# ON THE BREAKDOWN OF SOLUTIONS TO THE INCOMPRESSIBLE EULER EQUATIONS WITH FREE SURFACE BOUNDARY 

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#### Abstract

We prove a continuation critereon for incompressible liquids with free surface boundary. We combine the energy estimates of Christodoulou and Lindblad with an analog of the estimate due to Beale, Kato, and Majda for the gradient of the velocity in terms of the vorticity, and use this to show solution can be continued so long as the second fundamental form and injectivity radius of the free boundary, the vorticity, and one derivative of the velocity on the free boundary remain bounded, assuming that the Taylor sign condition holds.


## 1. Introduction

We consider the equations of motion for an incompressible, bounded fluid body with velocity $u=$ ( $u_{1}, u_{2}, u_{3}$ ) and pressure $p$ :

$$
\begin{array}{rlr}
\left(\partial_{t}+u^{k} \partial_{k}\right) u_{i} & =-\partial_{i} p, & \quad i=1, \ldots, 3, \text { in } \mathcal{D}, \\
\operatorname{div} u=\partial_{i} u^{i} & =0, & \text { in } \mathcal{D} . \tag{1.2}
\end{array}
$$

Here, we are summing over repeated upper and lower indices and writing $\partial_{t}=\frac{\partial}{\partial t}$ and $\partial_{k}=\frac{\partial}{\partial x^{k}}$ as well as $u^{i}=\delta^{i j} u_{j}$. The domain $\mathcal{D}=\bigcup_{0 \leq t \leq T}\{t\} \times \mathcal{D}_{t}$ is to be determined, and satisfies:

$$
\begin{equation*}
p=0 \text { on } \partial \mathcal{D}_{t}, \quad p>0 \text { in } \mathcal{D}_{t},\left.\quad\left(\partial_{t}+u^{k} \partial_{k}\right)\right|_{\partial \mathcal{D}} \in T(\partial \mathcal{D}), \tag{1.3}
\end{equation*}
$$

with $\mathcal{D}_{0}$ diffeomorphic to the unit sphere.
The problem (1.1)-(1.3) is ill-posed in Sobolev spaces unless the following condition holds, known as the "Taylor sign condition" (see [4):

$$
\left(-\nabla_{N} p\right)(t, \cdot)>0 \quad \text { on } \partial \mathcal{D}_{t} \quad \text { where } \nabla_{N}=N^{i} \nabla_{i}
$$

This and related problems have been studied by a wide variety of authors. In the case that $\mathcal{D}_{0}$ is diffeomorphic to the lower half plane and the vorticity $\omega=\operatorname{curl} v$ vanishes, this is known as the irrotational water waves problem. The literature on this problem is vast; let us just single out [6], [16] and [9, where the authors proved that in two and three dimensions, the water waves problem is globally well-posed in Sobolev spaces. See [10] for a comprehensive survey of this problem. See also [18], where the authors considered the water waves problem in two dimensions with constant vorticity and proved that the system has a cubic lifespan, and [17, where the authors considered the free boundary problem in two dimensions with self-gravitation but no vorticity and also proved a cubic lifespan bound.

In the case that $\mathcal{D}_{0}$ is a bounded domain and $\omega \neq 0$, Lindblad and Christodoulou proved energy estimates in 13. Local well-posedness was then shown in this case by Lindblad in 11 using a Nash-Moser iteration and later by Shkoller and Coutand in [2] using a tangential smoothing operator.

We now describe the energy estimates developed by Lindblad and Christodoulou. We let $N_{i}$ denote the unit conormal to $\partial \mathcal{D}_{t}$ and let $\Pi_{i}^{j}=\delta_{i}^{j}-N_{i} N^{j}$ denote the projection to the tangent space at the boundary. Writing $\nabla$ for the covariant derivative on $\mathcal{D}_{t}$, we let $\theta$ denote the second fundamental form of $\partial \mathcal{D}_{t}$ :

$$
\theta_{i j}=\Pi_{i}^{k} \Pi_{j}^{\ell} \nabla_{k} N_{\ell}
$$

The energies in [13] are of the form:

$$
\begin{equation*}
\mathcal{E}_{r}(t)=\int_{\mathcal{D}_{t}} \delta^{i j} Q\left(\nabla^{r} u_{i}, \nabla^{r} u_{j}\right)+\int_{\partial \mathcal{D}_{t}} Q\left(\nabla^{r} p, \nabla^{r} p\right)|\nabla p| d S+\int_{\mathcal{D}_{t}}\left|\operatorname{curl} \nabla^{r-1} u\right|^{2}, \tag{1.4}
\end{equation*}
$$

where $Q$ is a quadratic form which is the norm of the tangential components, $Q(\beta, \beta)=|\Pi \beta|^{2}$ when restricted to the boundary, and the full norm in the interior.

Let $\iota_{0}$ denote the injectivity radius of the normal exponential map. By definition this is the smallest number $\iota$ so that the map:

$$
\left(x, \iota^{\prime}\right) \rightarrow x+\iota^{\prime} N(x) \in \mathcal{D}_{t}
$$

defined for $x \in \partial \mathcal{D}_{t}, \iota^{\prime} \in(-\iota, \iota)$. is injective. Thus $\iota_{0}$ measures how far $\partial \mathcal{D}_{t}$ is from self-intersecting. The main theorem in [13] is:

Theorem 1. Let $(u, \mathcal{D})$ be a smooth solution to (1.1)-(1.3) for $0 \leq t \leq T$. Suppose that the following bounds hold:

$$
\begin{align*}
|\nabla u|+|\nabla p| \leq M & \text { in } \mathcal{D}_{t}  \tag{1.5}\\
|\theta|+\frac{1}{\iota_{0}} \leq K, & \text { on } \partial \mathcal{D}_{t}  \tag{1.6}\\
-\nabla_{N} p \geq \delta>0, & \text { on } \partial \mathcal{D}_{t}  \tag{1.7}\\
\left|\nabla^{2} p\right|+\left|\nabla D_{t} p\right| \leq L & \text { on } \partial \mathcal{D}_{t} \tag{1.8}
\end{align*}
$$

Then there are continuous functions $F_{r}$ with $\left.F_{r}\right|_{t=0}=1$ so that:

$$
\mathcal{E}_{r}(t) \leq F_{r}\left(t, M, K, \delta^{-1}, L, \mathcal{E}_{0}(0), \ldots, \mathcal{E}_{r-1}(0)\right) \mathcal{E}_{r}(0)
$$

By the local well-posedness results in e.g. [12] or [2], Theorem 1 and an approximation argument to go from smooth solutions to solutions in Sobolev spaces, this gives the following breakdown critereon for solutions to (1.1)-(1.3):
Corollary 1. Let $(u, \mathcal{D})$ be a solution to (1.1)-(1.3) with:

$$
\begin{equation*}
u(t) \in H^{s}\left(\mathcal{D}_{t}\right), \quad \mathcal{D}_{t} \in H^{s+1 / 2}\left(\mathbb{R}^{3}\right), \quad 0 \leq t \leq T \tag{1.9}
\end{equation*}
$$

for $s \geq 3$. Suppose that $T^{*}$ is the largest time so that $u$ can be continued as a solution to (1.1)-(1.3) in the class (1.9). Then either $T^{*}=\infty$ or at least one of the quantities on the left-hand sides of (1.5), (1.6), (1.8) go to infinity, or the quantity on the left-hand side of (1.7) goes to zero as $t \nearrow T^{*}$.

Here, $\mathcal{D}_{t} \in H^{s+1 / 2}\left(\mathbb{R}^{3}\right)$ means that locally, $\mathcal{D}_{t}$ can be written as the graph of a function in $H^{s+1 / 2}\left(\mathbb{R}^{3}\right)$. Our result is that the above breakdown condition can be replaced with a condition on the vorticity $\omega$ and some norms of $u$ on the free boundary $\partial \mathcal{D}_{t}$. Our approach follows the seminal article [1], where Beale, Kato, and Majda consider the incompressible Euler equations in $\mathbb{R}^{3}$ :

$$
\begin{align*}
\left(\partial_{t}+u^{k} \partial_{k}\right) u_{i} & =-\partial_{i} p, & & \text { in }[0, T] \times \mathbb{R}^{3},  \tag{1.10}\\
\operatorname{div} u & =0, & & \text { in }[0, T] \times \mathbb{R}^{3} . \tag{1.11}
\end{align*}
$$

with solutions in the following space:

$$
\begin{equation*}
u \in C\left([0, T] ; H^{3}\left(\mathbb{R}^{3}\right)\right) \cap C^{1}\left([0, T] ; H^{2}\left(\mathbb{R}^{3}\right)\right) \tag{1.12}
\end{equation*}
$$

The show that if $T^{*}<\infty$ is the first time for which the solution to (1.10)-(1.11) cannot be continued beyond $T=T^{*}$ in the class (1.12), then:

$$
\int_{0}^{T^{*}}\|\operatorname{curl} v(t, \cdot)\|_{L^{\infty}\left(\mathbb{R}^{3}\right)}=\infty
$$

See also [5] for a generalization to the case of a fixed, bounded domain with a Neumann boundary condition.
We now describe our main theorem. We write $\mathcal{N}$ for the Dirichlet-to-Neumann operator on $\partial \mathcal{D}_{t}$; if $\psi: \partial \mathcal{D}_{t} \rightarrow \mathbb{R}$ and $\psi_{\mathcal{H}}: \mathcal{D}_{t} \rightarrow \mathbb{R}$ denotes the harmonic extension of $\psi$ to $\mathcal{D}_{t}$, then:

$$
\mathcal{N} \psi=\left.\left(N \cdot \nabla \psi_{\mathcal{H}}\right)\right|_{\partial \mathcal{D}_{t}}
$$

