STEADY COLLISION OF TWO JETS ISSUING FROM TWO AXIALLY SYMMETRIC CHANNELS[†]

LILI $DU^{a,1}$, YONGFU WANG^{b,2}

^a Department of Mathematics, Sichuan University,

Chengdu 610064, P. R. China.

^b School of Economic Mathematics,

Southwestern University of Finance and Economics,

Chengdu 611130, P. R. China.

ABSTRACT. In the classical survey (Chapter 16.2, Mathematics in industrial problem, Vol. 24, Springer-Verlag, New York, 1989), A. Friedman proposed an open problem on the collision of two incompressible jets emerging from two axially symmetric nozzles. In this paper, we concerned with the mathematical theory on this collision problem, and establish the well-posedness theory on hydrodynamic impinging outgoing jets issuing from two coaxial axially symmetric nozzles. More precisely, we showed that for any given mass fluxes $M_1 > 0$ and $M_2 < 0$ in two nozzles respectively, that there exists an incompressible, inviscid impinging outgoing jet with contact discontinuity, which issues from two given semi-infinitely long axially symmetric nozzles and extends to infinity. Moreover, the constant pressure free stream surfaces of the impinging jet initiate smoothly from the mouths of the two nozzles and shrink to some asymptotic conical surface. There exists a smooth surface separating the two incompressible fluids and the contact discontinuity occurs on the surface. Furthermore, we showed that there is no stagnation point in the flow field and its closure, except one point on the symmetric axis. Some asymptotic behavior of the impinging jet in upstream and downstream, geometric properties of the free stream surfaces are also obtained. The main results in this paper solved the open problem on the collision of two incompressible axially symmetric jets in [24].

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Key words: Impinging outgoing jets, incompressible flows, free boundary, existence and uniqueness, contact discontinuity.

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 $^{^1}$ E-Mail: duli
li@scu.edu.cn. 2 E-mail: wyf1247@163.com. Corresponding author.

1. INTRODUCTION

The three-dimensional incompressible, stationary and inviscid flow is governed by the Euler equations

$$\begin{cases} \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \\ \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} + \frac{1}{\rho} \frac{\partial P}{\partial x_j} = 0, \quad \text{for} \quad j = 1, 2, 3, \end{cases}$$
(1.1)

with the irrotational condition

$$\nabla \times (u_1, u_2, u_3) = 0.$$

Here, (u_1, u_2, u_3) is the velocity, P denotes the pressure of the flow, and positive constant ρ denotes density.

In this paper, we shall be concerned with steady, irrotational incompressible impinging jets issuing from two semi-infinitely long axially symmetric nozzles with variable cross-section. We will investigate the well-posedness theory of the collision problem of two jets issuing from two general axially symmetric nozzles, and solve the open problem (1) proposed by A. Friedman in 1989.

Consider the axially symmetric flows in this paper and let U(x,r), V(x,r) and W(x,r) be the axial velocity, the radial velocity and the swirl velocity respectively, $x = x_1$ and $r = \sqrt{x_2^2 + x_3^2}$. Furthermore, we seek such an axially symmetric flow without swirl, one has

$$u_1 = U(x,r), \quad u_2 = V(x,r)\frac{x_2}{r}, \quad u_3 = V(x,r)\frac{x_3}{r}.$$
 (1.2)

Then, instead of (1.1), we have

$$\begin{cases} (rU)_{x} + (rV)_{r} = 0, \\ (r\rho U^{2})_{x} + (r\rho UV)_{r} + rP_{x} = 0, \\ (r\rho UV)_{x} + (r\rho V^{2})_{r} + rP_{r} = 0. \end{cases}$$
(1.3)

Consider the flow issuing from the two semi-infinitely long nozzles as (see Figure 1)

$$\mathcal{N}_1 = \left\{ (x, r) \in \mathbb{R}^2_+ \, | \, f_1(r) < x < -1, \ r < R_1 \right\},\,$$

and

$$\mathcal{N}_2 = \{(x, r) \in \mathbb{R}^2_+ | 1 < x < f_2(r), \ r < R_2 \},\$$

where $\mathbb{R}^2_+ = \mathbb{R}^1 \times [0, +\infty)$, R_1 , $R_2 > 0$, $f_1(r)$ and $f_2(r)$ are smooth functions and satisfy that

$$f_i(r) = (-1)^i \infty, \qquad r \le r_i, \quad (r_i > 0)$$
 (1.4)

and

$$f_i(R_i) = (-1)^i, (1.5)$$

for $R_i > r_i$ and i = 1, 2. Without loss of generality, we assume that $R_1 = R_2 = R$.

For convenience, we denote the symmetric axis

$$N_0 = \{(x, r) | r = 0, -\infty < x < +\infty\},\$$

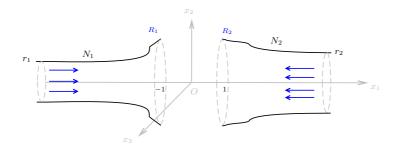


FIGURE 1. Two axially symmetric semi-infinitely long nozzles

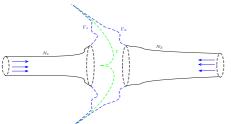




FIGURE 2. Collision of two jets

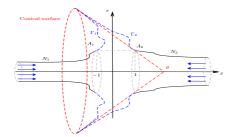


FIGURE 3. Impinging outgoing jet

the nozzle walls

$$N_1 = \{(x,r) | x = f_1(r), \ r_1 < r < R\}, \quad N_2 = \{(x,r) | x = f_2(r), \ r_2 < r < R\},$$

and the edge points of the nozzle walls $A_1 = (-1, R)$ and $A_2 = (1, R)$.

In this paper, we consider two ideal, nonmiscible, irrotational fluids (U_1, V_1, P_1, ρ_1) and (U_2, V_2, P_2, ρ_2) issuing from two semi-infinite axisymmetric nozzles. We designate by (U_1, V_1, P_1, ρ_1) and (U_2, V_2, P_2, ρ_2) be the axial velocity, the radial velocity and the pressure of the fluid I and the fluid II, respectively. Denote Ω_i as the fluid field of the *i*-th fluid for i = 1, 2, and

$$(U, V, P, \rho) = \begin{cases} (U_1, V_1, P_1, \rho_1) & \text{in } \Omega_1, \\ \\ (U_2, V_2, P_2, \rho_2) & \text{in } \Omega_2, \end{cases}$$
as the two-phase fluids.

In this paper, we seek a contact discontinuity (U, V, P, ρ) with a smooth interface Γ : $\{x = g(r)\}$ between the two fluids. And (U, V, P, ρ) is a week solution of (1.3) in the distributional sense and (U_i, V_i, P_i, ρ_i) solves the incompressible Euler system (1.3) classically in Ω_i for i = 1, 2. (Please see Figure 2).

Then the Rankine-Hugoniot jump conditions on Γ become

$$-\begin{bmatrix} \rho U\\ \rho U^{2} + P\\ \rho UV \end{bmatrix} + g'(r) \begin{bmatrix} \rho V\\ \rho UV\\ \rho VV \end{bmatrix} = 0, \qquad (1.6)$$

where $[\cdot]$ denotes the jump of a related function crossing the interface Γ .

Set $\mathfrak{m}_i = \rho_i \left(g'(r) V_i - U_i \right) (i = 1, 2)$ be the mass flux across the interface, if $\mathfrak{m}_1 = \mathfrak{m}_2 = 0$ on the interface Γ , then (U, V, P, ρ) is a contact discontinuity. The Rankine-Hugoniot conditions (1.6) read as

$$-U_1 + g'(r)V_1 = 0, \quad -U_2 + g'(r)V_2 = 0 \text{ and } P_1 = P_2.$$
 (1.7)

The condition (1.7) implies that the normal velocities on both sides of the interface Γ vanish, while the tangential velocity on both side of Γ may have nontrivial jump.

Furthermore, the well-known Bernoulli's law can be written as

$$(U_i, V_i) \cdot \nabla \left(\frac{1}{2}(U_i^2 + V_i^2) + \frac{P_i}{\rho_i}\right) = 0, \text{ for } i = 1, 2,$$

namely,

$$\frac{U_1^2 + V_1^2}{2} + \frac{P_1}{\rho_1} = \mathfrak{B}_1 \text{ and } \frac{U_2^2 + V_2^2}{2} + \frac{P_2}{\rho_2} = \mathfrak{B}_2,$$

where \mathfrak{B}_1 and \mathfrak{B}_2 denote the Bernoulli's constants of the two fluids, respectively, in view of (1.7), then

$$\rho_1 \left(U_1^2 + V_1^2 \right) - \rho_2 \left(U_2^2 + V_2^2 \right) = 2 \left(\rho_1 \mathfrak{B}_1 - \rho_2 \mathfrak{B}_2 \right) \triangleq \Lambda \quad \text{on} \quad \Gamma,$$
(1.8)

without loss of generality, we assume $\Lambda \geq 0$.

On another hand, on the free surfaces Γ_1 and Γ_2 , the pressure is assumed to be the constant atmosphere pressure P_{at} (in absence of gravity and surface tension), namely,

$$P = P_{at} \quad \text{on } \Gamma_1 \cup \Gamma_2. \tag{1.9}$$

Here is our problem of fluid mechanics: determine an impinging outgoing jet (U, V, P, ρ) issuing from two nozzles \mathcal{N}_1 and \mathcal{N}_2 with two mass fluxes M_1 and M_2 , bounded by two free surfaces Γ_1 and Γ_2 on which the pressure is a constant P_{at} . Furthermore, on the interface, the Rankine-Hugoniot conditions (1.7) and (1.8) hold.

On the solid walls N_1 and N_2 , the flow satisfies the slip-boundary condition,

$$(U_i, V_i) \cdot \vec{n}_i = 0, \quad \text{on } N_i, \tag{1.10}$$

where \vec{n}_i is the unit outer normal of the wall N_i , for i = 1, 2. Moreover, on the symmetry axis N_0 ,

$$V_i = 0.$$
 (1.11)

Denote M_1 and M_2 as the mass fluxes in nozzles \mathcal{N}_1 and \mathcal{N}_2 , respectively, then

$$\int_{\Sigma_i} (rU, rV, 0) \cdot \vec{l_i} dS = \frac{M_i}{2\pi}, \qquad (1.12)$$

where Σ_i is any curve transversal to the *x*-axis direction and \vec{l}_i is the normal of Σ_i in the positive *x*-axis direction for i = 1, 2.

1.1. **Impinging outgoing jet problem and main results.** We define the axially symmetric impinging outgoing jet problem as follows.

Axially symmetric impinging outgoing jet problem. For given any mass fluxes $M_1 > 0$ and $M_2 < 0$ in the two semi-infinitely long axially symmetric nozzles \mathcal{N}_1 and \mathcal{N}_2 , respectively, there exists an axially symmetric impinging outgoing jet extending to the infinity, the free stream surfaces initiate at the edges of the nozzles smoothly and shrink to some conical surface at the far field, and a smooth interface separates the two jets, furthermore, the pressure remains a constant on free stream surfaces (see Figure 3).

Next, we give the definition of the solution to the impinging outgoing jet problem.

A solution to the axially symmetric impinging outgoing jet problem. A quintuple $(U, V, P, \Gamma_1, \Gamma_2)$ is called a solution to the axially symmetric impinging outgoing jet problem, provided that

(1). The smooth surfaces Γ_1 and Γ_2 are given by two functions $x = g_1(r) \in C^1((R, +\infty))$ and $x = g_2(r) \in C^1((R, +\infty))$ with $g_1(r) < g_2(r)$, and

 $g_1(R+0) = f_1(R-0), \quad g_2(R+0) = f_2(R-0)$ (continuous fit conditions), (1.13) and

$$g'_1(R+0) = f'_1(R-0), \quad g'_2(R+0) = f'_2(R-0) \quad (smooth \ fit \ conditions).$$
(1.14)

Moreover, there exists an asymptotic direction $\nu = (\cos \theta, \sin \theta)$ with $\theta \in (0, \pi)$, such that g_1 and g_2 are close to the asymptotic direction ν at far field (See Figure 4), i.e.,

$$\lim_{r \to \infty} (g_2(r) - g_1(r)) = 0 \text{ and } \lim_{r \to \infty} g_1'(r) = \lim_{r \to \infty} g_2'(r) = \cot \theta, \quad (1.15)$$

the angle θ is called the asymptotic deflection angle of the impinging outgoing jet.

(2). Denote the flow field G bounded by the symmetric axis N_0 , the nozzle walls N_1, N_2 and the free boundaries Γ_1, Γ_2 . $(U, V, P) \in (C^{1,\alpha}(G) \cap C^0(\overline{G}))^3$ solves the steady incompressible Euler system (1.3), the boundary condition (1.10), the Rankine-Hugoniot conditions (1.7) and the mass flux conditions (1.12);

(3). The radial velocity V is positive in flow field and its closure, except the symmetric axis and interface Γ , namely, V > 0 in $\overline{G} \setminus (N_0 \cup \Gamma)$;

- (4). $P = P_{at}$ on $\Gamma_1 \cup \Gamma_2$;
- (5). The interface Γ satisfies the condition (1.8).

The first result in this paper is the existence of the impinging outgoing jet as follows.

Theorem 1.1. For any given atmosphere pressure P_{at} , mass fluxes $M_1 > 0$, $M_2 < 0$ and $\Lambda \geq 0$ issuing from the two axially symmetric nozzles \mathcal{N}_1 and \mathcal{N}_2 , respectively, there exists a solution $(U, V, P, \Gamma_1, \Gamma_2)$ to the axially symmetric impinging outgoing jet problem. Furthermore, there exists a C^1 -smooth surface x = g(r) satisfying $g_1(r) < g(r) < g_2(r)$ for $R < r < \infty$, which separates the two fluids and initiates at the branching point on the symmetric axis (Figure 5). Furthermore, there exists a positive constant λ , such that

$$r(g(r)-g_1(r)) \to \frac{M_1}{2\pi\sqrt{\rho_1(\Lambda+\lambda)}\sin\theta} \quad and \quad r(g(r)-g_2(r)) \to \frac{M_2}{2\pi\sqrt{\rho_2\lambda}\sin\theta} \qquad as \quad r \to +\infty$$
(1.16)

We would like to give the following comments on the existence theorem.

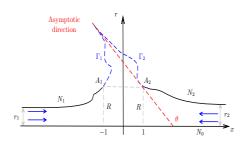


FIGURE 4. Axisymmetric impinging outgoing jet flow in cylindrical coordinates

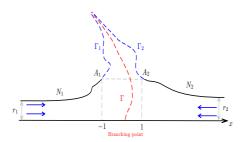


FIGURE 5. Impinging outgoing jet and the interface Γ

Remark 1.1. One of key points in this work is the appearance of the interface between the two fluids, which is also a free boundary and is determined by the solution itself. In this paper, the impinging outgoing jets possess a smooth surface separating the two immiscible fluids, which intersects the symmetric axis at a unique point, called the *branching point*. However, the appearance of the interface takes many essential difficulties to solve the free boundary problem in mathematics. The first one is the non-trivial jump of the velocity field on the interface (see (1.8)), which leads that we have to seek a non-smooth solution in the whole fluid field. The second one is that the interface is common boundaries of two fluids, which is totally free. And we shall define the interface as the level set of the stream function and show that it is indeed a smooth curve. The third one is the regularity of the two-phase fluids near the branching point.

Remark 1.2. There are many numerical results on the impinging free jets in absence of rigid nozzle walls, such as unsymmetrically impinging jets in [29], impinging free jets in [28], compressible impinging jets in [10]. However, here we have to consider the geometry of both solid boundaries and free boundaries, the one of main difficulties is to verify the continuous fit and smooth fit conditions. In present work, an essential point is that we can choose a suitable pair of parameters (λ, θ) , such that the continuous fit conditions are fulfilled. In other word, the parameters λ and θ can be determined by the continuous fit conditions, which is the main difference from the analysis of impinging free jets without rigid boundaries. Therefore, we first solve the free boundary problem for any λ and θ , and then show the existence of a suitable pair of parameters (λ, θ) to guarantee the continuous fit conditions imply the smooth fit conditions.

Theorem 1.1 gives that there exists a pair of parameters (λ, θ) to guarantee the existence of the axially symmetric impinging outgoing jet. However, to the best of our knowledge, the uniqueness of the jet with two free boundaries is totally open. Next, for $\Lambda = 0$, we give the uniqueness results on the axially symmetric impinging outgoing jet, the idea borrows from the recent work [11] on the uniqueness of the asymmetric incompressible jet.

Theorem 1.2. (Uniqueness of the axially impinging outgoing jet)(1) Given any parameters (λ, θ) , such that the continuous fit conditions (1.13) hold, then the axially symmetric impinging outgoing jet $(U, V, P, \Gamma_1, \Gamma_2)$ established in Theorem 1.1 is unique. (2) Suppose that there exist two pairs of the parameters (λ, θ) and (λ, θ) , such that the continuous fit conditions (1.13) to the axially symmetric impinging outgoing jet hold, then $\theta = \tilde{\theta}$.

Next, we give the asymptotic behaviors and the decay rate of the impinging outgoing jet in the far field.

Theorem 1.3. The impinging outgoing jet flow $(U, V, P, \Gamma_1, \Gamma_2)$ established in Theorem 1.1 satisfies the following asymptotic behavior in far fields,

$$(U(x,r), V(x,r), P(x,r)) \to \left(\frac{M_1}{\pi\rho_1 r_1^2}, 0, \frac{\lambda + \Lambda}{2\rho_1} + P_{at} - \frac{M_1^2}{2\rho_1 \pi^2 r_1^4}\right),$$
(1.17)

and

$$\nabla U \to 0, \quad \nabla V \to 0, \quad \nabla P \to 0,$$
 (1.18)

as $x \to -\infty$, in any compact subset of $(0, r_1)$ and

$$(U(x,r), V(x,r), P(x,r)) \to \left(\frac{M_2}{\pi\rho_2 r_1^2}, 0, \frac{\lambda}{2\rho_2} + P_{at} - \frac{M_2^2}{2\rho_2 \pi^2 r_2^4}\right),$$
(1.19)

and

$$\nabla U \to 0, \quad \nabla V \to 0, \quad \nabla P \to 0,$$
 (1.20)

as $x \to +\infty$, in any compact subset of $(0, r_2)$, and in the downstream,

$$(U(x,r), V(x,r), P(x,r)) \to \left(\sqrt{\frac{\Lambda+\lambda}{\rho_1}}\cos\theta, \sqrt{\frac{\Lambda+\lambda}{\rho_1}}\sin\theta, P_{at}\right),$$
(1.21)

uniformly in any compact subset of Ω_1 as $r \to +\infty$, and

$$(U(x,r), V(x,r), P(x,r)) \to \left(\sqrt{\frac{\lambda}{\rho_2}}\cos\theta, \sqrt{\frac{\lambda}{\rho_2}}\sin\theta, P_{at}\right), \tag{1.22}$$

uniformly in any compact subset of Ω_2 as $r \to +\infty$, and

 $\nabla U \to 0, \quad \nabla V \to 0, \quad \nabla P \to 0,$ (1.23)

uniformly in any compact subset of $\Omega_1 \cup \Omega_2$ as $r \to +\infty$.

