# A linear fixed parameter tractable algorithm for connected pathwidth ${ }^{1}$ 

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#### Abstract

The graph parameter of pathwidth can be seen as a measure of the topological resemblance of a graph to a path. A popular definition of pathwidth is given in terms of node search where we are given a system of tunnels (represented by a graph) that is contaminated by some infectious substance and we are looking for a search strategy that, at each step, either places a searcher on a vertex or removes a searcher from a vertex and where an edge is cleaned when both endpoints are simultaneously occupied by searchers. It was proved that the minimum number of searchers required for a successful cleaning strategy is equal to the pathwidth of the graph plus one. Two desired characteristics for a cleaning strategy is to be monotone (no recontamination occurs) and connected (clean territories always remain connected). Under these two demands, the number of searchers is equivalent to a variant of pathwidth called connected pathwidth. We prove that connected pathwidth is fixed parameter tractable, in particular we design a $2^{O\left(k^{2}\right)} \cdot n$ time algorithm that checks whether the connected pathwidth of $G$ is at most $k$. This resolves an open question by [Dereniowski, Osula, and Rzążewski, Finding small-width connected pathdecompositions in polynomial time. Theor. Comput. Sci., 794:85-100, 2019]. For our algorithm, we enrich the typical sequence technique that is able to deal with the connectivity demand. Typical sequences have been introduced in [Bodlaender and Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. J. Algorithms, 21(2):358-402, 1996] for the design of linear parameterized algorithms for treewidth and pathwidth. While this technique has been later applied to other parameters, none of its advancements was able to deal with the connectivity demand, as it is a "global" demand that concerns an unbounded number of parts of the graph of unbounded size. The proposed extension is based on an encoding of the connectivity property that is quite versatile and may be adapted so to deliver linear parameterized algorithms for the connected variants of other width parameters as well. An immediate consequence of our result is a $2^{O\left(k^{2}\right)} \cdot n$ time algorithm for the monotone and connected version of the edge search number.


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## 1 Introduction

Pathwidth. A path-decomposition of a graph $G=(V, E)$ is a sequence $\mathrm{Q}=\left\langle B_{1}, \ldots, B_{q}\right\rangle$ of vertex sets, called bags of Q, such that

1. $\bigcup_{i \in\{1, \ldots, q\}} B_{i}=V$,
2. every edge $e \in E$ is a subset of some member of Q, and
3. the trace of every vertex $v \in V$, that is the set $\left\{i \mid v \in B_{i}\right\}$, is a set of consecutive integers.

The width of a path-decomposition is $\max \left\{\left|B_{i}\right|-1 \mid i \in\{1, \ldots, q\}\right\}$ and the pathwidth of a graph $G$, denoted by $\mathrm{pw}(G)$, is the minimum width of a path-decomposition of $G$.

The above definition appeared for the first time in [42]. Pathwidth can be seen as a measure of the topological resemblance of a graph to a path. ${ }^{1}$ Pathwidth, along with its tree-analogue treewidth, have been used as key combinatorial tools in the Graph Minors series of Robertson and Seymour [41] and they are omnipresent in both structural and algorithmic graph theory. Apart from the above definition, pathwidth was also defined as the interval thickness [31] (in terms of interval graphs), as the vertex separation number [30] (in terms of graph layouts), as the maximum order of a blockage [8] (in terms of min-max dualities - see also [22]), and as the node search number $[7,9,31,36]$ (in terms of graph searching games).

Deciding whether the pathwidth of a graph is at most $k$ is an NP-complete problem [3]. This motivated the problem of the existence, or not, of a parameterized algorithm for this problem, and algorithm running in $f(k) \cdot n^{O(1)}$ time algorithm. An affirmative answer to this question was directly implied as a consequence of the algorithmic and combinatorial results of the Graph Minors series and the fact that, for every $k$, the class of graphs with pathwidth at most $k$ is closed under taking of minors ${ }^{2}$. On the negative side, this implication was purely existential. The challenge of constructing an $f(k) \cdot n^{O(1)}$ time algorithm for pathwidth (as well as for treewidth) was a consequence of the classic result of Bodlaender and Kloks in [12] (see also [19,34]). The main result in [12] implies a $2^{O\left(k^{3}\right)} \cdot n$ time algorithm. This was later improved to one running in $2^{O\left(k^{2}\right)} \cdot n$ time by Fürer in [24]).

Graph searching. In a graph searching game, the opponents are a group of searchers and an evading fugitive. The opponents move in turns in a graph. The objective of the searchers is to deploy a strategy of moves that leads to the capture of the fugitive. At each step of the node searching game, the searchers may either place a searcher at a vertex or remove a searcher from a vertex. The fugitive resides in the edges of the graph and is lucky, invisible, fast, and agile. The capture of the fugitive occurs when searchers occupy both endpoints of the edge where he currently resides. A node searching strategy is a sequence of moves of the searchers that can guarantee the eventual capture of the fugitive. ${ }^{3}$ The cost of a searching strategy is the maximum number of searchers simultaneously

[^1]present in the graph during the deployment of the strategy. The node search number of a graph $G$, denoted by $n s(G)$, is defined as the minimum cost of a node searching strategy. Node searching was defined by Kirousis and Papadimitriou in [32] who proved that the game is equivalent to its monotone variant where search strategies are monotone in the sense that they prevent the fugitive from pervading again areas from where he had been expelled. This result along with the results in $[30,31,36]$, imply that, for every $\operatorname{graph} G, \mathrm{~ns}(G)=\operatorname{pw}(G)+1$.

The connectivity issue. In several applications of graph searching it is important to guarantee secure communication channels between the searchers so that they can safely exchange information. This issue was treated for the first time in the area of distributed computing, in particular in [5], where the authors considered the problem of capturing an intruder by mobile agents (acting for example as antivirus programs). As agents deploy their cleaning strategy, they must guarantee that, at each moment of the search, the cleaned territories remain connected, so to permit the safe exchange of information between the coordinating agents.

The systematic study of connected graph searching was initiated in $[4,6]$. When, in node searching, we demand that the search strategies are monotone and connected, we define monotone connected node search number, denoted by $\operatorname{mcns}(G) .{ }^{4}$ The graph decomposition counterpart of this parameter was introduced by Dereniowski in [20]. He defined the connected pathwidth of a connected graph $G$, denoted by $\operatorname{cpw}(G)$, by considering connected path-decompositions $\mathrm{Q}=\left\{B_{1}, \ldots, B_{q}\right\}$ where the following additional property is satisfied:

- For every $i \in\{1, \ldots, q\}$, the subgraph of $G$ induced by $\bigcup_{h \in\{1, \ldots, i\}} B_{h}$ is connected.

As noticed in [20], for every connected graph $G$, $\operatorname{mons}(G)=\operatorname{cpw}(G)+1$ (see also [1]). Notice that the above demand results to a break of symmetry: the fact that $\left\langle B_{1}, \ldots, B_{q}\right\rangle$ is a connected path-decomposition does not imply that the same holds for $\left\langle B_{q}, \ldots, B_{1}\right\rangle$ (while this is always the case for conventional path-decompositions). This break of symmetry seems to be the source of all combinatorial particularities (and challenges) of connected pathwidth. This phenomenon was also observed in the context of connected treewidth [1, 35].

Computing connected pathwidth. It is easy to see that checking whether $\operatorname{cpw}(G) \leq k$ is an NP-complete problem: if we define $G^{*}$ as the graph obtained from $G$ after adding a new vertex adjacent with all the vertices of $G$, then observe that $\operatorname{pw}(G)=\operatorname{cpw}\left(G^{*}\right)-1$. This motivates the question on the parameterized complexity of the problem. The first progress in this direction was done recently in [21] by Dereniowski, Osula, and Rzążewski who gave an $f(k) \cdot n^{O\left(k^{2}\right)}$ time algorithm. In [21, Conjecture 1], they conjectured that there is a fixed parameter algorithm checking whether $\operatorname{cpw}(G) \leq k$. The general question on the parameterized complexity of the connected variants of
decontamination strategy [17,23]. The fact that the fugitive is invisible, fast, lucky, and agile permits us to see him as being omnipresent in any edge that has not yet been cleaned.
${ }^{4}$ As proved in [47], under the connectivity demand, the monotone and the non-monotone versions of graph searching are not any more equivalent.


Figure 1: A graph $G$ of connected pathwidth 2 with a subgraph of connected pathwidth 3.
graph search was raised as an open question by Fedor V. Fomin during the GRASTA 2017 workshop (see [2]).

A somehow dissuasive fact towards a parameterized algorithm for connected pathwidth is that connected pathwdith is not closed under minors and therefore it does not fit
in the powerful algorithmic framework of Graph Minors (which is the case with pathwidth). The removal of an edge may increase the parameter. For instance, the connected pathwidth of the graph in Figure 1 has connected pathwidth 2 while if we remove the edge $\{x, y\}$ its connected pathwidth increases to 3 . On the positive side, connected pathwidth is closed under contractions (see e.g., [1]), i.e, its value does not increase when we contract edges and, moreover, the yes-instances of the problem have bounded pathwidth, therefore they also have bounded treewidth. Based on these observations, the existence of a parameterized algorithm would be implied if we can prove that, for any $k$, the set $\mathcal{Z}_{k}$ of contraction-minimal ${ }^{5}$ graphs with connected pathwidth more than $k$ is finite: as contraction containment can be expressed in MSO logic, one should just apply Courcelle's theorem [18] to check whether some graph of $\mathcal{Z}_{k}$ is a contraction of $G$. The hurdle in this direction is that we have no idea whether $\mathcal{Z}_{k}$ is finite or not. The alternative pathway is to try to devise a linear parameterized algorithm by applying the algorithmic techniques that are already known for pathwidth.

The typical sequence technique. The main result of [12] was an algorithm that, given a pathdecomposition Q of $G$ of width at most $k$ and an integer $w$, outputs, if exists, a path-decomposition of $G$ of width at most $w$, in $2^{O(k(w+\log k))} \cdot n$ time. In this algorithm Bodlaender and Kloks introduced the celebrated typical sequence technique, a refined dynamic programming technique that encodes partial path/tree decompositions as a system of suitably compressed sequences of integers, able to encode all possible path-decompositions of width at most $w$. This technique was later extended/adapted for the design of parametrized algorithms for numerous graph parameters such as branchwidth [13], linearwidth [14], cutwidth [45], carving-width [44], modified cutwidth, and others [10,11,46]. Also a similar approach was used by Lagergren in [33] for bounding the sizes of minor obstruction sets. In [10] the typical sequence technique was viewed as a result of un-nondeterminization: a stepwise evolution of a trivial hypothetical non-deterministic algorithm towards a deterministic parameterized algorithm. A considerable generalization of the characteristic sequence technique was proposed in the PhD thesis of Soares [37] where this technique was implemented under the powerful meta-algorithmic framework of $q$-branched $\Phi$-width. Non-trivial extensions of the typical sequence technique where proposed

[^2]for devising parameterized algorithms for parameters on matroids such as matroid pathwidth [26], matroid branchwidth [28], as well as all the parameters on graphs or hypergraphs that can be expressed by them. Very recently Bodlaender, Jaffke, and Telle in [11] suggested refinements of the typical sequence technique that enabled the polynomial time computation of several width parameters on directed graphs. Finally, Bojańczyk and Pilipczuk suggested an alternative approach to the typical sequence technique, based on MSO transductions between decompositions [15].

Unfortunately, the above mentioned state of the art on the typical sequence technique is unable to encompass connected pathwidth. A reason for this is that the connectivity demand is a "global property" applying to every prefix of the path-decomposition, which corresponds to an unbounded number of subgraphs of arbitrary size.

Our result. In this paper we resolve affirmatively the conjecture that checking whether $\mathrm{cpw}(G) \leq k$ is fixed parameter tractable. Our main result is the following.

Theorem 1. One may construct an algorithm that given an n-vertex connected graph $G$, a pathdecomposition $\mathrm{Q}=\left\langle B_{1}, \ldots, B_{q}\right\rangle$ of $G$ of width at most $k$ and an integer $w$, outputs a connected path-decomposition of $G$ of width at most $w$ or reports correctly that such an algorithm does not exist in $2^{O(k(w+\log k))} \cdot n$ time.

To design an algorithm checking whether $\operatorname{cpw}(G) \leq k$ we first use the algorithms of [12] and [24], to build, if exists, a path decomposition of $G$ of width at most $k$, in $2^{O\left(k^{2}\right)} \cdot n$ time. In case of a negative answer we know than $\operatorname{cpw}(G)>k$, otherwise we apply the algorithm of Theorem 1. The overall running time is dominated by the algorithm of Fürer in [24] which is $2^{O\left(k^{2}\right)} \cdot n$.

Our techniques. We now give a brief description of our techniques by focusing on the novel issues that we introduce. This description demands some familiarity with the typical sequence technique. Otherwise, the reader can go directly to the next section.

Let $\mathrm{Q}=\left\langle B_{1}, \ldots, B_{q}\right\rangle$ be a (nice) path-decomposition of $G$ of width at most $k$. For every $i \in[q]$, we let $\mathbf{G}_{i}=\left(G_{i}, B_{i}\right)$ be the boundaried graph where $G_{i}=G\left[\bigcup_{h \in\{1, \ldots, i\}} B_{h}\right]$. We follow standard dynamic programming over a path-decomposition that consists in computing a representation of the set of partial solutions associated to $\mathbf{G}_{i}$, which in our case are connected path-decompositions of $\mathbf{G}_{i}$ of width at most $w$. The challenge is how to handle in a compact way the connectivity requirement of a path-decomposition of a graph that can be of arbitrarily large size.

A connected path-decomposition $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ of $\mathbf{G}_{i}$ is represented by means of a $\left(\mathbf{G}_{i}, \mathrm{P}\right)$ encoding sequence $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$. For every $j \in[\ell]$, the element $\mathrm{s}_{j}$ of the sequence S is a triple $\left(\mathbf{b d}\left(\mathbf{s}_{j}\right), \mathbf{c c}\left(\mathbf{s}_{j}\right), \operatorname{val}\left(\mathrm{s}_{j}\right)\right)$ where: $\mathbf{b d}\left(\mathrm{s}_{i}\right)=A_{j} \cap B_{i} ; \operatorname{val}\left(\mathrm{s}_{j}\right)=\left|A_{j} \backslash B_{i}\right|$; and $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ is the projection of the connected components of $G_{i}^{j}=G_{i}\left[\bigcup_{h \in\{1, \ldots, j\}} A_{h}\right]$ onto the subset of boundary vertices $B_{i} \cap V\left(G_{i}^{j}\right)$. To compress a $\left(\mathbf{G}_{i}, \mathrm{P}\right)$-encoding sequence S , we identify a subset $\mathbf{b p}(\mathrm{S})$ of indexes, called breakpoints, such that $j \in \mathbf{b p}(\mathrm{~S})$ if $\mathbf{b d}\left(\mathrm{s}_{j-1}\right) \neq \mathbf{b d}\left(\mathbf{s}_{j}\right)$ (type-1) or $\mathbf{c c}\left(\mathbf{s}_{j-1}\right) \neq \mathbf{c c}\left(\mathbf{s}_{j}\right)$ (type-2) or $j$ is an index belonging to a typical sequence of the integer sequence $\left\langle\operatorname{val}\left(\mathrm{s}_{b}\right), \ldots, \operatorname{val}\left(\mathrm{s}_{c-1}\right)\right\rangle$ where
$b, c \in[\ell]$ are consecutive type-1 or 2 - breakpoints. We define rep(S) as the induced subsequence $\mathrm{S}[\mathbf{b p}(\mathrm{S})]$.