We will also write $U=\left.u\right|_{\partial \mathcal{D}_{t}}$. Our main result is then:
Theorem 2. Let $u$ be a solution to (1.1)-(1.2) in the class (1.9) with $s>3$. Define:

$$
\begin{aligned}
\mathcal{A}(t) & =\|\omega(t)\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\|\nabla u(t)\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}+\|\mathcal{N} U(t)\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \\
\mathcal{K}(t) & =\|\theta(t)\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}+\frac{1}{\iota_{0}(t)}+\left\|\left(\nabla_{N} p(t)\right)^{-1}\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}
\end{aligned}
$$

Suppose that there is a time $T^{*}$ so that the solution cannot be continued past $T^{*}$ in the class (1.9) and that $T^{*}$ is the first such time. Then either:

$$
\limsup _{t \nearrow T^{*}} \mathcal{K}(t)=\infty
$$

or

$$
\begin{equation*}
\int_{0}^{T^{*}}(\mathcal{A}(t))^{2}+\left\|\nabla_{N} D_{t} p(t)\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)} d t=\infty \tag{1.13}
\end{equation*}
$$

In particular, if (1.13) occurs, then:

$$
\limsup _{t \nearrow T^{*}} \mathcal{A}(t)+\left\|\nabla_{N} D_{t} p(t)\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}=\infty
$$

The reason that this criterion requires a bound for $\mathcal{A}^{2}$ instead of $\mathcal{A}$ is ultimately due to the fact that the energy estimates we will use require a bound for $D_{t} p$ in the interior. Since $\Delta p=-\left(\nabla^{i} u_{j}\right)\left(\nabla^{j} u_{i}\right)$, it follows that $\Delta D_{t} p$ is cubic in $u$ (see Lemma (8).

As in [1] and [5], the proof relies on two ingredients. First, in Section 5we assume that $(u, \mathcal{D})$ is a solution to (1.1)-(1.3) in the class (1.9) and that the following bounds hold:

$$
\begin{gathered}
|\theta(t)|+\frac{1}{\iota_{0}(t)} \leq K, \quad \text { on } \partial \mathcal{D}_{t}, 0 \leq t \leq T \\
\left(-\nabla_{N} p(t)\right) \geq \delta>0, \quad \text { on } \partial \mathcal{D}_{t} .0 \leq t \leq T
\end{gathered}
$$

Then, with $\mathcal{E}_{r}$ defined by (1.4):

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{3}(t) \leq C\left(K, \delta^{-1}\right)\left(\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\mathcal{A}(t)+(\mathcal{A}(t))^{2}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}\right) \mathcal{E}_{3}(t) \tag{1.14}
\end{equation*}
$$

and for $s \geq 4$,

$$
\left.\frac{d}{d t} \mathcal{E}_{s}(t) \leq C\left(K, \delta^{-1}\right)\left(\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\mathcal{A}(t)+(\mathcal{A}(t))^{2}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}\right)\left(\mathcal{E}_{s}(t)+P\left(\mathcal{E}_{s-1}(t), \ldots, \mathcal{E}_{0}(t\rangle\right)\right), 15\right)
$$

Write $u=u_{0}+u_{1}$ where:

$$
\begin{align*}
\Delta u_{0} & =\operatorname{curl} \omega, & & \text { in } \mathcal{D}_{t},  \tag{1.16}\\
u_{0} & =0, & & \text { on } \partial \mathcal{D}_{t} \tag{1.17}
\end{align*}
$$

and $\Delta u_{1}=0$ in $\mathcal{D}_{t},\left.u_{1}\right|_{\partial \mathcal{D}_{t}}=\left.u\right|_{\partial \mathcal{D}_{t}}$. In section 4 we prove the following nonlinear estimate:

$$
\begin{equation*}
\left\|\nabla u_{0}(t)\right\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \leq C(K)\left(\left(1+\log ^{+}\left(\|\omega(t)\|_{H^{2}\left(\mathcal{D}_{t}\right)}\right)\|\omega(t)\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+1\right),\right. \tag{1.18}
\end{equation*}
$$

with $\log ^{+}(s)=\max (0, \log (s))$, which relies on properties of the Green's function for the Dirichlet problem in $\mathcal{D}_{t}$.

Assuming that (1.14)-(1.15) and (1.18) hold for the moment, we have:
Proof of Theorem 2. Write $y_{s}(t)=\mathcal{E}_{s}(t), y_{s}^{*}=\sum_{k \leq s} \mathcal{E}_{k}$. By the maximum principle (3.16), we have $\left\|\nabla u_{1}\right\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \leq \mathcal{A}(t)$. Therefore, by (1.14) and (1.18), we have that $y_{3}(t)$ satisfies the differential inequality:

$$
\begin{equation*}
\frac{d}{d t} y_{3} \leq C(K)\left(\mathcal{A}+\mathcal{A}^{2}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}\right) y_{3}\left(1+\log ^{+} y_{3}\right) \tag{1.19}
\end{equation*}
$$

and that for $s \geq 4$ :

$$
\frac{d}{d t} y_{s} \leq C(K)\left(\mathcal{A}+\mathcal{A}^{2}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}\right)\left(y_{s}\left(1+\log ^{+} y_{s}\right)+P\left(y_{s-1}^{*}\right)\right)
$$

As in the proof of Proposition 17.2.3 in [15, we now note that (1.19) implies:

$$
\begin{equation*}
\limsup _{t \nearrow T^{*}} \int_{y_{3}(0)}^{y_{3}(t)} \frac{d y}{y\left(1+\log ^{+} y\right)} \leq C(K) \limsup _{t \nearrow T^{*}} \int_{0}^{t} \mathcal{A}(s)+(\mathcal{A}(s))^{2}+\left\|\nabla_{N} D_{t} p(s)\right\|_{L^{\infty}\left(\partial \mathcal{D}_{s}\right)} d s \tag{1.20}
\end{equation*}
$$

and this is finite, by assumption.

On the other hand, since $T_{*}$ is the largest time for which (1.1)-(1.3) has a solution in the space (1.12), we have $\lim \sup _{t \nearrow T_{*}} y_{3}(t)=\infty$. However,

$$
\lim _{s \rightarrow \infty} \int_{a}^{s} \frac{d y}{y\left(1+\log ^{+} y\right)}=\infty
$$

for any $a>0$, which contradicts the fact that the right-hand side of (1.20) is finite. The result for $s \geq 4$ then follows from induction and (1.15).

## 2. The free boundary $\partial \mathcal{D}_{t}$ and the projection to the tangent space at the boundary

It is easiest to define the geometric quantities that we will need in terms of Lagrangian coordinates, which we now define. Let $\Omega \subset \mathbb{R}^{3}$ be diffeomorphic to the unit ball. We define $x^{i}(t)=x^{i}(t, \cdot): \Omega \rightarrow \mathcal{D}_{t}$ by:

$$
\begin{aligned}
\frac{d}{d t} x^{i}(t, y) & =u^{i}(t, x(t, y)), \\
x^{i}(0, y) & =f_{0}(y)
\end{aligned}
$$

where $f_{0}: \Omega \rightarrow \mathcal{D}_{0}$ is a volume preserving diffeomorphism. This change of coordinates and the metric $\delta_{i j}$ in $\mathbb{R}^{3}$ induce a time-dependent metric $g=g(t, y)$ on $\Omega$ :

$$
\begin{equation*}
g_{a b}(t, y)=\delta_{i j} \frac{\partial x^{i}}{\partial y^{a}} \frac{\partial x^{j}}{\partial y^{b}} . \tag{2.1}
\end{equation*}
$$

As in [13], we will work with the covariant derivative associated to $g$; if $\alpha=\alpha_{a_{1} \cdots a_{r}} d y^{a_{1}} \ldots d y^{a_{r}}$ is a ( $0, r$ ) tensor then $\nabla \alpha$ is a $(0, r+1)$ tensor with components:

$$
\nabla_{a} \alpha_{a_{1} \cdots a_{r}}=\partial_{a} \alpha_{a_{1} \cdots a_{r}}-\Gamma_{a_{1} a}^{b} \alpha_{b a_{2} \cdots a_{r}}-\cdots-\Gamma_{a a_{r}}^{b} \alpha_{a_{1} \cdots a_{r-1} b},
$$

where the Christoffel symbols $\Gamma_{a b}^{c}$ are defined by:

$$
\Gamma_{a b}^{c}=\frac{1}{2} g^{c d}\left(\frac{\partial}{\partial y^{a}} g_{b d}+\frac{\partial}{\partial y^{b}} g_{a d}-\frac{\partial}{\partial y^{d}} g_{a b}\right) .
$$

We will frequently make use of the fact that the covariant derivatives commute, which just follows from the fact that they commute when expressed in Eulerian coordinates. We will write $\nabla_{a_{1} \cdots a_{r}}^{r}=\nabla_{a_{1}} \cdots \nabla_{a_{r}}$ and will often omit the indices $a_{1}, \cdots a_{r}$.

