# LINEAR FILTERING WITH FRACTIONAL NOISES: LARGE TIME AND SMALL NOISE ASYMPTOTICS 

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#### Abstract

The classical state-space approach to optimal estimation of stochastic processes is efficient when the driving noises are generated by martingales. In particular, the weight function of the optimal linear filter, which solves a complicated operator equation in general, simplifies to the Riccati ordinary differential equation in the martingale case. This reduction lies in the foundations of the Kalman-Bucy approach to linear optimal filtering. In this paper we consider a basic Kalman-Bucy model with noises, generated by independent fractional Brownian motions, and develop a new method of asymptotic analysis of the integro-differential filtering equation arising in this case. We establish existence of the steady-state error limit and find its asymptotic scaling in the high signal-to-noise regime. Closed form expressions are derived in a number of important cases.


## 1. Introduction

1.1. The Kalman-Bucy problem. In its most basic form, the Kalman-Bucy filtering problem [12] is concerned with estimation of the state process, generated by the linear stochastic equation

$$
\begin{equation*}
X_{t}=\beta \int_{0}^{t} X_{s} d s+W_{t} \tag{1.1}
\end{equation*}
$$

given a trajectory of the observation process

$$
\begin{equation*}
Y_{t}=\mu \int_{0}^{t} X_{s} d s+\sqrt{\varepsilon} V_{t} \tag{1.2}
\end{equation*}
$$

Here $\beta$ and $\mu \neq 0$ are fixed real constants, $\varepsilon>0$ is the observation noise intensity parameter, and $W=\left(W_{t} ; t \in \mathbb{R}_{+}\right)$and $V=\left(V_{t} ; t \in \mathbb{R}_{+}\right)$are independent Brownian motions.

The filtering problem consists of finding the optimal estimator $\widehat{X}_{t}=\mathbb{E}\left(X_{t} \mid \mathcal{F}_{t}^{Y}\right)$, whose mean squared error $P_{t}=\mathbb{E}\left(X_{t}-\widehat{X}_{t}\right)^{2}$ is minimal among all functionals, measurable with respect to $\mathcal{F}_{t}^{Y}=\sigma\left\{Y_{s}, s \leq t\right\}$. For the linear Gaussian model (1.1)-(1.2), this problem has a famously elegant solution, discovered in [12]. The filtering estimator in this case can be generated by the stochastic differential equation

$$
d \widehat{X}_{t}=\beta \widehat{X}_{t} d t+\frac{\mu P_{t}}{\varepsilon}\left(d Y_{t}-\mu \widehat{X}_{t} d t\right)
$$

[^0]and the corresponding minimal error $P_{t}$ solves the Riccati o.d.e.
\[

$$
\begin{equation*}
\dot{P}_{t}=2 \beta P_{t}+1-(\mu / \sqrt{\varepsilon})^{2} P_{t}^{2} \tag{1.3}
\end{equation*}
$$

\]

subject to zero initial conditions. Elementary analysis of (1.3) shows that the filtering error converges to the steady-state limit

$$
\begin{equation*}
\lim _{T \rightarrow \infty} P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{\beta+\sqrt{\beta^{2}+\mu^{2} / \varepsilon}}{\mu^{2} / \varepsilon} \tag{1.4}
\end{equation*}
$$

and reveals its scaling with respect to the noise intensity

$$
\begin{equation*}
P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{\sqrt{\varepsilon}}{\mu}(1+o(1)), \quad \text { as } \varepsilon \rightarrow 0, \quad \forall T>0 . \tag{1.5}
\end{equation*}
$$

These limiting quantities are of considerable interest, as they exhibit the fundamental accuracy limitations in the problem.
1.2. Fractional noises. A natural generalization of the system (1.1)-(1.2) is obtained by replacing $W$ and $V$ with independent fractional Brownian motions ( fBm ) with the Hurst exponents $H_{1}$ and $H_{2}$, respectively. Recall that the fBm is a centred Gaussian process with covariance function

$$
\begin{equation*}
K(s, t)=\frac{1}{2}\left(s^{2 H}+t^{2 H}-|s-t|^{2 H}\right), \quad s, t \in \mathbb{R}_{+}, \tag{1.6}
\end{equation*}
$$

where $H \in(0,1)$ is its Hurst exponent.
This process coincides with the standard Brownian motion for $H=\frac{1}{2}$, but otherwise exhibits a rich diversity of properties, which makes it an interesting mathematical object and an important tool in modelling, [28]. In particular, it is neither a semi-martingale nor a Markov process. For $H>\frac{1}{2}$ increments of the fBm are positively correlated and have long range dependence

$$
\sum_{n=1}^{\infty} \mathbb{E} V_{1}\left(V_{n+1}-V_{n}\right)=\infty
$$

This property makes it useful in design and analysis of engineering systems, [2].
Consider the integro-differential equation

$$
\begin{equation*}
\frac{\partial}{\partial s} \int_{0}^{T} g_{T}(r) \frac{\partial}{\partial r} K_{V}(r, s) d r+\frac{\mu^{2}}{\varepsilon} \int_{0}^{T} K_{X}(r, s) g_{T}(r) d r=\frac{\mu}{\sqrt{\varepsilon}} K_{X}(s, T) \tag{1.7}
\end{equation*}
$$

where $K_{V}(s, t)$ and $K_{X}(s, t)$ are the covariance functions of the $\mathrm{fBm} V$ and the state process $X$. If this equation has a sufficiently regular solution $g_{T}(\cdot)$ (see Appendix A for some details) then standard calculations, see e.g. [23, Lemma 10.2], show the optimal estimator is given by the stochastic integral

$$
\widehat{X}_{T}=\frac{1}{\sqrt{\varepsilon}} \int_{0}^{T} g_{T}(s) d Y_{s}
$$

and the filtering error is determined by the solution to (1.7) through the formula

$$
\begin{equation*}
P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{\sqrt{\varepsilon}}{\mu}\left(\frac{\partial}{\partial s} \int_{0}^{T} g_{T}(r) \frac{\partial}{\partial r} K_{V}(r, s) d r\right)_{\mid s:=T} \tag{1.8}
\end{equation*}
$$

In the standard Kalman-Bucy case these equations simplify in two ways. First, when the observation noise is white, i.e., $H_{2}=\frac{1}{2}$, the integro-differential terms in (1.7) and (1.8) reduce merely to $g_{T}(s)$, and (1.7) takes the form of an integral equation. Further, when the state noise is white as well, $H_{1}=\frac{1}{2}$, the state process is Markov, and its covariance kernel $K_{X}(s, t)$ factorizes into a product of exponentials. This allows to express solutions to (1.7) in terms of the Riccati o.d.e. (1.3).

In the more general, fractional setting under consideration, equation (1.7) retains its integro-differential form, and questions of solvability and asymptotic behaviour of its solution remained till now mainly open.
(1) How does the filtering error scale with observation noise intensity as $\varepsilon \rightarrow 0$ ?
(2) Does it converge to a limit as $T \rightarrow \infty$ ?
(3) How are these two asymptotics related?
(4) Do the limits admit of reasonably explicit expressions?

Answering such questions requires an entirely different approach, which is the main focus of the paper.

## 2. The main Results

Let $X$ and $Y$ be the processes generated by equations (1.1) and (1.2), driven by independent fBm's $W$ and $V$ with the Hurst parameters $H_{1}, H_{2} \in(0,1)$, respectively. Define $P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right):=\mathbb{E}\left(X_{T}-\mathbb{E}\left(X_{T} \mid \mathcal{F}_{T}^{Y}\right)\right)^{2}$. To avoid trivialities $\mu \neq 0$ is assumed throughout, but the values of all other parameters can be arbitrary.
2.1. General asymptotics. Our principal result is the following theorem.

Theorem 2.1. The large time limit exists

$$
\begin{equation*}
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\lim _{T \rightarrow \infty} P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right), \tag{2.1}
\end{equation*}
$$

and, for any $T>0$, the filtering error satisfies the scaling property

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \varepsilon^{-\nu} P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=P_{\infty}(0, \mu) \quad \text { with } \nu=\frac{H_{1}}{1+H_{1}-H_{2}} \text {. } \tag{2.2}
\end{equation*}
$$

## Remark 2.2.

a) As in the standard Kalman-Bucy problem, the first order term of the small noise asymptotics (2.2) does not depend on the interval length $T$ or the drift of the state process $\beta$, cf. (1.5). This is not entirely intuitive in the fractional case, since the memory of the optimal filter in the non-Markov setup, and the more so for processes with long range dependence, does not have to be a priori negligible as $\varepsilon \rightarrow 0$.
b) The rate $\nu$ in (2.2) coincides with the optimal minimax rate in the nonparametric problem of estimating a deterministic function observed in fractional type noise, [34]. This agrees with the smoothness of the fBm paths, which are Holder continuous with an exponent arbitrarily close to $H$. In particular, the estimators suggested in [34] should be rate optimal for the filtering problem under consideration, however, with a suboptimal
constant. The dependence of $\nu$ on $H_{1}$ and $H_{2}$ agrees with the intuition, that the filtering accuracy should improve with path regularity of the processes which generate the noises.
2.2. Special cases. In principle, the limit in (2.1) is derived in the proof as an explicit but a rather cumbersome expression. It can be significantly simplified in a number of meaningful cases, as detailed in the theorems below. The key ingredient of the emerging formulas is the complex valued structural function

$$
\begin{equation*}
\Lambda\left(z ; H_{1}, H_{2}\right)=\left(z^{2}-\beta^{2}\right) \kappa\left(H_{2}\right)\left(\frac{z}{i}\right)^{1-2 H_{2}}-\frac{\mu^{2}}{\varepsilon} \kappa\left(H_{1}\right)\left(\frac{z}{i}\right)^{1-2 H_{1}} \tag{2.3}
\end{equation*}
$$

where $z$ takes values in the upper half of the complex plane and

$$
\begin{equation*}
\kappa(H)=\Gamma(2 H+1) \sin (\pi H) \tag{2.4}
\end{equation*}
$$

Its domain is extended to the lower half-plane through conjugation

$$
\Lambda\left(z ; H_{1}, H_{2}\right)=\overline{\Lambda\left(\bar{z} ; H_{1}, H_{2}\right)}
$$

The structure of the filtering problem turns out to be largely determined by the configuration of zeros of this function. A simple calculation shows that $\Lambda\left(z ; H_{1}, H_{2}\right)$ has the unique complex zero $z_{0}$ in the first quadrant when $H_{1}>H_{2}$. As $H_{1}$ approaches $H_{2}$ this zero moves towards positive real semiaxis, and, at $H_{1}=H_{2}$, degenerates to the purely real value

$$
t_{0}=\sqrt{\beta^{2}+\mu^{2} / \varepsilon}
$$

When $H_{1}<H_{2}$ it has no zeros at all.
2.2.1. State/observation noises of the same type. The following result details the limiting behaviour of the filtering error, when the state and observation noises have the same Hurst exponent.

Theorem 2.3. Let $H_{1}=H_{2}=: H \in(0,1)$, then

$$
\begin{equation*}
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{1}{2} \Gamma(2 H+1) t_{0}^{-2 H}\left(1+\sin (\pi H) \frac{t_{0}+\beta}{t_{0}-\beta}\right) \tag{2.5}
\end{equation*}
$$

and, consequently ${ }^{1}$,

$$
\begin{equation*}
P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) \asymp \frac{1}{2} \Gamma(2 H+1)(1+\sin (\pi H))\left(\varepsilon / \mu^{2}\right)^{H}, \quad \text { as } \varepsilon \rightarrow 0 \tag{2.6}
\end{equation*}
$$

Remark 2.4. Formula (2.5) was previously derived in [15] for $H \in\left(\frac{1}{2}, 1\right)$, using a completely different method, based on the innovation representation of the fBm from [25]. This approach does not easily extend to the complementary case $H \in\left(0, \frac{1}{2}\right)$, unlike the method suggested in this paper.

[^1]2.2.2. Fractional state/white observation noise. To formulate further results, define the limit
$$
\Lambda^{+}\left(t ; H_{1}, H_{2}\right)=\lim _{\operatorname{Im}(z)>0, z \rightarrow t} \Lambda\left(z ; H_{1}, H_{2}\right), \quad t \in \mathbb{R}_{+}
$$
which coincides with the expression in (2.3) after replacing $z$ with $t \in \mathbb{R}_{+}$. Let $\theta\left(t ; H_{1}, H_{2}\right)$ be the argument of $\Lambda^{+}\left(t ; H_{1}, H_{2}\right)$, chosen so that it varies continuously with $t \in \mathbb{R}_{+}$and $\lim _{t \rightarrow \infty} \theta\left(t ; H_{1}, H_{2}\right) \in[-\pi, \pi]$. This choice defines $\theta\left(t ; H_{1}, H_{2}\right)$ in the unique way, and it is a completely explicit function.

The following theorem details the precise error asymptotics in the filtering problem with fractional state process and white noise observations.

Theorem 2.5. Let $H:=H_{1} \in(0,1) \backslash\left\{\frac{1}{2}\right\}$ and $H_{2}=\frac{1}{2}$, then

$$
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{\varepsilon}{\mu^{2}}\left(\frac{1}{\pi} \int_{0}^{\infty} \theta\left(t ; H, \frac{1}{2}\right) d t+\beta+\left\{\begin{array}{ll}
2 \operatorname{Re}\left(z_{0}\right) & \text { if } H>\frac{1}{2}  \tag{2.7}\\
0 & \text { if } H<\frac{1}{2}
\end{array}\right\}\right)
$$

where $z_{0}$ is the unique zero of $\Lambda\left(z ; H, \frac{1}{2}\right)$ in the first quadrant. Consequently,

$$
\begin{equation*}
P_{T}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) \asymp \frac{\kappa(H)^{\frac{1}{2 H+1}}}{\sin \frac{\pi}{2 H+1}}\left(\varepsilon / \mu^{2}\right)^{\frac{2 H}{2 H+1}}, \quad \text { as } \varepsilon \rightarrow 0 \tag{2.8}
\end{equation*}
$$

## Remark 2.6.

a) In the stable case with $\beta<0$, the following alternative expression for the filtering error can be obtained, using the spectral theory of stationary processes,

$$
\begin{equation*}
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{\varepsilon}{\mu^{2}} \frac{1}{2 \pi} \int_{-\infty}^{\infty} \log \left(1+\frac{\mu^{2}}{\varepsilon} \kappa(H) \frac{|\omega|^{1-2 H}}{\beta^{2}+\omega^{2}}\right) d \omega \tag{2.9}
\end{equation*}
$$

The spectral approach is not applicable in the non-stationary case $\beta \geq 0$ and, in fact, this formula can be seen to coincide with (2.7) only for $\beta<0$, but not otherwise.
b) The expression in (2.7) has the right and the left limits at $H=\frac{1}{2}$, which coincide with the classic formula (1.4). While the root of $\Lambda\left(z ; H_{1}, H_{2}\right)$ and the integral in (2.7) do not seem to admit any closed form formulae, both are not hard to compute numerically for any concrete values of the parameters.
c) Formula (2.8) can also be obtained using asymptotic approximation of the eigenvalues and eigenfunctions of the covariance operator of the fractional Ornstein-Uhlenbeck (fOU) process, [18]. This approximation however is not uniform with respect to $T$, and therefore the large time limiting error (2.7) cannot be derived using the same method.
2.2.3. White state/fractional observation noise. To formulate the results in the complementary case of white state and fractional observation noises define

$$
\begin{equation*}
X(z)=(-z)^{3 / 2-H} \exp \left(\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta\left(t ; \frac{1}{2}, H\right)}{t-z} d t\right), \quad z \in \mathbb{C} \backslash \mathbb{R}_{+} \tag{2.10}
\end{equation*}
$$

with $H \in(0,1)$. This function is holomorphic on the cut plane with a jump discontinuity across the positive real semiaxis $\mathbb{R}_{+}$. In the course of the proof the limits $X^{+}(|\beta|)$ and $X^{-}(|\beta|)$ are shown to coincide, and their common value will be denoted by $X(|\beta|)$.

Theorem 2.7. Let $H_{1}=\frac{1}{2}$ and $H:=H_{2} \in(0,1) \backslash\left\{\frac{1}{2}\right\}$. Then

$$
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)=\frac{1}{2 \beta}\left(\frac{X(-\beta)}{X(\beta)}\left\{\begin{array}{ll}
\left|\frac{z_{0}+\beta}{z_{0}-\beta}\right|^{2} & \text { if } H<\frac{1}{2}  \tag{2.11}\\
1 & \text { if } H>\frac{1}{2}
\end{array}\right\}-1\right)
$$

where $z_{0}$ is the zero of $\Lambda\left(z ; \frac{1}{2}, H\right)$ in the first quadrant. Consequently,

$$
\begin{equation*}
P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right) \asymp \frac{\kappa(H)^{\frac{1}{3-2 H}}}{\sin \frac{\pi}{3-2 H}}\left(\varepsilon / \mu^{2}\right)^{\frac{1}{3-2 H}}, \quad \varepsilon \rightarrow 0 . \tag{2.12}
\end{equation*}
$$

Remark 2.8. Numerical evaluation of $X(z)$ at $z:=\beta>0$ involves computation of the Cauchy principal value of the integral in (2.10). The following identity, proved in Lemmas 7.1 and 7.4 below,

$$
X(\beta) X(-\beta)=\frac{1}{\kappa(H)} \frac{\mu^{2}}{\varepsilon}\left(\frac{1}{\left|\beta^{2}-z_{0}^{2}\right|^{2}}\right)^{\mathbf{1}_{\left\{H<\frac{1}{2}\right\}}},
$$

can be more convenient for this purpose.

## 3. Related literature

3.1. Integro-differential equations. To the best of our knowledge integro-differential equations such as (1.7) do not have a general theory. As mentioned above, for the white observation noise $H_{2}=\frac{1}{2}$, problem (1.7) reduces to integral equation of the second kind. Such equations have been studied since the pioneering works of Fredholm, and their unique solvability in various spaces is very well understood. Nevertheless, even in this relatively standard setting, quantifying dependence of the solutions on parameters, such as $T$ and $\varepsilon$ in our context, can be a highly nontrivial matter. Essentially, the only case in which a complete theory is available is that of Kalman-Bucy, when reduction of (1.7) to the Riccati o.d.e. is possible. This reduction has far reaching implications, way beyond the scalar problem considered in this paper. It leads to a complete characterisation of the limit behaviour of the optimal error in terms of such notions as controllability and observability (see, e.g., [21]).
3.2. Stationary problem. The stationary version of the filtering problem for (1.7) on the semi-infinite time horizon $[0, \infty)$ can be solved within the framework of the KolmogorovWiener spectral theory, [30]. In some cases it yields closed form formulas for the steadystate error in the form of integrals over spectral densities such as (2.9). However this approach is strictly limited to the stable state equation (1.1) with $\beta<0$, even in the standard Kalman-Bucy problem. In fact, overcoming this difficulty was the main impetus behind the state-space approach pioneered in [12].
3.3. Nonlinear filtering. Optimal error analysis in the more general, nonlinear filtering problem attracted much attention in the more recent past. Questions of existence and uniqueness of the large time limit of the filtering error was addressed first in [20] and continued to generate much research over the years; surveys of different approaches can be found in [1], [3], [6], [19], [32]. The steady state error is never explicit beyond the linear problem, and consequently various techniques of computing its lower bounds have
been suggested, see [36]. Exact small noise error asymptotics was derived for a number of models, including diffusions [27], [37], [26] and finite state chains [13], [31]. Let us stress, however, that all these results are concerned exclusively with the Markov case and therefore do not apply to the filtering models with fractional noises.
3.4. Filtering with fractional noises. Filtering in systems driven by the fractional Brownian motion have been addressed by many authors, including [22], [8], [14], [16], [35], [9], both in linear and nonlinear settings. However, most of the literature is concerned with derivation of the filtering equation, rather than evaluation of the optimal error, which remained mainly elusive so far.
3.5. Contribution of this paper. The contribution of this paper is twofold. From the perspective of stochastic filtering theory, it suggests a method of asymptotic error analysis in a non-Markov system with fractional type driving noises. Existence of the steady-state error beyond the Markov setting remained largely unexplored, and this is probably one of the first systematic takes on the subject. Besides a qualitative asymptotic picture as in Theorem 2.1, our method yields closed form expressions for the filtering error limits in several cases of interest, such as Theorems 2.3, 2.5 and 2.7.

Another contribution is on the more technical side, and it consists of constructing a solution to equation (1.7), which was previously known to be solvable in some special cases, see e.g. [4]. Our approach is inspired by a technique, introduced in the mathematical physics literature, [33], [11], and its recent applications to fractional stochastic processes, [7], [5], [24]. Previously it was used in eigenproblems, i.e., homogeneous integral equations of the second kind, with the objective of approximating the sequence of its solutions. The non-homogeneous problem under consideration in this paper requires a complete revision of this technique at least from two standpoints. First, equation (1.7) has the unique solution, and this time the goal is its asymptotic analysis with respect to parameters. Moreover, the main object of interest is not the solution itself, but its particular functional (1.8).

## 4. Preliminaries

4.1. Notations, conventions and tools. The proof uses some basic tools from complex analysis. Unless otherwise stated, the standard range $z \in(-\pi, \pi]$ will be used for principal branches of the common multivalued functions. We will frequently encounter functions, which are holomorphic on the cut planes $\mathbb{C} \backslash \mathbb{R}$ or $\mathbb{C} \backslash \mathbb{R}_{+}$, with a finite jump discontinuity across the cut. For such a sectionally holomorphic function $\Psi(z)$, the limits across the real line will be denoted by

$$
\begin{aligned}
\Psi^{+}(t) & :=\lim _{\operatorname{Im}(z)>0, z \rightarrow t} \Psi(z), \\
\Psi^{-}(t) & :=\lim _{\operatorname{Im}(z)<0, z \rightarrow t} \Psi(z), \quad t \in \mathbb{R} .
\end{aligned}
$$

Often we will need to solve the Hilbert problem of finding a function $\Psi(z)$, which is sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$and satisfies the boundary condition

$$
\Psi^{+}(t)-\Psi^{-}(t)=\phi(t), \quad t \in \mathbb{R}_{+}
$$

for a given function $\phi(\cdot)$. When $\phi(\cdot)$ is Hölder on $\mathbb{R}_{+} \cup\{\infty\}$, by the Sokhotski-Plemelj theorem, the unique solution to this problem has the form

$$
\Psi(z)=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{\phi(t)}{t-z} d t+P(z)
$$

where $P(z)$ is a polynomial of a finite degree, whose growth as $z \rightarrow \infty$ matches that of $\Psi(z)$. A comprehensive account of such boundary value problems can be found in, e.g., monograph [10].