The novelty in this representation is the $\mathbf{c c}(\cdot)$ component which is a near-partition of the subset $B_{i} \cap V\left(G_{i}^{j}\right)$ of boundary vertices. The critical observation is that for every $j \in[\ell-1]$, $\mathbf{c c}\left(\mathrm{s}_{j+1}\right)$ is coarser than $\mathbf{c c}\left(\mathrm{s}_{j}\right)$. This, together with the known results on typical sequences, allows us to prove that the size of $\operatorname{rep}(\mathrm{S})$ is $O(k w)$ and that the number of representative sequences is $2^{O(k(w+\log k))}$. Finally, as in the typical sequence technique, we define a domination relation over the set of representative sequences. The DP algorithm over the path-decomposition Q consists then in computing a domination set $\mathbf{D}_{w}\left(G_{i+1}\right)$ of the representative sequences of $\mathbf{G}_{i+1}$ from a domination set $\mathbf{D}_{w}\left(G_{i}\right)$ of the representative sequences of $\mathbf{G}_{i}$.

The above scheme extends the current state of the art on typical sequences as it further incorporates the encoding of the connectivity property. While this is indeed a "global property", it appears that its evolution with respect to the bags of the decomposition can be controlled by the second component of our encoding and this is done in terms of a sequence of a gradually coarsening partitions. This establishes a dynamic programming framework that can potentially be applied on the connected versions of most of the parameters where the typical sequence technique was used so far. Moreover, it may be the starting point of the algorithmic study of parameters where other, alternative to connectivity, global properties are imposed to the corresponding decompositions.

Consequences in connected graph searching. The original version of graph searching was the edge searching variant, defined ${ }^{6}$ by Parsons [38,39], where the only differences with node searching is that a searcher can additionally slide along an edge and sliding is the only way to clean an edge. The corresponding search number is called edge search number and is denoted by es $(G)$. If we additionally demand that the searching strategy is connected and monotone, then we define the monotone connected edge search number denoted by mces $(G)$. As proved in [32], es $(G)=\mathrm{pw}\left(G_{\mathrm{v}}\right)$, where $G_{\mathrm{v}}$ is the graph obtained if we subdivide twice each edge of $G$. Applying the same reduction as in [32] for the monotone and connected setting, one can prove that $\operatorname{mces}(G)=\operatorname{cpw}\left(G_{\mathrm{v}}\right)$. As we already mentioned, $\operatorname{mcns}(G)=\operatorname{cpw}\left(G_{\mathrm{v}}\right)+1$. These two reductions imply that the result of Theorem 1 holds also for mcns and mces, i.e., the search numbers for the monotone and connected versions of both node and edge searching.

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## 2 Preliminaries and definitions

Sets and near-partitions. For an integer $\ell>0$, we denote by $[\ell]$ the set $\{1, \ldots, \ell\}$. Let $S$ be a finite set. A near-partition Q of $S$ is a family of subsets $\left\{X_{1}, \ldots, X_{k}\right\}$ (with $k \leq|S|+1$ ) of subsets of $S$, called blocks, such that $\bigcup_{i \in[k]} X_{i}=S$ and for every $1 \leq i<j \leq k, X_{i} \cap X_{j}=\emptyset$. Observe that a near-partition may contain several copies of the empty set. A partition of $S$ is a near-partition with the additional constraint that if it contains the empty set, then this is the unique block. Let Q be a near-partition of a set $S$ and $\mathrm{Q}^{\prime}$ be a near-partition of a set $S^{\prime}$ such that $S \subseteq S^{\prime}$. We say that Q is thinner than $\mathrm{Q}^{\prime}$, or that $\mathrm{Q}^{\prime}$ is coarser than Q , which we denote by $\mathrm{Q} \sqsubseteq \mathrm{Q}^{\prime}$, if for every block $X$ of Q , there exists a block $X^{\prime}$ of $\mathrm{Q}^{\prime}$ such that $X \subseteq X^{\prime}$. For a near-partition $\mathrm{Q}=\left\{X_{1}, \ldots, X_{\ell}\right\}$ of $S$ and a subset $S^{\prime} \subseteq S$, we define the projection of Q onto $S^{\prime}$ as the near-partition $\mathrm{Q}_{\mid S^{\prime}}=\left\{X_{1} \cap S^{\prime}, \ldots, X_{\ell} \cap S^{\prime}\right\}$. Observe that if Q is a partition, then $\mathrm{Q}_{\mid S^{\prime}}$ may not be a partition: if several blocks of Q are subsets of $S \backslash S^{\prime}$, then $\mathrm{Q}_{\mid S^{\prime}}$ contains several copies of the emptyset.

Sequences. Let $S$ be a set. A sequence of elements of $S$, denoted by $\alpha=\left\langle a_{1}, \ldots, a_{\ell}\right\rangle$, is a subset of $S$ equipped with a total ordering: for $1 \leqslant i<j \leqslant \ell, a_{i}$ occurs before $a_{j}$ in the sequence $\alpha$. The length of a sequence is the number of elements that it contains. Let $X \subseteq[\ell]$ be a subset of indexes of $\alpha$. We define the subsequence of $\alpha$ induced by $X$ as the sequence $\alpha[X]$ on the subset $\left\{a_{i} \mid i \in X\right\}$ such that, for $i, j \in X, a_{i}$ occurs before $a_{j}$ in $\alpha[X]$ if and only if $i<j$. If $\alpha=\left\langle a_{1}, \ldots, a_{\ell}\right\rangle$ and $\beta=\left\langle b_{1}, \ldots, b_{p}\right\rangle$ are two sequences, we let $\alpha \circ \beta$ denote the concatenation of $\alpha$ and $\beta$, i.e., $\alpha \circ \beta$ is the sequence $\left\langle a_{1}, \ldots, a_{\ell}, b_{1}, \ldots, b_{p}\right\rangle$.

The duplication of the element $a_{j}$, with $j \in[\ell]$, in the sequence $\alpha=\left\langle a, \ldots, a_{\ell}\right\rangle$ yields the sequence $\alpha^{\prime}=\left\langle a_{1}, \ldots, a_{j-1}, a_{j}, a_{j}, a_{j+1}, \ldots, a_{\ell}\right\rangle$ of length $\ell+1$. A sequence $\beta$ is an extension of the sequence $\alpha$ if it is either $\alpha$ or it results from a series of duplications on $\alpha$. We define the set of extensions of $\alpha$ as: $\operatorname{Ext}(\alpha)=\left\{\alpha^{*} \mid \alpha^{*}\right.$ is an extension of $\left.\alpha\right\}$.

Let $\alpha=\left\langle a_{1}, \ldots, a_{\ell}\right\rangle$ be a sequence and $\alpha^{*}=\left\langle a_{1}, \ldots, a_{p}\right\rangle$ be an extension of $\alpha$. If $p \leq \ell+k$, then $\alpha^{*}$ results from a series of at most $k$ duplications and we say that $\alpha^{*}$ is a $(\leq k)$-extension of $\alpha$. With the definition of an extension, every element of $\alpha^{*}$ is a copy of some element of $\alpha$. We define the extension surjection as a surjective function $\delta_{\alpha^{*} \rightarrow \alpha}:[p] \rightarrow[\ell]$ such that if $\delta_{\alpha^{*} \rightarrow \alpha}(j)=i$, then $a_{j}^{*}=a_{i}$. An extension surjection $\delta_{\alpha^{*} \rightarrow \alpha}$ is a certificate that $\alpha^{*} \in \operatorname{Ext}(\alpha)$. Finally, we observe that if $\alpha^{*} \in \operatorname{Ext}(\alpha)$, then $\alpha$ is an induced subsequence of $\alpha^{*}$. Moreover, if $\alpha^{*} \in \operatorname{Ext}(\alpha)$ and $\beta \in \operatorname{Ext}\left(\alpha^{*}\right)$, then $\beta$ is an extension of $\alpha$.

Graphs and boundaried graphs. Given a graph $G=(V, E)$ and a vertex set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ that is induced by the vertices of $S$, i.e., the graph ( $S,\{e \in E \mid$ $e \subseteq S\}$ ). Also, if $x \in V$, we define $G \backslash x=G[V \backslash\{x\}]$. The neighborhood of a vertex $v$ in $G$ is the set of vertices that are adjacent to $v$ in $G$ and is denoted by $N_{G}(v)$.

A boundaried graph is a pair $\mathbf{G}=(G, B)$ such that $G$ is a graph over a vertex set $V$ and $B \subseteq V$ is a subset of distinguished vertices, called boundary vertices. The vertices of $V \backslash B$ are called inactive vertices. We say that a boundaried graph $\mathbf{G}=(G, B)$ is connected if either $G$ is connected


Figure 2: The sequence $\beta=\left\langle b_{1}, \ldots, b_{19}\right\rangle$ is an ( $\leq 8$ )-extension of the sequence $\alpha=\left\langle a_{1}, \ldots, a_{11}\right\rangle$. The element $a_{3}$ has been duplicated twice in $\beta$ yielding three copies $b_{5}, b_{6}$, and $b_{7}$, which are certified by $\delta_{\beta \rightarrow \alpha}(5)=\delta_{\beta \rightarrow \alpha}(6)=\delta_{\beta \rightarrow \alpha}(7)=3$.
and $B=\emptyset$ or, in case $B \neq \emptyset$, every connected component $C$ of $G$ contains some boundary vertex, that is $C \cap B \neq \emptyset$.

### 2.1 Connected pathwidth.

A path-decomposition of a graph $G=(V, E)$ is a sequence $\mathrm{P}=\left\langle A_{1}, \ldots, A_{p}\right\rangle$ of subsets of $V$ where:

1. for every vertex $x \in V$, there exists $i \in[p]$ such that $x \in A_{i}$;
2. for every edge $e \in E$, there exists $i \in[p]$ such that $e \subseteq A_{i}$;
3. for every vertex $x \in V$, the set $\mathcal{A}(x)=\left\{i \in P \mid x \in A_{i}\right\}$ is a subset of consecutive integers.

Hereafter, the subsets $A_{i}$ 's (for $i \in[p]$ ) are called the bags of the path-decomposition P and the set $\mathcal{A}(x)$ is the trace of $x$ in P . The width of a path-decomposition is width $(\mathrm{P})=\max \left\{\left|A_{i}\right|-1 \mid i \in[p]\right\}$. The pathwidth of a graph $G$, denoted by $\operatorname{pw}(G)$, is the least width of a path-decomposition of $G$. Finally, for every $i \in[p]$, we define $V_{i}=\bigcup_{j \leq i} A_{j}$ and $G_{i}=G\left[V_{i}\right]$.

A path-decomposition $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ of a graph $G$ is nice if $\left|A_{1}\right|=1$ and for every $1<i \leq p$, the symmetric difference $A_{i-1} \triangle A_{i}$ has size one. We distinguish two types of bags:

- if $A_{i-1} \subset A_{i}(1<i \leq p)$, then $A_{i}$ is an introduce bag ( $A_{1}$ is also defined as an introduce bag);
- if $A_{i} \subset A_{i-1}(1<i \leq p)$, then $A_{i}$ is a forget bag.

It is well-known that any path-decomposition can be turned in linear time into a nice pathdecomposition of same width (see e.g., [12]).

Definition 1 (Connected path-decomposition). A path-decomposition $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ of a connected graph $G$ is connected if, for every $i \in[\ell]$, the subgraph $G_{i}$ is connected. The connected pathwidth, denoted by $\mathrm{cpw}(G)$, is the smallest width of a connected path-decomposition of $G$.

Let us notice that if $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ is a path-decomposition of a graph $G$, then $\mathrm{P}^{\prime}=$ $\left\langle A_{p}, \ldots, A_{1}\right\rangle$ is also a path-decomposition of $G$. But the fact that P is a connected path-decomposition does not imply that $\mathrm{P}^{\prime}$ is a connected path-decomposition.

Observation 1. For every graph $G, \mathrm{pw}(G) \leqslant \mathrm{cpw}(G)$.
Let P be a path-decomposition of a graph $G=(V, E)$. Then for every subset $B \subseteq V, \mathrm{P}$ is a path-decomposition of the connected boundaried graph $\mathbf{G}=(G, B)$. The definition of a connected path-decomposition also naturally extends to boundaried graphs as follows.

Definition 2 (Connected path-decomposition of a boundaried graph). Let $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ be a path-decomposition of the boundaried graph $\mathbf{G}=(G, B)$. Then P is connected if, for every $i \in[p]$, the boundaried graph $G_{i}=\left(G_{i}, V_{i} \cap B\right)$ is connected.

Let $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ be a path-decomposition of $\mathbf{G}=(G, B)$. If $x$ is a vertex of $G$, then $\left\langle A_{1} \backslash\{x\}, \ldots, A_{\ell} \backslash\{x\}\right\rangle$, is a path-decomposition of ( $G \backslash x, B \backslash\{x\}$ ). Notice that we may have a bag $A_{i}$ of $P$ such that $A_{i} \backslash\{x\}=\emptyset$, but this does not contradict the definition of path-decomposition. However, the fact that P is a connected path-decomposition does not imply that $\left\langle A_{1} \backslash\{x\}, \ldots, A_{\ell} \backslash\{x\}\right\rangle$ is. The following lemma establishes a condition for the vertex $x$ to satisfy so that its removal preserves connectivity.

Lemma 1. Let $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ be a connected path-decomposition of the connected boundaried graph $(G, B)$. If $x$ is a vertex of $B$ such that $N_{G}(x) \subseteq B$, then $\left\langle A_{1} \backslash\{x\}, \ldots, A_{\ell} \backslash\{x\}\right\rangle$ is a connected path-decomposition of ( $G \backslash x, B \backslash\{x\}$ ).

Proof. As already observed, $\mathrm{P}^{\bar{x}}=\left\langle A_{1} \backslash\{x\}, \ldots, A_{\ell} \backslash\{x\}\right\rangle$ is a path-decomposition of $G \backslash x$. Suppose that $[f, l]$ with $1 \leq f \leq l \leq \ell$ is the trace of $x$ in P . As for every integer $i<f$ (supposing that $1<l)$, the boundaried graph $\left(G_{i} \backslash x,\left(V_{i} \cap B\right) \backslash\{x\}\right)$ is equal to $\left(G_{i}, V_{i} \cap B\right)$ and is thereby connected. So, let us consider an integer $i$ such that $f \leq i$. Let $C_{x}$ be the connected component of $G_{i}$ that contains $x$. As $\left(G_{i}, V_{i} \cap B\right)$ is connected, every connected component of $G_{i}$ intersects $B$. Observe that every connected component $C$ of $G_{i}$ distinct from $C_{x}$ (if any) is a connected component of $G\left[V_{i} \backslash\{x\}\right]$ which intersects $B \backslash\{x\}$. If $C_{x}=\{x\}$, by the previous observations, the statement holds. So, let $C_{1}, \ldots, C_{s}$, with $s \geq 1$, be the connected components of $G\left[V_{i} \backslash\{x\}\right]$ such that for every $j \in[s], C_{j} \subsetneq C_{x}$. As $C_{x} \neq\{x\}$, for every $j \in[s], C_{j}$ contains a neighbor of $x$ which by assumption belongs to $B \backslash\{x\}$. It follows that every connected component of $G_{i} \backslash x$ contains a vertex of $B \backslash\{x\}$. Thereby $\left(G_{i} \backslash x,\left(V_{i} \cap B\right) \backslash\{x\}\right)$ is a connected boundaried graph implying that $\mathrm{P}^{\bar{x}}$ is a connected path-decomposition of ( $G \backslash x, B \backslash\{x\}$ ).