We will write $D_{t}$ for time differentiation in Lagrangian coordinates:

$$
D_{t}=\left.\frac{\partial}{\partial t}\right|_{y=\mathrm{const}}=\left.\frac{\partial}{\partial t}\right|_{x=\mathrm{const}}+u^{k} \frac{\partial}{\partial x^{k}} .
$$

Note that $D_{t}$ does not commute with $x$ derivatives:

$$
\begin{equation*}
\left[D_{t}, \partial_{i}\right]=-\left(\partial_{i} u^{k}\right) \partial_{k} \tag{2.2}
\end{equation*}
$$

We will use the convention that the letters $a, b, c .$. refer to quantites expressed in Lagrangian coordinates and the letters $i, j, k, \ldots$ refer to quantites expressed in the Eulerian coordinates.
2.1. The geometry of $\partial \mathcal{D}_{t}$. We let $N^{a}$ denote the unit normal to $\partial \Omega$ with respect to the metric $g$ defined by (2.1), and write $N_{a}=g_{a b} N^{b}$ for the unit conormal. We will write $\gamma$ for the (co)-metric on $\partial \Omega$ :

$$
\gamma_{a b}=g_{a b}-N_{a} N_{b}, \quad \gamma^{a b}=g^{a b}-N^{a} N^{b} .
$$

We also write:

$$
\Pi_{b}^{a}=\delta_{b}^{a}-N^{a} N_{b}
$$

for the orthogonal projection to $T(\partial \Omega)$. More generally if $\alpha$ is a $(r, s)$ tensor, we set:

$$
(\Pi \alpha)_{a_{1} \cdots a_{s}}^{b_{1} \cdots b_{r}}=\Pi_{a_{1}}^{c_{1}} \cdots \Pi_{a_{s}}^{c_{s}} \Pi_{d_{1}}^{b_{1}} \cdots \Pi_{d_{r}}^{b_{r}} \alpha_{c_{1} \cdots c_{s}}^{d_{1} \cdots d_{r}} .
$$

We will write $\bar{\nabla}$ for covariant differentiation on $\partial \Omega$, defined by:

$$
\bar{\nabla} \alpha=\Pi \nabla \alpha,
$$

for an arbitrary tensor field $\alpha$. The second fundamental form $\theta_{a b}$ is given by:

$$
\theta_{a b}=\bar{\nabla}_{a} N_{b}=\Pi_{a}^{c} \Pi_{b}^{d} \nabla_{c} N_{d} .
$$

We let $\iota_{0}$ denote the injectivity radius of $\partial \mathcal{D}_{t}$. The fundamental geometric assumption that we will make is that:

$$
\begin{equation*}
|\theta|+\frac{1}{\iota_{0}} \leq K, \text { on } \partial \mathcal{D}_{t} . \tag{2.3}
\end{equation*}
$$

Among other things, this assumption ensures that the domain $\mathcal{D}_{t}$ satisfies the "uniform exterior sphere condition":

Lemma 1. If (2.3) holds, then there is a $r_{0}=r_{0}(K)$ with $r_{0}>0$ so that for each $x \in \partial \mathcal{D}_{t}$, there is an $r>r_{0}$ and balls $B_{1}, B_{2}$ with radius less than $r$ so that:

$$
B \cap \overline{\mathcal{D}_{t}}=\{x\}, \quad B_{2} \cap \overline{\mathbb{R}^{3} \backslash \mathcal{D}_{t}}=\{x\} .
$$

Proof. Fix $x \in \partial \mathcal{D}_{t}$ and take $r \leq K / 2$. With $z_{ \pm}=x \pm r N(x)$, where $N(x)$ is the normal vector to $\partial \mathcal{D}_{t}$ at $x$, the ball of radius $r$ centered at $z_{ \pm}$touches $x$ but does not cross $\partial \mathcal{D}_{t}$ (from the outside in the " + " case and the inside in the "-" case) since the largest principal curvature at $x$ does not exceed $K$. Also, by the bound for the injectivity radius, there are no other points of $\partial \mathcal{D}_{t}$ in this ball.

We now let $d=d(t, y)$ denote the geodesic distance to the boundary and define the extension of the normal to the interior of $\Omega$ by:

$$
\tilde{N}_{a}=\frac{\nabla_{a} d}{\sqrt{g^{a b} \nabla_{a} d \nabla_{b} d}},
$$

which is well-defined so long as $(t, y)$ are such that $d(t, y)<\iota / 2$, say. From now on we will abuse notation and just write $N$ instead of $\tilde{N}$. Note that with this choice of $d$, we have $\nabla_{N} N=0$ near the boundary. We now extend this definition to all of $\partial \Omega$. Let $\varphi \in C_{0}^{\infty}(\mathbb{R})$ be so that $\varphi(x)=1$ when $|x| \leq 1 / 4$ and $\varphi(x)=0$ when $|x| \geq 1 / 2$, and set $\varphi_{\iota}(x)=\varphi\left(\iota^{-1} x\right)$. We then abuse notation further and write:

$$
\begin{equation*}
\Pi_{b}^{a}=\delta_{b}^{a}-\varphi_{\iota} N^{a} N_{b}, \quad \bar{\nabla}_{b}=\Pi_{b}^{a} \nabla_{b} . \tag{2.4}
\end{equation*}
$$

Away from the boundary $\Pi$ is just the identity map and close to the boundary it agrees with the projection onto the tangent space to the level sets of $d$. Similarly, away from the boundary $\bar{\nabla}$ is the covariant derivative and on $\partial \Omega$ it is the covariant derivative on $\partial \Omega$.

If the assumption (2.3) holds, we can control derivatives of $\Pi$ :
Lemma 2. Let $d(y)=\operatorname{dist}_{g}(y, \partial \Omega)$ denote the geodesic distance in the metric $g$ from $y$ to $\partial \Omega$. If $n$ denotes the conormal $n=\nabla d$ to the sets $\{y \in \Omega: d(y)=a\}$ and $\Pi$ is defined by (2.4), then:

$$
|\nabla n(t, y)|+|\nabla \Pi(t, y)| \leq C| | \theta(t, \cdot)\left\|_{L^{\infty}(\partial \Omega)}, \quad\left|D_{t} n(t, y)\right|+\left|D_{t} \Pi(t, y)\right| \leq C\right\| \nabla u(t, \cdot) \|_{L^{\infty}(\Omega)},
$$

when $d(y)<\frac{\iota_{0}}{2}$, where $\iota_{0}$ is the normal injectivity radius of $\partial \mathcal{D}_{t}$.
Proof. See Lemmas 2.1, 3.10-3.11 in 13.

## 3. Elliptic estimates

We will need many of the elliptic estimates from [13]. The basic result we rely on is the following pointwise inequality. Let $\beta=\beta_{I i} \nabla_{I}^{r} \alpha_{i}$ for a ( 0,1 )-tensor $\alpha$. We will write:

$$
\operatorname{div} \beta=\nabla^{i} \beta_{I i}, \quad \operatorname{curl} \beta_{i j}=\nabla_{i} \beta_{I j}-\nabla_{j} \beta_{I i},
$$

and:

$$
(\Pi \beta)_{J i}=\Pi_{J}^{I} \beta_{I i}=\Pi_{j_{1}}^{i_{i}} \cdots \Pi_{j_{s}}^{i_{s}} \beta_{i_{1} \cdot i_{s} i} .
$$

We then have Lemma 5.5 from (13):
Lemma 3. With the above notation:

$$
\begin{equation*}
|\nabla \beta| \leq C(|\operatorname{div} \beta|+|\operatorname{curl} \beta|+|\Pi \nabla \beta|) . \tag{3.1}
\end{equation*}
$$

We will also use the following simple version of the trace inequality, which is estimate (5.19) in 13 .
Lemma 4. If (2.3) holds, then:

$$
\begin{equation*}
\|\beta\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(\|\nabla \beta\|_{L^{2}(\Omega)}^{2}+K\|\beta\|_{L^{2}(\Omega)}^{2}\right) . \tag{3.2}
\end{equation*}
$$

The following estimates are based on (3.1) and will be used to control both $p$ and $D_{t} p$ in the interior. The first estimate for $r \geq 2$ is (5.28) in [13], while the second estimate follows from (5.20) in [13] and the estimate $\|\nabla q\|_{L^{2}(\Omega)} \leq\|\Delta q\|_{L^{2}(\Omega)}$ if $q=0$ on $\partial \Omega$ which is just integration by parts.

Lemma 5. If (2.3) holds and $r \geq 2$, then:

$$
\begin{equation*}
\left\|\nabla^{r} q\right\|_{L^{2}(\partial \Omega)}+\left\|\nabla^{r} q\right\|_{L^{2}(\Omega)} \leq C\left\|\Pi \nabla^{r} q\right\|_{L^{2}(\partial \Omega)}+C(K, \operatorname{Vol}(\Omega)) \sum_{s \leq r-1}\left\|\nabla^{s} \Delta q\right\|_{L^{2}(\Omega)} \tag{3.3}
\end{equation*}
$$

If $q=0$ on $\partial \Omega$, then in addition:

$$
\begin{equation*}
\|\nabla q\|_{L^{2}(\partial \Omega)}+\|\nabla q\|_{L^{2}(\Omega)} \leq C(K)\|\Delta q\|_{L^{2}(\Omega)} \tag{3.4}
\end{equation*}
$$

We also need estimates to control $\Pi \nabla^{r} q$ when $q=0$ on $\partial \Omega$. Note that when $r=2$, we have:

$$
\begin{equation*}
0=\Pi^{i j} \nabla_{i}\left(\Pi^{k \ell} \nabla_{k} q\right)=\Pi^{i j} \Pi^{k \ell} \nabla_{i} \nabla_{k} q+\Pi^{i j}\left(\nabla_{i} \Pi^{k \ell}\right) \nabla_{k} q \tag{3.5}
\end{equation*}
$$

Recalling that $\Pi^{k \ell}=\delta^{k \ell}-N^{k} N^{\ell}$ and that since $q=0$ on $\partial \Omega, \nabla_{k} q=N_{k}\left(\nabla_{N} q\right)$, so the above implies that:

$$
\begin{equation*}
\Pi^{i j} \Pi^{k \ell} \nabla_{i} \nabla_{k} q=\theta_{i j} \nabla_{N} q \tag{3.6}
\end{equation*}
$$

We will need a higher-order version of this formula. As explained in 13, the idea is that since $q=0$ on $\partial \Omega, q / d$ is smooth up to the boundary (recall that $d$ is the distance to the boundary), and we can write:

$$
\begin{equation*}
\Pi \nabla^{r} q=\Pi \nabla^{r}(d q / d)=\sum_{\ell=0}^{r} \Pi\left(\nabla^{\ell} d\right) \otimes\left(\nabla^{r-\ell} q / d\right) \tag{3.7}
\end{equation*}
$$

