A linear fixed parameter tractable algorithm for connected pathwidth¹

Mamadou Moustapha Kanté²

Christophe Paul³

Dimitrios M. Thilikos³

Abstract

The graph parameter of *pathwidth* can be seen as a measure of the topological resemblance of a graph to a path. A popular definition of pathwidth is given in terms of node search where we are given a system of tunnels (represented by a graph) that is contaminated by some infectious substance and we are looking for a search strategy that, at each step, either places a searcher on a vertex or removes a searcher from a vertex and where an edge is cleaned when both endpoints are simultaneously occupied by searchers. It was proved that the minimum number of searchers required for a successful cleaning strategy is equal to the pathwidth of the graph plus one. Two desired characteristics for a cleaning strategy is to be *monotone* (no recontamination occurs) and connected (clean territories always remain connected). Under these two demands, the number of searchers is equivalent to a variant of pathwidth called *connected pathwidth*. We prove that connected pathwidth is fixed parameter tractable, in particular we design a $2^{O(k^2)} \cdot n$ time algorithm that checks whether the connected pathwidth of G is at most k. This resolves an open question by Dereniowski, Osula, and Rzażewski, Finding small-width connected pathdecompositions in polynomial time. Theor. Comput. Sci., 794:85–100, 2019]. For our algorithm, we enrich the typical sequence technique that is able to deal with the connectivity demand. Typical sequences have been introduced in [Bodlaender and Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. J. Algorithms, 21(2):358–402, 1996 for the design of linear parameterized algorithms for treewidth and pathwidth. While this technique has been later applied to other parameters, none of its advancements was able to deal with the connectivity demand, as it is a "global" demand that concerns an unbounded number of parts of the graph of unbounded size. The proposed extension is based on an encoding of the connectivity property that is quite versatile and may be adapted so to deliver linear parameterized algorithms for the connected variants of other width parameters as well. An immediate consequence of our result is a $2^{O(k^2)} \cdot n$ time algorithm for the monotone and connected version of the edge search number.

¹An extended abstract of this paper appeared in the proceedings of Annual European Symposium on Algorithms (ESA) [29]

²Université Clermont Auvergne, LIMOS, CNRS, France. Supported by projects DEMOGRAPH (ANR-16-CE40-0028) and ASSK (ANR-18-CE40-0025-01). Email: mamadou.kante@uca.fr.

³CNRS, LIRMM, Univ de Montpellier, Montpellier, France. Supported by projects DEMOGRAPH (ANR-16-CE40-0028), ESIGMA (ANR-17-CE23-0010) and the French-German Collaboration ANR/DFG Project UTMA (ANR-20-CE92-0027). Emails: christophe.paul@lirmm.fr, sedthilk@thilikos.info.

1 Introduction

Pathwidth. A path-decomposition of a graph G = (V, E) is a sequence $Q = \langle B_1, \ldots, B_q \rangle$ of vertex sets, called *bags* of Q, such that

- 1. $\bigcup_{i \in \{1, \dots, q\}} B_i = V$,
- 2. every edge $e \in E$ is a subset of some member of Q, and
- 3. the *trace* of every vertex $v \in V$, that is the set $\{i \mid v \in B_i\}$, is a set of consecutive integers.

The width of a path-decomposition is $\max\{|B_i|-1 \mid i \in \{1, \ldots, q\}\}$ and the pathwidth of a graph G, denoted by pw(G), is the minimum width of a path-decomposition of G.

The above definition appeared for the first time in [42]. Pathwidth can be seen as a measure of the topological resemblance of a graph to a path.¹ Pathwidth, along with its tree-analogue treewidth, have been used as key combinatorial tools in the Graph Minors series of Robertson and Seymour [41] and they are omnipresent in both structural and algorithmic graph theory. Apart from the above definition, pathwidth was also defined as the *interval thickness* [31] (in terms of interval graphs), as the vertex separation number [30] (in terms of graph layouts), as the maximum order of a blockage [8] (in terms of min-max dualities – see also [22]), and as the node search number [7,9,31,36] (in terms of graph searching games).

Deciding whether the pathwidth of a graph is at most k is an NP-complete problem [3]. This motivated the problem of the existence, or not, of a parameterized algorithm for this problem, and algorithm running in $f(k) \cdot n^{O(1)}$ time algorithm. An affirmative answer to this question was directly implied as a consequence of the algorithmic and combinatorial results of the Graph Minors series and the fact that, for every k, the class of graphs with pathwidth at most k is closed under taking of minors². On the negative side, this implication was purely existential. The challenge of constructing an $f(k) \cdot n^{O(1)}$ time algorithm for pathwidth (as well as for treewidth) was a consequence of the classic result of Bodlaender and Kloks in [12] (see also [19,34]). The main result in [12] implies a $2^{O(k^3)} \cdot n$ time algorithm. This was later improved to one running in $2^{O(k^2)} \cdot n$ time by Fürer in [24]).

Graph searching. In a *graph searching* game, the opponents are a group of *searchers* and an evading *fugitive*. The opponents move in turns in a graph. The objective of the searchers is to deploy a strategy of moves that leads to the capture of the fugitive. At each step of the *node searching* game, the searchers may either place a searcher at a vertex or remove a searcher from a vertex. The fugitive resides in the edges of the graph and is lucky, invisible, fast, and agile. The capture of the fugitive occurs when searchers occupy both endpoints of the edge where he currently resides. A *node* searching strategy is a sequence of moves of the searchers that can guarantee the eventual capture of the fugitive.³ The cost of a searching strategy is the maximum number of searchers simultaneously

^{1}Or, alternatively, to a caterpillar, as aptly remarked in [43].

²A graph H is a *minor* of a graph G if H can be obtained by some subgraph of G by contracting edges.

³An equivalent setting of graph searching is to see G as a system of pipelines or corridors that is contaminated by some poisonous gas or some highly infectious substance. The searchers can be seen as cleaners that deploy a

present in the graph during the deployment of the strategy. The node search number of a graph G, denoted by ns(G), is defined as the minimum cost of a node searching strategy. Node searching was defined by Kirousis and Papadimitriou in [32] who proved that the game is equivalent to its monotone variant where search strategies are *monotone* in the sense that they prevent the fugitive from pervading again areas from where he had been expelled. This result along with the results in [30, 31, 36], imply that, for every graph G, ns(G) = pw(G) + 1.

The connectivity issue. In several applications of graph searching it is important to guarantee secure communication channels between the searchers so that they can safely exchange information. This issue was treated for the first time in the area of distributed computing, in particular in [5], where the authors considered the problem of capturing an intruder by mobile agents (acting for example as antivirus programs). As agents deploy their cleaning strategy, they must guarantee that, at each moment of the search, the cleaned territories remain connected, so to permit the safe exchange of information between the coordinating agents.

The systematic study of connected graph searching was initiated in [4, 6]. When, in node searching, we demand that the search strategies are monotone and connected, we define *monotone* connected node search number, denoted by mcns(G).⁴ The graph decomposition counterpart of this parameter was introduced by Dereniowski in [20]. He defined the connected pathwidth of a connected graph G, denoted by cpw(G), by considering connected path-decompositions $Q = \{B_1, \ldots, B_q\}$ where the following additional property is satisfied:

▶ For every $i \in \{1, ..., q\}$, the subgraph of G induced by $\bigcup_{h \in \{1,...,i\}} B_h$ is connected.

As noticed in [20], for every connected graph G, mcns(G) = cpw(G) + 1 (see also [1]). Notice that the above demand results to a break of symmetry: the fact that $\langle B_1, \ldots, B_q \rangle$ is a connected path-decomposition does not imply that the same holds for $\langle B_q, \ldots, B_1 \rangle$ (while this is always the case for conventional path-decompositions). This break of symmetry seems to be the source of all combinatorial particularities (and challenges) of connected pathwidth. This phenomenon was also observed in the context of connected treewidth [1,35].

Computing connected pathwidth. It is easy to see that checking whether $\mathsf{cpw}(G) \leq k$ is an NP-complete problem: if we define G^* as the graph obtained from G after adding a new vertex adjacent with all the vertices of G, then observe that $\mathsf{pw}(G) = \mathsf{cpw}(G^*) - 1$. This motivates the question on the parameterized complexity of the problem. The first progress in this direction was done recently in [21] by Dereniowski, Osula, and Rzążewski who gave an $f(k) \cdot n^{O(k^2)}$ time algorithm. In [21, Conjecture 1], they conjectured that there is a fixed parameter algorithm checking whether $\mathsf{cpw}(G) \leq k$. The general question on the parameterized complexity of the connected variants of

decontamination strategy [17,23]. The fact that the fugitive is invisible, fast, lucky, and agile permits us to see him as being omnipresent in any edge that has not yet been cleaned.

⁴As proved in [47], under the connectivity demand, the monotone and the non-monotone versions of graph searching are not any more equivalent.

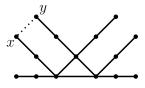


Figure 1: A graph G of connected pathwidth 2 with a subgraph of connected pathwidth 3.

graph search was raised as an open question by Fedor V. Fomin during the GRASTA 2017 workshop (see [2]).

A somehow dissuasive fact towards a parameterized algorithm for connected pathwidth is that connected pathwdith is not closed under minors and therefore it does not fit

in the powerful algorithmic framework of Graph Minors (which is the case with pathwidth). The removal of an edge may increase the parameter. For instance, the connected pathwidth of the graph in Figure 1 has connected pathwidth 2 while if we remove the edge $\{x, y\}$ its connected pathwidth increases to 3. On the positive side, connected pathwidth is closed under contractions (see e.g., [1]), i.e, its value does not increase when we contract edges and, moreover, the yes-instances of the problem have bounded pathwidth, therefore they also have bounded treewidth. Based on these observations, the existence of a parameterized algorithm would be implied if we can prove that, for any k, the set \mathcal{Z}_k of contraction-minimal⁵ graphs with connected pathwidth more than k is finite: as contraction containment can be expressed in MSO logic, one should just apply Courcelle's theorem [18] to check whether some graph of \mathcal{Z}_k is a contraction of G. The hurdle in this direction is that we have no idea whether \mathcal{Z}_k is finite or not. The alternative pathway is to try to devise a linear parameterized algorithm by applying the algorithmic techniques that are already known for pathwidth.

The typical sequence technique. The main result of [12] was an algorithm that, given a pathdecomposition \mathbb{Q} of G of width at most k and an integer w, outputs, if exists, a path-decomposition of G of width at most w, in $2^{O(k(w+\log k))} \cdot n$ time. In this algorithm Bodlaender and Kloks introduced the celebrated typical sequence technique, a refined dynamic programming technique that encodes partial path/tree decompositions as a system of suitably compressed sequences of integers, able to encode all possible path-decompositions of width at most w. This technique was later extended/adapted for the design of parametrized algorithms for numerous graph parameters such as branchwidth [13], linearwidth [14], cutwidth [45], carving-width [44], modified cutwidth, and others [10,11,46]. Also a similar approach was used by Lagergren in [33] for bounding the sizes of minor obstruction sets. In [10] the typical sequence technique was viewed as a result of un-nondeterminization: a stepwise evolution of a trivial hypothetical non-deterministic algorithm towards a deterministic parameterized algorithm. A considerable generalization of the characteristic sequence technique was proposed in the PhD thesis of Soares [37] where this technique was implemented under the powerful meta-algorithmic framework of q-branched Φ -width. Non-trivial extensions of the typical sequence technique where proposed

⁵For instance, the graph $G \setminus \{x, y\}$ from Figure 1 belongs in \mathbb{Z}_2 .

for devising parameterized algorithms for parameters on matroids such as matroid pathwidth [26], matroid branchwidth [28], as well as all the parameters on graphs or hypergraphs that can be expressed by them. Very recently Bodlaender, Jaffke, and Telle in [11] suggested refinements of the typical sequence technique that enabled the polynomial time computation of several width parameters on directed graphs. Finally, Bojańczyk and Pilipczuk suggested an alternative approach to the typical sequence technique, based on MSO transductions between decompositions [15].

Unfortunately, the above mentioned state of the art on the typical sequence technique is unable to encompass connected pathwidth. A reason for this is that the connectivity demand is a "global property" applying to *every* prefix of the path-decomposition, which corresponds to an unbounded number of subgraphs of arbitrary size.

Our result. In this paper we resolve affirmatively the conjecture that checking whether $cpw(G) \le k$ is fixed parameter tractable. Our main result is the following.

Theorem 1. One may construct an algorithm that given an n-vertex connected graph G, a pathdecomposition $\mathbf{Q} = \langle B_1, \ldots, B_q \rangle$ of G of width at most k and an integer w, outputs a connected path-decomposition of G of width at most w or reports correctly that such an algorithm does not exist in $2^{O(k(w+\log k))} \cdot n$ time.

To design an algorithm checking whether $\mathsf{cpw}(G) \leq k$ we first use the algorithms of [12] and [24], to build, if exists, a path decomposition of G of width at most k, in $2^{O(k^2)} \cdot n$ time. In case of a negative answer we know than $\mathsf{cpw}(G) > k$, otherwise we apply the algorithm of Theorem 1. The overall running time is dominated by the algorithm of Fürer in [24] which is $2^{O(k^2)} \cdot n$.

Our techniques. We now give a brief description of our techniques by focusing on the novel issues that we introduce. This description demands some familiarity with the typical sequence technique. Otherwise, the reader can go directly to the next section.

Let $\mathbf{Q} = \langle B_1, \ldots, B_q \rangle$ be a (nice) path-decomposition of G of width at most k. For every $i \in [q]$, we let $\mathbf{G}_i = (G_i, B_i)$ be the boundaried graph where $G_i = G[\bigcup_{h \in \{1,\ldots,i\}} B_h]$. We follow standard dynamic programming over a path-decomposition that consists in computing a representation of the set of partial solutions associated to \mathbf{G}_i , which in our case are *connected* path-decompositions of \mathbf{G}_i of width at most w. The challenge is how to handle in a compact way the connectivity requirement of a path-decomposition of a graph that can be of arbitrarily large size.