When dependence on parameters is important, they will be added to the notations: for example, $g(x), g_{T}(x)$ or $g(x ; \varepsilon, T)$ will denote the same function, depending on the context. It will also be convenient to use $\mu_{\varepsilon}:=\mu / \sqrt{\varepsilon}$ and reparameterize the problem by $\alpha_{1}:=2-2 H_{1}$ and $\alpha_{2}:=2-2 H_{2}$, which take values in $(0,2)$. Finally, we will write $r(u) \asymp q(u)$ when $r(u)=q(u)(1+o(1))$ for both $u:=T \rightarrow \infty$ and $u:=\varepsilon \rightarrow 0$.
4.2. Proof preview. In essence, the proof amounts to constructing the solution to (1.7) in a form, more amenable to asymptotic analysis. This is done by exploiting the structure of the Laplace transform of its solution,

$$
\begin{equation*}
\widehat{g}(z)=\int_{0}^{T} e^{-z x} g(x) d x, \quad z \in \mathbb{C} \tag{4.1}
\end{equation*}
$$

revealed by the representation formula, which is derived in Lemma 5.1 below,

$$
\begin{equation*}
\widehat{g}(z)=-\frac{1}{\Lambda(z)}\left((z+\beta)\left(\Phi_{0}(z)+e^{-z T} \Phi_{1}(-z)\right)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)\right) . \tag{4.2}
\end{equation*}
$$

This expression involves the following elements.
(i) The complex function

$$
N_{\alpha}(z)=\kappa_{\alpha} \begin{cases}(z / i)^{\alpha-1}, & \operatorname{Im}\{z\}>0  \tag{4.3}\\ (-z / i)^{\alpha-1}, & \operatorname{Im}\{z\}<0\end{cases}
$$

where, cf. (2.4),

$$
\kappa_{\alpha}=\kappa\left(1-\frac{\alpha}{2}\right)=\frac{(1-\alpha)(1-\alpha / 2)}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\alpha}{2} \pi}>0, \quad \alpha \in(0,2) \backslash\{1\} .
$$

This function is sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}$, and its limits across the real line satisfy the obvious symmetries

$$
\begin{equation*}
N_{\alpha}^{+}(t)=N_{\alpha}^{-}(-t) \quad \text { and } \quad N_{\alpha}^{+}(t)=\overline{N_{\alpha}^{-}(t)} . \tag{4.4}
\end{equation*}
$$

(ii) The structural function of the problem, cf. (2.3),

$$
\begin{equation*}
\Lambda(z)=\left(z^{2}-\beta^{2}\right) N_{\alpha_{2}}(z)-\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z), \tag{4.5}
\end{equation*}
$$

which inherits the discontinuity of $N_{\alpha_{j}}(z)$ 's along the real line and is holomorphic elsewhere. It does not vanish on the cut plane when $\alpha_{1}>\alpha_{2}$ and has four simple complex zeros, placed symmetrically in each quadrant, when $\alpha_{1}<\alpha_{2}$. In the case $\alpha_{1}=\alpha_{2}$, it has two purely real zeros. Configuration of zeros has a determining effect on the solution.
(iii) Functions $\Phi_{0}(z)$ and $\Phi_{1}(z)$ are sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$. They are defined explicitly as certain functionals of $g(\cdot)$, involving the Cauchy integrals, but their particular form is inessential, except for the growth (5.4) as $z \rightarrow 0$ and $z \rightarrow \infty$.
(iv) The quantity $\psi(0)$, also determined by $g(\cdot)$, is constant with respect to $z$, see (5.3) below.

The crucial feature of representation (4.2) is its singularities. Since the integration in (4.1) is carried out over a finite interval, the Laplace transform $\widehat{g}(z)$ is an entire function, and thus all singularities in (4.2) must be removable. This includes discontinuity across the real line, whose removal yields equations (5.23) below. These equations bind together the limits $\Phi_{0}^{ \pm}(t)$ and $\Phi_{1}^{ \pm}(t)$ at all $t \in \mathbb{R}_{+}$and can be viewed as boundary conditions on $\mathbb{R}_{+}$ for the functions $\Phi_{0}(z)$ and $\Phi_{1}(z)$, which are holomorphic elsewhere.

Finding all such functions satisfying the particular growth estimates, mentioned in (iii), is known as the Hilbert boundary value problem. In our case, all its solutions can be expressed in terms of auxiliary integral equations of the general form

$$
\begin{equation*}
p(t)=\left(A_{\varepsilon, T} p\right)(t)+f(t), \quad t \in \mathbb{R}_{+}, \tag{4.6}
\end{equation*}
$$

where $A_{\varepsilon, T}$ is an integral operator with an explicit kernel, and $f(\cdot)$ is either a specific function or a finite degree polynomial.

The functions $\Phi_{0}(z)$ and $\Phi_{1}(z)$ can be expressed in terms of solutions to these equations and several unknown constants. The number of these constants is determined by the configuration of zeros of $\Lambda(z)$ as mentioned in (ii). Substitution of the expressions for $\Phi_{0}(z)$ and $\Phi_{1}(z)$ into (4.2) yields an expression for the Laplace transform $\widehat{g}(z)$, which is therefore determined by solutions to (4.6) and the constants. The solution to (1.7) can then be found by inverting the Laplace transform. Consequently, the filtering error, determined by the functional (1.8), can also be expressed in terms of solutions to equations (4.6).

For example, when $\alpha_{1}>\alpha_{2}$, there are no zeros, and as it turns out, the only unknown constant in this case is $\psi(0)$ from (4.2). It can be found using the a priori condition

$$
\begin{equation*}
\left(\frac{\partial}{\partial s} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K_{V}(r, s) d r\right)_{\mid s:=0}=0 \tag{4.7}
\end{equation*}
$$

implied by (1.7) as $K_{X}(0, t)=0$ for all $t \in[0, T]$. When $\alpha_{1} \leq \alpha_{2}$, the function $\Lambda(z)$ has several zeros, which appear in (4.2) as simple poles. Removing these poles leads to a system of linear algebraic equations, which along with (4.7), determine all the unknown coefficients.

At the first glance, so constructed representation does not appear any simpler than the original problem itself, since equations (4.6) cannot be solved explicitly. Remarkably though, a significant simplification is possible due to the properties of operator $A_{\varepsilon, T}$, which force the first term in the right hand side of (4.6) to vanish asymptotically as either $T \rightarrow \infty$ or $\varepsilon \rightarrow 0$. Consequently, otherwise non-explicit function $p(t)$ can be approximated asymptotically by the forcing function $f(t)$. This is where the assertions of Theorem 2.1 come from. In fact, the limit $P_{\infty}\left(\beta, \frac{\mu}{\sqrt{\varepsilon}}\right)$ can be found in a closed, though rather complicated form.

Further simplifications are possible in the special cases, when either of the functions $N_{\alpha_{1}}(z)$ or $N_{\alpha_{2}}(z)$ in formula (4.2) degenerate to 1 , as in Theorem 2.5 and Theorem 2.7, or when they remain non-degenerate, but coincide as in Theorem 2.3. The ultimate expressions in all three cases are obtained by somewhat different calculations, which are detailed in Sections 6-7.

## 5. Proof of Theorem 2.1

In the notations introduced above, the covariance function of the fBm in (1.2) has the form, cf. (1.6),

$$
\begin{equation*}
K_{V}(s, t)=\frac{1}{2}\left(s^{2-\alpha_{2}}+t^{2-\alpha_{2}}-|s-t|^{2-\alpha_{2}}\right), \tag{5.1}
\end{equation*}
$$

with $\alpha_{2} \in(0,2)$. The covariance function of the fractional Ornstein-Uhlenbeck state process $X$, generated by (1.1), is

$$
\begin{equation*}
K_{X}(s, t)=\int_{0}^{s} e^{\beta(s-u)} \frac{\partial}{\partial u} \int_{0}^{t} e^{\beta(t-v)} \frac{\partial}{\partial v} K_{W}(u, v) d v d u \tag{5.2}
\end{equation*}
$$

where $K_{W}(s, t)$ is the kernel in (5.1) with $\alpha_{2}$ replaced by $\alpha_{1}$.
5.1. The Laplace transform. The following lemma details the structure of the Laplace transform of the solution to the main filtering equation and its relation to the filtering error.

Lemma 5.1. Let $g(\cdot)$ solve equation (1.7) with $K_{V}(s, t)$ and $K_{X}(s, t)$ as above.

1. The Laplace transform $\widehat{g}(z)$, defined in (4.1), satisfies representation (4.2) where

$$
\begin{equation*}
\psi(r)=e^{-\beta r} \int_{r}^{T} e^{\beta \tau} g(\tau) d \tau-\frac{1}{\mu_{\varepsilon}} e^{\beta(T-r)}, \tag{5.3}
\end{equation*}
$$

and the functions $\Phi_{0}(z)$ and $\Phi_{1}(z)$ are sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$, satisfying

$$
\left|\Phi_{1}(z)\right| \vee\left|\Phi_{0}(z)\right|= \begin{cases}O\left(z^{\left(\alpha_{1} \wedge \alpha_{2}-1\right) \wedge 0}\right), & z \rightarrow 0  \tag{5.4}\\ O\left(z^{\left(\alpha_{2}-1\right) \vee 0}\right), & z \rightarrow \infty\end{cases}
$$

2. The following condition holds

$$
\begin{equation*}
\lim _{\operatorname{Re}(z) \rightarrow \infty} z\left(N_{\alpha_{2}}(z) \widehat{g}(z)-\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t\right)=0, \tag{5.5}
\end{equation*}
$$

and the filtering error (1.8) is given by the limit

$$
\begin{align*}
& P_{T}=\frac{1}{\mu_{\varepsilon}} \lim _{\operatorname{Re}(z) \rightarrow \infty} z\left(N_{\alpha_{2}}(-z) e^{-z T} \widehat{g}(-z)\right. \\
&\left.-\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) e^{-t T} \widehat{g}(-t) d t\right) . \tag{5.6}
\end{align*}
$$

The proof of this lemma uses the operator

$$
\begin{equation*}
v_{f, \alpha}(s):=\frac{\partial}{\partial s} \int_{0}^{T}\left(1-\frac{\alpha}{2}\right)|s-r|^{1-\alpha} \operatorname{sign}(s-r) f(r) d r \tag{5.7}
\end{equation*}
$$

which acts on sufficiently regular functions $f$. Since for $\alpha \in(0,2)$

$$
|x-y|^{1-\alpha} \operatorname{sign}(x-y)=\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1}(x-y) e^{-t|x-y|} d t,
$$

we can write

$$
\begin{equation*}
v_{f, \alpha}(x)=\frac{1}{c_{\alpha}} \frac{d}{d x} \int_{0}^{\infty} t^{\alpha-1} u_{f}(x, t) d t \tag{5.8}
\end{equation*}
$$

where $c_{\alpha}:=\frac{\Gamma(\alpha)}{1-\frac{\alpha}{2}}$ and

$$
u_{f}(x, t):=\int_{0}^{T}(x-y) e^{-t|x-y|} f(y) d y
$$

In addition, let us define another auxiliary function

$$
w_{f}(x, t):=\int_{0}^{T} e^{-t|x-y|} f(y) d y
$$

Lemma 5.2. The Laplace transform of (5.7) satisfies

$$
\begin{equation*}
\widehat{v}_{f, \alpha}(z)=N_{\alpha}(z) \widehat{f}(z)+e^{-z T} \Psi_{f, 1}(-z)+\Psi_{f, 0}(z) \tag{5.9}
\end{equation*}
$$

where $N_{\alpha}(z)$ is defined in (4.3) and

$$
\begin{align*}
& \Psi_{f, 1}(z):=\frac{1}{c_{\alpha}} \int_{0}^{\infty} \frac{t^{\alpha}}{t-z} u_{f}(T, t) d t+\frac{1}{c_{\alpha}} z \int_{0}^{\infty} \frac{t^{\alpha-1}}{(t-z)^{2}} w_{f}(T, t) d t  \tag{5.10}\\
& \Psi_{f, 0}(z):=-\frac{1}{c_{\alpha}} \int_{0}^{\infty} \frac{t^{\alpha}}{t-z} u_{f}(0, t) d t+\frac{1}{c_{\alpha}} z \int_{0}^{\infty} \frac{t^{\alpha-1}}{(t-z)^{2}} w_{f}(0, t) d t .
\end{align*}
$$

Proof. Differentiating $u_{f}(x, t)$ once with respect to $x$ gives

$$
\begin{aligned}
u_{f}^{\prime}(x, t) & =\frac{d}{d x}\left(\int_{0}^{x}(x-y) e^{-t(x-y)} f(y) d y-\int_{x}^{T}(y-x) e^{-t(y-x)} f(y) d y\right)= \\
& w_{f}(x, t)-t \int_{0}^{x}(x-y) e^{-t(x-y)} f(y) d y-t \int_{x}^{T}(y-x) e^{-t(y-x)} f(y) d y
\end{aligned}
$$

and, consequently,

$$
\begin{align*}
u_{f}^{\prime}(0, t) & =w_{f}(0, t)+t u_{f}(0, t) \\
u_{f}^{\prime}(T, t) & =w_{f}(T, t)-t u_{f}(T, t) . \tag{5.11}
\end{align*}
$$

Similarly,

$$
\begin{aligned}
w_{f}^{\prime}(x, t)= & \frac{d}{d x} \int_{0}^{x} e^{-t(x-y)} f(y) d y+\frac{d}{d x} \int_{x}^{T} e^{-t(y-x)} f(y) d y= \\
& -t \int_{0}^{x} e^{-t(x-y)} f(y) d y+t \int_{x}^{T} e^{-t(y-x)} f(y) d y
\end{aligned}
$$

and

$$
\begin{align*}
w_{f}^{\prime}(0, t) & =t w_{f}(0, t)  \tag{5.12}\\
w_{f}^{\prime}(T, t) & =-t w_{f}(T, t) .
\end{align*}
$$

Taking further derivative gives the system of equations

$$
\begin{align*}
u_{f}^{\prime \prime}(x, t) & =t^{2} u_{f}(x, t)+2 w_{f}^{\prime}(x, t), \\
w_{f}^{\prime \prime}(x, t) & =t^{2} w_{f}(x, t)-2 t f(x) . \tag{5.13}
\end{align*}
$$

Applying the Laplace transform to the first equation we obtain

$$
\widehat{u}_{f}^{\prime \prime}(z, t)=2 \widehat{w}_{f}^{\prime}(z, t)+t^{2} \widehat{u}_{f}(z, t) .
$$

or, equivalently,

$$
\begin{aligned}
& e^{-z T} u_{f}^{\prime}(T, t)-u_{f}^{\prime}(0, t)+e^{-z T} z u_{f}(T, t)-z u_{f}(0, t)+z^{2} \widehat{u}_{f}(z, t)= \\
& 2 e^{-z T} w_{f}(T, t)-2 w_{f}(0, t)+2 z \widehat{w}_{f}(z, t)+t^{2} \widehat{u}_{f}(z, t)
\end{aligned}
$$

Collecting the terms and using the boundary conditions (5.11), this can be written as

$$
\begin{aligned}
& e^{-z T}\left((z-t) u_{f}(T, t)-w_{f}(T, t)\right)-(z+t) u_{f}(0, t)+w_{f}(0, t) \\
& -2 z \widehat{w}_{f}(z, t)+\left(z^{2}-t^{2}\right) \widehat{u}_{f}(z, t)=0
\end{aligned}
$$

Due to the usual relation between Laplace transforms of a function and its derivatives, and again, in view of the boundary conditions (5.11), we can further write

$$
\begin{align*}
\widehat{u}_{f}(z, t)= & \frac{2 z}{z^{2}-t^{2}} \widehat{w}_{f}(z, t)-e^{-z T}\left(\frac{u_{f}(T, t)}{z+t}-\frac{w_{f}(T, t)}{z^{2}-t^{2}}\right) \\
& +\frac{u_{f}(0, t)}{z-t}-\frac{w_{f}(0, t)}{z^{2}-t^{2}} . \tag{5.14}
\end{align*}
$$

A similar calculation shows that the second equation in (5.13) along with the corresponding boundary conditions (5.12) yields

$$
e^{-z T}(z-t) w_{f}(T, t)-(z+t) w_{f}(0, t)+\left(z^{2}-t^{2}\right) \widehat{w}_{f}(z, t)+2 t \widehat{f}(z)=0
$$

and

$$
\widehat{w}_{f}(z, t)=-e^{-z T} \frac{w_{f}(T, t)}{z+t}+\frac{w_{f}(0, t)}{z-t}-\frac{2 t}{z^{2}-t^{2}} \widehat{f}(z) .
$$

Combining this with (5.14), we obtain

$$
\widehat{u}_{f}(z, t)=-\frac{4 z t}{\left(z^{2}-t^{2}\right)^{2}} \widehat{f}(z)+\frac{u_{f}(0, t)}{z-t}+\frac{w_{f}(0, t)}{(z-t)^{2}}-e^{-z T}\left(\frac{u_{f}(T, t)}{z+t}+\frac{w_{f}(T, t)}{(z+t)^{2}}\right) .
$$

By definition (5.8),

$$
\begin{aligned}
c_{\alpha} \widehat{v}_{f, \alpha}(z)= & e^{-z T} \int_{0}^{\infty} t^{\alpha-1} u_{f}(T, t) d t-\int_{0}^{\infty} t^{\alpha-1} u_{f}(0, t) d t \\
& +z \int_{0}^{\infty} t^{\alpha-1} \widehat{u}_{f}(z, t) d t .
\end{aligned}
$$

Substituting the expression for $\widehat{u}_{f}(z, t)$, we arrive at (5.9) with

$$
N_{\alpha}(z)=-\frac{1}{c_{\alpha}} 4 z^{2} \int_{0}^{\infty} \frac{t^{\alpha}}{\left(t^{2}-z^{2}\right)^{2}} d t .
$$

The simpler expression (4.3) is derived by the standard contour integration.
We are now in position to proceed with the proof of Lemma 5.1.
Proof of Lemma 5.1.

1. Observe that $K_{X}(s, t)$ in (5.2) is differentiable in $s \in(0, T)$ and

$$
\frac{\partial}{\partial s} K_{X}(s, t)=\beta K_{X}(s, t)+\frac{\partial}{\partial s} \int_{0}^{t} e^{\beta(t-v)} \frac{\partial}{\partial v} K_{W}(s, v) d v
$$

Hence taking derivative of (1.7) we get

$$
\begin{aligned}
& \frac{\partial^{2}}{\partial s^{2}} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K_{V}(r, s) d r+\beta\left(\mu_{\varepsilon}^{2} \int_{0}^{T} K_{X}(s, r) g(r) d r-\mu_{\varepsilon} K_{X}(s, T)\right)+ \\
& \mu_{\varepsilon}^{2} \frac{\partial}{\partial s} \int_{0}^{T} g(r) \int_{0}^{r} e^{\beta(r-v)} \frac{\partial}{\partial v} K_{W}(s, v) d v d r=\mu_{\varepsilon} \frac{\partial}{\partial s} \int_{0}^{T} e^{\beta(T-v)} \frac{\partial}{\partial v} K_{W}(s, v) d v
\end{aligned}
$$

In view of (1.7), the expression in brackets here can be replaced with

$$
\mu_{\varepsilon}^{2} \int_{0}^{T} K_{X}(s, r) g(r) d r-\mu_{\varepsilon} K_{X}(s, T)=-\frac{\partial}{\partial s} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K_{V}(r, s) d r,
$$

and the last term in the left had side with

$$
\int_{0}^{T} \int_{0}^{r} g(r) e^{\beta(r-v)} \frac{\partial}{\partial v} K_{W}(s, v) d v d r=\int_{0}^{T} \frac{\partial}{\partial r} K_{W}(s, r) e^{-\beta r} \int_{r}^{T} e^{\beta u} g(u) d u d r,
$$

after integration by parts. Plugging these expressions, we arrive at

$$
\begin{aligned}
\frac{\partial^{2}}{\partial s^{2}} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K_{V}(s, r) d r & -\beta \frac{\partial}{\partial s} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K_{V}(s, r) d r \\
& +\mu_{\varepsilon}^{2} \frac{\partial}{\partial s} \int_{0}^{T} \psi(r) \frac{\partial}{\partial r} K_{W}(s, r) d r=0
\end{aligned}
$$

with $\psi(r)$ as defined in (5.3). In terms of the transformation introduced in (5.7), this equation is equivalent to

$$
\frac{\partial}{\partial s} v_{g, \alpha_{2}}(s)-\beta v_{g, \alpha_{2}}(s)+\mu_{\varepsilon}^{2} v_{\psi, \alpha_{1}}(s)=0
$$

Applying the Laplace transform to both sides and using the condition $v_{g, \alpha_{2}}(0)=0$, implied by (1.7), we obtain

$$
\begin{equation*}
e^{-z T} v_{g, \alpha_{2}}(T)+(z-\beta) \widehat{v}_{g, \alpha_{2}}(z)+\mu_{\varepsilon}^{2} \widehat{v}_{\psi, \alpha_{1}}(z)=0 . \tag{5.15}
\end{equation*}
$$

Similarly, the Laplace transform of (5.3) yields the relation

$$
\begin{equation*}
(z+\beta) \widehat{\psi}(z)=\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}-\widehat{g}(z) . \tag{5.16}
\end{equation*}
$$

Combining (5.15), (5.16) and (5.9) with $f:=g$ and $f:=\psi$ gives the representation, claimed in (4.2)

$$
\widehat{\psi}(z) \Lambda(z)=e^{-z T} \Phi_{1}(-z)+\Phi_{0}(z)+N_{\alpha_{2}}(z)(z-\beta)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)
$$

with

$$
\begin{align*}
& \Phi_{0}(z):=\Psi_{g, 0}(z)(z-\beta)+\mu_{\varepsilon}^{2} \Psi_{\psi, 0}(z),  \tag{5.17}\\
& \Phi_{1}(z):=-\Psi_{g, 1}(z)(z+\beta)+\mu_{\varepsilon}^{2} \Psi_{\psi, 1}(z)+v_{g, \alpha_{2}}(T) .
\end{align*}
$$

The Cauchy integrals in (5.10) define sectionally holomorphic functions on $\mathbb{C} \backslash \mathbb{R}_{+}$, and the estimates (5.4) are derived from (5.10) and (5.17) by standard calculations.
2. Subtracting the limits of equation

$$
\begin{equation*}
\widehat{v}_{g, \alpha_{2}}(z)=N_{\alpha_{2}}(z) \widehat{g}(z)+e^{-z T} \Psi_{g, 1}(-z)+\Psi_{g, 0}(z) \tag{5.18}
\end{equation*}
$$

as $z \rightarrow t \in \mathbb{R}_{+}$in the upper and lower half-planes, gives the boundary condition

$$
\Psi_{g, 0}^{+}(t)-\Psi_{g, 0}^{-}(t)=-\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t), \quad t>0
$$

Since the function in the right hand side is Hölder on $\mathbb{R}_{+} \cup\{\infty\}$, and $\Psi_{g, 0}(z)$ vanishes as $z \rightarrow \infty$, applying the Sokhotski-Plemelj formula gives

$$
\Psi_{g, 0}(z)=-\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t, \quad z \in \mathbb{C} \backslash \mathbb{R}_{+} .
$$

Condition (5.5) now follows, since $v_{g, \alpha_{2}}(0)=0$ and, in view of (5.18),

$$
v_{g, \alpha_{2}}(0)=\lim _{\operatorname{Re}(z) \rightarrow \infty} z \widehat{v}_{g, \alpha_{2}}(z)=\lim _{\operatorname{Re}(z) \rightarrow \infty} z\left(N_{\alpha_{2}}(z) \widehat{g}(z)+\Psi_{g, 0}(z)\right) .
$$

Formula (5.6) is obtained similarly, since by (1.8),

$$
P_{T}=\frac{1}{\mu_{\varepsilon}} v_{g, \alpha_{2}}(T)=\frac{1}{\mu_{\varepsilon}} \lim _{\operatorname{Re}(z) \rightarrow \infty} z e^{-z T} \widehat{v}_{g, \alpha_{2}}(-z) .
$$

Remark 5.3. For $\alpha_{2} \in(0,1)$, the first term in the brackets in both (5.5) and (5.6) vanishes, and these equations reduce to

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t=0 \tag{5.19}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{T}=\frac{1}{\mu_{\varepsilon}} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) e^{-t T} \widehat{g}(-t) d t, \tag{5.20}
\end{equation*}
$$

respectively. For $\alpha_{2} \in(1,2)$ this first term diverges to infinity as $z \rightarrow \infty$ and compensates by the leading asymptotic term of the integral. Hence the useful information is actually contained in the second order asymptotics of these expressions.

The next lemma reveals several important properties of the structural function.