On $\partial \Omega, d=\Pi \nabla d=0$ and $\Pi \nabla^{2} d=\theta$, so that $\Pi \nabla^{\ell} d \sim \bar{\nabla}^{\ell-2} \theta$, to highest order. We now want to replace the derivatives $\nabla^{r-\ell}$ with tangential derivatives in the above expression. Writing $\nabla^{i}=\bar{\nabla}^{i}+N^{i} N_{j} \nabla^{j}$, and inserting this into (3.7) generates derivatives of $N$. Repeatedly using that $\nabla_{N} N=0$, these can be turned into derivatives of $\theta$. Also, we have $q / d \sim \nabla_{N} q$, and so we should expect:

$$
\begin{equation*}
\Pi \nabla^{r} q \sim \sum_{\ell=2}^{r}\left(\bar{\nabla}^{\ell-2} \theta\right) \otimes\left(\nabla^{r-\ell} \nabla_{N} q\right) \tag{3.8}
\end{equation*}
$$

Note that this also gives a bound for $\bar{\nabla}^{r-2} \theta$, assuming that $\left|\nabla_{N} q\right|$ is bounded below. In particular, a direct calculation similar to (3.5) (see (4.21) in [13]) also shows that:

$$
\begin{equation*}
\bar{\nabla} \theta=\left(\nabla_{N} q\right)^{-1}\left(\Pi \nabla^{3} q-3 \theta \widetilde{\otimes} \bar{\nabla} \nabla_{N} q\right) \tag{3.9}
\end{equation*}
$$

where $\widetilde{\otimes}$ is a symmetrization over some of the indices appearing in the tensor product $\theta \otimes \bar{\nabla} \nabla_{N} q$ but for our purposes the exact form is not important.

The precise version of (3.8) and a version of (3.9) for higher derivatives of $\theta$ can be found in Proposition 5.9 in [13] (see also Proposition 4.3 there):

Proposition 1. We have:

$$
\begin{align*}
&\left\|\Pi \nabla^{r} q\right\|_{L^{2}(\partial \Omega)} \leq C(K)\left(\left\|\bar{\nabla}^{r-2} \theta\right\|_{L^{2}(\partial \Omega)}\left\|\nabla_{N} q\right\|_{L^{\infty}(\partial \Omega)}+\sum_{k=1}^{r-1}\|\theta\|_{L^{\infty}(\partial \Omega)}^{k}\left\|\nabla^{r-k} q\right\|_{L^{2}(\partial \Omega)}\right) \\
&\left.+\left(\|\theta\|_{L^{\infty}(\partial \Omega)}+\sum_{k \leq r-2}\left\|\bar{\nabla}^{k} \theta\right\|_{L^{2}(\partial \Omega)}\right) \sum_{k \leq r-2}\left\|\nabla^{k} q\right\|_{L^{2}(\partial \Omega)},\right) \tag{3.10}
\end{align*}
$$

and:

$$
\begin{align*}
& \left\|\nabla^{r-1} q\right\|_{L^{2}(\partial \Omega)} \leq C\left(\left\|\bar{\nabla}^{r-3} \theta\right\|_{L^{2}(\partial \Omega)}\left\|\nabla_{N} q\right\|_{L^{2}(\partial \Omega)}+\left\|\nabla^{r-2} \Delta q\right\|_{L^{2}(\Omega)}\right. \\
& \quad+C\left(K, \operatorname{Vol}(\Omega),\|\theta\|_{L^{2}(\partial \Omega)}, \ldots,\left\|\bar{\nabla}^{r-4} \theta\right\|_{L^{2}(\partial \Omega)}\right)\left(\left\|\nabla_{N} q\right\|_{L^{\infty}(\partial \Omega)}+\sum_{s \leq r-3}\left\|\nabla^{s} \Delta q\right\|_{L^{2}(\partial \Omega)}\right) \tag{3.11}
\end{align*}
$$

In addition, if $\left|\nabla_{N} q\right| \geq \varepsilon>0$ and $\left|\nabla_{N} q\right| \geq 2 \varepsilon| | \nabla_{N} q \|_{L^{\infty}(\partial \Omega)}$ on $\partial \Omega$ for some $\varepsilon>0$ then:

$$
\left.\left\|\bar{\nabla}^{r-2} \theta\right\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(\varepsilon^{-1}, K\right)\left(\left\|\Pi \nabla^{r} q\right\|_{L^{2}(\partial \Omega)}+\left(\|\theta\|_{L^{\infty}(\partial \Omega)}+\sum_{k \leq r-3}\left\|\bar{\nabla}^{k} \theta\right\|_{L^{2}(\partial \Omega)}\right) \sum_{k \leq r-1}\left\|\nabla^{k} q\right\|_{L^{2}(\partial \Omega)}\right) 12\right)
$$

The next set of estimates are more well-known. For example the first estimate is Theorem 9.14 in 7 combined with Lemma 1 and the second estimate follows from a modification of the proof of Theorem 6.4.8 [14 (see Lemma 2 in [3] for a concise proof). The third estimate is not mentioned explicitly in these references but it is clear that the arguments used to prove (3.14) can be used to prove (3.15) as well.
Proposition 2. If (2.3) holds, then for any $q \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega), 1<p<\infty$ :

$$
\begin{equation*}
\|q\|_{L^{p}(\Omega)}+\|\nabla q\|_{L^{p}(\Omega)}+\left\|\nabla^{2} q\right\|_{L^{p}(\Omega)} \leq C(K)\|\Delta q\|_{L^{p}(\Omega)} . \tag{3.13}
\end{equation*}
$$

If $\Delta q=\operatorname{div} F+g$ for a vector field $F$ and a function $g$, and $q \in W_{0}^{1, p}(\Omega)$, then:

$$
\begin{equation*}
\|\nabla q\|_{L^{p}(\Omega)} \leq C(K)\left(\|F\|_{L^{p}(\Omega)}+\|g\|_{L^{1}(\Omega)}\right) . \tag{3.14}
\end{equation*}
$$

Similarly, if $\beta$ is a vector field and $\Delta \beta=\operatorname{curl} \gamma+\rho$ for vector fields $\gamma, \rho$ and $\beta \in W_{0}^{1, p}(\Omega)$, then:

$$
\begin{equation*}
\|\nabla \beta\|_{L^{p}(\Omega)} \leq C(K)\left(\|\gamma\|_{L^{p}(\Omega)}+\|\rho\|_{L^{1}(\Omega)} .\right. \tag{3.15}
\end{equation*}
$$

We will also need the following "Bernstein" maximum principle, which is just the maximum principle for subharmonic functions combined with the fact that if $\Delta f=0$, then $\Delta|\nabla f|^{2}=2\left|\nabla^{2} f\right|^{2} \geq 0$ :

Proposition 3. Suppose $f \in C^{3}(\Omega)$ is harmonic. Then

$$
\begin{equation*}
\|\nabla f\|_{L^{\infty}(\Omega)} \leq\|\nabla f\|_{L^{\infty}(\partial \Omega)} . \tag{3.16}
\end{equation*}
$$

We now note the following simple consequence of (3.15) and (3.16). Writing $u=u_{0}+u_{1}$ as in (1.16)- (1.17) and using Hölder's inequality implies that:

$$
\begin{align*}
\|\nabla u\|_{L^{p}(\Omega)} & \leq C(K, \operatorname{Vol}(\Omega))\left(\|\omega\|_{L^{\infty}(\Omega)}+\|\bar{\nabla} U\|_{L^{\infty}(\partial \Omega)}+\|\mathcal{N} U\|_{L^{\infty}(\Omega)}\right) \\
& \leq C(K, \operatorname{Vol}(\Omega)) \mathcal{A} \tag{3.17}
\end{align*}
$$

with $\mathcal{A}$ defined by (5.4).

## 4. A nonlinear estimate

If $v \in L^{2}\left(\mathbb{R}^{3}\right)$ is a vector field with $\operatorname{div} v=0$ and $\operatorname{curl} v=\omega$, then $\Delta v=\operatorname{curl} \omega$ and using the fact that the Riesz transform is bounded on $L^{p}\left(\mathbb{R}^{3}\right)$ for $1<p<\infty$ leads to:

$$
\|\nabla v\|_{L^{p}\left(\mathbb{R}^{3}\right)} \leq C\|\omega\|_{L^{p}\left(\mathbb{R}^{3}\right)} .
$$

Unfortunately, such an estimate fails when $p=\infty$. The key idea in $[1$ is that there is a replacement for the $p=\infty$ case if we are willing to control more derivatives of $v$ :

$$
\|v\|_{L^{\infty}\left(\mathbb{R}^{3}\right)} \leq C\left(\left(1+\log ^{+}\|v\|_{H^{r}\left(\mathbb{R}^{3}\right)}\right)\|\omega\|_{L^{\infty}\left(\mathbb{R}^{2}\right)}+\|\omega\|_{L^{2}\left(\mathbb{R}^{3}\right)}\right),
$$

for any $r>3 / 2$, with $\log ^{+}(s)=\max (0, \log (s))$.
There is a version of this inequality that holds on our domain $\mathcal{D}_{t}$ as well, provided that we have control over the boundary:

Theorem 3. Suppose that (2.3) holds. If $\Delta v=\operatorname{curl} \alpha$ in $\mathcal{D}_{t}$ and $v=0$ on $\partial \mathcal{D}_{t}$, then for any $s>3$ :

$$
\|\nabla v\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \leq C(K)\left(\left(1+\log ^{+}\|\alpha\|_{H^{s-1}\left(\mathcal{D}_{t}\right)}\right)\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+1\right),
$$

where $\log ^{+}(q)=\max (0, \log (q))$.
As in [1] and [5, the proof relies on estimates for the Green's functions for the Laplacian. We will use the following result, which follows from Theorems 1.1, 1.3, and 3.3 in 8 . The result in [8 assumes that the domain satisfies the uniform exterior sphere condition, which holds if (2.3) holds, by Lemma

