Fine metrizable convex relaxations of parabolic optimal control problems¹

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Abstract. Nonconvex optimal-control problems governed by evolution problems in infinitedimensional spaces (as e.g. parabolic boundary-value problems) needs a continuous (and possibly also smooth) extension on some (preferably convex) compactification, called relaxation, to guarantee existence of their solutions and to facilitate analysis by relatively conventional tools. When the control is valued in some subsets of Lebesgue spaces, the usual extensions are either too coarse (allowing in fact only very restricted nonlinearities) or too fine (being nonmetrizable). To overcome these drawbacks, a compromising convex compactification is here devised, combining classical techniques for Young measures with Choquet theory. This is applied to parabolic optimal control problems as far as existence and optimality conditions concerns.

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1 Introduction

Relaxations in optimal control theory means usually a certain natural extension of optimization problems. The adjective "natural" means most often "by continuity". The essential attribute of the extended (called relaxed) problems is compactness of the set of admissible relaxed controls, which ensures existence and stability of solutions. An additional attribute is convexity of this set, which allows for further analysis leading to optimality conditions. A general theory of so-called convex compactifications can be found in [32].

A particular situation, which this paper is focused on, appears in optimal control of evolution problems in infinite-dimensional spaces. This abstract situation covers in particular optimal control of systems governed by parabolic partial differential equations.

Conventional relaxation in control theory of such evolution problems uses the original controls ranging over an abstract topological space S and works with continuous nonlinearities. After relaxation, this gives rise to a standard σ -additive functions (measures) on the Borel σ -algebra of Borel subsets. These measures are parameterized by time and possibly, in the parabolic systems, also by space. Such parameterized measures are called Young measures [38], although L.C. Young worked rather with functionals because the measure theory was rather only developing. The spirit

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of Young measures as functionals (allowing more straightforwardly for various generalizations or approximation) is accented in [32].

In abstract evolution problems, the set S where the controls are valued is a compact metric space or, a bit more generally, a so-called Polish space (separable completely metrizable topological space) which is compact. This leads to Young measures parameterized by time valued in conventional (i.e. σ -additive) probability measures supported on S, cf. e.g. [1,2,7,8,13,18,29,36,37]. A modification for metrizable locally compact sets was devised by [9].

Often, S is a subset of an infinite dimensional Banach space. In parabolic problems interpreted as evolution problems on Sobolev spaces over a domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, the set S where the controls are valued is typically a subset of some Lebesgue space over Ω . More specifically, let us consider

$$S_p = \{ u \in L^p(\Omega; \mathbb{R}^m); \ u(x) \in B \text{ for a.a. } x \in \Omega \},$$
(1)

which is compact in its weak (or weak^{*}) topology if $B \subset \mathbb{R}^m$ is bounded and closed; except Remark 4, the set B will always be bounded and p only denotes some (or equally any) number such that $1 \leq p < +\infty$ and wants to emphasize that (except Remark 4) this subset of $L^{\infty}(\Omega; \mathbb{R}^m)$ is considered in the L^p -topology with $p \neq \infty$. The nonlinearities occurring in concrete optimalcontrol problems have typically a local form of the type $u \mapsto h(x, u(x))$ with some Carathéodory mapping $h: \Omega \times \mathbb{R}^m \to \mathbb{R}^k$. Yet, in this space, such mappings are weakly continuous on S_p only if $h(x, \cdot)$ is affine for a.a. $x \in \Omega$. This hidden effect makes the approach from [7,8,29,36,37] in fact very restrictive, as functions of controls which are not affine do not admit continuous extension in terms of Young measures.

A finer convex compactification was devised by Fattorini [19–21], allowing for a general continuous nonlinearities on S_p but using the rather abstract concept of Young measures valued in probability regular finite additive measures "rba" on S_p , or equivalently [32] as standard probability regular countably additive measures "rca" but on the Čech-Stone compactification βS_p of S_p . Such compactification is not metrizable, and one cannot work with conventional sequences but, instead, the general-topological concept of nets and Moore-Smith convergence must be used.

The goal of this paper is to devised a compromising relaxation which admits a wider class of nonlinearities than only affine while still working with conventional σ -additive measures and conventional sequences. For this, a characterization of extreme Young measures together with celebrated Choquet-Bishop-de Leeuw [11,15] theory is used first in the "static" situation in Section 2, and then for the evolution situations parameterized by time in Section 3. Eventually, in Section 4, application to optimal control of parabolic partial differential equations is briefly shown.

2 Young measures and probability measures on them

Let us begin with some definitions and brief presentation of basic needed concepts and facts. An algebra on a set M is a collection of subsets of M closed on the complements and finite unions, including also an empty set. If it is also closed on union of countable number of sets, then it is called an σ -algebra. We denote by C(M) a space of continuous *bounded* function on a topological space M. In fact, it is algebraically also an algebra and, if M is compact, it is a Banach space and, by the classical Riesz theorem, its dual $C(M)^*$ is isometrically isomorphic to the Banach space of Borel measures denoted by rca(M), i.e. of regular bounded countably-additive (so-called σ -additive) set functions on the σ -algebra of Borel subsets of M.

If M is a (not necessarily compact) normal topological space, $C(M)^*$ is isometrically isomorphic to the Banach space of regular bounded finitely additive (not necessarily σ -additive) set functions on the algebra generated by all subsets of M, denoted by rba(M). Actually, $rba(M) = rca(\beta M)$ with βM the Čech-Stone compactification βM of M.

The subsets of rca(M) and rba(M) consisting from positive measures having a unit mass (i.e. probability measures) will be denoted by rca_1^+ and $rba_1^+(M)$, respectively. If C(M) is separable, then the weak* topology on $rca_1^+(M)$ or $rba_1^+(M)$ is metrizable.

If M is a domain in an Euclidean space equipped with the Lebesgue measure (denoted then mostly by $\Omega \subset \mathbb{R}^d$), then $L^p(M; \mathbb{R}^n)$ will denote the Lebesgue space of all measurable \mathbb{R}^n -valued whose *p*-power is integrable.

An important attribute of $\operatorname{rca}_1^+(M)$ and of $\operatorname{rba}_1^+(M)$ is convexity. Let us remind that a point z in a convex set K is called *extreme* in K if there is no open interval in K containing z; in other words, $z = az_1 + (1-a)z_2$ for some $a \in (0,1)$ and some $z_1, z_2 \in K$ implies $z_1 = z_2$. The set of the extreme points will be denoted as $\operatorname{ext} K$. Let us note that the set of all extreme points of a metrizable convex compact (K, ρ) is a Borel set (more precisely a G_{δ} -set), being a countable intersection of open sets as complements to the closed set $\{z \in K; \exists z_1, z_2 \in K : z = \frac{1}{2}z_1 + \frac{1}{2}z_2 \& \rho(z_1, z_2) \leq \epsilon\}$ for $\epsilon > 0$.

One of the important ingredients used below is that every point z of a convex compact set K is an average of the extreme points according to a certain probability measure μ supported on extreme points in the sense

$$\forall z \in K \; \exists \, \mu \in \operatorname{rca}_1^+(\operatorname{ext} K) \; \forall \, f \in \operatorname{Aff} K : \quad f\Big(\int_{\operatorname{ext} K} \widetilde{z} \, \mu(\mathrm{d} \widetilde{z})\Big) = \int_{\operatorname{ext} K} f(\widetilde{z}) \, \mu(\mathrm{d} \widetilde{z}) = f(z) \,, \quad (2)$$

where Aff K denotes the set of all affine continuous functions on A. In other words, any $z \in K$ is a so-called barycentre of a probability measure supported on ext K. This is known as a *Choquet-Bishop-de Leeuw representation theorem* [11,15]; cf. also e.g. [3,27]. Recall that $z \in K$ is called a barycentre of $\mu \in \operatorname{rca}_1^+(K)$ if $f(z) = \int_K f(\tilde{z}) \,\mu(d\tilde{z})$ for any affine continuous $f: K \to \mathbb{R}$, so that the last equation in (2) says that z is a barycentre of μ .

We now briefly recall the classical Young measures. We consider the set S_p from (1) with $B \subset \mathbb{R}^m$ compact, not necessarily convex. As B is bounded, the set S_p actually does not depend on $1 \leq p \leq +\infty$. When endowed with the norm topology from $L^p(\Omega; \mathbb{R}^m)$, it becomes a normal topological space. This topology is separable and does not depend on $1 \leq p < \infty$, but S_{∞} has a strictly finer (and non-separable) topology.

The notation $L^{\infty}_{w*}(\Omega; X^*)$ stands for the Banach space of weakly^{*} measurable mappings ν : $\Omega \to X^*$ for some Banach space X, i.e. $x \mapsto \langle \nu(x), h(x) \rangle$ is measurable for any $h \in L^1(\Omega; X)$. Here we use it for X = C(B) and later also for some subspaces of $C(S_p)$. By the Dunford-Pettis' theorem combined with the mentioned Riesz theorem, $L^1(\Omega; C(B))^* \cong L^{\infty}_{w*}(\Omega; \operatorname{rca}(B))$. For $\nu \in L^{\infty}_{w*}(\Omega; \operatorname{rca}(B))$, it is customary to write ν_x instead of $\nu(x)$. We define the set of Young measures

$$\mathcal{Y}(\Omega; B) := \left\{ \nu \in L^{\infty}_{w*}(\Omega; \operatorname{rca}(B)); \ \nu_x \in \operatorname{rca}^+_1(B) \text{ for a.a. } x \in \Omega \right\}.$$
(3)

It is obvious that $\mathcal{Y}(\Omega; B)$ is convex, weakly^{*} compact, and metrizable. The set S_p is embedded into $\mathcal{Y}(\Omega; B)$ by the mapping $[\delta(u)]_x = \delta_{u(x)}$ where $\delta_s \in \operatorname{rca}_1^+(B)$ denotes the Dirac measure supported at $s \in B$. By a direct construction of fast oscillating sequences, one can show that this embedding is weakly^{*} dense and thus, in particular, $\mathcal{Y}(\Omega; B)$ is separable. It is important that the embedding $\delta : S_p \to \mathcal{Y}(\Omega; B)$ is even (strong,weak^{*})-homeomorphical with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ for any $1 \leq p < +\infty$, although not for $p = +\infty$. Here we note that $\delta(u_k) \to \delta(u)$ weakly^{*} in $\mathcal{Y}(\Omega; B)$ implies, when tested by

$$\left(h:(x,z)\mapsto |z-u(x)|^p\right)\in L^1(\Omega;C(B))\,,\tag{4}$$

that $\langle \delta(u_k) - \delta(u), h \rangle = \int_{\Omega} |u_k - u|^p dx \to 0$. Let us remind that *B* is considered bounded (and closed) and $p < +\infty$, otherwise the inclusion in (4) would not hold.

