ for all $i \in [r]$, so \mathcal{V} is 2-balanced.

Let $\ell' = \ell - 2k$. Let H be a copy of $P_{\ell'}^k$ together with additional isolated vertices so that it has n^* vertices. Let w_1, \ldots, w_{n^*} be its vertices, with $w_1, \ldots, w_{\ell'}$ being the vertices of the copy of $P_{\ell'}^k$ in a path order. Let $P := \{w_i : i \in [\ell']\}$. Suppose that we have a vertex partition $\mathcal{X} := \{X_i\}_{i \in [r]}$ of H and a family $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$ of subsets of V(H) such that \mathcal{X} is size-compatible with \mathcal{V} , (H, \mathcal{X}) is an R-partition, and $\bar{\mathcal{X}}$ is an α -buffer for H. Suppose further that for each $j \in [k]$ we have $X_{1,j} = V_i$ and $X_{2,j} = V_h$ with i, h such that $w_j \in X_i$ and $w_{\ell'-j+1} \in X_h$. Define $\mathcal{I} := \{I_x\}_{x \in V(H)}$ and $\mathcal{J} := \{J_x\}_{x \in V(H)}$ as follows.

$$I_{w_j} = \begin{cases} V_i \cap \Gamma(u_{1,j}, \dots, u_{1,k}) & \text{for } j \in [k], \text{ with } w_j \in X_i \\ V_i & \text{for } k < j \le \ell' - k, \text{ with } w_j \in X_i \\ V_i \cap \Gamma(u_{2,k}, \dots, u_{2,(\ell'-j+1)}) & \text{for } \ell' - k < j \le \ell', \text{ with } w_j \in X_i; \end{cases}$$
$$J_{w_j} = \begin{cases} \{u_{1,j}, \dots, u_{1,k}\} & \text{for } j \in [k] \\ \varnothing & \text{for } k < j \le \ell' - k \\ \{u_{2,k}, \dots, u_{2,(\ell'-j+1)}\} & \text{for } \ell' - k < j \le \ell'. \end{cases}$$

Since $|\Gamma(u_{i,j},\ldots,u_{i,k}) \cap X_{i,j}| \geq \frac{2dn}{r} - \frac{2\varepsilon n}{r} \geq \frac{3dn}{2r}$ for each pair $(i,j) \in [2] \times [k]$ and $|V_i| \geq (1-2\varepsilon)\frac{n}{r} \geq \frac{2}{\rho_0}$, \mathcal{I} and \mathcal{J} form a $(\rho_0, \zeta, \Delta, \Delta_J)$ -restriction pair.

Then, by Lemma A.1 we will have an embedding $\phi: V(H) \to V(G^*)$ such that w_j is adjacent to $u_{1,j}, \ldots, u_{1,k}$ for $j \in [k]$ and w_j is adjacent to $u_{2,k}, \ldots, u_{2,(\ell-j+1)}$ for $\ell' - k < j \leq \ell'$. Together with $u_1, \ldots, u_k, v_1, \ldots, v_k$, this will yield a copy of P_ℓ^k which starts in u_1, \ldots, u_k and ends in v_1, \ldots, v_k (in those orders), contains no element of A and has at most $(d + \varepsilon)n$ vertices not in $\bigcup \mathcal{T}'$, which will then complete our proof of (iii). Therefore, for suitable values of ℓ' it remains to find a vertex partition \mathcal{X} of H and a family $\bar{\mathcal{X}}$ of subsets of V(H) such that \mathcal{X} is size-compatible with $\mathcal{V}, (H, \mathcal{X})$ is an R-partition, $\bar{\mathcal{X}}$ is an α -buffer for H and for each $j \in [k]$ we have $X_{1,j} = V_i$ and $X_{2,j} = V_h$ with i, h such that $w_j \in X_i$ and $w_{\ell'-j+1} \in X_h$. We consider $\ell \in \left(3r^{k+1}, \frac{(1-d)(k+1)tn}{r}\right]$. Let S be a copy of K_{k+2} in the same K_{k+1} -component