Theorem 4. If $\mathcal{D}_{t}$ satisfies (2.3), there is a unique function $G: \mathcal{D}_{t} \times \mathcal{D}_{t} \rightarrow \mathbb{R} \cup\{\infty\}$ so that $G \geq 0$, $G(x, y)=G(y, x)$ and so that the following properties hold:

- If $y \in \mathcal{D}_{t}$, then for any $r>0$ :

$$
G(\cdot, y) \in W^{2,1}\left(\mathcal{D}_{t} \backslash B(r, y)\right) \cap H_{0}^{1}\left(\mathcal{D}_{t}\right),
$$

- for each $\varphi \in C_{c}^{\infty}(\Omega)$ we have:

$$
\begin{equation*}
\int_{\mathcal{D}_{t}} G(x, y) \Delta \varphi(y)=-\int_{\mathcal{D}_{t}} \delta^{i j} \nabla_{y^{i}} G(x, y) \nabla_{y^{j}} \varphi(y) d y=\varphi(x), \tag{4.1}
\end{equation*}
$$

and

- if the bound (2.3) holds, then for all $x, y \in \mathcal{D}_{t}$ with $x \neq y$ :

$$
\begin{aligned}
|G(x, y)| & \leq C|x-y|^{-1}, \\
|\nabla G(x, y)| & \leq C(K)|x-y|^{-2}, \\
\left|\nabla_{x} \nabla_{y} G(x, y)\right| & \leq C(K)|x-y|^{-3} .
\end{aligned}
$$

Using this result, we can now provide the
Proof of Theorem 图. We argue nearly exactly as in [1. Fix $x \in \mathcal{D}_{t}$ and let $\delta>0$ be a parameter to be determined later. Let $\chi_{\delta}$ be a smooth function so that $\chi_{\delta}(z)=1$ when $|z-x|<\delta, \chi_{\delta}(z)=0$ when $|z-x| \geq 2 \delta$, and so that $\left|\nabla \chi_{\delta}\right| \leq \frac{C}{\delta}$. By (4.1) we have:

$$
\begin{equation*}
v_{i}(x)=\int_{\mathcal{D}_{t}} G(x, y) \operatorname{curl} \alpha_{i}(y) d y=\int_{\mathcal{D}_{t}} G(x, y)\left(1-\chi_{\delta}(y)\right) \operatorname{curl} \alpha_{i}(y) d y+\int_{\mathcal{D}_{t}} G(x, y) \chi_{\delta}(y) \operatorname{curl} \alpha_{i}(y) d y \tag{4.2}
\end{equation*}
$$

We now re-write (4.2) as:

$$
\begin{aligned}
v_{i}(x) & =\int_{\mathcal{D}_{t}} \mathcal{K}_{i}^{j}(x, y)\left(1-\chi_{\delta}(y)\right) \alpha_{j}(y) d y+\int_{\mathcal{D}_{t}} G(x, y) \delta_{i \ell} \epsilon^{\ell j k}\left(\nabla_{k} \chi_{\delta}(y)\right) \alpha_{j}(y) d y+\int_{\mathcal{D}_{t}} G(x, y) \chi_{\delta}(y) \operatorname{curl} \alpha_{i}(y) d y \\
& \equiv v_{A}(x)+v_{B}(x)+v_{C}(x),
\end{aligned}
$$

where $\epsilon$ is the fully anti-symmetric symbol normalized by $\epsilon^{123}=1$ and $\mathcal{K}_{i}^{j}(x, y)=\delta_{i \ell} \epsilon^{j j k} \nabla_{y^{k}} G(x, y)$.
Using $\left|\nabla_{x} \mathcal{K}(x, y)\right| \leq\left|\nabla_{x} \nabla_{y} G(x, y)\right| \leq C(K)|x-y|^{-3}$, we have:

$$
\begin{aligned}
\left|\nabla v_{A}(x)\right| & \leq C(K)\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}\left(\int_{\{\delta \leq|x-y| \leq 1\} \cap \mathcal{D}_{t}}|x-y|^{-1} d y+\int_{\{1 \leq|x-y|\} \cap \mathcal{D}_{t}}|x-y|^{-1} d y\right) \\
& \leq C(K)\left(C_{1}\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}-\log \delta\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}\right),
\end{aligned}
$$

where $C_{1}=C_{1}\left(\operatorname{Vol} \mathcal{D}_{t}\right)$.
Since $\left|\nabla \chi_{\delta}\right| \leq C \delta^{-1}$ and $\nabla \chi_{\delta}$ is supported on the annulus $\delta \leq|x-y| \leq 2 \delta$, and since $|\nabla G(x, y)| \leq$ $C(K)|x-y|^{-2}$ we have:

$$
\left|\nabla v_{B}(x)\right| \leq C(K) \delta^{-1}\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \int_{\delta \leq|x-y| \leq 2 \delta}|x-y|^{-2} d y \leq C(K)\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} .
$$

Finally, to control $\nabla v_{C}(x)$, we use Hölder's inequality and then Sobolev's inequality:

$$
\begin{aligned}
\left|\nabla v_{C}(x)\right| \leq\left|\int_{D_{t}} \nabla_{x} G(x, y) \chi_{\delta}(y) \nabla_{y} \alpha(y) d y\right| & \leq C(K)\left(\int_{\{0 \leq|x-y| \leq \delta\}}|x-y|^{-12 / 5} d y\right)^{5 / 6}\|\nabla \alpha\|_{L^{6}\left(\mathcal{D}_{t}\right)} \\
& \leq C(K) \delta^{1 / 2}\|\alpha\|_{H^{2}\left(\mathcal{D}_{t}\right)} .
\end{aligned}
$$

Therefore we have shown:

$$
|\nabla v(x)| \leq C\left(K, \operatorname{Vol} \mathcal{D}_{t}\right)\left((1-\log \delta)\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\delta^{1 / 2}\|\alpha\|_{H^{2}\left(\mathcal{D}_{t}\right)}\right) .
$$

We now take $\delta^{-1 / 2}=\min \left(1,\|\alpha\|_{H^{2}\left(\mathcal{D}_{t}\right)}\right)$, so that the above becomes:

$$
|\nabla v(x)| \leq C\left(K, \operatorname{Vol} \mathcal{D}_{t}\right)\left(\left(1+\log \|\alpha\|_{H^{2}\left(\mathcal{D}_{t}\right)}\right)\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+1\right)
$$

and noting that by (1.2), $\operatorname{Vol} \mathcal{D}_{t}=\operatorname{Vol} \mathcal{D}_{0}$, this completes the proof.

## 5. Energy estimates

The results in this section are nearly identical to those in [13]. The only difference is that we are slightly more explicit about the lower-order terms that occur in the computations.

We define:

$$
\begin{equation*}
E_{r}(t)=\int_{\Omega} \delta^{i j} \Pi^{I J}\left(\nabla_{I}^{r} u_{i}\right)\left(\nabla_{J}^{r} u_{j}\right) d x+\int_{\partial \Omega} \Pi^{I J}\left(\nabla_{I}^{r} p\right)\left(\nabla_{J}^{r} p\right)|\nabla p|^{-1} d S \tag{5.1}
\end{equation*}
$$

as well as:

$$
K_{r}(t)=\int_{\mathcal{D}_{t}}\left|\nabla^{r-1} \omega\right|^{2} d x
$$

and:

$$
\mathcal{E}_{r}(t)=E_{r}(t)+K_{r}(t)
$$

We also write:

$$
\mathcal{E}_{0}(t)=\int_{\mathcal{D}_{t}}|u|^{2} d x
$$

and a simple calculation shows that $\mathcal{E}_{0}$ is conserved. We will estimate these quantites assuming that the following bounds hold on $\partial \mathcal{D}_{t}$ :

$$
\begin{align*}
|\theta|+\frac{1}{\iota_{0}} & \leq K  \tag{5.2}\\
|\nabla p| & \geq \delta>0 \tag{5.3}
\end{align*}
$$

Our estimates will involve the following quantity:

$$
\begin{equation*}
\mathcal{A}=\|\omega\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\|\nabla u\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}+\|\mathcal{N} U\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)} \tag{5.4}
\end{equation*}
$$

where $U=\left.u\right|_{\partial \mathcal{D}_{t}}$. We then have the following energy estimates:
Proposition 4. If the assumptions (5.2) -(5.3) hold, then for $r=1,2,3$ :

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{r} \leq C\left(K, \delta^{-1}\right)\left(\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}+\mathcal{A}+\mathcal{A}^{2}\right) \sum_{r=0}^{3} \mathcal{E}_{r} \tag{5.5}
\end{equation*}
$$

and for $r \geq 4$, there is a polynomial $P$ so that:

$$
\begin{equation*}
\frac{d}{d t} \mathcal{E}_{r} \leq C\left(K, \delta^{-1}\right)\left(\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)}+\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)}+\mathcal{A}+\mathcal{A}^{2}\right) \mathcal{E}_{r} P\left(\mathcal{E}_{r-1}, \ldots, \mathcal{E}_{0}\right) \tag{5.6}
\end{equation*}
$$

The proof of this estimate is nearly identical to the proof of Theorem 7.1 in in [13], except that we need to ensure that the dependence on $\|\nabla u\|_{L^{\infty}}$ is linear (compare with (7.16) in [13]). The only part of the argument that needs to be changed is the proof of the estimates for derivatives of $D_{t} p$ on $\partial \mathcal{D}_{t}$. This is because we will ultimately bound derivatives of $D_{t} p$ by derivatives of $\Delta D_{t} p$ and this is cubic in $\nabla u$.