Lemma 5.4. The function $\Lambda(z)$ defined in (4.5) is sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}$ with a jump discontinuity across the real line, and its limits $\Lambda^{ \pm}(t), t \in \mathbb{R}$ satisfy

$$
\begin{equation*}
\Lambda^{+}(t)=\Lambda^{-}(-t) \quad \text { and } \quad \overline{\Lambda^{+}(t)}=\Lambda^{-}(t) \tag{5.21}
\end{equation*}
$$

It does not vanish on the cut plane, except, possibly, at simple zeros. More precisely,
(a) $\Lambda(z)$ has no zeros if $\alpha_{1}>\alpha_{2}$, or
(b) a pair of purely real zeros at $\pm t_{0}$ with $t_{0}=\sqrt{\beta^{2}+\mu_{\varepsilon}^{2}}$, if $\alpha_{1}=\alpha_{2}$, or
(c) complex zeros at $\pm z_{0}$ and $\pm \bar{z}_{0}$ for some $z_{0}$ with $\arg \left(z_{0}\right) \in\left(0, \frac{\pi}{2}\right)$, if $\alpha_{1}<\alpha_{2}$.

Proof. The analytic structure of $\Lambda(z)$, the discontinuity and properties (5.21) are inherited from $N_{\alpha}(z)$, cf. (4.4). The symmetric structure of zeros is obvious from the definition of $\Lambda(z)$, and hence it suffices to locate its zeros only in the first quadrant. Since $N_{\alpha}(z)$ may vanish only at the origin, for $z:=\rho e^{i \phi}$ with $\rho \in \mathbb{R}_{+}$and $\phi \in\left[0, \frac{\pi}{2}\right]$,

$$
-\frac{\Lambda(z)}{N_{\alpha_{2}}(z)}=\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}(z)}{N_{\alpha_{2}}(z)}-z^{2}+\beta^{2}=\mu_{\varepsilon}^{2} \frac{\kappa_{\alpha_{1}}}{\kappa_{\alpha_{2}}} \rho^{\alpha_{1}-\alpha_{2}} e^{i\left(\phi-\frac{\pi}{2}\right)\left(\alpha_{1}-\alpha_{2}\right)}-\rho^{2} e^{2 \phi i}+\beta^{2}
$$

Equating the imaginary and real parts of this expression to zero, we get

$$
\begin{aligned}
& \rho^{2} \sin (2 \phi)-\mu_{\varepsilon}^{2} \frac{\kappa_{\alpha_{1}}}{\kappa_{\alpha_{2}}} \rho^{-\delta} \sin \left(\frac{\pi}{2}-\phi\right) \delta=0 \\
& \rho^{2} \cos (2 \phi)-\mu_{\varepsilon}^{2} \frac{\kappa_{\alpha_{1}}}{\kappa_{\alpha_{2}}} \rho^{-\delta} \cos \left(\frac{\pi}{2}-\phi\right) \delta=\beta^{2}
\end{aligned}
$$

where $\delta:=\alpha_{2}-\alpha_{1}$. The angle $\phi=\frac{\pi}{2}$ is inconsistent with the second equation and $\phi=0$ with the first equation, unless $\delta=0$ as well. In this case, that is, when $\alpha_{1}=\alpha_{2}, \phi=0$ is the only possibility, and there are two real zeros as claimed.

If $\alpha_{1}>\alpha_{2}$ the first equation is inconsistent for any $\rho>0$, and hence $\Lambda(z)$ does not have zeros in this case. For $\alpha_{1}<\alpha_{2}$ the absolute value $\rho$ can be expressed in terms of $\phi$ using the first equation

$$
\begin{equation*}
\rho=\left(\mu_{\varepsilon}^{2} \frac{\kappa_{\alpha_{1}}}{\kappa_{\alpha_{2}}}\right)^{\frac{1}{2+\delta}}\left(\frac{\sin \left(\frac{\pi}{2}-\phi\right) \delta}{\sin (2 \phi)}\right)^{\frac{1}{2+\delta}} \tag{5.22}
\end{equation*}
$$

Plugging this into the second equation we get

$$
\sin (\widetilde{\phi} \delta)^{-\frac{\delta}{2+\delta}} \sin (2 \widetilde{\phi})^{-\frac{2}{2+\delta}} \sin (\widetilde{\phi}(2+\delta))=-\beta^{2}\left(\mu_{\varepsilon}^{2} \frac{\kappa_{\alpha_{1}}}{\kappa_{\alpha_{2}}}\right)^{-\frac{2}{2+\delta}}
$$

where $\widetilde{\phi}:=\frac{\pi}{2}-\phi \in\left(0, \frac{\pi}{2}\right)$ was defined for brevity. The left hand side is a continuous decreasing function of $\widetilde{\phi}$, it diverges to $-\infty$ as $\widetilde{\phi} \rightarrow \frac{\pi}{2}$ and has a positive finite limit at $\widetilde{\phi}=0$. Hence this equation has the unique root $\phi_{0}$ and, consequently, $\Lambda(z)$ has the unique zero in the first quadrant at $z_{0}:=\rho_{0} e^{i \phi_{0}}$ with $\rho_{0}$ given by (5.22) with $\phi$ replaced by $\phi_{0}$.
5.2. The equivalent problem. In this subsection we formulate a different problem, which is equivalent to solving equation (1.7). The key observation to this end is that all singularities in expression (4.2) must be removable, since the Laplace transform in (4.1) defines an entire function $\widehat{g}(z)$. In particular, its limits as $z \rightarrow t \in \mathbb{R}$ in the upper and lower half-planes must coincide, which implies

$$
\begin{aligned}
& (t+\beta) \frac{\Phi_{0}^{+}(t)+e^{-t T} \Phi_{1}(-t)}{\Lambda^{+}(t)}+\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right)= \\
& (t+\beta) \frac{\Phi_{0}^{-}(t)+e^{-t T} \Phi_{1}(-t)}{\Lambda^{-}(t)}+\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{-}(t)}{\Lambda^{-}(t)}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right), \quad t \in \mathbb{R}_{+}
\end{aligned}
$$

and

$$
\begin{aligned}
& (t+\beta) \frac{\Phi_{0}(t)+e^{-t T} \Phi_{1}^{-}(-t)}{\Lambda^{-}(t)}+\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{-}(t)}{\Lambda^{-}(t)}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right)= \\
& (t+\beta) \frac{\Phi_{0}(t)+e^{-t T} \Phi_{1}^{+}(-t)}{\Lambda(z)}+\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{-}(t)}{\Lambda^{-}(t)}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right), \quad t \in \mathbb{R}_{-}
\end{aligned}
$$

In view of symmetries (5.21) and formula (4.5), these equations can be written as

$$
\begin{align*}
& \Phi_{0}^{+}(t)-\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} \Phi_{0}^{-}(t)=e^{-t T} \Phi_{1}(-t)\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}-1\right) \\
& \quad+\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} N_{\alpha_{2}}^{-}(t)-N_{\alpha_{2}}^{+}(t)\right)(t-\beta)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right), \quad t \in \mathbb{R}_{+} \\
& \Phi_{1}^{+}(t)-\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} \Phi_{1}^{-}(t)=e^{-t T} \Phi_{0}(-t)\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}-1\right)  \tag{5.23}\\
& \quad-\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} N_{\alpha_{2}}^{-}(t)-N_{\alpha_{2}}^{+}(t)\right)(t+\beta)\left(e^{-t T} \psi(0)+\frac{1}{\mu_{\varepsilon}}\right), \quad t \in \mathbb{R}_{+}
\end{align*}
$$

In addition, removal of the poles in (4.2) implies that the expression in the brackets therein must vanish at the zeros of $\Lambda(z)$,

$$
\begin{align*}
& (z+\beta)\left(\Phi_{0}(z)+e^{-z T} \Phi_{1}(-z)\right)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z)\left(\psi(0)+\frac{e^{-z T}}{\mu_{\varepsilon}}\right)=0  \tag{5.24}\\
& \quad \forall z \in\{\zeta: \Lambda(\zeta)=0\}
\end{align*}
$$

At this point the proof splits into several cases, corresponding to the three possible zeros configurations of $\Lambda(z)$, described in Lemma 5.4, and the computation of filtering error, as explained in Remark 5.3. While the specific calculations are somewhat different in each case, they are based on the same technique, which we will detail for $\alpha_{1}>\alpha_{2} \in(0,1)$, omitting all other cases.

Define $\theta(t):=\arg \left(\Lambda^{+}(t)\right)$, choosing the argument branch so that $\theta(t)$ is continuous on $(0, \infty)$ and $\theta(\infty):=\lim _{t \rightarrow \infty} \theta(t)$ belongs to the interval $(-\pi, \pi)$. This defines $\theta(t)$ uniquely, and, for $\alpha_{1}>\alpha_{2}$,

$$
\theta(\infty)=\frac{1-\alpha_{2}}{2} \pi \quad \text { and } \quad \theta(0+)=\frac{1-\alpha_{2}}{2} \pi+\pi
$$

In what follows we will need a function $X(z)$, which is sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$, satisfies the boundary condition

$$
\begin{equation*}
\frac{X^{+}(t)}{X^{-}(t)}=\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}=e^{2 i \theta(t)}, \quad t \in \mathbb{R}_{+} \tag{5.25}
\end{equation*}
$$

and does not vanish on the cut plane. Finding all such functions is known as the Hilbert boundary value problem, whose solutions are given by the Sokhotski-Plemelj formula

$$
\begin{equation*}
X(z)=(-z)^{k-\theta(\infty) / \pi} \exp \left(\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)-\theta(\infty)}{t-z} d t\right) \tag{5.26}
\end{equation*}
$$

where $k$ is an arbitrary integer. The choice of $k$ controls the growth of $X(z)$ at the origin and at infinity

$$
X(z)= \begin{cases}O\left(z^{k-\theta(0+) / \pi}\right), & z \rightarrow 0  \tag{5.27}\\ O\left(z^{k-\theta(\infty) / \pi}\right), & z \rightarrow \infty\end{cases}
$$

Define a pair of auxiliary functions

$$
\begin{equation*}
S(z):=\frac{\Phi_{0}(z)+\Phi_{1}(z)}{2 X(z)} \quad \text { and } \quad D(z):=\frac{\Phi_{0}(z)-\Phi_{1}(z)}{2 X(z)} \tag{5.28}
\end{equation*}
$$

In view of (5.23) and (5.25), these functions satisfy the decoupled boundary conditions

$$
\begin{align*}
S^{+}(t)-S^{-}(t) & =2 i e^{-t T} h(t) S(-t)+f_{S}(t)  \tag{5.29}\\
D^{+}(t)-D^{-}(t) & =-2 i e^{-t T} h(t) D(-t)+f_{D}(t),
\end{align*} \quad t \in \mathbb{R}_{+}
$$

where we defined

$$
\begin{align*}
f_{S}(t):= & \frac{1}{2}\left(\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right) \\
& \left((t-\beta)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right)-(t+\beta)\left(e^{-t T} \psi(0)+\frac{1}{\mu_{\varepsilon}}\right)\right)  \tag{5.30}\\
f_{D}(t):= & \frac{1}{2}\left(\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right) \\
& \left((t-\beta)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right)+(t+\beta)\left(e^{-t T} \psi(0)+\frac{1}{\mu_{\varepsilon}}\right)\right)
\end{align*}
$$

and the real valued function

$$
h(t):=\frac{X(-t)}{X^{+}(t)} e^{i \theta(t)} \sin \theta(t)=\exp \left(-\frac{1}{\pi} \int_{0}^{\infty} \theta^{\prime}(s) \log \left|\frac{t+s}{t-s}\right| d s\right) \sin \theta(t)
$$

Due to estimates (5.4) and (5.27), the choice $k=1$ in (5.26) guarantees that $S(-t)$ and $D(-t)$ are integrable and, moreover, square integrable near the origin, and implies that $S(z)$ and $D(z)$ vanish as $z \rightarrow \infty$. Hence by the Sokhotski-Plemelj theorem, applied to (5.29), these functions must satisfy the equations

$$
\begin{align*}
& S(z)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h(t)}{t-z} S(-t) d t+F_{S}(z)  \tag{5.31}\\
& D(z)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h(t)}{t-z} D(-t) d t+F_{D}(z)
\end{align*}
$$

where we defined

$$
\begin{equation*}
F_{S}(z):=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{S}(t)}{t-z} d t \quad \text { and } \quad F_{D}(z):=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{D}(t)}{t-z} d t \tag{5.32}
\end{equation*}
$$

Consider now a pair of auxiliary integral equations

$$
\begin{align*}
& p(t)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h(\tau)}{\tau+t} p(\tau) d \tau+F_{S}(-t) \\
& q(t)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h(\tau)}{\tau+t} q(\tau) d \tau+F_{D}(-t) \tag{5.33}
\end{align*}
$$

In view of (5.30) the restrictions $F_{S}(-t)$ and $F_{D}(-t)$ are real valued functions. The operator in the right hand side

$$
(A f)(t)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h(\tau)}{\tau+t} f(\tau) d \tau
$$

is a contraction on $L^{2}\left(\mathbb{R}_{+}\right)$, see [7, Lemma 5.6], and a calculation as in [7, Lemma 5.7] shows that $F_{S}, F_{D} \in L^{2}\left(\mathbb{R}_{+}\right)$. Consequently, equations (5.33) have unique solutions $p, q \in$ $L^{2}\left(\mathbb{R}_{+}\right)$.

Comparing (5.31) and (5.33) shows that

$$
S(z)=p(-z) \quad \text { and } \quad D(z)=q(-z), \quad z \in \mathbb{C} \backslash \mathbb{R}_{+}
$$

where $p(z)$ and $q(z)$ are the analytic extensions. Then, by definition (5.28),

$$
\begin{align*}
& \Phi_{0}(z)=X(z)(p(-z)+q(-z)) \\
& \Phi_{1}(z)=X(z)(p(-z)-q(-z)) \tag{5.34}
\end{align*}
$$

Let us summarize our findings so far. Given the unique solutions to the integral equations (5.33), we can compute the functions $\Phi_{0}(z)$ and $\Phi_{1}(z)$ by means of (5.34) and plug them into (4.2). The constant $\psi(0)$ can be found by plugging the obtained expression for $\widehat{g}(z)$ into (5.5), or equivalently in this case, into (5.19). Applying the inverse Laplace transform to $\widehat{g}(z)$, gives a function which solves (1.7) and belongs to $L^{1}([0, T]) \cap \Lambda_{T}^{H-\frac{1}{2}}$, where $\Lambda_{T}^{H-\frac{1}{2}}$ is a space of nonrandom functions, on which the stochastic integral with respect to fBm can be defined and has suitable properties (see Appendix A). Thus the original equation is reduced to an equivalent problem of solving integral equations (5.33). The filtering error $P_{T}$ is found by substitution of the expression for $\widehat{g}(z)$ into (5.20).
5.3. Asymptotic analysis. While for any fixed values of the parameters, the equivalent problem derived above does not appear any simpler than the original equation, it does simplify drastically when either of the limits $T \rightarrow \infty$ or $\varepsilon \rightarrow 0$ is taken. The key to the asymptotic analysis are the estimates

$$
\begin{equation*}
\left|p(z)-F_{S}(-z)\right| \leq C \frac{1}{z} \frac{1}{T}, \quad\left|q(z)-F_{D}(-z)\right| \leq C \frac{1}{z} \frac{1}{T} \tag{5.35}
\end{equation*}
$$

where $C$ is a constant independent of $T$ and $\varepsilon$. These bounds are derived exactly as in [7, Lemma 5.7] and we omit the proof.
5.3.1. Large time asymptotics. Upon substitution of expression (4.2) into the integral in (5.19), the latter simplifies, asymptotically as $T \rightarrow \infty$, to

$$
\begin{aligned}
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t \asymp \\
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)\left((t+\beta) \frac{\Phi_{0}^{+}(t)}{\Lambda^{+}(t)} d t+\psi(0) \mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}\right) d t .
\end{aligned}
$$

Due to (5.34) and estimates (5.35), the first term satisfies

$$
\begin{aligned}
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)(t+\beta) \frac{\Phi_{0}^{+}(t)}{\Lambda^{+}(t)} d t \asymp \\
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)(t+\beta) \frac{X^{+}(t)}{\Lambda^{+}(t)}\left(p^{-}(-t)+q^{-}(-t)\right) d t= \\
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)(t+\beta) \frac{X^{+}(t)}{\Lambda^{+}(t)}\left(F_{S}^{+}(t)+F_{D}^{+}(t)\right) d t,
\end{aligned}
$$

where, by definitions (5.30),

$$
\begin{align*}
& F_{S}(z)+F_{D}(z)=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{S}(t)+f_{D}(t)}{t-z} d t= \\
& \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right)(t-\beta)\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t T}\right) d t \asymp  \tag{5.36}\\
& \psi(0) \frac{1}{2 \pi i} \int_{0}^{\infty}\left(\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right) \frac{t-\beta}{t-z} d t=: \psi(0) R\left(z ; \beta, \mu_{\varepsilon}\right) .
\end{align*}
$$

It follows that

$$
\begin{equation*}
\int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t \asymp \psi(0) I\left(\beta, \mu_{\varepsilon}\right) \tag{5.37}
\end{equation*}
$$

where the quantity

$$
\begin{aligned}
& I\left(\beta, \mu_{\varepsilon}\right):= \\
& \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)\left((t+\beta) \frac{X^{+}(t)}{\Lambda^{+}(t)} R^{+}\left(t ; \beta, \mu_{\varepsilon}\right)+\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}\right) d t
\end{aligned}
$$

does not depend on $T$. A lengthy but otherwise direct calculation shows that this expression is nonzero and therefore condition (5.19) implies that $\psi(0) \rightarrow 0$ as $T \rightarrow \infty$.

Similarly, we can simplify expression (5.20),

$$
\begin{aligned}
& P_{T}=\frac{1}{\mu_{\varepsilon}} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) e^{-t T} \widehat{g}(-t) d t \asymp \\
& \quad \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)\left((t-\beta) \frac{1}{\mu_{\varepsilon}} \frac{\Phi_{1}^{+}(t)}{\Lambda^{+}(t)}-\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}\right) d t \asymp \\
& \quad \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)(t-\beta) \frac{1}{\mu_{\varepsilon}} \frac{X^{+}(t)}{\Lambda^{+}(t)}\left(F_{S}^{+}(t)-F_{D}^{+}(t)\right) d t \\
& - \\
& -\frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)} d t,
\end{aligned}
$$

where $e^{-t T} \widehat{g}(-t)$ is computed using (4.2). Since $\psi(0)$ remains bounded as $T \rightarrow \infty,(5.30)$ implies

$$
\begin{align*}
F_{S}(z)-F_{D}(z)= & \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{S}(t)-f_{D}(t)}{t-z} d t \asymp  \tag{5.38}\\
& \frac{1}{2 \pi i} \frac{1}{\mu_{\varepsilon}} \int_{0}^{\infty}\left(\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}-\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}\right) \frac{t+\beta}{t-z} d t=: Q\left(z ; \beta, \mu_{\varepsilon}\right)
\end{align*}
$$

and hence, as claimed in (2.1), $P_{T}$ converges to the limit

$$
\begin{equation*}
P_{\infty}\left(\beta, \mu_{\varepsilon}\right):=\frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)\left(\frac{t-\beta}{\mu_{\varepsilon}} \frac{X^{+}(t)}{\Lambda^{+}(t)} Q^{+}\left(t ; \beta, \mu_{\varepsilon}\right)-\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(x t)}\right) d t \tag{5.39}
\end{equation*}
$$

5.3.2. Small noise asymptotics. To emphasise the dependence on $\varepsilon$ and other parameters we will add them to the notations, writing $\Lambda\left(z ; \beta, \mu_{\varepsilon}\right)$ for $\Lambda(z)$, etc. In view of definitions (4.3) and (4.5), the structural function satisfies the scaling property

$$
\Lambda\left(\varepsilon^{-\gamma} z ; \beta, \mu_{\varepsilon}\right)=\varepsilon^{-\gamma\left(1+\alpha_{2}\right)} \Lambda\left(z ; \varepsilon^{\gamma} \beta, \mu\right), \quad \gamma:=\frac{1}{2+\alpha_{2}-\alpha_{1}}>0
$$

Consequently, $\theta\left(\varepsilon^{-\gamma} t ; \beta, \mu_{\varepsilon}\right)=\theta\left(t ; \varepsilon^{\gamma} \beta, \mu\right)$ and, by definition (5.26),

$$
\begin{aligned}
X\left(\varepsilon^{-\gamma} z ; \beta, \mu_{\varepsilon}\right)= & \left(-\varepsilon^{-\gamma} z\right)^{1-\frac{1-\alpha_{2}}{2}} \exp \left(\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta\left(t ; \varepsilon^{\gamma} \beta, \mu\right)-\theta(\infty)}{t-z} d t\right)= \\
& \varepsilon^{-\frac{1}{2} \gamma\left(1+\alpha_{2}\right)} X\left(z ; \varepsilon^{\gamma} \beta, \mu\right)
\end{aligned}
$$

Substituting formula (4.2), expressions (5.34) and estimates (5.35) into the integral in (5.19) and changing the integration variable accordingly, we obtain

$$
\begin{aligned}
& \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t \asymp \\
& -\varepsilon^{-\gamma \frac{1+\alpha_{2}}{2}} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right)\left(t+\varepsilon^{\gamma} \beta\right) \frac{X^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)\left(F_{S}^{+}\left(\varepsilon^{-\gamma} t\right)+F_{D}^{+}\left(\varepsilon^{-\gamma} t\right)\right)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)} d t \\
& -\varepsilon^{-\gamma \alpha_{2}} \mu^{2} \psi(0) \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)} d t
\end{aligned}
$$

as $\varepsilon \rightarrow 0$. Here, in view of (5.30) and (5.32),

$$
\begin{aligned}
& F_{S}\left(\varepsilon^{-\gamma} z\right)+F_{D}\left(\varepsilon^{-\gamma} z\right)=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{S}\left(\varepsilon^{-\gamma} t\right)+f_{D}\left(\varepsilon^{-\gamma} t\right)}{t-z} d t \asymp \\
& \varepsilon^{\frac{1}{2} \gamma\left(1-\alpha_{2}\right)} \psi(0) \frac{1}{2 \pi i} \int_{0}^{\infty}\left(\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}\right) \frac{t-\varepsilon^{\gamma} \beta}{t-z} d t= \\
& \varepsilon^{\frac{1}{2} \gamma\left(1-\alpha_{2}\right)} \psi(0) R\left(z ; \varepsilon^{\gamma} \beta, \mu\right)
\end{aligned}
$$

where $R(\cdot)$ was defined in (5.36). Consequently, cf. (5.37),

$$
\int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t \asymp \varepsilon^{-\gamma \alpha_{2}} \psi(0) I\left(\varepsilon^{\gamma} \beta, \mu\right)
$$

and thus condition (5.19) implies $\psi(0)=o\left(\varepsilon^{\gamma \alpha_{2}}\right)$ as $\varepsilon \rightarrow 0$.