A connected path-decomposition $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ of \mathbf{G}_i is represented by means of a $(\mathbf{G}_i, \mathsf{P})$ encoding sequence $\mathsf{S} = \langle \mathsf{s}_1, \ldots, \mathsf{s}_\ell \rangle$. For every $j \in [\ell]$, the element s_j of the sequence S is a triple $(\mathbf{bd}(\mathsf{s}_j), \mathbf{cc}(\mathsf{s}_j), \mathbf{val}(\mathsf{s}_j))$ where: $\mathbf{bd}(\mathsf{s}_i) = A_j \cap B_i$; $\mathbf{val}(\mathsf{s}_j) = |A_j \setminus B_i|$; and $\mathbf{cc}(\mathsf{s}_j)$ is the projection of the connected components of $G_i^j = G_i[\bigcup_{h \in \{1,\ldots,j\}} A_h]$ onto the subset of boundary vertices $B_i \cap V(G_i^j)$. To compress a $(\mathbf{G}_i, \mathsf{P})$ -encoding sequence S , we identify a subset $\mathbf{bp}(\mathsf{S})$ of indexes, called breakpoints, such that $j \in \mathbf{bp}(\mathsf{S})$ if $\mathbf{bd}(\mathsf{s}_{j-1}) \neq \mathbf{bd}(\mathsf{s}_j)$ (type-1) or $\mathbf{cc}(\mathsf{s}_{j-1}) \neq \mathbf{cc}(\mathsf{s}_j)$ (type-2) or jis an index belonging to a typical sequence of the integer sequence $\langle \mathbf{val}(\mathsf{s}_b), \ldots, \mathbf{val}(\mathsf{s}_{c-1}) \rangle$ where $b, c \in [\ell]$ are consecutive type-1 or 2- breakpoints. We define rep(S) as the induced subsequence S[bp(S)].

The novelty in this representation is the $\mathbf{cc}(\cdot)$ component which is a near-partition of the subset $B_i \cap V(G_i^j)$ of boundary vertices. The critical observation is that for every $j \in [\ell - 1]$, $\mathbf{cc}(\mathbf{s}_{j+1})$ is coarser than $\mathbf{cc}(\mathbf{s}_j)$. This, together with the known results on typical sequences, allows us to prove that the size of $\mathbf{rep}(\mathsf{S})$ is O(kw) and that the number of representative sequences is $2^{O(k(w+\log k))}$. Finally, as in the typical sequence technique, we define a domination relation over the set of representative sequences. The DP algorithm over the path-decomposition Q consists then in computing a domination set $\mathbf{D}_w(G_{i+1})$ of the representative sequences of \mathbf{G}_{i+1} from a domination set $\mathbf{D}_w(G_i)$ of the representative sequences of \mathbf{G}_i .

The above scheme extends the current state of the art on typical sequences as it further incorporates the encoding of the connectivity property. While this is indeed a "global property", it appears that its evolution with respect to the bags of the decomposition can be controlled by the second component of our encoding and this is done in terms of a sequence of a gradually coarsening partitions. This establishes a dynamic programming framework that can potentially be applied on the connected versions of most of the parameters where the typical sequence technique was used so far. Moreover, it may be the starting point of the algorithmic study of parameters where other, alternative to connectivity, global properties are imposed to the corresponding decompositions.

Consequences in connected graph searching. The original version of graph searching was the edge searching variant, defined⁶ by Parsons [38, 39], where the only differences with node searching is that a searcher can additionally slide along an edge and sliding is the only way to clean an edge. The corresponding search number is called *edge search number* and is denoted by es(G). If we additionally demand that the searching strategy is connected and monotone, then we define the monotone connected edge search number denoted by mces(G). As proved in [32], $es(G) = pw(G_v)$, where G_v is the graph obtained if we subdivide twice each edge of G. Applying the same reduction as in [32] for the monotone and connected setting, one can prove that $mces(G) = cpw(G_v)$. As we already mentioned, $mcns(G) = cpw(G_v) + 1$. These two reductions imply that the result of Theorem 1 holds also for mcns and mces, i.e., the search numbers for the monotone and connected versions of both node and edge searching.

⁶An equivalent model was proposed independently by Petrov [40]. The models of Parsons and Petrov where different but also equivalent, as proved by Golovach in [25]. The model of Parsons was inspired by an earlier paper by Breisch [16], titled "An intuitive approach to speleotopology", where the aim was to rescue an (unlucky) speleologist lost in a system of caves. Notice that "unluckiness" cancels the speleologist's will of being rescued as, from the searchers' point of view, it imposes on him/her the status of an "evading entity". As a matter of fact, the connectivity issue appears even in the first inspiring model of the search game. In a more realistic scenario, the searchers cannot "teleport" themselves to non-adjacent territories of the caves while this was indeed permitted in the original setting of Parsons.

2 Preliminaries and definitions

Sets and near-partitions. For an integer $\ell > 0$, we denote by $[\ell]$ the set $\{1, \ldots, \ell\}$. Let S be a finite set. A near-partition \mathbb{Q} of S is a family of subsets $\{X_1, \ldots, X_k\}$ (with $k \leq |S| + 1$) of subsets of S, called blocks, such that $\bigcup_{i \in [k]} X_i = S$ and for every $1 \leq i < j \leq k$, $X_i \cap X_j = \emptyset$. Observe that a near-partition may contain several copies of the empty set. A partition of S is a near-partition with the additional constraint that if it contains the empty set, then this is the unique block. Let \mathbb{Q} be a near-partition of a set S and \mathbb{Q}' be a near-partition of a set S' such that $S \subseteq S'$. We say that \mathbb{Q} is thinner than \mathbb{Q}' , or that \mathbb{Q}' is coarser than \mathbb{Q} , which we denote by $\mathbb{Q} \sqsubseteq \mathbb{Q}'$, if for every block X of \mathbb{Q} , there exists a block X' of \mathbb{Q}' such that $X \subseteq X'$. For a near-partition $\mathbb{Q} = \{X_1, \ldots, X_\ell\}$ of S and a subset $S' \subseteq S$, we define the projection of \mathbb{Q} onto S' as the near-partition: if several blocks of \mathbb{Q} are subsets of $S \setminus S'$, then $\mathbb{Q}_{|S'}$ contains several copies of the emptyset.

Sequences. Let S be a set. A sequence of elements of S, denoted by $\alpha = \langle a_1, \ldots, a_\ell \rangle$, is a subset of S equipped with a total ordering: for $1 \leq i < j \leq \ell$, a_i occurs before a_j in the sequence α . The *length* of a sequence is the number of elements that it contains. Let $X \subseteq [\ell]$ be a subset of indexes of α . We define the subsequence of α induced by X as the sequence $\alpha[X]$ on the subset $\{a_i \mid i \in X\}$ such that, for $i, j \in X$, a_i occurs before a_j in $\alpha[X]$ if and only if i < j. If $\alpha = \langle a_1, \ldots, a_\ell \rangle$ and $\beta = \langle b_1, \ldots, b_p \rangle$ are two sequences, we let $\alpha \circ \beta$ denote the concatenation of α and β , *i.e.*, $\alpha \circ \beta$ is the sequence $\langle a_1, \ldots, a_\ell, b_1, \ldots, b_p \rangle$.

The duplication of the element a_j , with $j \in [\ell]$, in the sequence $\alpha = \langle a, \ldots, a_\ell \rangle$ yields the sequence $\alpha' = \langle a_1, \ldots, a_{j-1}, a_j, a_j, a_{j+1}, \ldots, a_\ell \rangle$ of length $\ell + 1$. A sequence β is an extension of the sequence α if it is either α or it results from a series of duplications on α . We define the set of extensions of α as: $\mathsf{Ext}(\alpha) = \{\alpha^* \mid \alpha^* \text{ is an extension of } \alpha\}$.

Let $\alpha = \langle a_1, \ldots, a_\ell \rangle$ be a sequence and $\alpha^* = \langle a_1, \ldots, a_p \rangle$ be an extension of α . If $p \leq \ell + k$, then α^* results from a series of at most k duplications and we say that α^* is a $(\leq k)$ -extension of α . With the definition of an extension, every element of α^* is a copy of some element of α . We define the extension surjection as a surjective function $\delta_{\alpha^* \to \alpha} : [p] \to [\ell]$ such that if $\delta_{\alpha^* \to \alpha}(j) = i$, then $a_j^* = a_i$. An extension surjection $\delta_{\alpha^* \to \alpha}$ is a certificate that $\alpha^* \in \text{Ext}(\alpha)$. Finally, we observe that if $\alpha^* \in \text{Ext}(\alpha)$, then α is an induced subsequence of α^* . Moreover, if $\alpha^* \in \text{Ext}(\alpha)$ and $\beta \in \text{Ext}(\alpha^*)$, then β is an extension of α .

Graphs and boundaried graphs. Given a graph G = (V, E) and a vertex set $S \subseteq V(G)$, we denote by G[S] the subgraph of G that is induced by the vertices of S, i.e., the graph $(S, \{e \in E \mid e \subseteq S\})$. Also, if $x \in V$, we define $G \setminus x = G[V \setminus \{x\}]$. The neighborhood of a vertex v in G is the set of vertices that are adjacent to v in G and is denoted by $N_G(v)$.

A boundaried graph is a pair $\mathbf{G} = (G, B)$ such that G is a graph over a vertex set V and $B \subseteq V$ is a subset of distinguished vertices, called *boundary vertices*. The vertices of $V \setminus B$ are called *inactive* vertices. We say that a boundaried graph $\mathbf{G} = (G, B)$ is *connected* if either G is connected

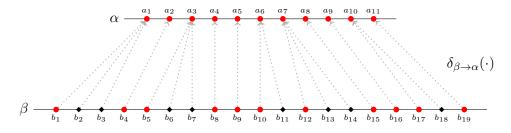


Figure 2: The sequence $\beta = \langle b_1, \ldots, b_{19} \rangle$ is an (≤ 8) -extension of the sequence $\alpha = \langle a_1, \ldots, a_{11} \rangle$. The element a_3 has been duplicated twice in β yielding three copies b_5 , b_6 , and b_7 , which are certified by $\delta_{\beta \to \alpha}(5) = \delta_{\beta \to \alpha}(6) = \delta_{\beta \to \alpha}(7) = 3$.

and $B = \emptyset$ or, in case $B \neq \emptyset$, every connected component C of G contains some boundary vertex, that is $C \cap B \neq \emptyset$.

2.1 Connected pathwidth.

A path-decomposition of a graph G = (V, E) is a sequence $\mathsf{P} = \langle A_1, \ldots, A_p \rangle$ of subsets of V where:

- 1. for every vertex $x \in V$, there exists $i \in [p]$ such that $x \in A_i$;
- 2. for every edge $e \in E$, there exists $i \in [p]$ such that $e \subseteq A_i$;
- 3. for every vertex $x \in V$, the set $\mathcal{A}(x) = \{i \in P \mid x \in A_i\}$ is a subset of consecutive integers.

Hereafter, the subsets A_i 's (for $i \in [p]$) are called the *bags* of the path-decomposition P and the set $\mathcal{A}(x)$ is the *trace* of x in P. The *width* of a path-decomposition is width(P) = max{ $|A_i| - 1 | i \in [p]$ }. The *pathwidth* of a graph G, denoted by pw(G), is the least width of a path-decomposition of G. Finally, for every $i \in [p]$, we define $V_i = \bigcup_{j \leq i} A_j$ and $G_i = G[V_i]$.

A path-decomposition $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ of a graph G is *nice* if $|A_1| = 1$ and for every $1 < i \leq p$, the symmetric difference $A_{i-1} \bigtriangleup A_i$ has size one. We distinguish two types of bags:

- if $A_{i-1} \subset A_i$ $(1 < i \le p)$, then A_i is an *introduce* bag $(A_1$ is also defined as an introduce bag);
- if $A_i \subset A_{i-1}$ $(1 < i \le p)$, then A_i is a forget bag.

It is well-known that any path-decomposition can be turned in linear time into a nice pathdecomposition of same width (see e.g., [12]).

Definition 1 (Connected path-decomposition). A path-decomposition $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ of a connected graph G is connected if, for every $i \in [\ell]$, the subgraph G_i is connected. The connected pathwidth, denoted by $\mathsf{cpw}(G)$, is the smallest width of a connected path-decomposition of G.

Let us notice that if $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ is a path-decomposition of a graph G, then $\mathsf{P}' = \langle A_p, \ldots, A_1 \rangle$ is also a path-decomposition of G. But the fact that P is a connected path-decomposition does not imply that P' is a connected path-decomposition.

Observation 1. For every graph G, $pw(G) \leq cpw(G)$.

Let P be a path-decomposition of a graph G = (V, E). Then for every subset $B \subseteq V$, P is a path-decomposition of the connected boundaried graph $\mathbf{G} = (G, B)$. The definition of a connected path-decomposition also naturally extends to boundaried graphs as follows.

Definition 2 (Connected path-decomposition of a boundaried graph). Let $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ be a path-decomposition of the boundaried graph $\mathbf{G} = (G, B)$. Then P is connected if, for every $i \in [p]$, the boundaried graph $G_i = (G_i, V_i \cap B)$ is connected.

Let $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ be a path-decomposition of $\mathbf{G} = (G, B)$. If x is a vertex of G, then $\langle A_1 \setminus \{x\}, \ldots, A_\ell \setminus \{x\} \rangle$, is a path-decomposition of $(G \setminus x, B \setminus \{x\})$. Notice that we may have a bag A_i of P such that $A_i \setminus \{x\} = \emptyset$, but this does not contradict the definition of path-decomposition. However, the fact that P is a connected path-decomposition does not imply that $\langle A_1 \setminus \{x\}, \ldots, A_\ell \setminus \{x\} \rangle$ is. The following lemma establishes a condition for the vertex x to satisfy so that its removal preserves connectivity.

Lemma 1. Let $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$ be a connected path-decomposition of the connected boundaried graph (G, B). If x is a vertex of B such that $N_G(x) \subseteq B$, then $\langle A_1 \setminus \{x\}, \ldots, A_\ell \setminus \{x\} \rangle$ is a connected path-decomposition of $(G \setminus x, B \setminus \{x\})$.