The other important ingredient used below is that each extreme point $\nu = {\nu_x}_{x \in \Omega}$ in the set of all Young measures $\mathcal{Y}(\Omega; B)$ is composed from Diracs, i.e. $\nu_x = \delta_{u(x)}$ for a.a. $x \in \Omega$ with some $u \in S_p$; see Berliocchi and Lasry [10, Proposition II.3] or Castaing and Valadier [14, Thm. IV.15], cf. also [24]. The extreme points of $\mathcal{Y}(\Omega; B)$ are thus a dense and, as mentioned above, G_{δ} -set in $\mathcal{Y}(\Omega; B)$.

Lemma 1. Let $B \subset \mathbb{R}^m$ is compact and $1 \leq p < +\infty$. Any Young measure $\nu \in \mathcal{Y}(\Omega; B)$ can be represented (in a non-unique way in general) by a probability measure μ supported on S_p . More specifically,

$$\forall \nu \in \mathcal{Y}(\Omega; B) \quad \exists \mu \in \operatorname{rca}_{1}^{+}(\mathcal{Y}(\Omega; B)), \quad \operatorname{supp} \mu \subset \mathfrak{d}(S_{p}) \quad \forall h \in L^{1}(\Omega; C(B)):$$
$$\int_{\Omega} \int_{B} h(x, z) \,\nu_{x}(\mathrm{d}z) \,\mathrm{d}x = \int_{S_{p}} \int_{\Omega} h(x, u(x)) \,\mathrm{d}x \,\mu(\mathrm{d}u) \,, \tag{5}$$

where we identified $\mu(\mathfrak{d}(A))$ and $\mu(A)$ for $A \subset S_p$; here $\mathfrak{d} : S_p \to \operatorname{rca}_1^+(S_p) : u \mapsto \mathfrak{d}_u$ with $\mathfrak{d}_u \in \operatorname{rca}_1^+(S_p)$ denoting the Dirac measure supported at $u \in S_p$. Thus, in fact, $\mu \in \operatorname{rca}_1^+(S_p)$.

This means that \mathfrak{d}_u as a functional on $C(S_p)$ defined by $v \mapsto v(u)$ for any $v \in C(S_p)$, in contrast to $\delta(u) = \{\delta_{u(x)}\}_{x \in \Omega} \in \mathcal{Y}(\Omega; B)$.

Proof of Lemma 1. In view of the abstract result (2), i.e. the Choquet-Bishop-de Leeuw representation theorem applied on the convex compact $K = \mathcal{Y}(\Omega; B)$. We thus obtain a probability measure $\mu \in \operatorname{rca}_1^+(\mathcal{Y}(\Omega; B))$ supported on $\operatorname{ext}(\mathcal{Y}(\Omega; B))$. As mentioned above, $\operatorname{ext}(\mathcal{Y}(\Omega; B)) = \mathfrak{d}(S_p)$. Since $\mathcal{Y}(\Omega; B)$ is metrizable, $\mathfrak{d}(S_p)$ is a Borel subset in $\mathcal{Y}(\Omega; B)$, and μ is a Borel measure on it. Also realize that any weakly* continuous affine function on $K = \mathcal{Y}(\Omega; B) \subset L^{\infty}_{w*}(\Omega; \operatorname{rca}(B)) \cong L^1(\Omega; C(B))^*$ is of the form $\nu \mapsto \int_{\Omega} \int_B h(x, z) \nu_x(\mathrm{d}z) \,\mathrm{d}x$ for some $h \in L^1(\Omega; C(B))$. Thus (2) yields (5).

It is important that, as mentioned above, the embedding \mathfrak{d} : S_p is (strong,weak*)homeomorphical so that the weak* topology on $\mathfrak{d}(S_p)$ induces just the strong topology on S_p . Thus the measure on $\mathfrak{d}(S_p)$ induces a Borel measure on S_p , referring to the Borel σ -algebra on S_p with respect to the L^p -norm, $p < +\infty$, such a measure on S_p being again denoted by μ . \Box

Let us still remind a canonical construction of compactifications, here applied to S_p . For this, we consider a general complete closed sub-ring \mathscr{R} of $C(S_p)$ containing constants. Every such a ring \mathscr{R} is also a commutative Banach algebra and determines a compactification $\gamma_{\mathscr{R}}S_p$ of S_p as a subset of \mathscr{R}^* endowed with the weak* topology consisting of multiplicative means, i.e.

$$\gamma_{\mathscr{R}}S_p := \left\{ \mu \in \mathscr{R}^*; \ \|\mu\| = 1, \ \mu(1) = 1, \ \forall v_1, v_2 \in \mathscr{R}: \ \left\langle \mu, v_1 v_2 \right\rangle = \left\langle \mu, v_1 \right\rangle \left\langle \mu, v_2 \right\rangle \right\}$$

This means $\gamma_{\mathscr{R}}S_p$ is compact when endowed by the weak* topology of \mathscr{R}^* and the embedding $e: S_p \to \gamma_{\mathscr{R}}S_p$ defined by $\langle e(u), v \rangle = v(u)$ is homeomorphical. Let us recall that a ring is called complete if it separates closed subsets of S_p from points in S_p . This means that, for any $A \subset S_p$ closed and $u_0 \notin S_p \setminus A$, there is $v \in C(\mathcal{Y}(\Omega; B))|_{S_p}$ such that $v(u_0) = 0$ and v(A) = 1. The functionals from $\gamma_{\mathscr{R}}S_p$ are positive in the sense that $\mu(v) \geq 0$ for any $v \in \mathscr{R}$ with $f(\cdot) \geq 0$ on S. Each function from \mathscr{R} admits a (uniquely determined) continuous extension on $\gamma_{\mathscr{R}}S_p$. Thus \mathscr{R} is isometrically isomorphic with the space $C(\gamma_{\mathscr{R}}S_p)$. By the Riesz theorem, $\mathscr{R}^* \cong \operatorname{rca}(\gamma_{\mathscr{R}}S_p)$ and

$$\langle \mu, v \rangle = \int_{\gamma_{\mathscr{R}} S_p} \overline{v}(s) \mu(\mathrm{d}s) \quad \text{with } \overline{v} \in C(\gamma_{\mathscr{R}} S_p) \text{ a continuous extension of } v \in \mathscr{R}.$$
 (6)

As already mentioned in Sect. 1, the construction $\operatorname{rba}_1^+(S_p) \cong \operatorname{rca}_1^+(\beta S_p)$ is non-metrizable (and thus rather constructive) because it is based on the nonseparable space of test functions $C(S_p)$. It is thus desirable to consider some subspace of $C(S_p)$ which would be separable but still bigger than $\operatorname{Aff}(S_p)$. Motivated by Lemma 1, we take the choice

$$\mathscr{R} = C(\mathcal{Y}(\Omega; B))\big|_{S_p} = \left\{ v \in C(S_p); \ \exists \, \overline{v} \in C(\mathcal{Y}(\Omega; B)) : \ v = \overline{v} \circ \delta \right\}.$$
(7)

This is obviously a sub-ring of $C(S_p)$ containing constants.

Lemma 2. The ring \mathscr{R} from (7) is complete and separable and the compactification $\gamma_{\mathscr{R}}S_p$ is metrizable and homeomorphical with $\mathcal{Y}(\Omega; B)$.

Proof. Since dist $(u, A) = \epsilon > 0$, one can take $v(u) = \min(\epsilon, ||u-u_0||_{L^p(\Omega;\mathbb{R}^m)}^p)$ which can indeed be continuously extended on $\mathcal{Y}(\Omega; B)$ as $v(\nu) = \min(\epsilon, \langle \nu, h \rangle)$ with h from (4); note that such his an integrand from $L^1(\Omega; C(B))$. In fact, $C(\mathcal{Y}(\Omega; B))|_{S_p}$ is the smallest closed ring containing $\Psi(L^1(\Omega; C(B)))$, where Ψ is a linear operator from $L^1(\Omega; C(B))$ to $C(S_p)$ defined by

$$\Psi h: u \mapsto \int_{\Omega} h(x, u(x)) \,\mathrm{d}x \,. \tag{8}$$

Let us also remind that $\mathcal{Y}(\Omega; B)$ is a metrizable separable compact. Hence $C(\mathcal{Y}(\Omega; B))$ itself is separable. The separability holds also for (7).

Since \mathscr{R} is separable, bounded sets in its dual endowed with the weak* topology (and in particular $\gamma_{\mathscr{R}}S_p$) are metrizable. The homeomorphism between $\gamma_{\mathscr{R}}S_p$ and $\mathcal{Y}(\Omega; B)$ is realized by the adjoint operator to the embedding of $\Psi(L^1(\Omega; C(B)))$ into $C(\mathcal{Y}(\Omega; B))|_{S_p}$; here it is important that $\Psi(L^1(\Omega; C(B)))$ is a so-called convexifying subspace of $C(S_p)$ in the sense that any $u_1, u_2 \in S_p$ admits a sequence $\{u_k\}_{k \in \mathbb{N}}$ such that $f(u_1) + f(u_2) = 2 \lim_{k \to \infty} f(u_k)$ for any $f \in \Psi(L^1(\Omega; C(B)))$, cf. [32, Sect. 2.2 and 3.1].

Example 1. Let us still illustrate Lemma 1 on a piece-wise homogeneous two-atomic Young measure

$$\nu_x = \begin{cases} \frac{1}{2} \delta_{u_1(x)} + \frac{1}{2} \delta_{u_2(x)} & \text{for } x \in A ,\\ \frac{1}{4} \delta_{u_1(x)} + \frac{3}{4} \delta_{u_2(x)} & \text{for } x \in \Omega \backslash A , \end{cases}$$
(9)

with some $u_1 \neq u_2$ and $A \subset \Omega$ measurable. Then μ from Lemma 1 takes (for example) the form

$$\mu = a\mathfrak{d}_{u_{11}} + \left(\frac{1}{2} - a\right)\mathfrak{d}_{u_{12}} + \left(\frac{1}{4} - a\right)\mathfrak{d}_{u_{21}} + \left(\frac{1}{4} + a\right)\mathfrak{d}_{u_{22}} \tag{10}$$

with an arbitrary parameter $0 \le a \le 1/4$ and with $u_{11} = u_1, u_{22} = u_2$,

$$u_{12}(x) = \begin{cases} u_1(x), & \text{and} & u_{21}(x) = \begin{cases} u_2(x) & \text{for } x \in A, \\ u_1(x) & \text{for } x \in \Omega \setminus A. \end{cases}$$

For a = 0 and for a = 1/4, the four-atomic measure (10) degenerates to only three-atomic ones. In particular, it illustrates non-uniqueness of the probability measure from Lemma 1. Even more, (10) does not cover all representations of ν from (9). Although these measures cannot be distinguished when tested by test functions from $\Psi(L^1(\Omega; C(B)))$, they can be distinguished from each other when tested by functions from $C(\mathcal{Y}(\Omega; B))|_{S_p}$; for example, if ρ is a metric of $\mathcal{Y}(\Omega; B)$, one can take $\varrho(\cdot, \mu)|_{S_p}$ with μ from (10) for some specific $0 \le a \le 1/4$. **Remark 1** (An approximation of Young measures). Various numerical schemes have been devised to numerical approximation of Young measures, cf. [32] for a survey. Lemma 1 inspires an approximation by a convex combination of elements from S_p . Actually, this sort of approximation is supported by arguments that each element of convex compact sets (i.e. here the set of Young measures) can be approximated by a convex combination of extreme points due to the celebrated Kreĭn-Milman theorem. Here one can consider a fixed countable collection $\{u_l\}_{l\in\mathbb{N}}$ dense in S_p and, for any $\ell \in \mathbb{N}$, define the finite-dimensional convex subset of $\operatorname{rca}_1^+(S_p)$ as

$$\left\{\mu = \sum_{l=1}^{\ell} a_l \mathfrak{d}_{u_l}; \ \exists \{a_l\}_{l=1}^{\ell}, \ a_l \ge 0, \ \sum_{l=1}^{\ell} a_l = 1 \right\}.$$
 (11)

This approximation, devised by V.M. Tikhomirov [35], was used e.g. in [5,6] under the name a mix of controls. Passing $\ell \to \infty$, the sets (11) increase and their union is dense in $\operatorname{rca}_1^+(S_p)$ due to the weak* density of $\{\mathfrak{d}_{u_l}\}_{l\in\mathbb{N}}$. This allows for the convergence proof behind this sort of convex approximation.