Before proving the above estimates, it is helpful to see what quantities the energies bound. The following lemma is Lemma 7.3 in [13, and relies on the elliptic estimates described in section 3. We need slightly different estimates to deal with the pressure and second fundamental form depending on how many derivatives are present, because in the estimates for $\mathcal{E}_{3}$, we need all of our estimates to be linear in $\|\nabla u\|_{L^{\infty}(\Omega)}$.

Lemma 6. We have:

$$
\begin{gather*}
\left\|\nabla^{r} u\right\|_{L^{2}(\Omega)}^{2} \leq C \mathcal{E}_{r}, \quad\left\|\Pi \nabla^{r} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq\|\nabla p\|_{L^{\infty}(\partial \Omega)} \mathcal{E}_{r}, \\
\|\nabla p\|_{L^{2}(\partial \Omega)}^{2}+\left\|\nabla^{2} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq C(K, \text { Vol } \Omega)\left(\|\nabla p\|_{L^{\infty}(\partial \Omega)}+\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\right) \sum_{s=0}^{2} \mathcal{E}_{s} \tag{5.7}
\end{gather*}
$$

and:

$$
\begin{equation*}
\left\|\nabla^{r} p\right\|_{L^{2}(\partial \Omega)}^{2}+\left\|\nabla^{r} p\right\|_{L^{2}(\Omega)}^{2} \leq C(K, \operatorname{Vol} \Omega)\left(\|\nabla p\|_{L^{\infty}(\partial \Omega)}+\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\right) \sum_{s=0}^{r} \mathcal{E}_{s} \tag{5.8}
\end{equation*}
$$

If the bound (5.3) holds then:

$$
\begin{equation*}
\|\theta\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(\delta^{-1}\right) \mathcal{E}_{2}, \quad \quad\|\bar{\nabla} \theta\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(K, \delta^{-1}, \operatorname{Vol} \Omega\right)\left(\mathcal{E}_{3}+\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \sum_{s=0}^{2} \mathcal{E}_{s}\right) \tag{5.9}
\end{equation*}
$$

and for $r \geq 4$, there is a polynomial $P$ so that:

$$
\begin{equation*}
\left\|\bar{\nabla}^{r-2} \theta\right\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(K, \delta^{-1}\right)\left(1+\|\nabla p\|_{L^{\infty}(\partial \Omega)}+\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\right) P\left(\mathcal{E}_{r-1}, \ldots \mathcal{E}_{0}\right) \mathcal{E}_{r} \tag{5.10}
\end{equation*}
$$

Proof. The first estimate follows from (3.1) and the second estimate follows from the definition of $\mathcal{E}_{r}$. The estimates in (5.7) follow from the inequality (3.4) along with (3.6) and (5.11), while the estimate (5.8) follows from (3.3) (3.10).

The first estimate in (5.9) is just (3.6) and the second estimate is (3.9) combined with the estimates (5.7). To prove the estimate (5.10), we combine (3.12), (5.8) and use the assumption (5.3):

$$
\left\|\bar{\nabla}^{r-2} \theta\right\|_{L^{2}(\partial \Omega)}^{2} \leq C\left(K, \delta^{-1}\right)\left(1+\|\nabla p\|_{L^{\infty}(\partial \Omega)}+\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\right)\left(\|\theta\|_{L^{\infty}(\partial \Omega)}+\sum_{k \leq r-3}\left\|\bar{\nabla}^{k} \theta\right\|_{L^{2}(\partial \Omega)}\right) \sum_{s=0}^{r} \mathcal{E}_{s}
$$

Taking the divergence of (1.1) and using the fact that $\left[D_{t}, \nabla_{i}\right]=-\left(\nabla_{i} u^{j}\right) \nabla_{j}$, we have:

$$
\begin{equation*}
\Delta p=-\left(\nabla_{i} u^{j}\right)\left(\nabla_{j} u^{i}\right) \tag{5.11}
\end{equation*}
$$

Applying $D_{t}$ to both sides of this equation, a calculation using (5.11) and (2.2) (see just below (6.14) in [13]) yields:

$$
\begin{equation*}
\Delta D_{t} p=p_{1}+p_{2}+p_{3} \tag{5.12}
\end{equation*}
$$

with:

$$
p_{1}=4 g^{a b} g^{c d}\left(\nabla_{a} u_{c}\right) \nabla_{b} \nabla_{d} p, \quad \quad p_{2}=2\left(\nabla_{a} u^{d}\right)\left(\nabla_{d} u^{c}\right)\left(\nabla_{c} u^{a}\right), \quad \quad p_{3}=-\left(\Delta u^{e}\right) \nabla_{e} p
$$

Differentiating (1.1), (5.11) and (5.12) gives Lemma 6.1 of [13]:

## Lemma 7.

$$
\begin{align*}
\left|D_{t} \nabla^{r} u+\nabla^{r+1} p\right|+\left|D_{t} \nabla^{r-1} \operatorname{curl} u\right|+\left|\nabla^{r-1} \Delta p\right| & \leq C \sum_{s=0}^{r-1}\left|\left(\nabla^{1+s} u\right)\left(\nabla^{r-s} u\right)\right|  \tag{5.13}\\
\left|\Pi\left(D_{t} \nabla^{r} p+\left(\nabla^{r} u\right) \cdot \nabla p-\nabla^{r} D_{t} p\right)\right| & \leq C \sum_{s=0}^{r-2}\left|\Pi\left(\left(\nabla^{1+s} u\right) \cdot \nabla^{r-s} p\right)\right| \tag{5.14}
\end{align*}
$$

and

$$
\begin{align*}
& \left|\nabla^{r-2} \Delta D_{t} p-\left(\nabla^{r-2} \Delta u\right) \cdot \nabla p\right| \\
& \leq C \sum_{s=0}^{r-2}\left|\left(\nabla^{1+s} u\right)\left(\nabla^{r-s} p\right)\right|+C \sum_{r_{1}+r_{2}+r_{3}=r-2}\left|\left(\nabla^{1+r_{1}} u\right)\left(\nabla^{1+r_{2}} u\right)\left(\nabla^{1+r_{3}} u\right)\right| \tag{5.15}
\end{align*}
$$

The next ingredient we will need are the following $L^{2}$ estimates for $\Delta D_{t} p$. These are similar to the estimates in [13] except that we need to ensure that $\|\nabla u\|_{L^{\infty}(\Omega)}$ appears with the same homogeneity as $\nabla^{r-2} \Delta D_{t} p$ :
Lemma 8. For $r=2,3$ :

$$
\begin{equation*}
\left\|\nabla^{r-2} \Delta D_{t} p\right\|_{L^{2}(\Omega)}^{2}+\leq C(K, \operatorname{Vol} \Omega)\|\nabla u\|_{L^{\infty}(\Omega)}^{2} \mathcal{A}^{2} \sum_{k=0}^{r} \mathcal{E}_{k}, \tag{5.16}
\end{equation*}
$$

and for $r \geq 4$ :

$$
\begin{equation*}
\left\|\nabla^{r-2} \Delta D_{t} p\right\|_{L^{2}(\Omega)}^{2} \leq C(K, \operatorname{Vol}(\Omega))\|\nabla u\|_{L^{\infty}(\Omega)}^{2} \mathcal{E}_{r} \sum_{k=0}^{r-1} \mathcal{E}_{k} \tag{5.17}
\end{equation*}
$$

Proof. Using (5.15), we need to control:

$$
\begin{array}{ll}
\left\|\left(\nabla^{r-2} \Delta u\right)(\nabla p)\right\|_{L^{2}(\Omega)}, & s=0, \ldots, r-2 \\
\left\|\left(\nabla^{1+s} u\right)\left(\nabla^{r-s} p\right)\right\|_{L^{2}(\Omega)}, & r_{1}+r_{2}+r_{3}=r-2 \\
\left\|\left(\nabla^{1+r_{1}} u\right)\left(\nabla^{1+r_{2}} u\right)\left(\nabla^{1+r_{3}} u\right)\right\|_{L^{2}(\Omega)}, & \tag{5.20}
\end{array}
$$

We first control (5.18) by $\left\|\nabla^{r} u\right\|_{L^{2}(\Omega)}\|\nabla p\|_{L^{\infty}(\Omega)}$. When $r \leq 3$, we use Sobolev embedding (A.4) and the elliptic estimate (3.13) with Lebsegue exponent $p=3$ :

$$
\begin{aligned}
\|\nabla p\|_{L^{\infty}(\Omega)} & \leq C(K)\left(\|\nabla p\|_{L^{3}(\Omega)}+\left\|\nabla^{2} p\right\|_{L^{3}(\Omega)}\right) \\
& \leq C(K)\|\nabla u\|_{L^{\infty}(\Omega)}\|\nabla u\|_{L^{3}(\Omega)}
\end{aligned}
$$

By (3.17), we have $\|\nabla u\|_{L^{3}(\Omega)} \leq C(K, \operatorname{Vol}(\Omega)) \mathcal{A}$. When $r \geq 4$, we can instead use the Sobolev inequality (A.4) with $k=p=2$ and the estimate (5.8):

$$
\begin{align*}
\|\nabla p\|_{L^{\infty}(\Omega)}^{2} & \leq C(K, \operatorname{Vol}(\Omega)) \sum_{k=1}^{3}\left\|\nabla^{k} p\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C(K, \operatorname{Vol}(\Omega))\left(\|\nabla p\|_{L^{\infty}(\partial \Omega)}+\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\right) \mathcal{E}_{3} \tag{5.21}
\end{align*}
$$