Proof. As already observed, $\mathsf{P}^{\overline{x}} = \langle A_1 \setminus \{x\}, \ldots, A_\ell \setminus \{x\} \rangle$ is a path-decomposition of $G \setminus x$. Suppose that [f, l] with $1 \leq f \leq l \leq \ell$ is the trace of x in P . As for every integer i < f (supposing that 1 < l), the boundaried graph $(G_i \setminus x, (V_i \cap B) \setminus \{x\})$ is equal to $(G_i, V_i \cap B)$ and is thereby connected. So, let us consider an integer i such that $f \leq i$. Let C_x be the connected component of G_i that contains x. As $(G_i, V_i \cap B)$ is connected, every connected component of G_i intersects B. Observe that every connected component C of G_i distinct from C_x (if any) is a connected component of $G[V_i \setminus \{x\}]$ which intersects $B \setminus \{x\}$. If $C_x = \{x\}$, by the previous observations, the statement holds. So, let C_1, \ldots, C_s , with $s \geq 1$, be the connected components of $G[V_i \setminus \{x\}]$ such that for every $j \in [s], C_j \subsetneq C_x$. As $C_x \neq \{x\}$, for every $j \in [s], C_j$ contains a neighbor of x which by assumption belongs to $B \setminus \{x\}$. It follows that every connected component of $G_i \setminus x, (V_i \cap B) \setminus \{x\}$ is a connected boundaried graph implying that $\mathsf{P}^{\overline{x}}$ is a connected path-decomposition of $(G \setminus x, B \setminus \{x\})$.

2.2 Integer sequences

Let us recall the notion of *typical sequences* introduced by Bodlaender and Kloks [12] (see also [19,34]).

Definition 3. Let $\alpha = \langle a_1, \ldots, a_\ell \rangle$ be an integer sequence. The typical sequence $\mathsf{Tseq}(\alpha)$ is obtained after iterating the following operations, until none is possible anymore:

- if for some $i \in [\ell 1]$, $a_i = a_{i+1}$, then remove a_{i+1} from α ;
- if there exists $i, j \in [\ell]$ such that $i \leq j-2$ and $\forall h, i < h < j, a_i \leq a_h \leq a_j$ or $\forall h, i < h < j, a_i \geq a_h \geq a_j$, then remove the subsequence $\langle a_{i+1}, \ldots, a_{j-1} \rangle$ from α .

As a typical sequence $\mathsf{Tseq}(\alpha) = \langle b_1, \ldots, b_i, \ldots, b_r \rangle$ is a subsequence of α , it follows that, for every $i \in [r]$, there exists $j_i \in [\ell]$ such that $b_i = a_{j_i}$. Herefater every such index j_i is called a *tip* of the sequence α .

Lemma 2 ([12]). Let $\alpha = \langle a_1, \ldots, a_\ell \rangle$ be an integer sequence. Then, $\mathsf{Tseq}(\alpha)$ is uniquely defined. If, moreover, for every $i \in [\ell]$, we have $a_i \in \{0, 1, \ldots, k\}$, then the length of $\mathsf{Tseq}(\alpha)$ is at most 2k + 1.

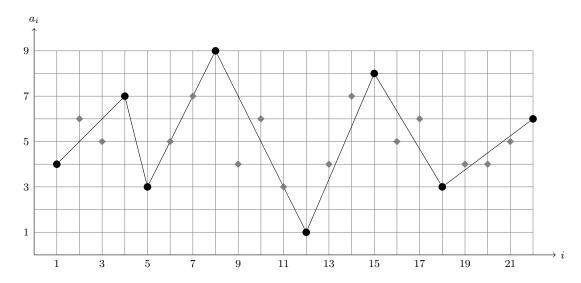


Figure 3: The black bullets forms the typical sequence $\mathsf{Tseq}(\alpha) = \langle 4, 7, 3, 9, 1, 8, 3, 6 \rangle$ of the sequence $\alpha = \langle 4, 6, 5, 7, 3, 5, 7, 9, 4, 6, 3, 1, 4, 7, 8, 5, 6, 3, 4, 4, 5, 6 \rangle$ represented by black bullets and gray diamonds.

Lemma 3 ([12]). The number of different typical sequences of integers in $\{0, 1, \ldots, k\}$ is at most $\frac{8}{3} \cdot 2^{2k}$.

A consequence of the next lemma is that every tip of the sequence $\alpha \circ \beta$ is a tip of α or of β .

Lemma 4 ([12]). Let α and β be two integer sequences. Then, $\mathsf{Tseq}(\alpha \circ \beta) = \mathsf{Tseq}(\mathsf{Tseq}(\alpha) \circ \mathsf{Tseq}(\beta))$.

If α and β are two integer sequences of same length ℓ , we say that $\alpha \leq \beta$ if for every $j \in [\ell]$, $a_j \leq b_j$.

Definition 4. Let α and β be two integer sequences. Then $\alpha \leq \beta$ if there are $\alpha^* \in \text{Ext}(\alpha)$ and $\beta^* \in \text{Ext}(\beta)$ such that $\alpha^* \leq \beta^*$. Whenever $\alpha \leq \beta$ and $\beta \leq \alpha$, we say that α and β are equivalent which is denoted by $\alpha \equiv \beta$.

We summarize in the following a set of known properties concerning duplications of integer sequences and the binary relation \leq we will need.

Lemma 5 ([12]). Let α and β be two integer sequences.

- 1. If α has length at most ℓ , then $\mathsf{Ext}(\alpha)$ contains at most $2^{\ell-1}$ sequences of length ℓ .
- 2. If $\alpha^* \in \mathsf{Ext}(\alpha)$, then $\alpha \equiv \alpha^*$.
- 3. If $\alpha^* \in \mathsf{Ext}(\alpha)$ and $\beta^* \in \mathsf{Ext}(\beta)$, then $\alpha^* \circ \beta^* \in \mathsf{Ext}(\alpha \circ \beta)$.
- 4. If $\alpha' \preceq \alpha$ and $\beta' \preceq \beta$, then $\alpha' \circ \beta' \preceq \alpha \circ \beta$.
- 5. The relation \leq is transitive, and \equiv is an equivalence relation.
- 6. For every integer sequence α , we have $\mathsf{Tseq}(\alpha) \equiv \alpha$. Moreover, there exist extensions α' and α'' of $\mathsf{Tseq}(\alpha)$ such that $\alpha' \leq \alpha \leq \alpha''$.
- 7. $\alpha \leq \beta$ if and only if $\mathsf{Tseq}(\alpha) \leq \mathsf{Tseq}(\beta)$.

We extend the definition of the \leq -relation and \leq -relation on integer sequences to sequences of integer sequences. Let $\mathsf{P} = \langle \mathsf{L}_1, \ldots, \mathsf{L}_r \rangle$ and $\mathsf{Q} = \langle \mathsf{K}_1, \ldots, \mathsf{K}_r \rangle$ be two sequences of integer sequences such that for every $i \in [r]$, L_i and K_i have the same length. We say that $\mathsf{P} \leq \mathsf{Q}$ if for every $i \in [r]$, $\mathsf{L}_i \leq \mathsf{K}_i$. The set of *extensions* of P is $\mathsf{Ext}(\mathsf{P}) = \{\langle \mathsf{L}'_1, \ldots, \mathsf{L}'_r \rangle \mid i \in [r], \mathsf{L}'_i \in \mathsf{Ext}(\mathsf{L}_i)\}$. Finally we say that $\mathsf{P} \leq \mathsf{Q}$ if there exist $\mathsf{P}' \in \mathsf{Ext}(\mathsf{P})$ and $\mathsf{Q}' \in \mathsf{Ext}(\mathsf{Q})$ such that $\mathsf{P}' \leq \mathsf{Q}'$. If $\mathsf{P} \leq \mathsf{Q}$ and $\mathsf{Q} \leq \mathsf{P}$, then we say that $\mathsf{P} \equiv \mathsf{Q}$. The relation \equiv is an equivalence relation.

3 Boundaried sequences

Definition 5 (*B*-boundaried sequence). Let *B* be a finite set. A *B*-boundaried sequence is a sequence $S = \langle s_1, \ldots, s_\ell \rangle$ such that for every $j \in [\ell]$, $s_j = (\mathbf{bd}(s_j), \mathbf{cc}(s_j), \mathbf{val}(s_j))$ is defined as follows:

- bd(s_j) ⊆ B with the property that for every x ∈ B, the indices j ∈ [l] such that x ∈ bd(s_j) are consecutive;
- cc(s_j) is a near-partition of ∪_{i≤j} bd(s_i) ⊆ B with the property that for every j < l, cc(s_j) ⊑ cc(s_{j+1});
- val(s_i) is a positive integer.

The width of S is defined as width(S) = $\max_{i \in \ell} (|\mathbf{bd}(\mathbf{s}_i)| + \mathbf{val}(\mathbf{s}_i))$.

Definition 6 (Connected *B*-boundaried sequence). Let $S = \langle s_1, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence for some finite set *B*. We say that *S* is connected if for every $i \in [\ell]$, $\mathbf{cc}(s_i)$ is a partition of $\bigcup_{i \leq j} \mathbf{bd}(s_i) \subseteq B$.

Observe that if $S = \langle s_1, \dots, s_\ell \rangle$ is a connected *B*-boundaried sequence and if there exists some $i \in [\ell]$ such that $\mathbf{cc}(s_i) = \{\emptyset\}$, then, for every $j \leq i$, $\mathbf{bd}(s_j) = \emptyset$ and $\mathbf{cc}(s_j) = \{\emptyset\}$.

As we will see in subsection 4.1, the *B*-boundaried sequences will allow us to encode partial connected path-decompositions. Intuitively, if $P = \langle A_1, \ldots, A_\ell \rangle$ is a path-decomposition, a triple s_j

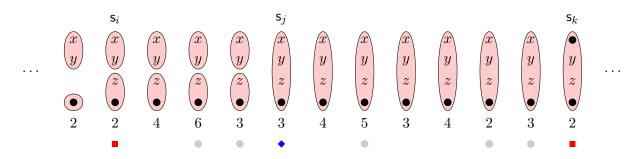


Figure 4: The part $\langle \mathbf{s}_{i-1}, \ldots, \mathbf{s}_k \rangle$ of a *B*-boundaried sequence **S** where the boundary set *B* contains among others the vertices x, y and z. A bullet • at some index j represents an element of $\bigcup_{h < j} \mathbf{bd}(\mathbf{s}_h)$. Observe that at index k, x is indeed represented by a black bullet. For the index i, we have $\mathbf{bd}(\mathbf{s}_i) = \{x, y, z\}$, $\mathbf{cc}(\mathbf{s}_i) = \{\{x, y\}, \{z, \bullet\}\}$ and $\mathbf{val}(\mathbf{s}_i) = 2$. At every position j, only named elements belong to $\mathbf{bd}(\mathbf{s}_j)$. The red squares mark the type-1 breakpoints: at position i, element z is new, while at position k, element x is forgotten. The blue diamond at index j marks a type-2 breakpoint which corresponds to the merge of two parts of $\mathbf{cc}(\mathbf{s}_{i+4})$ into a single part. Finally, the grey bullets mark type-3 breakpoints corresponding to tips of the integer sequences $\langle \mathbf{val}(\mathbf{s}_i), \ldots, \mathbf{val}(\mathbf{s}_{j-1}) \rangle$ and $\langle \mathbf{val}(\mathbf{s}_j), \ldots, \mathbf{val}(\mathbf{s}_{k-1}) \rangle$.

will represent the informations about bag A_j : $\mathbf{bd}(\mathbf{s}_j)$ contains the active vertices of the boundary set B; $\mathbf{val}(\mathbf{s}_j)$ the number of boundary vertices that appear in prior bags A_i (i < j) but not in A_j ; and $\mathbf{cc}(\mathbf{s}_j)$ encodes how the connected components of the graph induced by $\bigcup_{i < j} A_i$ project on B.

3.1 Breakpoints, representatives and domination relation

Definition 7 (Breakpoints). Let $S = \langle s_1, \ldots, s_j, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence for some finite set *B*. Then the index *j*, with $1 \le j \le \ell$, is a breakpoint of:

- type-1 if j = 1 or $\mathbf{bd}(\mathbf{s}_j) \neq \mathbf{bd}(\mathbf{s}_{j-1})$ or $j = \ell$;
- type-2 if it is not a type-1 breakpoint and $\mathbf{cc}(\mathbf{s}_j) \neq \mathbf{cc}(\mathbf{s}_{j-1})$;
- type-3 if it is not a type-1 nor a type-2 and j is a tip of the integer sequence ⟨val(s_{lj}),..., val(s_{rj-1})⟩ where l_j and r_j are respectively the largest and smallest type-1 or type-2 breakpoints such that l_j < j < r_j.

We denote by $\mathbf{bp}(S)$ the set of breakpoints of S and by $\mathbf{bp}_t(S)$ the set of type-t breakpoints of S, for $t \in \{1, 2, 3\}$. We define the representative sequence $\operatorname{rep}(S)$ of S as the induced subsequence of $S[\mathbf{bp}(S)]$.

Figure 4 illustrates the notions of *B*-boundaried sequence and breakpoints. Observe that rep(S) can be computed from the *B*-boundaried sequence S by an algorithm similar to the one described in Definition 3 and as in Lemma 2 rep(S) is uniquely defined. Notice that, as an induced subsequence of S, rep(S) is a *B*-boundaried sequence. Let ℓ be the length of S. It is worth to remark that if

 $1 < j \leq \ell$ belongs to $\mathbf{bp}_1(\mathsf{S}) \cup \mathbf{bp}_2(\mathsf{S})$, then j - 1 is also a breakpoint. This is the case because the last index of an integer sequence is by definition a tip.

We define the set of representative B-boundaried sequences of width at most w as

 $\operatorname{\mathbf{Rep}}_w(B) = \{\operatorname{\mathsf{rep}}(\mathsf{S}) \mid \mathsf{S} \text{ is a } B \text{-boundaried sequence of width } \leq w\}.$

Definition 8 (*B*-boundary model). Let $S = \langle s_1, \ldots, s_j, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence. For every $j \in [\ell]$, we set $\dot{s}_j = (\mathbf{bd}(s_j), \mathbf{cc}(s_j), \mathbf{t}(s_j))$ with $\mathbf{t}(s_j) = 1$ if $j \in \mathbf{bp}_1(S)$, $\mathbf{t}(s_j) = 2$ if $j \in \mathbf{bp}_2(S)$ and $\mathbf{t}(s_j) = 0$ otherwise. The *B*-boundary model of S, denoted by model(S), is the subsequence of $\dot{S} = \langle \dot{s}_1, \ldots, \dot{s}_j, \ldots, \dot{s}_\ell \rangle$ induced by $\mathbf{bp}_1(S) \cup \mathbf{bp}_2(S)$.