Furthermore, for any $\alpha \in (0,2)$, one has

$$r^{\alpha} \left(\left| U_1(x,r) - \sqrt{\frac{\Lambda + \lambda}{\rho_1}} \cos \theta \right| + \left| V_1(x,r) - \sqrt{\frac{\Lambda + \lambda}{\rho_1}} \sin \theta \right| \right) \to 0, \quad (1.24)$$

$$r^{\alpha}\left(\left|U_{2}(x,r)-\sqrt{\frac{\lambda}{\rho_{2}}}\cos\theta\right|+\left|V_{2}(x,r)-\sqrt{\frac{\lambda}{\rho_{2}}}\sin\theta\right|\right)\to0,$$
(1.25)

as $r \to +\infty$.

Remark 1.3. The one of main differences between the impinging outgoing jet in twodimensional and axially symmetric case is that the two-dimensional outgoing jet possesses a uniform positive width in far field, and however, the distance of free streamlines goes to zero in downstream in axially symmetric case. Here, we have to establish the decay estimates of outgoing jets and the free boundaries in far field. Indeed, the facts (1.16) and (1.24) give the decay rates of the velocity field and the distance of the two free stream surfaces in downstream. In particular, (1.16) implies that the optimal decay rate of the distance of two free stream surfaces is $\frac{1}{r}$ in downstream. 1.2. Motivation and history of the problem. The motivation to investigate the impinging outgoing jets from two nozzles comes from Chapter V. § 5 in the classical book [9] by G. Birkhoff and E. H. Zarantonello, in which the impinging outgoing jets from two plane symmetric cylinders were considered. Except for some simple channel geometries, the impinging problem of two jets can not be solved analytically, as was shown in the monographs [9] and [27]. Here, we consider the general case that the impinging jet issuing from two axially symmetric nozzles with variable cross-section.

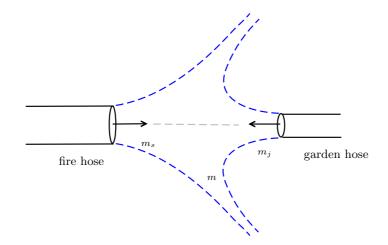


FIGURE 6. Collision of two jets (Figure 16.7 in [24])

Another motivation to investigate the impinging outgoing jets issuing from two nozzles comes from the Chapter 16 in famous survey [24]. The physical problem is also related to the shaped charge question in [13]. As mentioned in Page 152 [24], "... we can formulate this problem as a collision of two jets, say a garden hose and a fire hose; see Figure 16.7." (see Figure 6) and A. Friedman proposed an open problem that

"Problem (1). Analyze the axially symmetric free boundary problem associated with the flow in Figure 16.7 in incompressible case."

On another side, there are many numerical results on this impinging outgoing jet problem, such as the incompressible problem for an arbitrary polygonal nozzle in [14], and the incompressible jet with gravity in [15] and so on. Moreover, Hurean and Weber in [28] considered the impinging of two incompressible ideal free jets (in the absence of rigid nozzle walls) numerically, and some existence results on two compressible free jets were also investigated in [10]. However, we also would like to mention the numerical result on asymmetric impinging free jets in [29].

The study of liquid jets issuing from containers is centuries old. As far back as 1868, Helmholtz and Kirchhoff introduced the classical theory of free streamlines in twodimensional jets. The steady irrotational flows of ideal incompressible fluid, bounded by nozzle walls and free streamlines were investigated. The following decades saw extensions of a great many different kinds of two-dimensional flows, on the basis of the complex analysis methods by Planck, Joukowsky, Réthy, Levi-Civita, Greenhill and others.

Some substantial post-war monographs are those of Birkhoff-Zarantonello [9], Gurevich [27], Milne-Thomson [30]. For two-dimensional irrotational case, a generalized Schwarz-Christoffel transformation, combined with a Fourier technique to formulate a free boundary

problem into a nonlinear integral-differential equation, some existence results on jets in special nozzles have been constructed. However, two-dimensional jets have been given most of the attention in the existence theory, and the limited amount of work on axisymmetric jets has been confined. The reason is that the complex analysis method which has been adapted to two-dimensional jets has noneffective in the axially symmetric case. A first breakthrough work on the axially symmetric free streamline was due to Garabedian, Lewy and Schiffer in [25] in 1952, in which some existence results on the axially symmetric cavity were established by variational approach. Furthermore, Alt, Caffarelli and Friedman developed the variational method to deal the free streamlines problem in their elegant works [1, 8]. Based on their framework, some well-posedness results on axially symmetric jet in [4], asymmetric jet in [2], jet with gravity in [3], and jets with two fluids in [6, 7] have been established. In this paper, some fundamental ideas on the existence theory are still borrowed from the variational method in [1]. Recently, if we assume that the fluid is smooth across the interface Γ (i.e. $\Lambda = 0$) apriority, some existence results for incompressible plane symmetric impinging outgoing jets has been obtained in [17]. In fact, the interface Γ is a contact discontinuity and the jump Λ is always non-trivial and non-zero generally. As we mentioned before, we have to investigate the non-zero jump $\Lambda \neq 0$, which is one of main differences between this paper to the previous paper [17]. As a first step to attack the original problem on impinging outgoing jets with nontrivial jump, Wang and Xiang in [31] considered a toy model on the incompressible fluids issuing from two infinity long co-axis and symmetric nozzles without jet free boundary and established some properties on the contact discontinuity between the two fluids. Therefore, the objective of the present paper is to establish the well-posedness theory on the impinging outgoing jet problem and solve the open problem proposed by A. Frideman in 1989.

1.3. **Methodology.** From the physical problem here, the main difference and difficulty here stems from the shape of nozzle walls, we have to find a mechanism, such that the free boundaries of the jets connecting smoothly at the edge of the nozzle walls (so-called *continuous fit and smooth fit conditions*). Another main difficulty is how to analyze the interface with contact discontinuity between the two incompressible ideal fluids.

We would like to comment the main ideas of the proof as follows. The variational method developed by H. Alt, L. Caffarelli and A. Friedman in 1980s has been shown to be powerful and effective to solve the free streamline theory for more general models. In two-dimensional case, the stream function is harmonic in the fluid domain, while in axially symmetric case, it solves a linear elliptic equation with some lower-order term, and we have to deal with the regularity near the symmetric axis. This is the first difficulty in this paper. The second one is that the distance of the two free boundaries converges to a positive constant in two-dimensional case (see [17]), and however, the distance goes to zero in axially symmetric jet here. Therefore, we can not borrow some uniform special flow to show some fundamental properties of the free boundaries as in 2D case, such as the vanishing and non-vanishing of free boundary, and the asymptotic behaviors of the jet in far fields, and so on. Here, our strategy is to establish firstly the decay rate of distance of the two free boundaries and the impinging outgoing jet in far field, and then to obtain the desired properties via some rescaling arguments. The third principal difficulty in this paper centers about how to choose the suitable parameters (λ, θ) to assure the continuous fit conditions in impinging outgoing jets. Some continuous dependent relationships and monotonic properties to the impinging outgoing jets with respect to the parameters (λ, θ) are established and guarantee the fact to be fulfilled. The fourth difficulty here is the occurrence of three free boundaries Γ_1 , Γ_2 and Γ , which is main difference to the previous results, such as [17, 31]. In particular, the solution is not smooth across the interface Γ and it is a contact discontinuity between the two fluids. To our knowledge, this is the first well-posedness work on the jet flows problem with three free boundaries.

The remain of this paper is organized as follows. First, we establish the free boundary value problem to the physical problem in Section 2. The solvability of the free boundary value problem follows from the standard variational approach, which has been developed by Alt, Caffarelli and Friedman in the celebrated works [1, 2, 4]. Moreover, some properties of the free boundaries will be obtained and we can verify the continuous fit and smooth fit conditions for suitable parameters λ and θ . Additionally, we will investigate the existence and properties of the interface between the two fluids. Hence, we can obtain the existence of impinging outgoing jet in Section 3. In Section 4, we will give the uniqueness of the impinging outgoing jet is obtained along the blow-up argument, which has been used to deal with the subsonic compressible flows in infinitely long nozzles in [16, 18, 19, 20, 21, 22, 32, 33, 34]. Some results on the variational problem are given in the Appendix.

2. MATHEMATICAL SETTINGS OF THE FREE BOUNDARY PROBLEM

Due to the continuity equation (1.3), one can introduce stream functions $\Psi_i(x,r)$ (i = 1, 2) such that

$$\partial_x \Psi_i = -r\rho_i V_i, \quad \partial_r \Psi_i = r\rho_i U_i. \tag{2.1}$$

In order to convenient to the analysis, we introduce the scaled stream function $\psi_i = \frac{\Psi_i}{\sqrt{\rho_i}}$ (i = 1, 2), and denote ψ as

$$\psi = \begin{cases} \psi_1 & \text{in } \Omega_1, \\ \\ \psi_2 & \text{in } \Omega_2. \end{cases}$$

This together with the irrotational condition gives

$$\Delta \psi - \frac{1}{r} \frac{\partial \psi}{\partial r} = 0 \quad \text{in } \Omega_1 \cup \Omega_2.$$
(2.2)

Here and after, Ω_i denotes the flow field, bounded by the nozzle walls N_i , the symmetric axis N_0 , the interface Γ and the free boundaries Γ_i (i = 1, 2).

In this paper, we expect to seek an axially symmetric impinging outgoing jet flow with positive vertical velocity, and thus denote Ω bounded by N_i , N_0 and L_i (i = 1, 2) as the possible flow field of impinging outgoing jet (see Figure 7), where

$$L_1 = \{(x,r) \mid r = R, x < -1\}$$
 and $L_2 = \{(x,r) \mid r = R, x > 1\}.$

Moreover, the nozzles $N_1 \cup N_2$ and the free boundaries $\Gamma_1 \cup \Gamma_2$ are streamlines, thus ψ remains some constant on those boundaries, without loss of generality, we can impose

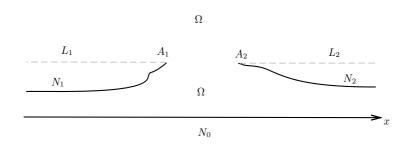


FIGURE 7. The possible flow field Ω

 $\psi = m_1$ on $N_1 \cup \Gamma_1$ and $\psi = m_2$ on $N_2 \cup \Gamma_2$. Thus, the free boundaries Γ_1 and Γ_2 can be defined as

$$\Gamma_1 = \Omega \cap \partial \left\{ \psi < m_1 \right\}, \ \ \Gamma_2 = \Omega \cap \partial \left\{ \psi > m_2 \right\},$$

respectively, where Ω is the possible flow field defined before and $m_i = \frac{M_i}{2\pi\sqrt{\rho_i}}$ (i = 1, 2). And the constant pressure boundary condition on free boundaries can be rewritten as

$$\left|\frac{1}{r}\frac{\partial\psi}{\partial\nu}\right| = \sqrt{\Lambda + \lambda} \text{ on } \Gamma_1, \qquad \left|\frac{1}{r}\frac{\partial\psi}{\partial\nu}\right| = \sqrt{\lambda} \text{ on } \Gamma_2, \tag{2.3}$$

where ν is unit outward normal to the free stream surfaces Γ_1 and Γ_2 .

Hence, we formulate the boundary value problem to the stream function ψ ,

$$\begin{aligned}
\Delta \psi - \frac{1}{r} \frac{\partial \psi}{\partial r} &= 0, & \text{in } \Omega_1 \cup \Omega_2, \\
\left| \frac{1}{r} \frac{\partial \psi}{\partial \nu} \right| &= \sqrt{\Lambda + \lambda}, & \text{on } \Gamma_1, \quad \left| \frac{1}{r} \frac{\partial \psi}{\partial \nu} \right| &= \sqrt{\lambda}, & \text{on } \Gamma_2, \\
\left| \frac{\nabla \psi^+}{r} \right|^2 - \left| \frac{\nabla \psi^-}{r} \right|^2 &= \Lambda, & \text{on } \Gamma, \\
\psi &= m_1, & \text{on } N_1 \cup \Gamma_1, \quad \psi = m_2, & \text{on } N_2 \cup \Gamma_2, \\
\psi &= 0, & \text{on } N_0 \cup \Gamma,
\end{aligned}$$
(2.4)

where $\psi^{\pm}(X_0)$ $(X_0 \in \Gamma)$ denotes the limit of $\psi(X)$ with $X \in \{\pm \psi > 0\}$, as $X \to X_0$.

We would like to emphasize that the undetermined constants λ and θ are regarded as two parameters to solve the free boundary problem. We will solve the free boundary problem for any λ and θ , and then to show the existence of suitable parameters to guarantee the continuous fit conditions.

3. EXISTENCE OF THE IMPINGING OUTGOING JETS

3.1. Truncated variational problem. In order to solve the free boundary value problem (2.4), we first introduce some notations and two auxiliary functions as follows. Define domain D as

$$D = \Omega \cap \{r < R\}.$$

Next, we define two bounded functions Φ_1 and Φ_2 as follows,

$$\Delta \Phi_{1} - \frac{1}{r} \frac{\partial \Phi_{1}}{\partial r} = 0 \text{ in } D, \text{ and } m_{2} < \Phi_{1} < m_{1} \text{ in } D,$$

$$\Phi_{1}(x,r) = \begin{cases} 0, & \text{if } (x,r) \in N_{0}, \\ m_{2}, & \text{if } r_{2} < r \leq R, \ (x,r) \text{ lies right } N_{2}, \\ m_{1}, & \text{if } r_{1} < r \leq R, \ (x,r) \text{ lies right } N_{1}, \\ m_{1}, & \text{if } (x,r) \in \Omega \cap \{r \geq R\}, \end{cases}$$
(3.1)

and

$$\Delta \Phi_2 - \frac{1}{r} \frac{\partial \Phi_2}{\partial r} = 0 \text{ in } D, \ m_2 < \Phi_2 < m_1 \text{ in } D,$$

$$\Phi_2(x, r) = \begin{cases} 0, & \text{if } (x, r) \in N_0, \\ m_2, & \text{if } r_2 < r \le R, \ (x, r) \text{ lies right } N_2, \\ m_1, & \text{if } r_1 < r \le R, \ (x, r) \text{ lies left } N_1, \\ m_2, & \text{if } (x, r) \in \Omega \cap \{r \ge R\}. \end{cases}$$
(3.2)

We introduce the admissible set as

$$K = \left\{ \psi \in H^1_{loc}(\Omega) | \Phi_2 \le \psi \le \Phi_1 \right\},$$

set $e = (-\sin\theta, \cos\theta)$ with $\theta \in [0, \pi]$, and a functional

$$J_{\lambda,\theta}(\psi) = \int_{\Omega} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dx dr.$$
(3.3)

Since the functional $J_{\lambda,\theta}$ is unbounded for any $\psi \in K$, we have to truncate the possible flow field Ω and formulate truncated problems as follows.

For any $\mu > 1$ and i = 1, 2, we define

$$r_{i,\mu} = \min\{r \mid (-1)^{i} \mu = f_{i}(r)\}, \quad H_{i,\mu} = \{((-1)^{i} \mu, r) \mid 0 \le r \le r_{i,\mu}\}, \\ N_{0,\mu} = N_{0} \cap \{-\mu < x < \mu\}, \text{ and } N_{i,\mu} = \{(x,r) \mid x = f_{i}(r), r_{i,\mu} < r \le R\}.$$
(3.4)

Moreover, we introduce a cutoff domain Ω_{μ} (see Figure 8) as

$$\Omega_{\mu}$$
 is bounded by $N_{i,\mu}$, L_i , $N_{0,\mu}$ and $H_{i,\mu}$, (3.5)

、

and denote

$$D_{\mu} = \Omega_{\mu} \cap \{r < R\}.$$

We also introduce an admissible set

$$K_{\mu} = \left\{ \psi \in K \left| \psi = \frac{m_1}{r_{1,\mu}^2} r^2 \text{ on } H_{1,\mu}, \ \psi = \frac{m_2}{r_{2,\mu}^2} r^2 \text{ on } H_{2,\mu} \right\},\$$

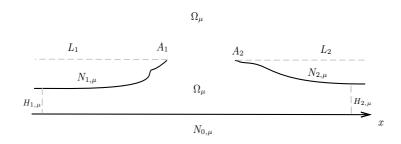


FIGURE 8. The truncated domain Ω_{μ}

and an auxiliary functional

$$J_{\lambda,\theta,\mu}(\psi) = \int_{\Omega_{\mu}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dX, \quad \psi \in K_{\mu},$$

where χ_E is the indicator function of the set *E*. Here and after, we denote dX = dxdr and X = (x, r) for simplicity.

The truncated variational problem $(P_{\lambda,\theta,\mu})$: Find a $\psi_{\lambda,\theta,\mu} \in K_{\mu}$, such that

$$J_{\lambda,\theta,\mu}(\psi_{\lambda,\theta,\mu}) = \min_{\psi \in K_{\mu}} J_{\lambda,\theta,\mu}(\psi).$$
(3.6)

Furthermore, the free boundaries of the truncated variational problem $(P_{\lambda,\theta,\mu})$ are defined as follows.

Definition 3.1. The set

$$\Gamma_{1,\mu} = \Omega_{\mu} \cap \partial \left\{ \psi_{\lambda,\theta,\mu} < m_1 \right\},\,$$

is called the left free boundary, and

$$\Gamma_{2,\mu} = \Omega_{\mu} \cap \partial \left\{ \psi_{\lambda,\theta,\mu} > m_2 \right\},$$

is called the right free boundary.

Furthermore, define

$$\Gamma_{\mu} = \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} = 0\},\$$

be the interface separating the two fluids.

3.2. Existence of minimizer to the truncated variational problem. First, we give the existence of the minimizer to the truncated variational problem.

Proposition 3.1. For any $\lambda > 0$, $\theta \in [0, \pi]$ and $\mu > 1$, there exists a minimizer $\psi_{\lambda,\theta,\mu} \in K_{\mu}$ to the truncated variational problem $(P_{\lambda,\theta,\mu})$.

Proof. Due to Theorem 1.3 in [1], it suffices to construct a function $\psi_0 \in K_{\mu}$ such that $J_{\lambda,\theta,\mu}(\psi_0) < +\infty$.

Case 1: For $\theta \in (0, \pi)$ in Ω_{μ} .