### 2.2 Integer sequences

Let us recall the notion of typical sequences introduced by Bodlaender and Kloks [12] (see also [19, 34]).
Definition 3. Let $\alpha=\left\langle a_{1}, \ldots, a_{\ell}\right\rangle$ be an integer sequence. The typical sequence $\operatorname{Tseq}(\alpha)$ is obtained after iterating the following operations, until none is possible anymore:

- if for some $i \in[\ell-1], a_{i}=a_{i+1}$, then remove $a_{i+1}$ from $\alpha$;
- if there exists $i, j \in[\ell]$ such that $i \leqslant j-2$ and $\forall h, i<h<j, a_{i} \leq a_{h} \leq a_{j}$ or $\forall h, i<h<j$, $a_{i} \geq a_{h} \geq a_{j}$, then remove the subsequence $\left\langle a_{i+1}, \ldots, a_{j-1}\right\rangle$ from $\alpha$.

As a typical sequence $\operatorname{Tseq}(\alpha)=\left\langle b_{1}, \ldots, b_{i}, \ldots, b_{r}\right\rangle$ is a subsequence of $\alpha$, it follows that, for every $i \in[r]$, there exists $j_{i} \in[\ell]$ such that $b_{i}=a_{j_{i}}$. Herefater every such index $j_{i}$ is called a tip of the sequence $\alpha$.

Lemma 2 ( [12]). Let $\alpha=\left\langle a_{1}, \ldots, a_{\ell}\right\rangle$ be an integer sequence. Then, $\operatorname{Tseq}(\alpha)$ is uniquely defined. If, moreover, for every $i \in[\ell]$, we have $a_{i} \in\{0,1, \ldots, k\}$, then the length of $\operatorname{Tseq}(\alpha)$ is at most $2 k+1$.


Figure 3: The black bullets forms the typical sequence $\operatorname{Tseq}(\alpha)=\langle 4,7,3,9,1,8,3,6\rangle$ of the sequence $\alpha=\langle 4,6,5,7,3,5,7,9,4,6,3,1,4,7,8,5,6,3,4,4,5,6\rangle$ represented by black bullets and gray diamonds.

Lemma 3 ( [12]). The number of different typical sequences of integers in $\{0,1, \ldots, k\}$ is at most $\frac{8}{3} \cdot 2^{2 k}$.

A consequence of the next lemma is that every tip of the sequence $\alpha \circ \beta$ is a tip of $\alpha$ or of $\beta$.
Lemma 4 ([12]). Let $\alpha$ and $\beta$ be two integer sequences. Then, $\operatorname{Tseq}(\alpha \circ \beta)=\operatorname{Tseq}(\operatorname{Tseq}(\alpha) \circ \operatorname{Tseq}(\beta))$.
If $\alpha$ and $\beta$ are two integer sequences of same length $\ell$, we say that $\alpha \leq \beta$ if for every $j \in[\ell]$, $a_{j} \leq b_{j}$.

Definition 4. Let $\alpha$ and $\beta$ be two integer sequences. Then $\alpha \preceq \beta$ if there are $\alpha^{*} \in \operatorname{Ext}(\alpha)$ and $\beta^{*} \in \operatorname{Ext}(\beta)$ such that $\alpha^{*} \leq \beta^{*}$. Whenever $\alpha \preceq \beta$ and $\beta \preceq \alpha$, we say that $\alpha$ and $\beta$ are equivalent which is denoted by $\alpha \equiv \beta$.

We summarize in the following a set of known properties concerning duplications of integer sequences and the binary relation $\preceq$ we will need.

Lemma 5 ( [12]). Let $\alpha$ and $\beta$ be two integer sequences.

1. If $\alpha$ has length at most $\ell$, then $\operatorname{Ext}(\alpha)$ contains at most $2^{\ell-1}$ sequences of length $\ell$.
2. If $\alpha^{*} \in \operatorname{Ext}(\alpha)$, then $\alpha \equiv \alpha^{*}$.
3. If $\alpha^{*} \in \operatorname{Ext}(\alpha)$ and $\beta^{*} \in \operatorname{Ext}(\beta)$, then $\alpha^{*} \circ \beta^{*} \in \operatorname{Ext}(\alpha \circ \beta)$.
4. If $\alpha^{\prime} \preceq \alpha$ and $\beta^{\prime} \preceq \beta$, then $\alpha^{\prime} \circ \beta^{\prime} \preceq \alpha \circ \beta$.
5. The relation $\preceq$ is transitive, and $\equiv$ is an equivalence relation.
6. For every integer sequence $\alpha$, we have $\operatorname{Tseq}(\alpha) \equiv \alpha$. Moreover, there exist extensions $\alpha^{\prime}$ and $\alpha^{\prime \prime}$ of $\operatorname{Tseq}(\alpha)$ such that $\alpha^{\prime} \leq \alpha \leq \alpha^{\prime \prime}$.
7. $\alpha \preceq \beta$ if and only if $\operatorname{Tseq}(\alpha) \preceq \operatorname{Tseq}(\beta)$.

We extend the definition of the $\leq$-relation and $\preceq$-relation on integer sequences to sequences of integer sequences. Let $\mathrm{P}=\left\langle\mathrm{L}_{1}, \ldots, L_{r}\right\rangle$ and $\mathrm{Q}=\left\langle\mathrm{K}_{1}, \ldots, \mathrm{~K}_{r}\right\rangle$ be two sequences of integer sequences such that for every $i \in[r], \mathrm{L}_{i}$ and $\mathrm{K}_{i}$ have the same length. We say that $\mathrm{P} \leq \mathrm{Q}$ if for every $i \in[r]$, $\mathrm{L}_{i} \leq \mathrm{K}_{i}$. The set of extensions of P is $\operatorname{Ext}(\mathrm{P})=\left\{\left\langle\mathrm{L}_{1}^{\prime}, \ldots, \mathrm{L}_{r}^{\prime}\right\rangle \mid i \in[r], \mathrm{L}_{i}^{\prime} \in \operatorname{Ext}\left(\mathrm{L}_{i}\right)\right\}$. Finally we say that $P \preceq Q$ if there exist $P^{\prime} \in \operatorname{Ext}(P)$ and $Q^{\prime} \in \operatorname{Ext}(Q)$ such that $P^{\prime} \leq Q^{\prime}$. If $P \preceq Q$ and $Q \preceq P$, then we say that $\mathrm{P} \equiv \mathrm{Q}$. The relation $\equiv$ is an equivalence relation.

## 3 Boundaried sequences

Definition 5 ( $B$-boundaried sequence). Let $B$ be a finite set. $A B$-boundaried sequence is a sequence $\mathbf{S}=\left\langle\mathbf{s}_{1}, \ldots, \mathbf{s}_{\ell}\right\rangle$ such that for every $j \in[\ell], \mathbf{s}_{j}=\left(\mathbf{b d}\left(\mathbf{s}_{j}\right), \mathbf{c c}\left(\mathbf{s}_{j}\right), \operatorname{val}\left(\mathbf{s}_{j}\right)\right)$ is defined as follows:

- $\mathbf{b d}\left(\mathbf{s}_{j}\right) \subseteq B$ with the property that for every $x \in B$, the indices $j \in[l]$ such that $x \in \mathbf{b d}\left(\mathbf{s}_{j}\right)$ are consecutive;
- $\mathbf{c c}\left(\mathbf{s}_{j}\right)$ is a near-partition of $\bigcup_{i \leq j} \mathbf{b d}\left(\mathbf{s}_{i}\right) \subseteq B$ with the property that for every $j<\ell, \mathbf{c c}\left(\mathbf{s}_{j}\right) \sqsubseteq$ $\mathbf{c c}\left(\mathrm{s}_{j+1}\right)$;
- $\operatorname{val}\left(\mathrm{s}_{j}\right)$ is a positive integer.

The width of S is defined as width $(\mathrm{S})=\max _{j \in \ell}\left(\left|\mathbf{b d}\left(\mathrm{~s}_{j}\right)\right|+\operatorname{val}\left(\mathrm{s}_{j}\right)\right)$.
Definition 6 (Connected $B$-boundaried sequence). Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence for some finite set $B$. We say that S is connected if for every $i \in[\ell], \mathbf{c c}\left(\mathbf{s}_{i}\right)$ is a partition of $\bigcup_{i \leq j} \mathbf{b d}\left(s_{i}\right) \subseteq B$.

Observe that if $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ is a connected $B$-boundaried sequence and if there exists some $i \in[\ell]$ such that $\mathbf{c c}\left(\mathbf{s}_{i}\right)=\{\emptyset\}$, then, for every $j \leq i, \mathbf{b d}\left(\mathbf{s}_{j}\right)=\emptyset$ and $\mathbf{c c}\left(\mathbf{s}_{j}\right)=\{\emptyset\}$.

As we will see in subsection 4.1, the $B$-boundaried sequences will allow us to encode partial connected path-decompositions. Intuitively, if $P=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ is a path-decomposition, a triple $\mathrm{s}_{j}$


Figure 4: The part $\left\langle\mathrm{s}_{i-1}, \ldots, \mathrm{~s}_{k}\right\rangle$ of a $B$-boundaried sequence S where the boundary set $B$ contains among others the vertices $x, y$ and $z$. A bullet • at some index $j$ represents an element of $\bigcup_{h<j} \mathbf{b d}\left(s_{h}\right)$. Observe that at index $k, x$ is indeed represented by a black bullet. For the index $i$, we have $\mathbf{b d}\left(\mathbf{s}_{i}\right)=\{x, y, z\}, \mathbf{c c}\left(\mathbf{s}_{i}\right)=\{\{x, y\},\{z, \bullet\}\}$ and $\operatorname{val}\left(\mathbf{s}_{i}\right)=2$. At every position $j$, only named elements belong to $\mathbf{b d}\left(\mathbf{s}_{j}\right)$. The red squares mark the type- 1 breakpoints: at position $i$, element $z$ is new, while at position $k$, element $x$ is forgotten. The blue diamond at index $j$ marks a type-2 breakpoint which corresponds to the merge of two parts of $\mathbf{c c}\left(\mathbf{s}_{i+4}\right)$ into a single part. Finally, the grey bullets mark type-3 breakpoints corresponding to tips of the integer sequences $\left\langle\operatorname{val}\left(\mathrm{s}_{i}\right), \ldots, \operatorname{val}\left(\mathrm{s}_{j-1}\right)\right\rangle$ and $\left\langle\operatorname{val}\left(\mathrm{s}_{j}\right), \ldots, \operatorname{val}\left(\mathrm{s}_{k-1}\right)\right\rangle$.
will represent the informations about bag $A_{j}: \mathbf{b d}\left(\mathbf{s}_{j}\right)$ contains the active vertices of the boundary set $B$; $\mathbf{v a l}\left(\mathrm{s}_{j}\right)$ the number of boundary vertices that appear in prior bags $A_{i}(i<j)$ but not in $A_{j}$; and $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ encodes how the connected components of the graph induced by $\cup_{i \leq j} A_{i}$ project on $B$.

### 3.1 Breakpoints, representatives and domination relation

Definition 7 (Breakpoints). Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence for some finite set $B$. Then the index $j$, with $1 \leq j \leq \ell$, is a breakpoint of:

- type-1 if $j=1$ or $\mathbf{b d}\left(\mathbf{s}_{j}\right) \neq \mathbf{b d}\left(\mathbf{s}_{j-1}\right)$ or $j=\ell$;
- type-2 if it is not a type-1 breakpoint and $\mathbf{c c}\left(\mathrm{s}_{j}\right) \neq \mathbf{c c}\left(\mathrm{s}_{j-1}\right)$;
- type-3 if it is not a type-1 nor a type-2 and $j$ is a tip of the integer sequence $\left\langle\operatorname{val}\left(s_{l_{j}}\right), \ldots\right.$, $\left.\operatorname{val}\left(\mathbf{s}_{r_{j}-1}\right)\right\rangle$ where $l_{j}$ and $r_{j}$ are respectively the largest and smallest type-1 or type-2 breakpoints such that $l_{j}<j<r_{j}$.

We denote by $\mathbf{b p}(\mathrm{S})$ the set of breakpoints of S and by $\mathbf{b p}_{t}(\mathrm{~S})$ the set of type-t breakpoints of S , for $t \in\{1,2,3\}$. We define the representative sequence $\operatorname{rep}(S)$ of $S$ as the induced subsequence of $\mathrm{S}[\mathbf{b} \mathbf{p}(\mathrm{S})]$.

Figure 4 illustrates the notions of $B$-boundaried sequence and breakpoints. Observe that rep(S) can be computed from the $B$-boundaried sequence S by an algorithm similar to the one described in Definition 3 and as in Lemma $2 \operatorname{rep}(S)$ is uniquely defined. Notice that, as an induced subsequence of S , $\operatorname{rep}(\mathrm{S})$ is a $B$-boundaried sequence. Let $\ell$ be the length of S . It is worth to remark that if
$1<j \leq \ell$ belongs to $\mathbf{b p}_{1}(\mathrm{~S}) \cup \mathbf{b p}_{2}(\mathrm{~S})$, then $j-1$ is also a breakpoint. This is the case because the last index of an integer sequence is by definition a tip.

We define the set of representative $B$-boundaried sequences of width at most $w$ as

$$
\operatorname{Rep}_{w}(B)=\{\operatorname{rep}(\mathrm{S}) \mid \mathrm{S} \text { is a } B \text {-boundaried sequence of width } \leq w\} .
$$

Definition 8 ( $B$-boundary model). Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence. For every $j \in[\ell]$, we set $\dot{\mathbf{s}}_{j}=\left(\mathbf{b d}\left(\mathbf{s}_{j}\right), \mathbf{c c}\left(\mathbf{s}_{j}\right), \mathbf{t}\left(\mathbf{s}_{j}\right)\right)$ with $\mathbf{t}\left(\mathbf{s}_{j}\right)=1$ if $j \in \mathbf{b} \mathbf{p}_{1}(\mathrm{~S}), \mathbf{t}\left(\mathbf{s}_{j}\right)=2$ if $j \in \mathbf{b} \mathbf{p}_{2}(\mathrm{~S})$ and $\mathbf{t}\left(\mathrm{s}_{j}\right)=0$ otherwise. The B-boundary model of S , denoted by model $(\mathrm{S})$, is the subsequence of $\dot{\mathrm{S}}=\left\langle\dot{\mathbf{s}}_{1}, \ldots, \dot{\mathrm{~s}}_{j}, \ldots, \dot{\mathrm{~s}}_{\ell}\right\rangle$ induced by $\mathbf{b p}_{1}(\mathrm{~S}) \cup \mathbf{b p}_{2}(\mathrm{~S})$.

As in $[12,27]$, we will bound the number of representatives of $B$-boundaried sequences, and for doing so we bound the number of $B$-boundaried models and then use Lemma 3 which gives an upper bound on the number of typical sequences.