As in [12, 27], we will bound the number of representatives of *B*-boundaried sequences, and for doing so we bound the number of *B*-boundaried models and then use Lemma 3 which gives an upper bound on the number of typical sequences.

Lemma 6. Let S be a B-boundaried sequence. If $S^* \in Ext(S)$, then $model(S^*) = model(S)$.

Proof. This follows from the observation that the duplication of an element of a B-boundaried sequence does not generate a new breakpoint nor kill any existant breakpoint.

Lemma 7. Let B be a set of size k. Then, there are at most 2k + 1 type-1 breakpoints and at most k + 1 type-2 breakpoints.

Proof. Let $S = \langle s_1, \ldots, s_j, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence. By definition, for every $x \in B$, the subset $\{j \in [\ell] \mid x \in \mathbf{bd}(s_j)\}$ forms a set of consecutive integers. So every element $x \in B$ may generate 2 type-1 breakpoints. This implies that S contains at most 2k + 1 breakpoints.

Let's now consider the number of type-2 breakpoints. By definition of a *B*-boundaried sequence, for every $i < \ell$, we have $\mathbf{cc}(\mathbf{s}_i) \sqsubseteq \mathbf{cc}(\mathbf{s}_{i+1})$. Moreover if $i, j \in [\ell]$ are two consecutive type-2 breakpoints with i < j, then $\mathbf{cc}(\mathbf{s}_i) \neq \mathbf{cc}(\mathbf{s}_j)$. Observe that if $\mathbf{cc}(\mathbf{s}_i) \neq \mathbf{cc}(\mathbf{s}_j)$, then either several blocks of $\mathbf{cc}(\mathbf{s}_i)$ are joined into one block in $\mathbf{cc}(\mathbf{s}_j)$ or some new block X appears in $\mathbf{cc}(\mathbf{s}_j)$ such that $X \cap \mathbf{bd}(\mathbf{s}_i) = \emptyset$. Because |B| = k and a near-partition contains at most k + 1 blocks, by the previous argument we can have at most k + 1 type-2 breakpoints.

Lemma 8. Let B be a set of size k. Then, there are $2^{O(k \log k)}$ different B-boundary models.

Proof. By Lemma 7, the length of a *B*-boundary model is at most 3k + 2. By definition, each vertex $x \in B$ appears in an interval. Therefore, to build a *B*-boundary model, we have to choose, for each vertex $x \in B$, 2 positions among 3k + 2 ones, therefore there are $(3k + 2)^{2k} = 2^{O(k \log k)}$ possibilities for choosing the positions of the elements $\mathbf{bd}(\mathbf{s}_j)$ in *B*. Since each type-2 breakpoint is assigned a near-partition of at most k blocks on a set of size at most k and these near-partitions are gradually coarsening, the possibilities of assigning them correspond to the number of rooted trees on 3k + 2 levels and k leaves. As this is bounded by $2^{O(k)}$, the number of *B*-boundary models is $2^{O(k \log k)}$. \Box

Lemma 9. Let B be a set of size k. Then, $|\mathbf{Rep}_w(B)| = 2^{O(k(w + \log k))}$.

Proof. We only need to bound the number of possible representatives of width w having the same B-boundary model. By Lemma 7, there are at most 3k + 2 type-1 or type-2 breakpoints. Because $\operatorname{rep}(S)$ has size $\operatorname{bp}(S)$ and a type-3 breakpoint is between two type-1 or type-2 breakpoints, we have to bound the number of typical sequences. By Lemma 3, the number of typical sequences with integers $\{0, 1, \ldots, w\}$ is at most $\frac{8}{3} \cdot 2^{2w} = 2^{O(w)}$. Since there are at most 3k + 2 = O(k) intervals where we can locate type-3 breakpoints, we have $2^{O(wk)}$ possible ways to assign them. The lemma now follows if we take into account the upper bound by Lemma 8.

Notice that the notion of a *B*-boundary model corresponds to the one of *interval model* in [12]. Besides the *B*-boundary model of a sequence S, we introduce the *profile* of S, which corresponds to the concept of *list representation* in [12].

Definition 9 (Profile). Let S be a B-boundaried sequence of length ℓ and let $1 = j_1 < \cdots < j_i < \cdots < j_r = \ell$ be the subset of indices of $[\ell]$ that belong to $\mathbf{bp}_1(\mathsf{S}) \cup \mathbf{bp}_2(\mathsf{S})$. Then we set profile(S) = $\langle \mathsf{L}_1, \ldots, \mathsf{L}_r \rangle$ with, for $i \in [r]$, $\mathsf{L}_j = \langle \mathbf{val}(\mathsf{s}_{j_i}), \ldots, \mathbf{val}(\mathsf{s}_{j_{i+1}-1}) \rangle$.

Let us now introduce the domination relation over B-boundaried sequences. This relation will allow us to compare B-boundaried sequences having the same model by means of their B-profiles.

Definition 10 (Domination relation). Let $S = \langle s_1, ..., s_j, ..., s_\ell \rangle$ and $T = \langle t_1, ..., t_j, ..., t_\ell \rangle$ be two *B*-boundaried sequences such that model(S) = model(T). If profile(S) \leq profile(T), then we write $S \leq T$. And, we say that S dominates T, denoted by $S \leq T$, if profile(S) \leq profile(T). If we have profile(S) \leq profile(T) and profile(T) \leq profile(S), then we say that S and T are equivalent, which is denoted by $S \equiv T$.

Lemma 10. Let S and T be two B-boundaried sequences such that model(S) = model(T). If $S \leq T$, then there exist S^* an extension of S and T^* an extension of T such that $S^* \leq T^*$.

Proof. This is a direct consequence of the definitions.

We observe that some properties on integer sequences from Lemma 5 transfer to *B*-boundaried sequences, and we state in the following some of them that we refer to implicitly most of the time (to avoid overloading the text).

Lemma 11. Let S be a B-boundaried sequence. Then,

- 1. $rep(S) \equiv S$,
- 2. if $S^* \in Ext(S)$, then $S^* \equiv S$,
- 3. $S \leq T$ if and only if $rep(S) \leq rep(T)$.
- 4. If T is a B-boundaried sequence such that $S \leq T$, then there exist an extension S^* of S and an extension T^* of T such that $S^* \leq T^*$.
- 5. The relation \leq is transitive, and \equiv is an equivalence relation (referring to boundary sequences).

Proof. Let's prove (1). By definition S and rep(S) have the same *B*-boundary model. Let profile(S) = $\langle L_1, \ldots, L_p \rangle$. By definition, profile(rep(S)) = $\langle \mathsf{Tseq}(L_1), \ldots, \mathsf{Tseq}(L_p) \rangle$, and by Lemma 5(6), we know that $\mathsf{Tseq}(L_i) \equiv L_i$, for $i \in [p]$. We can therefore conclude that $\mathsf{profile}(\mathsf{S}) \equiv \mathsf{profile}(\mathsf{rep}(\mathsf{S}))$, *i.e.*, $\mathsf{S} \equiv \mathsf{rep}(\mathsf{S})$. For (2), if $\mathsf{S}^* \in \mathsf{Ext}(\mathsf{S})$, then clearly $\mathsf{S}^* \preceq \mathsf{S}$ and $\mathsf{S} \preceq \mathsf{S}^*$ by taking as an extension of S its extension S^* , and for an extension of S^* itself. Finally, (4) follows directly from the definitions, (5) follows from Lemma 5(5), and (3) follows from (1) and (5).

3.2 Operations on *B*-boundaried sequences

Given a finite set B, we define two operations on B-boundaried sequences that will be later used in the DP algorithm. The first operation, *projection*, will be used in the case of forget bags where we need to transform a B-boundaried sequence representing a connected path-decomposition of a boundaried graph $\mathbf{G} = (G, B)$ into a $B \setminus \{x\}$ -boundaried sequence representing a connected path-decomposition of the boundaried graph $\mathbf{G}^{\overline{x}} = (G, B \setminus \{x\})$. The second operation deals with the insertion in a B-boundaried sequence of a new boundary element x with respect to a subset $X \subseteq B$. This will be used by the DP algorithm when handling insertion bags.

3.2.1 Projection of *B*-boundaried sequences

The projection of a B-boundaried sequence S onto $B' \subseteq B$ aims at moving the vertices of $B \setminus B'$ from the status of boundary vertices to the status of inactive vertices.

Definition 11 (Projection). Let $S = \langle s_1, \ldots, s_i, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence. For a subset $B' \subseteq B$, the projection of S onto B' is the *B'*-boundaried sequence $S_{|B'} = \langle s_{1|B'}, \ldots, s_{i|B'}, \ldots, s_{\ell|B'} \rangle$ such that for every $i \in [\ell]$:

- $\mathbf{bd}(\mathbf{s}_{i|B'}) = \mathbf{bd}(\mathbf{s}_i) \cap B';$
- $\mathbf{cc}(\mathbf{s}_{i|B'}) = \mathbf{cc}(\mathbf{s}_i)_{|B'};$
- $\operatorname{val}(\mathsf{s}_{i|B'}) = \operatorname{val}(\mathsf{s}_i) + |\mathbf{bd}(\mathsf{s}_i) \setminus B'|.$

We observe that when the *B*-boundaried sequence S is connected, its projection $S_{|B'}$ onto $B' \subseteq B$ may not be connected. This is the case if for some $j \in [\ell]$, the partition $\mathbf{cc}(\mathbf{s}_j)$ contains several blocks and at least one of them is a subset of $B \setminus B'$.

Lemma 12. Let B be a finite set and $B' \subseteq B$. Then, the width of $S_{|B'}$ is equal to the width of S, for every B-boundaried sequence S.

Proof. Let $S = \langle s_1, \ldots, s_j, \ldots, s_\ell \rangle$ and $S_{|B'} = \langle s'_1, \ldots, s'_j, \ldots, s'_\ell \rangle$. By definition, for each $1 \le j \le \ell$, $|\mathbf{bd}(s_j)| + \mathbf{val}(s_j) = |\mathbf{bd}(s_j) \cap B'| + |\mathbf{bd}(s_j) \setminus B'| + \mathbf{val}(s_j)$, the latter being exactly $|\mathbf{bd}(s'_j)| + \mathbf{val}(s'_j)$.

Lemma 13. Let B be a finite set and $B' \subsetneq B$. If S^* is an extension of a B-boundaried sequence S, then $S^*_{|B'}$ is an extension of $S_{|B'}$.

Proof. Let $S = \langle s_1, \ldots, s_\ell \rangle$. As by Lemma 6, model(S) = model(S^{*}), duplicating s_i and then computing $s_{i|B'}$ is the same as computing $s_{i|B'}$ and then duplicating the latter.

Lemma 14. Let B be a finite set and $B' \subseteq B$. If S and T are B-boundaried sequences such that $S \leq T$, then $S_{|B'} \leq T_{|B'}$.

Proof. Let $S = \langle s_1, \ldots, s_\ell \rangle$ and let $T = \langle t_1, \ldots, t_\ell \rangle$. Because model(S) = model(T), we also have that model($S_{|B'}$) = model($T_{|B'}$). Because model(S) = model(T), we can check that $val(s_{i|B'})$ and $val(t_{i|B'})$ are both obtained by adding the same value to $val(s_i)$ and to $val(t_i)$, respectively. Hence, we can conclude that $S_{|B'} \leq T_{|B'}$ because profile(S) \leq profile(T).

Lemma 15. Let B be a finite set and $B' \subseteq B$. If S and T are B-boundaried sequences such that $S \preceq T$, then $S_{|B'} \preceq T_{|B'}$.

Proof. Let S^* and T^* be extensions of S and T, respectively, such that $S^* \leq T^*$. By Lemma 14, $S^*_{|B'} \leq T^*_{|B'}$. By Lemma 13, $S^*_{|B'}$ is an extension of $S_{|B'}$, *i.e.*, $S_{|B'} \equiv S^*_{|B'}$ by Lemma 11(2). Similarly, we have $T^*_{|B'} \equiv T_{|B'}$. Hence, we can conclude that $S_{|B'} \preceq T_{|B'}$.

3.2.2 Insertion into a *B*-boundaried sequence

Let $S = \langle s_1, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence and let *X* be a subset of *B*. An *insertion position* is a pair of indices (f_x, l_x) such that $1 \leq f_x \leq l_x \leq \ell$. An insertion position is valid with respect to *X* in *S* if $X \subseteq \bigcup_{f_x \leq j < l_x} \mathbf{bd}(s_j)$. Let us now formally describe the insertion operation.

Definition 12. Let $S = \langle s_1, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence and (f_x, l_x) be a valid insertion position with respect to $X \subseteq B$. Then $S^x = \text{Ins}(S, x, X, f_x, l_x) = \langle s_1^x, \ldots, s_\ell^x \rangle$ is the $(B \cup \{x\})$ -boundaried sequence such that for every $j \in [\ell]$:

- if $j < f_x$, then $\mathbf{bd}(\mathbf{s}_i^x) = \mathbf{bd}(\mathbf{s}_j)$; $\mathbf{cc}(\mathbf{s}_i^x) = \mathbf{cc}(\mathbf{s}_j)$ and $\mathbf{val}(\mathbf{s}_i^x) = \mathbf{val}(\mathbf{s}_j)$.
- if f_x ≤ j ≤ l_x, then bd(s^x_j) = bd(s_j) ∪ {x}; cc(s^x_j) is obtained by adding a new block {x} to cc(s_j) and then merging that new block with all the blocks of cc(s_j) that contain an element of X (if any); val(s^x_j) = val(s_j).
- and otherwise, bd(s^x_j) = bd(s_j); cc(s^x_j) is obtained by adding a new block {x} to cc(s_j) and then merging that new block with all the blocks of cc(s_j) that contains an element of X (if any); val(s^x_j) = val(s_j).

It is worth to notice that a type-2 breakpoint j in a B-boundaried sequence S may disappear in $lns(S, x, X, f_x, l_x)$, because the insertion of x with respect to X may merge in $cc(s_{j-1}^x)$ distinct blocks of $cc(s_{j-1})$ that are joined in $cc(s_j)$. However one can prove that if $j \in bp_2(S^x)$, then $j \in bp_2(S)$ (see Figure 5 for an illustration of this property) and if $j \in bp_3(S^x)$, then $j \in bp_3(S)$.