Indeed, for some sufficiently large

$$R_0 > \max\left\{1, \frac{m_1}{\sqrt{\Lambda + \lambda}(R+1)\sin\theta} - (R+1)\cot\theta, -\frac{m_2}{\sqrt{\lambda}(R+1)\sin\theta} + (R+1)\cot\theta\right\},\$$

and taking $\overline{\psi}$ be a smooth function such that $\psi_0 \in K_{\mu}$. Define ψ_0 in Ω_{μ} as follows

$$\psi_{0}(X) = \begin{cases} \min\left\{\max\left\{\sqrt{\Lambda + \lambda r}\left(r\cos\theta - x\sin\theta\right), 0\right\}, m_{1}\right\}, & \text{if } r \ge R + 1, \ r\cos\theta - x\sin\theta \ge 0, \\ \max\left\{\min\left\{\sqrt{\lambda r}\left(r\cos\theta - x\sin\theta\right), 0\right\}, m_{2}\right\}, & \text{if } r \ge R + 1, \ r\cos\theta - x\sin\theta \le 0, \\ m_{1}, & \text{if } x \le R + 1, \ r\cos\theta - x\sin\theta \le 0, \\ m_{2}, & \text{if } x \le -R_{0}, \ R \le r \le R + 1, \\ m_{2}, & \text{if } x \ge R_{0}, \ R \le r \le R + 1, \\ \overline{\psi}(X), & \text{if } (x, r) \in \Omega_{\mu, R_{0}}, \\ \eta(x)\frac{m_{1}}{r_{1, \mu}^{2}}r^{2} + (1 - \eta(x))\frac{m_{2}}{r_{2, \mu}^{2}}r^{2}, & \text{if } (x, r) \in \Omega_{\mu}. \end{cases}$$

Here, $\Omega'_{\mu} = \Omega_{\mu} \cap \{r \leq \min\{r_{1,\mu}, r_{2,\mu}\}\}, \eta(x)$ be a cut-off function satisfying

$$\eta(x) = 1 \text{ for } x \le -\mu, \ \eta(x) = \frac{\mu - x}{2\mu} \text{ for } -\mu \le x \le \mu, \ \eta(x) = 0 \text{ for } x \ge \mu,$$
 (3.7)

and

$$\Omega_{\mu,R_0} = \Omega_{\mu} \cap \{\min\{r_{1,\mu}, r_{2,\mu}\} \le r \le R\} \cup \{-R_0 \le x \le R_0, R \le r \le R+1\}.$$

It is easy to check that $J_{\lambda,\theta,\mu}(\psi_0) < +\infty$.

Case 2: For $\theta = 0$ or π .

Without loss of generality, assume $\theta = 0$. It suffice to define a function $\psi_0(X)$ as follows. Set

$$\tilde{\Omega}_{\mu,R_0} = \Omega_{\mu} \cap \{\min\{r_{1,\mu}, r_{2,\mu}\} \le r \le R\} \cup \{-2 \le x \le 2, R \le r \le R_0\},\$$

for some sufficiently large $R_0 > \sqrt{R^2 + \frac{2m_1}{\sqrt{\Lambda + \lambda}} - \frac{2m_2}{\sqrt{\lambda}}}$, and define a function ψ_0 as

$$\psi_{0}(X) = \begin{cases} \frac{\sqrt{\lambda}(r^{2} - R^{2})}{2} + m_{2}, & \text{if } x \ge 2, \quad R \le r \le \sqrt{R^{2} - \frac{2m_{2}}{\sqrt{\lambda}}}, \\ \min\left\{\frac{\sqrt{\Lambda + \lambda}}{2}\left(r^{2} + \frac{2m_{2}}{\sqrt{\lambda}} - R^{2}\right), m_{1}\right\}, & \text{if } x \ge 2, \quad r \ge \sqrt{R^{2} - \frac{2m_{2}}{\sqrt{\lambda}}}, \\ m_{1}, & \text{if } x \le 2, \quad r \ge R_{0}, \\ m_{1}, & \text{if } x \le 2, \quad r \ge R_{0}, \\ m_{1}, & \text{if } x \le -2, \quad R \le r \le R_{0}, \\ \overline{\psi}(X), & \text{if } (x, r) \in \tilde{\Omega}_{\mu, R_{0}}, \\ \eta(x)\frac{m_{1}}{r_{1, \mu}^{2}}r^{2} + (1 - \eta(x))\frac{m_{2}}{r_{2, \mu}^{2}}r^{2}, & \text{if } (x, r) \in \Omega_{\mu} \cap \{r \le \{r_{1, \mu}, r_{2, \mu}\}\}, \end{cases}$$

where $\eta(x)$ is defined as (3.7), and $\overline{\psi}$ be a smooth function such that $\psi_0 \in K_{\mu}$.

Therefore, we finish the proof of Proposition 3.1.

Next, we will obtain the regularity of the minimizer.

Proposition 3.2. Let $\psi_{\lambda,\theta,\mu}$ be a minimizer to the truncated variational problem $(P_{\lambda,\theta,\mu})$, and for any open subset $\Omega_0 \subset \subset \Omega_\mu \cap \{m_2 < \psi_{\lambda,\theta,\mu} < m_1\} \cap \{\psi_{\lambda,\theta,\mu} \neq 0\}$, then $\psi_{\lambda,\theta,\mu} \in C^{0,1}(\Omega_\mu)$, $\psi_{\lambda,\theta,\mu} \in C^{2,\sigma}(\Omega_0)$ and $\psi_{\lambda,\theta,\mu} \in C^{1,\sigma}(\Omega_0 \cup N_{1,\mu} \cup N_{2,\mu})$ for some $0 < \sigma < 1$.

Proof. Firstly, $\psi_{\lambda,\theta,\mu} \in C^{0,1}(\Omega_{\mu})$ follows in the same manner as Corollary 4.4 in [6].

Next, the standard interior estimates to linear elliptic equation in Chapter 8 in [26] gives $\psi_{\lambda,\theta,\mu} \in C^{2,\sigma}(\Omega_0)$ and $\psi_{\lambda,\theta,\mu} \in C^{1,\sigma}(N_{1,\mu} \cup N_{2,\mu})$.

The regularity of $\psi_{\lambda,\theta,\mu}$ near the axis $N_{0,\mu}$ can be obtained by the standard arguments as in [16] and [33].

Therefore, we finish the proof of Lemma 3.2.

3.3. Uniqueness and monotonicity of the minimizer. Firstly, we will give a lower bound and an upper bound to the minimizer $\psi_{\lambda,\theta,\mu}$.

Lemma 3.3. For any minimizer $\psi_{\lambda,\theta,\mu}$ to the variational problem $(P_{\lambda,\theta,\mu})$, one has

$$\max\left\{\frac{m_2}{r_{2,\mu}^2}r^2, m_2\right\} \le \psi_{\lambda,\theta,\mu}(x,r) \le \min\left\{\frac{m_1}{r_{1,\mu}^2}r^2, m_1\right\} \quad in \quad \Omega_{\mu},$$
(3.8)

where $r_{1,\mu}$ and $r_{2,\mu}$ are defined in (3.4).

Proof. Firstly, consider the lower bound of $\psi_{\lambda,\theta,\mu}$.

Set
$$\phi_1 = \max\left\{\frac{m_2}{r_{2,\mu}^2}r^2, m_2\right\}, \phi_2 = \min\left\{\frac{m_1}{r_{1,\mu}^2}r^2, m_1\right\}$$
 and $\psi = \psi_{\lambda,\theta,\mu}$ for simplicity.
Firstly, since $\psi \in K_{\mu}$, one has

$$m_2 \le \psi \le m_1$$
 in Ω_{μ} . (3.9)

Next, we shall prove that

$$\phi_1 \leq \psi \quad \text{in } \Omega_\mu.$$

Due to the fact that $\max{\{\psi, \phi_1\}} \in K_{\mu}$, we have

 $J_{\lambda,\theta,\mu}(\psi) \le J_{\lambda,\theta,\mu}(\max\left\{\psi,\phi_1\right\}).$

Furthermore, the fact

$$\phi_1 = m_2 \text{ for } r \ge r_{2,\mu},$$
 (3.10)

gives that

$$\psi \ge \phi_1$$
 in $\Omega_\mu \cap \{r \ge r_{2,\mu}\}.$

Therefore, we obtain

$$0 \ge \int_{\Omega_{\mu,r_{2,\mu}}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) \cdot e \right|^2 dX - \int_{\Omega_{\mu},r_{2,\mu}} r \left| \frac{\nabla \max\left\{\psi, \phi_1\right\}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \max\{\psi, \phi_1\} < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \max\{\psi, \phi_1\} \le 0\}} \right) \cdot e \right|^2 dX$$

$$(3.11)$$

here, $\Omega_{\mu,r_{2,\mu}} = \Omega_{\mu} \cap \{r < r_{2,\mu}\}$. This together with the similar arguments as Lemma 3.4 in [31] yields to

$$\int_{\Omega_{\mu,r_{2,\mu}}} \left|\nabla \min\left(\psi - \phi_1, 0\right)\right|^2 dX \le 0,$$

which implies that

$$\psi - \phi_1 = \text{constant} \quad \text{in } \Omega_{\mu, r_2 \mu}$$

Since $\psi \ge \phi_1$ on $\partial \Omega_{\mu, r_{2,\mu}}$, we conclude that

 $\psi \ge \phi_1$ in $\Omega_{\mu, r_{2,\mu}}$.

Therefore, we obtain the lower bound of $\psi_{\lambda,\theta,\mu}$ in (3.8).

Next, we can now proceed as before to show the upper bound of $\psi_{\lambda,\theta,\mu}$.

Similarly, it suffices to prove that the upper bound holds in $\Omega_{\mu,r_{1,\mu}} = \Omega_{\mu} \cap \{r < r_{1,\mu}\}$. Due to (6.2), one has

$$\Delta \psi_{\lambda,\theta,\mu} - \frac{1}{r} \frac{\partial \psi_{\lambda,\theta,\mu}}{\partial r} \ge 0 \text{ in } \Omega_{\mu,r_{1,\mu}} \text{ in a weak sense,}$$

and $\psi_{\lambda,\theta,\mu} \leq \frac{m_1}{r_{1,\mu}^2} r^2$ on $\partial \Omega_{\mu,r_{1,\mu}}$, then the maximum principle implies $\psi_{\lambda,\theta,\mu}(x,r) \leq \frac{m_1}{r_{1,\mu}^2} r^2$ in $\Omega_{\mu,r_{1,\mu}}$. We complete the proof of Lemma 3.3.

Next, in view of Lemma 3.3, using the similar arguments Proposition 3.5 in [31], we will establish the uniqueness and some monotonicity of the minimizer $\psi_{\lambda,\theta,\mu}$ to the variational problem $(P_{\lambda,\theta,\mu})$, and we omit the proof here.

Proposition 3.4. For any $\lambda \in (0, +\infty)$ and $\theta \in [0, \pi]$, the minimizer $\psi_{\lambda,\theta,\mu}$ to the truncated variational problem $(P_{\lambda,\theta,\mu})$ is unique. Furthermore, the solution $\psi_{\lambda,\theta,\mu}$ is monotonic with respect to x, namely

$$\psi_{\lambda,\theta,\mu}(x,r) \le \psi_{\lambda,\theta,\mu}(\tilde{x},r) \quad \text{for any } x \ge \tilde{x}.$$

$$(3.12)$$

3.4. Some properties of the free boundaries.

3.4.1. *Preliminaries.* Before we investigate the properties of the free boundaries, we give some important auxiliary lemmas, and we refer the proofs in [1, 2, 23]. So we only state the result and omit the proof as follows.

Lemma 3.5. There exists a universal constant c > 0, such that for $X_0 = (x_0, r_0) \in \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} < 0\}$ and $B_r(X_0) \subset \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} < 0\}$ with

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \left(\psi_{\lambda,\theta,\mu} - m_2 \right) dS \ge \sqrt{\lambda} c r_0,$$

then we have $\psi_{\lambda,\theta,\mu} > m_2$ in $B_r(X_0)$; Similarly, $X_0 = (x_0, r_0) \in \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} > 0\}$ and $B_r(X_0) \subset \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} > 0\}$, if

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \left(m_1 - \psi_{\lambda,\theta,\mu} \right) dS \ge \sqrt{\lambda + \Lambda} cr_0,$$

then we have $\psi_{\lambda,\theta,\mu} < m_1$ in $B_r(X_0)$. Here and after, $B_r(X)$ denotes some ball with radius r > 0 and center $X \in \Omega_{\mu}$.

Next, we will establish a non-degeneracy lemma to $\psi_{\lambda,\theta,\mu} - m_2$ and $m_1 - \psi_{\lambda,\theta,\mu}$ as follows.

Lemma 3.6. (Non-degeneracy lemma) For any $0 < \kappa_1 < 1$, there exists a positive constant c (depending on κ_1), if $B_r(X_0) \subset \Omega_{\mu} \cap \{\psi_{\lambda,\theta,\mu} < 0\}$ ($X_0 = (x_0, r_0)$) and

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \left(\psi_{\lambda,\theta,\mu} - m_2 \right) dS \le \sqrt{\lambda} cr_0, \text{ and } \psi_{\lambda,\theta,\mu} < 0 \text{ in } B_r(X_0),$$

then $\psi_{\lambda,\theta,\mu} = m_2$ in $B_{\kappa_1 r}(X_0)$; Similarly, for any $0 < \kappa_2 < 1$, there exists a positive constant c (depending on κ_2) and

$$\frac{1}{r} \oint_{\partial B_r(X_0)} (m_1 - \psi_{\lambda,\theta,\mu}) \, dS \le \sqrt{\lambda + \Lambda} cr_0, \text{ and } \psi_{\lambda,\theta,\mu} > 0 \text{ in } B_r(X_0),$$

then $\psi_{\lambda,\theta,\mu} = m_1$ in $B_{\kappa_2 r}(X_0)$.

A direct application of Lemma 3.6 gives the following lemma.

Lemma 3.7. Suppose that $X_0 = (x_0, r_0) \in \overline{\{\psi_{\lambda,\theta,\mu} > m_2\} \cap (\Omega_\mu \setminus D_\mu)}$ and $\psi_{\lambda,\theta,\mu} < 0$ in $B_r(X_0)$ for some r > 0, then

$$\frac{1}{r} \oint_{\partial B_r(X_0)} \left(\psi_{\lambda,\theta,\mu} - m_2 \right) dS \ge \sqrt{\lambda} c r_0.$$
(3.13)

In particular,

$$\sup_{\partial B_r(X_0)} \left(\psi_{\lambda,\theta,\mu} - m_2 \right) \ge \sqrt{\lambda} c r_0 r.$$
(3.14)

We shall establish a non-oscillation lemma, which implies that the free boundary $\Gamma_{i,\mu}$ for i = 1, 2 cannot oscillate near the solid boundaries. Without loss of generality, consider the right free boundary $\Gamma_{2,\mu}$, and introduce a domain $G \subset \Omega_{\mu} \setminus D_{\mu}$ bounded by

$$x = x_1, \quad x = x_1 + h \ (h > 0)$$

and

$$\gamma_1 : X = X^1(t) = (x^1(t), r^1(t)), \quad \gamma_2 : X = X^2(t) = (x^2(t), r^2(t))$$

where $0 \le t \le T$ with

$$x_1 < x^i(t) < x_1 + h$$
 for $0 < t < T_1$

and

$$x^{i}(0) = x_{1}, x^{i}(T) = x_{1} + h, r_{1} \le r^{i}(t) \le r_{1} + \delta, i = 1, 2.$$

Furthermore, the arc γ_2 lies above the arc γ_1 , this implies that $r^1(0) < r^2(0)$, γ_1 and γ_2 do not intersect, γ_2 is contained in $\Gamma_{2,\mu}$, either

Case 1.
$$\gamma_1$$
 is contained in $\Gamma_{2,\mu}$, (see Figure 9)

or

Case 2. γ_1 lies on $\{r = R, x > 1\}$, and then $r_1 = R, x_1 \ge 1$. (see Figure 10)

Let the domain $G \subset \{\psi_{\lambda,\theta,\mu} > m_2\}$ be a neighborhood of γ_1 and γ_2 , and $\psi_{\lambda,\theta,\mu} < 0$ in G and for some $c^* > 0$, we have

$$\operatorname{dist}(G, \overline{A_1 A_2}) > c^*.$$

Lemma 3.8. (Non-oscillation lemma) Under the foregoing assumptions, there exists a positive constant C depending only on λ , m_2 and c^* such that

$$h \le C\delta. \tag{3.15}$$

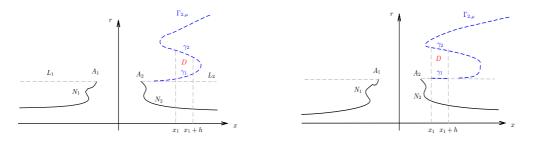


FIGURE 9. Case 1

FIGURE 10. Case 2

The proof is similar to Lemma 4.1 in [2] and Lemma 5.6 [4], we omit it here.

Finally, we give the uniform bound of the gradient to the minimizer, which is independent of m_1 and m_2 . Please see Lemma 8.1 in [4] and Lemma 5.2 in [2] for the proof.

Lemma 3.9. Let $X_0 = (x_0, r_0)$ be a free boundary point in $\Omega_{\mu} \setminus \overline{D}_{\mu}$ and G be a bounded domain with $X_0 \in G$, $\overline{G} \subset \Omega_{\mu} \setminus \overline{D}_{\mu}$. There exists a constant C > 0 depending only on λ , G and Λ , such that

$$\frac{\nabla\psi_{\lambda,\theta,\mu}|}{r} \le C \text{ in } G. \tag{3.16}$$

3.4.2. Some properties of the free boundaries. It follows from the monotonicity of $\psi_{\lambda,\theta,\mu}$ with respect to x that the free boundaries are r-graph, namely, the free boundaries $\Gamma_{i,\mu}$ (i = 1, 2) intersect $r = r_0$ either one single point or a segment for any $r_0 \in (R, +\infty)$. Thus, there exist four mappings $g_{1,\lambda,\theta,\mu}(r)$ with r > R, $g_{2,\lambda,\theta,\mu}(r)$ with r > R, $g_{\lambda,\theta,\mu}(r)$ with r > 0 and $\tilde{g}_{\lambda,\theta,\mu}(r)$ with r > 0 such that

$$\{0 < \psi_{\lambda,\theta,\mu} < m_1\} \cap \Omega_\mu = \{\tilde{g}_{1,\lambda,\theta,\mu}(r) < x < g_{\lambda,\theta,\mu}(r)\} \cap \Omega_\mu, \tag{3.17}$$

and

$$\{m_2 < \psi_{\lambda,\theta,\mu} < 0\} \cap \Omega_{\mu} = \{\tilde{g}_{\lambda,\theta,\mu}(r) < x < \tilde{g}_{2,\lambda,\theta,\mu}(r)\} \cap \Omega_{\mu}, \tag{3.18}$$

where

$$\tilde{g}_{1,\lambda,\theta,\mu}(r) = \begin{cases} f_1(r) & \text{for } 0 < r \le R, \\ g_{1,\lambda,\theta,\mu}(r) & \text{for } R < r < +\infty, \end{cases}$$

and

$$\tilde{g}_{2,\lambda,\theta,\mu}(r) = \begin{cases} f_2(r) & \text{for } 0 < r \le R, \\ g_{2,\lambda,\theta,\mu}(r) & \text{for } R < r < +\infty. \end{cases}$$

Indeed, along similar arguments as in [2], we obtain that $g_{i,\lambda,\theta,\mu}(r)$ is indeed a general continuous function in $[R, +\infty)$, and $g_{i,\lambda,\theta,\mu}(R)$ is defined as $\lim_{r\to R^+} g_{i,\lambda,\theta,\mu}(r)$ for i = 1, 2. Furthermore, due to Lemma 3.3 in [7] and Proposition 4.1 in [31], the interface $g_{\lambda,\theta,\mu}(r) \equiv \tilde{g}_{\lambda,\theta,\mu}(y)$ is indeed a continuous function in $[0, +\infty)$, and we omit the proof here.