Remark 2 (Special probability measures on S_p). We do not claim that each $\mu \in \operatorname{rca}_1^+(S_p)$ corresponds to some $\nu \in \mathcal{Y}(\Omega; B)$ via (5). For further purposes, let us denote the set of such "special" μ 's by

$$\operatorname{srca}_{1}^{+}(S_{p}) := \left\{ \mu \in \operatorname{rca}_{1}^{+}(S_{p}); \exists \nu \in \mathcal{Y}(\Omega; B) \quad \forall h \in L^{1}(\Omega; C(B)) : \int_{S_{p}} [\Psi h](u) \, \mu(\mathrm{d}u) = \int_{\Omega} \int_{B} h(x, z) \, \nu_{x}(\mathrm{d}z) \, \mathrm{d}x \right\}.$$
(12)

Although this set is defined only very implicitly, we can nevertheless see that the set $\operatorname{srca}_1^+(S_p)$ is convex. Indeed, for $\mu_1, \mu_2 \in \operatorname{rca}_1^+(S_p)$, we can take $\nu_1, \nu_2 \in \mathcal{Y}(\Omega; B)$ such that, for i = 1, 2, it holds

$$\forall h \in L^1(\Omega; C(B)): \quad \int_{S_p} [\Psi h](u) \,\mu_i(\mathrm{d}u) = \int_{\Omega} \int_B h(x, z) \, [\nu_i]_x(\mathrm{d}z) \,\mathrm{d}x \,. \tag{13}$$

As $\mathcal{Y}(\Omega; B)$ is convex, also $\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 \in \mathcal{Y}(\Omega; B)$. Thus $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies the identity in (12).

Example 2 (Special functions from $C(\mathcal{Y}(\Omega; B))|_{S_p}$). Using Ψ from (8), let us consider

$$\mathscr{R}_0 := \Big\{ v \in C(S_p); \ \exists m, n \in \mathbb{N}, \ f_{ij} \in C(\mathbb{R}), \ h_{ij} \in L^1(\Omega; C(B)): \ v = \sum_{i=1}^m \prod_{j=1}^n f_{ij} \circ \Psi h_{ij} \Big\}.$$
(14)

This is a complete separable ring. Each $v \in \mathscr{R}_0$ admits a weakly^{*} continuous extension \overline{v} on $\mathcal{Y}(\Omega; B)$, given explicitly by

$$\overline{v}(\nu) = \sum_{i=1}^{m} \prod_{j=1}^{n} f_{ij} \left(\int_{\Omega} \int_{B} h_{ij}(x, z) \,\nu_x(\mathrm{d}z) \mathrm{d}x \right)$$

for all $\nu \in \mathcal{Y}(\Omega; B)$. Thus $\mathscr{R}_0 \subset C(\mathcal{Y}(\Omega; B))|_{S_p}$. Moreover, by Lemma 1, there is $\mu \in \operatorname{rca}_1^+(S_p)$ depending on ν such that

$$\overline{v}(\nu) = \sum_{i=1}^{m} \prod_{j=1}^{n} f_{ij} \left(\int_{S_p} \left[\Psi h_{ij} \right](u) \, \mu(\mathrm{d}u) \right).$$

Since $C(\mathcal{Y}(\Omega; B))|_{S_p}$ is the smallest closed ring containing the linear space $\Psi(L^1(\Omega; C(B))), \mathscr{R}_0$ is dense in $C(\mathcal{Y}(\Omega; B))|_{S_p}$.

3 Young measures parameterized by time

We will now extend the construction from Sect. 2 to be applicable for evolution problems. To this goal, we consider a time interval I = [0, T] with some fixed time horizon T > 0. We will use the standard notation $L^p(I; X)$ for the Lebesgue-Bochner space of abstract functions $I \to X$ whose X-norm is valued in $L^p(I)$. From now on, let us agree to use boldface fonts for functions of time valued in spaces functions (or their duals) on Ω , or also functions of such arguments. We will consider the set of "admissible controls"

$$\boldsymbol{U}_{\mathrm{ad}} = \left\{ u \in L^{p}(I \times \Omega; \mathbb{R}^{m}); \ u(\cdot) \in B \text{ a.e. on } I \times \Omega \right\} \\
\cong \left\{ \boldsymbol{u} \in \boldsymbol{L}^{p}(I; L^{p}(\Omega; \mathbb{R}^{m})); \ \boldsymbol{u}(\cdot) \in S_{p} \text{ a.e. on } I \right\}$$
(15)

with S_p from (1) and with identifying $\boldsymbol{u}(t, \cdot)$ and $\boldsymbol{u}(t) \in S_p$.

Let us start with a general construction, advancing the scheme devised by H. Fattorini who used the non-separable space $L^1(I; C(S_p))$ of test functions. Here, instead of the whole (nonseparable) ring $C(S_p)$, we consider a general complete closed sub-ring \mathscr{R} of $C(S_p)$) containing constants, and then we can consider the test-function space $L^1(I;\mathscr{R})$. Instead of $L^1(I; C(S_p))$, we now suggest to use $L^1(I;\mathscr{R})$. If \mathscr{R} is also separable, also $L^1(I;\mathscr{R})$ is separable and both $\gamma_{\mathscr{R}}S_p$ and bounded sets in $L^1(I;\mathscr{R})^*$ are metrizable. Then, by Dunford-Pettis' theorem (as used in [21, Thms. 12.2.4 and 12.2.11]), it holds

$$\boldsymbol{L}^{1}(I;\mathscr{R})^{*} \cong \boldsymbol{L}^{\infty}_{w*}(I;\mathscr{R}^{*}) \cong \boldsymbol{L}^{\infty}_{w*}(I;\operatorname{rca}(\gamma_{\mathscr{R}}S_{p})).$$
(16)

Like before for $\nu(x) \equiv \nu_x$, we use the convention $\nu(t) \equiv \nu_t$. The duality between $L^{\infty}_{w*}(I; \operatorname{rca}(\gamma_{\mathscr{R}}S_p))$ and $L^1(I; \mathscr{R})$ is then

$$\langle \boldsymbol{\nu}, \boldsymbol{h} \rangle = \int_0^T \int_{\gamma_{\mathscr{R}} S_p} \overline{\boldsymbol{h}}(t, \boldsymbol{s}) \, \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \mathrm{d}t \quad \text{for } \boldsymbol{h} \in \boldsymbol{L}^1(I; \mathscr{R}),$$
(17)

where, like in (6), $\overline{h}(t, \cdot) \in C(\gamma_{\mathscr{R}}S_p)$ is the (uniquely defined) continuous extension of $h(t, \cdot) \in \mathscr{R}$ on $\gamma_{\mathscr{R}}S_p$.

Like in (3), we further define the set of Young measures with values supported on $\gamma_{\mathscr{R}}S_p$ as

$$\boldsymbol{\mathcal{Y}}(I;\gamma_{\mathscr{R}}S_p) := \left\{ \boldsymbol{\nu} \in \boldsymbol{L}^{\infty}_{w*}(I;\operatorname{rca}(\gamma_{\mathscr{R}}S_p)); \ \boldsymbol{\nu}_t \in \operatorname{rca}^+_1(\gamma_{\mathscr{R}}S_p) \ \text{for a.a.} \ t \in I \right\}.$$
(18)

Like (17), the embedding $\boldsymbol{\delta} : \boldsymbol{U}_{ad} \to \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)$ is defined as $\boldsymbol{\delta}(\boldsymbol{u}) = \{\boldsymbol{\delta}_{\boldsymbol{u}(t)}\}_{t \in I}$, i.e.

$$\langle \boldsymbol{\delta}(\boldsymbol{u}), \boldsymbol{h} \rangle = \int_0^T \boldsymbol{h}(t, \boldsymbol{u}(t)) \, \mathrm{d}t \quad \text{for } \boldsymbol{h} \in \boldsymbol{L}^1(I; \mathscr{R}) \,.$$
 (19)

Proposition 1. Let $\mathscr{R} \subset C(S_p)$ be a separable complete closed sub-ring \mathscr{R} of $C(S_p)$ containing constants. Then the set of Young measures $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ from (18) is convex, weakly* sequentially compact and separable. Moreover, the embedding δ from (19) is (strong, weak*)-continuous if U_{ad} is equipped with the strong topology of $L^1(I \times \Omega; \mathbb{R}^m)$ and $\delta(U_{ad})$ is sequentially weakly* dense in $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$.

Proof. The convexity and weak*-compactness of $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p) \subset \mathbf{L}_{w*}^{\infty}(I; \operatorname{rca}(\gamma_{\mathscr{R}}S_p))$ is obvious from the definition of the convex, closed, bounded set (18). The continuity of $\boldsymbol{\delta}$ follows from the continuity of the Nemytskiĭ mappings induced by the integrands $\boldsymbol{h} \in \mathbf{L}^1(I; \mathscr{R})$ in (19).

The metrizability of the weak^{*} topology on $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ follows from the separability of $L^1(I; \mathscr{R})$, relying on the separability of \mathscr{R} .

The density of $\delta(S_p)$, i.e. the attainability of a general $\boldsymbol{\nu} \in \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}}S_p)$ in the sense that $\boldsymbol{\nu} = \mathbf{w}^*$ -lim_{k\to\infty} $\delta(\boldsymbol{u}_k)$ for some sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset \boldsymbol{U}_{ad}$, follows by the standard arguments used for conventional Young measures, i.e. by an explicit construction of a sequence oscillating fast in time, cf. [21, Thm. 12.6.7] or [32, Thm. 3.6]. For this, an essential fact is that the Lebesgue measure on I is non-atomic. In particular, as S_p is separable, $\boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}}S_p)$ is separable, too.