Lemma 16. Let B and B' be finite sets with $B = B' \setminus \{x\}$ for some $x \in B'$. Let S be a B-boundaried sequence and let (f_x, l_x) be a valid insertion position with respect to subset $X \subseteq B$ in S. Then, the width of S is at most the width of $\mathsf{Ins}(\mathsf{S}, x, X, f_x, l_x)$.

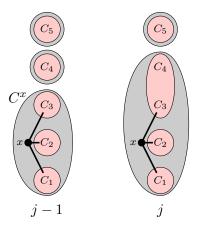


Figure 5: In red, the partitions $\mathbf{cc}(\mathbf{s}_{j-1}) = \{C_1, C_2, C_3, C_4, C_5\}$ and $\mathbf{cc}(\mathbf{s}_j) = \{C_1, C_2, C_3 \cup C_4, C_5\}$ certifying that $j \in \mathbf{bp}_2(\mathsf{S})$. In grey, the partitions $\mathbf{cc}(\mathbf{s}_{j-1}^x) = \{C^x, C_4, C_5\}$ and $\mathbf{cc}(\mathbf{s}_j^x) = \{C^x \cup C_4, C_5\}$ certifying that $j \in \mathbf{bp}_2(\mathsf{S}^x)$.

Proof. Suppose that $S = \langle s_1, \ldots, s_\ell \rangle$ and $Ins(S, x, X, f_x, l_x) = \langle s_1^x, \ldots, s_\ell^x \rangle$. By Definition 12 we have that: for each $1 \leq j \leq \ell$, $val(s_j^x) = val(s_j)$; if $j \notin [f_x, l_x]$, then $bd(s_j^x) = bd(s_j)$, otherwise $bd(s_j^x) = bd(s_j) \cup \{x\}$. The statement follows therefore by definition of width of *B*-boundaried sequences.

Let us remind that if a *B*-boundaried sequence T of length p is an extension of S of length ℓ , then the extension surjection $\delta_{\mathsf{T}\to\mathsf{S}}:[p]\to[\ell]$ associates each element of T with its original copy in S (see Section 2).

Lemma 17. Let *B* and *B'* be finite sets with $B = B' \setminus \{x\}$ for some $x \in B'$. Let $S = \langle s_1, \ldots, s_\ell \rangle$ be a *B*-boundaried sequence, and let $T \in Ext(S)$ that has length *p* and is certified by the surjective function $\delta_{T\to S} : [p] \to [\ell]$. For every valid insertion position (f_x, l_x) with respect to some subset $X \subseteq B$ in S, (f_x^*, l_x^*) is a valid insertion position with respect to *X* in T, where $f_x^* = \min\{h \in [p] \mid f_x = \delta_{T\to S}(h)\}$ and $l_x^* = \max\{h \in [p] \mid f_x = \delta_{T\to S}(h)\}$. Moreover, $Ins(T, x, X, f_x^*, l_x^*)$ is an extension of $Ins(S, x, X, f_x, l_x)$.

Proof. Let us prove the statement for a 1-extension T of S. Inductively applying the proof $p - \ell$ times leads to the statement.

Let us denote $\mathsf{T} = \langle \mathsf{t}_1, \ldots, \mathsf{t}_{\ell+1} \rangle$. Suppose that s_i , for $1 \leq i \leq \ell$, is duplicated, that is for every $j \leq i, \, \delta_{\mathsf{T} \to \mathsf{S}}(j) = j$ and for every $i < j \leq \ell+1, \, \delta_{\mathsf{T} \to \mathsf{S}}(j) = j-1$. It is clear that if $i > l_x$ then (f_x, l_x) is still a valid insertion position with respect to X in T, and similarly for $(f_x + 1, l_x + 1)$ if $i < f_x$. If $f_x \leq i \leq l_x$, then $(f_x, l_x + 1)$ is a valid insertion position with respect to X in T because $\mathsf{t}_j = \mathsf{s}_j$ for $f_x \leq j \leq i$, and $\mathsf{s}_j^* = \mathsf{s}_{j-1}$ for $i+1 \leq j \leq \ell+1$.

We claim now that $Ins(T, x, X, f_x^*, l_x^*)$ is an extension of $Ins(S, x, X, f_x, l_x)$ certified by the surjective function $\delta_{T\to S}$. Indeed, observe that for every $j \in [\ell+1]$, $t_j = s_{\delta_{T\to S}(j)}$. So, if we duplicate s_i^x in S^x , we will obtain $Ins(T, x, X, f_x^*, l_x^*)$.

Lemma 17 shows that if T is an extension of S, then, to every valid insertion position (f_x, l_x) with respect to some subset $X \subseteq B$ in S, one can associate a valid insertion position (f_x^*, l_x^*) with respect to X in T. As shown by the example of Figure 6, the reverse is not true. The following lemma states that it is indeed possible to associate a valid insertion position (f_x^*, l_x^*) with respect to X in T to some valid insertion position with respect to X in some (≤ 2)-extension of S.

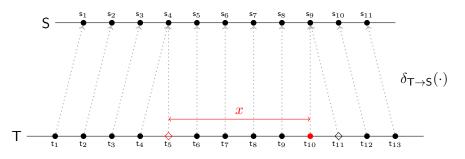


Figure 6: Let T be a 2-extension of the *B*-boundaried sequence S. Suppose that (5, 10) is a valid insertion position with respect to some for $X \subseteq B$ in T. Observe that as $4 = \delta_{T\to S}(5)$ and $9 = \delta_{T\to S}(10)$, (4, 9) is also a valid insertion position with respect to some for $X \subseteq B$ in S. However, lns(T, x, X, 5, 10) is not an extension of lns(S, x, X, 4, 9).

Lemma 18. Let B and B' be finite sets with $B = B' \setminus \{x\}$ for some $x \in B'$. Let T be an extension of a B-boundaried sequence S. If (f_x^*, l_x^*) is a valid insertion position with respect to a subset $X \subseteq B$ in T, then there is a (≤ 2) -extension R of S and a valid insertion position (f_x, l_x) with respect to X in R such that $lns(T, x, X, f_x^*, l_x^*)$ is an extension of $lns(R, x, X, f_x, l_x)$.

Proof. Suppose that $S = \langle s_1, \ldots, s_\ell \rangle$ and $T = \langle t_1, \ldots, t_p \rangle$. Let $\delta_{T \to S} : [p] \to [\ell]$ be the surjection certifying that $T \in Ext(S)$, that is for every $j \in [p]$, if $\delta_{T \to S}(j) = i$, then t_j is a copy originating from s_i . Let us denote $f = \delta_{T \to S}(f_x^*)$ and $l = \delta_{T \to S}(l_x^*)$. We also define $f'_x = \min\{j \in [p] \mid \delta_{T \to S}(j) = f\}$ and $l'_x = \max\{j \in [p] \mid \delta_{T \to S}(j) = l\}$. The (≤ 2)-extension R of S is built as follows: if $f'_x < f^*_x$, then we duplicate s_f and if $l^*_x < l'_x$, then we duplicate s_l . Let r be the size of R and let $\delta_{R \to S} : [r] \to [\ell]$ certifying that R is a (≤ 2)-extension of S.

Let us build a surjection $\delta_{\mathsf{T}\to\mathsf{R}}: [p] \to [r]$ certifying that T is an extension of R . To that aim, we define $f_x = \max\{h \in [r] \mid \delta_{\mathsf{R}\to\mathsf{S}}(h) = f\}$ and $l_x = \min\{h \in [r] \mid \delta_{\mathsf{R}\to\mathsf{S}}(h) = l\}$. Then:

$$\delta_{\mathsf{T}\to\mathsf{R}}(j) = \begin{cases} \delta_{\mathsf{T}\to\mathsf{S}}(j) & \text{if } j < f_x^*, \\ \delta_{\mathsf{T}\to\mathsf{S}}(j) - f + f_x & \text{if } f_x^* \le j \le l_x^*, \\ \delta_{\mathsf{T}\to\mathsf{S}}(j) - l + l' & \text{if } l_x^* < j. \end{cases}$$

where as in Figure 7 $l' = \max\{h \in [r] \mid \delta_{\mathsf{R} \to \mathsf{S}}(h) = l\}.$

Observe that as $T \in Ext(S)$ and $R \in Ext(S)$, by Lemma 6, we have model(R) = model(T). Thereby $\delta_{T\to S}(f_x^*) = f_x$ and $\delta_{T\to S}(l_x^*) = l_x$ implies that (f_x, l_x) is a valid insertion position with respect to X in R. It remains to prove that $T^x = Ins(T, x, X, f_x^*, l_x^*)$ is an extension of

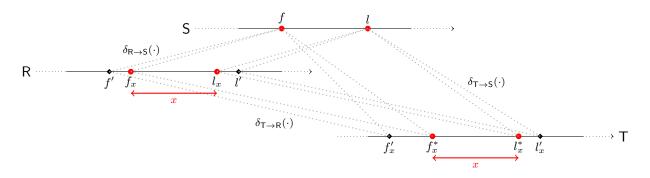


Figure 7: The three surjective functions $\delta_{\mathsf{T}\to\mathsf{S}}(\cdot)$, $\delta_{\mathsf{R}\to\mathsf{S}}(\cdot)$ and $\delta_{\mathsf{T}\to\mathsf{R}}(\cdot)$ respectively certifying that $\mathsf{T}\in\mathsf{Ext}(\mathsf{S})$, $\mathsf{R}\in\mathsf{Ext}(\mathsf{S})$ and $\mathsf{T}\in\mathsf{Ext}(\mathsf{R})$ in the case $f'_x\neq f^*_x$ and $l'_x\neq l^*_x$. In this case, as R is a 2-extension of S , $f'=\min\{h\in[r]\mid\delta_{\mathsf{R}\to\mathsf{S}}(h)=f\}$ and $l'=\max\{h\in[r]\mid\delta_{\mathsf{R}\to\mathsf{S}}(h)=l\}$.

 $\mathsf{R}^x = \mathsf{Ins}(\mathsf{R}, x, X, f_x, l_x)$. Observe that, by construction of R , $f_x^* = \min\{j \in [p] \mid \delta_{\mathsf{T} \to \mathsf{R}}(j) = f_x\}$ and $l_x^* = \max\{j \in [p] \mid \delta_{\mathsf{T} \to \mathsf{R}}(j) = l_x\}$. This implies that we can certify $\mathsf{T}^x \in \mathsf{Ext}(\mathsf{R}^x)$ by Lemma 17. \Box

Lemma 19. Let B and B' be finite sets with $B = B' \setminus \{x\}$ for some $x \in B'$. Let S and T be B-boundaried sequences such that $S \leq T$. If (f_x, l_x) is a valid insertion position with respect to a subset $X \subseteq B$ in T, then (f_x, l_x) is a valid insertion position with respect to X in S and $lns(S, x, X, f_x, l_x) \leq lns(T, x, X, f_x, l_x)$.

Proof. Suppose that $\operatorname{profile}(S) = \langle L_1, \ldots, L_r \rangle$ and $\operatorname{profile}(T) = \langle L'_1, \ldots, L'_r \rangle$. By Definition 10, as $S \leq T$, T and S have the same *B*-model. It follows that (f_x, l_x) is a valid insertion position with respect to X in S as well. And it implies that for every $i \in [r]$, $i \in \operatorname{bp}_1(S)$ if and only if $i \in \operatorname{bp}_1(T)$ and that $i \in \operatorname{bp}_2(S)$ if and only if $i \in \operatorname{bp}_2(T)$. Thereby, if we denote $S^x = \operatorname{lns}(S, x, X, f_x, l_x)$ and $T^x = \operatorname{lns}(T, x, X, f_x, l_x)$, by Definition 12, we obtain that, for every $i \in [r]$, $i \in \operatorname{bp}_1(S^x)$ if and only if $i \in \operatorname{bp}_2(S^x)$ if and only if $i \in \operatorname{bp}_2(T^x)$. Thereby we have $\operatorname{model}(S^x) = \operatorname{model}(T^x)$. Observe moreover that $S \leq T$ implies that for every $i \in [r]$, $\operatorname{val}(s_i) \leq \operatorname{val}(t_i)$. As for every $i \in [r]$, we have that $\operatorname{val}(s_i) = \operatorname{val}(s_i^x)$ and $\operatorname{val}(t_i) = \operatorname{val}(t_i^x)$, we obtain that $\operatorname{val}(s_i^x) \leq \operatorname{val}(t_i^x)$. It follows that $\operatorname{profile}(S^x) \leq \operatorname{profile}(T^x)$, in other words $S^x \leq T^x$.

Lemma 20. Let B and B' be finite sets with $B = B' \setminus \{x\}$ for some $x \in B'$. Let S and T be B-boundaried sequences such that $S \leq T$. If (f_x, l_x) is a valid insertion position with respect to a subset $X \subseteq B$ in T, then there is a valid insertion position (f'_x, l'_x) in a (≤ 2) -extension R of S such that $lns(R, x, X, f'_x, l'_x) \leq lns(T, x, X, f_x, l_x)$.

Proof. Let S^* and T^* be extensions of S and T, respectively, such that $S^* \leq T^*$. Suppose that T^* has size p^* . Let $\delta_{T^* \to T}$ be the surjective function certifying that $T^* \in Ext(T)$. Let us denote $f_x^* = \min\{h \in [p^*] \mid f_x = \delta_{T^* \to T}(h)\}$ and $l_x^* = \max\{h \in [p^*] \mid l_x = \delta_{T^* \to T}(h)\}$. As (f_x, l_x) is a valid insertion position with respect to X in T, then by Lemma 17, (f_x^*, l_x^*) is also a valid insertion position with respect to X in T^* and $Ins(T^*, x, X, f_x^*, l_x^*)$ is an extension of $Ins(T, x, X, f_x, l_x)$.

By Lemma 19, (f_x^*, l_x^*) is a valid insertion position with respect to X in S^{*} and by Lemma 18 there is a (≤ 2)-extension R of S and a valid insertion position (f'_x, l'_x) in R such that $Ins(S^*, x, X, f_x^*, l_x^*)$ is an extension of $Ins(R, x, X, f'_x, l'_x)$. Then, by Lemma 19, we have

$$\ln(\mathsf{S}^*, x, X, f_x^*, l_x^*) \le \ln(\mathsf{T}^*, x, X, f_x^*, l_x^*).$$

By using Lemma 11(2), it follows that $lns(T, x, X, f_x, l_x) \equiv lns(T^*, x, X, f_x^*, l_x^*)$ and $lns(R, x, X, f_x', l_x') \equiv lns(S^*, x, X, f_x^*, l_x^*)$, implying the statement by Lemma 11(5).