Lemma 3.10. The free boundary $\Gamma_{i,\mu} : x = g_{i,\lambda,\theta,\mu}(r)$ is a generalized continuous function in $R \leq r < +\infty$ with values in $[-\infty, +\infty]$ (i = 1, 2), respectively. Furthermore, the interface $\Gamma_{\mu} : x = g_{\lambda,\theta,\mu}(r)$ is bounded continuous functions in $0 < r < +\infty$, $g_{\lambda,\theta,\mu}(0+0) \triangleq \lim_{r \to 0^+} g_{\lambda,\theta,\mu}(r)$ exists and is finite. In order to study the limit behavior of the solution as $r \to +\infty$, we first establish the decay estimate of the minimizer in far field as follows. This is one of the crucial parts in this paper.

Lemma 3.11. For any $\theta \in (0, \pi)$ and $r_0 > 2R$, there exists a constant C (independent of r_0) such that

$$\int_{\Omega_{\mu} \cap \{r > r_0\}} r \left| \frac{\nabla \psi_{\lambda,\theta,\mu}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi_{\lambda,\mu,\theta} < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi_{\lambda,\theta,\mu} \le 0\}} \right) e \right|^2 dX \le \frac{C}{r_0^3}.$$
(3.19)

Proof. Denote $\psi(x,r) = \psi_{\lambda,\theta,\mu}(x,r)$, $g(r) = g_{\lambda,\theta,\mu}(r)$ and $g_i(r) = g_{i,\lambda,\theta,\mu}(r)$ (i = 1, 2) for simplicity.

For any $r_0 > 2R$, define

$$S(r_0) = \int_{\Omega_{\mu} \cap \{\frac{r_0}{2} < r < r_0\}} r \left| \frac{\nabla \psi_{\lambda,\theta,\mu}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{m_2 < \psi_{\lambda,\mu,\theta} < 0\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi_{\lambda,\theta,\mu} \le 0\}} \right) e \right|^2 dX$$

Taking advantage of the mean value theorem, there exists some $\tilde{r} \in \left(\frac{r_0}{2}, r_0\right)$ such that

$$S(r_0) = \frac{r_0}{2\tilde{r}} \left\{ \int_{g_1(\tilde{r})}^{g(\tilde{r})} \left| \nabla \psi(x,\tilde{r}) - \sqrt{\Lambda + \lambda} \tilde{r} e \right|^2 dx + \int_{g(\tilde{r})}^{g_2(\tilde{r})} \left| \nabla \psi(x,\tilde{r}) - \sqrt{\lambda} \tilde{r} e \right|^2 dx \right\}$$

$$\geq \frac{1}{2} \left\{ \int_{g_1(\tilde{r})}^{g(\tilde{r})} \left| \nabla \psi(x,\tilde{r}) - \sqrt{\Lambda + \lambda} \tilde{r} e \right|^2 dx + \int_{g(\tilde{r})}^{g_2(\tilde{r})} \left| \nabla \psi(x,\tilde{r}) - \sqrt{\lambda} \tilde{r} e \right|^2 dx \right\}.$$
(3.20)

We choose a function w(x, r) as follows

$$w(x,r) = \begin{cases} \psi(x,r), & \text{in } \Omega_{\mu} \cap \{r \le \tilde{r}\}, \\ \eta(r)\bar{\psi}(x,r) + (1-\eta(r))\phi(x,r), & \text{in } \Omega_{\mu} \cap \{r \ge \tilde{r}\}, \end{cases}$$
(3.21)

where $\eta(r) = \max\left\{0, \frac{\bar{r} - r}{\bar{r} - \tilde{r}}\right\}$ with $\bar{r} = \tilde{r} + \frac{1}{\tilde{r}}$,

$$\psi(x,r) = \psi(x - (r - \tilde{r})\cot\theta, \tilde{r}),$$

and

$$\phi(x,r) = \min\left\{\max\left\{\sqrt{\Lambda + \lambda}r\left((r-\tilde{r})\cos\theta - (x-g(\tilde{r}))\sin\theta\right), 0\right\}, m_1\right\} + \max\left\{\min\left\{\sqrt{\lambda}r\left((r-\tilde{r})\cos\theta - (x-g(\tilde{r}))\sin\theta\right), 0\right\}, m_2\right\}.$$

It's easy to check that $w(x,r) \in K_{\mu}$, then $J_{\lambda,\theta,\mu}(\psi) \leq J_{\lambda,\theta,\mu}(w)$, which implies

$$\int_{\Omega_{\mu} \cap \{r > \tilde{r}\}} r \left| \frac{\nabla \psi_{\lambda,\theta,\mu}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{m_2 < \psi_{\lambda,\mu,\theta} < 0\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi_{\lambda,\theta,\mu} \le 0\}} \right) e \right|^2 dX$$

$$\leq \int_{\Omega_{\mu} \cap \{r > \tilde{r}\}} r \left| \frac{\nabla w}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{m_2 < w < 0\}} + \sqrt{\lambda} \chi_{\{m_2 < w \le 0\}} \right) e \right|^2 dX$$

$$= \int_{\Omega_{\mu} \cap \{\tilde{r} < r \le \tilde{r}\}} r \left| \frac{\nabla w}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{m_2 < w < 0\}} + \sqrt{\lambda} \chi_{\{m_2 < w \le 0\}} \right) e \right|^2 dX$$

$$+ \int_{\Omega_{\mu} \cap \{r \ge \tilde{r}\}} r \left| \frac{\nabla w}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{m_2 < w < 0\}} + \sqrt{\lambda} \chi_{\{m_2 < w \le 0\}} \right) e \right|^2 dX.$$
(3.22)

First, similar arguments as Lemma 3.10 in [10] and Lemma 4.1 in [11]. We obtain

$$\int_{\Omega_{\mu} \cap \{r > \tilde{r}\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dX \le \frac{C}{r_0^3} + \frac{S(r_0)}{16}, \quad (3.23)$$

where C is independent of r_0 .

Next, $S(2r_0)$ can be calculated as follows,

$$S(2r_0) = \int_{\Omega_{\mu} \cap \{r_0 < r < 2r_0\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dX$$

$$\leq \int_{\Omega_{\mu} \cap \{r > \tilde{r}\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dX \qquad (3.24)$$

$$\leq \frac{C}{r_0^3} + \frac{S(r_0)}{16},$$

where we have used $\tilde{r} \in \left(\frac{r_0}{2}, r_0\right)$ and (3.23).

Using mathematical induction for any $n \in \mathbb{N}$ and (3.24), one has

$$S(2^{n+1}R) \le \frac{2C}{(2^n R)^3}, \quad n = 0, 1....$$
 (3.25)

Indeed, (3.25) holds for n = 0 when choose C large enough. If (3.25) holds for n, one has

$$S(2^{n+2}R) = S(2 \cdot 2^{n+1}R) \le \frac{C}{(2^{n+1}R)^3} + \frac{S(2^{n+1}R)}{16} \le \frac{C}{(2^{n+1}R)^3} + \frac{1}{16}\frac{2C}{(2^nR)^3} = \frac{2C}{(2^{n+1}R)^3},$$

which implies (3.25) holds for n + 1.

Therefore, for any $r_0 > 2R$, there exists a n_0 such that $2^{n_0}R \leq r_0 \leq 2^{n_0+1}R$, this together with (3.25) yields to

$$\begin{split} &\int_{\Omega_{\mu} \cap \{r > r_{0}\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2} < \psi \le 0\}} \right) e \right|^{2} dX \\ &\leq \int_{\Omega_{\mu} \cap \{r > 2^{n_{0}} R\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2} < \psi \le 0\}} \right) e \right|^{2} dX \\ &\leq \sum_{j=n_{0}}^{+\infty} S(2^{j+1}R) \leq \frac{\tilde{C}}{R^{3} r_{0}^{3}}, \end{split}$$

where have used the following fact

$$\sum_{j=n_0}^{+\infty} S(2^{j+1}R) \le 2C \sum_{j=n_0}^{+\infty} \frac{1}{(2^j R)^3} \le \frac{32C}{(2^{n_0+1}R)^3} \le \frac{\tilde{C}}{r_0^3}.$$

This completes the proof of Lemma 3.11.

Firstly, we will show that some convergence of the minimizer in far field and the free boundaries approach to the asymptotic direction $\theta \in (0, \pi)$ as $r \to +\infty$ in the far field.

Lemma 3.12. Let $\theta \in (0, \pi)$, $\psi_n(\tilde{x}, \tilde{r}) = \psi_{\lambda, \theta, \mu} \left(x_n + \frac{\tilde{x}}{r_n}, r_n + \frac{\tilde{r}}{r_n} \right)$ with $X_n = (x_n, r_n) \in \Gamma_{1, \mu}$ and $r_n \to +\infty$, $\tilde{X} = (\tilde{x}, \tilde{r}) \in \mathbb{R}^2$, then for a subsequence

$$\psi_{n}(\tilde{x},\tilde{r}) \to \Theta(\tilde{x},\tilde{r}) \triangleq \begin{cases} m_{1}, & \text{if } \tilde{r}\cos\theta - \tilde{x}\sin\theta \ge 0, \\ m_{1} + \sqrt{\Lambda + \lambda}(\tilde{r}\cos\theta - \tilde{x}\sin\theta), & \text{if } -\frac{m_{1}}{\sqrt{\Lambda + \lambda}} \le \tilde{r}\cos\theta - \tilde{x}\sin\theta \le 0, \\ \frac{m_{1}\sqrt{\lambda}}{\sqrt{\Lambda + \lambda}} + \sqrt{\lambda}(\tilde{r}\cos\theta - \tilde{x}\sin\theta), & \text{if } \frac{m_{2}}{\sqrt{\lambda}} - \frac{m_{1}}{\sqrt{\Lambda + \lambda}} \le \tilde{r}\cos\theta - \tilde{x}\sin\theta \le -\frac{m_{1}}{\sqrt{\Lambda + \lambda}}, \\ m_{2}, & \text{if } \tilde{r}\cos\theta - \tilde{x}\sin\theta \le \frac{m_{2}}{\sqrt{\lambda}} - \frac{m_{1}}{\sqrt{\Lambda + \lambda}}, \\ (3.26) \end{cases}$$

uniformly in any compact subset of \mathbb{R}^2 . Furthermore,

 $g'_{1,\lambda,\theta,\mu}(r) \to \cot \theta \quad as \quad r \to +\infty.$

The similar conclusion holds for $X_n \in \Gamma_{2,\mu}$.

Proof. Set $\psi = \psi_{\lambda,\theta,\mu}$, $g(r) = g_{\lambda,\theta,\mu}(r)$ and $g_i(r) = g_{i,\lambda,\theta,\mu}(r)$ (i = 1, 2) for simplicity. Define $x = x_n + \frac{\tilde{x}}{r_n}$ and $r = r_n + \frac{\tilde{r}}{r_n}$.

For any $R_0 > 0$, one has

$$\int_{\{|r-r_n|
$$= \int_{\{|\tilde{r}|(3.27)$$$$

where $\tilde{\nabla} = (\partial_{\tilde{x}}, \partial_{\tilde{r}}).$

For large $r_n > R_0 + 2R$, in view of (3.19), we obtain

$$\int_{\{|r-r_n|$$

which together with (3.27) implies that

$$\int_{\Omega_{\mu} \cap \{ |\tilde{r}| < R_0 r_n \}} \left(1 + \frac{\tilde{r}}{r_n^2} \right) \left| \frac{\tilde{\nabla}\psi_n}{1 + \frac{\tilde{r}}{r_n^2}} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi_n < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi_n \le 0\}} \right) e \right|^2 d\tilde{X} \le \frac{Cr_n}{(r_n - R_0)^3}$$
(3.28)

Recalling Proposition 3.2 and $\Theta(\tilde{x}, \tilde{r}) \in H^1_{loc}(\mathbb{R}^2)$, then there exist a subsequence ψ_{n_k} and two functions $\gamma_1, \gamma_2 \in [0, 1]$ such that

$$\psi_{n_k}(\tilde{x}, \tilde{r}) \to \Theta(\tilde{x}, \tilde{r}) \text{ weakly in } H^1_{loc}(\mathbb{R}^2),$$

 $\psi_{n_k}(\tilde{x}, \tilde{r}) \to \Theta(\tilde{x}, \tilde{r}) \text{ a.e. in } \mathbb{R}^2,$

 $\chi_{\{0 < \psi_{n_k} < m_1\}} \to \gamma_1 \quad \text{weak-star in } L^{\infty}_{loc}(\mathbb{R}^2), \ \gamma_1 = 1 \text{ a.e. on } \{0 < \Theta(\tilde{x}, \tilde{r}) < m_1\},$

and

$$\chi_{\{m_2 < \psi_{n_k} \le 0\}} \to \gamma_2 \text{ weak-star in } L^{\infty}_{loc}(\mathbb{R}^2), \, \gamma_2 = 1 \text{ a.e. on } \{m_2 < \Theta(\tilde{x}, \tilde{r}) \le 0\},$$

as $k \to +\infty$. This together with (3.28) gives that

$$\tilde{\nabla}\Theta = \sqrt{\Lambda + \lambda} e \chi_{\{0 < \psi_0 < m_1\}} + \sqrt{\lambda} e \chi_{\{m_2 < \psi_0 \le 0\}} \quad \text{a.e.}, \tag{3.29}$$

in any compact subset Ω' of \mathbb{R}^2 .

Lemma 3.9 implies that for sufficiently large n

$$\left|\tilde{\nabla}\psi_n(\tilde{x},\tilde{r})\right| = \left|\frac{1}{r_n}\nabla\psi\left(x_n + \frac{\tilde{x}}{\tilde{r}_n}, r_n + \frac{\tilde{r}}{r_n}\right)\right| \le c_0,$$

where the constant c_0 is independent of R_0 . Hence, we conclude that there exists a subsequence $\psi_{n_k} \to \Theta(\tilde{x}, \tilde{r})$ uniformly in any compact subset of \mathbb{R}^2 and

$$m_1 - \psi_n(\tilde{x}, \tilde{r}) = \psi(x_n, r_n) - \psi\left(x_n + \frac{\tilde{x}}{\tilde{r}_n}, r_n + \frac{\tilde{r}}{r_n}\right) \le |\tilde{\nabla}\psi_n| |\tilde{X}|, \text{ for } |\tilde{X}| < \frac{m_1}{c_0},$$

which implies $\psi_n(\tilde{x}, \tilde{r}) > 0$.

The non-degeneracy lemma 3.6 implies that

$$\frac{1}{r} \oint_{\partial B_r(0)} (m_1 - \psi_n(\tilde{x}, \tilde{r})) d\tilde{S} = \frac{1}{r_n} \frac{1}{\frac{r}{r_n}} \oint_{\partial B_{\frac{r}{r_n}}(X_n)} (m_1 - \psi(x, r)) dS \ge c\sqrt{\Lambda + \lambda}, \text{ for } r < \frac{m_1}{c_0},$$

taking $n \to +\infty$, which implies that $\Theta \not\equiv m_1$ in $B_r(0)$ and $\Theta(0) = m_1$.

Define

$$t = \tilde{x}\cos\theta + \tilde{r}\sin\theta$$
 and $s = \tilde{r}\cos\theta - \tilde{x}\sin\theta$, (3.30)

and $w(t,s) = \Theta(\tilde{x}, \tilde{r})$, then (3.29) implies that

$$\frac{\partial w}{\partial t} = 0, \quad \frac{\partial w}{\partial s} = \sqrt{\Lambda + \lambda} e \chi_{\{0 < w < m_1\}} + \sqrt{\lambda} e \chi_{\{m_2 < w \le 0\}} \quad \text{a.e. in } \Omega'_{\mu} \quad \text{and} \quad w(0) = m_1.$$

A direction computation gives that

$$w(t,s) = \begin{cases} m_1 & \text{if } s \ge 0, \\ m_1 + \sqrt{\Lambda + \lambda}s, & \text{if } -\frac{m_1}{\sqrt{\Lambda + \lambda}} \le s \le 0, \\ \frac{m_1\sqrt{\lambda}}{\sqrt{\Lambda + \lambda}} + \sqrt{\lambda}s, & \text{if } \frac{m_2}{\sqrt{\lambda}} - \frac{m_1}{\sqrt{\Lambda + \lambda}} \le s \le -\frac{m_1}{\sqrt{\Lambda + \lambda}}, \\ m_2, & \text{if } s \le \frac{m_2}{\sqrt{\lambda}} - \frac{m_1}{\sqrt{\Lambda + \lambda}}, \end{cases}$$

which yields (3.26).

Next, let
$$X_0 = (t, s)$$
 with $s > 0$ or $s < \frac{m_2}{\sqrt{\lambda}} - \frac{m_1}{\sqrt{\Lambda + \lambda}}$, for small $r > 0$, then

$$\lim_{n \to +\infty} \frac{1}{r} \oint_{\partial B_r(X_0)} (m_1 - \psi_n) dS = 0, \text{ or } \lim_{n \to +\infty} \frac{1}{r} \oint_{\partial B_r(X_0)} (\psi_n - m_2) dS = 0,$$

respectively. Then, the non-degeneracy lemma implies that X_0 is not a free boundary point for sufficiently large n.

Similarly, for the case $\frac{m_2}{\sqrt{\lambda}} - \frac{m_1}{\sqrt{\Lambda + \lambda}} < s < 0$, one gets

$$\lim_{n \to +\infty} \frac{1}{r} \oint_{\partial B_r(X_0)} (m_1 - \psi_n) dS \to +\infty, \text{ or } \lim_{n \to +\infty} \frac{1}{r} \oint_{\partial B_r(X_0)} (\psi_n - m_2) dS \to +\infty, \text{ as } r \to 0,$$

and then X_0 is not a free boundary point for sufficiently large n.

Then, one has

$$\partial\{\psi_n > 0\} \rightarrow \left\{s = \frac{m_2}{\sqrt{\lambda}} - \frac{m_1}{\sqrt{\Lambda + \lambda}}\right\} \text{ and } \partial\{\psi_n < m_1\} \rightarrow \{s = 0\},$$
 (3.31)

locally in Hausdorff distance (see Definition 3.1 in [23]).

Noticing the flatness conditions in Section 7 in [1] for the free boundaries, there exists
a
$$\xi_{1,n} \in \left(\min\left\{r_n, r_n + \frac{\tilde{r}}{r_n}\right\}, \max\left\{r_n, r_n + \frac{\tilde{r}}{r_n}\right\}\right)$$
 such that
 $\tilde{x} = r_n \left(g_1 \left(r_n + \frac{\tilde{r}}{r_n}\right) - x_n\right) = r_n \left(g_1 \left(r_n + \frac{\tilde{r}}{r_n}\right) - g_1(r_n)\right) = g_1'(\xi_{1,n})\tilde{r},$

thus, we obtain

$$g_1'\left(r_n + \frac{\tilde{r}}{r_n}\right) \to \cot\theta$$
, as $n \to +\infty$.