Lemma 6. Let S be a $B$-boundaried sequence. If $\mathrm{S}^{*} \in \operatorname{Ext}(\mathrm{~S})$, then $\operatorname{model}\left(S^{*}\right)=\operatorname{model}(S)$.
Proof. This follows from the observation that the duplication of an element of a $B$-boundaried sequence does not generate a new breakpoint nor kill any existant breakpoint.

Lemma 7. Let $B$ be a set of size $k$. Then, there are at most $2 k+1$ type- 1 breakpoints and at most $k+1$ type-2 breakpoints.

Proof. Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence. By definition, for every $x \in B$, the subset $\left\{j \in[\ell] \mid x \in \mathbf{b d}\left(\mathrm{~s}_{j}\right)\right\}$ forms a set of consecutive integers. So every element $x \in B$ may generate 2 type- 1 breakpoints. This implies that $S$ contains at most $2 k+1$ breakpoints.

Let's now consider the number of type-2 breakpoints. By definition of a $B$-boundaried sequence, for every $i<\ell$, we have $\mathbf{c c}\left(\mathbf{s}_{i}\right) \sqsubseteq \mathbf{c c}\left(\mathbf{s}_{i+1}\right)$. Moreover if $i, j \in[\ell]$ are two consecutive type- 2 breakpoints with $i<j$, then $\mathbf{c c}\left(\mathbf{s}_{i}\right) \neq \mathbf{c c}\left(\mathrm{s}_{j}\right)$. Observe that if $\mathbf{c c}\left(\mathrm{s}_{i}\right) \neq \mathbf{c c}\left(\mathrm{s}_{j}\right)$, then either several blocks of $\mathbf{c c}\left(\mathbf{s}_{i}\right)$ are joined into one block in $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ or some new block $X$ appears in $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ such that $X \cap \mathbf{b d}\left(s_{i}\right)=\emptyset$. Because $|B|=k$ and a near-partition contains at most $k+1$ blocks, by the previous argument we can have at most $k+1$ type- 2 breakpoints.

Lemma 8. Let $B$ be a set of size $k$. Then, there are $2^{O(k \log k)}$ different $B$-boundary models.
Proof. By Lemma 7, the length of a $B$-boundary model is at most $3 k+2$. By definition, each vertex $x \in B$ appears in an interval. Therefore, to build a $B$-boundary model, we have to choose, for each vertex $x \in B, 2$ positions among $3 k+2$ ones, therefore there are $(3 k+2)^{2 k}=2^{O(k \log k)}$ possibilities for choosing the positions of the elements $\mathbf{b d}\left(\mathbf{s}_{j}\right)$ in $B$. Since each type-2 breakpoint is assigned a near-partition of at most $k$ blocks on a set of size at most $k$ and these near-partitions are gradually coarsening, the possibilities of assigning them correspond to the number of rooted trees on $3 k+2$ levels and $k$ leaves. As this is bounded by $2^{O(k)}$, the number of $B$-boundary models is $2^{O(k \log k)}$.

Lemma 9. Let $B$ be a set of size $k$. Then, $\left|\boldsymbol{R e p}_{w}(B)\right|=2^{O(k(w+\log k)}$.

Proof. We only need to bound the number of possible representatives of width $w$ having the same $B$-boundary model. By Lemma 7, there are at most $3 k+2$ type- 1 or type- 2 breakpoints. Because rep $(S)$ has size $\mathbf{b p}(S)$ and a type- 3 breakpoint is between two type- 1 or type- 2 breakpoints, we have to bound the number of typical sequences. By Lemma 3, the number of typical sequences with integers $\{0,1, \ldots, w\}$ is at most $\frac{8}{3} \cdot 2^{2 w}=2^{O(w)}$. Since there are at most $3 k+2=O(k)$ intervals where we can locate type-3 breakpoints, we have $2^{O(w k)}$ possible ways to assign them. The lemma now follows if we take into account the upper bound by Lemma 8 .

Notice that the notion of a $B$-boundary model corresponds to the one of interval model in [12]. Besides the $B$-boundary model of a sequence S , we introduce the profile of S , which corresponds to the concept of list representation in [12].

Definition 9 (Profile). Let S be a $B$-boundaried sequence of length $\ell$ and let $1=j_{1}<\cdots<$ $j_{i}<\cdots<j_{r}=\ell$ be the subset of indices of $[\ell]$ that belong to $\mathbf{b p}_{1}(\mathrm{~S}) \cup \mathbf{b p}_{2}(\mathrm{~S})$. Then we set profile $(\mathrm{S})=\left\langle\mathrm{L}_{1}, \ldots, \mathrm{~L}_{r}\right\rangle$ with, for $i \in[r], \mathrm{L}_{j}=\left\langle\operatorname{val}\left(\mathrm{s}_{j_{i}}\right), \ldots, \operatorname{val}\left(\mathrm{s}_{j_{i+1}-1}\right)\right\rangle$.

Let us now introduce the domination relation over $B$-boundaried sequences. This relation will allow us to compare $B$-boundaried sequences having the same model by means of their $B$-profiles.

Definition 10 (Domination relation). Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ and $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{j}, \ldots, \mathrm{t}_{\ell}\right\rangle$ be two $B$-boundaried sequences such that $\operatorname{model}(\mathrm{S})=\operatorname{model}(\mathrm{T})$. If profile $(\mathrm{S}) \leq \operatorname{profile}(\mathrm{T})$, then we write $\mathrm{S} \leq \mathrm{T}$. And, we say that S dominates T , denoted by $\mathrm{S} \preceq \mathrm{T}$, if profile( S$) \preceq$ profile( T$)$. If we have profile(S) $\preceq$ profile( T ) and profile( T$) \preceq$ profile(S), then we say that S and T are equivalent, which is denoted by $\mathrm{S} \equiv \mathrm{T}$.

Lemma 10. Let S and T be two $B$-boundaried sequences such that model $(\mathrm{S})=\operatorname{model}(\mathrm{T})$. If $\mathrm{S} \preceq \mathrm{T}$, then there exist $\mathrm{S}^{*}$ an extension of S and $\mathrm{T}^{*}$ an extension of T such that $\mathrm{S}^{*} \leq \mathrm{T}^{*}$.

Proof. This is a direct consequence of the definitions.
We observe that some properties on integer sequences from Lemma 5 transfer to $B$-boundaried sequences, and we state in the following some of them that we refer to implicitly most of the time (to avoid overloading the text).

Lemma 11. Let S be a B-boundaried sequence. Then,

1. $\operatorname{rep}(\mathrm{S}) \equiv \mathrm{S}$,
2. if $\mathrm{S}^{*} \in \operatorname{Ext}(\mathrm{~S})$, then $\mathrm{S}^{*} \equiv \mathrm{~S}$,
3. $\mathrm{S} \preceq \mathrm{T}$ if and only if $\operatorname{rep}(\mathrm{S}) \preceq \operatorname{rep}(\mathrm{T})$.
4. If T is a $B$-boundaried sequence such that $\mathrm{S} \preceq \mathrm{T}$, then there exist an extension $\mathrm{S}^{*}$ of S and an extension $\mathrm{T}^{*}$ of T such that $\mathrm{S}^{*} \leq \mathrm{T}^{*}$.
5. The relation $\preceq$ is transitive, and $\equiv$ is an equivalence relation (refering to boundary sequences).

Proof. Let's prove (1). By definition S and rep(S) have the same $B$-boundary model. Let profile $(\mathrm{S})=$ $\left\langle L_{1}, \ldots, L_{p}\right\rangle$. By definition, profile $(\operatorname{rep}(\mathrm{S}))=\left\langle\operatorname{Tseq}\left(L_{1}\right), \ldots, \operatorname{Tseq}\left(L_{p}\right)\right\rangle$, and by Lemma $5(6)$, we know that $\operatorname{Tseq}\left(L_{i}\right) \equiv L_{i}$, for $i \in[p]$. We can therefore conclude that profile $(\mathrm{S}) \equiv \operatorname{profile}(\operatorname{rep}(\mathrm{S})$ ), i.e., $S \equiv \operatorname{rep}(S)$. For (2), if $S^{*} \in \operatorname{Ext}(S)$, then clearly $S^{*} \preceq S$ and $S \preceq S^{*}$ by taking as an extension of $S$ its extension $S^{*}$, and for an extension of $S^{*}$ itself. Finally, (4) follows directly from the definitions, (5) follows from Lemma 5(5), and (3) follows from (1) and (5).

### 3.2 Operations on $B$-boundaried sequences

Given a finite set $B$, we define two operations on $B$-boundaried sequences that will be later used in the DP algorithm. The first operation, projection, will be used in the case of forget bags where we need to transform a $B$-boundaried sequence representing a connected path-decomposition of a boundaried graph $\mathbf{G}=(G, B)$ into a $B \backslash\{x\}$-boundaried sequence representing a connected path-decomposition of the boundaried graph $\mathbf{G}^{\bar{x}}=(G, B \backslash\{x\})$. The second operation deals with the insertion in a $B$-boundaried sequence of a new boundary element $x$ with respect to a subset $X \subseteq B$. This will be used by the DP algorithm when handling insertion bags.

### 3.2.1 Projection of $B$-boundaried sequences

The projection of a $B$-boundaried sequence S onto $B^{\prime} \subseteq B$ aims at moving the vertices of $B \backslash B^{\prime}$ from the status of boundary vertices to the status of inactive vertices.

Definition 11 (Projection). Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{i}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence. For a subset $B^{\prime} \subseteq B$, the projection of S onto $B^{\prime}$ is the $B^{\prime}$-boundaried sequence $\mathrm{S}_{\mid B^{\prime}}=\left\langle\mathrm{s}_{1 \mid B^{\prime}}, \ldots, \mathrm{s}_{i \mid B^{\prime}}, \ldots, \mathrm{s}_{\ell \mid B^{\prime}}\right\rangle$ such that for every $i \in[\ell]$ :

- $\mathbf{b d}\left(\mathbf{s}_{i \mid B^{\prime}}\right)=\mathbf{b d}\left(\mathbf{s}_{i}\right) \cap B^{\prime} ;$
- $\mathbf{c c}\left(\mathbf{s}_{i \mid B^{\prime}}\right)=\mathbf{c c}\left(\mathbf{s}_{i}\right)_{\mid B^{\prime}} ;$
- $\operatorname{val}\left(\mathrm{s}_{i \mid B^{\prime}}\right)=\operatorname{val}\left(\mathrm{s}_{i}\right)+\left|\mathbf{b d}\left(\mathrm{s}_{i}\right) \backslash B^{\prime}\right|$.

We observe that when the $B$-boundaried sequence S is connected, its projection $\mathrm{S}_{\mid B^{\prime}}$ onto $B^{\prime} \subseteq B$ may not be connected. This is the case if for some $j \in[\ell]$, the partition $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ contains several blocks and at least one of them is a subset of $B \backslash B^{\prime}$.

Lemma 12. Let $B$ be a finite set and $B^{\prime} \subseteq B$. Then, the width of $\mathrm{S}_{\mid B^{\prime}}$ is equal to the width of S , for every $B$-boundaried sequence S .

Proof. Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ and $\mathrm{S}_{\mid B^{\prime}}=\left\langle\mathrm{s}^{\prime}{ }_{1}, \ldots, \mathrm{~s}^{\prime}{ }_{j}, \ldots, \mathrm{~s}^{\prime} \ell\right\rangle$. By definition, for each $1 \leq j \leq \ell$, $\left|\mathbf{b d}\left(\mathbf{s}_{j}\right)\right|+\operatorname{val}\left(\mathbf{s}_{j}\right)=\left|\mathbf{b d}\left(\mathbf{s}_{j}\right) \cap B^{\prime}\right|+\left|\mathbf{b d}\left(\mathbf{s}_{j}\right) \backslash B^{\prime}\right|+\operatorname{val}\left(\mathbf{s}_{j}\right)$, the latter being exactly $\left|\mathbf{b d}\left(\mathbf{s}_{j}^{\prime}\right)\right|+$ $\operatorname{val}\left(s^{\prime}{ }_{j}\right)$.

Lemma 13. Let $B$ be a finite set and $B^{\prime} \subsetneq B$. If $\mathrm{S}^{*}$ is an extension of a $B$-boundaried sequence S , then $\mathrm{S}_{\mid B^{\prime}}^{*}$ is an extension of $\mathrm{S}_{\mid B^{\prime}}$.

Proof. Let $S=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$. As by Lemma $6, \operatorname{model}(\mathrm{~S})=\operatorname{model}\left(\mathrm{S}^{*}\right)$, duplicating $\mathrm{s}_{i}$ and then computing $\mathrm{s}_{i \mid B^{\prime}}$ is the same as computing $\mathrm{s}_{i \mid B^{\prime}}$ and then duplicating the latter.

Lemma 14. Let $B$ be a finite set and $B^{\prime} \subseteq B$. If $S$ and $T$ are $B$-boundaried sequences such that $\mathrm{S} \leq \mathrm{T}$, then $\mathrm{S}_{\mid B^{\prime}} \leq \mathrm{T}_{\mid B^{\prime}}$.

Proof. Let $S=\left\langle s_{1}, \ldots, s_{\ell}\right\rangle$ and let $T=\left\langle t_{1}, \ldots, t_{\ell}\right\rangle$. Because $\operatorname{model}(S)=\operatorname{model}(T)$, we also have that $\operatorname{model}\left(\mathrm{S}_{\mid B^{\prime}}\right)=\operatorname{model}\left(\mathrm{T}_{\mid B^{\prime}}\right)$. Because $\operatorname{model}(\mathrm{S})=\operatorname{model}(\mathrm{T})$, we can check that $\operatorname{val}\left(\mathrm{s}_{i \mid B^{\prime}}\right)$ and $\operatorname{val}\left(\mathrm{t}_{i \mid B^{\prime}}\right)$ are both obtained by adding the same value to $\boldsymbol{v a l}\left(\mathrm{s}_{i}\right)$ and to $\boldsymbol{v a l}\left(\mathrm{t}_{i}\right)$, respectively. Hence, we can conclude that $S_{\mid B^{\prime}} \leq \mathrm{T}_{\mid B^{\prime}}$ because profile $(\mathrm{S}) \leq$ profile $(\mathrm{T})$.

Lemma 15. Let $B$ be a finite set and $B^{\prime} \subseteq B$. If $S$ and $T$ are $B$-boundaried sequences such that $\mathrm{S} \preceq \mathrm{T}$, then $\mathrm{S}_{\mid B^{\prime}} \preceq \mathrm{T}_{\mid B^{\prime}}$.

Proof. Let $\mathrm{S}^{*}$ and $\mathrm{T}^{*}$ be extensions of S and T , respectively, such that $\mathrm{S}^{*} \leq \mathrm{T}^{*}$. By Lemma 14, $\mathrm{S}_{\mid B^{\prime}}^{*} \leq \mathrm{T}_{\mid B^{\prime}}^{*}$. By Lemma $13, \mathrm{~S}_{\mid B^{\prime}}^{*}$ is an extension of $\mathrm{S}_{\mid B^{\prime}}$, i.e., $\mathrm{S}_{\mid B^{\prime}} \equiv \mathrm{S}_{\mid B^{\prime}}^{*}$ by Lemma $11(2)$. Similarly, we have $\mathrm{T}_{\mid B^{\prime}}^{*} \equiv \mathrm{~T}_{\mid B^{\prime}}$. Hence, we can conclude that $\mathrm{S}_{\mid B^{\prime}} \preceq \mathrm{T}_{\mid B^{\prime}}$.