Next, to control (5.19), when $r=2$ it is bounded by:

$$
\begin{equation*}
\|\nabla u\|_{L^{\infty}(\Omega)}\left\|\nabla^{2} p\right\|_{L^{2}(\Omega)} \leq C(K)\|\nabla u\|_{L^{\infty}(\Omega)}\|\nabla u\|_{L^{4}(\Omega)}^{2} \leq\|\nabla u\|_{L^{\infty}(\Omega)} \mathcal{A}^{2} \tag{5.22}
\end{equation*}
$$

and when $r=3$ we instead bound it by:

$$
\|\nabla u\|_{L^{\infty}(\Omega)}\left\|\nabla^{2} p\right\|_{L^{2}(\Omega)}+\left\|\nabla^{2} u\right\|_{L^{2}(\Omega)}\|\nabla p\|_{L^{\infty}(\Omega)}
$$

and then control the first term as in (5.22) and use (5.21) to bound $\|\nabla p\|_{L^{\infty}(\Omega)}$.
For $r \geq 4$, we use the interpolation inequality (A.2):

$$
\left\|\left(\nabla^{1+s} u\right)\left(\nabla^{r-s} p\right)\right\|_{L^{2}(\Omega)} \leq\|\nabla u\|_{L^{\infty}(\Omega)} \sum_{k=1}^{r}\left\|\nabla^{k} p\right\|_{L^{2}(\Omega)}+\|\nabla p\|_{L^{\infty}(\Omega)} \sum_{k=1}^{r}\left\|\nabla^{k} u\right\|_{L^{2}(\Omega)} .
$$

These terms can be bounded by the right-hand side of (5.17) by arguing as above and using (5.8).
Finally we bound (5.20). For $r=2$ we use (3.17) and Sobolev embedding (A.4):

$$
\begin{aligned}
\|\nabla u\|_{L^{\infty}(\Omega)}\|\nabla u\|_{L^{4}(\Omega)}^{2} & \leq C(K, \operatorname{Vol} \Omega)\|\nabla u\|_{L^{\infty}(\Omega)} \mathcal{A}\left(\|\nabla u\|_{L^{2}(\Omega)}+\left\|\nabla^{2} u\right\|_{L^{2}(\Omega)}\right) \\
& \leq C(K, \operatorname{Vol} \Omega)\|\nabla u\|_{L^{\infty}(\Omega)} \mathcal{A}\left(\mathcal{E}_{2}+\mathcal{E}_{1}\right)
\end{aligned}
$$

and for $r=3$ the same strategy gives that (5.20) is bounded by:

$$
\|\nabla u\|_{L^{\infty}(\Omega)}\|\nabla u\|_{L^{4}(\Omega)}\left\|\nabla^{2} u\right\|_{L^{4}(\Omega)} \leq C(K, \operatorname{Vol} \Omega)\|\nabla u\|_{L^{\infty}(\Omega)} \mathcal{A}\left(\mathcal{E}_{3}+\mathcal{E}_{2}\right)
$$

When $r \geq 4$, we use the interpolation inequality (A.2) and Sobolev embedding (A.4) to bound it by:

$$
\begin{aligned}
\|\nabla u\|_{L^{\infty}(\Omega)}^{2} \sum_{k=0}^{r-1}\left\|\nabla^{k} u\right\|_{L^{2}(\Omega)} & \leq C(K)\|\nabla u\|_{L^{\infty}(\Omega)}\left(\sum_{k=0}^{3}\left\|\nabla^{k} u\right\|_{L^{2}(\Omega)}\right)\left(\sum_{k=0}^{r-1}\left\|\nabla^{k} u\right\|_{L^{2}(\Omega)}\right) \\
& \leq C(K)\|\nabla u\|_{L^{\infty}(\Omega)}\left(\sum_{s=0}^{r-1} \mathcal{E}_{s}\right)^{2}
\end{aligned}
$$

Combining the previous two results, we have:
Corollary 2. With $\mathcal{A}$ defined by (5.4), for $r=2,3$ :

$$
\begin{equation*}
\left\|\Pi \nabla^{r} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2}+\left\|\nabla^{r-1} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq C(K, \operatorname{Vol}(\Omega))\left(\|\nabla u\|_{L^{\infty}(\Omega)}^{2}+1\right) \mathcal{A} \mathcal{E}_{3} \tag{5.23}
\end{equation*}
$$

and for $r \geq 4$, there is a polynomial $P$ so that:

$$
\begin{equation*}
\left\|\Pi \nabla^{r} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2}+\left\|\nabla^{r-1} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq C(K, \operatorname{Vol}(\Omega))\left(\|\nabla u\|_{L^{\infty}(\Omega)}^{2}+1\right) \mathcal{A}\left(\mathcal{E}_{r}+P\left(\mathcal{E}_{r-1}\right)\right) \tag{5.24}
\end{equation*}
$$

Proof. The $r=2$ case follows by first applying (3.11) and (5.16):

$$
\begin{align*}
\left\|\nabla D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq C(K, \operatorname{Vol}(\Omega))\left\|\Delta D_{t} p\right\|_{L^{2}(\Omega)}^{2} & \leq C(K, \operatorname{Vol}(\Omega))\|\nabla u\|_{L^{\infty}(\Omega)}^{2}\left(\|\omega\|_{L^{\infty}(\Omega)}^{2}+\|\bar{\nabla} u\|_{L^{\infty}(\partial \Omega)}^{2}\right) \mathcal{E}_{3} \\
& \leq C(K, \operatorname{Vol}(\Omega))\|\nabla u\|_{L^{\infty}(\Omega)} \mathcal{A}^{2} \mathcal{E}_{3}, \tag{5.25}
\end{align*}
$$

and then applying (3.6):

$$
\begin{equation*}
\left\|\Pi \nabla^{2} D_{t} p\right\|_{L^{2}(\partial \Omega)} \leq\|\theta\|_{L^{2}(\partial \Omega)}\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}(\partial \Omega)} \tag{5.26}
\end{equation*}
$$

When $r=3$, we have:

$$
\left\|\nabla^{2} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2} \leq C(K, \operatorname{Vol}(\Omega))\left(\left\|\Pi \nabla^{2} D_{t} p\right\|_{L^{2}(\partial \Omega)}^{2}+\left\|\Delta D_{t} p\right\|_{L^{2}(\Omega)}^{2}+\left\|\nabla \Delta D_{t} p\right\|_{L^{2}(\Omega)}^{2}\right),
$$

and using (5.26), (5.25) and (5.16), the right-hand side is bounded by (5.23).
Using (3.10), we also have:

$$
\begin{aligned}
\left\|\Pi \nabla^{3} D_{t} p\right\|_{L^{2}(\partial \Omega)} & \leq C(K)\left(\|\bar{\nabla} \theta\|_{L^{2}(\partial \Omega)}\left\|\nabla_{N} D_{t} p\right\|_{L^{\infty}(\partial \Omega)}+\left\|\nabla D_{t} p\right\|_{L^{2}(\partial \Omega)}+\left\|\nabla^{2} D_{t} p\right\|_{L^{2}(\partial \Omega)}\right. \\
& \left.+\left(\|\bar{\nabla} \theta\|_{L^{2}(\partial \Omega)}+\|\theta\|_{L^{2}(\partial \Omega)}+\|\theta\|_{L^{\infty}(\partial \Omega)}\right)\left\|\nabla D_{t} p\right\|_{L^{\infty}(\partial \Omega)}\right)
\end{aligned}
$$

Using (3.6) and (3.9) with $q=p$ and then (5.8), this is bounded by the right-hand side of (5.24).
For $r \geq 4$, the argument is the same, except we use the bound (5.17) in place of the bound (5.16).
Proof of Proposition 4 By the Reynolds transport theorem,

$$
\frac{1}{2} \frac{d}{d t} \int_{\mathcal{D}_{t}}\left|\nabla^{r-1} \omega(t)\right|^{2} d x=\int_{\mathcal{D}_{t}} D_{t} \nabla^{r-1} \omega \cdot \nabla^{r-1} \omega d x .
$$

By (5.13) and the interpolation inequality (A.2), this gives:

$$
\frac{d}{d t} K_{r}(t) \leq C(K)\|\nabla u\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \mathcal{E}_{r}
$$

To control the time derivative of $E_{r}$ (defined in (5.1)), we use Proposition 5.12 of [13] with $\alpha=\nabla^{r} p$ and $\beta=\nabla^{r-1} u$ and $\nu=(-\nabla p)^{-1}$. This gives:

$$
\begin{align*}
\frac{d}{d t} E_{r}(t) & \leq C \sqrt{E_{r}}\left(\left\|\Pi\left(D_{t} \nabla^{r} p-(\nabla p) N^{k} \nabla^{r} u_{k}\right)\right\|_{L^{2}(\partial \Omega)}+\left\|D_{t} \nabla^{r} u+\nabla \nabla^{r} p\right\|_{L^{2}(\Omega)}\right) \\
& +C K E_{r}+C \sqrt{E_{r}}\left\|\operatorname{curl} \nabla^{r-1} u\right\|_{L^{2}(\Omega)}+\left(K\left\|\nabla^{r} p\right\|_{L^{2}(\Omega)}+\left\|\operatorname{curl} \nabla^{r-1} u\right\|_{L^{2}(\Omega)}\right)^{2} \tag{5.27}
\end{align*}
$$

By Lemma 6 the terms on the second line are all bounded by $C(K, \operatorname{Vol}(\Omega)) \mathcal{A} \mathcal{E}_{r}$. Also, using (5.13), the interpolation inequality (A.2) and Lemma 6e the second term in (5.27) is bounded by the right-hand side of (5.5) (resp. (5.6)). It remains to control the first term in (5.27). Using (5.14):

$$
\left\|\Pi\left(D_{t} \nabla^{r} p+\left(\nabla^{r} u\right) \cdot \nabla p\right)\right\|_{L^{2}(\partial \Omega)} \leq C\left(\left\|\Pi \nabla^{r} D_{t} p\right\|_{L^{2}(\partial \Omega)}+\sum_{s=0}^{r-2}\left\|\Pi\left(\left(\nabla^{1+s} u\right) \cdot \nabla^{r-s} p\right)\right\|_{L^{2}(\partial \Omega)}\right) .
$$