Therefore, we complete the proof of Lemma 3.12.

Next, for the critical cases $\theta = 0$ or $\theta = \pi$, we have the following facts.

Proposition 3.13. Assume that there exist some free boundary points $X_n = (x_n, r_n) \in \Gamma_{2,\mu}$, such that $r_n \to \xi > R$ and $x_n \to +\infty$, where ξ is a finite positive number, then $\theta = 0$. Moreover, let $\psi_n(\tilde{X}) = \psi_{\lambda,\theta,\mu}(x_n + \tilde{x}, r_n + \tilde{r})$, then

$$\psi_n(\tilde{X}) \to \min\left\{\max\left\{\sqrt{\lambda}\left(\frac{m_2}{\sqrt{\lambda}} + \frac{\tilde{r}^2}{2} + \xi\tilde{r}\right), m_2\right\}, 0\right\} + \max\left\{\min\left\{\sqrt{\Lambda + \lambda}\left(\frac{m_2}{\sqrt{\lambda}} + \frac{\tilde{r}^2}{2} + \xi\tilde{r}\right), m_1\right\}, 0\right\}$$

uniformly in any compact subset of $\{(\tilde{x}, \tilde{r}) \mid \tilde{r} > R - \xi\}$. If $r_n \to \xi > \sqrt{R^2 + \frac{2m_1}{\sqrt{\Lambda + \lambda}} - \frac{2m_2}{\sqrt{\lambda}}}$ and $x_n \to -\infty$, ξ is a finite positive number, then $\theta = \pi$ and

$$\psi_n(\tilde{X}) \to \min\left\{ \max\left\{ \sqrt{\lambda} \left(\frac{m_2}{\sqrt{\lambda}} - \frac{\tilde{r}^2}{2} - \xi \tilde{r} \right), m_2 \right\}, 0 \right\} + \max\left\{ \min\left\{ \sqrt{\Lambda + \lambda} \left(\frac{m_2}{\sqrt{\lambda}} - \frac{\tilde{r}^2}{2} - \xi \tilde{r} \right), m_1 \right\}, 0 \right\}$$

uniformly in any compact subset of $\{(\tilde{x}, \tilde{r}) \mid \tilde{r} > R - \xi\}$. The similar assertion holds for $X_n = (x_n, r_n) \in \Gamma_{1,\mu}$.

Proof. If $X_n = (x_n, r_n) \in \Gamma_{2,\mu}$ with $r_n \to \xi$ (ξ is a finite positive number) and $x_n \to +\infty$. Set $x = x_n + \tilde{x}$ and $r = r_n + \tilde{r}$. For any large $R_0 > 0$, the boundedness of $J_{\lambda,\theta,\mu}(\psi_{\lambda,\theta,\mu})$ gives that

$$\int_{\Omega_{\mu} \cap \{|x-x_{n}| < R_{0}\} \cap \{R-\xi < r-r_{n} < R_{0}\}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2} < \psi \le 0\}} \right) e \right|^{2} dX$$

$$= \int_{\tilde{\Omega}_{\mu} \cap \{|\tilde{x}| < R_{0}\} \cap \{R-\xi < \tilde{r} < R_{0}\}} (r_{n} + \tilde{r}) \left| \frac{\tilde{\nabla} \psi_{n}}{r_{n} + \tilde{r}} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi_{n} < m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2} < \psi_{n} \le 0\}} \right) e \right|^{2} d\tilde{X} \to 0,$$
as $n \to +\infty$

as $n \to +\infty$.

Along the similar arguments in Lemma 3.12, then there exists a subsequence ψ_{n_k} such that

 $\psi_{n_k} \to \psi_0$ weakly in $H_{loc}(\Omega'_{\mu})$,

and

$$\psi_{n_k} \to \psi_0$$
 a.e. in Ω'_{μ_k}

as $k \to +\infty$, and

$$\nabla \psi_0 = (\tilde{r} + \xi) \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi_0 < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi_0 \le 0\}} \right) (-\sin\theta, \cos\theta) \quad \text{a.e.},$$

in any compact subset Ω' of $\{(\tilde{x}, \tilde{r}) \mid \tilde{r} > R - \xi\}$. Furthermore, it's easy to see that $\psi_0(0,0) = m_2, \ \psi_0 \not\equiv m_2$ in any neighborhood of (0,0) and

$$\psi(x_n + \tilde{x}, R - \xi + r_n) \to \psi_0(\tilde{x}, R - \xi) = m_2 \quad \text{if} \quad x_n \to +\infty.$$
(3.32)

Next, we claim $\theta = 0$. Suppose not, if $\theta = \pi$, indeed, similar arguments as Lemma 3.12, we obtain

$$\psi_0(\tilde{x},\tilde{r}) = \min\left\{\max\left\{\sqrt{\lambda}\left(\frac{m_2}{\sqrt{\lambda}} - \frac{\tilde{r}^2}{2} - \xi\tilde{r}\right), m_2\right\}, 0\right\} + \max\left\{\min\left\{\sqrt{\Lambda + \lambda}\left(\frac{m_2}{\sqrt{\lambda}} - \frac{\tilde{r}^2}{2} - \xi\tilde{r}\right), m_1\right\}, 0\right\}$$

which contradicts with (3.32).

If $\theta \in (0, \pi)$, since ψ_0 is smooth in any compact subset of $G \subset \{m_2 < \psi_0 < 0\} \cap \{\tilde{r} \geq 0\}$ $R-\xi$, one has

$$\frac{\partial^2 \psi_0}{\partial \tilde{x} \partial \tilde{r}} = -\sqrt{\lambda} \sin \theta, \quad \frac{\partial^2 \psi_0}{\partial \tilde{r} \partial \tilde{x}} = 0,$$

which derives a contradiction with $\theta \in (0, \pi)$.

Therefore, we obtain $\theta = 0$. Along the similar arguments in Lemma 3.12, one has

$$\psi_n(\tilde{X}) \to \min\left\{\max\left\{\sqrt{\lambda}\left(\frac{m_2}{\sqrt{\lambda}} + \frac{\tilde{r}^2}{2} + \xi\tilde{r}\right), m_2\right\}, 0\right\} + \max\left\{\min\left\{\sqrt{\Lambda + \lambda}\left(\frac{m_2}{\sqrt{\lambda}} + \frac{\tilde{r}^2}{2} + \xi\tilde{r}\right), m_1\right\}, 0\right\}$$

uniformly in any compact subset of $\{(\tilde{x}, \tilde{r}) \mid \tilde{r} > R - \xi\}$. The similar conclusion holds if $X_n = (x_n, r_n) \in \Gamma_{1,\mu}$ with $r_n \to \xi > R$ and $x_n \to -\infty$.

Similarly, we can obtain the conclusion for $\theta = \pi$. Thus, we complete the proof of Proposition 3.13.

Now, we can obtain the convergence rate of distance of the two free boundaries and the minimizer as follows.

Lemma 3.14. For any $\theta \in (0, \pi)$ and $\alpha \in (0, 2)$, the free boundaries $x = g_{1,\lambda,\theta,\mu}(r)$, $x = g_{2,\lambda,\theta,\mu}(r)$, the interface $x = g_{\lambda,\theta,\mu}(r)$ and the minimizer $\psi_{\lambda,\theta,\mu}$ satisfy

$$r(g_{\lambda,\theta,\mu}(r) - g_{1,\lambda,\theta,\mu}(r)) \to \frac{m_1}{\sqrt{\Lambda + \lambda}\sin\theta},$$
(3.33)

$$r(g_{\lambda,\theta,\mu}(r) - g_{2,\lambda,\theta,\mu}(r)) \to \frac{m_2}{\sqrt{\lambda}\sin\theta},$$
(3.34)

and

$$r^{\alpha}\left(\frac{\nabla\psi_{\lambda,\theta,\mu}}{r} - \left(\sqrt{\Lambda + \lambda}\chi_{\{0 < \psi_{\lambda,\theta,\mu} < m_1\}} + \sqrt{\lambda}\chi_{\{m_2 < \psi_{\lambda,\theta,\mu} \le 0\}}\right)e\right) \to 0, \tag{3.35}$$

as $r \to +\infty$.

Proof. Define $\psi_n(\tilde{x}, \tilde{r}) = \psi_{\lambda,\theta,\mu} \left(x_n + \frac{\tilde{x}}{r_n}, r_n + \frac{\tilde{r}}{r_n} \right)$ with $(x_n, r_n) \in \Gamma_{1,\lambda,\theta,\mu}, r_n \to +\infty$. Set $x = x_n + \frac{\tilde{x}}{r_n}$ and $r = r_n + \frac{\tilde{r}}{r_n}$. The free boundaries and interface of $\psi_n(\tilde{x}, \tilde{r})$ are given by

$$\left\{ \left(\tilde{x}, \tilde{r}\right) \mid x_n + \frac{\tilde{x}}{r_n} = g_{i,\lambda,\theta,\mu}\left(r_n + \frac{\tilde{r}}{r_n}\right), i = 1,2 \right\} \text{ and } \left\{ \left(\tilde{x}, \tilde{r}\right) \mid x_n + \frac{\tilde{x}}{r_n} = g_{\lambda,\theta,\mu}\left(r_n + \frac{\tilde{r}}{r_n}\right) \right\}$$
(3.36)

Thanks to Lemma 3.12, we have

 $\partial \{0 < \psi_n < m_1\}$ converges to $\partial \{0 < \Theta < m_1\}$ locally in Hausdorff distance.

This together with (3.36) and (3.26), taking $\tilde{r} = 0$, yields that

$$\tilde{x}\sin\theta = r_n \left(g_{\lambda,\theta,\mu}(r_n) - x_n\right)\sin\theta = r_n \left(g_{\lambda,\theta,\mu}(r_n) - g_{1,\lambda,\theta,\mu}(r_n)\right)\sin\theta \to \frac{m_1}{\sqrt{\Lambda + \lambda}}.$$

Furthermore, set $\psi_n(\tilde{x}, \tilde{r}) = \psi_{\lambda, \theta, \mu} (x_n + \tilde{x}, r_n + \tilde{r})$ with $(x_n, r_n) \in \Gamma_{1, \lambda, \theta, \mu}, r_n \to +\infty$, for any $\tilde{R} > 0$, and large $r_n > \tilde{R} + 2R$, similarly in Lemma 3.11, one has

$$\begin{split} &\int_{\Omega_{\mu} \cap\{|r-r_{n}|<\tilde{R}\}} r^{2\alpha} \left| \frac{\nabla \psi_{\lambda,\theta,\mu}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0<\psi_{\lambda,\theta,\mu}< m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2}<\psi_{\lambda,\theta,\mu}\leq 0\}} \right) e \right|^{2} dX \\ &= \int_{\{|\tilde{r}|<\tilde{R}\}} (\tilde{r}+r_{n})^{2\alpha} \left| \frac{\tilde{\nabla}\psi_{n}}{\tilde{r}+r_{n}} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0<\psi_{n}< m_{1}\}} + \sqrt{\lambda} \chi_{\{m_{2}<\psi_{n}\leq 0\}} \right) e \right|^{2} d\tilde{X} \\ &\leq \frac{C(r_{n}+\tilde{R})^{2\alpha-1}}{\left(r_{n}-\tilde{R}\right)^{3}}. \end{split}$$

Hence for any $\alpha \in (0,2)$ and $(\tilde{x}, \tilde{r}) \in \mathbb{R}^2$, one has

$$\left(\tilde{r}+r_n\right)^{\alpha} \left| \frac{\tilde{\nabla}\psi_n}{\tilde{r}+r_n} - \left(\sqrt{\Lambda+\lambda}\chi_{\{0<\psi_n< m_1\}} + \sqrt{\lambda}\chi_{\{m_2<\psi_n\le 0\}}\right) e \right| \to 0 \text{ as } n \to +\infty,$$

taking $\tilde{r} = 0$, $x = x_n + \tilde{x}$ and $r = r_n$ yields to the desired estimate (3.35).

Therefore, we complete the proof of Lemma 3.14.

Next, we will prove that one of free boundaries will vanish, provided that the asymptotic direction of the outgoing jet is horizontal. We call that $\Gamma_{1,\mu}$ vanishes in $\Omega_{\mu} \cap \{r > R\}$, means $\psi_{\lambda,\theta,\mu} < m_1$ in $\Omega_{\mu} \cap \{r > R\}$, and similarly, we call that the free boundary $\Gamma_{2,\mu}$ vanishes in $\Omega_{\mu} \cap \{r > R\}$ means that $\psi_{\lambda,\theta,\mu} > m_2$ in $\Omega_{\mu} \cap \{r > R\}$.

Proposition 3.15. (1). If $\theta = \pi$, then the left free boundary $\Gamma_{1,\mu}$ vanishes in $\Omega_{\mu} \cap \{r > R\}$; (2). If $\theta = 0$, then the right free boundary $\Gamma_{2,\mu}$ vanishes in $\Omega_{\mu} \cap \{r > R\}$.

Proof. Denote $\psi = \psi_{\lambda,\theta,\mu}$ for simplicity.

For $\theta = \pi$, then e = (0, -1), for $(x, r) \in \Omega_{\mu}$, set

$$\psi_0(x,r) = \max\left\{\min\left\{m_1 - \frac{\sqrt{\Lambda + \lambda}(r^2 - R^2)}{2}, m_1\right\}, 0\right\} + \min\left\{\max\left\{\frac{\sqrt{\lambda}(R^2 + \frac{2m_1}{\sqrt{\Lambda + \lambda}} - r^2)}{2}, m_2\right\}, 0\right\}$$

Next, we claim that

$$\psi(x,r) \le \psi_0(x,r) \quad \text{in } \Omega_\mu. \tag{3.37}$$

Suppose that the assertion (3.37) is not true, recalling that $\min\{\psi, \psi_0\} \in K_{\mu}$ and the uniqueness of minimizer, we obtain

$$J_{\lambda,\theta,\mu}(\psi) < J_{\lambda,\theta,\mu}(\min\{\psi,\psi_0\}).$$

This implies that there exists some sufficiently large $R_0 > \max\left\{\sqrt{\frac{2m_1}{\sqrt{\Lambda+\lambda}} - \frac{2m_2}{\sqrt{\lambda}} + R^2}, 1\right\}$, and

$$0 > \int_{\Omega_{\mu,R_0}} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \le 0\}} \right) e \right|^2 dX - \int_{\Omega_{\mu,R_0}} r \left| \frac{\nabla \min\{\psi,\psi_0\}}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \min\{\psi,\psi_0\} < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \min\{\psi,\psi_0\} \le 0\}} \right) e \right|^2 dX = \int_{\Omega_{\mu,R_0}} \frac{\nabla \max\{\psi - \psi_0, 0\} \cdot \nabla(\psi + \psi_0)}{r} dX - 2\sqrt{\Lambda + \lambda} \int_{\Omega_{\mu,R_0}} \nabla \psi \cdot e\chi_{\{0 < \psi < m_1\}} - \nabla \min\{\psi,\psi_0\} \cdot e\chi_{\{0 < \min\{\psi,\psi_0\} < m_1\}} dX - 2\sqrt{\lambda} \int_{\Omega_{\mu,R_0}} \nabla \psi \cdot e\chi_{\{m_2 < \psi \le 0\}} - \nabla \min\{\psi,\psi_0\} \cdot e\chi_{\{m_2 < \min\{\psi,\psi_0\} \le 0\}} dX + \int_{\Omega_{\mu,R_0}} (\Lambda + \lambda) r \left(\chi_{\{0 < \psi < m_1\}} - \chi_{\{0 < \min\{\psi,\psi_0\} < m_1\}} \right) + \lambda r \left(\chi_{\{m_2 < \psi \le 0\}} - \chi_{\{m_2 < \min\{\psi,\psi_0\} \le 0\}} \right) dX = I_1 + I_2 + I_3 + I_4,$$
(3.38)

where Ω_{μ,R_0} is bounded by $N_{i,\mu}$, $N_{0,\mu}$, L_i , $H_{i,\mu}$, $\{((-1)^i, r) \mid R \leq r \leq R_0\}$ and $\{(x,R_0) \mid -R_0 \leq x \leq R_0\}$ for i = 1, 2.

The first term I_1 can be estimated as follows,

$$I_{1} = \int_{\Omega_{\mu,R_{0}}} \frac{|\nabla \max\{\psi - \psi_{0}, 0\}|^{2}}{r} dX + 2 \int_{\Omega_{\mu,R_{0}}} \frac{\nabla \max\{\psi - \psi_{0}, 0\} \cdot \nabla \psi_{0}}{r} dX$$

$$= \int_{\Omega_{\mu,R_{0}}} \frac{|\nabla \max\{\psi - \psi_{0}, 0\}|^{2}}{r} dX - 2\sqrt{\Lambda + \lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0} = 0\}} \max\{\psi - \psi_{0}, 0\} dx$$

$$+ 2\sqrt{\lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0} = 0\}} \max\{\psi - \psi_{0}, 0\} dx - 2\sqrt{\lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0} < 0\} \cap \{r = R_{0}\}} \max\{\psi - \psi_{0}, 0\} dx.$$

(3.39)

Furthermore, for the second term I_2 , one has

$$I_{2} = -2\sqrt{\Lambda + \lambda} \left\{ \int_{\Omega_{\mu,R_{0}} \cap \{0 < \psi_{0} < m_{1}\} \cap \{0 < \psi < m_{1}\}} \nabla \max \left\{\psi - \psi_{0}, 0\right\} \cdot edX + \int_{\Omega_{\mu,R_{0}} \cap \{\psi_{0} < 0\} \cap \{0 < \psi < m_{1}\}} \nabla (\psi - \psi_{0}) \cdot edX - \int_{\Omega_{\mu,R_{0}} \cap \{0 < \psi_{0} < m_{1}\} \cap \{\psi = m_{1}\}} \nabla (m_{1} - \psi_{0}) \cdot edX \right\}$$
$$= 2\sqrt{\Lambda + \lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0} = 0\}} \max\{\psi - \psi_{0}, 0\} dx.$$
(3.40)

Similarly, we obtain

$$I_{3} = -2\sqrt{\lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0}=0\}} \max\{\psi - \psi_{0}, 0\} dx + 2\sqrt{\lambda} \int_{\bar{\Omega}_{\mu,R_{0}} \cap \{\psi_{0}<0\} \cap \{r=R_{0}\}} \max\{\psi - \psi_{0}, 0\} dx.$$
(3.41)

Finally, we have

$$I_{4} \ge (\Lambda + \lambda) \int_{\Omega_{\mu,R_{0}}} r\left(\chi_{\{0 < \psi < m_{1}\} \cap \{\psi_{0} \le 0\}} - \chi_{\{0 < \psi_{0} < m_{1}\} \cap \{\psi = m_{1}\}}\right) dX - \lambda \int_{\Omega_{\mu,R_{0}}} r\chi_{\{m_{2} < \psi_{0} \le 0\} \cap \{\psi > 0\}} dX$$

$$\ge - (\Lambda + \lambda) \int_{\Omega_{\mu,R_{0}}} r\chi_{\{0 < \psi_{0} < m_{1}\} \cap \{\psi = m_{1}\}} dX - \lambda \int_{\Omega_{\mu,R_{0}}} \Gamma_{\{m_{2} < \psi_{0} \le 0\}} r\left(\chi_{\{\psi > 0\}} - \chi_{\{0 < \psi < m_{1}\}}\right) dX$$

$$\ge - (\Lambda + \lambda) \int_{\Omega_{\mu,R_{0}}} r\chi_{\{0 < \psi_{0} < m_{1}\} \cap \{\psi = m_{1}\}} dX - \lambda \int_{\Omega_{\mu,R_{0}}} r\chi_{\{m_{2} < \psi_{0} \le 0\} \cap \{\psi = m_{1}\}} dX.$$

(3.42)

Inserting (3.39)-(3.42) into (3.38) yields

$$\begin{split} 0 &> \int_{\Omega_{\mu,R_0}} \frac{|\nabla \max\{\psi - \psi_0, 0\}|^2}{r} dX - (\Lambda + \lambda) \int_{\Omega_{\mu,R_0}} r\chi_{\{0 < \psi_0 < m_1\} \cap \{\psi = m_1\}} dX \\ &- \lambda \int_{\Omega_{\mu,R_0}} r\chi_{\{m_2 < \psi_0 \le 0\} \cap \{\psi = m_1\}} dX \\ &= \int_{\Omega_{\mu,R_0} \cap \{0 < \psi_0 < m_1\} \cap \{\psi = m_1\}} \left(\frac{|\nabla \psi_0|^2}{r} - (\Lambda + \lambda)r\right) dX + \int_{\Omega_{\mu,R_0} \cap \{0 < \psi_0 < \psi < m_1\}} \frac{|\nabla (\psi - \psi_0)|^2}{r} dX \\ &+ \int_{\Omega_{\mu,R_0} \cap \{m_2 < \psi_0 \le 0\} \cap \{\psi = m_1\}} \left(\frac{|\nabla \psi_0|^2}{r} - \lambda r\right) dX + \int_{\Omega_{\mu,R_0} \cap \{m_2 < \psi_0 \le 0\} \cap \{\psi = m_1\}} \frac{|\nabla (\psi - \psi_0)|^2}{r} dX \\ &= \int_{\Omega_{\mu,R_0} \cap \{m_2 < \psi_0 < \psi < m_1\}} \frac{|\nabla (\psi - \psi_0)|^2}{r} dX, \end{split}$$

which derives a contradiction. Hence, (3.37) holds, it implies that

$$\psi(x,r) < m_1 \quad \text{in } \Omega_\mu \cap \{r > R\},$$

and thus, this gives that the free boundaries $\Gamma_{1,\mu}$ vanishes.