### 3.2.2 Insertion into a $B$-boundaried sequence

Let $S=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence and let $X$ be a subset of $B$. An insertion position is a pair of indices $\left(f_{x}, l_{x}\right)$ such that $1 \leq f_{x} \leq l_{x} \leq \ell$. An insertion position is valid with respect to $X$ in $S$ if $X \subseteq \bigcup_{f_{x} \leq j \leq l_{x}} \mathbf{b d}\left(s_{j}\right)$. Let us now formally describe the insertion operation.

Definition 12. Let $S=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a B-boundaried sequence and $\left(f_{x}, l_{x}\right)$ be a valid insertion position with respect to $X \subseteq B$. Then $\mathrm{S}^{x}=\operatorname{lns}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)=\left\langle\mathrm{s}_{1}^{x}, \ldots, \mathrm{~s}_{\ell}^{x}\right\rangle$ is the $(B \cup\{x\})$ boundaried sequence such that for every $j \in[\ell]$ :

- if $j<f_{x}$, then $\mathbf{b d}\left(\mathrm{s}_{j}^{x}\right)=\mathbf{b d}\left(\mathrm{s}_{j}\right) ; \mathbf{c c}\left(\mathrm{s}_{j}^{x}\right)=\mathbf{c c}\left(\mathrm{s}_{j}\right)$ and $\operatorname{val}\left(\mathrm{s}_{j}^{x}\right)=\operatorname{val}\left(\mathrm{s}_{j}\right)$.
- if $f_{x} \leq j \leq l_{x}$, then $\mathbf{b d}\left(\mathrm{s}_{j}^{x}\right)=\mathbf{b d}\left(\mathrm{s}_{j}\right) \cup\{x\} ; \mathbf{c c}\left(\mathrm{s}_{j}^{x}\right)$ is obtained by adding a new block $\{x\}$ to $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ and then merging that new block with all the blocks of $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ that contain an element of $X$ (if any); $\operatorname{val}\left(\mathrm{s}_{j}^{x}\right)=\operatorname{val}\left(\mathrm{s}_{j}\right)$.
- and otherwise, $\mathbf{b d}\left(\mathrm{s}_{j}^{x}\right)=\mathbf{b d}\left(\mathrm{s}_{j}\right) ; \mathbf{c c}\left(\mathrm{s}_{j}^{x}\right)$ is obtained by adding a new block $\{x\}$ to $\mathbf{c c}\left(\mathbf{s}_{j}\right)$ and then merging that new block with all the blocks of $\mathbf{c c}\left(\mathrm{s}_{j}\right)$ that contains an element of $X$ (if any) $; \operatorname{val}\left(\mathrm{s}_{j}^{x}\right)=\operatorname{val}\left(\mathrm{s}_{j}\right)$.

It is worth to notice that a type- 2 breakpoint $j$ in a $B$-boundaried sequence $S$ may disappear in $\operatorname{lns}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)$, because the insertion of $x$ with respect to $X$ may merge in $\mathbf{c c}\left(\mathrm{s}_{j-1}^{x}\right)$ distinct blocks of $\mathbf{c c}\left(\mathrm{s}_{j-1}\right)$ that are joined in $\mathbf{c c}\left(\mathrm{s}_{j}\right)$. However one can prove that if $j \in \mathbf{b} \mathbf{p}_{2}\left(\mathrm{~S}^{x}\right)$, then $j \in \mathbf{b p}_{2}(\mathrm{~S})$ (see Figure 5 for an illustration of this property) and if $j \in \mathbf{b p}_{3}\left(\mathrm{~S}^{x}\right)$, then $j \in \mathbf{b} \mathbf{p}_{3}(\mathrm{~S})$.

Lemma 16. Let $B$ and $B^{\prime}$ be finite sets with $B=B^{\prime} \backslash\{x\}$ for some $x \in B^{\prime}$. Let S be a $B$-boundaried sequence and let $\left(f_{x}, l_{x}\right)$ be a valid insertion position with respect to subset $X \subseteq B$ in S . Then, the width of S is at most the width of $\operatorname{Ins}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)$.


Figure 5: In red, the partitions $\mathbf{c c}\left(\mathbf{s}_{j-1}\right)=\left\{C_{1}, C_{2}, C_{3}, C_{4}, C_{5}\right\}$ and $\mathbf{c c}\left(\mathbf{s}_{j}\right)=\left\{C_{1}, C_{2}, C_{3} \cup C_{4}, C_{5}\right\}$ certifying that $j \in \mathbf{b p}_{2}(\mathrm{~S})$. In grey, the partitions $\mathbf{c c}\left(\mathrm{s}_{j-1}^{x}\right)=\left\{C^{x}, C_{4}, C_{5}\right\}$ and $\mathbf{c c}\left(\mathrm{s}_{j}^{x}\right)=\left\{C^{x} \cup\right.$ $\left.C_{4}, C_{5}\right\}$ certifying that $j \in \mathbf{b p}_{2}\left(\mathrm{~S}^{x}\right)$.

Proof. Suppose that $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ and $\operatorname{Ins}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)=\left\langle\mathrm{s}_{1}^{x}, \ldots, \mathrm{~s}_{\ell}^{x}\right\rangle$. By Definition 12 we have that: for each $1 \leq j \leq \ell, \operatorname{val}\left(\mathbf{s}_{j}^{x}\right)=\operatorname{val}\left(\mathbf{s}_{j}\right)$; if $j \notin\left[f_{x}, l_{x}\right]$, then $\mathbf{b d}\left(\mathbf{s}_{j}^{x}\right)=\mathbf{b d}\left(\mathbf{s}_{j}\right)$, otherwise $\mathbf{b d}\left(\mathrm{s}_{j}^{x}\right)=\mathbf{b d}\left(\mathrm{s}_{j}\right) \cup\{x\}$. The statement follows therefore by definition of width of $B$-boundaried sequences.

Let us remind that if a $B$-boundaried sequence T of length $p$ is an extension of S of length $\ell$, then the extension surjection $\delta_{\mathrm{T} \rightarrow \mathrm{S}}:[p] \rightarrow[\ell]$ associates each element of T with its original copy in S (see Section 2).

Lemma 17. Let $B$ and $B^{\prime}$ be finite sets with $B=B^{\prime} \backslash\{x\}$ for some $x \in B^{\prime}$. Let $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ be a $B$-boundaried sequence, and let $\mathrm{T} \in \operatorname{Ext}(\mathrm{S})$ that has length $p$ and is certified by the surjective function $\delta_{\mathrm{T} \rightarrow \mathrm{S}}:[p] \rightarrow[\ell]$. For every valid insertion position $\left(f_{x}, l_{x}\right)$ with respect to some subset $X \subseteq B$ in $\mathrm{S},\left(f_{x}^{*}, l_{x}^{*}\right)$ is a valid insertion position with respect to $X$ in T , where $f_{x}^{*}=\min \{h \in[p] \mid$ $\left.f_{x}=\delta_{\mathrm{T} \rightarrow \mathrm{S}}(h)\right\}$ and $l_{x}^{*}=\max \left\{h \in[p] \mid f_{x}=\delta_{\mathrm{T} \rightarrow \mathrm{S}}(h)\right\}$. Moreover, $\operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of $\operatorname{Ins}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)$.

Proof. Let us prove the statement for a 1-extension T of S. Inductively applying the proof $p-\ell$ times leads to the statement.

Let us denote $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{\ell+1}\right\rangle$. Suppose that $\mathrm{s}_{i}$, for $1 \leq i \leq \ell$, is duplicated, that is for every $j \leq i, \delta_{\mathrm{T} \rightarrow \mathrm{S}}(j)=j$ and for every $i<j \leq \ell+1, \delta_{\mathrm{T} \rightarrow \mathrm{S}}(j)=j-1$. It is clear that if $i>l_{x}$ then $\left(f_{x}, l_{x}\right)$ is still a valid insertion position with respect to $X$ in T , and similarly for $\left(f_{x}+1, l_{x}+1\right)$ if $i<f_{x}$. If $f_{x} \leq i \leq l_{x}$, then $\left(f_{x}, l_{x}+1\right)$ is a valid insertion position with respect to $X$ in $\mathbf{T}$ because $\mathrm{t}_{j}=\mathrm{s}_{j}$ for $f_{x} \leq j \leq i$, and $\mathrm{s}_{j}^{*}=\mathrm{s}_{j-1}$ for $i+1 \leq j \leq \ell+1$.

We claim now that $\operatorname{lns}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of $\operatorname{lns}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)$ certified by the surjective function $\delta_{\mathrm{T} \rightarrow \mathrm{s}}$. Indeed, observe that for every $j \in[\ell+1], \mathrm{t}_{j}=\mathrm{s}_{\delta_{\mathrm{T} \rightarrow \mathrm{s}}(j)}$. So, if we duplicate $\mathrm{s}_{i}^{x}$ in $\mathrm{S}^{x}$, we will obtain $\operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$.

Lemma 17 shows that if $\mathbf{T}$ is an extension of S , then, to every valid insertion position $\left(f_{x}, l_{x}\right)$ with respect to some subset $X \subseteq B$ in S , one can associate a valid insertion position $\left(f_{x}^{*}, l_{x}^{*}\right)$ with respect to $X$ in T. As shown by the example of Figure 6, the reverse is not true. The following lemma states that it is indeed possible to associate a valid insertion position $\left(f_{x}^{*}, l_{x}^{*}\right)$ with respect to $X$ in T to some valid insertion position with respect to $X$ in some $(\leq 2)$-extension of S .


Figure 6: Let T be a 2 -extension of the $B$-boundaried sequence $S$. Suppose that $(5,10)$ is a valid insertion position with respect to some for $X \subseteq B$ in T . Observe that as $4=\delta_{\mathrm{T} \rightarrow \mathrm{S}}(5)$ and $9=\delta_{\mathrm{T} \rightarrow \mathrm{S}}(10),(4,9)$ is also a valid insertion position with respect to some for $X \subseteq B$ in S . However, $\operatorname{Ins}(\mathrm{T}, x, X, 5,10)$ is not an extension of $\operatorname{Ins}(\mathrm{S}, x, X, 4,9)$.

Lemma 18. Let $B$ and $B^{\prime}$ be finite sets with $B=B^{\prime} \backslash\{x\}$ for some $x \in B^{\prime}$. Let T be an extension of a $B$-boundaried sequence S . If $\left(f_{x}^{*}, l_{x}^{*}\right)$ is a valid insertion position with respect to a subset $X \subseteq B$ in T , then there is a $(\leq 2)$-extension R of S and a valid insertion position $\left(f_{x}, l_{x}\right)$ with respect to $X$ in R such that $\operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of $\operatorname{Ins}\left(\mathrm{R}, x, X, f_{x}, l_{x}\right)$.

Proof. Suppose that $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ and $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{p}\right\rangle$. Let $\delta_{\mathrm{T} \rightarrow \mathrm{S}}:[p] \rightarrow[\ell]$ be the surjection certifying that $\mathrm{T} \in \operatorname{Ext}(\mathrm{S})$, that is for every $j \in[p]$, if $\delta_{\mathrm{T} \rightarrow \mathrm{S}}(j)=i$, then $\mathrm{t}_{j}$ is a copy originating from $\mathrm{s}_{i}$. Let us denote $f=\delta_{\mathrm{T} \rightarrow \mathrm{S}}\left(f_{x}^{*}\right)$ and $l=\delta_{\mathrm{T} \rightarrow \mathrm{S}}\left(l_{x}^{*}\right)$. We also define $f_{x}^{\prime}=\min \left\{j \in[p] \mid \delta_{\mathrm{T} \rightarrow \mathrm{S}}(j)=f\right\}$ and $l_{x}^{\prime}=\max \left\{j \in[p] \mid \delta_{\mathrm{T} \rightarrow \mathrm{S}}(j)=l\right\}$. The $(\leq 2)$-extension R of S is built as follows: if $f_{x}^{\prime}<f_{x}^{*}$, then we duplicate $\mathrm{s}_{f}$ and if $l_{x}^{*}<l_{x}^{\prime}$, then we duplicate $\mathrm{s}_{l}$. Let $r$ be the size of R and let $\delta_{\mathrm{R} \rightarrow \mathrm{s}}:[r] \rightarrow[\ell]$ certifying that R is a $(\leq 2)$-extension of S .

Let us build a surjection $\delta_{\mathrm{T} \rightarrow \mathrm{R}}:[p] \rightarrow[r]$ certifying that T is an extension of R . To that aim, we define $f_{x}=\max \left\{h \in[r] \mid \delta_{\mathrm{R} \rightarrow \mathrm{S}}(h)=f\right\}$ and $l_{x}=\min \left\{h \in[r] \mid \delta_{\mathrm{R} \rightarrow \mathrm{S}}(h)=l\right\}$. Then:

$$
\delta_{\mathrm{T} \rightarrow \mathrm{R}}(j)= \begin{cases}\delta_{\mathrm{T} \rightarrow \mathrm{~s}}(j) & \text { if } j<f_{x}^{*} \\ \delta_{\mathrm{T} \rightarrow \mathrm{~s}}(j)-f+f_{x} & \text { if } f_{x}^{*} \leq j \leq l_{x}^{*} \\ \delta_{\mathrm{T} \rightarrow \mathrm{~S}}(j)-l+l^{\prime} & \text { if } l_{x}^{*}<j\end{cases}
$$

where as in Figure $7 l^{\prime}=\max \left\{h \in[r] \mid \delta_{\mathrm{R} \rightarrow \mathrm{S}}(h)=l\right\}$.
Observe that as $T \in \operatorname{Ext}(\mathrm{~S})$ and $\mathrm{R} \in \operatorname{Ext}(\mathrm{S})$, by Lemma 6, we have $\operatorname{model}(\mathrm{R})=\operatorname{model}(\mathrm{T})$. Thereby $\delta_{\mathrm{T} \rightarrow \mathrm{S}}\left(f_{x}^{*}\right)=f_{x}$ and $\delta_{\mathrm{T} \rightarrow \mathrm{S}}\left(l_{x}^{*}\right)=l_{x}$ implies that $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in R . It remains to prove that $\mathrm{T}^{x}=\operatorname{lns}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of


Figure 7: The three surjective functions $\delta_{\mathrm{T} \rightarrow \mathrm{S}}(\cdot), \delta_{\mathrm{R} \rightarrow \mathrm{S}}(\cdot)$ and $\delta_{\mathrm{T} \rightarrow \mathrm{R}}(\cdot)$ respectively certifying that $\mathrm{T} \in \operatorname{Ext}(\mathrm{S}), \mathrm{R} \in \operatorname{Ext}(\mathrm{S})$ and $\mathrm{T} \in \operatorname{Ext}(\mathrm{R})$ in the case $f_{x}^{\prime} \neq f_{x}^{*}$ and $l_{x}^{\prime} \neq l_{x}^{*}$. In this case, as R is a 2-extension of $\mathbf{S}, f^{\prime}=\min \left\{h \in[r] \mid \delta_{\mathbf{R} \rightarrow \mathrm{S}}(h)=f\right\}$ and $l^{\prime}=\max \left\{h \in[r] \mid \delta_{\mathrm{R} \rightarrow \mathrm{S}}(h)=l\right\}$.
$\mathrm{R}^{x}=\operatorname{Ins}\left(\mathrm{R}, x, X, f_{x}, l_{x}\right)$. Observe that, by construction of $\mathrm{R}, f_{x}^{*}=\min \left\{j \in[p] \mid \delta_{\mathrm{T} \rightarrow \mathrm{R}}(j)=f_{x}\right\}$ and $l_{x}^{*}=\max \left\{j \in[p] \mid \delta_{\mathrm{T} \rightarrow \mathrm{R}}(j)=l_{x}\right\}$. This implies that we can certify $\mathrm{T}^{x} \in \operatorname{Ext}\left(\mathrm{R}^{x}\right)$ by Lemma 17.