The filtering error asymptotics is deduced from (5.20) by similar calculations,

$$
\begin{aligned}
& P_{T}\left(\beta, \mu_{\varepsilon}\right) \asymp \varepsilon^{\gamma\left(2-\alpha_{1}\right)} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \\
& \left(\varepsilon^{-\frac{\gamma}{2}\left(3-\alpha_{1}\right)}\left(t-\varepsilon^{\gamma} \beta\right) \frac{1}{\mu} \frac{X^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}\left(F_{S}^{+}\left(\varepsilon^{-\gamma} t\right)-F_{D}^{+}\left(\varepsilon^{-\gamma} t\right)\right)-\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}\right) d t
\end{aligned}
$$

Here, cf. (5.38),

$$
\begin{aligned}
& F_{S}\left(\varepsilon^{-\gamma} z\right)-F_{D}^{+}\left(\varepsilon^{-\gamma} z\right)=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{f_{S}\left(\varepsilon^{-\gamma} t\right)-f_{D}\left(\varepsilon^{-\gamma} t\right)}{t-z} d t= \\
& \varepsilon^{\frac{1}{2}+\frac{1}{2} \gamma\left(1-\alpha_{2}\right)} \frac{1}{2 \pi i} \frac{1}{\mu} \int_{0}^{\infty}\left(\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}-\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}\right) \frac{t+\varepsilon^{\gamma} \beta}{t-z} d t
\end{aligned}
$$

and consequently

$$
\begin{aligned}
& P_{T}\left(\beta, \mu_{\varepsilon}\right) \asymp \varepsilon^{\gamma\left(2-\alpha_{1}\right)} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \\
& \left(\left(t-\varepsilon^{\gamma} \beta\right) \frac{1}{\mu} \frac{X^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)} Q\left(t ; \varepsilon^{\gamma} \beta, \mu\right)-\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}\left(t ; \varepsilon^{\gamma} \beta, \mu\right)}\right) d t \asymp \varepsilon^{\gamma\left(2-\alpha_{1}\right)} P_{\infty}(0, \mu)
\end{aligned}
$$

where $P_{\infty}(\cdot)$ is exactly the function obtained in (5.39). This proves the asymptotics claimed in (2.2).

## 6. Proof of Theorem 2.3

In this section we derive the large time limit (2.5), from which small noise asymptotics (2.6) follows by Theorem 2.1 in the obvious way.
6.1. The equivalent problem. For $\alpha_{1}=\alpha_{2}=: \alpha \in(0,2)$, expression (4.2) for the Laplace transform reduces to

$$
\begin{equation*}
\widehat{g}(z)=-\frac{z+\beta}{z^{2}-t_{0}^{2}} \frac{\Phi_{0}(z)+e^{-z T} \Phi_{1}(-z)}{N_{\alpha}(z)}-\frac{\mu_{\varepsilon}^{2}}{z^{2}-t_{0}^{2}}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right) \tag{6.1}
\end{equation*}
$$

where $t_{0}^{2}=\beta^{2}+\mu_{\varepsilon}^{2}$, cf. (4.5). In this case,

$$
\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}=\frac{N_{\alpha}^{+}(t)}{N_{\alpha}^{-}(t)}=e^{(1-\alpha) \pi i}, \quad t \in \mathbb{R}
$$

and equations (5.23) simplify to

$$
\begin{aligned}
& \Phi_{0}^{+}(t)-e^{(1-\alpha) \pi i} \Phi_{0}^{-}(t)=e^{-t T} \Phi_{1}(-t)\left(e^{(1-\alpha) \pi i}-1\right) \\
& \Phi_{1}^{+}(t)-e^{(1-\alpha) \pi i} \Phi_{1}^{-}(t)=e^{-t T} \Phi_{0}(-t)\left(e^{(1-\alpha) \pi i}-1\right)
\end{aligned}
$$

The sectionally holomorphic function in (5.26) reduces to

$$
\begin{equation*}
X(z)=(-z)^{\frac{\alpha-1}{2}}, \quad z \in \mathbb{C} \backslash \mathbb{R}_{+} \tag{6.2}
\end{equation*}
$$

with the constant jump across the real line

$$
\frac{X^{+}(t)}{X^{-}(t)}=e^{(1-\alpha) \pi i}, \quad t \in \mathbb{R}_{+} .
$$

In this case the functions defined in (5.28) satisfy, cf. (5.29),

$$
\begin{aligned}
S^{+}(t)-S^{-}(t) & =2 i e^{-t T} h S(-t), \\
D^{+}(t)-D^{-}(t) & =-2 i e^{-t T} h D(-t),
\end{aligned} \quad t \in \mathbb{R}_{+},
$$

with the constant $h=\sin \left(\frac{1-\alpha}{2} \pi\right)$. In view of estimates (5.4) and expression (6.2), functions $S(z)$ and $D(z)$ grow sublinearly as $z \rightarrow \infty$ and their restrictions to negative reals are (square) integrable near the origin. Consequently, by the Sokhotski-Plemelj theorem, cf. (5.31),

$$
\begin{aligned}
S(z) & =\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h}{t-z} S(-t) d t+k_{0}^{S} \\
D(z) & =-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h}{t-z} D(-t) d t+k_{0}^{D}
\end{aligned}
$$

where $k_{0}^{S}$ and $k_{0}^{D}$ are some constants, yet to be determined. The relevant auxiliary integral equations in this case are

$$
\begin{aligned}
& p_{0}(t)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h}{\tau+t} p_{0}(\tau) d \tau+1 \\
& q_{0}(t)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h}{\tau+t} q_{0}(\tau) d \tau+1
\end{aligned}
$$

They have unique solutions, such that $A p_{0}, A q_{0} \in L^{2}\left(\mathbb{R}_{+}\right)$and, by linearity, $S(z)=$ $k_{0}^{S} p_{0}(-z)$ and $D(z)=k_{0}^{D} q_{0}(-z)$, so that, cf. (5.34),

$$
\begin{align*}
& \Phi_{0}(z)=X(z)\left(k_{0}^{S} p_{0}(-z)+k_{0}^{D} q_{0}(-z)\right), \\
& \Phi_{1}(z)=X(z)\left(k_{0}^{S} p_{0}(-z)-k_{0}^{D} q_{0}(-z)\right) . \tag{6.3}
\end{align*}
$$

Substituting these formulas into (6.1), we obtain an expression for the Laplace transform, which depends on the unknown constants $\psi(0), k_{0}^{S}$ and $k_{0}^{D}$. These constants can be found from the linear algebraic system, consisting of (5.5) and the two additional equations, obtained by the poles removal in (6.1),

$$
\begin{align*}
& \left(t_{0}+\beta\right) \frac{\Phi_{0}^{+}\left(t_{0}\right)+e^{-t_{0} T} \Phi_{1}\left(-t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)}+\mu_{\varepsilon}^{2}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-t_{0} T}\right)=0, \\
& \left(t_{0}-\beta\right) \frac{e^{-t_{0} T} \Phi_{0}\left(-t_{0}\right)+\Phi_{1}^{-}\left(t_{0}\right)}{N_{\alpha}^{-}\left(t_{0}\right)}-\mu_{\varepsilon}^{2}\left(e^{-t_{0} T} \psi(0)+\frac{1}{\mu_{\varepsilon}}\right)=0 . \tag{6.4}
\end{align*}
$$

Once this system is solved, the Laplace transform $\widehat{g}(z)$ becomes completely specified and the filtering error can be computed by means of equation (5.6).
6.2. Large time limit $\alpha \in(\mathbf{0}, \mathbf{1})$. The main element of the asymptotic analysis is the estimates similar to (5.35),

$$
\left|p_{0}(z)-1\right| \leq C \frac{1}{z} \frac{1}{T}, \quad\left|q_{0}(z)-1\right| \leq C \frac{1}{z} \frac{1}{T}
$$

Due to these bounds and equations (6.3), conditions (6.4) simplify as $T \rightarrow \infty$ to

$$
\begin{align*}
& \left(t_{0}+\beta\right) \frac{X^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)} k_{0}^{S}+\left(t_{0}+\beta\right) \frac{X^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)} k_{0}^{D}+\mu_{\varepsilon}^{2} \psi(0) \asymp 0,  \tag{6.5}\\
& \left(t_{0}-\beta\right) \frac{X^{-}\left(t_{0}\right)}{N_{\alpha}^{-}\left(t_{0}\right)} k_{0}^{S}-\left(t_{0}-\beta\right) \frac{X^{-}\left(t_{0}\right)}{N_{\alpha}^{-}\left(t_{0}\right)} k_{0}^{D} \asymp \mu_{\varepsilon} .
\end{align*}
$$

Further calculations are carried out somewhat differently, depending on the values of $\alpha$, as explained in Remark 5.3.

Let us first consider the case $\alpha \in(0,1)$. The restriction of $\widehat{g}(z)$ to the real line, needed in (5.19), is found by taking the limit $z \rightarrow t \in \mathbb{R}_{+}$in (6.1), either in the upper or lower half planes. By subtracting from $\widehat{g}(t)$ the first equation in (6.4), plugging the result into (5.19) and taking $T \rightarrow \infty$ we obtain the asymptotics

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}(t) d t \asymp \\
& \left(k_{0}^{S}+k_{0}^{D}\right) \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t^{2}-t_{0}^{2}}\left(\left(t_{0}+\beta\right) \frac{X_{0}^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)}-(t+\beta) \frac{X_{0}^{+}(t)}{N_{\alpha}^{+}(t)}\right) d t . \tag{6.6}
\end{align*}
$$

The latter integral is well defined, since singularity at $t_{0}$ is integrable, and it does not vanish for all $\alpha \in(0,1)$. Therefore (5.19) implies that $k_{0}^{S}+k_{0}^{D} \rightarrow 0$ as $T \rightarrow \infty$ and, due to (6.5), we also have $\psi(0) \rightarrow 0$ and

$$
\begin{equation*}
k_{0}^{S}-k_{0}^{D} \underset{T \rightarrow \infty}{\longrightarrow} \frac{\mu_{\varepsilon}}{t_{0}-\beta} \frac{N_{\alpha}^{-}\left(t_{0}\right)}{X^{-}\left(t_{0}\right)} . \tag{6.7}
\end{equation*}
$$

The expression (2.5) can now be derived by using (5.20):

$$
\begin{align*}
& P_{\infty}\left(\beta, \mu_{\varepsilon}\right) \stackrel{(\mathrm{a})}{=} \\
& \frac{k_{0}^{S}-k_{0}^{D}}{\mu_{\varepsilon}} \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t^{2}-t_{0}^{2}}\left((t-\beta) \frac{X^{-}(t)}{N_{\alpha}^{-}(t)}-\left(t_{0}-\beta\right) \frac{X^{-}\left(t_{0}\right)}{N_{\alpha}^{-}\left(t_{0}\right)}\right) d t \stackrel{(\mathrm{~b})}{=} \\
& \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t^{2}-t_{0}^{2}}\left(\frac{t-\beta}{t_{0}-\beta} \frac{N_{\alpha}^{-}\left(t_{0}\right)}{X^{-}\left(t_{0}\right)} \frac{X^{-}(t)}{N_{\alpha}^{-}(t)}-1\right) d t \stackrel{(\mathrm{c})}{=}  \tag{6.8}\\
& \kappa_{\alpha} \frac{\cos \frac{\alpha}{2} \pi}{\pi} \int_{0}^{\infty} \frac{t^{\alpha-1}}{t^{2}-t_{0}^{2}}\left(\frac{t-\beta}{t_{0}-\beta}\left(t / t_{0}\right)^{\frac{1-\alpha}{2}}-1\right) d t \stackrel{(\mathrm{~d})}{=} \\
& \frac{\Gamma(3-\alpha)}{2} t_{0}^{\alpha-2}\left(1+\sin \left(\frac{\alpha}{2} \pi\right) \frac{t_{0}+\beta}{t_{0}-\beta}\right),
\end{align*}
$$

where (a) is obtained by plugging $e^{-T t} \widehat{g}(-t)$ from (6.1) and subtracting the second equation from (6.4), the limit (b) holds due to (6.7), equality (c) follows by by substitution of the explicit formulas from (4.3) and (6.2) and (d) is computed by the standard contour integration and simplified using elementary trigonometry.
6.3. Large time limit $\boldsymbol{\alpha} \in(\mathbf{1 , 2})$. In view of (6.1) and (6.3), the first term in the brackets in (5.5) satisfies

$$
\begin{align*}
z N_{\alpha}(z) \widehat{g}(z)= & -\Phi_{0}(z)+O\left(z^{\alpha-2}\right)=-X(z)\left(k_{0}^{S}+k_{0}^{D}\right)+O\left(z^{\alpha-2}\right)= \\
& -(-z)^{\frac{\alpha-1}{2}}\left(k_{0}^{S}+k_{0}^{D}\right)+O\left(z^{\alpha-2}\right), \quad \text { as } \operatorname{Re}(z) \rightarrow \infty \tag{6.9}
\end{align*}
$$

Similarly to (6.6), the second term satisfies, as $T \rightarrow \infty$,

$$
\begin{aligned}
& \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) \widehat{g}(t) d t \asymp \\
& \frac{k_{0}^{S}+k_{0}^{D}}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \frac{1}{t^{2}-t_{0}^{2}}\left(\left(t_{0}+\beta\right) \frac{X^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)}-(t+\beta) \frac{X^{+}(t)}{N_{\alpha}^{+}(t)}\right) d t
\end{aligned}
$$

The latter integral can be written as the sum of three parts,

$$
\begin{aligned}
J_{1}(z) & :=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \frac{t_{0}+\beta}{t^{2}-t_{0}^{2}}\left(\frac{X^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)}-\frac{X^{+}(t)}{N_{\alpha}^{+}(t)}\right) d t \\
J_{2}(z) & :=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \frac{t_{0}}{t\left(t+t_{0}\right)} \frac{X^{+}(t)}{N_{\alpha}^{+}(t)} d t \\
J_{3}(z) & :=-\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \frac{1}{t} \frac{X^{+}(t)}{N_{\alpha}^{+}(t)} d t
\end{aligned}
$$

where both $z J_{1}(z)$ and $z J_{2}(z)$ converge to finite limits as $z \rightarrow \infty$ and

$$
\begin{aligned}
-z J_{3}(z)= & z \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \frac{1}{t} \frac{X^{+}(t)}{N_{\alpha}^{+}(t)} d t= \\
& -z \frac{\sin \left(\frac{\alpha-1}{2} \pi\right)}{\pi} \int_{0}^{\infty} \frac{t^{\frac{\alpha-1}{2}-1}}{t-z} d t=(-z)^{\frac{\alpha-1}{2}} .
\end{aligned}
$$

This term cancels out with (6.9) in the limit (5.5), which therefore, takes the form

$$
\left(k_{0}^{S}+k_{0}^{D}\right) \lim _{\operatorname{Re}(z) \rightarrow \infty}\left(z J_{1}(z)+z J_{2}(z)\right)=0
$$

A calculation shows that the limit here remains non-zero for all $\alpha \in(1,2)$. Consequently, $k_{0}^{S}+k_{0}^{D} \rightarrow 0$ as $T \rightarrow \infty$ and (6.7) remains true. Similarly, the first term in (5.6) has the asymptotics

$$
z N_{\alpha}(-z) e^{-z T} \widehat{g}(-z)=(-z)^{\frac{\alpha-1}{2}}\left(k_{0}^{S}-k_{0}^{D}\right)+O\left(z^{\alpha-2}\right), \quad z \rightarrow \infty
$$

which compensates the leading order term in the integral. Hence we obtain

$$
\begin{aligned}
& P_{T}\left(\beta, \mu_{\varepsilon}\right) \asymp \\
& \left(k_{0}^{S}-k_{0}^{D}\right) \frac{t_{0}-\beta}{\mu_{\varepsilon}} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) \frac{1}{t^{2}-t_{0}^{2}}\left(\frac{X^{+}(t)}{N_{\alpha}^{+}(t)}-\frac{X^{+}\left(t_{0}\right)}{N_{\alpha}^{+}\left(t_{0}\right)}\right) d t \\
& -\left(k_{0}^{S}-k_{0}^{D}\right) \frac{t_{0}}{\mu_{\varepsilon}} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) \frac{1}{t\left(t+t_{0}\right)} \frac{X^{+}(t)}{N_{\alpha}^{+}(t)} d t, \quad T \rightarrow \infty
\end{aligned}
$$

Upon substitution of (4.3) and (6.2) these integrals can be evaluated explicitly by means of standard contour integration. Then plugging the limit (6.7) and simplifying the obtained trigonometric formulas, we arrive at the very same expression, derived in (6.8).

## 7. Proof of Theorem 2.7

7.1. The equivalent problem. The equivalent problem in Subsection 5.2 was derived for $\alpha_{2} \in(0,1)$, and in the complementary case $\alpha_{2} \in(1,2)$ it takes a somewhat different form. The function $\theta(t)=\arg \left(\Lambda^{+}(t)\right)$ is now negative with $\theta(0+)=-\pi$ and $\theta(\infty)=\frac{1-\alpha_{2}}{2} \pi$, and, consequently, in view of estimates (5.4) and (5.27), the appropriate choice of the factor in (5.26) is $k=-1$. The functions $S(z)$ and $D(z)$, defined in (5.28), grow at most linearly as $z \rightarrow \infty$ and, therefore,

$$
\begin{align*}
& S(z)=-k_{1}^{S} p_{1}(-z)+k_{0}^{S} p_{0}(-z)+p(-z), \\
& D(z)=-k_{1}^{D} q_{1}(-z)+k_{0}^{D} q_{0}(-z)+q(-z), \tag{7.1}
\end{align*}
$$

where $p_{j}(z), q_{j}(z)$ and $p(z), q(z)$ are solutions to auxiliary integral equations (8.11) and (5.33), respectively. Combining (5.28) with (7.1) yields the expressions for $\Phi_{0}(z)$ and $\Phi_{1}(z)$ and, in turn, for the Laplace transform $\widehat{g}(z)$ in (4.2), specified up to unknown constants $k_{j}^{S}, k_{j}^{D}$ and $\psi(0)$. These constants are found using (5.5) and the conditions, implied by removal of the poles,

$$
\begin{align*}
& \left(z_{0}+\beta\right)\left(\Phi_{0}\left(z_{0}\right)+e^{-z_{0} T} \Phi_{1}\left(-z_{0}\right)\right)+\mu_{\varepsilon}^{2}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z_{0} T}\right)=0 \\
& \left(z_{0}-\beta\right)\left(\Phi_{1}\left(z_{0}\right)+e^{-z_{0} T} \Phi_{0}\left(-z_{0}\right)\right)-\mu_{\varepsilon}^{2}\left(\frac{1}{\mu_{\varepsilon}}+e^{-z_{0} T} \psi(0)\right)=0 \tag{7.2}
\end{align*}
$$

Finally, the limit filtering error can be computed using (5.6).
7.2. Large time limit $\boldsymbol{\alpha} \in(\mathbf{0}, \mathbf{1})$. Let $\alpha_{1}=1$ and $\alpha_{2}=\alpha \in(0,1)$. Our starting point is the expression for the limiting error (5.39). Using the special form of the structural function in this case,

$$
\begin{equation*}
\Lambda(z)=\left(z^{2}-\beta^{2}\right) N_{\alpha}(z)-\mu_{\varepsilon}^{2}, \tag{7.3}
\end{equation*}
$$

and property (5.25), the integral in (5.38) simplifies to

$$
\begin{aligned}
Q(z)= & \frac{1}{2 \pi i} \frac{1}{\mu_{\varepsilon}} \int_{0}^{\infty} \frac{t+\beta}{t-z}\left(\frac{N_{\alpha}^{+}(t)}{X^{+}(t)}-\frac{N_{\alpha}^{-}(t)}{X^{-}(t)}\right) d t= \\
& \mu_{\varepsilon} \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z} \frac{1}{t-\beta}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right) d t=\frac{\mu_{\varepsilon}}{z-\beta}(H(z)-H(\beta)),
\end{aligned}
$$

where we defined

$$
\begin{equation*}
H(z):=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right) d t \tag{7.4}
\end{equation*}
$$

and $H(\beta)$ stands for the common value of $H^{+}(\beta)=H^{-}(\beta)$. These two limits coincide, since $\Lambda(z)$ in (7.3) satisfies $\Lambda^{+}(\beta)=\Lambda^{-}(\beta)$ and hence, in view of (5.25),

$$
X^{+}(\beta)=X^{-}(\beta)=: X(\beta) \in \mathbb{R}, \quad \beta>0
$$

By the Sokhotski-Plemelj theorem

$$
H^{+}(t)-H^{-}(t)=\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}, \quad t \in \mathbb{R}_{+}
$$

and hence $H(z)-1 / X(z)$ is an entire function. Since it vanishes as $z \rightarrow \infty$, it must coincide with the zero function and hence

$$
\begin{equation*}
Q(z)=\frac{\mu_{\varepsilon}}{z-\beta}\left(\frac{1}{X(z)}-\frac{1}{X(\beta)}\right) \tag{7.5}
\end{equation*}
$$

Plugging this formula into (5.39) yields

$$
\begin{equation*}
P_{\infty}\left(\beta, \mu_{\varepsilon}\right)=-\frac{1}{X(\beta)} \frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) \frac{X^{+}(t)}{\Lambda^{+}(t)} d t \tag{7.6}
\end{equation*}
$$

Further simplification is possible due to the following lemmas.
Lemma 7.1.

$$
\begin{equation*}
X(z) X(-z)=-\frac{1}{\kappa_{\alpha}} \Lambda(z), \quad z \in \mathbb{C} \backslash \mathbb{R} \tag{7.7}
\end{equation*}
$$

Proof. By definition (5.26) with $k=1$,

$$
\begin{equation*}
X(z) X(-z)=(-z)^{1-\theta(\infty) / \pi} z^{1-\theta(\infty) / \pi} \exp (\Upsilon(z)) \tag{7.8}
\end{equation*}
$$

where

$$
\Upsilon(z):=\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)-\theta(\infty)}{t-z} d t+\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)-\theta(\infty)}{t+z} d t
$$

Define the function, cf. (7.3),

$$
\widetilde{\Lambda}(z):=\frac{\Lambda(z)}{z^{2} N_{\alpha}(z)}=1-\beta^{2} z^{-2}-\frac{\mu_{\varepsilon}^{2}}{z^{2} N_{\alpha}(z)}
$$

Since $\arg \left(N_{\alpha}^{+}(t)\right)=\theta(\infty)$ and in view of symmetries (4.4) and (5.21),

$$
\theta(t)-\theta(\infty)=\arg \left(\widetilde{\Lambda}^{+}(t)\right)=\frac{1}{2 i} \log \frac{\widetilde{\Lambda}^{+}(t)}{\widetilde{\Lambda}^{-}(t)}=: \widetilde{\theta}(t)
$$

The angle function $\widetilde{\theta}(t)$ is odd, $\widetilde{\theta}(t)=-\widetilde{\theta}(-t)$, and hence we can write

$$
\begin{aligned}
\Upsilon(z)= & \frac{1}{\pi} \int_{0}^{\infty} \frac{\tilde{\theta}(t)}{t-z} d t+\frac{1}{\pi} \int_{0}^{\infty} \frac{\tilde{\theta}(t)}{t+z} d t=\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{\theta}(t)}{t-z} d t= \\
& \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{+}(t)}{t-z} d t-\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{-}(t)}{t-z} d t
\end{aligned}
$$

The latter integrals are well defined since $\log \widetilde{\Lambda}^{ \pm}(t)=O\left(|t|^{-1-\alpha}\right)$ as $|t| \rightarrow \infty$. An elementary calculation shows that $\widetilde{\Lambda}(z)$ does not cross the branch cut of the logarithm for any $z \in \mathbb{C} \backslash \mathbb{R}$. Hence these integrals can be computed by integrating the function $f(\zeta)=\log \widetilde{\Lambda}(\zeta) /(\zeta-z)$ over the circular arcs in the lower and upper half planes. Standard residue calculus then shows that $\Upsilon(z)=\log \widetilde{\Lambda}(z)$, and the claimed formula is obtained by plugging this into (7.8).
The large time limit (2.11) for $H>\frac{1}{2}$ follows from (7.6) and the following expression.