For $\theta = 0$, taking

$$\psi_0 = \min\left\{ \max\left\{ \frac{\sqrt{\lambda}(r^2 - R^2)}{2} + m_2, m_2 \right\}, 0 \right\} + \max\left\{ \min\left\{ \frac{\sqrt{\Lambda + \lambda}(r^2 - R^2 + \frac{2m_2}{\sqrt{\lambda}})}{2}, m_1 \right\}, 0 \right\}.$$

Similar arguments as before, yield that

$$\psi(x,r) \ge \psi_0(x,r)$$
 in $\Omega_\mu \cap \{r > R\}$

which implies that the free boundary $\Gamma_{2,\mu}$ is empty.

Therefore, we complete the proof of Proposition 3.15.

Remark 3.1. Furthermore, we define $g_{1,\lambda,\theta,\mu}(R) = -\infty$ for $\theta = \pi$, $g_{2,\lambda,\theta,\mu}(R) = +\infty$ for $\theta = 0$, respectively.

Proposition 3.15 implies that the one of free boundaries vanishes for horizontal asymptotic direction, and on another side, we will show that the both of two free boundaries are non-empty, for non-horizontal asymptotic direction.

Lemma 3.16. If $\theta \in (0,\pi)$, then $\Gamma_{i,\mu}$ is non-empty and a connected curve, $x = g_{i,\lambda,\theta,\mu}(r)$ is continuous in $(R, +\infty)$. And $\lim_{r \to R^+} g_{i,\lambda,\theta,\mu}(r)$ exists and denoted as $g_{i,\lambda,\theta,\mu}(R+0)$ for i = 1, 2.

Proof. Step 1. We will show that $\Gamma_{i,\mu}$ is non-empty for i = 1, 2.

Firstly, we claim that there exists a constant $R_0 > 0$, such that

$$B_{R_0}(X_0) \subset \Omega_\mu \cap \{r > R_0\}$$

contains a free boundary point $X_0 = (x_0, R + R_0) \in \Omega_\mu$ for any $0 < \psi_{\lambda,\theta,\mu}(X_0) < m_1$.

Indeed, suppose not, we have $B_{R_0}(X_0) \cap \Gamma_{1,\mu} = \emptyset$. Similar arguments as Lemma 3.7, one gets

$$\sup_{\partial B_{R_0}(X_0)} (m_1 - \psi_{\lambda,\theta,\mu}) \ge c\sqrt{\Lambda + \lambda}R(R + R_0),$$

which implies $R_0 \leq \frac{m_1}{c\sqrt{\Lambda + \lambda}R}$. This is impossible for sufficiently large R_0 . Hence, the claim holds.

Without loss of generality, we assume that $\Gamma_{1,\mu}$ is empty, then we obtain $\psi_{\lambda,\theta,\mu} < m_1$ in $\Omega_{\mu} \cap \{r > R\}$.

In view of the claim, there is a sequence $X_n = (x_n, r_n) \in \Gamma_{2,\mu}$ such that $R < r_n \leq c$ and $x_n \to -\infty$. Hence, there exists a subsequence $X_{n_k} = (x_{n_k}, r_{n_k}) \in \Gamma_{2,\mu}$ and $r_{n_k} \to \xi$, $x_{n_k} \to -\infty$ as $k \to +\infty$. Due to Proposition 3.13, we can prove that $\psi_{\lambda,\theta,\mu}(X + X_{n_k}) \to \psi_0(X)$ uniformly in any compact subset of $\{(x, r) | r > R - \xi\}$ as $k \to +\infty$, where ψ_0 is a constant flow with deflection angle $\theta = \pi$. This contradicts with $\theta \in (0, \pi)$. Thus, the free boundaries $\Gamma_{1,\mu}$ and $\Gamma_{2,\mu}$ are non-empty.

Step 2. We will verify that $\Gamma_{i,\mu}$ is a connected curve and $x = g_{i,\lambda,\theta,\mu}(r)$ is a continuous function in $[R, +\infty)$, i = 1, 2.

Without loss of generality, we consider the left free boundary. Let (α, β) be the maximal interval such that $x = g_{1,\lambda,\theta,\mu}(r)$ is finite-valued for all $[R, +\infty)$.

Similar arguments as Section 5 in [2], we obtain $\alpha = R$, and the limit $\lim_{r \to R} g_{1,\lambda,\theta,\mu}(r)$ exists.

If $\beta < +\infty$, one has

 $x = g_{1,\lambda,\theta,\mu}(r) \to +\infty$ or $x = g_{1,\lambda,\theta,\mu}(r) \to -\infty$ as $r \to \beta$,

which together with Proposition 3.13 implies $\theta = 0$ or $\theta = \pi$. This leads a contradiction to the assumption $\theta \in (0, \pi)$.

Therefore, we complete the proof of Lemma 3.16.

3.5. Monotonicity with respect to the parameter θ . Next, we will establish a fact that the minimizer $\psi_{\lambda,\theta,\mu}$ and free boundary $x = g_{i,\lambda,\theta,\mu}(r)$ (i = 1, 2) are monotonic with respect to the asymptotic deflection angle θ .

Proposition 3.17. Suppose that $\theta_1, \theta_2 \in [0, \pi]$ with $\theta_1 < \theta_2$, $\psi_{\lambda,\theta_1,\mu}$ and $\psi_{\lambda,\theta_2,\mu}$ are minimizers to the truncated variational problem $(P_{\lambda,\theta_1,\mu})$ and $(P_{\lambda,\theta_2,\mu})$, and $x = g_{i,\lambda,\theta_1,\mu}(r)$ and $x = g_{i,\lambda,\theta_2,\mu}(r)$ be the free boundary of $\psi_{\lambda,\theta_1,\mu}$ and $\psi_{\lambda,\theta_2,\mu}$, respectively, then

$$\psi_{\lambda,\theta_1,\mu} \ge \psi_{\lambda,\theta_2,\mu} \quad for \ (x,r) \in \Omega_{\mu},$$
(3.43)

and

$$g_{i,\lambda,\theta_1,\mu}(r) > g_{i,\lambda,\theta_2,\mu}(r) \text{ for } r \ge R, \ i = 1, 2.$$
 (3.44)

Proof. Denote $\psi_1 = \psi_{\lambda,\theta_1,\mu}$ and $\psi_2 = \psi_{\lambda,\theta_2,\mu}$ for simplicity, and set $v_1 = \max{\{\psi_1,\psi_2\}}$ and $v_2 = \min{\{\psi_1,\psi_2\}}$.

For $\theta_1 < \theta_2$, as is customary Lemma 8.1 in [2], we obtain

$$J_{\lambda,\theta_1,\mu}(\psi_1) = J_{\lambda,\theta_1,\mu}(v_1)$$
 and $J_{\lambda,\theta_2,\mu}(\psi_2) = J_{\lambda,\theta_2,\mu}(v_2)$.

Since ψ_1 and ψ_2 are the minimizers to the functionals $J_{\lambda,\theta_1,\mu}$ and $J_{\lambda,\theta_2,\mu}$, respectively, we can now proceed as in Theorem 7.1 in [4] to obtain that

either
$$\psi_1 \geq \psi_2$$
 or $\psi_1 \leq \psi_2$ in Ω_{μ} .

However, noticing that $\psi_1 \ge \psi_2$ in $\Omega_{\mu} \cap \{r > R_0\}$ for some sufficiently large $R_0 > R$, we conclude that the case $\psi_1 \ge \psi_2$ in Ω_{μ} .

Next, without loss of generality, we prove that (3.44) holds for i = 1, namely

$$g_{1,\lambda,\theta_1,\mu}(r) > g_{1,\lambda,\theta_2,\mu}(r) \quad \text{for } r \ge R.$$

$$(3.45)$$

Indeed, in view of (3.43), one has

$$g_{1,\lambda,\theta_1,\mu}(r) \ge g_{1,\lambda,\theta_2,\mu}(r) \quad \text{for} \quad r \ge R.$$

$$(3.46)$$

For any r > R, suppose not, there exists a point $X_0 = (x_0, r_0)$ with $r_0 > R$ such that

$$x_0 = g_{1,\lambda,\theta_1,\mu}(r_0) = g_{1,\lambda,\theta_2,\mu}(r_0)$$

Since the free boundary $x = g_{1,\lambda,\theta_1,\mu}(r)$ is analytic in r > R, and applying Hopf's lemma yields that

$$\frac{\partial}{\partial\nu} \left(\psi_{\lambda,\theta_1,\mu} - \psi_{\lambda,\theta_2,\mu} \right) < 0 \text{ at } X_0,$$

where ν is the unit outward normal vector of $x = g_{1,\lambda,\theta_1,\mu}(r)$ at X_0 . This contradicts to the free boundary conditions $\sqrt{\lambda + \Lambda} = \frac{1}{r} \frac{\partial \psi_{\lambda,\theta_1,\mu}}{\partial \nu} < \frac{1}{r} \frac{\partial \psi_{\lambda,\theta_2,\mu}}{\partial \nu} = \sqrt{\lambda + \Lambda}$ at X_0 .

On another side, for r = R, suppose that $g_{1,\lambda,\theta_1,\mu}(R) = g_{1,\lambda,\theta_2,\mu}(R)$ and $X_0 = (g_{1,\lambda,\theta_1,\mu}(R), R)$. If $g_{1,\lambda,\theta_1,\mu}(R) \leq -1$, let G_{δ} be a domain bounded by N_1 , L_1 , $\Gamma_{1,\lambda,\theta_1,\mu}$ and $\partial B_{\delta}(X_0)$, and $G_{\delta} \subset \{0 < \psi_{\lambda,\theta_1,\mu} < m_1\}$. If $g_{1,\lambda_{\mu},\theta_1,\mu}(R) > -1$, set G_{δ} be a domain bounded by N_1 , $\{(x,R) \mid -1 \leq x \leq g_{1,\lambda_{\mu},\theta_1,\mu}(R)\}$, $\Gamma_{1,\lambda,\theta_1,\mu}$ and $\partial B_{\delta}(X_0)$, and $G_{\delta} \subset \{0 < \psi_{\lambda,\theta_1,\mu} < m_1\}$.

 Set

$$\psi_{\varepsilon} = (1+\varepsilon)(m_1 - \psi_{\lambda,\theta_1,\mu}) - (m_1 - \psi_{\lambda,\theta_2,\mu})$$
 for some $\varepsilon > 0$.

Recalling the fact $\psi_1 \geq \psi_2$ in Ω_{μ} , we can choose $\delta > 0$ sufficiently small such that

$$G_{\delta} \subset \{ 0 < \psi_{\lambda, \theta_1, \mu} < m_1 \} \cap \{ 0 < \psi_{\lambda, \theta_2, \mu} < m_1 \}.$$

It follows from the similar arguments as in Corollary 11.5 [23] that there exists a small $\varepsilon > 0$ such that

$$\psi_{\varepsilon} < 0 \quad \text{in } G_{\delta}. \tag{3.47}$$

Hence, we obtain

$$\frac{1}{R}\frac{\partial\psi_{\varepsilon}\left(g_{1,\lambda,\theta_{1},\mu}(R),R\right)}{\partial\nu} \ge 0.$$

where ν is the unit normal vector of the left free boundary $\Gamma_{1,\lambda,\theta_1,\mu}$ at $(g_{1,\lambda,\theta_1,\mu}(R), R)$, then

$$(1+\varepsilon)\sqrt{\lambda+\Lambda} \le \sqrt{\lambda+\Lambda}.$$

This leads a contradiction and then the inequality (3.45) holds for r = R.

Therefore, we finish the proof of the Proposition 3.17.

3.6. Continuous dependence to the parameters λ and θ . In this subsection, a convergence result to the parameters λ and θ will be stated as follows.

Proposition 3.18. For any $\lambda > 0$ and $\theta \in [0, \pi]$, and sequences $\lambda_n \to \lambda$, $\theta_n \to \theta$ with $\theta_n \in [0, \pi]$, let $\psi_{\lambda_n, \theta_n, \mu}$ be the minimizer to the variational problem $(P_{\lambda_n, \theta_n, \mu})$, $x = g_{i,\lambda_n, \theta_n, \mu}(r)$ and $x = g_{\lambda_n, \theta_n, \mu}(r)$ be the free boundary of $\psi_{\lambda_n, \theta_n, \mu}$ and interface, respectively. Then there exist three subsequences still labeled as $\psi_{\lambda_n, \theta_n, \mu}$, $g_{i,\lambda_n, \theta_n, \mu}(r)$ and $g_{\lambda_n, \theta_n, \mu}(r)$ such that

$$\psi_{\lambda_n,\theta_n,\mu} \to \psi_{\lambda,\theta,\mu}$$
 weakly in $H^1_{loc}(\Omega_\mu)$ and pointwise in Ω_μ , (3.48)

$$g_{i,\lambda_n,\theta_n,\mu}(r) \to g_{i,\lambda,\theta,\mu}(r) \quad uniformly \text{ for } r \ge R,$$

$$(3.49)$$

and

$$g_{\lambda_n,\theta_n,\mu}(r) \to g_{\lambda,\theta,\mu}(r)$$
 uniformly for $r \ge 0.$ (3.50)

Here, $\psi_{\lambda,\theta,\mu}$ is the minimizer to the variational problem $(P_{\lambda,\theta,\mu})$ and $x = g_{i,\lambda,\theta,\mu}(r)$ and $x = g_{\lambda,\theta,\mu}(r)$ are the free boundary and interface of $\psi_{\lambda,\theta,\mu}$ for i = 1, 2, respectively.

Proof. Firstly, recalling the following facts

$$\psi_{\lambda_n,\theta_n,\mu} \in H^1_{loc}(\Omega_\mu), \quad |\nabla \psi_{\lambda_n,\theta_n,\mu}| \le C,$$

and using diagonal procedure gives that there exists a subsequence $\{\psi_{\lambda_n,\theta_n,\mu}\}_{n=1}^{\infty}$ and a function $\omega \in H^1_{loc}(\Omega_{\mu})$ for some $0 < \alpha < 1$ such that

 $\psi_{\lambda_n,\theta_n,\mu} \to \omega$ weakly in $H^1_{loc}(\Omega_\mu)$, $C^{\alpha}_{loc}(\Omega_\mu)$ and pointwise in Ω_μ .

Along the similar arguments as Lemma 9.2 in [2], we obtain that ω is indeed a minimizer to the truncated variational problem $(P_{\lambda,\theta,\mu})$. Due to the uniqueness of minimizer to the truncated variational problem $(P_{\lambda,\theta,\mu})$, we have $\omega = \psi_{\lambda,\theta,\mu}$. Therefore, we obtain the convergence of (3.48).

Secondly, we will show the statement (3.49) for $\Gamma_{1,\lambda_n,\theta_n,\mu}$. Indeed, for any $r_n > R$, let

$$X_n = (g_{1,\lambda_n,\theta_n,\mu}(r_n), r_n) \in \Gamma_{1,\lambda_n,\theta_n,\mu}, \text{ and } X_n \to X_0 = (x_0, r_0), \text{ as } n \to +\infty$$

Then, for any small r > 0, the non-degeneracy lemma implies that there exist two positive constants C_1 and C_2 , such that

$$C_1\lambda_n r_n \le \frac{1}{r} \oint_{\partial B_r(X_n)} \left(m_1 - \psi_{\lambda_n, \theta_n, \mu} \right) dS \le C_2\lambda_n r_n.$$

Letting $n \to +\infty$ gives

$$C_1 \lambda r_0 \leq \frac{1}{r} \oint_{\partial B_r(X_0)} \left(m_1 - \psi_{\lambda,\theta,\mu} \right) dS \leq C_2 \lambda r_0.$$

Moreover, recalling the non-degeneracy Lemma 3.5 and Lemma 3.6 yields that $X_0 \in \Gamma_{1,\mu}$. Hence, we obtain the assertion (3.49) for $r \in (R, +\infty)$.

Using Lemma 10.4 in [23], we can obtain the result for r = R, namely,

$$g_{1,\lambda_n,\theta_n,\mu}(R) \to g_{1,\lambda,\theta,\mu}(R) \text{ as } n \to +\infty.$$

Similarly, (3.49) holds for the right free boundary $\Gamma_{2,\lambda_n,\theta_n,\mu}$.

Finally, similar arguments as Theorem 7.1 in [6], we can obtain (3.50).