Lemma 19. Let $B$ and $B^{\prime}$ be finite sets with $B=B^{\prime} \backslash\{x\}$ for some $x \in B^{\prime}$. Let S and T be $B$-boundaried sequences such that $\mathrm{S} \leq \mathrm{T}$. If $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to a subset $X \subseteq B$ in T , then $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in S and $\operatorname{lns}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right) \leq \operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}, l_{x}\right)$.

Proof. Suppose that profile $(\mathrm{S})=\left\langle L_{1}, \ldots, L_{r}\right\rangle$ and profile $(\mathrm{T})=\left\langle L_{1}^{\prime}, \ldots, L_{r}^{\prime}\right\rangle$. By Definition 10, as $\mathrm{S} \leq \mathrm{T}, \mathrm{T}$ and S have the same $B$-model. It follows that $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in $S$ as well. And it implies that for every $i \in[r], i \in \mathbf{b} \mathbf{p}_{1}(\mathrm{~S})$ if and only if $i \in \mathbf{b} \mathbf{p}_{1}(\mathbf{T})$ and that $i \in \mathbf{b p}_{2}(\mathrm{~S})$ if and only if $i \in \mathbf{b p}_{2}(\mathrm{~T})$. Thereby, if we denote $\mathrm{S}^{x}=\operatorname{lns}\left(\mathrm{S}, x, X, f_{x}, l_{x}\right)$ and $\mathrm{T}^{x}=\operatorname{lns}\left(\mathrm{T}, x, X, f_{x}, l_{x}\right)$, by Definition 12 , we obtain that, for every $i \in[r], i \in \mathbf{b} \mathbf{p}_{1}\left(\mathrm{~S}^{x}\right)$ if and only if $i \in \mathbf{b} \mathbf{p}_{1}\left(\mathbf{T}^{x}\right)$ and that $i \in \mathbf{b} \mathbf{p}_{2}\left(\mathrm{~S}^{x}\right)$ if and only if $i \in \mathbf{b} \mathbf{p}_{2}\left(\mathrm{~T}^{x}\right)$. Thereby we have model $\left(\mathrm{S}^{x}\right)=$ $\operatorname{model}\left(\mathbf{T}^{x}\right)$. Observe moreover that $\mathrm{S} \leq \mathrm{T}$ implies that for every $i \in[r], \operatorname{val}\left(\mathrm{s}_{i}\right) \leq \operatorname{val}\left(\mathrm{t}_{i}\right)$. As for every $i \in[r]$, we have that $\operatorname{val}\left(\mathrm{s}_{i}\right)=\operatorname{val}\left(\mathrm{s}_{i}^{x}\right)$ and $\operatorname{val}\left(\mathrm{t}_{i}\right)=\operatorname{val}\left(\mathrm{t}_{i}^{x}\right)$, we obtain that $\operatorname{val}\left(\mathrm{s}_{i}^{x}\right) \leq \operatorname{val}\left(\mathrm{t}_{i}^{x}\right)$. It follows that profile $\left(\mathrm{S}^{x}\right) \leq \operatorname{profile}\left(\mathrm{T}^{x}\right)$, in other words $\mathrm{S}^{x} \leq \mathrm{T}^{x}$.

Lemma 20. Let $B$ and $B^{\prime}$ be finite sets with $B=B^{\prime} \backslash\{x\}$ for some $x \in B^{\prime}$. Let S and T be $B$-boundaried sequences such that $\mathrm{S} \preceq \mathrm{T}$. If $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $a$ subset $X \subseteq B$ in T , then there is a valid insertion position $\left(f_{x}^{\prime}, l_{x}^{\prime}\right)$ in a $(\leq 2)$-extension R of S such that $\operatorname{Ins}\left(\mathrm{R}, x, X, f_{x}^{\prime}, l_{x}^{\prime}\right) \preceq \operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}, l_{x}\right)$.

Proof. Let $\mathrm{S}^{*}$ and $\mathrm{T}^{*}$ be extensions of S and T , respectively, such that $\mathrm{S}^{*} \leq \mathrm{T}^{*}$. Suppose that $\mathrm{T}^{*}$ has size $p^{*}$. Let $\delta_{\mathrm{T}^{*} \rightarrow \mathrm{~T}}$ be the surjective function certifying that $\mathrm{T}^{*} \in \operatorname{Ext}(\mathrm{~T})$. Let us denote $f_{x}^{*}=\min \left\{h \in\left[p^{*}\right] \mid f_{x}=\delta_{\mathbf{T}^{*} \rightarrow \mathbf{T}}(h)\right\}$ and $l_{x}^{*}=\max \left\{h \in\left[p^{*}\right] \mid l_{x}=\delta_{\mathbf{T}^{*} \rightarrow \mathbf{T}}(h)\right\}$. As $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in T , then by Lemma $17,\left(f_{x}^{*}, l_{x}^{*}\right)$ is also a valid insertion position with respect to $X$ in $\mathrm{T}^{*}$ and $\operatorname{Ins}\left(\mathrm{T}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of $\operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}, l_{x}\right)$.

By Lemma $19,\left(f_{x}^{*}, l_{x}^{*}\right)$ is a valid insertion position with respect to $X$ in $\mathrm{S}^{*}$ and by Lemma 18 there is a $(\leq 2)$-extension R of S and a valid insertion position $\left(f_{x}^{\prime}, l_{x}^{\prime}\right)$ in R such that $\operatorname{Ins}\left(\mathrm{S}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ is an extension of $\operatorname{Ins}\left(\mathrm{R}, x, X, f_{x}^{\prime}, l_{x}^{\prime}\right)$. Then, by Lemma 19, we have

$$
\operatorname{Ins}\left(\mathrm{S}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right) \leq \operatorname{Ins}\left(\mathrm{T}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right)
$$

By using Lemma 11(2), it follows that $\operatorname{lns}\left(\mathrm{T}, x, X, f_{x}, l_{x}\right) \equiv \operatorname{Ins}\left(\mathrm{T}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right)$ and $\operatorname{Ins}\left(\mathrm{R}, x, X, f_{x}^{\prime}, l_{x}^{\prime}\right) \equiv$ $\operatorname{Ins}\left(\mathrm{S}^{*}, x, X, f_{x}^{*}, l_{x}^{*}\right)$, implying the statement by Lemma 11(5).

## 4 Computing the connected pathwidth

We first explain how $B$-boundaried sequence are natural combinatorial objects to encode a connected path-decomposition. We describe and analyze the time complexity of the Forget Routine and the Insertion Routine that allow us to respectively process forget and insertion bags of the nice path-decomposition given as input to the DP algorithm.

### 4.1 Encoding a connected path-decomposition

Let us explain how to represent a path-decomposition of a boundaried graph $(G, B)$ by means of a $B$-boundaried sequence.

Definition 13 ((G,P)-encoding sequence). Let $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$ be a path-decomposition of the boundaried graph $\mathbf{G}=(G, B)$. A B-boundaried sequence $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{j}, \ldots, \mathrm{~s}_{\ell}\right\rangle$ is a $(\mathbf{G}, \mathrm{P})$-encoding sequence, if for every $j \in[\ell]$ :

- $\mathbf{b d}\left(\mathrm{s}_{j}\right)=A_{j} \cap B$ : the set of boundary vertices of $(G, B)$ belonging to the bag $A_{j}$;
- $\mathbf{c c}\left(\mathrm{s}_{j}\right)=\left\{V(C) \cap B \mid C\right.$ is a connected component of $\left.G_{j}\right\}$;
- $\operatorname{val}\left(\mathrm{s}_{j}\right)=\left|A_{j} \backslash B\right|$ : the number of inactive vertices in the bag $A_{j}$.

It is worth to observe that $\mathbf{c c}\left(\mathbf{s}_{j}\right)$ is, in general, not a partition of $A_{j}$ (see Figure 4). Also, notice that if $G_{j}$ is connected and $B \cap V_{j}=\emptyset$, then $\mathbf{c c}\left(\mathrm{s}_{j}\right)=\{\emptyset\}$.

Lemma 21. Let P be a path-decomposition of a connected boundaried graph $\mathbf{G}=(G, B)$. If P is a connected path-decomposition, then its $(\mathbf{G}, \mathrm{P})$-encoding sequence is a connected $B$-boundaried sequence.

Proof. Follows directly from the definitions.
Definition 14. Let $\mathbf{G}=(G, B)$ be a connected boundaried graph and $S$ a $B$-boundaried sequence. We say that S is realizable in $\mathbf{G}$ if there is an extension $\mathrm{S}^{*}$ of S that is the $(\mathbf{G}, \mathrm{P})$-encoding sequence of some connected path-decomposition $\mathbf{P}$ of $\mathbf{G}$.

Let us observe that if a $B$-boundaried sequence S is realizable, then by Lemma 21 S is connected. The set of representative $B$-boundaried sequences of a connected boundaried graph $\mathbf{G}=(G, B)$ of width $\leq w$ is defined as:

$$
\operatorname{Rep}_{w}(\mathbf{G})=\{\operatorname{rep}(\mathrm{S}) \mid \mathrm{S} \text { of width } \leq w \text { is realizable in } \mathbf{G}=(G, B)\} .
$$

To compute the connected pathwidth of a graph, rather than computing $\operatorname{Rep}_{w}(\mathbf{G})$, we compute a subset $\mathbf{D}_{w}(\mathbf{G}) \subseteq \operatorname{Rep}_{w}(\mathbf{G})$, called domination set, such that for every representative $B$-boundaried sequence $\mathrm{S} \in \boldsymbol{R e p}_{w}(\mathbf{G})$, there exists a representative $B$-boundaried sequence $\mathrm{R} \in \mathbf{D}_{w}(\mathbf{G})$ such that $R \preceq S$.

Proposition 1. A connected boundaried graph $\mathbf{G}=(G, B)$ has connected pathwidth at most $w$ if and only if $\mathbf{D}_{w+1}(\mathbf{G}) \neq \emptyset$.

Proof. Let $\mathbf{P}$ be a connected path-decomposition of width at most $w$ of $\mathbf{G}$. Recall the the bags of such decomposition have size at most $w+1$. By definition, the ( $\mathbf{G}, \mathbf{P}$ )-encoding sequence is realizable in $\mathbf{G}$, implying that $\operatorname{Rep}_{w+1}(\mathbf{G})$ and thereby $\mathbf{D}_{w+1}(\mathbf{G})$ is not empty. Conversely, suppose that $\operatorname{Rep}_{w+1}(\mathbf{G})$ is non-empty and consider $S \in \mathbf{D}_{w+1}(\mathbf{G})$. As $S \in \operatorname{Rep}_{w+1}(\mathbf{G})$, there exists a connected path-decomposition P of width at most $w$ of $\mathbf{G}$ and $\mathbf{S}^{*}$ the ( $\mathbf{G}, \mathbf{P}$ )-encoding sequence with $\operatorname{rep}\left(\mathrm{S}^{*}\right)=S$, implying that $\operatorname{cpw}(\mathbf{G}) \leq w$.

### 4.2 Forget Routine

Let $\mathbf{G}=(G, B)$ be a boundaried graph. If $x \in B$ is a boundary vertex, we denote by $B^{\bar{x}}=B \backslash\{x\}$. We define $\mathbf{G}^{\bar{x}}=\left(G, B^{\bar{x}}\right)$, that is, while the graph $G$ is left unchanged, we remove $x$ from the set of boundary vertices. Given $\mathbf{D}_{w}(\mathbf{G})$ and $x \in B$, Forget Routine aims at computing a domination set $\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)$. The routine is described in Algorithm 1.

```
Algorithm 1: Forget Routine
    Input: A boundaried graph \(\mathbf{G}=(G, B)\), a vertex \(x \in B\), and \(\mathbf{D}_{w}(\mathbf{G})\).
    Output: \(\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)\), a domination set of \(\boldsymbol{\operatorname { R e p }}_{w}\left(\mathbf{G}^{\bar{x}}\right)\).
    \(\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right) \leftarrow \emptyset ;\)
    foreach \(\mathbf{S} \in \mathbf{D}_{w}(\mathbf{G})\) do
        if \(\mathrm{S}_{\mid B \backslash\{x\}}\) is connected, then add \(\operatorname{rep}\left(\mathrm{S}_{\mid B \backslash\{x\}}\right)\) to \(\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)\);
    end
    return \(\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)\).
```

To prove the correctness of Forget Routine, we proceed in two steps. We first establish the completeness of the algorithm. More precisely, Proposition 2 states that, for every connected path-decomposition $\mathbf{P}$ of $\mathbf{G}^{\bar{x}}$, there exists some $B$-boundaried sequence $\mathbf{S} \in \mathbf{D}_{w}(\mathbf{G})$ such that $\operatorname{rep}\left(\mathrm{S}_{\mid B \backslash\{x\}}\right) \preceq \operatorname{rep}(\mathbf{T})$ where T is the $\left(\mathbf{G}^{\bar{x}}, \mathbf{P}\right)$-encoding sequence. Then Proposition 3 proves the soundness of the routine: for every $B$-boundaried sequence $\mathbf{S} \in \mathbf{D}_{w}(\mathbf{G}), \operatorname{rep}\left(\mathrm{S}_{\mid B \backslash\{x\}}\right) \in \mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)$ if $\mathrm{S}_{\mid B \backslash\{x\}}$ is connected.

Proposition 2 (Forget completeness). Let $\mathbf{G}=(G, B)$ be a boundaried graph and $x \in B$ be $a$ boundary vertex. If P is a connected path-decomposition of width at most $w$ of $\mathbf{G}^{\bar{x}}$, then there exists $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$ such that $\mathrm{S}_{\mid B^{\bar{x}}}$ is connected and $\operatorname{rep}\left(\mathrm{S}_{\mid B^{\bar{x}}}\right) \preceq \operatorname{rep}(\mathrm{T})$ where T is the $\left(\mathbf{G}^{\bar{x}}, \mathrm{P}\right)$-encoding sequence.

Proof. Suppose that $\mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$. Observe that P is also a connected path-decomposition of $\mathbf{G}$ of width at most $w$. Let $\mathrm{R}=\left\langle\mathrm{r}_{1}, \ldots, \mathrm{r}_{\ell}\right\rangle$ be the $(\mathbf{G}, \mathrm{P})$-encoding sequence.