We consider $\ell \in \left(3r^{k+1}, \frac{(1-d)(k+1)tn}{r}\right]$. Let S be a copy of K_{k+2} in the same K_{k+1} -component of R as the copies of K_{k+1} in \mathcal{T} and let Z_1, \ldots, Z_{k+2} be the vertices of S. Fix a K_{k+1} -walk W_0 whose first copy of K_k is $X_{1,1} \ldots X_{1,k}$ and whose last is in T_1 , which is of minimal length. For each $i \in [t-1]$, fix a K_{k+1} -walk W_i whose first copy of K_k is in T_i and whose last is in T_{i+1} , which is of minimal length. Fix a K_{k+1} -walk W_t whose first copy of K_k is in T_t , whose last is $X_{2,1} \ldots X_{2,k}$, which includes a copy of K_k from S and is one of minimal length satisfying these conditions. We have $|W_i| \leq {r \choose k}$ for $i \in [t-1] \cup \{0\}$ and $|W_t| \leq 2{r \choose k}$. Let W' be the K_{k+1} -walk obtained by concatenating W_0, \ldots, W_t .

because obtained by concatenating W_0, \ldots, W_t . We construct the sequence $Q(W, \overline{U_{11}} \ldots \overline{U_{1k}})$ for any K_{k+1} -walk $W = (E_1, E_2, \ldots)$ in Rand any orientation $\overrightarrow{U_{11}} \ldots \overrightarrow{U_{1k}}$ of E_1 , its first copy of K_k , identically to that in (i). Orient $X_{1,1} \ldots X_{1,k}$ as $\overrightarrow{X_{1,1}} \ldots \overrightarrow{X_{1,k}}$. Construct $Q(W', \overrightarrow{X_{1,1}} \ldots \overrightarrow{X_{1,k}})$. Then, the first copy of K_k $\overrightarrow{U_{i1}} \ldots \overrightarrow{U_{ik}}$ and the last copy of $K_k U'_{i1} \ldots U'_{ik}$ of each W_i obtain orientations, say $\overrightarrow{U_{i1}} \ldots \overrightarrow{U_{ik}}$ and $\overrightarrow{U'_{i1}} \ldots \overrightarrow{U'_{ik}}$. Construct $Q(W_i, \overrightarrow{U_{i1}} \ldots \overrightarrow{U_{ik}})$ for $i \in [t] \cup \{0\}$. Clearly, there are sequences $\overline{Q_i}$ of vertices in T_i for $i \in [t]$, such that $Q(W', \overrightarrow{X_{1,1}} \ldots \overrightarrow{X_{1,k}})$ is the concatenation of

$$Q(W_0, \overrightarrow{X_{1,1} \dots X_{1,k}}), \overline{Q}_1, Q(W_1, \overrightarrow{U_{11} \dots U_{1k}}), \dots, \overline{Q}_t, Q(W_t, \overrightarrow{U_{t1} \dots U_{tk}}).$$

Define $f_i \equiv |\bar{Q}_i| \mod k + 1$ for $i \in [t]$. Let Q_0^* denote $Q(W_0, \overline{X_{1,1} \dots X_{1,k}})$ and let Q_i^* denote $Q(W_i, \overline{U_{i1} \dots U_{ik}})$ for $i \in [t]$. Define $q_i := |Q_i^*|$ for $i \in [t] \cup \{0\}$. For a sequence Q of vertices of R, let $(Q)_h$ denote the hth term of Q. For each $i \in [t]$ let $T_i = Y_{i1} \dots Y_{i(k+1)}$ be such that $\overrightarrow{Y_{i2} \dots Y_{i(k+1)}}$ is the oriented last copy of K_k of W_{i-1} in Q_{i-1}^* .