## Lemma 7.2.

$$
\frac{1}{2 \pi i} \int_{0}^{\infty}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) \frac{X^{+}(t)}{\Lambda^{+}(t)} d t=-\frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{1}{2 \beta}\left(\frac{1}{X(\beta)}-\frac{1}{X(-\beta)}\right)
$$

Proof. In view of (4.3) and identity (7.7), this integral equals $-\frac{1}{2} \frac{1}{\sin \left(\frac{\alpha}{2} \pi\right)} I$ with

$$
I=\frac{\sin (\pi \alpha)}{\pi} \int_{0}^{\infty} \frac{t^{\alpha-1}}{X(-t)} d t
$$

Integrating the function $f(z)=z^{\alpha-1} / X(z)$ over semicircular contours in the upper and lower half-planes, applying Jordan's lemma and subtracting the results, we obtain an alternative expression

$$
I=\frac{1}{2 \pi i} \int_{0}^{\infty} t^{\alpha-1}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right) d t
$$

This can also be viewed as the limit $I=-\lim _{z \rightarrow \infty} z F(z)$ for

$$
F(z):=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{t^{\alpha-1}}{t-z}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right) d t
$$

Now define the sectionally holomorphic function

$$
G(z):=\left(z^{2}-\beta^{2}\right)\left(F(z)+\frac{(-z)^{\alpha-1}}{X(z)}\right), \quad z \in \mathbb{C} \backslash \mathbb{R}_{+}
$$

Its limits across the positive real semiaxis satisfy

$$
\begin{aligned}
& G^{+}(t)-G^{-}(t)=\left(t^{2}-\beta^{2}\right)\left(F^{+}(t)-F^{-}(t)+\frac{\left(e^{-\pi i} t\right)^{\alpha-1}}{X^{+}(t)}-\frac{\left(e^{\pi i} t\right)^{\alpha-1}}{X^{-}(t)}\right)= \\
& 2 \frac{\cos \left(\frac{\alpha-1}{2} \pi\right)}{\kappa_{\alpha}}\left(\frac{\left(t^{2}-\beta^{2}\right) N_{\alpha}^{+}(t)}{X^{+}(t)}-\frac{\left(t^{2}-\beta^{2}\right) N_{\alpha}^{-}(t)}{X^{-}(t)}\right)= \\
& 2 \frac{\sin \left(\frac{\alpha}{2} \pi\right)}{\kappa_{\alpha}}\left(\frac{\Lambda^{+}(t)+\mu_{\varepsilon}^{2}}{X^{+}(t)}-\frac{\Lambda^{-}(t)+\mu_{\varepsilon}^{2}}{X^{-}(t)}\right)=2 \frac{\sin \left(\frac{\alpha}{2} \pi\right) \mu_{\varepsilon}^{2}}{\kappa_{\alpha}}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right)
\end{aligned}
$$

where the last equality holds by virtue of (5.25). Since $G(z)=-I z(1+o(1))$ and $1 / X(z) \rightarrow$ 0 as $z \rightarrow \infty$, by the Sokhotski-Plemelj theorem

$$
G(z)=2 \mu_{\varepsilon}^{2} \frac{\sin \left(\frac{\alpha}{2} \pi\right)}{\kappa_{\alpha}} \frac{1}{X(z)}-I z+C
$$

with a constant $C$. By definition of $G(z)$ it must vanish at $\pm \beta$ and hence

$$
I=\frac{\mu_{\varepsilon}^{2}}{\beta} \frac{\sin \left(\frac{\alpha}{2} \pi\right)}{\kappa_{\alpha}}\left(\frac{1}{X(\beta)}-\frac{1}{X(-\beta)}\right)
$$

7.3. Small noise asymptotics $\boldsymbol{\alpha} \in(\mathbf{0}, \mathbf{1})$. The formula in (2.12) is obtained using (2.2), continuity of (2.11) with respect to $\beta$ and the following limit.

Lemma 7.3. For $\varepsilon=1$,

$$
\lim _{\beta \rightarrow 0+} \frac{1}{\beta} \log \frac{X(-\beta)}{X(\beta)}=\frac{2}{\sin \frac{\pi}{1+\alpha}}\left(\frac{\kappa_{\alpha}}{\mu^{2}}\right)^{\frac{1}{1+\alpha}}
$$

Proof. Let $\widetilde{\theta}(t):=\theta(t)-\theta(\infty)$ and note that $\widetilde{\theta}(t)=\arg \left(\widetilde{\Lambda}^{+}(t)\right)$ where, c.f. (4.5),

$$
\widetilde{\Lambda}(z)=\frac{\Lambda(z)}{N_{\alpha}(z)}=z^{2}-\beta^{2}-\frac{\mu^{2}}{N_{\alpha}(z)} .
$$

By definition (5.26), for $\beta>0$,

$$
\begin{gathered}
\frac{X(\beta)}{X(-\beta)}=-\exp \left(-\frac{1}{\pi} \int_{0}^{\infty} \frac{\widetilde{\theta}(t)}{t+\beta} d t+\frac{1}{\pi} f_{0}^{\infty} \frac{\widetilde{\theta}(t)}{t-\beta} d t+i \theta(\beta)\right)= \\
\\
\exp \left(\frac{2 \beta}{\pi} \int_{0}^{\infty} \frac{\widetilde{\theta}(t)}{t^{2}-\beta^{2}} d t\right)=: \exp (J),
\end{gathered}
$$

where the integral is in the sense of the Cauchy principal value and the second equality holds since $\theta(\beta)=\pi$. Due to symmetry (5.21),

$$
\widetilde{\theta}(t)=\frac{1}{2 i} \log \frac{\widetilde{\Lambda}^{+}(t)}{\widetilde{\Lambda}^{-}(t)},
$$

and since $\int_{0}^{\infty} \frac{1}{t^{2}-\beta^{2}} d t=0$, we can write

$$
J=\frac{\beta}{\pi i} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{+}(t)-\log \tilde{\Lambda}^{+}(\beta)}{t^{2}-\beta^{2}} d t-\frac{\beta}{\pi i} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{-}(t)-\log \widetilde{\Lambda}^{-}(\beta)}{t^{2}-\beta^{2}} d t .
$$

Integrating the function

$$
f(z)=\frac{\log \widetilde{\Lambda}(z)-\log \widetilde{\Lambda}^{+}(\beta)}{z^{2}-\beta^{2}}
$$

over the closed contour in the first quadrant, formed by the axes and a circular arc, and applying Jordan's lemma, we find that

$$
\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{+}(t)-\log \widetilde{\Lambda}^{+}(\beta)}{t^{2}-\beta^{2}} d t=-\frac{1}{2 \pi} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{r}(i t)-\log \widetilde{\Lambda}^{+}(\beta)}{t^{2}+\beta^{2}} d t
$$

where $\widetilde{\Lambda}^{r}(i t)$ stands for the limit of $\widetilde{\Lambda}(z)$ as $z \rightarrow i t$ in the right half-plane. Integrating the function

$$
h(z)=\frac{\log \widetilde{\Lambda}(z)-\log \widetilde{\Lambda}^{-}(\beta)}{z^{2}-\beta^{2}}
$$

over similar contour in the fourth quadrant, we get

$$
\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{-}(t)-\log \widetilde{\Lambda}^{-}(\beta)}{t^{2}-\beta^{2}} d t=\frac{1}{2 \pi} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{r}(-i t)-\log \widetilde{\Lambda}^{-}(\beta)}{t^{2}+\beta^{2}} d t
$$

Subtracting, we obtain

$$
\begin{aligned}
J & =-\frac{\beta}{\pi} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{r}(i t)-\log \widetilde{\Lambda}^{+}(\beta)}{t^{2}+\beta^{2}} d t-\frac{\beta}{\pi} \int_{0}^{\infty} \frac{\log \widetilde{\Lambda}^{r}(-i t)-\log \widetilde{\Lambda}^{-}(\beta)}{t^{2}+\beta^{2}} d t \\
& =\frac{1}{2}\left(\log \widetilde{\Lambda}^{+}(\beta)+\log \widetilde{\Lambda}^{-}(\beta)\right)-\frac{2 \beta}{\pi} \int_{0}^{\infty} \frac{\operatorname{Re}\left(\log \widetilde{\Lambda}^{r}(i t)\right)}{t^{2}+\beta^{2}} d t .
\end{aligned}
$$

A standard calculation, which uses the explicit expressions

$$
\operatorname{Re}\left(\log \widetilde{\Lambda}^{r}(i t)\right)=\log \left(t^{2}+\beta^{2}+\frac{\mu^{2}}{\kappa_{\alpha}} t^{1-\alpha}\right)
$$

and

$$
\log \widetilde{\Lambda}^{+}(\beta)+\log \widetilde{\Lambda}^{-}(\beta)=2 \log \left(\frac{\mu^{2}}{\kappa_{\alpha}} \beta^{1-\alpha}\right)
$$

yields the claimed asymptotics

$$
J=-\frac{2 \beta}{\sin \frac{\pi}{1+\alpha}}\left(\frac{\kappa_{\alpha}}{\mu^{2}}\right)^{\frac{1}{1+\alpha}}(1+o(1)), \quad \beta \rightarrow 0
$$

7.4. Large time limit $\boldsymbol{\alpha} \in(\mathbf{1}, \mathbf{2})$. We can use (5.28) and (7.1) to express $\Phi_{0}(z)$ and $\Phi_{1}(z)$ in terms of solutions to (5.33) and (8.11). If we plug the obtained expressions into equations (7.2) and apply the estimates (5.35) and (8.12), we arrive at the large time asymptotic relations, $T \rightarrow \infty$,

$$
\begin{align*}
& \left(z_{0}+\beta\right) X\left(z_{0}\right)\left(\left(k_{1}^{S}+k_{1}^{D}\right) z_{0}+k_{0}^{S}+k_{0}^{D}+F_{S}\left(z_{0}\right)+F_{D}\left(z_{0}\right)\right)+\mu_{\varepsilon}^{2} \psi(0) \asymp 0  \tag{7.9}\\
& \left(z_{0}-\beta\right) X\left(z_{0}\right)\left(\left(k_{1}^{S}-k_{1}^{D}\right) z_{0}+k_{0}^{S}-k_{0}^{D}+F_{S}\left(z_{0}\right)-F_{D}\left(z_{0}\right)\right)-\mu_{\varepsilon} \asymp 0
\end{align*}
$$

In the case $\alpha \in(1,2)$ the function $H(z)$ in (7.4) can no longer be defined, but nevertheless the formula in (7.5) remains valid, as can be checked directly using the Sokhotski-Plemelj theorem. Hence the second equation in (7.9) is equivalent to

$$
\left(k_{1}^{S}-k_{1}^{D}\right) z_{0}+k_{0}^{S}-k_{0}^{D} \asymp \frac{\mu_{\varepsilon}}{z_{0}-\beta} \frac{1}{X(\beta)} .
$$

Since $X(\beta)$ is purely real, this implies

$$
\begin{align*}
& k_{1}^{S}-k_{1}^{D} \asymp-\frac{\mu_{\varepsilon}}{X(\beta)} \frac{1}{\left|z_{0}-\beta\right|^{2}},  \tag{7.10}\\
& k_{0}^{S}-k_{0}^{D} \asymp-\frac{\mu_{\varepsilon}}{X(\beta)} \frac{\beta-z_{0}-\bar{z}_{0}}{\left|z_{0}-\beta\right|^{2}} .
\end{align*}
$$

The filtering error can now be computed using (5.6). To this end, note that, in view of (4.2) and (7.1), the first term satisfies

$$
\begin{equation*}
N_{\alpha}(-z) e^{-z T} \widehat{g}(-z)=\left(k_{1}^{S}-k_{1}^{D}\right)(-z)^{\frac{\alpha-1}{2}-1}+O\left(z^{\frac{\alpha-1}{2}-2}\right), \quad \operatorname{Re}(z) \rightarrow \infty \tag{7.11}
\end{equation*}
$$

where

$$
(-z)^{\frac{\alpha-1}{2}-1}=-\frac{\cos \left(\frac{\alpha}{2} \pi\right)}{\pi} \int_{0}^{\infty} \frac{t^{\frac{\alpha-1}{2}-1}}{t-z} d t
$$

Due to (4.2) and (4.3), the integral in (5.6) takes the form

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{t-z}\left(N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)\right) e^{-t T} \widehat{g}(-t) d t= \\
& \kappa_{\alpha} \frac{\cos \left(\frac{\alpha}{2} \pi\right)}{\pi} \int_{0}^{\infty} \frac{t^{\alpha-1}}{t-z}\left((t-\beta) \frac{\Phi_{1}^{+}(t)}{\Lambda^{+}(t)}-\frac{\mu_{\varepsilon}}{\Lambda^{+}(t)}\right) d t+z^{-1} R(z, T) \tag{7.12}
\end{align*}
$$

where the residual $R(z, T)$ vanishes as $T \rightarrow \infty$, uniformly over $z$. The value of the latter integral will not change asymptotically as $T \rightarrow \infty$, if we replace

$$
\Phi_{1}^{+}(t) \asymp\left(k_{1}^{S}-k_{1}^{D}\right) t+k_{0}^{S}-k_{0}^{D}+Q^{+}(t) .
$$

Thus, substituting approximations (7.11) and (7.12) into (5.6) and using formula (7.5), we arrive at

$$
\begin{equation*}
P_{T}\left(\beta, \mu_{\varepsilon}\right) \asymp \frac{\kappa_{\alpha}}{\mu_{\varepsilon}} \frac{\cos \left(\frac{\alpha}{2} \pi\right)}{\pi}\left(\left(k_{1}^{S}-k_{1}^{D}\right) I_{2}+\left(k_{0}^{S}-k_{0}^{D}\right) I_{1}+I_{0}\right), \tag{7.13}
\end{equation*}
$$

with

$$
\begin{align*}
& I_{0}:=-\frac{\mu_{\varepsilon}}{X(\beta)} \int_{0}^{\infty} t^{\alpha-1} \frac{X^{+}(t)}{\Lambda^{+}(t)} d t \\
& I_{1}:=\int_{0}^{\infty} t^{\alpha-1}(t-\beta) \frac{X^{+}(t)}{\Lambda^{+}(t)} d t  \tag{7.14}\\
& I_{2}:=\int_{0}^{\infty}\left(t^{\alpha}(t-\beta) \frac{X^{+}(t)}{\Lambda^{+}(t)}+\frac{1}{\kappa_{\alpha}} t^{\frac{\alpha-1}{2}-1}\right) d t
\end{align*}
$$

To simplify the expression in (7.13) we will need the following identity.
Lemma 7.4.

$$
\begin{equation*}
X(z) X(-z)=-\frac{1}{\kappa_{\alpha}} \frac{\Lambda(z)}{\left(z^{2}-z_{0}^{2}\right)\left(z^{2}-\bar{z}_{0}^{2}\right)}, \quad z \in \mathbb{C} \backslash \mathbb{R} \tag{7.15}
\end{equation*}
$$

Proof. For $X(z)$ defined in (5.26) with $k=-1$,

$$
X(z) X(-z)=(-z)^{-1+\frac{\alpha-1}{2}} z^{-1+\frac{\alpha-1}{2}} \exp (\Upsilon(z))
$$

where

$$
\Upsilon(z):=\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)-\theta(\infty)}{t-z} d t+\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)-\theta(\infty)}{t+z} d t .
$$

Define the function

$$
\widetilde{\Lambda}(z):=\frac{\Lambda(z)}{z^{2} N_{\alpha}(z)}=1-\beta^{2} z^{-2}-\frac{\mu_{\varepsilon}^{2}}{z^{2} N_{\alpha}(z)} .
$$

Since $\arg \left(N_{\alpha}^{+}(t)\right)=\theta(\infty)$ for all $t \in \mathbb{R}_{+}$,

$$
\theta(t)-\theta(\infty)=\arg \left(\widetilde{\Lambda}^{+}(t)\right)=\frac{1}{2 i} \log \frac{\widetilde{\Lambda}^{+}(t)}{\widetilde{\Lambda}^{-}(t)}=: \widetilde{\theta}(t)
$$

and, therefore,

$$
\begin{align*}
\Upsilon(z)= & \frac{1}{\pi} \int_{0}^{\infty} \frac{\widetilde{\theta}(t)}{t-z} d t+\frac{1}{\pi} \int_{0}^{\infty} \frac{\widetilde{\theta}(t)}{t+z} d t \stackrel{\dagger}{=} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\widetilde{\theta}(t)}{t-z} d t=  \tag{7.16}\\
& \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{+}(t)}{t-z} d t-\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{-}(t)}{t-z} d t,
\end{align*}
$$

where the equality $\dagger$ holds by the antisymmetry $\widetilde{\theta}(t)=-\widetilde{\theta}(-t)$. The last two integrals in (7.16) are well defined since $\lim _{t \rightarrow \pm \infty} \widetilde{\Lambda}^{ \pm}(t)=1$. They can be evaluated by contour integration of the function

$$
\begin{equation*}
f(\zeta):=\frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} \tag{7.17}
\end{equation*}
$$

which must take into account the branch cut of the logarithm,

$$
\begin{equation*}
C=\left\{\zeta \in \mathbb{C}: \widetilde{\Lambda}(\zeta) \in \mathbb{R}_{-}\right\}=C_{1} \cup C_{2} \cup C_{3} \cup C_{4}, \tag{7.18}
\end{equation*}
$$

where $C_{j}$ denotes the intersection of $C$ with the $j$-th quadrant.
Let us determine the geometric shapes of each curve $C_{j}$ 's starting with $C_{1}$. For $z=\rho e^{i \phi}$ in the first quadrant, with $\rho \in \mathbb{R}_{+}$and $\phi \in\left(0, \frac{\pi}{2}\right)$,

$$
\widetilde{\Lambda}(z)=1+\beta^{2} \rho^{-2} e^{2 \widetilde{\phi} i}+\frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \rho^{-1-\alpha} e^{(\alpha+1) \widetilde{\phi} i}
$$

where $\widetilde{\phi}:=\frac{\pi}{2}-\phi \in\left(0, \frac{\pi}{2}\right)$. Hence $\operatorname{Im}(\widetilde{\Lambda}(z))=0$ holds if and only if either $\widetilde{\varphi}=0$ or

$$
\begin{equation*}
\rho^{\alpha-1}=-\frac{1}{\beta^{2}} \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{\sin ((\alpha+1) \widetilde{\phi})}{\sin (2 \widetilde{\phi})} . \tag{7.19}
\end{equation*}
$$

This equation has a solution only if the right hand side is positive, that is, when $\widetilde{\phi} \in$ $\left(\frac{\pi}{\alpha+1}, \frac{\pi}{2}\right)$. For all such $\widetilde{\phi}$ and with $\rho$ as in (7.19),

$$
\begin{aligned}
\operatorname{Re}(\widetilde{\Lambda}(z))= & 1+\beta^{2} \rho^{-2} \cos (2 \widetilde{\phi})+\frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \rho^{-1-\alpha} \cos ((\alpha+1) \widetilde{\phi})= \\
& 1+\rho^{-2} \beta^{2} \frac{\sin ((\alpha-1) \widetilde{\phi})}{\sin ((\alpha+1) \widetilde{\phi})}=1+\left(\beta^{2}\right)^{\frac{\alpha+1}{\alpha-1}}\left(\frac{\kappa_{\alpha}}{\mu_{\varepsilon}^{2}}\right)^{\frac{2}{\alpha-1}} g(\widetilde{\phi}),
\end{aligned}
$$

where we defined

$$
g(\widetilde{\phi}):=\left(-\frac{\sin (2 \widetilde{\phi})}{\sin ((\alpha+1) \widetilde{\phi})}\right)^{\frac{2}{\alpha-1}} \frac{\sin ((\alpha-1) \widetilde{\phi})}{\sin ((\alpha+1) \widetilde{\phi})}
$$

This function is strictly increasing on the interval $\left(\frac{\pi}{\alpha+1}, \frac{\pi}{2}\right)$ and maps it onto $(-\infty, 0)$. Hence $\operatorname{Re}(\widetilde{\Lambda}(z))$ vanishes at the unique angle $\widetilde{\phi}_{0} \in\left(\frac{\pi}{\alpha+1}, \frac{\pi}{2}\right)$, and $\operatorname{Re}(\widetilde{\Lambda}(z))<0$ if and only if $\widetilde{\phi} \in\left(\frac{\pi}{\alpha+1}, \widetilde{\phi}_{0}\right)$. Therefore $C_{1}$ is the curve, which starts with $\widetilde{\phi}=\frac{\pi}{\alpha+1}$ at the origin and terminates at $z_{0}=\rho_{0} e^{i \phi_{0}}$, where $\phi_{0}=\frac{\pi}{2}-\widetilde{\phi}_{0}$ and the absolute value $\rho_{0}$ is determined by (7.19). The terminal point $z_{0}$ is precisely the zero of $\Lambda(z)$, and hence also of $\widetilde{\Lambda}(z)$, in the first quadrant.

The imaginary part $\operatorname{Im}(\widetilde{\Lambda}(z))$ vanishes on the positive imaginary semiaxis and on the continuation of $C_{1}$ corresponding to $\widetilde{\phi} \in\left[\tilde{\phi}_{0}, \frac{\pi}{2}\right)$, where $\operatorname{Re}(\widetilde{\Lambda}(z))$ remains positive. Hence $\operatorname{Im}(\widetilde{\Lambda}(z))$ preserves its sign on the subset of the first quadrant, which lies between these curves, and it is readily checked to be positive. The rest of $C_{j}$ 's have similar forms, starting at the origin and terminating at the other zeros of $\Lambda(z)$, as shown on Figure 1. Along with the real and imaginary axes they divide the plane into eight subsets, on which the sign of $\operatorname{Im}(\widetilde{\Lambda}(z))$ is constant.


Figure 1. The branch cut $C$ in (7.18) is depicted in solid blue; $\operatorname{Im}(\widetilde{\Lambda}(z))$ changes signs across the blue lines, both dashed and solid, being encircled over the corresponding regions; the two integration contours are coloured in red.