We claim that $\mathrm{R}_{\mid B^{\bar{x}}}$ is the $\left(\mathbf{G}^{\bar{x}}, P\right)$-encoding sequence. To see this, we apply Definition 11 on the projection of R onto $B^{\bar{x}}$. Consider an index $j \in[\ell]$. First, we have that $\mathbf{b d}\left(\mathrm{r}_{j \mid B^{\bar{x}}}\right)=\mathbf{b d}\left(\mathrm{r}_{j}\right) \cap B^{\bar{x}}$. As by construction of $\mathrm{R}, \mathbf{b d}\left(\mathrm{r}_{j}\right)=A_{j} \cap B^{\bar{x}}$ and as $B^{\bar{x}} \subset B$, we obtain $\mathbf{b d}\left(\mathrm{r}_{j \mid B^{\bar{x}}}\right)=A_{j} \cap B^{\bar{x}}$. For the same arguments, observe that $\operatorname{val}\left(\mathrm{r}_{j \mid B^{\bar{x}}}\right)=\operatorname{val}\left(\mathrm{r}_{j}\right)+\left|\mathbf{b d}\left(\mathrm{r}_{j}\right) \backslash B^{\bar{x}}\right|=\left|A_{j} \backslash B^{\bar{x}}\right|$. Let us now examine $\mathbf{c c}\left(\mathrm{r}_{j_{\mid B^{\bar{x}}}}\right)=\mathbf{c c}\left(\mathrm{r}_{j}\right)_{\mid B^{\bar{x}}}$. By Definition 11, every block $X \in \mathbf{c c}\left(\mathrm{r}_{j_{\mid B^{\bar{x}}}}\right)$ is obtained as $X=X^{\prime} \cap B^{\bar{x}}$ for some block $X^{\prime}$ of $\mathbf{c c}\left(\mathrm{r}_{j}\right)$. Since R is connected, $X^{\prime}=C \cap B$ for some connected component $C$ of $G_{j}=G\left[V_{j}\right]$, and thereby $X=C \cap B^{\bar{x}}$. The assumption that $\mathbf{G}^{\bar{x}}$ is connected implies that if $X=\emptyset$, then $G_{j}$ is connected (that is $C=V_{j}$ ) and $B^{\bar{x}} \cap V_{j}=\emptyset$ (that is $B=\{x\}$ ). This implies that $\mathbf{c c}\left(\mathrm{r}_{j \mid B^{\bar{x}}}\right)$ is a partition and fulfills the requirements of Definition 13. It follows that $\mathrm{R}_{\mid B^{\bar{x}}}$ is indeed the $\left(\mathbf{G}^{\bar{x}}, \mathbf{P}\right)$-encoding sequence and we can thereby set $\mathrm{T}=\mathrm{R}_{\mid B^{\bar{x}}}$.

Since $\mathbf{D}_{w}(\mathbf{G})$ is a domination set of $\operatorname{Rep}_{w}(\mathbf{G})$, there exists a $B$-boundaried sequence $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$ such that $\mathrm{S} \preceq \operatorname{rep}(\mathrm{R})$. As model $(\mathrm{R})=\operatorname{model}(\mathrm{S})$, by Lemma 15 we can conclude that $\mathrm{S}_{\mid B^{\bar{x}}} \preceq \mathrm{R}_{\mid B^{\bar{x}}}=\mathrm{T}$. Lemma 11(3) allows to conclude that $\operatorname{rep}\left(\mathrm{S}_{\mid B^{\bar{x}}}\right) \preceq \operatorname{rep}(\mathrm{T})$.

Proposition 3 (Forget soundness). Let $\mathbf{G}=(G, B)$ be a boundaried graph and $x \in B$ be a boundary vertex. If $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$ and $\mathrm{S}_{\mid B^{\bar{x}}}$ is connected, then $\operatorname{rep}\left(\mathrm{S}_{\mid B^{\bar{x}}}\right) \in \operatorname{Rep}_{w}\left(\mathbf{G}^{\bar{x}}\right)$.

Proof. As $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G}) \subseteq \operatorname{Rep}_{w}(\mathbf{G})$, there exists a connected path-decomposition $\mathbf{P}$ of $\mathbf{G}$ of width at most $w$ such that the ( $\mathbf{G}, \mathrm{P}$ )-encoding sequence $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{p}\right\rangle$ satisfies $\mathrm{S}=\operatorname{rep}(\mathbf{T})$. Since $\operatorname{model}(\mathrm{S})=\operatorname{model}(\mathrm{T})$, the hypothesis that $\mathrm{S}_{\mid B^{\bar{x}}}$ is connected implies that $\mathrm{T}_{\mid B^{\bar{x}}}$ is also connected. It follows that P is also a connected path-decomposition of $\mathbf{G}^{\bar{x}}$. One can check that $\mathrm{T}_{\mid B^{\bar{x}}}$ is the $\left(\mathbf{G}^{\bar{x}}, \mathbf{P}\right)$-encoding sequence (for this, one may just copy the corresponding argument of Proposition 2). As $S=\operatorname{rep}(T)$, we have that $S \equiv \mathrm{~T}$ by Lemma $11(1)$ and then $\operatorname{model}(\mathrm{S})=\operatorname{model}(\mathrm{T})$. Then, Lemma 15 implies that $\mathrm{S}_{\mid B^{\bar{x}}} \equiv \mathrm{~T}_{\mid B^{\bar{x}}}$ and so rep $\left(\mathrm{S}_{\mid B^{\bar{x}}}\right)=\operatorname{rep}\left(\mathrm{T}_{\mid B^{\bar{x}}}\right)$ by Lemma 11 and the fact that the representative is uniquely defined. Finally, as $\mathbf{S}$ has width at most $w$ (it belongs to $\mathbf{D}_{w}(\mathbf{G})$ ), by Lemma $12, \mathrm{~S}_{\mid B^{\bar{x}}}$ has width at most $w$ as well. It follows that $\operatorname{rep}\left(\mathrm{S}_{\mid B^{\bar{x}}}\right) \in \boldsymbol{\operatorname { R e p }}_{w}\left(\mathbf{G}^{\bar{x}}\right)$.

Theorem 2. Algorithm 1 computes $\mathbf{D}_{w}\left(\mathbf{G}^{\bar{x}}\right)$ in $2^{O(k(w+\log k))}$-time, where $k=|B|$.
Proof. The correctness of Algorithm 1 is proved by Proposition 2 and Proposition 3. These two propositions imply that by applying Forget Routine on a domination set of $\mathbf{G}$ included in the set of representatives of $\mathbf{G}$, we indeed compute a domination set of $\mathbf{G}^{\bar{x}}$ that is a subset of the set of representatives of $\mathbf{G}^{\bar{x}}$. As performing the projection of $B$-boundaried sequence onto $B^{\bar{x}}$ can be performed in polynomial time in the size of the sequence, the complexity of the algorithm is dominated by the size of $\mathbf{D}_{w}(\mathbf{G})$ that is $2^{O(k(w+\log k))}$, because of Lemma 9.

### 4.3 Insertion Routine

In this subsection, we present the Insertion Routine. Suppose that $\mathbf{G}=(G, B)$ is a boundaried graph with $G=(V, E)$. For a subset $X \subseteq B$, we set $G^{x}=(V \cup\{x\}, E \cup\{x y \mid y \in X\})$ and $\mathbf{G}^{x}=\left(G^{x}, B^{x}\right)$ where $B^{x}=B \cup\{x\}$. Given a domination set $\mathbf{D}_{w}(\mathbf{G})$ of $\operatorname{Rep}_{w}(\mathbf{G})$, the task of Insertion Routine is to compute a domination set $\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)$ of $\operatorname{Rep}_{w}\left(\mathbf{G}^{x}\right)$. Algorithm 2 is describing Insertion Routine.

```
Algorithm 2: Insertion Routine
    Input: A boundaried graph \(\mathbf{G}=(G, B)\), a subset \(X \subset B\), and \(\mathbf{D}_{w}(\mathbf{G})\).
    Output: \(\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)\), a domination set of \(\boldsymbol{\operatorname { R e p }}_{w}\left(\mathbf{G}^{x}\right)\).
    \(\mathbf{D}_{w}\left(\mathbf{G}^{x}\right) \leftarrow \emptyset ;\)
    foreach \(\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle \in \mathbf{D}_{w}(\mathbf{G})\) do
        foreach \(f, l \in[\ell]\) such that \(X \subseteq \bigcup_{f \leq j \leq l} \mathbf{b d}\left(\mathbf{s}_{j}\right)\) do
            foreach \((\leq 2)\)-extension \(\mathrm{S}^{\prime}\) of S duplicating none, one or both of \(\mathrm{s}_{f}\) and \(\mathrm{s}_{l}\) do
                let \(\ell^{\prime}\) be the length of \(S^{\prime}\);
                set \(f_{x}=\max \left\{j \in\left[\ell^{\prime}\right] \mid \delta_{\mathrm{s}^{\prime} \rightarrow \mathrm{s}}(j)=f\right\}\) and \(l_{x}=\min \left\{j \in\left[\ell^{\prime}\right] \mid \delta_{\mathrm{s}^{\prime} \rightarrow \mathrm{s}}(j)=l\right\}\);
                set \(\mathrm{S}^{x}=\operatorname{Ins}\left(\mathrm{S}^{\prime}, x, X, f_{x}, l_{x}\right)\);
                (observe that by construction \(\left(f_{x}, l_{x}\right)\) is valid with respect to \(X\) in \(\mathrm{S}^{\prime}\) );
                if width \(\left(\mathbf{S}^{x}\right) \leq w\), then add \(\operatorname{rep}\left(\mathbf{S}^{x}\right)\) to \(\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)\);
            end
        end
    end
    return \(\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)\).
```

To prove the correctness of Insertion Routine, we proceed in two steps. We first establish the completeness of the algorithm. More precisely, Proposition 4 aims at proving that for every connected path-decomposition $\mathrm{P}^{x}$ of $\mathbf{G}^{x}$, the $\left(\mathbf{G}^{x}, \mathrm{P}^{x}\right)$-encoding sequence $T^{x}$ is dominated by some $B^{x}$-boundaried sequence $\mathrm{S}^{x}$ that can be computed from a $B$-boundaried sequence S belonging to $\mathbf{D}_{w}(\mathbf{G})$. Then we argue about the soundness of Insertion Routine. Proposition 5 shows that if $\mathbf{S}^{x}$ is generated from a $B$-boundaried sequence $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$, then rep $\left(\mathrm{S}^{x}\right)$ belongs to $\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)$.

Proposition 4 (Insertion completeness). Let $\mathbf{G}=(G, B)$ be a boundaried graph and let $X \subseteq B$ be a subset of boundary vertices. Let $\mathrm{P}^{x}$ be a connected path-decomposition of width at most $w$ of the boundaried graph $\mathbf{G}^{x}=\left(G^{x}, B^{x}\right)$ and let $\mathbf{T}^{x}$ be the $\left(\mathbf{G}^{x}, \mathrm{P}^{x}\right)$-encoding sequence. Then there exist a $B$ boundaried sequence $\mathrm{S}^{\prime}$ such that $\mathrm{S}^{\prime}$ is a $(\leq 2)$-extension of some $B$-boundaried sequence $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$ and an insertion position $\left(f_{x}, l_{x}\right)$ valid with respect to $X$ in $\mathrm{S}^{\prime}$ such that the $B^{x}$-boundaried sequence $\mathrm{S}^{x}=\operatorname{Ins}\left(\mathrm{S}^{\prime}, x, X, f_{x}, l_{x}\right)$ satisfies $\operatorname{rep}\left(\mathrm{S}^{x}\right) \preceq \operatorname{rep}\left(\mathrm{T}^{x}\right)$.

Proof. Suppose that $\mathrm{P}^{x}=\left\langle A_{1}^{x}, \ldots, A_{\ell}^{x}\right\rangle$ and that $\mathrm{T}^{x}=\left\langle\mathrm{t}_{1}^{x}, \ldots, \mathrm{t}_{p}^{x}\right\rangle$. Let $\left[f_{x}^{*}, l_{x}^{*}\right]$ be the trace of $x$ in $\mathrm{P}^{x}$. By the definition of a path-decomposition and of an encoding sequence, $X \subseteq \bigcup_{f_{x} \leq j \leq l_{x}} \mathbf{b d}\left(\mathrm{t}_{j}^{x}\right)$. By Lemma $1, \mathrm{P}=\left\langle A_{1}, \ldots, A_{\ell}\right\rangle$, with $A_{i}=A_{i}^{x} \backslash\{x\}$ for every $1 \leq i \leq \ell$, is a connected pathdecomposition of $\mathbf{G}$. Let $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{\ell}\right\rangle$ be the $(\mathbf{G}, \mathrm{P})$-encoding sequence. Observe that by the construction of $\mathbf{G}^{x}$, if $y \in X$, then $y \in A_{j}$ for some $f_{x}^{*} \leq j \leq l_{x}^{*}$. As by assumption, $X \subseteq B$, we have
that $y \in \mathbf{b d}\left(\mathrm{t}_{j}\right)$. Therefore, $\left(f_{x}^{*}, l_{x}^{*}\right)$ is a valid insertion position with respect to $X$ in T . One can easily check that $\operatorname{Ins}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)=\mathrm{T}^{x}$. Observe that, as the width of $\mathrm{T}^{x}$ is at most $w$, the width of T is at most $w$ as well, because of Lemma 16. Since $\mathbf{D}_{w}(\mathbf{G})$ is a domination set of $\boldsymbol{\operatorname { R e p }} \mathbf{p}_{w}(\mathbf{G})$, there exists a $B$-boundaried sequence $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G})$ such that $\mathrm{S} \preceq \mathrm{T}$. By Lemma 20, there exists a $(\leq 2)$-extension $\mathrm{S}^{\prime}$ of S and a valid insertion position $\left(f_{x}, l_{x}\right)$ with respect to $X$ in $\mathrm{S}^{\prime}$ such that $\operatorname{Ins}\left(\mathrm{S}^{\prime}, x, X, f_{x}, l_{x}\right) \preceq \operatorname{lns}\left(\mathrm{T}, x, X, f_{x}^{*}, l_{x}^{*}\right)$. By Lemma 11(3), we have rep $\left(\mathrm{S}^{x}\right) \preceq \operatorname{rep}\left(\mathrm{T}^{x}\right)$.

We let the reader observe that the completeness of Insertion Routine relies on Lemma 20 and thereby on Lemma 18. And the reason we compute a domination set of $\operatorname{Rep}_{w}\left(\mathbf{G}^{x}\right)$ rather than the set $\boldsymbol{\operatorname { R e p }}_{w}\left(\mathbf{G}^{x}\right)$, is the issue discussed in Figure 6.

Proposition 5 (Insertion soundness). Let $\mathbf{G}=(G, B)$ be a boundaried graph and let $X \subseteq B$ be $a$ subset of boundary vertices. If $\mathrm{S}^{\prime}=\left\langle\mathrm{s}_{1}^{\prime}, \ldots, \mathrm{s}_{\ell^{\prime}}^{\prime}\right\rangle$ is a $(\leq 2)$-extension of a $B$-boundaried sequence $\mathrm{S}=\left\langle\mathrm{s}_{1}, \ldots, \mathrm{~s}_{\ell}\right\rangle \in \mathbf{D}_{w}(\mathbf{G})$ and if $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in $\mathrm{S}^{\prime}$ such that $\mathrm{S}^{x}=\operatorname{lns}\left(\mathrm{S}^{\prime}, x, X, f_{x}, l_{x}\right)$ has width at most $w$, then $\operatorname{rep}\left(\mathrm{S}^{x}\right) \in \operatorname{Rep}_{w}\left(\mathbf{G}^{\bar{x}}\right)$.