We now use this general construction for the special choice $\mathscr{R} = C(\mathcal{Y}(\Omega; B))|_{S_p}$ as in (7). We already showed in Lemma 2 that this ring is complete. We thus consider the Banach space of test functions $\mathbf{L}^1(I; C(\mathcal{Y}(\Omega; B))|_{S_p})$ and embed \mathbf{U}_{ad} into the dual of this test-function space as in (16) with $(C(\mathcal{Y}(\Omega; B))|_{S_p})^* \cong \operatorname{rca}(\mathcal{Y}(\Omega; B))$, i.e. into

$$\left(\boldsymbol{L}^{1}(I; C(\mathcal{Y}(\Omega; B))|_{S_{p}}\right)^{*} \cong \boldsymbol{L}_{w*}^{\infty}(I; C(\mathcal{Y}(\Omega; B))|_{S_{p}}^{*}) \cong \boldsymbol{L}_{w*}^{\infty}(I; \operatorname{rca}(\mathcal{Y}(\Omega; B))).$$
(20)

By exploitation of Proposition 1 with Lemma 2, we have

$$\boldsymbol{\mathcal{Y}}(I; \mathcal{Y}(\Omega; B)) = \{ \boldsymbol{\nu} \in \boldsymbol{L}_{w*}^{\infty}(I; \operatorname{rca}(\mathcal{Y}(\Omega; B))); \quad \boldsymbol{\nu}(t) \in \operatorname{rca}_{1}^{+}(\mathcal{Y}(\Omega; B)) \text{ for a.a. } t \in I \}.$$
(21)

Proposition 2. The set of Young measures $\mathcal{Y}(I; \mathcal{Y}(\Omega; B))$ is convex and a separable metrizable compact, the embedding $\boldsymbol{\delta} : \boldsymbol{U}_{ad} \to \mathcal{Y}(I; \mathcal{Y}(\Omega; B))$ again defined as (19) is (strong,weak*)homeomorphical if \boldsymbol{U}_{ad} is equipped with the strong topology of $L^p(I \times \Omega; \mathbb{R}^m)$ with any $1 \le p < +\infty$ and $\boldsymbol{\delta}(\boldsymbol{U}_{ad})$ is weakly* dense in $\mathcal{Y}(I; \mathcal{Y}(\Omega; B))$.

Proof. Most of the assertion follows from Proposition 1 as a special case for the choice $\mathscr{R} = C(\mathcal{Y}(\Omega; B))|_{S_n}$.

Considering a sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}} \subset \boldsymbol{U}_{ad}$ and $\boldsymbol{u} \in \boldsymbol{U}_{ad}$ such that $\boldsymbol{\delta}(\boldsymbol{u}_k) \to \boldsymbol{\delta}(\boldsymbol{u})$ weakly^{*}, we can prove that $\boldsymbol{u}_k \to \boldsymbol{u}$ strongly in $L^p(I \times \Omega; \mathbb{R}^m)$. Indeed, we can take $\boldsymbol{h} \in \boldsymbol{L}^1(I; C(\mathcal{Y}(\Omega; B))|_{S_p})$ defined as

$$\boldsymbol{h}(t,\boldsymbol{s}) = \int_{\Omega} \left| \boldsymbol{s}(x) - [\boldsymbol{u}(t)](x) \right|^p \mathrm{d}x.$$

This gives $\langle \boldsymbol{\delta}(\boldsymbol{u}_k) - \boldsymbol{\delta}(\boldsymbol{u}), \boldsymbol{h} \rangle = \|\boldsymbol{u}_k - \boldsymbol{u}\|_{L^p(I \times \Omega; \mathbb{R}^m)}^p \to 0$. Here it is important that $\boldsymbol{h}(t, \boldsymbol{s}) = \int_{\Omega} h(t, x, \boldsymbol{s}(x)) dx$ for $h(t, x, z) = |z - [\boldsymbol{u}(t)](x)|^p$ with $h(t, \cdot, \cdot) \in L^1(\Omega; C(B))$ so that the functional $\boldsymbol{h}(t, \cdot) : S_p \to \mathbb{R}$ can be continuously extended on $\mathcal{Y}(\Omega; B)$, namely $\nu \mapsto \int_{\Omega} \int_B h(t, x, z) \nu_x(dz) dt$.

The convex compactification of U_{ad} from Proposition 2 is coarser than the (non-metrizable) Fattorini's construction mentioned in Sect. 1.

Remark 3 (Numerical approximation). The explicit characterization of convex compactifications may suggest some approximation strategies. Just as an example in the particular case (21), one can apply the mentioned extreme-point-characterization arguments to see that $ext(\mathcal{Y}(I;\mathcal{Y}(\Omega;B)) = \{\nu : I \mapsto \mathcal{Y}(\Omega;B) \text{ weakly}^* \text{ measurable}\}$ and then to combine the Kreĭn-Milman theorem yielding an approximation of $\boldsymbol{\nu}$ in the form $\boldsymbol{\nu}(t) \sim \sum_i \alpha_i \nu_i(t)$ with some $\nu_i : I \to \mathcal{Y}(\Omega;B)$ and Remark 1 yielding an approximation $\nu_i(t) \sim \sum_j \beta_{ij} \mathfrak{d}_{u_{ij}(t)}$ with some $u_{ij}(t) \in S_p$. The non-negative coefficients satisfies $\sum_i \alpha_i = 1$ and $\sum_j \beta_{ij} = 1$. Altogether,

$$\boldsymbol{\nu}(t) \sim \sum_{i} \alpha_{i} \Big(\sum_{j} \alpha_{i} \beta_{ij} \boldsymbol{\mathfrak{d}}_{u_{ij}(t)} \Big) = \sum_{ij} a_{ij} \boldsymbol{\mathfrak{d}}_{u_{ij}(t)} \quad \text{with} \quad a_{ij} = \alpha_{i} \beta_{ij} \,.$$

Note that $\sum_{ij} a_{ij} = \sum_i \alpha_i (\sum_j \beta_{ij}) = \sum_i \alpha_i = 1$ and we obtain a mix of controls $[u_{ij}(t)](x) \equiv u_{ij}(t,x)$ in the spirit of Remark 1. This approximation is convex, the analytical details about such

approximation deserving still some investigation. Of course, one can also think about combination of some interpolation over time of some convex combination with coefficients depending on $t \in I$. This expectedly opens wide menagerie of possible numerical strategies, which remains out of the scope of this article, however.

Remark 4 (One generalization). The above construction can be generalized for $B \subset \mathbb{R}^m$ unbounded by considering a general S_p bounded in $L^p(\Omega; \mathbb{R}^m)$ with some specific $1 \leq p < \infty$ fixed but not necessarily bounded in $L^{\infty}(\Omega; \mathbb{R}^m)$. Instead of the conventional Young measures $\mathcal{Y}(\Omega; B)$, we can then consider so-called DiPerna-Majda measures $\mathrm{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$ induced by test-functions of the form $g(x)v(z)(1+|z|^p)$ with $g \in C(\overline{\Omega})$ and v ranging over some complete separable ring $\mathcal{R} \subset C(\mathbb{R}^m)$ containing constants. More specifically, $\mathrm{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$ is a convex subset of the Radon measures on $\overline{\Omega} \times \gamma_{\mathcal{R}} \mathbb{R}^m$ attainable from S_p when embedded into $\overline{\Omega} \times \gamma_{\mathcal{R}} \mathbb{R}^m$ via the mapping $\delta : u \mapsto (h \mapsto \int_{\Omega} h(x, u(x))(1+|u(x)|^p) \, dx)$ with $h \in C(\Omega) \otimes \mathcal{R}$. The embedding $\delta : L^p(\Omega; \mathbb{R}^m) \to \mathrm{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$ is homeomorphical and $\mathrm{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$ is convex, metrizable, and locally compact, having all extreme points of the form of Diracs δ_s with $s \in S_p$, cf. [25, 26] for $B = \mathbb{R}^m$. As S_p is bounded in $L^p(\Omega; \mathbb{R}^m)$, the closure of $\delta(S_p)$ is compact. Like $\mathcal{Y}(\Omega; B)$, we thus obtained a convex, metrizable separable compact with extreme points being Diracs, so that the above arguments can be adopted to this situation, too.

4 Application: optimal control of parabolic systems

Let us briefly outline application to an optimal control of a system of n semilinear parabolic differential equations. We confine ourselves on homogeneous Dirichlet conditions, and use the standard notation $H_0^1(\Omega; \mathbb{R}^n)$ for the Sobolev space of functions $\Omega \to \mathbb{R}^n$ whose distributional derivative is in $L^2(\Omega; \mathbb{R}^{d \times n})$ and traces on the boundary Γ of Ω are zero, and similarly $H^1(I; X)$ is a Bochner-Sobolev space of functions $I \to X$ whose distributional derivative is in the Bochner space $L^2(I; X)$. The dual of $H_0^1(\Omega; \mathbb{R}^n)$ is denoted standardly as $H^{-1}(\Omega; \mathbb{R}^n)$. Moreover, we will use the abbreviation $H^1(I; V, V^*) = L^2(I; V) \cap H^1(I; V^*)$ for a Banach space V. We will use it for $V = H_0^1(\Omega; \mathbb{R}^n)$. We will also use the notation $\mathcal{L}(V)$ for the space of linear bounded operators from V to $V \cong V^*$. We then consider an initial-boundary value problem (with Γ denoting the boundary of Ω with the unit outward normal \vec{n}):

with $\varphi: I \times L^2(\Omega; \mathbb{R}^n) \times L^{\infty}(\Omega; \mathbb{R}^m) \to \mathbb{R}$ and $f: I \times L^2(\Omega; \mathbb{R}^n) \times L^{\infty}(\Omega; \mathbb{R}^m) \to L^2(\Omega; \mathbb{R}^n)$. In view of Remark 2, the spatially nonlocal right-hand side of the controlled system can involve an integral over Ω so that we could speak rather about a parabolic integro-differential system.