For definiteness, suppose $\operatorname{Im}(z)>0$. Then integrating $f(\zeta)$ from (7.17) along the closed contour in the upper half plane, applying Jordan's lemma and Cauchy's residue theorem, we obtain

$$
\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{+}(t)}{t-z} d t=\log \widetilde{\Lambda}(z)-\frac{1}{2 \pi i} \oint_{C_{1}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta-\frac{1}{2 \pi i} \oint_{C_{2}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta
$$

where the last two terms stand for the limiting values of the integrals over the shrinking contours around $C_{1}$ and $C_{2}$. Since $|\widetilde{\Lambda}(\zeta)|$ is continuous across $C_{j}$ 's and taking into account the signs of $\widetilde{\Lambda}(\zeta)$,

$$
\frac{1}{2 \pi i} \oint_{C_{1}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta=\frac{2 \pi i}{2 \pi i}[\log (\zeta-z)]_{0}^{z_{0}}=\log \frac{z-z_{0}}{z}
$$

and

$$
\frac{1}{2 \pi i} \oint_{C_{2}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta=\frac{2 \pi i}{2 \pi i}[\log (\zeta-z)]_{0}^{-\bar{z}_{0}}=\log \frac{z+\bar{z}_{0}}{z}
$$

Similarly, integration over the contour in the lower half plane gives

$$
\begin{aligned}
\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{\log \widetilde{\Lambda}^{-}(t)}{t-z} d t= & \frac{1}{2 \pi i} \oint_{C_{3}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \oint_{C_{4}} \frac{\log \widetilde{\Lambda}(\zeta)}{\zeta-z} d \zeta= \\
& \log \frac{z+z_{0}}{z}+\log \frac{z-\bar{z}_{0}}{z}
\end{aligned}
$$

Plugging this into (7.16) we obtain

$$
\Upsilon(z)=\log \widetilde{\Lambda}(z) \frac{z^{4}}{\left(z^{2}-z_{0}^{2}\right)\left(z^{2}-\bar{z}_{0}^{2}\right)}
$$

and, consequently,

$$
\begin{aligned}
X(z) X(-z) & =(-z)^{-1+\frac{\alpha-1}{2}} z^{-1+\frac{\alpha-1}{2}} \frac{\Lambda(z)}{z^{2} N_{\alpha}(z)} \frac{z^{4}}{\left(z^{2}-z_{0}^{2}\right)\left(z^{2}-\bar{z}_{0}^{2}\right)}= \\
& -\frac{(-z)^{\frac{\alpha-1}{2}} z^{\frac{\alpha-1}{2}}}{N_{\alpha}(z)} \frac{\Lambda(z)}{\left(z^{2}-z_{0}^{2}\right)\left(z^{2}-\bar{z}_{0}^{2}\right)}=-\frac{1}{\kappa_{\alpha}} \frac{\Lambda(z)}{\left(z^{2}-z_{0}^{2}\right)\left(z^{2}-\bar{z}_{0}^{2}\right)} .
\end{aligned}
$$

Using formula (7.15), the integrals in (7.14) can now be written as

$$
\begin{equation*}
I_{0}=\frac{\mu_{\varepsilon}}{X(\beta)} \frac{J_{0}}{\kappa_{\alpha}}, \quad I_{1}=\frac{\beta J_{0}-J_{1}}{\kappa_{\alpha}}, \quad I_{2}=\frac{\beta J_{1}-J_{2}}{\kappa_{\alpha}}, \tag{7.20}
\end{equation*}
$$

where the basic elements are

$$
\begin{align*}
J_{0} & :=\int_{0}^{\infty} \frac{1}{\left|t^{2}-z_{0}^{2}\right|^{2}} \frac{t^{\alpha-1}}{X(-t)} d t, \\
J_{1} & :=\int_{0}^{\infty} \frac{t}{\left|t^{2}-z_{0}^{2}\right|^{2}} \frac{t^{\alpha-1}}{X(-t)} d t,  \tag{7.21}\\
J_{2} & :=\int_{0}^{\infty}\left(\frac{t^{2}}{\left|t^{2}-z_{0}^{2}\right|^{2}} \frac{t^{\alpha-1}}{X(-t)}-t^{\frac{\alpha-1}{2}-1}\right) d t .
\end{align*}
$$

Closed form expressions for these integrals are derived in the following lemma.
Lemma 7.5. The integrals in (7.21) satisfy

$$
\begin{align*}
& J_{0}=\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(L\left(\frac{\alpha+1}{2}\right)-\widetilde{b}_{0} L\left(\frac{\alpha-1}{2}\right)\right. \\
&\left.+z_{0}^{-1} M\left(z_{0}\right)-z_{0}^{-1} M\left(-z_{0}\right)-\bar{z}_{0}^{-1} M\left(\bar{z}_{0}\right)+\bar{z}_{0}^{-1} M\left(-\bar{z}_{0}\right)\right) \\
& J_{1}=\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(L\left(\frac{\alpha+3}{2}\right)-\widetilde{b}_{0} L\left(\frac{\alpha+1}{2}\right)\right. \\
&\left.+M\left(z_{0}\right)+M\left(-z_{0}\right)-M\left(\bar{z}_{0}\right)-M\left(-\bar{z}_{0}\right)\right),  \tag{7.22}\\
& J_{2}=\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(L\left(\frac{\alpha+5}{2}\right)-\widetilde{b}_{0} L\left(\frac{\alpha+3}{2}\right)\right. \\
&\left.+z_{0} M\left(z_{0}\right)-z_{0} M\left(-z_{0}\right)-\bar{z}_{0} M\left(\bar{z}_{0}\right)+\bar{z}_{0} M\left(-\bar{z}_{0}\right)\right)
\end{align*}
$$

where

$$
\begin{equation*}
L(\gamma)=\frac{\pi}{\sin \pi \gamma}\left(z_{0}^{\gamma-1}+\left(-z_{0}\right)^{\gamma-1}-\bar{z}_{0}^{\gamma-1}-\left(-\bar{z}_{0}\right)^{\gamma-1}\right) \tag{7.23}
\end{equation*}
$$

and

$$
\begin{align*}
M(z)= & \frac{1}{\kappa_{\alpha}} \frac{\pi}{\cos \left(\frac{\alpha}{2} \pi\right)}\left(\frac{1}{X(z)} \frac{\Lambda(z)}{z^{2}-\beta^{2}}-N_{\alpha}(z) z^{\frac{3-\alpha}{2}}\left(1-\widetilde{b}_{0} z^{-1}\right)\right.  \tag{7.24}\\
& \left.+\frac{1}{2} \frac{\mu_{\varepsilon}^{2}}{z^{2}-\beta^{2}}\left(\frac{1}{X(\beta)}+\frac{1}{X(-\beta)}\right)-\frac{1}{2} \frac{\mu_{\varepsilon}^{2}}{z^{2}-\beta^{2}} \frac{z}{\beta}\left(\frac{1}{X(\beta)}-\frac{1}{X(-\beta)}\right)\right)
\end{align*}
$$

Proof. Define the function

$$
\begin{equation*}
Y(z):=\frac{1}{X(z)}-(-z)^{\frac{3-\alpha}{2}}\left(1+\widetilde{b}_{0} z^{-1}\right) \tag{7.25}
\end{equation*}
$$

where

$$
\widetilde{b}_{0}:=\frac{1}{\pi} \int_{0}^{\infty}(\theta(t)-\theta(\infty)) d \tau<0
$$

The integrals in (7.21) can be written as

$$
\begin{align*}
J_{0} & =\int_{0}^{\infty} \frac{1}{\left|t^{2}-z_{0}^{2}\right|^{2}} t^{\alpha-1} Y(-t) d t+U\left(\frac{\alpha+1}{2}\right)-\widetilde{b}_{0} U\left(\frac{\alpha-1}{2}\right) \\
J_{1} & =\int_{0}^{\infty} \frac{t}{\left|t^{2}-z_{0}^{2}\right|^{2}} t^{\alpha-1} Y(-t) d t+U\left(\frac{\alpha+3}{2}\right)-\widetilde{b}_{0} U\left(\frac{\alpha+1}{2}\right)  \tag{7.26}\\
J_{2} & =\int_{0}^{\infty} \frac{t^{2}}{\left|t^{2}-z_{0}^{2}\right|^{2}} t^{\alpha-1} Y(-t) d t+V\left(\frac{\alpha-3}{2}\right)-\widetilde{b}_{0} U\left(\frac{\alpha+3}{2}\right)
\end{align*}
$$

where

$$
\begin{aligned}
U(\gamma) & :=\int_{0}^{\infty} \frac{t^{\gamma}}{\left|t^{2}-z_{0}^{2}\right|^{2}} d t=\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}} L(\gamma) \\
V(\gamma) & :=\int_{0}^{\infty} t^{\gamma}\left(\frac{t^{4}}{\left|t^{2}-z_{0}^{2}\right|^{2}}-1\right) d t=\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}} L(\gamma+4)
\end{aligned}
$$

Here $L(\gamma)$ is the function defined in (7.23) and the latter integrals are evaluated by standard contour integration. The formulas in (7.22) are derived from (7.26), using the partial fraction decompositions

$$
\begin{aligned}
\frac{1}{\left|t^{2}-z_{0}^{2}\right|^{2}} & =\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(\frac{z_{0}^{-1}}{t-z_{0}}-\frac{z_{0}^{-1}}{t+z_{0}}-\frac{\bar{z}_{0}^{-1}}{t-\bar{z}_{0}}+\frac{\bar{z}_{0}^{-1}}{t+\bar{z}_{0}}\right) \\
\frac{t}{\left|t^{2}-z_{0}^{2}\right|^{2}}= & \frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(\frac{1}{t-z_{0}}+\frac{1}{t+z_{0}}-\frac{1}{t-\bar{z}_{0}}-\frac{1}{t+\bar{z}_{0}}\right) \\
\frac{t^{2}}{\left|t^{2}-z_{0}^{2}\right|^{2}} & =\frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(\frac{z_{0}}{t-z_{0}}-\frac{z_{0}}{t+z_{0}}-\frac{\bar{z}_{0}}{t-\bar{z}_{0}}+\frac{\bar{z}_{0}}{t+\bar{z}_{0}}\right)
\end{aligned}
$$

and the notation

$$
M(z):=\int_{0}^{\infty} \frac{t^{\alpha-1}}{t-z} Y(-t) d t
$$

It is left to show that $M(z)$ satisfies the claimed formula. To this end, integrating the function $f(\zeta)=\frac{\zeta^{\alpha-1}}{\zeta-z} Y(\zeta)$ over semicircular contours in the upper and lower half planes
and summing the obtained equations, we get

$$
\begin{equation*}
\frac{\sin (\alpha \pi)}{\pi} M(-z)=z^{\alpha-1} Y(z)-P(z) \tag{7.27}
\end{equation*}
$$

where

$$
P(z):=\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{t^{\alpha-1}}{t-z}\left(Y^{+}(t)-Y^{-}(t)\right) d t
$$

To evaluate this integral, define

$$
\begin{equation*}
H(z)=\left(z^{2}-\beta^{2}\right)\left(P(z)+(-z)^{\alpha-1} Y(z)\right) \tag{7.28}
\end{equation*}
$$

This function is sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$and for $t>0$

$$
\begin{aligned}
& H^{+}(t)-H^{-}(t)= \\
& \left(t^{2}-\beta^{2}\right)\left(P^{+}(t)-P^{-}(t)+\left(e^{-\pi i} t\right)^{\alpha-1} Y^{+}(t)-\left(e^{\pi i} t\right)^{\alpha-1} Y^{-}(t)\right)= \\
& 2 \cos \left(\frac{\alpha-1}{2} \pi\right)\left(t^{2}-\beta^{2}\right) t^{\alpha-1}\left(e^{-\frac{\alpha-1}{2} \pi i} Y^{+}(t)-e^{\frac{\alpha-1}{2} \pi i} Y^{-}(t)\right)= \\
& 2 \sin \left(\frac{\alpha}{2} \pi\right)\left(t^{2}-\beta^{2}\right) t^{\alpha-1}\left(e^{-\frac{\alpha-1}{2} \pi i} \frac{1}{X^{+}(t)}-e^{\frac{\alpha-1}{2} \pi i} \frac{1}{X^{-}(t)}\right)= \\
& 2 \sin \left(\frac{\alpha}{2} \pi\right) \frac{1}{\kappa_{\alpha}}\left(\frac{\left(t^{2}-\beta^{2}\right) N_{\alpha}^{+}(t)}{X^{+}(t)}-\frac{\left(t^{2}-\beta^{2}\right) N_{\alpha}^{+}(t)}{X^{-}(t)}\right)= \\
& 2 \sin \left(\frac{\alpha}{2} \pi\right) \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}}\left(\frac{1}{X^{+}(t)}-\frac{1}{X^{-}(t)}\right)
\end{aligned}
$$

Since $H(z)$ grows not faster than linearly, it follows that

$$
\begin{equation*}
H(z)=2 \sin \left(\frac{\alpha}{2} \pi\right) \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{1}{X(z)}+c_{1} z+c_{0} \tag{7.29}
\end{equation*}
$$

where constants $c_{1}$ and $c_{0}$ are identified using the equations $H^{+}( \pm \beta)=0$,

$$
\begin{aligned}
& c_{0}=-\sin \left(\frac{\alpha}{2} \pi\right) \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}}\left(\frac{1}{X(\beta)}+\frac{1}{X(-\beta)}\right) \\
& c_{1}=-\sin \left(\frac{\alpha}{2} \pi\right) \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{1}{\beta}\left(\frac{1}{X(\beta)}-\frac{1}{X(-\beta)}\right)
\end{aligned}
$$

Plugging (7.25), (7.28) and (7.29) into (7.27) we obtain (7.24) since

$$
N_{\alpha}(z)=\frac{\kappa_{\alpha}}{2 \sin \left(\frac{\alpha}{2} \pi\right)}\left(z^{\alpha-1}+(-z)^{\alpha-1}\right)
$$

Now we are ready to find the ultimate expression for the filtering error in this case. Substituting (7.10), (7.20) and (7.22) into (7.13), we get

$$
\begin{aligned}
P_{\infty}\left(\beta, \mu_{\varepsilon}\right)= & \frac{1}{X(\beta)} \frac{\cos \left(\frac{\alpha}{2} \pi\right)}{\pi} \frac{1}{\left|z_{0}-\beta\right|^{2}}\left(J_{2}-J_{1}\left(z_{0}+\bar{z}_{0}\right)+J_{0} z_{0} \bar{z}_{0}\right)= \\
& \frac{1}{X(\beta)} \frac{\cos \left(\frac{\alpha}{2} \pi\right)}{\pi} \frac{1}{\left|z_{0}-\beta\right|^{2}}\left(B_{1}+B_{2}+B_{3}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
B_{1}:= & \frac{1}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(L\left(\frac{\alpha+5}{2}\right)-\left(z_{0}+\bar{z}_{0}\right) L\left(\frac{\alpha+3}{2}\right)+z_{0} \bar{z}_{0} L\left(\frac{\alpha+1}{2}\right)\right)= \\
& \frac{1}{z_{0}-\bar{z}_{0}} \frac{\pi}{\cos \frac{\alpha}{2} \pi}\left(z_{0}^{\frac{\alpha+1}{2}}-\bar{z}_{0}^{\frac{\alpha+1}{2}}\right), \\
B_{2}:=- & \frac{\widetilde{b}_{0}}{2} \frac{1}{z_{0}^{2}-\bar{z}_{0}^{2}}\left(L\left(\frac{\alpha+3}{2}\right)-\left(z_{0}+\bar{z}_{0}\right) L\left(\frac{\alpha+1}{2}\right)+z_{0} \bar{z}_{0} L\left(\frac{\alpha-1}{2}\right)\right)= \\
& \widetilde{b}_{0} \frac{1}{z_{0}-\bar{z}_{0}} \frac{\pi}{\cos \frac{\alpha}{2} \pi}\left(z_{0}^{\frac{\alpha-1}{2}}-\bar{z}_{0}^{\frac{\alpha-1}{2}}\right), \\
B_{3}:= & \frac{1}{z_{0}-\bar{z}_{0}}\left(M\left(-\bar{z}_{0}\right)-M\left(-z_{0}\right)\right) .
\end{aligned}
$$

Since $-\bar{z}_{0}$ and $-z_{0}$ are zeros of $\Lambda(z)$, equation (7.24) yields

$$
\begin{aligned}
& M\left(-\bar{z}_{0}\right)-M\left(-z_{0}\right)=\frac{\pi}{\cos \left(\frac{\alpha}{2} \pi\right)}\left\{\bar{z}_{0}^{\frac{\alpha+1}{2}}-z_{0}^{\frac{\alpha+1}{2}}+\widetilde{b}_{0}\left(\bar{z}_{0}^{\frac{\alpha-1}{2}}-z_{0}^{\frac{\alpha-1}{2}}\right)\right. \\
& \left.+\frac{1}{2} \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{1}{\beta}\left(\frac{1}{X(-\beta)} \frac{1}{z_{0}+\beta}-\frac{1}{X(\beta)} \frac{1}{z_{0}-\beta}+\frac{1}{X(\beta)} \frac{1}{\bar{z}_{0}-\beta}-\frac{1}{X(-\beta)} \frac{1}{\bar{z}_{0}+\beta}\right)\right\}
\end{aligned}
$$

and, collecting all parts together, we finally get

$$
\begin{aligned}
P_{\infty}\left(\beta, \mu_{\varepsilon}\right)= & \frac{1}{X(\beta)} \frac{1}{\left|z_{0}-\beta\right|^{2}} \frac{\mu_{\varepsilon}^{2}}{\kappa_{\alpha}} \frac{1}{2 \beta}\left(\frac{1}{X(\beta)\left|z_{0}-\beta\right|^{2}}-\frac{1}{X(-\beta)\left|z_{0}+\beta\right|^{2}}\right)= \\
& \frac{1}{2 \beta}\left(\frac{\left|z_{0}+\beta\right|^{2}}{\left|z_{0}-\beta\right|^{2}} \frac{X(-\beta)}{X(\beta)}-1\right)
\end{aligned}
$$

where identity (7.15) was used in the last equality. This is the large time limit claimed in (2.11) for $H<\frac{1}{2}$.
7.5. Small noise asymptotics $\alpha \in(\mathbf{1}, \mathbf{2})$. The corresponding small noise asymptotics (2.12) is derived as in Lemma 7.3.

## 8. Proof of Theorem 2.5

The limit expression for the filtering error (5.39) was derived when $\alpha_{2}<\alpha_{1}<1$, in which case the structural function does not have zeros. To demonstrate how the method works in presence of zeros, in this section we will also detail the proof for $\alpha_{1}=\alpha \in(0,1)$ and $\alpha_{2}=1$. To derive the analog of formula (5.39) in this case, we will have to return to the point, where the proof splits into cases, and reformulate the equivalent problem accordingly.
8.1. The equivalent problem. For $\alpha_{1}=: \alpha \in(0,2)$ and $\alpha_{2}=1$, cf. (4.5),

$$
\begin{equation*}
\Lambda(z)=z^{2}-\beta^{2}-\mu_{\varepsilon}^{2} N_{\alpha}(z) . \tag{8.1}
\end{equation*}
$$

Since $N_{\alpha_{2}}(z)=1$, equations (5.23) can be written as

$$
\begin{align*}
& \widetilde{\Phi}_{0}^{+}(t)-\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} \widetilde{\Phi}_{0}^{-}(t)=e^{-t T} \widetilde{\Phi}_{1}(-t)\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}-1\right), \\
& \widetilde{\Phi}_{1}^{+}(t)-\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)} \widetilde{\Phi}_{1}^{-}(t)=e^{-t T} \widetilde{\Phi}_{0}(-t)\left(\frac{\Lambda^{+}(t)}{\Lambda^{-}(t)}-1\right), \tag{8.2}
\end{align*}
$$

where we defined, cf. (5.17),

$$
\begin{align*}
& \widetilde{\Phi}_{0}(z):=\Phi_{0}(z)+\psi(0)(z-\beta), \\
& \widetilde{\Phi}_{1}(z):=\Phi_{1}(z)-\frac{1}{\mu_{\varepsilon}}(z+\beta) . \tag{8.3}
\end{align*}
$$

Unlike (5.23) equations (8.2) do not contain additional free term in the right hand side.
When $\alpha \in(0,1)$ the structural function $\Lambda(z)$ has four zeros, see Lemma 5.4. In view of (8.1) and definitions (8.3), the expression (4.2) takes the form

$$
\widehat{g}(z)=-(z+\beta) \frac{\widetilde{\Phi}_{0}(z)+e^{-z T} \widetilde{\Phi}_{1}(-z)}{\Lambda(z)}+\psi(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T},
$$

and hence removal of the poles implies the conditions, cf. (5.24),

$$
\begin{align*}
& \widetilde{\Phi}_{0}\left(z_{0}\right)+e^{-z_{0} T} \widetilde{\Phi}_{1}\left(-z_{0}\right)=0, \\
& \widetilde{\Phi}_{1}\left(z_{0}\right)+e^{-z_{0} T} \widetilde{\Phi}_{0}\left(-z_{0}\right)=0 . \tag{8.4}
\end{align*}
$$

Finally, when $\alpha_{2}=1$, the integral terms in (5.5) and (5.6) vanish, and in view of (4.2), $\lim _{z \rightarrow \infty} \Phi_{0}(z)=0$ and $\lim _{z \rightarrow \infty} \Phi_{1}(z)=v_{g, 1}(T)$, or equivalently,

$$
\begin{align*}
& \widetilde{\Phi}_{0}(z) \asymp \psi(0)(z-\beta), \\
& \widetilde{\Phi}_{1}(z) \asymp v_{g, 1}(T)-\frac{1}{\mu_{\varepsilon}}(z+\beta), \quad z \rightarrow \infty . \tag{8.5}
\end{align*}
$$

8.2. Large time limit $\boldsymbol{\alpha} \in(\mathbf{0}, \mathbf{1})$. For $\alpha \in(0,1)$ the angle $\theta(t)=\arg \left(\Lambda^{+}(t)\right)$ is negative with the limits

$$
\theta(0+)=\frac{1-\alpha}{2} \pi-\pi \quad \text { and } \quad \theta(\infty)=0 .
$$

Define, cf. (5.28),

$$
\begin{equation*}
\widetilde{S}(z):=\frac{\widetilde{\Phi}_{0}(z)+\widetilde{\Phi}_{1}(z)}{2 X(z)} \quad \text { and } \quad \widetilde{D}(z):=\frac{\widetilde{\Phi}_{0}(z)-\widetilde{\Phi}_{1}(z)}{2 X(z)} \tag{8.6}
\end{equation*}
$$

Since the functions in (8.3) have the same growth near the origin as in (5.4) and in view of (5.27), the choice $k=-1$ in (5.26) guarantees (square) integrability of the restrictions $\widetilde{S}(-t)$ and $\widetilde{D}(-t), t \in \mathbb{R}_{+}$near the origin. Due to the additional linear terms in (8.3), it also implies that $\widetilde{S}(z)$ and $\widetilde{D}(z)$ are asymptotic to polynomials of degree two as $z \rightarrow \infty$
and therefore, by the Sokhotski-Plemelj theorem, cf. (5.31),

$$
\begin{align*}
& \widetilde{S}(z)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h(t)}{t-z} \widetilde{S}(-t) d t+P_{S}(-z), \\
& \widetilde{D}(z)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-t T} h(t)}{t-z} \widetilde{D}(-t) d t+P_{D}(-z), \tag{8.7}
\end{align*}
$$

with polynomials

$$
\begin{equation*}
P_{S}(z)=k_{2}^{S} z^{2}+k_{1}^{S} z+k_{0}^{S} \quad \text { and } \quad P_{D}(z)=k_{2}^{D} z^{2}+k_{1}^{D} z+k_{0}^{D}, \tag{8.8}
\end{equation*}
$$

where the coefficients are constants, possibly dependent on $T$ and $\varepsilon$.
In this case $\theta(t)=O\left(t^{\alpha-3}\right)$ as $t \rightarrow \infty$ and hence the exponent in (5.26) satisfies

$$
\begin{align*}
X_{c}(z):= & \exp \left(\frac{1}{\pi} \int_{0}^{\infty} \frac{\theta(t)}{t-z} d t\right)=\exp \left(-\frac{1}{z} m_{0}-\frac{1}{z^{2}} m_{1}+O\left(z^{\alpha-3}\right)\right)= \\
& 1-\frac{1}{z} b_{0}-\frac{1}{z^{2}} b_{1}+O\left(z^{\alpha-3}\right), \quad z \rightarrow \infty, \tag{8.9}
\end{align*}
$$

where $b_{0}=m_{0}, b_{1}=m_{1}-\frac{1}{2} m_{0}^{2}$ and

$$
m_{j}=\frac{1}{\pi} \int_{0}^{\infty} t^{j} \theta(t) d t
$$

Consequently, in view of (8.5), the asymptotic terms in (8.6) and (8.7) match, if the coefficients in (8.8) satisfy

$$
\begin{align*}
k_{2}^{S} & =-\frac{1}{2}\left(\psi(0)-\frac{1}{\mu_{\varepsilon}}\right), \quad k_{1}^{S}=-k_{2}^{S} b_{0}-\frac{\beta}{2}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}}\right)+\frac{v_{g, 1}(T)}{2}, \\
k_{2}^{D} & =-\frac{1}{2}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}}\right), \quad k_{1}^{D}=-k_{2}^{D} b_{0}-\frac{\beta}{2}\left(\psi(0)-\frac{1}{\mu_{\varepsilon}}\right)-\frac{v_{g, 1}(T)}{2} . \tag{8.10}
\end{align*}
$$

As in the previous sections, the auxiliary integral equations

$$
\begin{align*}
& p_{j}(t)=\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h(\tau)}{\tau+t} p_{j}(\tau) d \tau+t^{j} \\
& q_{j}(t)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{e^{-\tau T} h(\tau)}{\tau+t} q_{j}(\tau) d \tau+t^{j} \tag{8.11}
\end{align*}
$$

have unique solutions, whose analytic extensions satisfy, cf. (5.35),

$$
\begin{equation*}
\left|p_{j}(z)-z^{j}\right| \leq C \frac{1}{z} \frac{1}{T} \quad \text { and } \quad\left|q_{j}(z)-z^{j}\right| \leq C \frac{1}{z} \frac{1}{T} . \tag{8.12}
\end{equation*}
$$

By linearity

$$
\begin{align*}
& \widetilde{S}(z)=k_{2}^{S} p_{2}(-z)+k_{1}^{S} p_{1}(-z)+k_{0}^{S} p_{0}(-z), \\
& \widetilde{D}(z)=k_{2}^{D} q_{2}(-z)+k_{1}^{D} q_{1}(-z)+k_{0}^{D} q_{0}(-z) . \tag{8.13}
\end{align*}
$$

We can now use (8.6) and (8.13) to express $\widetilde{\Phi}_{0}(z)$ and $\widetilde{\Phi}_{1}(z)$ in terms of the above constants and solutions to integral equations (8.11). Then plugging these expressions into (8.4) and
using the estimates (8.12) we obtain, as $T \rightarrow \infty$,

$$
\begin{align*}
& z_{0}^{2} k_{2}^{S}-z_{0} k_{1}^{S}+k_{0}^{S}+z_{0}^{2} k_{2}^{D}-z_{0} k_{1}^{D}+k_{0}^{D} \asymp 0, \\
& z_{0}^{2} k_{2}^{S}-z_{0} k_{1}^{S}+k_{0}^{S}-z_{0}^{2} k_{2}^{D}+z_{0} k_{1}^{D}-k_{0}^{D} \asymp 0 . \tag{8.14}
\end{align*}
$$

Powers of $z_{0}$ have nonzero complex parts and hence these are, in fact, four equations with real valued coefficients. Thus we arrive at a system of eight linear equations (8.10) and (8.14) for the limiting values of the eight unknowns, namely $\psi(0), v_{g, 1}(T)$ and $k_{j}^{S}$ and $k_{j}^{D}, j=0,1,2$.