 $\overrightarrow{Y_{i2} \ldots Y_{i(k+1)}}$ is the oriented last copy of K_k of W_{i-1} in Q_{i-1}^* . Let $\alpha := q_0 + \sum_{i=1}^t (q_i + f_i)$. Pick $x \in [k] \cup \{0\}$ such that $\ell' - \alpha \equiv x \mod k+1$. Let Z_3, \ldots, Z_{k+2} be the first k consecutive terms of Q_t^* which correspond to a copy of K_k in S. Define Q' as the result of inserting x copies of Z_1, \ldots, Z_{k+2} into Q_t^* right after the first occurrence of Z_3, \ldots, Z_{k+2} in Q_t^* . Let q' := |Q'|. Define $\alpha_x := q_0 + \sum_{i=1}^{t-1} (q_i + f_i) + f_t + q' = \alpha + x(k+2)$. Define the following.

$$p_{0} := \max\left\{p \in \mathbb{Z} \mid \ell' \ge p(1-d)(k+1)\frac{n}{r} + \alpha_{x}\right\};$$

$$t_{i} = \begin{cases} (1-d)\frac{n}{r} & \text{if } i \in [p_{0}] \\ \frac{\ell'-\alpha_{x}}{k+1} - p_{0}(1-d)\frac{n}{r} & \text{if } i = p_{0} + 1 \\ 0 & \text{if } i > p_{0} + 1; \end{cases}$$

$$L_{0} = q_{0}, L_{j} = q_{0} + \sum_{i=1}^{j} [t_{i}(k+1) + q_{i} + f_{i}] \text{ for } j \in [t-1];$$

$$M_{0} = 0, M_{j} = L_{j-1} + t_{j}(k+1) + f_{j} \text{ for } j \in [t].$$

Define $\phi: V(H) \to V(R)$ as follows. For $i \leq \ell'$, set

$$\phi(v_i) = \begin{cases} Y_{jh} & \text{if } L_{j-1} < i \le M_j, \text{ with } h \equiv i - L_{j-1} \mod k+1 \\ (Q_j^*)_{i-M_j} & \text{if } M_j < i \le L_j \\ (Q')_{i-M_t} & \text{if } M_t < i \le \ell'. \end{cases}$$

For $i > \ell'$, given $\phi(v_1), \dots, \phi(v_{i-1})$, set $\phi(v_i) = V_j$ with $j = \min\{h : |\{b < i : \phi(v_b) = V_h\}| < |V_h|\}.$

Set $X_i := \phi^{-1}(V_i)$, $\bar{X}_i := \phi^{-1}(V_i) \setminus P$ for $i \in [r]$. Define $\mathcal{X} := \{X_i\}_{i \in [r]}$ and $\bar{\mathcal{X}} := \{\bar{X}_i\}_{i \in [r]}$. Since all edges in H have pairs of vertices in P at most k apart in the path order as endpoints and any k + 1 consecutive vertices in the cyclic order are mapped to a copy of K_{k+1} in R, it follows that (H, \mathcal{X}) is an R-partition. Furthermore, for each $i \in [r]$ at most $(1 - d)\frac{n}{r} + \frac{3r\binom{n}{k} + k(k+2)}{k+1} \leq (1 - d + \varepsilon)\frac{n}{r} \leq (1 - 2\varepsilon)\frac{n}{r} \leq |V_i|$ vertices in P are mapped to V_i , so \mathcal{X} is a vertex partition of Hwhich is size-compatible with \mathcal{V} . Finally, \bar{X}_i is a set of isolated vertices in H by definition and

$$|\bar{X}_i| = |X_i| - |P \cap X_i| \ge \left(1 - \frac{1 - d + \varepsilon}{1 - 3\varepsilon}\right) |X_i| \ge \alpha |X_i|$$

for each $i \in [r]$, so $\bar{\mathcal{X}}$ is an α -buffer for H. This completes the proof for (iii).