4 Computing the connected pathwidth

We first explain how *B*-boundaried sequence are natural combinatorial objects to encode a connected path-decomposition. We describe and analyze the time complexity of the *Forget Routine* and the *Insertion Routine* that allow us to respectively process forget and insertion bags of the nice path-decomposition given as input to the DP algorithm.

4.1 Encoding a connected path-decomposition

Let us explain how to represent a path-decomposition of a boundaried graph (G, B) by means of a *B*-boundaried sequence.

Definition 13 ((**G**, P)-encoding sequence). Let $P = \langle A_1, \ldots, A_\ell \rangle$ be a path-decomposition of the boundaried graph $\mathbf{G} = (G, B)$. A B-boundaried sequence $S = \langle s_1, \ldots, s_j, \ldots, s_\ell \rangle$ is a (**G**, P)-encoding sequence, if for every $j \in [\ell]$:

- $\mathbf{bd}(\mathbf{s}_i) = A_i \cap B$: the set of boundary vertices of (G, B) belonging to the bag A_i ;
- $\mathbf{cc}(\mathbf{s}_j) = \{V(C) \cap B \mid C \text{ is a connected component of } G_j\};$
- $\operatorname{val}(s_j) = |A_j \setminus B|$: the number of inactive vertices in the bag A_j .

It is worth to observe that $\mathbf{cc}(\mathbf{s}_j)$ is, in general, not a partition of A_j (see Figure 4). Also, notice that if G_j is connected and $B \cap V_j = \emptyset$, then $\mathbf{cc}(\mathbf{s}_j) = \{\emptyset\}$.

Lemma 21. Let P be a path-decomposition of a connected boundaried graph $\mathbf{G} = (G, B)$. If P is a connected path-decomposition, then its (\mathbf{G}, P) -encoding sequence is a connected B-boundaried sequence.

Proof. Follows directly from the definitions.

Definition 14. Let $\mathbf{G} = (G, B)$ be a connected boundaried graph and S a B-boundaried sequence. We say that S is realizable in \mathbf{G} if there is an extension S^* of S that is the (\mathbf{G}, P) -encoding sequence of some connected path-decomposition P of \mathbf{G} .

Let us observe that if a *B*-boundaried sequence S is realizable, then by Lemma 21 S is connected. The set of representative *B*-boundaried sequences of a connected boundaried graph $\mathbf{G} = (G, B)$ of width $\leq w$ is defined as:

$$\operatorname{\mathbf{Rep}}_w(\mathbf{G}) = \{\operatorname{rep}(\mathsf{S}) \mid \mathsf{S} \text{ of width } \leq w \text{ is realizable in } \mathbf{G} = (G, B)\}.$$

To compute the connected pathwidth of a graph, rather than computing $\operatorname{Rep}_w(\mathbf{G})$, we compute a subset $\mathbf{D}_w(\mathbf{G}) \subseteq \operatorname{Rep}_w(\mathbf{G})$, called *domination set*, such that for every representative *B*-boundaried sequence $\mathsf{S} \in \operatorname{Rep}_w(\mathbf{G})$, there exists a representative *B*-boundaried sequence $\mathsf{R} \in \mathbf{D}_w(\mathbf{G})$ such that $\mathsf{R} \preceq \mathsf{S}$.

Proposition 1. A connected boundaried graph $\mathbf{G} = (G, B)$ has connected pathwidth at most w if and only if $\mathbf{D}_{w+1}(\mathbf{G}) \neq \emptyset$.

Proof. Let P be a connected path-decomposition of width at most w of **G**. Recall the the bags of such decomposition have size at most w + 1. By definition, the (\mathbf{G}, P) -encoding sequence is realizable in **G**, implying that $\operatorname{\mathbf{Rep}}_{w+1}(\mathbf{G})$ and thereby $\mathbf{D}_{w+1}(\mathbf{G})$ is not empty. Conversely, suppose that $\operatorname{\mathbf{Rep}}_{w+1}(\mathbf{G})$ is non-empty and consider $\mathsf{S} \in \mathbf{D}_{w+1}(\mathbf{G})$. As $\mathsf{S} \in \operatorname{\mathbf{Rep}}_{w+1}(\mathbf{G})$, there exists a connected path-decomposition P of width at most w of **G** and S^* the (G,P) -encoding sequence with $\operatorname{\mathbf{rep}}(\mathsf{S}^*) = S$, implying that $\operatorname{\mathbf{cpw}}(\mathbf{G}) \leq w$.

4.2 Forget Routine

Let $\mathbf{G} = (G, B)$ be a boundaried graph. If $x \in B$ is a boundary vertex, we denote by $B^{\overline{x}} = B \setminus \{x\}$. We define $\mathbf{G}^{\overline{x}} = (G, B^{\overline{x}})$, that is, while the graph G is left unchanged, we remove x from the set of boundary vertices. Given $\mathbf{D}_w(\mathbf{G})$ and $x \in B$, Forget Routine aims at computing a domination set $\mathbf{D}_w(\mathbf{G}^{\overline{x}})$. The routine is described in Algorithm 1.

Algorithm 1: Forget Routine
Input: A boundaried graph $\mathbf{G} = (G, B)$, a vertex $x \in B$, and $\mathbf{D}_w(\mathbf{G})$.
Output: $\mathbf{D}_w(\mathbf{G}^{\overline{x}})$, a domination set of $\mathbf{Rep}_w(\mathbf{G}^{\overline{x}})$.
1 $\mathbf{D}_w(\mathbf{G}^{\overline{x}}) \leftarrow \emptyset;$
2 foreach ${\sf S}\in {f D}_w({f G})$ do
3 if $S_{ B\setminus\{x\}}$ is connected, then add $\operatorname{rep}(S_{ B\setminus\{x\}})$ to $D_w(\mathbf{G}^{\overline{x}})$;
4 end
5 return $\mathbf{D}_w(\mathbf{G}^{\overline{x}})$.

To prove the correctness of Forget Routine, we proceed in two steps. We first establish the *completeness* of the algorithm. More precisely, Proposition 2 states that, for every connected path-decomposition P of $\mathbf{G}^{\overline{x}}$, there exists some B-boundaried sequence $S \in \mathbf{D}_w(\mathbf{G})$ such that $\operatorname{rep}(S_{|B\setminus\{x\}}) \leq \operatorname{rep}(\mathsf{T})$ where T is the $(\mathbf{G}^{\overline{x}}, \mathsf{P})$ -encoding sequence. Then Proposition 3 proves the soundness of the routine: for every B-boundaried sequence $S \in \mathbf{D}_w(\mathbf{G})$, $\operatorname{rep}(S_{|B\setminus\{x\}}) \in \mathbf{D}_w(\mathbf{G}^{\overline{x}})$ if $S_{|B\setminus\{x\}}$ is connected.

Proposition 2 (Forget completeness). Let $\mathbf{G} = (G, B)$ be a boundaried graph and $x \in B$ be a boundary vertex. If P is a connected path-decomposition of width at most w of $\mathbf{G}^{\overline{x}}$, then there exists $\mathsf{S} \in \mathbf{D}_w(\mathbf{G})$ such that $\mathsf{S}_{|B^{\overline{x}}}$ is connected and $\mathsf{rep}(\mathsf{S}_{|B^{\overline{x}}}) \preceq \mathsf{rep}(\mathsf{T})$ where T is the $(\mathbf{G}^{\overline{x}}, \mathsf{P})$ -encoding sequence.

Proof. Suppose that $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$. Observe that P is also a connected path-decomposition of \mathbf{G} of width at most w. Let $\mathsf{R} = \langle \mathsf{r}_1, \ldots, \mathsf{r}_\ell \rangle$ be the (\mathbf{G}, P) -encoding sequence.

We claim that $\mathsf{R}_{|B^{\overline{x}}}$ is the $(\mathbf{G}^{\overline{x}}, P)$ -encoding sequence. To see this, we apply Definition 11 on the projection of R onto $B^{\overline{x}}$. Consider an index $j \in [\ell]$. First, we have that $\mathbf{bd}(\mathsf{r}_{j|B^{\overline{x}}}) = \mathbf{bd}(\mathsf{r}_j) \cap B^{\overline{x}}$. As by construction of R , $\mathbf{bd}(\mathsf{r}_j) = A_j \cap B^{\overline{x}}$ and as $B^{\overline{x}} \subset B$, we obtain $\mathbf{bd}(\mathsf{r}_{j|B^{\overline{x}}}) = A_j \cap B^{\overline{x}}$. For the same arguments, observe that $\mathbf{val}(\mathsf{r}_{j|B^{\overline{x}}}) = \mathbf{val}(\mathsf{r}_j) + |\mathbf{bd}(\mathsf{r}_j) \setminus B^{\overline{x}}| = |A_j \setminus B^{\overline{x}}|$. Let us now examine $\mathbf{cc}(\mathsf{r}_{j|B^{\overline{x}}}) = \mathbf{cc}(\mathsf{r}_j)_{|B^{\overline{x}}}$. By Definition 11, every block $X \in \mathbf{cc}(\mathsf{r}_{j|B^{\overline{x}}})$ is obtained as $X = X' \cap B^{\overline{x}}$ for some block X' of $\mathbf{cc}(\mathsf{r}_j)$. Since R is connected, $X' = C \cap B$ for some connected component Cof $G_j = G[V_j]$, and thereby $X = C \cap B^{\overline{x}}$. The assumption that $\mathbf{G}^{\overline{x}}$ is connected implies that if $X = \emptyset$, then G_j is connected (that is $C = V_j$) and $B^{\overline{x}} \cap V_j = \emptyset$ (that is $B = \{x\}$). This implies that $\mathbf{cc}(\mathsf{r}_{j|B^{\overline{x}}})$ is a partition and fulfills the requirements of Definition 13. It follows that $\mathsf{R}_{|B^{\overline{x}}}$ is indeed the ($\mathbf{G}^{\overline{x}}, \mathsf{P}$)-encoding sequence and we can thereby set $\mathsf{T} = \mathsf{R}_{|B^{\overline{x}}}$.

Since $\mathbf{D}_w(\mathbf{G})$ is a domination set of $\mathbf{Rep}_w(\mathbf{G})$, there exists a *B*-boundaried sequence $\mathsf{S} \in \mathbf{D}_w(\mathbf{G})$ such that $\mathsf{S} \preceq \mathsf{rep}(\mathsf{R})$. As $\mathsf{model}(\mathsf{R}) = \mathsf{model}(\mathsf{S})$, by Lemma 15 we can conclude that $\mathsf{S}_{|B^{\overline{x}}} \preceq \mathsf{R}_{|B^{\overline{x}}} = \mathsf{T}$. Lemma 11(3) allows to conclude that $\mathsf{rep}(\mathsf{S}_{|B^{\overline{x}}}) \preceq \mathsf{rep}(\mathsf{T})$.

Proposition 3 (Forget soundness). Let $\mathbf{G} = (G, B)$ be a boundaried graph and $x \in B$ be a boundary vertex. If $S \in \mathbf{D}_w(\mathbf{G})$ and $S_{|B^{\overline{x}}}$ is connected, then $\operatorname{rep}(S_{|B^{\overline{x}}}) \in \operatorname{Rep}_w(\mathbf{G}^{\overline{x}})$.

Proof. As $S \in D_w(G) \subseteq \operatorname{Rep}_w(G)$, there exists a connected path-decomposition P of G of width at most w such that the (G, P)-encoding sequence $T = \langle t_1, \ldots, t_p \rangle$ satisfies $S = \operatorname{rep}(T)$. Since $\operatorname{model}(S) = \operatorname{model}(T)$, the hypothesis that $S_{|B^{\overline{x}}}$ is connected implies that $T_{|B^{\overline{x}}}$ is also connected. It follows that P is also a connected path-decomposition of $G^{\overline{x}}$. One can check that $T_{|B^{\overline{x}}}$ is the $(G^{\overline{x}}, P)$ -encoding sequence (for this, one may just copy the corresponding argument of Proposition 2). As $S = \operatorname{rep}(T)$, we have that $S \equiv T$ by Lemma 11(1) and then $\operatorname{model}(S) = \operatorname{model}(T)$. Then, Lemma 15 implies that $S_{|B^{\overline{x}}} \equiv T_{|B^{\overline{x}}}$ and so $\operatorname{rep}(S_{|B^{\overline{x}}}) = \operatorname{rep}(T_{|B^{\overline{x}}})$ by Lemma 11 and the fact that the representative is uniquely defined. Finally, as S has width at most w (it belongs to $D_w(G)$), by Lemma 12, $S_{|B^{\overline{x}}}$ has width at most w as well. It follows that $\operatorname{rep}(S_{|B^{\overline{x}}}) \in \operatorname{Rep}_w(G^{\overline{x}})$. □

Theorem 2. Algorithm 1 computes $\mathbf{D}_w(\mathbf{G}^{\overline{x}})$ in $2^{O(k(w+\log k))}$ -time, where k = |B|.

Proof. The correctness of Algorithm 1 is proved by Proposition 2 and Proposition 3. These two propositions imply that by applying Forget Routine on a domination set of \mathbf{G} included in the set of representatives of \mathbf{G} , we indeed compute a domination set of $\mathbf{G}^{\overline{x}}$ that is a subset of the set of representatives of $\mathbf{G}^{\overline{x}}$. As performing the projection of *B*-boundaried sequence onto $B^{\overline{x}}$ can be performed in polynomial time in the size of the sequence, the complexity of the algorithm is dominated by the size of $\mathbf{D}_w(\mathbf{G})$ that is $2^{O(k(w+\log k))}$, because of Lemma 9.

4.3 Insertion Routine

In this subsection, we present the *Insertion Routine*. Suppose that $\mathbf{G} = (G, B)$ is a boundaried graph with G = (V, E). For a subset $X \subseteq B$, we set $G^x = (V \cup \{x\}, E \cup \{xy \mid y \in X\})$ and $\mathbf{G}^x = (G^x, B^x)$ where $B^x = B \cup \{x\}$. Given a domination set $\mathbf{D}_w(\mathbf{G})$ of $\mathbf{Rep}_w(\mathbf{G})$, the task of Insertion Routine is to compute a domination set $\mathbf{D}_w(\mathbf{G}^x)$ of $\mathbf{Rep}_w(\mathbf{G}^x)$. Algorithm 2 is describing Insertion Routine.