Taking the imaginary part of the equations above we get rid of $k_{0}^{S}$ and $k_{0}^{D}$ :

$$
\begin{aligned}
& \operatorname{Im}\left(z_{0}^{2}\right)\left(k_{2}^{S}+k_{2}^{D}\right)-\operatorname{Im}\left(z_{0}\right)\left(k_{1}^{S}+k_{1}^{D}\right) \asymp 0, \\
& \operatorname{Im}\left(z_{0}^{2}\right)\left(k_{2}^{S}-k_{2}^{D}\right)-\operatorname{Im}\left(z_{0}\right)\left(k_{1}^{S}-k_{1}^{D}\right) \asymp 0 .
\end{aligned}
$$

Now from (8.10)

$$
\begin{array}{ll}
k_{2}^{S}+k_{2}^{D}=-\psi(0), & k_{1}^{S}+k_{1}^{D}=\left(b_{0}-\beta\right) \psi(0) \\
k_{2}^{S}-k_{2}^{D}=\frac{1}{\mu_{\varepsilon}}, & k_{1}^{S}-k_{1}^{D}=-\frac{1}{\mu_{\varepsilon}}\left(b_{0}+\beta\right)+v_{g, 1}(T)
\end{array}
$$

and hence

$$
\begin{aligned}
& -\operatorname{Im}\left(z_{0}^{2}\right) \psi(0)-\operatorname{Im}\left(z_{0}\right)\left(b_{0}-\beta\right) \psi(0) \asymp 0, \\
& \operatorname{Im}\left(z_{0}^{2}\right) \frac{1}{\mu_{\varepsilon}}+\operatorname{Im}\left(z_{0}\right) \frac{1}{\mu_{\varepsilon}}\left(b_{0}+\beta\right)-\operatorname{Im}\left(z_{0}\right) v_{g, 1}(T) \asymp 0 .
\end{aligned}
$$

This implies that $\psi(0) \asymp 0$ and

$$
P_{T}\left(\beta, \mu_{\varepsilon}\right)=\frac{1}{\mu_{\varepsilon}} v_{g, 1}(T) \asymp \frac{1}{\mu_{\varepsilon}^{2}}\left(b_{0}+\beta+\frac{\operatorname{Im}\left(z_{0}^{2}\right)}{\operatorname{Im}\left(z_{0}\right)}\right)=\frac{1}{\mu_{\varepsilon}^{2}}\left(b_{0}+\beta+2 \operatorname{Re}\left(z_{0}\right)\right),
$$

which is the formula claimed in (2.7).
8.3. Small noise asymptotics $\boldsymbol{\alpha} \in(\mathbf{0}, \mathbf{1})$. The expression in (2.8) follows from Theorem 2.1, since for $\beta=0$ and $\varepsilon=1$, the zero of $\Lambda(z)$ in the first quadrant can be found explicitly,

$$
z_{0}=\left(\mu^{2} \kappa_{\alpha}\right)^{\frac{1}{3-\alpha}} \exp \left(\frac{1-\alpha}{3-\alpha} \frac{\pi}{2} i\right),
$$

and the first moment of $\theta(t)$ can be computed in the closed form

$$
b_{0}=-\left(\mu^{2} \kappa_{\alpha}\right)^{\frac{1}{3-\alpha}} \frac{\sin \frac{\pi}{2} \frac{1+\alpha}{3-\alpha}}{\sin \frac{\pi}{3-\alpha}} .
$$

8.4. Large time limit $\boldsymbol{\alpha} \in(\mathbf{1}, \mathbf{2})$. In this case, $\theta(t)=\arg \left(\Lambda^{+}(t)\right)$ is positive and

$$
\theta(0+)=\pi \quad \text { and } \quad \theta(\infty)=0 .
$$

In view of estimates (5.4) and (5.27), the suitable choice of the power factor in (5.26) is $k=1$, which guarantees that the restrictions of (8.6) to $\mathbb{R}_{-}$are (square) integrability near the origin. This choice and (8.3) imply that $\widetilde{S}(z)$ and $\widetilde{D}(z)$ are of order $O(1)$ as $z \rightarrow \infty$. Hence representation (8.7) hold with $P_{S}(z)=k_{0}^{S}$ and $P_{S}(z)=k_{0}^{D}$, and hence, by linearity,

$$
\widetilde{S}(z)=k_{0}^{S} p_{0}(-z) \quad \text { and } \quad \widetilde{D}(z)=k_{0}^{D} q_{0}(-z) .
$$

Comparing this with (8.6) implies

$$
\begin{equation*}
k_{0}^{S}=-\frac{1}{2}\left(\psi(0)-\frac{1}{\mu_{\varepsilon}}\right) \quad \text { and } \quad k_{0}^{D}=-\frac{1}{2}\left(\psi(0)+\frac{1}{\mu_{\varepsilon}}\right) . \tag{8.15}
\end{equation*}
$$

The filtering error can now be found from (5.6), where for $\alpha_{2}=1$ the last term vanishes. Plugging (8.3) into (4.2) yields

$$
\begin{aligned}
P_{T}= & \frac{1}{\mu_{\varepsilon}} \lim _{\operatorname{Re}(z) \rightarrow 0} z e^{-z T} \widehat{g}(-z)=\frac{1}{\mu_{\varepsilon}} \lim _{\operatorname{Re}(z) \rightarrow 0} z(z-\beta) \frac{\widetilde{\Phi}_{1}(z)+\frac{1}{\mu_{\varepsilon}}(z+\beta)}{\Lambda(-z)}= \\
& \frac{1}{\mu_{\varepsilon}} \lim _{\operatorname{Re}(z) \rightarrow 0} \frac{z(z-\beta)}{\Lambda(-z)}\left(X(z)\left(k_{0}^{S} p_{0}(-z)-k_{0}^{D} q_{0}(-z)\right)+\frac{1}{\mu_{\varepsilon}}(z+\beta)\right) \asymp \\
& \frac{1}{\mu_{\varepsilon}^{2}}\left(b_{0}+\beta\right)=P_{\infty}\left(\beta, \mu_{\varepsilon}\right),
\end{aligned}
$$

where we used (8.15) and the approximation, cf. (8.9),

$$
X(z)=-z\left(1-b_{0} z^{-1}+o\left(z^{-1}\right)\right), \quad z \rightarrow \infty .
$$

8.5. Small noise asymptotics $\alpha \in(\mathbf{1}, \mathbf{2})$. In this case, for $\beta=0$ and $\varepsilon=1$,

$$
b_{0}=\left(\mu^{2} \kappa_{\alpha}\right)^{\frac{1}{3-\alpha}} \frac{1}{\sin \frac{\pi}{3-\alpha}}
$$

and the small noise asymptotics follows by virtue of Theorem 2.1.

## Appendix A. More on solvability of (1.7)

Solvability of equation (1.7) in a space, suitable for our purposes, is a subtle matter. In essence, we construct such a solution which, moreover, turns out to be amenable to asymptotic analysis. The roadmap of our construction is outlined in Section 4.2. and this section gives an extended discussion of the solvability question, outlines several alternative approaches and elaborates on the construction in this paper.
A.1. Integration of nonrandom functions with respect to fBm . Let us briefly recall a construction of stochastic integrals with respect to $\mathrm{fBm} B^{H}$ for nonrandom integrands (see, e.g., [17]). For any $H \in(0,1) \backslash\left\{\frac{1}{2}\right\}$, consider the function space,

$$
\Lambda_{T}^{H-\frac{1}{2}}=\left\{f:[0, T] \mapsto \mathbb{R} \text { such that } \int_{0}^{T}\left(s^{\frac{1}{2}-H}\left(\Psi_{T} f\right)(s)\right)^{2} d s<\infty\right\},
$$

where $\Psi_{T}$ is the operator

$$
\left(\Psi_{T} f\right)(s):=-2 H \frac{d}{d s} \int_{s}^{T} f(r) r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}} d r .
$$

The bilinear form

$$
\begin{equation*}
\langle f, g\rangle_{\Lambda_{T}^{H-\frac{1}{2}}}=\frac{2-2 H}{\lambda_{H}} \int_{0}^{T} s^{1-2 H}\left(\Psi_{T} f\right)(s)\left(\Psi_{T} g\right)(s) d s, \tag{A.1}
\end{equation*}
$$

where $\lambda_{H}$ is an explicit constant, defines a scalar product on $\Lambda_{T}^{H-\frac{1}{2}}$. It can be shown that

$$
\begin{align*}
\langle f, g\rangle_{\Lambda_{T}^{H-\frac{1}{2}}}= & \int_{0}^{T} f(s) \frac{\partial}{\partial s} \int_{0}^{T} g(t) \frac{\partial}{\partial t} K(s, t) d t d s=  \tag{A.2}\\
& H \int_{0}^{T} f(s) \frac{\partial}{\partial s} \int_{0}^{T} g(t)|s-t|^{2 H-1} \operatorname{sign}(s-t) d t d s,
\end{align*}
$$

where $K(s, t)$ is the covariance function (1.6) of the fBm . These and other related formulas can be found in, e.g., [4, Subsection 3.3] for a quick reference.

Remark A.1. Various useful relations of $\Lambda_{T}^{H-\frac{1}{2}}$ to other spaces are known. In particular, it can be shown that $L^{2}([0, T]) \subset \Lambda_{T}^{H-\frac{1}{2}}$ for $H>\frac{1}{2}$, and $\Lambda_{T}^{H-\frac{1}{2}} \subset L^{2}([0, T])$ for $H<\frac{1}{2}$, see [28]. Also these inclusions follow from the eigenvalues asymptotics of the covariance operator of the fractional noise, the formal derivative of the fBm , see $[7]$.

Let $\mathcal{E}$ be the space of all simple functions. For $g \in \mathcal{E}$, the stochastic integral $I_{T}(f):=$ $\int_{0}^{T} g(s) d B_{s}^{H}$ is defined as the Riemann-Stieltjes sum. In view of (A.2), for $f, g \in \mathcal{E}$,

$$
\begin{equation*}
\mathbb{E} \int_{0}^{T} f(s) d B_{s}^{H} \int_{0}^{T} g(s) d B_{s}^{H}=\langle f, g\rangle_{\Lambda_{T}^{H-\frac{1}{2}}} . \tag{A.3}
\end{equation*}
$$

Hence the stochastic integral $I_{T}$ defines an isometry between $\mathcal{E}$ with the scalar product (A.1) and the linear subspace $\operatorname{sp}_{T}\left(B^{H}\right) \subset L^{2}(\Omega)$ of the finite linear combinations of increments of $B^{H}$ with the scalar product $\langle\xi, \eta\rangle=\mathbb{E} \xi \eta$ for $\xi, \eta \in \operatorname{sp}_{T}\left(B^{H}\right)$.

It can be shown that $\mathcal{E}$ is dense in $\Lambda_{T}^{H-\frac{1}{2}}$ and this allows to extend the isometry $I_{T}$ to any $g \in \Lambda_{T}^{H-\frac{1}{2}}$ by means of the $L^{2}(\Omega)$ limit. Namely, let $g^{n}$ be any sequence of simple functions, such that $\left\|g^{n}-g\right\|_{\Lambda_{T}^{H-\frac{1}{2}}} \rightarrow 0$, then, by the isometry property (A.3),

$$
\mathbb{E}\left(I_{T}\left(g^{n}\right)-I_{T}\left(g^{m}\right)\right)^{2}=\left\|g^{n}-g^{m}\right\|_{\Lambda_{T}^{H-\frac{1}{2}}} \xrightarrow[n, m \rightarrow \infty]{ } 0
$$

which means that $I_{T}\left(g^{n}\right)$ is Cauchy in $L^{2}(\Omega)$. By completeness of $L^{2}(\Omega)$, the limit

$$
I_{T}(g):=\lim _{n \rightarrow \infty} I_{T}\left(g^{n}\right)=: \int_{0}^{T} g(s) d B_{s}^{H}
$$

exists. This limit does not depend on the choice of approximating sequence $\left(g^{n}\right) \subset \mathcal{E}$ and hence defines the stochastic integral of any $g \in \Lambda_{T}^{H-\frac{1}{2}}$ unambiguously. Moreover, the extended map $I_{T}: \Lambda_{T}^{H-\frac{1}{2}} \mapsto L^{2}(\Omega)$ preserves the isometric property (A.3).

Let $\overline{\operatorname{sp}}_{T}\left(B^{H}\right)$ be the closure of $\operatorname{sp}_{T}\left(B^{H}\right)$ in $L^{2}(\Omega)$, that is, the subspace of all $L^{2}(\Omega)$ limits of linear combinations of increments of $B^{H}$. The image of the extended isometry $I_{T}$ is some linear subspace of $\overline{\operatorname{sp}}_{T}\left(B^{H}\right)$. Does it coincide with the whole closure $\overline{\operatorname{sp}}_{T}\left(B^{H}\right)$ ? Since $\overline{\mathrm{Sp}}_{T}\left(B^{H}\right)$ is a complete subspace and $I_{T}$ is an isometry, the answer to this question is affirmative if and only if the space $\Lambda_{T}^{H-\frac{1}{2}}$ is complete. Indeed, by definition, for any
$\xi \in \overline{\operatorname{sp}}_{T}\left(B^{H}\right)$ there is a sequence of random variables $\xi^{n}=I_{T}\left(g^{n}\right)$ such that $\xi_{n} \rightarrow \xi$ in $L^{2}(\Omega)$. Since $L^{2}(\Omega)$ is complete, $\xi_{n}$ is a Cauchy sequence and thus

$$
\left\|g^{n}-g^{m}\right\|_{\Lambda_{T}^{H-\frac{1}{2}}}=\mathbb{E}\left(I_{T}\left(g^{n}\right)-I_{T}\left(g^{m}\right)\right)^{2} \xrightarrow[n, m \rightarrow \infty]{ } 0
$$

that is, $g^{n}$ is Cauchy in $\Lambda_{T}^{H-\frac{1}{2}}$. Now if $\Lambda_{T}^{H-\frac{1}{2}}$ is complete, then $g=\lim _{n} g^{n} \in \Lambda_{T}^{H-\frac{1}{2}}$ exists and $I_{T}(g)=\xi$, $\mathbb{P}$-a.s. Conversely, let $g^{n}$ be a Cauchy sequence in $\Lambda_{T}^{H-\frac{1}{2}}$, then $I_{T}\left(g^{n}\right)$ is Cauchy in $L^{2}(\Omega)$. By completeness of $L\left(\Omega^{2}\right)$ there is $\xi=\lim _{n} \xi_{n} \in \overline{\operatorname{sp}}_{T}\left(B^{H}\right)$. Assume now that any random variable in $\overline{\mathrm{Sp}}_{T}\left(B^{H}\right)$ is an image of $I_{T}$, then there exists $g \in \Lambda_{T}^{H-\frac{1}{2}}$ such that $\xi=I_{T}(g)$ and

$$
\left\|g^{n}-g\right\|_{\Lambda_{T}^{H-\frac{1}{2}}}=\mathbb{E}\left(I_{T}\left(g^{n}\right)-I_{T}(g)\right)^{2}=\mathbb{E}\left(\xi_{n}-\xi\right)^{2} \rightarrow 0
$$

which means that $g^{n}$ is convergent in $\Lambda_{T}^{H-\frac{1}{2}}$ and hence the latter is complete.
It was shown in [28] that the space $\Lambda_{T}^{H-\frac{1}{2}}$ is complete for $H<\frac{1}{2}$, but it is incomplete for $H>\frac{1}{2}$. An important implication is that there are random variables in $\overline{\operatorname{sp}}_{T}\left(B^{H}\right)$ which cannot be represented as stochastic integrals of functions in $\Lambda_{T}^{H-\frac{1}{2}}$ with respect to fBm with $H>\frac{1}{2}$.

An additional insight is given by the canonical innovation representation of the fBm , see [17]. The innovation Brownian motion

$$
W_{t}=\int_{0}^{t} \sqrt{\frac{2-2 H}{\lambda_{H}}}\left(\Psi_{t}^{-1} u^{H-\frac{1}{2}}\right)(s) d B_{s}^{H}
$$

generates the same filtration as $B^{H}$. Then by the martingale representation theorem for any $\xi \in \overline{\operatorname{sp}}_{T}\left(B^{H}\right)$ there exists a function $f \in L^{2}([0, T])$ such that

$$
\begin{equation*}
\xi=\int_{0}^{T} f(s) d W_{s} \tag{A.4}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\int_{0}^{T} f(s) d W_{s}=\int_{0}^{T} \sqrt{\frac{2-2 H}{\lambda_{H}}}\left(\Psi_{T}^{-1} u^{H-\frac{1}{2}} f(u)\right)(s) d B_{s}^{H} \tag{A.5}
\end{equation*}
$$

whenever the integrand in the right hand side is well defined and belongs to $\Lambda_{T}^{H-\frac{1}{2}}$. For $H>\frac{1}{2}$ it is possible to find a function $f \in L^{2}([0, T])$ such that (A.5) fails and for such a function the random variable (A.4) will not be representable by the stochastic integral with respect to the fBm .
A.2. The linear filtering problem. Let $X$ be a zero mean Gaussian process with integrable cadlag paths and $B^{H}$ an independent fractional Brownian motion with Hurst parameter $H \in(0,1)$. Define the process

$$
Y_{t}=\int_{0}^{t} X_{s} d s+B_{t}^{H}, \quad t \geq 0
$$

Since all the processes are Gaussian, for any $T>0$, the conditional expectation $\widehat{X}_{T}=$ $\mathbb{E}\left(X_{T} \mid \mathcal{F}_{T}^{Y}\right)$ is a random variable, which belongs to $\overline{\operatorname{sp}}_{T}(Y)$, the closure in $L^{2}(\Omega)$ of all linear
combinations of increments of $Y$ on the interval $[0, T]$. In view of the discussion in the previous section, the following question arises.

Does there exist a non-random function $g \in \Lambda_{T}^{H-\frac{1}{2}} \cap L^{1}([0, T])$ such that

$$
\widehat{X}_{T}=\int_{0}^{T} g(s) d Y_{s}, \quad \mathbb{P}-a . s . ?
$$

If so how can it be found?
A.3. Refresh on the standard Brownian case ( $\mathbf{H}=\frac{1}{2}$ ). Let us briefly recall how an affirmative answer to the above question is given in [23] in the standard Brownian case. Let $\mathcal{F}_{T}^{Y, n}$ be the $\sigma$-algbera generated by the increments of process $Y$ on the $2^{n}$-dyadic partition of $[0, T]$. Since $Y$ is a cadlag process, $\mathcal{F}_{T}^{Y}=\bigvee_{n=1}^{\infty} \mathcal{F}_{T}^{Y, n}$ and by Levy's zero-one law,

$$
\widehat{X}_{T}^{n}=\mathbb{E}\left(X_{T} \mid \mathcal{F}_{T}^{Y, n}\right) \xrightarrow[n \rightarrow \infty]{\text { a.s. }} \mathbb{E}\left(X_{T} \mid \mathcal{F}_{T}^{Y}\right)=\widehat{X}_{T},
$$

where the convergence holds in $L^{2}(\Omega)$ as well. By the normal correlation theorem

$$
\widehat{X}_{T}^{n}=\int_{0}^{T} g^{n}(t) d Y_{t},
$$

where $g^{n}(\cdot)$ is a simple function. Being convergent, the sequence $\widehat{X}_{T}^{n}$ is Cauchy in $L^{2}(\Omega)$. On the other hand, by independence of $X$ and $B^{1 / 2}$,

$$
\mathbb{E}\left(\widehat{X}_{T}^{n}-\widehat{X}_{T}^{m}\right)^{2} \geq \int_{0}^{T}\left(g^{n}(s)-g^{m}(s)\right)^{2} d s
$$

and hence $g^{n}(\cdot)$ is Cauchy in $L^{2}([0, T])$. Since this space is complete, $g^{n}(\cdot)$ converges to some $g \in L^{2}([0, T])$. For such a function the stochastic integral $\int_{0}^{T} g(s) d Y_{s}$ is well defined, and

$$
\mathbb{E}\left(\widehat{X}_{T}-\int_{0}^{T} g(t) d Y_{t}\right)^{2} \leq 2 \mathbb{E}\left(\widehat{X}_{T}-\widehat{X}_{T}^{n}\right)^{2}+2 \mathbb{E}\left(\int_{0}^{T}\left(g^{n}(t)-g(t)\right) d Y_{t}\right)^{2} \rightarrow 0,
$$

and hence

$$
\widehat{X}_{T}=\int_{0}^{T} g(t) d Y_{t}, \quad \mathbb{P}-a . s .
$$

Now using the orthogonality property of the conditional expectation, the standard calculation shows that $g(\cdot)$ solves the equation

$$
\begin{equation*}
g(t)+\int_{0}^{T} g(s) K_{X}(s, t) d s=K_{X}(t, T), \quad \text { for a.a. } t \in[0, T] \tag{A.6}
\end{equation*}
$$

where $K_{X}(s, t)$ is the covariance function of $X$.
Conversely, we could have started with the equation (A.6): if it has a solution $g \in$ $L^{2}([0, T])$, then the stochastic integral $\widehat{X}_{T}:=\int_{0}^{T} g(s) d Y_{s}$ is well defined and satisfies the usual properties. Then (A.6) implies that $X_{T}-\widehat{X}_{T}$ is orthogonal to any random variable in $\overline{\operatorname{sp}}_{T}(Y)$ and hence $\widehat{X}_{T}=\mathbb{E}\left(X_{T} \mid \mathcal{F}_{T}^{Y}\right)$. In fact the Fredholm equation (A.6) does have such a solution if the kernel $K_{X}(s, t)$ is Hilbert-Schmidt.
A.4. The fractional case $\mathbf{H} \in(0,1) \backslash\left\{\frac{1}{2}\right\}$. The arguments from the previous section apply to $\Lambda_{T}^{H-\frac{1}{2}}$ with $H<\frac{1}{2}$, since this subspace is complete and, moreover, $\Lambda_{T}^{H-\frac{1}{2}} \subset$ $L^{2}([0, T]) \subset L^{1}([0, T])$. The latter inclusion ensures that the limit weight function $g$ is not only integrable with respect to fBm , but also with respect to the time variable, so that $\int_{0}^{T} g(s) d Y_{s}$ is indeed well defined.