3.7. Continuous and smooth fit conditions of the free boundaries. In this subsection, we will verify that there exist two parameters λ and θ , such that the free boundaries $\Gamma_{i,\mu}$ connect smoothly at the end points A_i of the nozzles N_i (i = 1, 2), respectively. Namely, for any $\mu > 0$, there exists a pair of parameters $(\lambda_{\mu}, \theta_{\mu})$ with $\lambda_{\mu} > 0$, $\theta_{\mu} \in (0, \pi)$, such that

$$g_{1,\lambda_{\mu},\theta_{\mu},\mu}(R) = -1$$
 and $g_{2,\lambda_{\mu},\theta_{\mu},\mu}(R) = 1$.

As already mentioned before, this is the main difference to the impinging free jet without rigid nozzle walls.

To see this, we first define a set Σ_{μ} as

$$\Sigma_{\mu} = \{\lambda \mid \lambda \ge 0, \text{ there exists a } \theta \in (0,\pi), \text{ such that } g_{1,\lambda,\theta,\mu}(R) < -1 \text{ and } g_{2,\lambda,\theta,\mu}(R) > 1\}.$$
(3.51)

The following lemma implies that Σ_{μ} is non-empty.

Lemma 3.19. There exists $\theta_0 \in (0, \pi)$ such that

$$g_{1,\lambda,\theta_0,\mu}(R) < -1 \quad and \quad g_{2,\lambda,\theta_0,\mu}(R) > 1,$$
(3.52)

for sufficiently small $\lambda > 0$.

Proof. For any $\Omega_0 \subset \subset \Omega_\mu \cap \{r < R\} \cap \{m_2 < \psi_{\lambda,\theta,\mu} < 0\}$, firstly, it follows from Lemma 5.2 in [2] that there exists a positive constant C (depending only on Ω_0), such that

$$|\nabla \psi_{\lambda,\theta,\mu}| \le C\lambda \quad \text{in} \quad \Omega_0, \tag{3.53}$$

provided that Ω_0 contains a free boundary point.

For $\theta \in (0, \pi)$, suppose not, without loss of generality, suppose $g_{2,\lambda,\theta,\mu}(R) \leq 1$.

Indeed, it follows from the monotonicity of $\psi_{\lambda,\theta,\mu}(x,r)$ with respect to x, that there exists a point $X_1 \in \Omega_{\mu}$, such that

$$\psi_{\lambda,\theta,\mu}(X_1) = \frac{m_2}{2}$$
, with $X_1 = (x_1, R)$, and $g_{\lambda,\theta,\mu}(R) < x_1 < 1$.

Denote $X_2 = (x_2, R)$ as the initial point of the right free boundary $x = g_{2,\lambda,\theta,\mu}(r)$, due to the monotonicity of $\psi_{\lambda,\theta,\mu}$ with respect to x, one has $x_2 > x_1$. Taking $X_3 =$

 $\left(\frac{3x_2-x_1}{2},R\right)$, an arc $\gamma \in \Omega_{\mu} \cap \{r > R\}$ connecting X_1 to X_3 and $|\gamma| \leq C|X_2-X_1| = C|x_2-x_1|$, which intersects $\Gamma_{2,\mu}$ at X_4 , γ_0 denotes the arc part γ from X_1 to X_4 .

Let Ω_0 be bounded by γ_0 , r = R and $\Gamma_{2,\mu}$, it follows from (3.53) that

$$|\nabla \psi_{\lambda,\theta,\mu}| \le C\lambda \text{ in } \Omega_0 \setminus B_{\delta}(A_2), \text{ for sufficiently small } \delta > 0.$$
(3.54)

Hence, it follows from (3.54) that

$$-\frac{m_2}{2} = \psi_{\lambda,\theta,\mu}(X_1) - \psi_{\lambda,\theta,\mu}(X_4) \le \int_{\gamma_0} |\nabla \psi_{\lambda,\theta,\mu}| dl \le C\lambda |X_1 - X_2| \le C\lambda \left(1 - g_{\lambda,\theta,\mu}(R)\right),$$

which is impossible with sufficiently small λ . Therefore, for some sufficiently small $\lambda > 0$, one has $g_{2,\lambda,\theta,\mu}(R) > 1$.

Indeed, due to Proposition 3.18 and Remark 3.1, we obtain that there exists an θ_0 $(\pi - \theta_0 \ll 1)$ such that (3.52) holds.

Hence, we complete the proof of this lemma.

Next, the following lemma implies that the set Σ_{μ} has a uniform positive lower bound.

Lemma 3.20. If $\theta \in (0, \pi)$, we have

$$\min\{-g_{1,\lambda,\theta,\mu}(R), g_{2,\lambda,\theta,\mu}(R)\} < 1, \tag{3.55}$$

for sufficiently large λ .

Proof. Indeed, it suffices to prove that the free boundaries $\Gamma_{1,\mu}$: $x = g_{1,\lambda,\theta,\mu}(r)$ and $\Gamma_{2,\mu}: x = g_{2,\lambda,\theta,\mu}(r)$ with $R \leq r \leq 2R$ are contained in a neighborhood of $\Gamma_{\mu}: x = g_{\lambda,\theta,\mu}(r)$ for sufficiently large λ .

Firstly, we prove that the free boundary $\Gamma_{1,\mu}$ with $R \leq r \leq 2R$ is contained in a neighborhood of Γ_{μ} for sufficiently large λ .

Suppose not, then there exists a small and fixed $r_0 > 0$ and $\tilde{X} = (\tilde{x}, \tilde{r}) \in \Gamma_{1,\mu} \cap \{R \leq r \leq 2R\}$, and $B_{r_0}(\tilde{X}) \subset \Omega_{\mu} \cap \{r > R\} \cap \{0 < \psi_{\lambda,\theta,\mu} < m_1\}$, such that for any $\lambda > 0$,

$$B_{r_0}(X) \cap \Gamma_{\mu} = \emptyset.$$

Thus, the non-degeneracy Lemma 3.6 implies that

$$\sqrt{\lambda + \Lambda} C\tilde{r} \le \frac{1}{r_0} \oint_{\partial B_{r_0}(\tilde{X})} \left(m_1 - \psi_{\lambda,\theta,\mu} \right) dS \le \frac{m_1}{r_0},$$

which yields

$$\sqrt{\lambda + \Lambda} \le \frac{m_1}{Cr_0 R}.$$

This leads to a contradiction for sufficiently large $\lambda > 0$.

Similarly, we can prove that the free boundaries $\Gamma_{2,\mu}$ with $R \leq r \leq 2R$ is contained in a neighborhood of Γ_{μ} for sufficiently large λ .

Therefore, we finish the proof of Lemma 3.20.

Define

$$\lambda_{\mu} = \sup\left\{\lambda \mid \lambda \in \Sigma_{\mu}\right\},\tag{3.56}$$

Lemma 3.20 implies that there exists a positive constant C independent of μ , such that

$$\lambda_{\mu} \leq C.$$

Finally, we will check that there exists a $\theta_{\mu} \in (0, \pi)$ such that

$$g_{1,\lambda_{\mu},\theta_{\mu},\mu}(R) = -1 \text{ and } g_{2,\lambda_{\mu},\theta_{\mu},\mu}(R) = 1.$$
 (3.57)

Proposition 3.21. There exists a $\theta_{\mu} \in (0, \pi)$ such that (3.57) holds. Furthermore, $N_i \cup \Gamma_{i,\lambda_{\mu},\theta_{\mu},\mu}$ is C^1 -smooth in a neighborhood of A_i , for i = 1, 2.

Proof. Taking a sequence (λ_n, θ_n) such that

$$g_{1,\lambda_n,\theta_n,\mu}(R) < -1, \quad g_{2,\lambda_n,\theta_n,\mu}(R) > 1,$$

and

$$\lambda_n \to \lambda_\mu > 0, \quad \theta_n \to \theta_\mu \in [0,\pi].$$

Noticing the fact that $x = g_{i,\lambda,\theta,\mu}(r)$ (i = 1, 2) is continuous with respect to the parameters λ and θ , then we have

$$g_{1,\lambda_{\mu},\theta_{\mu},\mu}(R) \le -1 \text{ and } g_{2,\lambda_{\mu},\theta_{\mu},\mu}(R) \ge 1.$$
 (3.58)

Firstly, we claim that

$$0 < \theta_{\mu} < \pi. \tag{3.59}$$

Without loss of generality, we suppose $\theta_{\mu} = 0$, then for any $\tilde{\theta} > 0$, the monotonicity in Proposition 3.17 gives that

$$\psi_{\lambda_{\mu},\tilde{\theta},\mu} \le \psi_{\lambda_{\mu},0,\mu},\tag{3.60}$$

and

$$g_{1,\lambda_{\mu},\tilde{\theta},\mu}(r) < g_{1,\lambda_{\mu},0,\mu}(r) \quad \text{for } r \ge R. \tag{3.61}$$

It follows from $\theta_{\mu} = 0$ that $g_{2,\lambda_{\mu},0,\mu}(R) = +\infty$, choosing $\tilde{\theta} > 0$ be sufficiently small, due to the convergence of the free boundaries (3.49), one gets

 $g_{2,\lambda_{\mu},\tilde{\theta},\mu}(R) \geq 2.$

Furthermore, (3.61) implies that

 $g_{1,\lambda_{\mu},\tilde{\theta},\mu}(R)<-1.$

Then, we can choose a $\lambda_0 > \lambda_\mu$ and $\lambda_0 - \lambda_\mu$ suitably small such that

$$g_{1,\lambda_0,\tilde{\theta},\mu}(R) < -1$$
 and $g_{2,\lambda_0,\tilde{\theta},\mu}(R) > 1$,

which implies $\lambda_0 \in \Sigma_{\mu}$. This contradicts with the definition of λ_{μ} .

Consequently, we obtain $\theta_{\mu} > 0$. Similarly, we can prove $\theta_{\mu} < \pi$. Hence, the claim (3.59) holds.

Moreover, we will verify the continuous fit conditions (3.57). Indeed, suppose not, without loss of generality, we assume that

$$g_{1,\lambda_{\mu},\theta_{\mu},\mu}(R) < -1.$$

Taking $\tilde{\theta} \in (0, \theta_{\mu})$ with $\theta_{\mu} - \tilde{\theta}$ being suitably small, then the continuity of $g_{1,\lambda_{\mu},\theta_{\mu},\mu}(R)$ with respect to θ gives

$$g_{1,\lambda_{\mu},\tilde{\theta},\mu}(R) < -1$$

Similar to (3.61), we have

$$1 \le g_{2,\lambda_{\mu},\theta_{\mu},\mu}(R) < g_{2,\lambda_{\mu},\tilde{\theta},\mu}(R).$$

Hence,

$$g_{1,\lambda_{\mu},\tilde{\theta},\mu}(R) < -1$$
 and $g_{2,\lambda_{\mu},\tilde{\theta},\mu}(R) > 1$.

Therefore, similar to the above arguments for $\theta = \tilde{\theta}$, we can choose a $\lambda_0 > \lambda_{\mu}$ and $\lambda_0 - \lambda_{\mu}$ being sufficiently small, and $\lambda_0 \in \Sigma_{\mu}$, it leads a contradiction to the definition of λ_{μ} .

Thus, we obtain the continuous fit conditions (3.57).

Furthermore, the similar proof to the jet flow problem in [4] implies that the free boundaries are C^1 -smooth at the end points of the nozzles A_i (i = 1, 2), we omit it here.

3.8. Existence of the impinging outgoing jet. In order to obtain the existence of the impinging outgoing jet, we take a sequence $\mu = \mu_n \to +\infty$, and the corresponding $(\lambda_{\mu_n}, \theta_{\mu_n})$ with $\lambda_{\mu_n} > 0$ and $\theta_{\mu_n} \in (0, \pi)$,

$$g_{1,\lambda_{\mu_n,\theta_{\mu_n},\mu_n}}(R) = -1, \text{ and } g_{2,\lambda_{\mu_n,\theta_{\mu_n},\mu_n}}(R) = 1,$$

then there exist a $\lambda \geq 0$ and $\theta \in [0, \pi]$ and a subsequence μ_n , such that $\lambda_{\mu_n} \to \lambda$, $\theta_{\mu_n} \to \theta$ and

 $\psi_{\lambda_{\mu_n},\theta_{\mu_n},\mu_n} \to \psi_{\lambda,\theta}$ weakly in $H^1_{loc}(\Omega)$ and a.e in Ω .

The similar arguments as in Proposition 3.18 imply that $\psi_{\lambda,\theta}$ is a local minimizer to the variational problem $J_{\lambda,\theta}$, namely,

$$J_{\Omega_0}(\psi_{\lambda,\theta}) \leq J_{\Omega_0}(v) \text{ for any } \Omega_0 \subset \subset \Omega \text{ and } v - \psi_{\lambda,\theta} \in H^1_0(\Omega_0),$$

where $J_{\Omega_0}(v) = \int_{\Omega_0} r \left| \frac{\nabla \psi}{r} - \left(\sqrt{\Lambda + \lambda} \chi_{\{0 < \psi < m_1\}} + \sqrt{\lambda} \chi_{\{m_2 < \psi \leq 0\}} \right) e \right|^2 dX.$

Furthermore, along the similar arguments in Proposition 6.1, we can check that $\psi_{\lambda,\theta}$ is a weak solution to the boundary value problem (2.4).

Since

$$m_2 \leq \psi_{\lambda,\theta} \leq m_1$$
 in Ω ,

and

$$\psi_{\lambda,\theta}(x,r) \ge \psi_{\lambda,\theta}(\tilde{x},r) \quad \text{for any } x < \tilde{x},$$
(3.62)

using the same arguments as before, there exist two C^1 -smooth functions $x = g_{1,\lambda,\theta}(r)$ and $x = g_{2,\lambda,\theta}(r)$ such that

$$g_{1,\lambda\mu_n,\theta\mu_n,\mu_n}(r) \to g_{1,\lambda,\theta}(r) \text{ for any } r \in [R,+\infty),$$

$$(3.63)$$

and

$$g_{2,\lambda\mu_n,\theta\mu_n,\mu_n}(r) \to g_{2,\lambda,\theta}(r) \text{ for any } r \in [R,+\infty).$$
 (3.64)

$$g_{1,\lambda,\theta}(R) = -1, \quad \text{and} \quad g_{2,\lambda,\theta}(R) = 1.$$

$$(3.65)$$

Furthermore, along the similar arguments as Lemma 3.19 and 3.20, we assert that

$$\lambda > 0 \quad \text{and} \quad 0 < \theta < \pi, \tag{3.66}$$

and the smooth fit condition of $x = g_{i,\lambda,\theta}(r)$ at A_i follows immediately from the arguments in Proposition 3.21 for i = 1, 2.

Using the standard elliptic estimates yields that $\psi_{\lambda,\theta} \in C^{2,\sigma}(\Omega_1 \cup \Omega_2) \cap C^0(\overline{\Omega_1 \cup \Omega_2})$ for some $\sigma \in (0,1)$ and it solves the boundary value problem (2.4). Hence, the existence of the impinging outgoing jets in Theorem 1.1 has been established.

Next, we will show the positivity of radial velocity to the axially symmetric impinging outgoing jets.

Proposition 3.22. Let $\psi_{\lambda,\theta}$ be the solution to the boundary value problem (2.4), then

$$m_2 < \psi_{\lambda,\theta} < m_1 \quad in \ G, \tag{3.67}$$

and

$$V = -\frac{1}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial x} > 0 \quad in \ \overline{G} \setminus (N_0 \cup \Gamma) , \qquad (3.68)$$

where G is bounded by N_i , Γ_i and N_0 for i = 1, 2.

Proof. Noting that

 $\Delta \psi_{\lambda,\theta} - \frac{1}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial r} = 0$ in any bounded connected smooth open subdomain $G_0 \subset G \setminus \Gamma$, and $m_2 \leq \psi_{\lambda,\theta} \leq m_1$ on ∂G_0 , then, the strong maximum principle implies that

$$m_2 < \psi_{\lambda,\theta} < m_1$$
 in G_0 .

The arbitrariness of domain $G_0 \subset G$ yields to (3.67).

Next, since $N_1 \cup \Gamma_1 \in C^1$, there exists a bounded smooth subdomain $G_0 \subset \{0 < \psi_{\lambda,\theta} < m_1\}$ with $\overline{G_0} \cap N_1 = X_0$ (or $G_0 \subset \{m_2 < \psi_{\lambda,\theta} < 0\}$ with $\overline{G_0} \cap N_2 = X_0$). Then, $w = -\frac{\partial \psi_{\lambda,\theta}}{\partial x}$ satisfies

$$\Delta w - \frac{1}{r} \frac{\partial w}{\partial r} = 0 \quad \text{in } G_0.$$

Since $\psi_{\lambda,\theta} = m_1$ on N_1 , the slip boundary condition (1.10) implies that

 $\partial_x \psi_{\lambda,\theta}(f_1(r), r) f_1'(r) + \partial_r \psi_{\lambda,\theta}(f_1(r), r) = 0.$

This implies that the unit outward normal derivative satisfies

$$\frac{\partial \psi_{\lambda,\theta}}{\partial \nu} \left(f_1(r), r \right) = -\partial_x \psi_{\lambda,\theta} \left(f_1(r), r \right) \sqrt{1 + f_1'(r)^2}.$$

On another hand, $\psi_{\lambda,\theta}$ attains its maximum on N_1 , and Hopf's lemma gives that

 $w = -\partial_x \psi_{\lambda,\theta} > 0, \quad \text{on } N_1.$

Similarly, one gets

$$w = -\partial_x \psi_{\lambda,\theta} > 0, \text{ on } N_2.$$

Then, in view of (3.62), one has

$$w \ge 0$$
 on ∂G_0 , and $w(X_0) > 0$,

and applying the maximum principle to $w = -\frac{\partial \psi_{\lambda,\theta}}{\partial x}$ yields that

w > 0 in any subdomain $G_0 \subset G$.

Finally, we claim that (3.68) holds on $\Gamma_1 \cup \Gamma_2$.

Recalling the fact

$$w = \frac{\sqrt{\lambda + \Lambda r}}{\sqrt{1 + (g'_{1,\lambda,\theta}(r))^2}} \ge 0 \quad \text{on } \Gamma_1, \quad \text{and} \quad w = \frac{\sqrt{\lambda r}}{\sqrt{1 + (g'_{2,\lambda,\theta}(r))^2}} \ge 0 \quad \text{on } \Gamma_2,$$

Suppose that the claim can not hold, then, without loss of generality, there exists a $r_0 \ge R$ such that $g'_{1,\lambda,\theta}(r_0) = +\infty$ or $-\infty$, $\partial_x \psi_{\lambda,\theta}(g_{1,\lambda,\theta}(r_0), r_0) = 0$ and $w(g_{1,\lambda,\theta}(r_0), r_0) = 0$.