Algorithm 2: Insertion Routine	_
Input: A boundaried graph $\mathbf{G} = (G, B)$, a subset $X \subset B$, and $\mathbf{D}_w(\mathbf{G})$.	
Output: $\mathbf{D}_w(\mathbf{G}^x)$, a domination set of $\mathbf{Rep}_w(\mathbf{G}^x)$.	
1 $\mathbf{D}_w(\mathbf{G}^x) \leftarrow \emptyset;$	
2 foreach $S = \langle s_1, \dots, s_\ell \rangle \in \mathbf{D}_w(\mathbf{G})$ do	
3 for each $f, l \in [\ell]$ such that $X \subseteq \bigcup_{f \leq j \leq l} \mathbf{bd}(s_j)$ do	
4 for each (≤ 2)-extension S' of S duplicating none, one or both of s_f and s_l do	
5 let ℓ' be the length of S' ;	
6 set $f_x = \max\{j \in [\ell'] \mid \delta_{S' \to S}(j) = f\}$ and $l_x = \min\{j \in [\ell'] \mid \delta_{S' \to S}(j) = l\};$	
7 set $S^x = lns(S', x, X, f_x, l_x);$	
8 (observe that by construction (f_x, l_x) is valid with respect to X in S');	
9 if width(S^x) $\leq w$, then add rep(S^x) to $\mathbf{D}_w(\mathbf{G}^x)$;	
10 end	
11 end	
12 end	
13 return $\mathbf{D}_w(\mathbf{G}^x)$.	

To prove the correctness of Insertion Routine, we proceed in two steps. We first establish the *completeness* of the algorithm. More precisely, Proposition 4 aims at proving that for every connected path-decomposition P^x of \mathbf{G}^x , the $(\mathbf{G}^x, \mathsf{P}^x)$ -encoding sequence T^x is dominated by some B^x -boundaried sequence S^x that can be computed from a *B*-boundaried sequence S belonging to $\mathbf{D}_w(\mathbf{G})$. Then we argue about the *soundness* of Insertion Routine. Proposition 5 shows that if S^x is generated from a *B*-boundaried sequence $\mathsf{S} \in \mathbf{D}_w(\mathbf{G})$, then $\mathsf{rep}(\mathsf{S}^x)$ belongs to $\mathbf{D}_w(\mathbf{G}^x)$.

Proposition 4 (Insertion completeness). Let $\mathbf{G} = (G, B)$ be a boundaried graph and let $X \subseteq B$ be a subset of boundary vertices. Let P^x be a connected path-decomposition of width at most w of the boundaried graph $\mathbf{G}^x = (G^x, B^x)$ and let T^x be the $(\mathbf{G}^x, \mathsf{P}^x)$ -encoding sequence. Then there exist a Bboundaried sequence S' such that S' is a (≤ 2) -extension of some B-boundaried sequence $\mathsf{S} \in \mathbf{D}_w(\mathbf{G})$ and an insertion position (f_x, l_x) valid with respect to X in S' such that the B^x -boundaried sequence $\mathsf{S}^x = \mathsf{Ins}(\mathsf{S}', x, X, f_x, l_x)$ satisfies $\mathsf{rep}(\mathsf{S}^x) \preceq \mathsf{rep}(\mathsf{T}^x)$.

Proof. Suppose that $\mathsf{P}^x = \langle A_1^x, \ldots, A_\ell^x \rangle$ and that $\mathsf{T}^x = \langle \mathsf{t}_1^x, \ldots, \mathsf{t}_p^x \rangle$. Let $[f_x^*, l_x^*]$ be the trace of x in P^x . By the definition of a path-decomposition and of an encoding sequence, $X \subseteq \bigcup_{f_x \leq j \leq l_x} \mathbf{bd}(\mathsf{t}_j^x)$. By Lemma 1, $\mathsf{P} = \langle A_1, \ldots, A_\ell \rangle$, with $A_i = A_i^x \setminus \{x\}$ for every $1 \leq i \leq \ell$, is a connected path-decomposition of \mathbf{G} . Let $\mathsf{T} = \langle \mathsf{t}_1, \ldots, \mathsf{t}_\ell \rangle$ be the (\mathbf{G}, P) -encoding sequence. Observe that by the construction of \mathbf{G}^x , if $y \in X$, then $y \in A_j$ for some $f_x^* \leq j \leq l_x^*$. As by assumption, $X \subseteq B$, we have that $y \in \mathbf{bd}(\mathbf{t}_j)$. Therefore, (f_x^*, l_x^*) is a valid insertion position with respect to X in T. One can easily check that $\mathsf{Ins}(\mathsf{T}, x, X, f_x^*, l_x^*) = \mathsf{T}^x$. Observe that, as the width of T^x is at most w, the width of T is at most w as well, because of Lemma 16. Since $\mathbf{D}_w(\mathbf{G})$ is a domination set of $\mathbf{Rep}_w(\mathbf{G})$, there exists a *B*-boundaried sequence $\mathsf{S} \in \mathbf{D}_w(\mathbf{G})$ such that $\mathsf{S} \preceq \mathsf{T}$. By Lemma 20, there exists a (≤ 2)-extension S' of S and a valid insertion position (f_x, l_x) with respect to X in S' such that $\mathsf{Ins}(\mathsf{S}', x, X, f_x, l_x) \preceq \mathsf{Ins}(\mathsf{T}, x, X, f_x^*, l_x^*)$. By Lemma 11(3), we have $\mathsf{rep}(\mathsf{S}^x) \preceq \mathsf{rep}(\mathsf{T}^x)$.

We let the reader observe that the completeness of Insertion Routine relies on Lemma 20 and thereby on Lemma 18. And the reason we compute a domination set of $\operatorname{\mathbf{Rep}}_w(\mathbf{G}^x)$ rather than the set $\operatorname{\mathbf{Rep}}_w(\mathbf{G}^x)$, is the issue discussed in Figure 6.

Proposition 5 (Insertion soundness). Let $\mathbf{G} = (G, B)$ be a boundaried graph and let $X \subseteq B$ be a subset of boundary vertices. If $\mathsf{S}' = \langle \mathsf{s}'_1, \ldots, \mathsf{s}'_{\ell'} \rangle$ is a (≤ 2) -extension of a B-boundaried sequence $\mathsf{S} = \langle \mathsf{s}_1, \ldots, \mathsf{s}_\ell \rangle \in \mathbf{D}_w(\mathbf{G})$ and if (f_x, l_x) is a valid insertion position with respect to X in S' such that $\mathsf{S}^x = \mathsf{Ins}(\mathsf{S}', x, X, f_x, l_x)$ has width at most w, then $\mathsf{rep}(\mathsf{S}^x) \in \mathsf{Rep}_w(\mathbf{G}^{\overline{x}})$.

Proof. As $S \in D_w(G) \subseteq \operatorname{Rep}_w(G)$, there exists a connected path- decomposition P of G of width at most w such that the (G, P)-encoding sequence $T = \langle t_1, \ldots, t_p \rangle$ satisfies $\operatorname{rep}(T) = S$. Let $\delta_{S' \to S} : [\ell'] \to [\ell]$ be the extension surjection certifying that S' is a (≤ 2) -extension of S. Let us denote $f = \delta_{S' \to S}(f_x)$ and $l = \delta_{S' \to S}(l_x)$. As $S = \operatorname{rep}(T)$, with every $j \in [\ell]$, we can associate a $\iota_j \in [p]$ such that S is the subsequence of T induced by $\operatorname{bp}(T) = \{\iota_j \in [p] \mid j \in [\ell]\}$. We build a (≤ 2) -extension $T' = \langle t_1, \ldots, t_{p'} \rangle$ of T, in the same way as S' is obtained from S, that is: we duplicate t_{i_f} if and only if s_f is duplicated, and we duplicate t_{i_j} if and only if s_l is duplicated. Observe that S' is the subsequence of T' induced by $\{i_j \in [p'] \mid j \in [\ell']\}$ (see Figure 8). By construction of T', (i_{f_x}, i_{l_x}) is a valid insertion position with respect to X in T'. Thereby, we can define $T^x = \operatorname{Ins}(T', x, X, i_{f_x}, i_{l_x})$ and $S^x = \operatorname{Ins}(S', x, X, f_x, l_x)$. Let P' be the connected path-decomposition obtained from P by duplicating the bags corresponding to t_{ι_f} and t_{ι_l} and adding x to all bags between the bags associated with $t'_{i_{f_x}}$ and $t'_{i_{I_x}}$. We remark that T^x is the $(\mathbf{G}^x, \mathbf{P}')$ -encoding sequence and is thereby realizable.

We claim now that $\operatorname{rep}(S^x) = \operatorname{rep}(T^x)$. Because $S = \operatorname{rep}(T)$, one can prove, in the same way as the second statement of Lemma 5(6), that there are S_1 and S_2 , extensions of S, such that $S_1 \leq T \leq S_2$, $\delta_{S_1 \to S}(i_j) = \delta_{S_2 \to S}(i_j) = j \in [\ell]$, and $i_{f_x} = \min\{h \in [p] \mid f = \delta_{S_1 \to S}(h) = \delta_{S_2 \to S}(h)\}$ and $i_{l_x} = \max\{h \in [p] \mid l = \delta_{S_1 \to S}(h) = \delta_{S_2 \to S}(h)\}$. By making the same duplications in S' as in S to obtain S_1 and S_2 , one can construct extensions S'_1 and S'_2 of S' such that $S'_1 \leq T' \leq S'_2$, $\delta_{S'_1 \to S'}(i_j) =$ $\delta_{S'_2 \to S'}(i_j) = j \in [\ell']$, and $i_{f_x} = \min\{h \in [p'] \mid f_x = \delta_{S'_1 \to S'}(h) = \delta_{S'_2 \to S'}(h)\}$ and $i_{l_x} = \max\{h \in [p'] \mid l_x = \delta_{S'_1 \to S'}(h) = \delta_{S'_2 \to S'}(h)\}$. Therefore, (i_{f_x}, i_{l_x}) is a valid insertion position with respect to X in both S'_1 and S'_2 . By Lemma 19, we have $\operatorname{Ins}(S'_1, x, X, i_{f_x}, i_{l_x}) \leq T^x \leq \operatorname{Ins}(S'_2, x, X, i_{f_x}, i_{l_x})$. Because S'_1 and S'_2 are both extensions of S', $i_{f_x} = \min\{h \in [p'] \mid f_x = \delta_{S'_1 \to S'}(h) = \delta_{S'_2 \to S'}(h)\}$, and $i_{l_x} = \max\{h \in [p'] \mid l_x = \delta_{S'_1 \to S'}(h)\} = \delta_{S'_2 \to S'}(h)\}$, we can conclude by Lemma 17 that $\operatorname{Ins}(S'_1, x, X, i_{f_x}, i_{l_x})$ and $\operatorname{Ins}(S'_2, x, X, i_{f_x}, i_{l_x})$ are both extensions of S^x . We can therefore conclude that $S^x \equiv T^x$, i.e., $\operatorname{rep}(S^x) = \operatorname{rep}(T^x)$. Finally, as T^x is realisable, we can conclude that $\operatorname{rep}(S^x) \in \operatorname{Rep}_w(\mathbf{G}^{\overline{x}})$.

Theorem 3. Algorithm 2 computes $\mathbf{D}_w(\mathbf{G}^x)$ in $2^{O(k(w+\log k))}$ -time, where k = |B|.

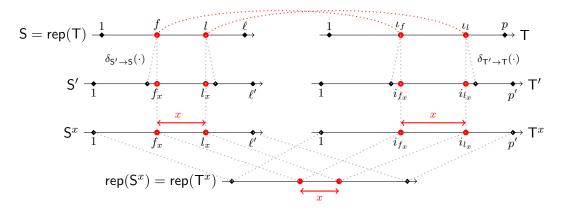


Figure 8: Soundness of the insertion routine: if $S' = \langle s'_1, \ldots, s'_{\ell'} \rangle$ is a (≤ 2)-extension of a *B*-boundaried sequence $S = \operatorname{rep}(T) \in \mathbf{D}_w(\mathbf{G})$ and (f_x, l_x) is a valid insertion position with respect to X in S', then $\operatorname{rep}(S^x) \in \mathbf{D}_w(\mathbf{G}^x)$.

Proof. The correctness of Algorithm 2 is proved by Proposition 4 and Proposition 5. These two propositions imply that by applying Insertion Routine on a domination set of **G** that is a subset of the representatives of **G**, we indeed compute a domination set of \mathbf{G}^x that is a subset of the set of representatives of \mathbf{G}^x . Let us analyse its time complexity. By Lemma 9, the size of $\operatorname{Rep}_w(\mathbf{G})$ (and so the size of $\mathbf{D}_w(\mathbf{G})$) depends on k and w. By Lemma 7, the length of a representative *B*-boundaried sequence of $\operatorname{Rep}_w(\mathbf{G})$ depends on k. As performing the insertion in a *B*-boundaried sequence can be performed in polynomial time in the size of the sequence, the time complexity of Algorithm 2 is dominated by the size of $\mathbf{D}_w(\mathbf{G})$ that is $2^{O(k(w+\log k))}$, because of Lemma 9.