On the other hand, the space $\Lambda_{T}^{H-\frac{1}{2}}$ for $H>\frac{1}{2}$, being incomplete, must be treated differently. It suffices to show that equation, cf. (1.7),

$$
\begin{equation*}
\frac{\partial}{\partial s} \int_{0}^{T} g(r) \frac{\partial}{\partial r} K(r, s) d r+\int_{0}^{T} K_{X}(r, s) g(r) d r=K_{X}(s, T), \quad s \in(0, T) \tag{A.7}
\end{equation*}
$$

has a solution in $L^{1}([0, T])$. Note that in this case it also belongs to $\Lambda_{T}^{H-\frac{1}{2}}$ : indeed, taking scalar product of (A.7) with $g$ implies

$$
\begin{equation*}
\|g\|_{\Lambda_{T}^{H-\frac{1}{2}}}^{2} \leq \int_{0}^{T} g(s) K_{X}(s, T) d s \leq\left\|K_{X}(\cdot, T)\right\|_{\infty}\|g\|_{1} . \tag{A.8}
\end{equation*}
$$

This is a special feature of the solution to the above equation, since $L^{1}([0, T]) \nsubseteq \Lambda_{T}^{H-\frac{1}{2}}$. For $g \in L^{1}([0, T]) \cap \Lambda_{T}^{H-\frac{1}{2}}$, the stochastic integral

$$
\widehat{X}_{T}=\int_{0}^{T} g(s) d Y_{s}:=\int_{0}^{T} g(s) d B_{s}^{H}+\int_{0}^{T} g(s) X_{s} d s
$$

is well defined, if, e.g. $X$ has continuous paths. Moreover, equation (A.7) implies that

$$
\mathbb{E}\left(X_{T}-\int_{0}^{T} g(s) d Y_{s}\right) \int_{0}^{T} h(s) d Y_{s}=0
$$

for any $h \in \Lambda_{T}^{H-\frac{1}{2}}$. Since $\xi \in \overline{\operatorname{sp}}_{T}(Y)$ can be approximated in $L^{2}(\Omega)$ by a sequence of stochastic integrals of simple functions, in fact, we have

$$
\mathbb{E}\left(X_{T}-\int_{0}^{T} g(s) d Y_{s}\right) \xi=0
$$

for any such $\xi$ and, therefore, $\widehat{X}_{T}$ is a version of conditional expectation. In Section A. 6 we detail the calculations which show that the solution to (A.7) constructed in this paper does indeed belong to $L^{1}([0, T])$.
A.5. Solvability through explicit inverse. For simplicity, let $T=1$ and omit it from the notations. Define the operators

$$
(A f)(s)=\frac{\partial}{\partial s} \int_{0}^{1} f(s) \frac{\partial}{\partial r} K(r, s) d r=H \frac{\partial}{\partial s} \int_{0}^{1} f(t)|s-t|^{2 H-1} \operatorname{sign}(s-t) d t
$$

and

$$
(R f)(s)=\int_{0}^{1} K_{X}(r, s) f(r) d r .
$$

Then equation (A.7) reads

$$
\begin{equation*}
A g+R g=f \tag{A.9}
\end{equation*}
$$

where $f(s)=K_{X}(s, 1)$. The operator $A$ is invertible, see e.g. [22],

$$
\begin{align*}
& \left(A^{-1} f\right)(s)=  \tag{A.10}\\
& -c_{H} s^{\frac{1}{2}-H} \frac{d}{d s} \int_{s}^{1} d w w^{2 H-1}(w-s)^{\frac{1}{2}-H} \frac{d}{d w} \int_{0}^{w} z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H} f(z) d z,
\end{align*}
$$

where $c_{H}$ is an explicit constant. If $A^{-1}$ is applicable to both sides of (A.9), then $g$ solves the equation

$$
\begin{equation*}
g+A^{-1} R g=A^{-1} f \tag{A.11}
\end{equation*}
$$

If moreover, the operator $A^{-1} R$ and the function $A^{-1} f$ are sufficiently regular, then (A.11) is a Fredholm equation of the second kind, and its solvability in an appropriate space follows from the general theory.
A.5.1. Case $H<\frac{1}{2}$. In this case, the derivatives and integrals in (A.10) are interchangeable and the inverse of $A$ turns out to be a weakly singular integral operator, see [4, Theorem 5.1 (iv)],

$$
\left(A^{-1} f\right)(s)=\int_{0}^{1} L(u, v) f(v) d u
$$

where $L(u, v)=|u-v|^{-2 H} N(u, v)$ with $N \in C\left([0,1]^{2}\right)$. Since $K_{X} \in C\left([0,1]^{2}\right)$ it follows that $A^{-1} R$ in this case is an integral operator with continuos kernel. Similarly, $A^{-1} f$ is a continuous function. This reduces (A.7) to a Fredholm equation with continuous kernel. The homogeneous equation $g+A^{-1} R g=0$, or equivalently, $A g+R g=0$ has only trivial solutions, since all eigenvalues of $A$ are positive, see [7]. Hence by the Fredholm alternative, the non-homogeneous equation (A.11) has the unique (continuous) solution (see, e.g., [29]). Then, in view of (A.8), $g \in \Lambda_{T}^{H-\frac{1}{2}} \cap L^{1}([0, T])$.
A.5.2. Case $H>\frac{1}{2}$. Note that $f(z)=\int_{0}^{z} f^{\prime}(r) d r$ and hence (A.10) in this case can be rewritten as

$$
\left(A^{-1} f\right)(x)=\int_{0}^{1} f^{\prime}(r) p(x, r) d r
$$

where

$$
p(x, u)=-c_{H} x^{\frac{1}{2}-H}\left((2-2 H) \int_{r}^{1} z^{\frac{1}{2}-H} \rho(x, z) d z+r^{\frac{3}{2}-H} \rho(x, r)\right),
$$

and

$$
\rho(x, r)=\frac{d}{d x} \int_{x \vee r}^{1} w^{2 H-2}(w-x)^{\frac{1}{2}-H}(w-r)^{\frac{1}{2}-H} d w .
$$

A calculation shows that

$$
|\rho(x, r)| \leq C|x-r|^{1-2 H}(x \vee r)^{2 H-2}+(1-x)^{\frac{1}{2}-H}(1-r)^{\frac{1}{2}-H} \mathbf{1}_{\{x>r\}} .
$$

Further computations become cumbersome, but if pushed, seem to lead to a Fredholm equation of the second kind, with $L^{2}\left([0, T]^{2}\right)$ kernel and $L^{1}([0, T])$ forcing function. Then it remains to be checked that such equations have $L^{1}([0, T])$ solution. We will not pursue this direction here.
A.6. Solvability by construction in this paper. Let us recap the main steps of the construction in this paper, considering, for definiteness, the case $\alpha_{2} \in(0,1)$ and $\alpha_{1}>\alpha_{2}$, which corresponds to $H_{2}>\frac{1}{2}$ and $H_{1}<H_{2}$; all other cases can be treated similarly. In order to avoid confusion with the notations used in the text, all the objects produced in the course of construction will be marked by asterisk.
(a) Solve the integral equations (5.33), treating the constant $\psi(0)$, which appears in (5.30) and thus also in (5.32), as a free parameter and denote it by $c$. As explained in the text, these equations have unique solution in $L^{2}\left(\mathbb{R}_{+}\right)$, at least for all sufficiently large $T$ or all sufficiently small $\varepsilon>0$.
(b) Form the functions $\Phi_{0}^{*}(z)$ and $\Phi_{1}^{*}(z)$ using the formulas (5.34), where $X(z)$ is defined by (5.26).
(c) Compute the function $\widehat{g}^{*}(z)$ by the formula (4.2) using $\Phi_{0}^{*}(z)$ and $\Phi_{1}^{*}(z)$ and the parameter $c$ in place of $\psi(0)$. Substitute this expression into condition (5.19) and solve the obtained equation for $c$. Denote the obtained value by $c^{*}$. Thus we obtain

$$
\begin{equation*}
\widehat{g}^{*}(z)=-\frac{1}{\Lambda(z)}\left((z+\beta)\left(\Phi_{0}^{*}(z)+e^{-z T} \Phi_{1}^{*}(-z)\right)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z)\left(c^{*}+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)\right) . \tag{A.12}
\end{equation*}
$$

(d) Compute $g^{*}(x)$ by applying the inverse Laplace transform to $\widehat{g}^{*}(z)$.

Our goal is to show that
(i) $g^{*} \in L^{1}([0, T])$ and
(ii) $g^{*}$ does indeed solve equation (A.7).

Note that if (i) and (ii) hold, then the bound (A.8) implies $g^{*} \in \Lambda_{T}^{H-1 / 2}$ as well.
A.6.1. Proof of (i). The proof is by a careful inspection of all the functions involved in the construction. Let us estimate the growth of the functions $f_{S}(t)$ and $f_{D}(t)$ defined in (5.30) at infinity and near the origin. To this end, write

$$
\begin{aligned}
& \left|\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right|=\left|\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}\right|\left|\frac{N_{\alpha_{2}}^{-}(t) \Lambda^{+}(t)-N_{\alpha_{2}}^{+}(t) \Lambda^{-}(t)}{N_{\alpha_{2}}^{-}(t) \Lambda^{+}(t)}\right|= \\
& \mu_{\varepsilon}^{2} \frac{1}{\left|X^{-}(t)\right|} \frac{1}{\left|\Lambda^{+}(t)\right|}\left|N_{\alpha_{2}}^{+}(t) N_{\alpha_{1}}^{-}(t)-N_{\alpha_{2}}^{-}(t) N_{\alpha_{1}}^{+}(t)\right|
\end{aligned}
$$

where the last equality is obtained by plugging the expression (4.5). In view of the formula (4.3) and the estimates (5.27) (with $k=1$ ) and since $\alpha_{1}>\alpha_{2}$, it follows that

$$
\left|\frac{N_{\alpha_{2}}^{-}(t)}{X^{-}(t)}-\frac{N_{\alpha_{2}}^{+}(t)}{X^{+}(t)}\right|= \begin{cases}O\left(t^{\alpha_{1}-4+\frac{1-\alpha_{2}}{2}}\right), & t \rightarrow \infty \\ O\left(t^{\alpha_{1}-1+\frac{1-\alpha_{2}}{2}}\right), & t \rightarrow 0\end{cases}
$$

Consequently, cf. (5.30),

$$
\left\{f_{S}(t), f_{D}(t)\right\}= \begin{cases}O\left(t^{\alpha_{1}-3+\frac{1-\alpha_{2}}{2}}\right), & t \rightarrow \infty \\ O\left(t^{\alpha_{1}-1+\frac{1-\alpha_{2}}{2}}\right), & t \rightarrow 0\end{cases}
$$

and, by definitions (5.32),

$$
\left\{F_{S}(-t), F_{D}(-t)\right\}= \begin{cases}O\left(t^{-1 \vee\left(\alpha_{1}-3+\frac{1-\alpha_{2}}{2}\right)}\right), & t \rightarrow \infty \\ O\left(t^{0 \wedge\left(\alpha_{1}-1+\frac{1-\alpha_{2}}{2}\right)}\right), & t \rightarrow 0\end{cases}
$$

Note that $F_{S}, F_{D} \in L^{2}\left(\mathbb{R}_{+}\right)$and hence equations (5.33) have unique solutions in $L^{2}([0, T])$, as explained in the text. Then in view of estimates (5.35), the formulas (5.34) imply that $\Phi_{0}^{* \pm}(t), \Phi_{1}^{* \pm}(t)$ are locally square integrable on $\mathbb{R}_{+}$and

$$
\left\{\Phi_{0}^{* \pm}(t), \Phi_{1}^{* \pm}(t)\right\}=O\left(t^{\alpha_{1}-2}\right) \quad \text { as } t \rightarrow \infty
$$

By construction $\widehat{g}^{*}(z)$ is an entire function, the Laplace transform inversion can be carried out by integration on the imaginary axis

$$
\begin{equation*}
g^{*}(x)=\frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} \widehat{g}^{*}(z) e^{z x} d z \tag{A.13}
\end{equation*}
$$

Since $\widehat{g}^{*}(z)$ is analytic and $\widehat{g}^{*}(z) \rightarrow 0$ as $z \rightarrow \infty$ and $\operatorname{Re}(z)>0$, its inverse Laplace transform vanishes on $\mathbb{R}_{-}$, i.e., $g^{*}(x)=0$ for $x<0$. Similarly, since $e^{z T} \widehat{g}^{*}(z) \rightarrow 0$ as $z \rightarrow \infty$ and $\operatorname{Re}(z)<0$, it follows that $g^{*}(x)=0$ for $x>T$. For $x \in(0, T)$, (A.13) can be evaluated by plugging the expression (A.12) and integrating along arc sector counters in each quarter of the complex plane. After a rearrangement this gives

$$
\begin{align*}
g^{*}(x)= & -\frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} \frac{1}{\Lambda(z)}\left((z+\beta) \Phi_{0}^{*}(z)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z) c^{*}\right) e^{z x} d z \\
& -\frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} \frac{1}{\Lambda(z)}\left((z+\beta) \Phi_{1}^{*}(-z)+\mu_{\varepsilon} N_{\alpha_{1}}(z)\right) e^{z(x-T)} d z= \\
& -\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{\Lambda^{-}(t)}\left((-t+\beta) \Phi_{0}^{*}(-t)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}^{-}(t) c^{*}\right) e^{-t x} d t \\
& +\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{\Lambda^{+}(t)}\left((-t+\beta) \Phi_{0}^{*}(-t)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}^{+}(t) c^{*}\right) e^{-t x} d t  \tag{A.14}\\
& -\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{\Lambda^{+}(t)}\left((t+\beta) \Phi_{1}^{*}(-t)+\mu_{\varepsilon} N_{\alpha_{1}}^{+}(t)\right) e^{-t(T-x)} d t \\
& +\frac{1}{2 \pi i} \int_{0}^{\infty} \frac{1}{\Lambda^{-}(t)}\left((t+\beta) \Phi_{1}^{*}(-t)+\mu_{\varepsilon} N_{\alpha_{1}}^{-}(t)\right) e^{-t(T-x)} d t= \\
& \frac{1}{2 \pi i} \int_{0}^{\infty} e^{-t x} R_{0}(t) d t-\frac{1}{2 \pi i} \int_{0}^{\infty} e^{-t(T-x)} R_{1}(t) d t
\end{align*}
$$

where we defined

$$
\begin{aligned}
& R_{0}(t)=(\beta-t) \Phi_{0}^{*}(-t)\left(\frac{1}{\Lambda^{+}(t)}-\frac{1}{\Lambda^{-}(t)}\right)+\mu_{\varepsilon}^{2}\left(\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}-\frac{N_{\alpha_{1}}^{-}(t)}{\Lambda^{-}(t)}\right) c^{*} \\
& R_{1}(t)=(\beta+t) \Phi_{1}^{*}(-t)\left(\frac{1}{\Lambda^{+}(t)}-\frac{1}{\Lambda^{-}(t)}\right)+\mu_{\varepsilon}\left(\frac{N_{\alpha_{1}}^{+}(t)}{\Lambda^{+}(t)}-\frac{N_{\alpha_{1}}^{-}(t)}{\Lambda^{-}(t)}\right)
\end{aligned}
$$

In view of the above estimates these functions are bounded and decay to zero as a power function as $t \rightarrow \infty$. This implies the desired claim:

$$
\begin{aligned}
\left\|g^{*}\right\|_{1} \leq & \int_{0}^{\infty}\left|R_{0}(t)\right| \int_{0}^{T} e^{-t x} d x d t+\int_{0}^{\infty}\left|R_{1}(t)\right| \int_{0}^{T} e^{-t(T-x)} d x d t \\
& \int_{0}^{\infty}\left|R_{0}(t)\right| \frac{1-e^{-T t}}{t} d t+\int_{0}^{\infty}\left|R_{1}(t)\right| \frac{1-e^{-T t}}{t} d t<\infty .
\end{aligned}
$$

A.6.2. Proof of (ii). The expression (A.14) reveals that $g^{*}$ is continuous on $(0, T)$, possibly with integrable singularities at the endpoints. Thus $g^{*}$ is in the domain of both operators in equation (A.7) and can be substituted into its left hand side. We want to show that

$$
\begin{equation*}
\frac{\partial}{\partial s} \int_{0}^{T} g^{*}(r) \frac{\partial}{\partial r} K(r, s) d r+\int_{0}^{T} K_{X}(r, s) g^{*}(r) d r=K_{X}(s, T), \quad \forall s \in(0, T) \tag{A.15}
\end{equation*}
$$

To do so we can apply the Laplace transform to both sides, extending them by zero outside the interval $(0, T)$, and check that equality is obtained on the whole plane. This amounts to repeating the calculations in the proof of Lemma 5.1. Thus proving (A.15) is equivalent to showing that, cf. (4.2),

$$
\begin{align*}
& \int_{0}^{T} g^{*}(x) e^{-z x} d x+\frac{1}{\Lambda(z)}\left((z+\beta)\left(\Xi_{0}(z)+e^{-z T} \Xi_{1}(-z)\right)+\right. \\
&\left.\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z)\left(\psi^{*}(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)\right)=0 \tag{A.16}
\end{align*}
$$

where $\Xi_{0}(z), \Xi_{1}(z)$ are the functions computed by plugging $g^{*}(x)$ and the corresponding function $\psi^{*}(x)$, defined by (5.3), into (5.17).

As explained in the proof of (i) (see the paragraph following (A.13)), the function $g^{*}(x)$ vanishes outside $[0, T]$. Thus the first term in (A.16) coincides with the Laplace transform $\widehat{g}_{*}(z)$ and hence (A.16) is equivalent to

$$
\widehat{g}^{*}(z)+\frac{1}{\Lambda(z)}\left((z+\beta)\left(\Xi_{0}(z)+e^{-z T} \Xi_{1}(-z)\right)+\mu_{\varepsilon}^{2} N_{\alpha_{1}}(z)\left(\psi^{*}(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)\right)=0 .
$$

After plugging the expression for $\widehat{g}^{*}(z)$ constructed in the steps (a)-(c), we see that this equality holds if $\Xi_{0}(z)$ and $\Xi_{1}(z)$ coincide with $\Phi_{0, *}(z)$ and $\Phi_{1, *}(z)$, and $\psi^{*}(0)$ coincides with the constant $c^{*}$ found in step (c).

Let us first argue that $\psi^{*}(0)=c^{*}$. Since $\widehat{g}^{*}(z)$ is an entire function, so is $\widehat{\psi}^{*}(z)$. Letting $z:=-\beta$ in (5.16), shows that $\psi^{*}(0)=\widehat{g}^{*}(-\beta)-\frac{1}{\mu_{\varepsilon}} e^{\beta T}$. On the other hand, by taking $z \rightarrow \beta$ in the upper half plane in (A.12) implies $c^{*}=\widehat{g}^{*}(-\beta)-\frac{1}{\mu_{\varepsilon}} e^{\beta T}$. Thus indeed $\psi^{*}(0)=c^{*}$. Now we can show that $\Xi_{0}(z)=\Phi_{0, *}(z)$. The functions in the right hand side of (5.17) are sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$and the first equation implies that for $t>0$,

$$
\begin{equation*}
\Xi_{0}^{+}(t)-\Xi_{0}^{-}(t)=(t-\beta)\left(\Psi_{g^{*}, 0}^{+}(t)-\Psi_{g^{*}, 0}^{-}(t)\right)+\mu_{\varepsilon}^{2}\left(\Psi_{\psi_{T, *}, 0}^{+}(t)-\Psi_{\psi_{T, *}, 0}^{-}(t)\right) \tag{A.17}
\end{equation*}
$$

A direct calculation shows that the operator defined in (5.10) takes the form

$$
\Psi_{f, 0}(z)=-\int_{0}^{\infty} \frac{N_{\alpha}^{+}(t)-N_{\alpha}^{-}(t)}{t-z} \widehat{f}(t) d t
$$

when either $\alpha \in(0,1)$ and $\widehat{f}$ is locally bounded or $\alpha \in(1,2)$ and $\widehat{f}$ vanishes at the origin at a suitable rate. Hence this formula is valid for both $f:=g$ and $f:=\psi$.

Since $\Phi_{0, *}(z)$ and $\Phi_{1, *}(z)$ satisfy, by construction, the boundary conditions (5.23), the function $\widehat{g}_{*}(z)$ is holomorphic and hence (A.17) becomes

$$
\begin{aligned}
& \Xi_{0}^{+}(t)-\Xi_{0}^{-}(t)= \\
& -(t-\beta)\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}^{*}(t)-\mu_{\varepsilon}^{2}\left(N_{\alpha_{1}}^{+}(t)-N_{\alpha_{1}}^{-}(t)\right) \widehat{\psi}^{*}(t) \stackrel{\dagger}{=} \\
& -(t-\beta)\left(N_{\alpha_{2}}^{+}(t)-N_{\alpha_{2}}^{-}(t)\right) \widehat{g}^{*}(t)-\mu_{\varepsilon}^{2} \frac{N_{\alpha_{1}}^{+}(t)-N_{\alpha_{1}}^{-}(t)}{t+\beta}\left(\psi^{*}(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}-\widehat{g}^{*}(z)\right)= \\
& -\frac{1}{t+\beta}\left(\left(\Lambda^{+}(t)-\Lambda^{-}(t)\right) \widehat{g}^{*}(t)+\mu_{\varepsilon}^{2}\left(N_{\alpha_{1}}^{+}(t)-N_{\alpha_{1}}^{-}(t)\right)\left(\psi^{*}(0)+\frac{1}{\mu_{\varepsilon}} e^{-z T}\right)\right) \stackrel{\ddagger}{=} \\
& \Phi_{0, *}^{+}(t)-\Phi_{0, *}^{-}(t), \quad t>0,
\end{aligned}
$$

where in $\dagger$ we used the expression for $\widehat{\psi}^{*}(z)$ in (5.16) and in $\ddagger$ the definition of $\widehat{g}^{*}(z)$ through (A.12) (where, as we already showed, $\left.c^{*}=\psi^{*}(0)\right)$.

By construction, both $\Xi_{0}(z)$ and $\Phi_{0, *}(z)$ are sectionally holomorphic on $\mathbb{C} \backslash \mathbb{R}_{+}$, share the same asymptotic as $z \rightarrow \infty$ and, by the above calculations, have the same jump on the boundary. Hence they coincide, which is what we wanted to show. The same arguments apply to $\Xi_{1}(z)$ and $\Phi_{1, *}(z)$.

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[^1]:    ${ }^{1}$ here and below, $g(\varepsilon) \asymp h(\varepsilon)$ stands for $\lim _{\varepsilon \rightarrow 0} g(\varepsilon) / h(\varepsilon)=1$