The relaxation by means of the Young-type measures from Section 3 (similarly as from [19,21]) records fast oscillations in time but not in space, in contrast to conventional the conventional Young measures on $I \times \Omega$ which record fast oscillations simultaneously in time and in space. Also,

the former relaxation allows for a bit more comprehensive optimality conditions than conventional Young measures on $I \times \Omega$, cf. Sect. 5 below. To perform our relaxation, we consider a separable sub-ring \mathscr{R} of $C(S_p)$ with S_p from (1) as in Sect. 3 and qualify the nonlinearities involving the control variable as

$$\forall \boldsymbol{y} \in C(I; L^{2}(\Omega; \mathbb{R}^{m})), \quad \boldsymbol{v} \in H^{1}(\Omega; \mathbb{R}^{m}):$$

$$\varphi \circ \boldsymbol{y} : (t, \boldsymbol{s}) \mapsto \varphi(t, \boldsymbol{y}(t), \boldsymbol{s}) \in \boldsymbol{L}^{1}(I; \mathscr{R}) \quad \text{and}$$
(23a)

$$\langle \boldsymbol{f} \circ \boldsymbol{y}, \boldsymbol{v} \rangle : (t, \boldsymbol{s}) \mapsto \langle \boldsymbol{f}(t, \boldsymbol{y}(t), \boldsymbol{s}), \boldsymbol{v} \rangle \in \boldsymbol{L}^{1}(I; \mathscr{R}).$$
 (23b)

Then (22) allows for a continuous extension on the set of the relaxed controls $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ from Proposition 2 as:

$$\begin{array}{ll}
\text{Minimize} & \int_{0}^{T} \int_{\gamma_{\mathscr{R}} S_{p}} \overline{\varphi}(t, \boldsymbol{y}(t, \cdot), \boldsymbol{s}) \boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) \mathrm{d}t + \int_{\Omega} \phi(\boldsymbol{y}(T)) \, \mathrm{d}x \\
\text{subject to} & \forall \boldsymbol{v} \in \boldsymbol{H}^{1}(I; L^{2}(\Omega; \mathbb{R}^{m})) \cap \boldsymbol{L}^{2}(I; H_{0}^{1}(\Omega; \mathbb{R}^{m})), \, \boldsymbol{v}(T) = 0: \\
& \int_{0}^{T} \left(\int_{\Omega} A \nabla \boldsymbol{y} : \nabla \boldsymbol{v} - \boldsymbol{y} \cdot \frac{\partial \boldsymbol{v}}{\partial t} \, \mathrm{d}x \\
& \quad - \int_{\gamma_{\mathscr{R}} S_{p}} \langle \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \boldsymbol{s}), \boldsymbol{v}(t) \rangle \boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) \right) \mathrm{d}t = \int_{\Omega} \boldsymbol{y}_{0} \cdot \boldsymbol{v}(0) \, \mathrm{d}x, \\
& \quad \boldsymbol{y} \in \boldsymbol{H}^{1}(I; H_{0}^{1}(\Omega; \mathbb{R}^{n}), H^{-1}(\Omega; \mathbb{R}^{n})), \quad \boldsymbol{\nu} \in \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_{p}),
\end{array} \right\}$$

$$(24)$$

where $\overline{\varphi}(t, \boldsymbol{y}(t, \cdot), \cdot)$ and $\overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \cdot)$ denote the (uniquely defined) continuous extension of $\varphi(t, \boldsymbol{y}(t, \cdot), \cdot)$ and $\boldsymbol{f}(t, \boldsymbol{y}(t), \cdot)$, respectively. The integral identity in (24) is a weak formulation of the initial-boundary value in (22) arisen by applying once Green formula in space with using the boundary conditions and by-part integration in time with using the initial condition.

We will further assume the following "semi-monotonicity" condition for $-f(t, \cdot, s)$:

$$\exists a_1 \in L^2(I) \quad \forall t \in I \quad \forall \boldsymbol{r}_1, \boldsymbol{r}_2 \in H^1_0(\Omega; \mathbb{R}^n) \quad \forall \boldsymbol{s} \in L^\infty(\Omega; \mathbb{R}^m) : \\ \int_{\Omega} \left(\boldsymbol{f}(t, \boldsymbol{r}_1, \boldsymbol{s}) - \boldsymbol{f}(t, \boldsymbol{r}_2, \boldsymbol{s}) \right) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2) \, \mathrm{d}\boldsymbol{x} \le a_1(t) \|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^2_{L^2(\Omega; \mathbb{R}^n)}.$$
(25)

The metrizability and separability of $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ allows for stating well-posedness of the relaxed scheme (24) conventionally in terms of sequences:

Proposition 3 (Well-posedness and correctness of (24)). Let (23), (25), $\phi \in C(L^2(\Omega; \mathbb{R}^n))$, $A \in \mathbb{R}^{(n \times n)^2}$ be positive definite, and $\mathbf{y}_0 \in L^2(\Omega; \mathbb{R}^n)$. Then:

- 1. (24) possesses a solution and $\min(24) = \inf(22)$.
- 2. Any infinizing sequence $\{u_k\}_{k\in\mathbb{N}}$ for (22) contains a subsequence which, when embedded into $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ by δ , converges to some ν . Any such limit ν solves the relaxed problem (24).
- 3. Any solution $\boldsymbol{\nu}$ to (24) is attainable by an infinizing sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{N}}$ for (22) in the sense $\boldsymbol{\nu} = w^* \lim_{k \to \infty} \boldsymbol{\delta}(\boldsymbol{u}_k)$.

Sketch of the proof. From positive definiteness of A, (23), and $\mathbf{y}_0 \in L^2(\Omega; \mathbb{R}^n)$, we get existence of weak solution \mathbf{y} of the initial-boundary value in (22). Note that (23b) ensures that all integrals in the integral identity in (24) have a good sense. From (25), we get also uniqueness of this response.

This unique solution thus determines a control-to-state mapping $\boldsymbol{\pi} : \boldsymbol{u} \mapsto \boldsymbol{y}$ from $\boldsymbol{L}^p(I; L^p(\Omega; \mathbb{R}^m))$ to $\boldsymbol{H}^1(I; H^1_0(\Omega; \mathbb{R}^n), H^{-1}(\Omega; \mathbb{R}^n))$. Thanks to (23b), this mapping admits a (weak*,weak)-continuous extension $\boldsymbol{\pi} : \boldsymbol{\nu} \mapsto \boldsymbol{y}$ from $\boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)$ to $\boldsymbol{H}^1(I; H^1_0(\Omega; \mathbb{R}^n), H^{-1}(\Omega; \mathbb{R}^n))$ with \boldsymbol{y} being the unique weak solution from the integral identity in (24). By the positive definitences of A, the mapping $\boldsymbol{\pi}$ is also (weak*,strong)-continuous from $\boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)$ to $\boldsymbol{L}^2(I; H^1_0(\Omega; \mathbb{R}^n))$.

We can then view the problems (22) and (24) as minimization problems in terms of the controls only, involving composed functionals

$$\boldsymbol{u} \mapsto \int_{0}^{T} \int_{\Omega} \varphi(t, [\boldsymbol{\pi}(\boldsymbol{u})](t), \boldsymbol{u}(t)) \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega} \phi(\boldsymbol{y}(T)) \, \mathrm{d}x \tag{26a}$$

and its continuous extension

$$\boldsymbol{\nu} \mapsto \int_0^T \int_{\gamma_{\mathscr{R}} S_p} \overline{\varphi}(t, [\overline{\boldsymbol{\pi}}(\boldsymbol{\nu})](t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \mathrm{d}t + \int_{\Omega} \phi(\boldsymbol{y}(T)) \,\mathrm{d}x\,,$$
(26b)

respectively. By density of U_{ad} from (15) in $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ (cf. Proposition 2) and metrizability of $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$, all the assertions 1.–3. follow.

The convexity of $\mathcal{Y}(I; \gamma_{\mathscr{R}}S_p)$ allows for derivation of optimality conditions essentially by standard methods of smooth/convex analysis. This convex geometry directly determines the resulting so-called maximum principle. For using the standard smooth analysis and adjoint-equation technique for evaluation of the Gâteaux derivative of the composed functional (26b), we assume that, for any $\mathbf{r}, \tilde{\mathbf{r}} \in L^2(\Omega; \mathbb{R}^n), \mathbf{s} \in S_p$, and $t \in I$, it holds

$$\varphi(t,\cdot,\boldsymbol{s}) : L^{2}(\Omega;\mathbb{R}^{n}) \to \mathbb{R} \text{ is Gâteaux differentiable,}
\forall \, \widetilde{\boldsymbol{y}} \in \boldsymbol{L}^{2}(I;L^{2}(\Omega;\mathbb{R}^{n})): \langle \varphi_{\boldsymbol{r}}^{\prime} \circ \boldsymbol{y}, \widetilde{\boldsymbol{y}} \rangle : (t,\boldsymbol{s}) \mapsto \langle \varphi_{\boldsymbol{r}}^{\prime}(t,\boldsymbol{y}(t),\boldsymbol{s}), \widetilde{\boldsymbol{y}}(t) \rangle \in \boldsymbol{L}^{1}(I;\mathscr{R}),
\|\varphi_{\boldsymbol{r}}^{\prime}(t,\boldsymbol{r},\boldsymbol{s})\|_{L^{2}(\Omega;\mathbb{R}^{n})} \leq a_{1}(t)\|\boldsymbol{r}\|_{L^{2}(\Omega;\mathbb{R}^{n})}, \text{ and}
\|\varphi_{\boldsymbol{r}}^{\prime}(t,\boldsymbol{r},\boldsymbol{s}) - \varphi_{\boldsymbol{r}}^{\prime}(t,\widetilde{\boldsymbol{r}},\boldsymbol{s})\|_{L^{2}(\Omega;\mathbb{R}^{n})} \leq a_{1}(t)\|\boldsymbol{r} - \widetilde{\boldsymbol{r}}\|_{L^{2}(\Omega;\mathbb{R}^{n})},$$

$$\phi: L^{2}(\Omega;\mathbb{R}^{n}) \to \mathbb{R} \text{ is Gâteaux differentiable,}$$

$$(27a)$$

$$\begin{aligned} \boldsymbol{f}(t,\cdot,\boldsymbol{s}) &: L^{2}(\Omega;\mathbb{R}^{n}) \to L^{2}(\Omega;\mathbb{R}^{n}) \text{ is Gateaux differentiable,} \\ &\forall \widetilde{\boldsymbol{z}} \in \boldsymbol{L}^{2}(I; \mathcal{L}(L^{2}(\Omega;\mathbb{R}^{n}))): \quad \langle \boldsymbol{f}_{\boldsymbol{r}}' \circ \boldsymbol{y}, \widetilde{\boldsymbol{z}} \rangle : (t,\boldsymbol{s}) \mapsto \langle \boldsymbol{f}_{\boldsymbol{r}}'(t,\boldsymbol{y}(t),\boldsymbol{s}), \widetilde{\boldsymbol{z}}(t) \rangle \in \boldsymbol{L}^{1}(I;\mathscr{R}), \\ &\| \boldsymbol{f}_{\boldsymbol{r}}'(t,\boldsymbol{r},\boldsymbol{s}) \|_{\mathcal{L}(L^{2}(\Omega;\mathbb{R}^{n}))} \leq a_{2}(t) \| \boldsymbol{r} \|_{L^{2}(\Omega;\mathbb{R}^{n})}, \quad \text{and} \\ &\| \boldsymbol{f}_{\boldsymbol{r}}'(t,\boldsymbol{r},\boldsymbol{s}) - \boldsymbol{f}_{\boldsymbol{r}}'(t,\widetilde{\boldsymbol{r}},\boldsymbol{s}) \|_{\mathcal{L}(L^{2}(\Omega;\mathbb{R}^{n}))} \leq a_{2}(t) \| \boldsymbol{r} - \widetilde{\boldsymbol{r}} \|_{L^{2}(\Omega;\mathbb{R}^{n})}, \end{aligned}$$

with $a_1 \in L^1(I)$ and $a_2 \in L^2(I)$; actually, a bit more general assumptions would work, too, cf. [32, Sect. 4.5].