Thanks to the fact that
$$\frac{|\nabla \psi_{\lambda,\theta}|}{r} = \sqrt{\lambda + \Lambda}$$
 on Γ_1 (or $\frac{|\nabla \psi_{\lambda,\theta}|}{r} = \sqrt{\lambda}$ on Γ_2), then
 $\frac{\partial}{\partial s} \left(\frac{|\nabla \psi_{\lambda,\theta}|^2}{r^2} \right) = 0$ on $\Gamma_1 \cup \Gamma_2$,

where s = (1,0) is the tangential vector of $\Gamma_1 \cup \Gamma_2$. This implies that

$$\left(\frac{\partial}{\partial x}\left(\frac{|\nabla\psi_{\lambda,\theta}|^2}{r^2}\right), \frac{\partial}{\partial r}\left(\frac{|\nabla\psi_{\lambda,\theta}|^2}{r^2}\right)\right) \cdot (1,0) = 0 \text{ on } \Gamma_1 \cup \Gamma_2.$$

Then, one has

$$\partial_{xr}\psi_{\lambda,\theta}(g_{1,\lambda,\theta}(r_0), r_0) = \partial_r w(g_{1,\lambda,\theta}(r_0), r_0) = 0.$$
(3.69)

However, the Hopf's Lemma gives that

$$\left|\frac{\partial w}{\partial \nu}\right| = \left|\frac{\partial w}{\partial r}\right| > 0 \text{ at } (g_{1,\lambda,\theta}(r_0), r_0),$$

which contradicts with (3.69). Then, we prove that (3.68) holds on $\Gamma_1 \cup \Gamma_2$.

Therefore, we obtain the positivity of the vertical velocity and complete the proof of Proposition 3.22. $\hfill \Box$

3.9. The properties of the interface. In this subsection, we will show that there exists a C^1 -smooth curve Γ : { $\psi_{\lambda,\theta} = 0$ } \cap {r > 0} separating the two fluids, and the axially symmetric impinging outgoing jet established here possesses a unique branching point on the symmetric axis N_0 . For $\Lambda = 0$, the proof is similar to Section 4.10 in [17], we omit it here.

Next, it suffices to prove that the results hold for $\Lambda > 0$.

Indeed, taking subsequence $\mu_n \to +\infty$, one has

$$g_{\lambda\mu_n,\theta\mu_n,\mu_n}(r) \to g_{\lambda,\theta}(r)$$
 for any $r > 0$.

Then, the similar arguments as Lemma 3.10 implies that $x = g_{\lambda,\theta}(r) \in [-\infty, +\infty]$ is generalized continuous function in $[0, +\infty)$, we need to prove that $x = g_{\lambda,\theta}(r)$ is finite valued for any $r \in [0, +\infty)$.

$$g_{\lambda\mu_n,\theta\mu_n,\mu_n}(r) \to g_{\lambda,\theta}(r) \text{ for any } r \in (0,+\infty).$$
 (3.70)

Since $g_{\lambda,\theta}(r) < g_{2,\lambda,\theta}(r)$ for any r > R and $g_{\lambda,\theta}(r) > g_{1,\lambda,\theta}(r)$ for any r > R, then it suffices to prove that $g_{\lambda,\theta}(r)$ is finite valued for $0 \le r \le M_0$, where $\max\{r_1, r_2\} \le M_0 \le R$. Denote by $(\alpha_i, \beta_i) \subset [0, M_0]$ $(i = 1, 2, ..., \alpha_i < \beta_i$ and $\beta_i \le \alpha_{i+1})$ the maximum intervals where $x = g_{\lambda,\theta}(r)$ is finite valued.

Similar arguments as Proposition 5.1 in [31], we can prove that the number of intervals (α_i, β_i) is one, denote $(\alpha, +\infty)$ for simplicity.

Next, we will prove that $\alpha = 0$.

Suppose $\alpha > 0$ and $\lim g_{\lambda,\theta}(r) = -\infty$. For some sufficiently large M > 0, Set

 $R_0 = \max\{x \mid x = g_{\lambda,\theta}(r), \alpha < r < r_1\}, r_3 = \min\{r \mid g_{\lambda,\theta}(r) = R_0 - M\},\$

 $r_4 = \max\{r \mid g_{\lambda,\theta}(r) = R_0, \alpha < r < r_1\}, \text{such that } R_0 - M \le g_{\lambda,\theta}(r) \le R_0 \text{ with } r_3 \le r \le r_4.$

It follows from Lemma 6.1 in [6] that

$$M \le C(r_4 - r_3) \le Cr_1,$$

and this is impossible when M sufficiently large.

Furthermore, the case $\alpha > 0$ and $\lim_{r \to \alpha} g_{\lambda,\theta}(r) = +\infty$, there exists a sufficiently large $R_0 > 0$ and for any M > 0, define

 $H_{R_0} = \min\{r \mid g_{\lambda,\theta}(r) = R_0\}, \ l_{R_0} = \{(R_0, r) \mid 0 \le r \le H_{R_0}\},\$

 $H_{R_0+M} = \min\{r \mid g_{\lambda,\theta}(r) = R_0 + M\}, \text{ and } l_{R_0+M} = \{(R_0,r) \mid 0 \le r \le H_{R_0+M}\}.$

We define a domain $G_{R_0,M}$, which is bounded by l_{R_0} , l_{R_0+M} , Γ and x-axis.

Applying Green's formula in $G_{R_0,M}$ and $|\psi_{\lambda,\theta}| \leq Cr^2$ in $\Omega \cap \{r < \min\{r_1, r_2\}\}$ (using the fact (3.8)), we find that

$$-\int_{\partial G_{R_0,M}} \frac{x - R_0}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} dS = -\int_{\partial G_{R_0,M}} \frac{\partial (x - R_0)}{\partial \nu} \frac{\psi_{\lambda,\theta}}{r} dS$$
$$= \int_{\partial G_{R_0,M} \cap \{x = R_0\}} \frac{\psi_{\lambda,\theta}}{r} dS - \int_{\partial G_{R_0,M} \cap \{x = R_0 + M\}} \frac{\psi_{\lambda,\theta}}{r} dS$$
$$\leq CH_{R_0}^2, \tag{3.71}$$

where ν is the unit normal vector.

Indeed, in view of $\frac{|\nabla \psi_{\lambda,\theta}^+|}{r} \ge \sqrt{\Lambda}$ on $\Gamma \cap \{R_0 \le x \le R_0 + M\}$, and $\frac{\partial \psi_{\lambda,\theta}}{\partial r} \ge 0$ on x-axis with $x \ge R_0$, the left hand side of (3.71) is estimated as

$$-\int_{\partial G_{R_0,M}} \frac{x - R_0}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} dS = -\int_{\partial G_{R_0,M} \cap (\Gamma \cup \{r=0\} \cup \{x=R_0+M\})} \frac{x - R_0}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} dS$$
$$\geq \sqrt{\Lambda} \int_{\partial G_{R_0,M} \cap \Gamma} (x - R_0) dS - M \int_{\partial G_{R_0,M} \cap \{x=R_0+M\}} \frac{1}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} dS$$
$$\geq c \sqrt{\Lambda} M^2 - C M H_{R_0},$$

due to (3.16). Here ν is parallel to $\nabla \psi_{\lambda,\theta}$ on free streamlines, the positive constants c, C are independent of M and H_{R_0} .

This together with (3.71) gives that

$$M \leq CH_{R_0},$$

then we derives a contradiction for sufficiently large M.

Hence, we obtain $\alpha = 0$.

Finally, it suffices to prove that $|g_{\lambda,\theta}(0+0)| < +\infty$.

Suppose not, without loss of generality, we assume $g_{\lambda,\theta}(0+0) = +\infty$. Similar to the above arguments, we only need to construct domain G bounded by $x = R_0$ with some $R_0 > 0$ sufficiently large, $x = R_0$, $x = R_0 + M$, r = 0 and interface Γ , then

$$M \leq C,$$

which derives a contradiction for sufficiently large M.

Similarly, we can exclude that $g_{\lambda,\theta}(0) = -\infty$. Therefore, we conclude that $x = g_{\lambda,\theta}(r)$ is finite for any $r \in [0, +\infty)$.

Now, collecting all results obtained above, we complete the proof of Theorem 1.1.

4. UNIQUENESS OF THE IMPINGING OUTGOING JET

In this section, we will investigate the uniqueness of the impinging outgoing jet and the parameters when $\Lambda = 0$.

Proof of Theorem 1.2. Let $\psi_{\lambda,\theta}$ and $\psi_{\lambda,\theta}$ be two solutions to the boundary value problem (2.4), and $\Gamma_{i,\lambda,\theta}: x = g_{i,\lambda,\theta}(r)$ and $\tilde{\Gamma}_{i,\lambda,\theta}: x = \tilde{g}_{i,\lambda,\theta}(r)$ be the corresponding free boundaries for i = 1, 2. Due to the continuous fit conditions, one has

$$f_1(R) = g_{1,\lambda,\theta}(R) = \tilde{g}_{1,\lambda,\theta}(R)$$
 and $f_2(R) = g_{2,\lambda,\theta}(R) = \tilde{g}_{2,\lambda,\theta}(R).$

Without loss of generality, we assume

$$\lim_{r \to +\infty} (g_{1,\lambda,\theta}(r) - \tilde{g}_{1,\lambda,\theta}(r)) \ge 0.$$

Set $\psi^{\varepsilon} = \psi_{\lambda,\theta}(x + \varepsilon, r)$ for some $\varepsilon \ge 0$ and choose a smallest $\varepsilon_0 \ge 0$ such that

$$\psi^{\varepsilon_0} \leq \tilde{\psi}_{\lambda,\theta}$$
 in Ω and $\psi^{\varepsilon_0}(X_0) = \tilde{\psi}_{\lambda,\theta}(X_0),$

for some $X_0 \in \overline{\{m_2 < \tilde{\psi}_{\lambda,\theta} < m_1\}}.$

We claim that

$$X_0 \notin \{m_2 < \psi^{\varepsilon_0} < m_1\} \cap \{m_2 < \psi_{\lambda,\theta} < m_1\}.$$

Suppose not and there exists a point $X_0 \in \{m_2 < \psi^{\varepsilon_0} < m_1\} \cap \{m_2 < \tilde{\psi}_{\lambda,\theta} < m_1\}$, such that

$$m_2 < \psi^{\varepsilon_0}(X_0) = \tilde{\psi}_{\lambda,\theta}(X_0) < m_1$$

The continuity of $\psi_{\lambda,\theta}$ in Ω implies that there exists a ball $B_r(X_0) \subset \{m_2 < \psi^{\varepsilon_0} < m_1\} \cap \{m_2 < \tilde{\psi}_{\lambda,\theta} < m_1\}$ such that

$$\begin{cases} \Delta \tilde{\psi}_{\lambda,\theta} - \frac{1}{r} \frac{\partial \tilde{\psi}_{\lambda,\theta}}{\partial r} = 0, \quad \Delta \psi^{\varepsilon_0} - \frac{1}{r} \frac{\partial \psi^{\varepsilon_0}}{\partial r} = 0 & \text{in } B_r(X_0), \\ \psi^{\varepsilon_0}(X) \le \tilde{\psi}_{\lambda,\theta}(X) & \text{on } \partial B_r(X_0). \end{cases}$$
(4.1)

Therefore, it follows from the strong maximum principle that

$$\psi^{\varepsilon_0}(X) = \tilde{\psi}_{\lambda,\theta}(X)$$
 in $B_r(X_0)$.

Applying the strong maximum principle in Ω again, we obtain a contradiction to the boundary condition of $\tilde{\psi}_{\lambda,\theta}$.

Then, one has

$$\psi^{\varepsilon_0}(X_0) = \tilde{\psi}_{\lambda,\theta}(X_0) = m_1, \text{ or } \psi^{\varepsilon_0}(X_0) = \tilde{\psi}_{\lambda,\theta}(X_0) = m_2.$$

Hence, the following two cases may occur.

Case 1. $\varepsilon_0 > 0$, then $X_0 \in \Gamma_{1,\lambda,\theta} \cap \tilde{\Gamma}_{1,\lambda,\theta}$ or $X_0 \in \Gamma_{2,\lambda,\theta} \cap \tilde{\Gamma}_{2,\lambda,\theta}$ and $X_0 \neq A_1, A_2$. Then,

$$\Delta \psi^{\varepsilon_0} - \frac{1}{r} \frac{\partial \psi^{\varepsilon_0}}{\partial r} = \Delta \tilde{\psi}_{\lambda,\theta} - \frac{1}{r} \frac{\partial \tilde{\psi}_{\lambda,\theta}}{\partial r} = 0 \text{ in } \Omega \cap \{m_2 < \psi^{\varepsilon_0} < m_1\} \cap \{m_2 < \tilde{\psi}_{\lambda,\theta} < m_1\}.$$

The C^1 -smoothness of the free boundaries implies that $\Gamma_{1,\lambda,\theta}$ (or $\Gamma_{2,\lambda,\theta}$) is tangent to $\tilde{\Gamma}_{1,\lambda,\theta}$ (or $\tilde{\Gamma}_{2,\lambda,\theta}$) at the point X_0 . Then, it follows from the maximum principle that

$$\sqrt{\lambda} = \frac{1}{r} \frac{\partial \psi^{\varepsilon_0}}{\partial \nu} > \frac{1}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = \sqrt{\lambda} \quad \text{or} \quad \sqrt{\lambda} = \frac{1}{r} \frac{\partial \psi^{\varepsilon_0}}{\partial \nu} < \frac{1}{r} \frac{\partial \psi_{\lambda,\theta}}{\partial \nu} = \sqrt{\lambda} \quad \text{at} \quad X_0,$$

where ν is outer normal vector, we derive a contradiction.

Case 2. $\varepsilon_0 = 0$, then $X_0 = A_1$ or A_2 . Without loss of generality, suppose $X_0 = A_1$, similar to the proof of Proposition 3.21, construct a domain G_{δ} with $\delta > 0$ and $\bar{\psi} = (1+\zeta)(m_1 - \tilde{\psi}_{\lambda,\theta}) - (m_1 - \psi^{\varepsilon_0})$ as (3.47), then for some sufficiently small $\zeta > 0$, which gives

$$\psi < 0$$
 in G_{δ} ,

we have

$$(1+\zeta)\sqrt{\lambda} = \frac{1+\zeta}{r} \left| \frac{\partial \tilde{\psi}_{\lambda,\theta}}{\partial \nu} \right| \le \frac{1}{r} \left| \frac{\partial \psi^{\varepsilon_0}}{\partial \nu} \right| = \sqrt{\lambda} \text{ at } A_1, \text{ for small } \zeta > 0,$$

a contradiction. Similarly, we obtain $X_0 \neq A_2$.

Hence, we obtain the uniqueness of the minimizer $\psi_{\lambda,\theta}$ for given λ and θ .

Next, we will prove $\theta = \tilde{\theta}$ for given λ .

Suppose not, without loss of generality, we assume $\theta < \theta$.

Let $\psi_{\lambda,\theta}$ and $\psi_{\lambda,\tilde{\theta}}$ be the two solutions to the boundary value problem (2.4) corresponding to the pairs of the parameters (λ, θ) and $(\lambda, \tilde{\theta})$, respectively. Due to Proposition 3.17, one has

$$\psi_{\lambda,\theta}(X) \ge \psi_{\lambda,\tilde{\theta}}(X)$$
 in Ω .

Similar arguments as above, we take $X_0 = A_1$ and derive a contradiction.

Hence, we obtain $\theta = \tilde{\theta}$ as desired.

5. Asymptotic behavior of impinging outgoing jet

In this section, we will establish the asymptotic behaviors of axially symmetric impinging outgoing jets in far fields which are stated in Theorem 1.3.

Proof of Theorem 1.3. Due to the standard elliptic estimates, there exists a constant C depending only on m_1 , m_2 and λ such that

$$\|\nabla\psi_{\lambda,\theta}\|_{C^{1,\sigma}(G)} \le C, \quad \text{for some} \quad 0 < \sigma < 1, \tag{5.1}$$

where $G \subset \{0 < \psi_{\lambda,\theta} < m_1\} \cup \{m_2 < \psi_{\lambda,\theta} < 0\}.$

Set $\psi_n(x,r) = \psi_{\lambda,\theta}(x-n,r)$ and a strip $E = \{-\infty < x < +\infty\} \times \{0 < r < r_1\}$, there exists a subsequence still labeled as $\psi_n(x,r)$ such that

$$\psi_n(x,r) \to \psi_0(x,r)$$
, uniformly in $C^{2,\sigma_0}(S), \ 0 < \sigma_0 < \sigma$,

for any compact set $S \subset E$ and $\psi_0(x, r)$ solves the following boundary value problem in the strip E,

$$\Delta \psi_0(x,r) - \frac{1}{r} \frac{\partial \psi_0}{\partial r} = 0, \quad \text{in } E,$$

$$\psi_0(x,0) = 0, \quad \psi_0(x,r_1) = m_1, \quad \text{for } -\infty < x < +\infty, \quad (5.2)$$

$$\max\left\{\frac{m_2}{r_2^2}r^2, m_2\right\} \le \psi_0(x,r) \le \frac{m_1}{r_1^2}r^2, \quad \text{in } E,$$

where we have used the Lemma 3.3. Obviously, the problem (5.2) has a unique solution as

$$\psi_0(x,r) = \frac{m_1}{r_1^2} r^2, \ r \in [0,r_1].$$
 (5.3)

Hence, we obtain

$$\nabla \psi_{\lambda,\theta}(x,r) \to \left(0, \frac{2m_1r}{r_1^2}\right)$$
 in $C^{1,\sigma_0}(S)$ as $x \to -\infty$.

Using the Bernoulli's law yields the asymptotic behavior (1.17) and (1.18) of flow field in the upstream.

Along the similar arguments as before, we obtain the asymptotic behavior in the upstream

$$\nabla \psi_{\lambda,\theta}(x,r) \to \left(0, \frac{2m_2r}{r_2^2}\right) \text{ in } C^{1,\sigma_0}(S') \text{ as } x \to +\infty,$$

where $S' \subset E' = \{-\infty < x < +\infty\} \times \{0 < r < r_2\}.$

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Finally, it follows from Lemma 3.14, we obtain (1.16), (1.24) and (1.25). Hence, the proof of Theorem 1.3 is done.

6. Appendix

The minimizer $\psi_{\lambda,\theta,\mu}$ satisfies the following elliptic equation in a weak sense. The similar proofs of these results can be found in Theorem 2.2-2.3 in [5], we omit here.

Proposition 6.1. Let $\psi_{\lambda,\theta,\mu}$ be a minimizer to the truncated variational problem $(P_{\lambda,\theta,\mu})$, and $\mathfrak{L}^2(\{\psi_{\lambda,\theta,\mu}=0\})=0$ (\mathfrak{L}^2 is the two dimensional Lebesgue measure), then

$$\Delta\psi_{\lambda,\theta,\mu} - \frac{1}{r}\frac{\partial\psi_{\lambda,\theta,\mu}}{\partial r} = 0, \quad in \quad \Omega_{\mu} \cap \{m_2 < \psi_{\lambda,\theta,\mu} < m_1\} \cap \{\psi_{\lambda,\theta,\mu} \neq 0\}, \tag{6.1}$$

and

$$\Delta \psi_{\lambda,\theta,\mu} - \frac{1}{r} \frac{\partial \psi_{\lambda,\theta,\mu}}{\partial r} \ge 0, \quad in \quad D_{\mu} = \Omega_{\mu} \cap \{r < R\}, \tag{6.2}$$

in a weak sense.

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