Proof. As $\mathrm{S} \in \mathbf{D}_{w}(\mathbf{G}) \subseteq \operatorname{Rep}_{w}(\mathbf{G})$, there exists a connected path- decomposition P of $\mathbf{G}$ of width at most $w$ such that the $(\mathbf{G}, \mathrm{P})$-encoding sequence $\mathrm{T}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{p}\right\rangle$ satisfies rep $(\mathrm{T})=\mathrm{S}$. Let $\delta_{S^{\prime} \rightarrow S}:\left[\ell^{\prime}\right] \rightarrow[\ell]$ be the extension surjection certifying that $\mathrm{S}^{\prime}$ is a $(\leq 2)$-extension of S . Let us denote $f=\delta_{\mathrm{S}^{\prime} \rightarrow \mathrm{S}}\left(f_{x}\right)$ and $l=\delta_{\mathrm{S}^{\prime} \rightarrow \mathrm{S}}\left(l_{x}\right)$. As $\mathrm{S}=\operatorname{rep}(\mathrm{T})$, with every $j \in[\ell]$, we can associate a $\iota_{j} \in[p]$ such that S is the subsequence of T induced by $\mathbf{b p}(\mathrm{T})=\left\{\iota_{j} \in[p] \mid j \in[\ell]\right\}$. We build a ( $\leq 2$ )-extension $\mathrm{T}^{\prime}=\left\langle\mathrm{t}_{1}, \ldots, \mathrm{t}_{p^{\prime}}\right\rangle$ of T , in the same way as $\mathrm{S}^{\prime}$ is obtained from S , that is: we duplicate $\mathrm{t}_{i_{f}}$ if and only if $s_{f}$ is duplicated, and we duplicate $\mathrm{t}_{i_{j}}$ if and only if $\mathrm{s}_{l}$ is duplicated. Observe that $\mathrm{S}^{\prime}$ is the subsequence of $\mathrm{T}^{\prime}$ induced by $\left\{i_{j} \in\left[p^{\prime}\right] \mid j \in\left[\ell^{\prime}\right]\right\}$ (see Figure 8). By construction of $\mathrm{T}^{\prime},\left(i_{f_{x}}, i_{l_{x}}\right)$ is a valid insertion position with respect to $X$ in $\mathrm{T}^{\prime}$. Thereby, we can define $\mathrm{T}^{x}=\operatorname{lns}\left(\mathrm{T}^{\prime}, x, X, i_{f_{x}}, i_{l_{x}}\right)$ and $\mathrm{S}^{x}=\operatorname{Ins}\left(\mathrm{S}^{\prime}, x, X, f_{x}, l_{x}\right)$. Let $\mathrm{P}^{\prime}$ be the connected path-decomposition obtained from P by duplicating the bags corresponding to $\mathrm{t}_{l_{f}}$ and $\mathrm{t}_{l_{l}}$ and adding $x$ to all bags between the bags associated with $\mathrm{t}_{i_{f_{x}}}^{\prime}$ and $\mathrm{t}_{i_{l_{x}}}^{\prime}$. We remark that $\mathrm{T}^{x}$ is the $\left(\mathbf{G}^{x}, \mathrm{P}^{\prime}\right)$-encoding sequence and is thereby realizable.

We claim now that $\operatorname{rep}\left(S^{x}\right)=\operatorname{rep}\left(T^{x}\right)$. Because $S=\operatorname{rep}(T)$, one can prove, in the same way as the second statement of Lemma $5(6)$, that there are $S_{1}$ and $S_{2}$, extensions of S , such that $\mathrm{S}_{1} \leq \mathrm{T} \leq \mathrm{S}_{2}, \delta_{\mathrm{S}_{1} \rightarrow \mathrm{~S}}\left(i_{j}\right)=\delta_{\mathrm{S}_{2} \rightarrow \mathrm{~S}}\left(i_{j}\right)=j \in[\ell]$, and $i_{f_{x}}=\min \left\{h \in[p] \mid f=\delta_{\mathrm{S}_{1} \rightarrow \mathrm{~S}}(h)=\delta_{\mathrm{S}_{2} \rightarrow \mathrm{~S}}(h)\right\}$ and $i_{l_{x}}=\max \left\{h \in[p] \mid l=\delta_{\mathrm{S}_{1} \rightarrow \mathrm{~S}}(h)=\delta_{\mathrm{S}_{2} \rightarrow \mathrm{~S}}(h)\right\}$. By making the same duplications in $\mathrm{S}^{\prime}$ as in S to obtain $\mathrm{S}_{1}$ and $\mathrm{S}_{2}$, one can construct extensions $\mathrm{S}_{1}^{\prime}$ and $\mathrm{S}_{2}^{\prime}$ of $\mathrm{S}^{\prime}$ such that $\mathrm{S}_{1}^{\prime} \leq \mathrm{T}^{\prime} \leq \mathrm{S}_{2}^{\prime}, \delta_{\mathrm{S}_{1}^{\prime} \rightarrow \mathrm{S}^{\prime}}\left(i_{j}\right)=$ $\delta_{\mathrm{S}_{2}^{\prime} \rightarrow \mathrm{S}^{\prime}}\left(i_{j}\right)=j \in\left[\ell^{\prime}\right]$, and $i_{f_{x}}=\min \left\{h \in\left[p^{\prime}\right] \mid f_{x}=\delta_{\mathrm{S}_{1}^{\prime} \rightarrow \mathbf{S}^{\prime}}(h)=\delta_{\mathrm{S}_{2}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)\right\}$ and $i_{l_{x}}=\max \left\{h \in\left[p^{\prime}\right] \mid\right.$ $\left.l_{x}=\delta_{\mathrm{S}_{1}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)=\delta_{\mathrm{S}_{2}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)\right\}$. Therefore, $\left(i_{f_{x}}, i_{l_{x}}\right)$ is a valid insertion position with respect to $X$ in both $\mathrm{S}_{1}^{\prime}$ and $\mathrm{S}_{2}^{\prime}$. By Lemma 19, we have $\operatorname{lns}\left(\mathrm{S}_{1}^{\prime}, x, X, i_{f_{x}}, i_{l_{x}}\right) \leq \mathrm{T}^{x} \leq \operatorname{lns}\left(\mathrm{S}_{2}^{\prime}, x, X, i_{f_{x}}, i_{l_{x}}\right)$. Because $\mathrm{S}_{1}^{\prime}$ and $\mathrm{S}_{2}^{\prime}$ are both extensions of $\mathrm{S}^{\prime}, i_{f_{x}}=\min \left\{h \in\left[p^{\prime}\right] \mid f_{x}=\delta_{\mathrm{S}_{1}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)=\delta_{\mathrm{S}_{2}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)\right\}$, and $i_{l_{x}}=$ $\left.\max \left\{h \in\left[p^{\prime}\right] \mid l_{x}=\delta_{\mathrm{S}_{1}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)\right\}=\delta_{\mathrm{S}_{2}^{\prime} \rightarrow \mathrm{S}^{\prime}}(h)\right\}$, we can conclude by Lemma 17 that $\operatorname{Ins}\left(\mathrm{S}_{1}^{\prime}, x, X, i_{f_{x}}, i_{l_{x}}\right)$ and $\operatorname{Ins}\left(\mathrm{S}_{2}^{\prime}, x, X, i_{f_{x}}, i_{l_{x}}\right)$ are both extensions of $\mathrm{S}^{x}$. We can therefore conclude that $\mathrm{S}^{x} \equiv \mathrm{~T}^{x}$, i.e., $\operatorname{rep}\left(\mathbf{S}^{x}\right)=\operatorname{rep}\left(\mathbf{T}^{x}\right)$. Finally, as $\mathbf{T}^{x}$ is realisable, we can conclude that $\operatorname{rep}\left(\mathbf{S}^{x}\right) \in \boldsymbol{\operatorname { R e p }} \mathbf{p}_{w}\left(\mathbf{G}^{\bar{x}}\right)$.

Theorem 3. Algorithm 2 computes $\mathbf{D}_{w}\left(\mathbf{G}^{x}\right)$ in $2^{O(k(w+\log k))}$-time, where $k=|B|$.


Figure 8: Soundness of the insertion routine: if $S^{\prime}=\left\langle\mathbf{s}_{1}^{\prime}, \ldots, \mathbf{s}_{\ell^{\prime}}^{\prime}\right\rangle$ is a $(\leq 2)$-extension of a $B$ boundaried sequence $\mathbf{S}=\operatorname{rep}(\mathbf{T}) \in \mathbf{D}_{w}(\mathbf{G})$ and $\left(f_{x}, l_{x}\right)$ is a valid insertion position with respect to $X$ in $\mathrm{S}^{\prime}$, then $\operatorname{rep}\left(\mathrm{S}^{x}\right) \in \mathbf{D}_{w}\left(\mathbf{G}^{x}\right)$.

Proof. The correctness of Algorithm 2 is proved by Proposition 4 and Proposition 5. These two propositions imply that by applying Insertion Routine on a domination set of $\mathbf{G}$ that is a subset of the representatives of $\mathbf{G}$, we indeed compute a domination set of $\mathbf{G}^{x}$ that is a subset of the set of representatives of $\mathbf{G}^{x}$. Let us analyse its time complexity. By Lemma 9, the size of $\boldsymbol{R e p}_{w}(\mathbf{G})$ (and so the size of $\mathbf{D}_{w}(\mathbf{G})$ ) depends on $k$ and $w$. By Lemma 7, the length of a representative $B$-boundaried sequence of $\operatorname{Rep}_{w}(\mathbf{G})$ depends on $k$. As performing the insertion in a $B$-boundaried sequence can be performed in polynomial time in the size of the sequence, the time complexity of Algorithm 2 is dominated by the size of $\mathbf{D}_{w}(\mathbf{G})$ that is $2^{O(k(w+\log k))}$, because of Lemma 9 .

### 4.4 The dynamic programming algorithm

We are now in position to prove Theorem 1. We first explain an algorithm that decides whether $\operatorname{cpw}(G) \leq w$. Suppose that we are given a path-decompositon $\mathrm{Q}=\left\langle B_{1}, \ldots, B_{q}\right\rangle$ of $G$ of width at most $k$. Our algorithm performs dynamic programming over $\mathbf{Q}$. For each $i \in[q]$, we consider the boundaried graph $\mathbf{G}_{i}=\left(G\left[V_{i}\right], B_{i}\right)$, where $V_{i}=\bigcup_{1 \leq h \leq i} B_{h}$. The task is to compute for every $i \in[q]$, a domination set $\mathbf{D}_{w+1}\left(\mathbf{G}_{i}\right)$. Let us describe $\mathbf{D}_{w+1}\left(\mathbf{G}_{1}\right)$. As $\mathbf{Q}$ is a nice path-decomposition, $B_{1}=\{x\}$ for some $x \in V$. The representative set $\operatorname{Rep}_{w+1}\left(\mathbf{G}_{1}\right)$ consists for the following four possible connected $B_{1}$-boundaried sequences:

- $\mathrm{S}_{1}=\langle(\{x\},\{\{x\}\}, 0)\rangle$,
- $\mathrm{S}_{2}=\langle(\emptyset,\{\emptyset\}, 0),(\{x\},\{\{x\}\}, 0)\rangle$,
- $\mathrm{S}_{3}=\langle(\emptyset,\{\emptyset\}, 0),(\{x\},\{\{x\}\}, 0),(\emptyset,\{\{x\}\}, 0)\rangle$, and
- $\mathrm{S}_{4}=\langle(\{x\},\{\{x\}\}, 0),(\emptyset,\{\{x\}\}, 0)\rangle$.

We use $\operatorname{Rep}_{w+1}\left(\mathbf{G}_{1}\right)$, as $\mathbf{D}_{w+1}\left(\mathbf{G}_{1}\right)$ as none of the above sequence is dominating the other. Now Algorithm 2 and Algorithm 1 describe how to compute for every $1<i \leq q, \mathbf{D}_{w+1}\left(\mathbf{G}_{i}\right)$ depending
on whether $B_{i}$ is an insertion or a forgetting bag. We obtain that $\operatorname{cpw}(G) \leq w$ if and only if $\mathbf{D}_{w+1}\left(\mathbf{G}_{q}\right) \neq \emptyset$, because of Proposition 1. The correctness of the DP algorithm described above follows from Theorem 2, Theorem 3. The time complexity depends on the running time of Insertion Routine (Algorithm 2) and Forget Routine (Algorithm 1) described respectively in Theorem 2 and Theorem 3. We just proved the decision version of Theorem 1. In [12, Section 6] Bodlaender and Kloks explained how to turn their decision algorithm for pathwidth and treewidth to one that is able to construct, in case of a positive answer, the corresponding decomposition. Following the same arguments, it is straightforward to transform the above decision algorithm for connected pathwidth to one that also constructs the connected path-decomposition, if it exists. This completes the proof of Theorem 1.

Theorem 4. One may construct an algorithm that, given an $n$-connected graph $G$ and a nonnegative integer $k$, either outputs a connected path-decomposition of $G$ of width at most $k$ or correctly reports that such a decomposition does not exist in $2^{O\left(k^{2}\right)} \cdot n$ time.

Proof. According to the result of Fürer [24] there is an algorithm that, given a graph $G$ and an integer $k$, outputs, if exists, a path-decomposition of width at most $k$ in $2^{O\left(k^{2}\right)} \cdot n$ time. We run this algorithm and if the answer is negative, we report that $\operatorname{cpw}(G)>k$ and we are done (here we use Observation 1). Otherwise we use the provided path-decomposition in order to solve the problem in $2^{O(w(k+\log w))} \cdot n$ time using the algorithm of Theorem 1 where $w \leq k$ is the width of the constructed path-decomposition in the first step.

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[^0]:    ${ }^{1}$ An extended abstract of this paper appeared in the proceedings of Annual European Symposium on Algorithms (ESA) [29]
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[^1]:    ${ }^{1}$ Or, alternatively, to a caterpillar, as aptly remarked in [43].
    ${ }^{2} \mathrm{~A}$ graph $H$ is a minor of a graph $G$ if $H$ can be obtained by some subgraph of $G$ by contracting edges.
    ${ }^{3}$ An equivalent setting of graph searching is to see $G$ as a system of pipelines or corridors that is contaminated by some poisonous gas or some highly infectious substance. The searchers can be seen as cleaners that deploy a

[^2]:    ${ }^{5}$ For instance, the graph $G \backslash\{x, y\}$ from Figure 1 belongs in $\mathcal{Z}_{2}$.

[^3]:    ${ }^{6}$ An equivalent model was proposed independently by Petrov [40]. The models of Parsons and Petrov where different but also equivalent, as proved by Golovach in [25]. The model of Parsons was inspired by an earlier paper by Breisch [16], titled "An intuitive approach to speleotopology", where the aim was to rescue an (unlucky) speleologist lost in a system of caves. Notice that "unluckiness" cancels the speleologist's will of being rescued as, from the searchers' point of view, it imposes on him/her the status of an "evading entity". As a matter of fact, the connectivity issue appears even in the first inspiring model of the search game. In a more realistic scenario, the searchers cannot "teleport" themselves to non-adjacent territories of the caves while this was indeed permitted in the original setting of Parsons.