Proposition 4 (Maximum principle for (24)). Let (23) and (27) hold. Then, any solution $\boldsymbol{\nu} \in \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}}S_p)$ to (24) satisfies

$$\int_{\gamma_{\mathscr{R}}S_{p}} h_{\boldsymbol{y},\boldsymbol{\chi}}(t,\boldsymbol{s}) \,\boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) = \sup_{\boldsymbol{s}\in S_{p}} h_{\boldsymbol{y},\boldsymbol{\chi}}(t,\boldsymbol{s}) \quad \text{for a.a.} \quad t \in I$$
with $h_{\boldsymbol{y},\boldsymbol{\chi}}(t,\boldsymbol{s}) = \langle \boldsymbol{f}(t,\boldsymbol{y}(t),\boldsymbol{s}),\boldsymbol{\chi}(t) \rangle - \varphi(t,\boldsymbol{r},\boldsymbol{s}),$
(28)

with $\boldsymbol{y} = \overline{\boldsymbol{\pi}}(\boldsymbol{\nu})$ and with $\boldsymbol{\chi} \in \boldsymbol{H}^1(I; H_0^1(\Omega; \mathbb{R}^n), H^{-1}(\Omega; \mathbb{R}^n))$ being a weak solution to the adjoint terminal-boundary-value parabolic problem

$$\frac{\partial \boldsymbol{\chi}}{\partial t} + \operatorname{div}(\boldsymbol{A}^{\top} \nabla \boldsymbol{\chi}) + \int_{\gamma_{\mathscr{R}} S_{p}} [h_{\boldsymbol{y}, \boldsymbol{\chi}}]_{\boldsymbol{r}}'(t, \boldsymbol{s})^{\top} \boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) = 0 \qquad \text{in } I \times \Omega , \qquad (29a)$$

$$\chi = 0 \qquad \qquad on \ I \times I^{+}, \tag{29b}$$

$$\boldsymbol{\chi}(T) = \phi_{\boldsymbol{r}}'(\boldsymbol{y}(T)) \qquad on \ \Omega.$$
(29c)

Sketch of the proof. Let us define the extensions $\overline{\varphi} : I \times H^1_0(\Omega; \mathbb{R}^n) \times \operatorname{rca}(\gamma_{\mathscr{R}}S_p) \to \mathbb{R} \cup \{+\infty\}$ and $\overline{f} : I \times H^1_0(\Omega; \mathbb{R}^n) \times \operatorname{rca}(\gamma_{\mathscr{R}}S_p) \to H^{-1}(\Omega; \mathbb{R}^n)$ of φ and f by

$$\overline{\varphi}(t, \boldsymbol{r}, \boldsymbol{\nu}) = \begin{cases} \int_{\gamma_{\mathscr{R}} S_p} \varphi(t, \boldsymbol{r}, \boldsymbol{s}) \boldsymbol{\nu}(\mathrm{d}\boldsymbol{s}) & \text{if } \boldsymbol{\nu} \in \mathrm{rca}_1^+(\gamma_{\mathscr{R}} S_p), \\ +\infty & \text{if } \boldsymbol{\nu} \in \mathrm{rca}(\gamma_{\mathscr{R}} S_p) \setminus \mathrm{rca}_1^+(\gamma_{\mathscr{R}} S_p), \end{cases}$$
(30a)

$$\langle \overline{f}(t, r, \nu), v \rangle = \int_{\gamma_{\mathscr{R}} S_p} \langle f(t, r, s), v \rangle \nu(\mathrm{d}s)$$
 (30b)

for any $\boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^n)$, respectively. By the assumptions (27a,b), the functional $\boldsymbol{y} \mapsto \int_0^T \int_{\gamma_{\mathscr{R}} S_p} \overline{\varphi}(t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \, \mathrm{d}t + \phi(\boldsymbol{y}(T))$ on $\boldsymbol{H}^1(I; H_0^1(\Omega; \mathbb{R}^n), H^{-1}\Omega; \mathbb{R}^n))$ is Gâteaux differentiable. Similarly, (27c) gives smoothness (namely continuous Gâteaux differentiability) of $\boldsymbol{y} \mapsto \overline{\boldsymbol{f}}(\boldsymbol{y}, \boldsymbol{\nu})$. Let us further define the extended composed cost functional $J : \boldsymbol{L}_{\mathrm{w*}}^{\infty}(I; \operatorname{rca}(\gamma_{\mathscr{R}} S_p)) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$J(\boldsymbol{\nu}) = \int_0^T \overline{\varphi}(t, \boldsymbol{y}_{\boldsymbol{\nu}}(t), \boldsymbol{\nu}_t) \, \mathrm{d}t + \phi(\boldsymbol{y}_{\boldsymbol{\nu}}(T))$$

with $\boldsymbol{y}_{\boldsymbol{\nu}}$ being the solution to the controlled system $\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \mathcal{A}\boldsymbol{y} = \overline{\boldsymbol{f}}(\boldsymbol{r},\boldsymbol{\nu})$ with $\mathcal{A} = -\mathrm{div}(A\nabla\boldsymbol{y})$ and with $\boldsymbol{y}(0) = \boldsymbol{y}_0$. The functional J has a smooth part determined by φ , \boldsymbol{f} , \boldsymbol{b} , and ϕ , and a nonsmooth but convex part as an indicator function of the convex subset $\boldsymbol{\mathcal{Y}}(I;\gamma_{\mathscr{R}}S_p)$ of $\boldsymbol{L}_{w*}^{\infty}(I;\mathrm{rca}(\gamma_{\mathscr{R}}S_p))$. The subdifferential ∂J of J can be calculated by the adjoint-equation techniques, leading to $\partial J(\boldsymbol{\nu}) = N_{\boldsymbol{\mathcal{Y}}(I;\gamma_{\mathscr{R}}S_p)}(\boldsymbol{\nu}) - h_{\boldsymbol{y}_{\boldsymbol{\nu}},\boldsymbol{\chi}}$ with $h_{\boldsymbol{y}_{\boldsymbol{\nu}},\boldsymbol{\chi}} \in \boldsymbol{L}^1(I;\mathscr{R})$ from (28) and with $N_{\boldsymbol{\mathcal{Y}}(I;\gamma_{\mathscr{R}}S_p)}$ denoting the normal cone to $\mathrm{rca}_1^+(\gamma_{\mathscr{R}}S_p)$ and with $\boldsymbol{\chi}$ satisfying the integral identity

$$\int_{0}^{T} \left(\left\langle A^{\top} \nabla \boldsymbol{\chi}(t), \nabla \boldsymbol{v}(t) \right\rangle + \left\langle \boldsymbol{\chi}(t), \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right\rangle + \int_{\gamma_{\mathscr{R}} S_{p}} \left\langle [\boldsymbol{f} \circ \boldsymbol{y}]_{\boldsymbol{r}}'(t, \boldsymbol{s})^{\top} \boldsymbol{\chi}(t), \boldsymbol{v}(t) \right\rangle \boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) \right) \mathrm{d}t \\
= \int_{0}^{T} \int_{\gamma_{\mathscr{R}} S_{p}} \left\langle [\varphi \circ \boldsymbol{y}]_{\boldsymbol{r}}'(t, \boldsymbol{s}), \boldsymbol{v}(t) \right\rangle \boldsymbol{\nu}_{t}(\mathrm{d}\boldsymbol{s}) \mathrm{d}t + \left\langle \phi_{\boldsymbol{r}}'(\boldsymbol{y}(T)), \boldsymbol{v}(T) \right\rangle. \quad (31)$$

This is the weak formulation of the terminal-boundary-value problem (29). The optimality condition $\partial J(\boldsymbol{\nu}) \ni 0$ reads as $\langle \tilde{\boldsymbol{\nu}} - \boldsymbol{\nu}, h_{\boldsymbol{y}_{\boldsymbol{\nu}}, \boldsymbol{\chi}} \rangle \leq 0$ for any $\tilde{\boldsymbol{\nu}} \in \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)$, i.e. $\langle \boldsymbol{\nu}, h_{\boldsymbol{y}_{\boldsymbol{\nu}}, \boldsymbol{\chi}} \rangle = \max_{\tilde{\boldsymbol{\nu}} \in \boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)} \langle \tilde{\boldsymbol{\nu}}, h_{\boldsymbol{y}_{\boldsymbol{\nu}}, \boldsymbol{\chi}} \rangle$. By the density of $\boldsymbol{\delta}(\boldsymbol{U}_{ad})$ in $\boldsymbol{\mathcal{Y}}(I; \gamma_{\mathscr{R}} S_p)$, this condition just gives (28).

Exploiting the maximum principle, one can weaken the convexity condition (37) by considering a smaller set than S_p in (40) excluding arguments which surely cannot satisfy the maximum principle, cf. [28] where the relaxed problems were exploited but for optimal control of ordinary differential equations. Thus existence for (22) can be proved even for nonconvex orientor fields, cf. [33].

Remark 5 (*Constancy of the Hamiltonian along optimal trajectories*). Still one more condition is sometimes completing the maximum principle for evolution systems, namely that the Hamiltonian is constant in time. Here, it is expected that the augmented Hamiltonian

$$h_{\boldsymbol{y},\boldsymbol{\chi}}^{\mathcal{A}}(t,\boldsymbol{\nu}) := \langle \overline{\boldsymbol{f}}(t,\boldsymbol{y}(t),\boldsymbol{\nu}) - \mathcal{A}\boldsymbol{y}(t),\boldsymbol{\chi}(t) \rangle - \overline{\varphi}(t,\boldsymbol{y}(t),\boldsymbol{\nu})$$

is constant in time for any optimal pair $(\boldsymbol{y}, \boldsymbol{\nu})$ with $\boldsymbol{\chi}$ solving (29), i.e. the function $t \mapsto \int_{S_r} h_{\boldsymbol{y},\boldsymbol{\chi}}^{\mathcal{A}}(t, \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s})$ is constant on I. This actually holds only for autonomous systems, i.e. φ, \boldsymbol{f} ,

and b independent of time. Then, by the following (formal) calculations (with the *t*-variable not explicitly written), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}h_{\boldsymbol{y},\boldsymbol{\chi}}^{\mathcal{A}}(t,\boldsymbol{\nu}) = \left\langle \overline{\boldsymbol{f}}(\boldsymbol{y},\boldsymbol{\nu}) - \mathcal{A}\boldsymbol{y}, \frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}t} \right\rangle + \left\langle \overline{\boldsymbol{f}}_{t}'(\boldsymbol{y},\boldsymbol{\nu}), \boldsymbol{\chi} \right\rangle - \overline{\varphi}_{t}'(\boldsymbol{y},\boldsymbol{\nu}) - \left\langle \overline{\varphi}_{\boldsymbol{r}}'(\boldsymbol{y},\boldsymbol{\nu}), \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} \right\rangle \\
+ \left\langle \left(\overline{\boldsymbol{f}}_{\boldsymbol{r}}'(\boldsymbol{y},\boldsymbol{\nu}) - \mathcal{A} \right) \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}, \boldsymbol{\chi} \right\rangle + \left\langle \frac{\mathrm{d}\boldsymbol{\nu}}{\mathrm{d}t}, \underbrace{h_{\boldsymbol{\chi}} \circ \boldsymbol{y} - N_{\mathrm{rea}_{1}^{+}(\gamma_{\mathscr{R}}S_{p})}(\boldsymbol{\nu})}_{=0 \text{ by } (28)} \right\rangle \\
= \left\langle \overline{\boldsymbol{f}}_{t}'(\boldsymbol{y},\boldsymbol{\nu}), \boldsymbol{\chi} \right\rangle - \overline{\varphi}_{t}'(\boldsymbol{y},\boldsymbol{\nu}), \qquad (32)$$