4.4 The dynamic programming algorithm

We are now in position to prove Theorem 1. We first explain an algorithm that decides whether $\mathsf{cpw}(G) \leq w$. Suppose that we are given a path-decompositon $\mathsf{Q} = \langle B_1, \ldots, B_q \rangle$ of G of width at most k. Our algorithm performs dynamic programming over Q . For each $i \in [q]$, we consider the boundaried graph $\mathbf{G}_i = (G[V_i], B_i)$, where $V_i = \bigcup_{1 \leq h \leq i} B_h$. The task is to compute for every $i \in [q]$, a domination set $\mathbf{D}_{w+1}(\mathbf{G}_i)$. Let us describe $\mathbf{D}_{w+1}(\mathbf{G}_1)$. As Q is a nice path-decomposition, $B_1 = \{x\}$ for some $x \in V$. The representative set $\operatorname{\mathbf{Rep}}_{w+1}(\mathbf{G}_1)$ consists for the following four possible connected B_1 -boundaried sequences:

- $S_1 = \langle (\{x\}, \{\{x\}\}, 0) \rangle$,
- $S_2 = \langle (\emptyset, \{\emptyset\}, 0), (\{x\}, \{\{x\}\}, 0) \rangle,$
- $S_3 = \langle (\emptyset, \{\emptyset\}, 0), (\{x\}, \{\{x\}\}, 0), (\emptyset, \{\{x\}\}, 0) \rangle$, and
- $S_4 = \langle (\{x\}, \{\{x\}\}, 0), (\emptyset, \{\{x\}\}, 0) \rangle.$

We use $\operatorname{\mathbf{Rep}}_{w+1}(\mathbf{G}_1)$, as $\mathbf{D}_{w+1}(\mathbf{G}_1)$ as none of the above sequence is dominating the other. Now Algorithm 2 and Algorithm 1 describe how to compute for every $1 < i \leq q$, $\mathbf{D}_{w+1}(\mathbf{G}_i)$ depending on whether B_i is an insertion or a forgetting bag. We obtain that $\mathsf{cpw}(G) \leq w$ if and only if $\mathbf{D}_{w+1}(\mathbf{G}_q) \neq \emptyset$, because of Proposition 1. The correctness of the DP algorithm described above follows from Theorem 2, Theorem 3. The time complexity depends on the running time of Insertion Routine (Algorithm 2) and Forget Routine (Algorithm 1) described respectively in Theorem 2 and Theorem 3. We just proved the decision version of Theorem 1. In [12, Section 6] Bodlaender and Kloks explained how to turn their decision algorithm for pathwidth and treewidth to one that is able to construct, in case of a positive answer, the corresponding decomposition. Following the same arguments, it is straightforward to transform the above decision algorithm for connected pathwidth to one that also constructs the connected path-decomposition, if it exists. This completes the proof of Theorem 1.

Theorem 4. One may construct an algorithm that, given an n-connected graph G and a nonnegative integer k, either outputs a connected path-decomposition of G of width at most k or correctly reports that such a decomposition does not exist in $2^{O(k^2)} \cdot n$ time.

Proof. According to the result of Fürer [24] there is an algorithm that, given a graph G and an integer k, outputs, if exists, a path-decomposition of width at most k in $2^{O(k^2)} \cdot n$ time. We run this algorithm and if the answer is negative, we report that $\mathsf{cpw}(G) > k$ and we are done (here we use Observation 1). Otherwise we use the provided path-decomposition in order to solve the problem in $2^{O(w(k+\log w))} \cdot n$ time using the algorithm of Theorem 1 where $w \leq k$ is the width of the constructed path-decomposition in the first step.

References

- Isolde Adler, Christophe Paul, and Dimitrios M. Thilikos. Connected search for a lazy robber. Journal of Graph Theory, 97:510–552, 2021. doi:10.1002/jgt.22669.
- [2] Spyros Angelopoulos, Pierre Fraigniaud, Fedor V. Fomin, Nicolas Nisse, and Dimitrios M. Thilikos. Report on GRASTA 2017, 6th workshop on graph searching, theory and applications. Technical Report HAL limm-01645614, CNRS, Université Montpellier, LIRMM, 2017. URL: https://hal-limm.ccsd.cnrs.fr/limm-01645614/document.
- [3] Stefan Arnborg, Derek G. Corneil, and Andrzej Proskurowski. Complexity of finding embeddings in a k-tree. SIAM Journal on Algebraic and Discrete Methods, 8(2):277-284, 1987. doi: 10.1137/0608024.
- [4] Lali Barrière, Paola Flocchini, Fedor V. Fomin, Pierre Fraigniaud, Nicolas Nisse, Nicola Santoro, and Dimitrios M. Thilikos. Connected graph searching. *Information and Computation*, 219:1–16, 2012. doi:10.1016/j.ic.2012.08.004.
- [5] Lali Barrière, Paola Flocchini, Pierre Fraigniaud, and Nicola Santoro. Capture of an intruder by mobile agents. In Annual ACM Symposium on Parallel Algorithms and Architectures (SPAA), pages 200–209, 2002. doi:10.1145/564870.564906.

- [6] Lali Barrière, Pierre Fraigniaud, Nicola Santoro, and Dimitrios M. Thilikos. Searching is not jumping. In International Workshop Graph-Theoretic Concepts in Computer Science, (WG), volume 2880 of Lecture Notes in Computer Science, pages 34–45, 2003. doi:10.1007/ 978-3-540-39890-5_4.
- [7] D. Bienstock and Paul D. Seymour. Monotonicity in graph searching. *Journal of Algorithms*, 12(2):239-245, 1991. doi:10.1016/0196-6774(91)90003-H.
- [8] Dan Bienstock, Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a forest. Journal of Combinatorial Theory, Series B, 52(2):274–283, 1991. doi:10.1016/ 0095-8956(91)90068-U.
- [9] Daniel Bienstock. Graph searching, path-width, tree-width and related problems (a survey). In Reliability of computer and communication networks, volume 5 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, pages 33–50, 1991.
- [10] Hans L. Bodlaender, Michael R. Fellows, and Dimitrios M. Thilikos. Derivation of algorithms for cutwidth and related graph layout parameters. *Journal of Computer and System Sciences*, 75(4):231–244, 2009. doi:10.1016/j.jcss.2008.10.003.
- [11] Hans L. Bodlaender, Lars Jaffke, and Jan Arne Telle. Typical sequences revisited computing width parameters of graphs. In 37th International Symposium on Theoretical Aspects of Computer Science, (STACS), volume 154 of Leibniz International Proceedings in Informatics, pages 57:1–57:16, 2020. doi:10.4230/LIPIcs.STACS.2020.57.
- [12] Hans L. Bodlaender and Ton Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. *Journal of Algorithms*, 21(2):358-402, 1996. doi:10.1006/jagm. 1996.0049.
- [13] Hans L. Bodlaender and Dimitrios M. Thilikos. Constructive linear time algorithms for branchwidth. In International Colloquium Automata, Languages and Programming, (ICALP), volume 1256 of Lecture Notes in Computer Science, pages 627–637, 1997. doi:10.1007/ 3-540-63165-8_217.
- [14] Hans L. Bodlaender and Dimitrios M. Thilikos. Computing small search numbers in linear time. In International Workshop on Parameterized and Exact Computation, (IWPEC), volume 3162 of Lecture Notes in Computer Science, pages 37–48, 2004. doi:10.1007/978-3-540-28639-4_4.
- [15] Mikołaj Bojańczyk and Michał Pilipczuk. Optimizing tree decompositions in MSO. In International Symposium on Theoretical Aspects of Computer Science, (STACS), volume 66 of Leibniz International Proceedings in Informatics, pages 15:1–15:13, 2017. doi:10.4230/ LIPIcs.STACS.2017.15.
- [16] R. Breisch. An intuitive approach to speleotopology. Southwestern Cavers (A publication of the Southwestern Region of the National Speleological Society), VI(5):72–78, 1967.

- [17] Gary Chartrand, Ping Zhang, Teresa W. Haynes, Michael A. Henning, Fred R. McMorris, and Robert C. Brigham. *Graphical Measurement*, chapter 9, pages 872–951. Discrete Mathematics and Its Applications. Chapman & Hall / Taylor & Francis, 2003. doi:10.1201/9780203490204.
- [18] Bruno Courcelle. The monadic second-order logic of graphs. I. recognizable sets of finite graphs. Information and Computation, 85(1):12–75, 1990. doi:10.1016/0890-5401(90)90043-H.
- [19] Bruno Courcelle and Jens Lagergren. Equivalent definitions of recognizability for sets of graphs of bounded tree-width. *Mathematical Structures in Computer Science*, 6(2):141–165, 1996. doi:10.1017/S096012950000092X.
- [20] Dariusz Dereniowski. From pathwidth to connected pathwidth. SIAM Journal on Discrete Mathematics, 26(4):1709–1732, 2012. doi:10.1137/110826424.
- [21] Dariusz Dereniowski, Dorota Osula, and Paweł Rzążewski. Finding small-width connected path decompositions in polynomial time. *Theoretical Computer Science*, 794:85–100, 2019. doi:10.1016/j.tcs.2019.03.039.
- [22] Fedor V. Fomin and Dimitrios M. Thilikos. On the monotonicity of games generated by symmetric submodular functions. *Discrete Applied Mathematics*, 131(2):323-335, 2003. doi: 10.1016/S0166-218X(02)00459-6.
- [23] Fedor V. Fomin and Dimitrios M. Thilikos. An annotated bibliography on guaranteed graph searching. *Theoretical Computer Science*, 399(3):236-245, 2008. doi:10.1016/j.tcs.2008.02.040.
- [24] Martin Fürer. Faster computation of path-width. In International Workshop on Combinatorial Algorithms, (IWOCA), volume 9843 of Lecture Notes in Computer Science, pages 385–396, 2016. doi:10.1007/978-3-319-44543-4_30.
- [25] Petr A. Golovach. Equivalence of two formalizations of a search problem on a graph (Russian). Vestnik Leningrad. Univ. Mat. Mekh. Astronom., vyp. 1:10–14, 122, 1989. translation in Vestnik Leningrad Univ. Math. 22 (1989), no. 1, 13–19.
- [26] Jisu Jeong, Eun Jung Kim, and Sang-il Oum. Constructive algorithm for path-width of matroids. In Annual ACM-SIAM Symposium on Discrete Algorithms, (SODA), pages 1695–1704, 2016. doi:10.1137/1.9781611974331.ch116.
- [27] Jisu Jeong, Eun Jung Kim, and Sang-il Oum. The "art of trellis decoding" is fixed-parameter tractable. *IEEE Transactions on Information Theory*, 63(11):7178–7205, 2017. doi:10.1109/ TIT.2017.2740283.
- [28] Jisu Jeong, Eun Jung Kim, and Sang-il Oum. Finding branch-decomposition of matroids, hypergraphs and more. In International Colloquium Automata, Languages and Programming, (ICALP), volume 107 of Leibniz International Proceedings in Informatics, pages 80:1–80:14, 2018. doi:10.4230/LIPIcs.ICALP.2018.80.

- [29] Mamadou Moustapha Kanté, Christophe Paul, and Dimitrios M. Thilikos. A linear fixed parameter tractable algorithm for connected pathwidth. In Annual European Symposium on Algorithms, ESA, volume 173 of Leibniz International Proceedings in Informatics, pages 64:1–64:16, 2020. doi:10.4230/LIPIcs.ESA.2020.64.
- [30] Nancy G. Kinnersley. The vertex separation number of a graph equals its path-width. Information Processing Letters, 42(6):345–350, 1992. doi:10.1016/0020-0190(92)90234-M.
- [31] Lefteris M. Kirousis and Christos H. Papadimitriou. Interval graphs and searching. Discrete Mathematics, 55(2):181–184, 1985. doi:10.1016/0012-365X(85)90046-9.
- [32] Lefteris M. Kirousis and Christos H. Papadimitriou. Searching and pebbling. Theoretical Computer Science, 47(2):205-218, 1986. doi:10.1016/0304-3975(86)90146-5.
- [33] J. Lagergren. Upper bounds on the size of obstructions and intertwines. *Journal of Combinatorial Theory, Series B*, 73:7–40, 1998. doi:10.1006/jctb.1997.1788.
- [34] Jens Lagergren and Stefan Arnborg. Finding minimal forbidden minors using a finite congruence. In International Colloquium on Automata, Languages and Programming, (ICALP), volume 510 of Lecture Notes in Computer Science, pages 532–543, 1991. doi:10.1007/3-540-54233-7\ _161.
- [35] Guillaume Mescoff, Christophe Paul, and Dimitrios M. Thilikos. A polynomial time algorithm to compute the connected treewidth of a series-parallel graph. *Discrete Applied Mathematics*, 2021. doi:10.1016/j.dam.2021.02.039.
- [36] Rolf H. Möhring. Graph problems related to gate matrix layout and PLA folding. In Computational graph theory, volume 7 of Computing Supplementum, pages 17–51. Springer, 1990. doi:10.1007/978-3-7091-9076-0_2.
- [37] Ronan Pardo Soares. Pursuit-Evasion, Decompositions and Convexity on Graphs. PhD thesis, Université Nice Sophia Antipolis, 2013. URL: https://tel.archives-ouvertes.fr/ tel-00908227.
- [38] Torrence D. Parsons. Pursuit-evasion in a graph. In International Conference on the Theory and Applications of Graphs, volume 642 of Lecture Notes in Mathematics, pages 426–441, 1978.
- [39] Torrence D. Parsons. The search number of a connected graph. In Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium, XXI, pages 549–554. Utilitas Mathematica, 1978.
- [40] Nicolai N. Petrov. A problem of pursuit in the absence of information on the pursued. Differentsial'nye Uravneniya, 18(8):1345–1352, 1468, 1982.
- [41] Neil Robertson and P. D. Seymour. Graph Minors. XX. Wagner's conjecture. Journal of Combinatorial Theory, Series B, 92(2):325–357, 2004. doi:10.1016/j.jctb.2004.08.001.

- [42] Neil Robertson and Paul D. Seymour. Graph Minors. I. Excluding a forest. Journal of Combinatorial Theory, Series B, 35(1):39–61, 1983. doi:10.1016/0095-8956(83)90079-5.
- [43] Jan Arne Telle. Tree-decomposition of small pathwidth. Discrete Applied Mathematics, 145(2):210-218, 2005. doi:10.1016/j.dam.2004.01.012.
- [44] Dimitrios M. Thilikos, Maria J. Serna, and Hans L. Bodlaender. Constructive linear time algorithms for small cutwidth and carving-width. In *International Symposium on Algorithms* and computation (ISAAC), volume 1969 of Lecture Notes in Computer Science, pages 192–203, 2000. doi:10.1007/3-540-40996-3_17.
- [45] Dimitrios M. Thilikos, Maria J. Serna, and Hans L. Bodlaender. Cutwidth. I. A linear time fixed parameter algorithm. *Journal of Algorithms*, 56(1):1–24, 2005. doi:10.1016/j.jalgor. 2004.12.001.
- [46] Dimitrios M. Thilikos, Maria J. Serna, and Hans L. Bodlaender. Cutwidth II: algorithms for partial w-trees of bounded degree. *Journal of Algorithms*, 56(1):25–49, 2005. doi:10.1016/j. jalgor.2004.12.003.
- [47] Boting Yang, Danny Dyer, and Brian Alspach. Sweeping graphs with large clique number. Discrete Mathematics, 309(18):5770–5780, 2009. doi:10.1016/j.disc.2008.05.033.