By (5.23) (resp. (5.24)), the first term here is bounded by the right-hand side of (5.5) (resp. (5.6)).
To control $\left\|\Pi\left(\nabla^{1+s} u\right) \cdot\left(\nabla^{r-s} p\right)\right\|_{L^{2}(\partial \Omega)}$ for $s=0, \ldots, r-2$, we note that when $r=2$ the result is bounded by:

$$
\begin{equation*}
\|\nabla u\|_{L^{\infty}(\partial \Omega)}\left\|\nabla^{2} p\right\|_{L^{\infty}(\partial \Omega)} \tag{5.28}
\end{equation*}
$$

and for $r=3$, the result is bounded by:

$$
\begin{equation*}
\|\nabla u\|_{L^{\infty}(\partial \Omega)}\left\|\nabla^{3} p\right\|_{L^{2}(\partial \Omega)}+\left\|\nabla^{2} u\right\|_{L^{2}(\partial \Omega)}\left\|\nabla^{2} p\right\|_{L^{\infty}(\partial \Omega)} . \tag{5.29}
\end{equation*}
$$

By the pointwise estimate (3.1), the equation (5.11) and (3.6), we have:

$$
\left\|\nabla^{2} p\right\|_{L^{\infty}(\partial \Omega)} \leq C\left(\|\nabla u\|_{L^{\infty}(\partial \Omega)}^{2}+\|\theta\|_{L^{\infty}(\partial \Omega)}\left\|\nabla_{N} p\right\|_{L^{\infty}(\partial \Omega)}\right),
$$

and combining this with the trace inequality (3.2) and Lemma 6 shows that (5.28) and (5.29) are controlled by the right-hand side of (5.5).

To control $\left.\| \Pi\left(\nabla^{1+s} u\right) \cdot \nabla^{r-s} p\right) \|_{L^{2}(\partial \Omega)}$ for $r \geq 4$, we note that we could use Sobolev embedding, the trace inequality (3.2) and Lemma 6 to show that this is bounded by interior terms, but this would lead to estimates
that are not linear in the highest order norm $\mathcal{E}_{r}$. The idea from [13] is to use the fact that because of the prescence of the projection $\Pi$, the derivatives $\nabla^{1+s}, \nabla^{r-s}$ are nearly tangential derivatives and for tangential derivatives we can use the interpolation inequality (A.1) to control the intermediate terms. Arguing as in (7.25) in 13 this gives:

$$
\begin{aligned}
& \left\|\Pi\left(\left(\nabla^{1+s} u\right) \cdot\left(\nabla^{r-s} p\right)\right)\right\|_{L^{2}(\partial \Omega)} \\
& \quad \leq C(K)\left(\|\nabla u\|_{L^{\infty}(\partial \Omega)}+\sum_{k=0}^{r-2}\left\|\nabla^{k} u\right\|_{L^{2}(\partial \Omega)}\right)\left\|\nabla^{r} p\right\|_{L^{2}(\partial \Omega)} \\
& \quad+C(K)\left(\left\|\nabla^{2} p\right\|_{L^{\infty}(\partial \Omega)}+\sum_{k=0}^{r-1}\left\|\nabla^{k} p\right\|_{L^{2}(\partial \Omega)}\right)\left\|\nabla^{r-1} u\right\|_{L^{2}(\partial \Omega)} \\
& \quad+C(K)\left(\|\theta\|_{L^{\infty}(\partial \Omega)}+\sum_{k=0}^{r-2}\left\|\bar{\nabla}^{k} \theta\right\|_{L^{2}(\partial \Omega)}\right)\left(\|\nabla u\|_{L^{\infty}(\partial \Omega)}+\sum_{k=0}^{r-2}\left\|\nabla^{k} u\right\|_{L^{2}(\partial \Omega)}\right) \\
& \quad \times\left(\left\|\nabla^{2} p\right\|_{L^{\infty}(\partial \Omega)}+\sum_{k=0}^{r-1}\left\|\nabla^{k} p\right\|_{L^{2}(\partial \Omega)}\right)
\end{aligned}
$$

Using the Sobolev estimates (A.3), (A.4) along with Lemma 6 and arguing as above proves (5.6).

## Appendix A. Sobolev Estimates

Here we collect the various Sobolev embeddings that we will rely on. These are all well-known, but what is important is that the constants in the various inequalities depend only on bounds for the second fundamental form and the volume of $\mathcal{D}_{t}$ (which is constant if (1.2) holds). The proofs of these theorems with these constants appear in the appendix to [13].
A.1. Interpolation inequalities. We will require interpolation inequalities both on $\partial \mathcal{D}_{t}$ and $\mathcal{D}_{t}$.

Lemma 9. Suppose that:

$$
\frac{m}{s}=\frac{k}{p}+\frac{m-k}{q}, 2 \leq p \leq s \leq q \leq \infty
$$

and let $a=k / m$. Then there is a constant $C$ depending only on $m$ so that for any $(0, r)$ tensor $\alpha$ :

$$
\begin{equation*}
\left\|\bar{\nabla}^{k} \alpha\right\|_{L^{s}\left(\partial \mathcal{D}_{t}\right)} \leq C\|\alpha\|_{L^{q}\left(\partial \mathcal{D}_{t}\right)}^{1-a}\left\|\bar{\nabla}^{m} \alpha\right\|_{L^{p}\left(\partial \mathcal{D}_{t}\right)}^{a} \tag{A.1}
\end{equation*}
$$

In addition, if $\iota_{0} \geq \frac{1}{K}$, then:

$$
\sum_{j=0}^{k}\left\|\nabla^{j} \alpha\right\|_{L^{s}\left(\mathcal{D}_{t}\right)} \leq C\|\alpha\|_{L^{q}\left(\mathcal{D}_{t}\right)}^{1-a}\left(\sum_{i=0}^{m}\left\|\bar{\nabla}^{i} \alpha\right\|_{L^{p}\left(\mathcal{D}_{t}\right)} K^{m-i}\right)^{a}
$$

In particular, if $\ell+m=k$ then:

$$
\left\|\bar{\nabla}^{\ell} \alpha \bar{\nabla}^{m} \beta\right\|_{L^{2}\left(\partial \mathcal{D}_{t}\right)} \leq C\left(\|\alpha\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)} \sum_{\ell=0}^{k}\left\|\bar{\nabla}^{\ell} \beta\right\|_{L^{2}\left(\partial \mathcal{D}_{t}\right)}+\|\beta\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)} \sum_{\ell=0}^{k}\left\|\bar{\nabla}^{\ell} \alpha\right\|_{L^{2}\left(\partial \mathcal{D}_{t}\right)}\right)
$$

and

$$
\begin{equation*}
\left\|\nabla^{\ell} \alpha \nabla^{m} \beta\right\|_{L^{2}\left(\mathcal{D}_{t}\right)} \leq C(K)\left(\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \sum_{\ell=0}^{k}\left\|\nabla^{\ell} \beta\right\|_{L^{2}\left(\mathcal{D}_{t}\right)}+\|\beta\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} \sum_{\ell=0}^{k}\left\|\nabla^{\ell} \alpha\right\|_{L^{2}\left(\mathcal{D}_{t}\right)}\right) \tag{A.2}
\end{equation*}
$$

## A.2. Sobolev and Poincaré inequalities.

Lemma 10. Suppose that $1 / \iota_{0} \leq K$. Then for any $(0, r)$-tensor:

$$
\begin{array}{rlrl}
\|\alpha\|_{L^{2 p / 2-k p}\left(\partial \mathcal{D}_{t}\right)} & \leq C(K) \sum_{\ell=0}^{k}\left\|\bar{\nabla}^{\ell} \alpha\right\|_{L^{p}\left(\partial \mathcal{D}_{t}\right)}, & & 1 \leq p \leq \frac{2}{k} \\
\|\alpha\|_{L^{\infty}\left(\partial \mathcal{D}_{t}\right)} & \leq C(K) \sum_{0 \leq \ell \leq k-1}\left\|\bar{\nabla}^{\ell} \alpha\right\|_{L^{p}\left(\partial \mathcal{D}_{t}\right)}, & k>\frac{2}{p} \tag{A.3}
\end{array}
$$

and

$$
\begin{array}{rlrl}
\|\alpha\|_{L^{3 p / 3-k p}\left(\mathcal{D}_{t}\right)} & \leq C(K) \sum_{\ell=0}^{k}\left\|\nabla^{\ell} \alpha\right\|_{L^{p}\left(\mathcal{D}_{t}\right)}, & & 1 \leq p \leq \frac{3}{k}, \\
\|\alpha\|_{L^{\infty}\left(\mathcal{D}_{t}\right)} & \leq C(K) \sum_{0 \leq \ell \leq k-1}\left\|\nabla^{\ell} \alpha\right\|_{L^{p}\left(\mathcal{D}_{t}\right)}, & k>\frac{3}{p} . \tag{A.4}
\end{array}
$$

We will also need the following version of the Poincaré inequality, whose proof is also in 13.
Lemma 11. If $q=0$ on $\partial \mathcal{D}_{t}$ then:

$$
\begin{aligned}
\|q\|_{L^{2}\left(\mathcal{D}_{t}\right)} & \leq C\left(\operatorname{Vol}_{t}\right)^{1 / 3}\|\nabla q\|_{L^{2}\left(\mathcal{D}_{t}\right)}, \\
\|\nabla q\|_{L^{2}\left(\mathcal{D}_{t}\right)} & \leq C\left(\operatorname{Vol}_{\mathcal{D}_{t}}\right)^{1 / 6}\|\Delta q\|_{L^{2}\left(\mathcal{D}_{t}\right)} .
\end{aligned}
$$

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