where we used that $\langle \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nu}, N_{\mathrm{rca}_{1}^{+}(\gamma_{\mathscr{R}}S_{p})}(\boldsymbol{\nu}) \rangle = 0$. Also we used $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} + \mathcal{A}\boldsymbol{y} = \overline{\boldsymbol{f}}(\boldsymbol{y}, \boldsymbol{\nu})$ and the adjoint equation (29a,b) in the form $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\chi} - \mathcal{A}^{*}\boldsymbol{\chi} = \overline{\varphi}_{\boldsymbol{r}}'(\boldsymbol{y}, \boldsymbol{\nu}) - \overline{\boldsymbol{f}}_{\boldsymbol{r}}'(\boldsymbol{y}, \boldsymbol{\nu})^{*}\boldsymbol{\chi}$, which yields

$$\left\langle \overline{f}(\boldsymbol{y},\boldsymbol{\nu}) - \mathcal{A}\boldsymbol{y}, \frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}t} \right\rangle + \left\langle \left(\overline{f}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu}) - \mathcal{A}\right) \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}, \boldsymbol{\chi} \right\rangle - \left\langle \overline{\varphi}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu}), \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} \right\rangle$$
$$= \left\langle \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}, \mathcal{A}^{*}\boldsymbol{\chi} + \overline{\varphi}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu}) - \overline{f}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu})^{*}\boldsymbol{\chi} \right\rangle + \left\langle \left(\overline{f}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu}) - \mathcal{A}\right) \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}, \boldsymbol{\chi} \right\rangle - \left\langle \overline{\varphi}'_{\boldsymbol{r}}(\boldsymbol{y},\boldsymbol{\nu}), \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} \right\rangle = 0.$$

From (32), we can see that $h_{y,\chi}^{\mathcal{A}}$ is constant in time if both $f'_t = 0$, $b'_t = 0$, and $\varphi'_t = 0$.

5 Some other relaxation schemes

Sometimes, the relaxed problem uses the conventional Young measures from $\mathcal{Y}(I \times \Omega; B)$. This coarser compactification may naturally record fast oscillations of infinizing controls both in time and space simultaneously.

In view of Example 2, we can consider rather general nonlinearities. To avoid too many notational complications, we consider for example the problem:

$$\begin{array}{ll}
\text{Minimize} & \int_{0}^{T} \sum_{i=1}^{k} \prod_{j=1}^{l} \widehat{\varphi}_{ij} \left(t, \int_{\Omega} h_{ij}(t, x, y(t, x), u(t, x)) \, \mathrm{d}x \right) \mathrm{d}t \\ & \quad + \int_{\Omega} \phi(x, y(T, x)) \, \mathrm{d}x \\
\text{subject to} & \frac{\partial y}{\partial t} - \operatorname{div}(A \nabla y) = f\left(t, x, y(t, x), u(t, x) \right) & \text{in } I \times \Omega , \\ & y = 0 & \text{on } I \times \Gamma , \\ & y(0, \cdot) = y_{0} & \text{on } \Omega , \\ & u(t, x) \in B & \text{for } (t, x) \in I \times \Omega , \\ & y \in \mathbf{H}^{1}(I; H_{0}^{1}(\Omega; \mathbb{R}^{n}), H^{-1}(\Omega; \mathbb{R}^{n})), \quad u \in L^{p}(I \times \Omega; \mathbb{R}^{m}) \end{array}\right)$$

$$(33)$$

with $\widehat{\varphi}_{ij}: I \times \mathbb{R} \to \mathbb{R}, h_{ij}: I \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \phi: \Omega \times \mathbb{R}^n \to \mathbb{R}, \text{ and } f: I \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, i = 1, ..., k \text{ and } j = 1, ..., l \text{ with } k, l \in \mathbb{N}.$ This falls into the form (22) when taking

$$\varphi(t, \boldsymbol{r}, \boldsymbol{s}) = \sum_{i=1}^{k} \prod_{j=1}^{l} \widehat{\varphi}_{ij} \left(t, \int_{\Omega} h_{ij}(t, x, \boldsymbol{r}(x), \boldsymbol{s}(x)) \, \mathrm{d}x \right), \tag{34a}$$

$$[\boldsymbol{f}(t,\boldsymbol{r},\boldsymbol{s})](x) = f(t,x,\boldsymbol{r}(x),\boldsymbol{s}(x)).$$
(34b)

The natural (although not the weakest possible) qualification of these data is

$$h_{ij} \in L^1(I \times \Omega; C(\mathbb{R}^n \times B)), \quad \phi \in L^1(\Omega; C(\mathbb{R}^n)),$$
(35a)

$$\widehat{\varphi}_{ij} \in \boldsymbol{L}^1(I; C(\mathbb{R})), \quad \text{and} \quad f \in L^1(I \times \Omega; C(\mathbb{R}^n \times B)^n).$$
 (35b)

Under these assumptions, (33) bears an extension to the conventional Young measures $\mathcal{Y}(I \times \Omega; B)$, which leads to the relaxed problem

This extension is however not weakly^{*} continuous unless l = 1 and all $\widehat{\varphi}_{i1}(t, \cdot) : \mathbb{R} \to \mathbb{R}$ are affine.

The resulted (Pontryagin-type) maximum principle is then formulated pointwise for a.a. $(t,x) \in I \times \Omega$. For a very special case k = 1 = l and $\widehat{\varphi}_{11}(t, \cdot)$ affine, such relaxation scheme has been used e.g. in [16, 17] or also [32, Sect.4.5.b]. In this special case, one can prove also existence of solutions, i.e. optimal relaxed controls from $\mathcal{Y}(I \times \Omega; B)$. For a derivation of the mentioned pointwise maximum principle for the original problem without relaxation in this special case we refer e.g. to [12, 23, 30]. The pointwise constancy of the Hamiltonian on $I \times \Omega$ however does not seem to hold, in contrast to the finer relaxation examined before in Remark 5.

In general, the existence of solutions to (36) is however not granted by usual direct-method arguments unless l = 1 and $\hat{\varphi}_{i1}(t, \cdot)$ are convex. In view of (34), we can exploit also the relaxation scheme from Section 4. The metrizability and separability of $\gamma_{\mathscr{R}}S_p$ allows for a generalization of the (originally finite-dimensional) Filippov-Roxin [22,34] existence theory for nonconvex problems. Here we exploit the relaxed problem (24) similarly as it was done for finite-dimensional systems in [28,31].

Proposition 5 (Filippov-Roxin existence for (24)). Let the assumptions of Proposition 3 with φ and **f** from (34) be fulfilled and let the so-called orientor field

$$\boldsymbol{Q}(t,\boldsymbol{r}) := \left\{ (\alpha, \boldsymbol{f}(t,\boldsymbol{r},\boldsymbol{s})) \in \mathbb{R} \times H_0^1(\Omega;\mathbb{R}^n)^*; \ \alpha \ge \varphi(t,\boldsymbol{r},\boldsymbol{s}), \ \boldsymbol{s} \in S_p \right\}$$
(37)

be convex for a.a. $t \in I$ and all $\mathbf{r} \in H_0^1(\Omega; \mathbb{R}^n)$. Then the following relaxed problem possesses a solution:

$$\begin{array}{ll}
 Minimize & \int_{0}^{T} \overline{\varphi}(t, \boldsymbol{y}(t, \cdot), \overline{\boldsymbol{u}}(t, \cdot)) \, \mathrm{d}t + \int_{\Omega} \phi(\boldsymbol{y}(T)) \, \mathrm{d}x \\
 subject to & \forall \boldsymbol{v} \in \boldsymbol{H}^{1}(I; L^{2}(\Omega; \mathbb{R}^{m})) \cap \boldsymbol{L}^{2}(I; H_{0}^{1}(\Omega; \mathbb{R}^{m})), \quad \boldsymbol{v}(T) = 0: \\
 \int_{0}^{T} \left(\int_{\Omega} A \nabla \boldsymbol{y} : \nabla \boldsymbol{v} - \boldsymbol{y} \cdot \frac{\partial \boldsymbol{v}}{\partial t} \, \mathrm{d}x \\
 - \langle \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \overline{\boldsymbol{u}}(t, \cdot)), \boldsymbol{v}(t) \rangle \right) \mathrm{d}t = \int_{\Omega} \boldsymbol{y}_{0} \cdot \boldsymbol{v}(0) \, \mathrm{d}x, \\
 \boldsymbol{y} \in \boldsymbol{H}^{1}(I; H_{0}^{1}(\Omega; \mathbb{R}^{n}), H^{-1}(\Omega; \mathbb{R}^{n})), \\
 \overline{\boldsymbol{u}} \in \boldsymbol{L}_{w*}^{\infty}(I; \mathscr{R}^{*}), \quad \overline{\boldsymbol{u}}(t) \in \gamma_{\mathscr{R}} S_{p} \quad for \ a.a. \quad t \in I. \end{array}$$

$$(38)$$

Sketch of the proof. Let us define

$$\overline{\boldsymbol{Q}}(t,\boldsymbol{r}) := \left\{ (\alpha, \overline{\boldsymbol{f}}(t,\boldsymbol{r},\boldsymbol{s})) \in \mathbb{R} \times H^{-1}(\Omega;\mathbb{R}^n); \ \alpha \ge \overline{\varphi}(t,\boldsymbol{r},\boldsymbol{s}), \ \boldsymbol{s} \in \gamma_{\mathscr{R}} S_p \right\}.$$
(39)

For a.a. $t \in I$ and all $\boldsymbol{r} \in H_0^1(\Omega; \mathbb{R}^n)$, the convexity and closedness of $\overline{\boldsymbol{Q}}(t, \boldsymbol{r})$ just means $\overline{\operatorname{co}} \left[\overline{\boldsymbol{\varphi}} \times \overline{\boldsymbol{f}} \right](t, \boldsymbol{r}, \gamma_{\mathscr{R}} S_p) \subset \overline{\boldsymbol{Q}}(t, \boldsymbol{r})$ (40)

with " \overline{co} " denoting the closed convex full. By (40), we get

$$\int_{S_p} \left[\varphi \times \boldsymbol{f} \right] (t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \in \overline{\mathrm{co}} \left[\overline{\varphi} \times \overline{\boldsymbol{f}} \right] (t, \boldsymbol{y}(t), \gamma_{\mathscr{R}} S_p) \subset \overline{\boldsymbol{Q}}(t, \boldsymbol{y}(t)) \,. \tag{41}$$

Taking a solution ν to the relaxed problem (24), we put

$$\boldsymbol{S}(t) = \left\{ \boldsymbol{s} \in \gamma_{\mathscr{R}} S_p; \ \overline{\varphi}(t, \boldsymbol{y}(t), \boldsymbol{s}) \leq \int_{\gamma_{\mathscr{R}} S_p} \varphi(t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}), \\ \boldsymbol{f}(t, \boldsymbol{y}(t), \boldsymbol{s}) = \int_{\gamma_{\mathscr{R}} S_p} \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \right\},$$
(42)

Obviously, S(t) is closed for a.a. $t \in I$. We further show that it is also non-empty. Indeed, by (40), for any $(\alpha, q) \in \overline{Q}(t, y(t))$ there is $s \in \gamma_{\mathscr{R}}S_p$ such that $\alpha \geq \overline{\varphi}(t, y(t), s)$ and $q = \overline{f}(t, y(t), s)$. Hence, for the particular choice

$$(\alpha, \boldsymbol{q}) = (\alpha(t), \boldsymbol{q}(t)) := \int_{\gamma_{\mathscr{R}} S_p} [\overline{\varphi} \times \overline{\boldsymbol{f}}](t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}),$$
(43)

the inclusion (41) implies that $\alpha(t) \geq \overline{\varphi}(t, \boldsymbol{y}(t), \boldsymbol{s})$ and $\boldsymbol{q}(t) = \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \boldsymbol{s})$ for some $\boldsymbol{s} \in \gamma_{\mathscr{R}} S_p$, hence $\boldsymbol{S}(t) \neq \emptyset$.

Moreover, the multi-valued mapping $\boldsymbol{S}: I \Rightarrow \gamma_{\mathscr{R}} S_p$ defined by (42) is measurable. Indeed, $\boldsymbol{\nu}$ weakly^{*} measurable and $\overline{\boldsymbol{\varphi}}$ and $\overline{\boldsymbol{f}}$ Carathéodory mappings imply that q from (43) is measurable. Furthermore, by [4, Thm. 8.2.9], the level sets $t \mapsto \{\boldsymbol{s} \in \gamma_{\mathscr{R}} S_p; \ \overline{\boldsymbol{\varphi}}(t, \boldsymbol{y}(t), \boldsymbol{s}) \leq \alpha(t)\}$ and $t \mapsto \{\boldsymbol{s} \in \gamma_{\mathscr{R}} S_p; \ \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \boldsymbol{s}) = \boldsymbol{q}(t)\}$ are measurable. By [4, Thm. 8.2.4], the intersection of these level sets, which is just $\boldsymbol{S}(t)$, is also a measurable multi-valued mapping.

Then, by [4, Thm. 8.1.4], the multi-valued mapping \boldsymbol{S} possesses a measurable selection $\overline{\boldsymbol{u}}(t) \in \boldsymbol{S}(t)$; here separability and metrizability of $\gamma_{\mathscr{R}}S_p$ were used.

In view of (42), $\overline{f}(t, y(t), \overline{u}(t)) = q(t) = \int_{\gamma_{\mathscr{R}}S_p} \overline{f}(t, y(t), s) \nu_t(\mathrm{d}s)$ so that the pair (\overline{u}, y) is admissible for (24), and moreover

$$\int_0^T \overline{\varphi}(t, \boldsymbol{y}(t), \overline{\boldsymbol{u}}(t)) \, \mathrm{d}t \le \int_0^T \alpha(t) \, \mathrm{d}t = \int_0^T \int_{\gamma_{\mathscr{R}} S_p} \overline{\varphi}(t, \boldsymbol{y}(t), \boldsymbol{s}) \boldsymbol{\nu}_t(\mathrm{d}\boldsymbol{s}) \, \mathrm{d}t = \min\left(24\right) = \inf\left(38\right).$$

This \overline{u} thus solves (38).

The particular choice (7) allows for usage of Lemma 2. In this case, Proposition 5 with Lemma 2 gives $\overline{u} : I \to \mathcal{Y}(\Omega; B)$ as a solution to (38). Then, in view of special nonlinearities involved in (33), we can use Lemma 1, which leads to a relaxation using a certain Young measure valued on the original set S_p from (1) as actually used in (22), provided we weaken a bit the measurability of Young measures. More specifically, we define:

$$\boldsymbol{w} \cdot \boldsymbol{\mathcal{Y}}(I; S_p) := \left\{ \boldsymbol{\mu} : I \to \operatorname{rca}_1^+(S_p); \, \forall \boldsymbol{h} \in \boldsymbol{L}^1(I; C(\boldsymbol{\mathcal{Y}}(\Omega; B))|_{S_p}) : t \mapsto \left\langle \boldsymbol{\mu}_t, \boldsymbol{h}(t, \cdot) \right\rangle \text{ is measurable} \right\},$$
(44)

where we used again the convention $\boldsymbol{\mu}_t := \boldsymbol{\mu}(t)$, so that we will write $\boldsymbol{\mu} = \{\boldsymbol{\mu}_t\}_{t \in I}$ in what follows. We call elements of $\boldsymbol{w} \cdot \boldsymbol{\mathcal{Y}}(I; S_p)$ as weak-Young measures. Note that the set of test functions in (44) is smaller than the nonseparable space $\boldsymbol{L}^1(I; C(S_p))$ and thus weak-Young measures do not live in $\boldsymbol{\mathcal{Y}}(I; S_p)$ in general. **Corollary 1.** Let the assumptions of Proposition 3 with \mathscr{R} from (7) and (37) hold for φ and f from (34). Then there exists a solution to the following relaxed problem:

where Ψ is from (8). Moreover, if also

$$[h_{ij}]'_r \in L^1(I \times \Omega; C(\mathbb{R}^n \times B)^n), \quad \phi'_r \in L^1(\Omega; C(\mathbb{R}^n)^n),$$
(46a)

$$[\widehat{\varphi}_{ij}]'_r \in \boldsymbol{L}^1(I; C(\mathbb{R})), \quad and \quad [f]'_r \in L^1(I \times \Omega; C(\mathbb{R}^n \times B)^{n \times n}), \tag{46b}$$

then this solution satisfies, for a.a. $t \in I$, the maximum principle

$$\int_{S_p} h_{\boldsymbol{y},\boldsymbol{\chi}}(t,u) \,\boldsymbol{\mu}_t(\mathrm{d}u) = \sup_{u \in S_p} h_{\boldsymbol{y},\boldsymbol{\chi}}(t,u) \tag{47}$$

with $h_{y,\chi}$ from (28) with χ satisfying the adjoint terminal-boundary-value parabolic problem, written in the weak form here as

$$\int_{0}^{T} \left(\left\langle A^{\top} \nabla \boldsymbol{\chi}(t), \nabla \boldsymbol{v}(t) \right\rangle + \left\langle \boldsymbol{\chi}(t), \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \right\rangle + \int_{S_{p}} \left[\boldsymbol{\Psi}(\left\langle [\boldsymbol{f} \circ \boldsymbol{y}]_{\boldsymbol{r}}'(t)^{\top} \boldsymbol{\chi}(t), \boldsymbol{v}(t) \right\rangle) \right](u) \boldsymbol{\mu}_{t}(\mathrm{d}u) \right) \mathrm{d}t \\
= \int_{0}^{T} \int_{S_{p}} \left[\boldsymbol{\Psi}(\left\langle [\varphi \circ \boldsymbol{y}]_{\boldsymbol{r}}'(t), \boldsymbol{v}(t) \right\rangle) \right](u) \boldsymbol{\mu}_{t}(\mathrm{d}u) \mathrm{d}t + \left\langle \phi_{\boldsymbol{r}}'(\boldsymbol{y}(T)), \boldsymbol{v}(T) \right\rangle \tag{48}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(I; L^2(\Omega; \mathbb{R}^m)) \cap \boldsymbol{L}^2(I; H^1_0(\Omega; \mathbb{R}^m))$ with $\boldsymbol{v}(0) = 0$.

Proof. Recall that now, for the choice (7), $\gamma_{\mathscr{R}}S_p \cong \mathcal{Y}(\Omega; B)$, cf. Lemma 2. Take $\overline{\boldsymbol{u}} : I \to \mathcal{Y}(\Omega; B)$ a solution to (38). In particular, for any $\boldsymbol{h} \in \boldsymbol{L}^1(I; \mathscr{R})$, the function $t \mapsto \langle \overline{\boldsymbol{u}}(t), \boldsymbol{h}(t) \rangle = \int_{\Omega} \int_{B} \boldsymbol{h}(t, x, z) \overline{\boldsymbol{u}}(t, \mathrm{d}z) \mathrm{d}x = \overline{\boldsymbol{h}}(t, \overline{\boldsymbol{u}}(t))$ is measurable (and integrable); here $\overline{\boldsymbol{h}}(t, \cdot)$ denotes the weakly* continuous extension of $\boldsymbol{h}(t, \cdot) : S_p \to \mathbb{R}$ on $\mathcal{Y}(\Omega; B)$.

The functions $[h_{ij} \circ \boldsymbol{y}](t) : (x, z) \mapsto h(t, x, y(t, x), z)$ and $\langle [\boldsymbol{f} \circ \boldsymbol{y}](t), \boldsymbol{v}(t) \rangle : (x, z) \mapsto \langle f(t, x, y(t, x), z), v(t, x) \rangle$ belong to $L^1(\Omega; C(B))$. Therefore, by Lemma 1, for some $\boldsymbol{\mu}_t \in \operatorname{rca}_1^+(S_p)$, we have

$$\widehat{\varphi}_{ij}(t, \langle \overline{\boldsymbol{u}}(t, \cdot), [h_{ij} \circ \boldsymbol{y}](t, \cdot) \rangle) = g_{ij}(t, \langle \boldsymbol{\mu}_t, \Psi([h_{ij} \circ \boldsymbol{y}](t)) \rangle) \text{ and}$$
(49a)

$$\left\langle \overline{\boldsymbol{f}}(t, \boldsymbol{y}(t), \overline{\boldsymbol{u}}(t, \cdot)), \boldsymbol{v}(t) \right\rangle = \left\langle \boldsymbol{\mu}_t, \Psi(\left\langle [\boldsymbol{f} \circ \boldsymbol{y}](t), \boldsymbol{v}(t) \right\rangle) \right\rangle.$$
(49b)

Thus $\min(38) \ge \min(45)$.

On the other hand, also min (38) $\leq \min(45)$ because, for any μ admissible for (45), there is some $\overline{\boldsymbol{u}}: I \to \mathcal{Y}(\Omega; B)$ such that (49) holds. Thus such $\overline{\boldsymbol{u}}$ is admissible for (38), yielding the cost not lower than min (45). Here the definition (12) of srca₁⁺(S_p) has been used.

By (46), it can be seen that (27) for φ and f from (34) is satisfied with \mathscr{R} from (7). Then one can use Proposition 4. By this way, (28) results to (47) while (31) gives (48).

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