

# INERTIAL MANIFOLDS VIA SPATIAL AVERAGING REVISITED

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**ABSTRACT.** The paper gives a comprehensive study of inertial manifolds for semilinear parabolic equations and their smoothness using the spatial averaging method suggested by G. Sell and J. Mallet-Paret. We present a universal approach which covers the most part of known results obtained via this method as well as gives a number of new ones. Among our applications are reaction-diffusion equations, various types of generalized Cahn-Hilliard equations, including fractional and 6th order Cahn-Hilliard equations and several classes of modified Navier-Stokes equations including the Leray- $\alpha$  regularization, hyperviscous regularization and their combinations. All of the results are obtained in 3D case with periodic boundary conditions.

## CONTENTS

1. Introduction	1
2. Preliminaries and an abstract model	6
3. Inertial Manifolds and cone property	10
4. Verification of the cone property via spatial averaging	13
5. The truncation procedure	16
6. Spatial averaging in the case of periodic boundary conditions	23
7. Applications	27
7.1. Scalar Reaction-Diffusion equation	27
7.2. Cahn-Hilliard type equations	28
7.3. Modified Navier-Stokes equations	30
References	32

## 1. INTRODUCTION

It is believed that in many cases the longtime behavior of trajectories of a dissipative system, say, generated by a partial differential equation (PDE) is essentially finite-dimensional. In other words, despite of the infinite-dimensionality of the initial phase space, the generated dynamics is governed, up to some "non-essential" transient effects, by finitely many parameters, the so-called *order* parameters in the terminology of I. Prigogine, see [41]. Ideally, it is expected that these order parameters obey a system of ordinary differential equations (ODEs) which is called an inertial form (IF) of the initial dissipative system. Thus, the IF if it exists allows us to reduce the study of the dynamics generated by PDEs to the study of the corresponding system of ODEs which in turn can be done using the methods of classical dynamics. In particular, the dream to understand the nature of turbulence using the ideas and methods of classical dynamics permanently inspires the development of the dynamical theory of dissipative systems during the

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last 50 years, see [3, 7, 13, 16, 15, 33, 43, 50, 51] and references therein. We only mention here that the key concepts of the theory like *inertial* form or *inertial* manifold were initially related with the so-called *inertial* scale in the theory of turbulence and the corresponding *inertial* term in Navier-Stokes equations.

However, despite a lot of progress done by prominent researches, the nature of the above mentioned finite-dimensional reduction and its rigorous justification somehow remains a mystery. Moreover, as recent examples and counterexamples show, there are deep obstacles to effective realization of this program, e.g., related with the *smoothness* of the IF and related finite-dimensional reduction, see [53] and references therein.

Indeed, the most popular way to justify this finite-dimensional reduction is related with the theory of attractors. By definition, a global attractor of a dynamical system (DS) is a compact invariant set in the phase space which attracts the images of bounded sets as time tends to infinity. The main achievement of the attractors theory is that a global attractor  $\mathcal{A}$  exists under rather weak assumptions on a dissipative system considered and in many cases has finite Hausdorff and box-counting dimensions, see [3, 7, 43, 51, 39] and references therein. The class of such systems includes reaction-diffusion and 2D Navier-Stokes systems, pattern formation equations (like Cahn-Hilliard or Swift-Hohenberg ones), damped wave equations and many others. This result in turn allows us to build up the desired finite-dimensional reduction as well as the IF using the Mané projection theorem, see [44] and references therein. In this approach the box-counting dimension of the attractor  $\mathcal{A}$  is usually interpreted as a number of "degrees of freedom" in the reduced IF. In particular, this explains the permanent interest to various upper and lower bounds for the box-counting dimension of  $\mathcal{A}$ .

On the other hand, the obtained in such a way IF is only Hölder continuous and it can be not even Lipschitz continuous in general. In a fact, there are natural examples where the box-counting dimension of the attractor is low (e.g, 3), but a Lipschitz IF does not exist. Moreover, the reduced dynamics on the attractor contains features which can hardly be interpreted as "finite-dimensional" (like limit cycles with super-exponential rate of attraction, traveling waves in Fourier space, etc.), see [11, 35, 46, 26, 53] for more details. In these cases, the "finite-dimensionality" obtained via Mané projections looks artificial and controversial and it seems more natural to accept that the dynamics here is infinite-dimensional despite the finiteness of box-counting dimension.

The above mentioned problems motivate an increasing interest to alternative methods of constructing IFs, not related with box-counting dimension and Mané projection theorem. One of the most natural alternative approaches is based on the concept of an *inertial* manifold (IM) suggested in [18]. Roughly speaking, an IM  $\mathcal{M}$  is a sufficiently smooth (at least Lipschitz) finite-dimensional invariant submanifold of the phase space which is normally-hyperbolic and exponentially stable. If such an object exists, then the finite-dimensional reduction is ideally justified. Indeed, the reduction of the initial PDE to the manifold  $\mathcal{M}$  gives us the desired IF and the normal hyperbolicity gives us the so-called asymptotic phase or exponential tracking property which in turn gives us a nice rigorous interpretation in what sense the transient features are "non-essential".

However, being a sort of a center manifold, an IM requires strong separation of the phase space on slow and fast variables which is usually stated in the form of *spectral gap* conditions or/and invariant cone properties, see [8, 10, 18, 17, 14, 37] and references therein for more details. In particular, for the simplest model of a semilinear parabolic equation in a real Hilbert space  $H$ :

$$\partial_t u + Au + F(u) = 0, \tag{1.1}$$

where  $A : D(A) \rightarrow H$  is a positive self-adjoint operator with compact inverse and  $F : H \rightarrow H$  is a globally Lipschitz map with Lipschitz constant  $L$ , the spectral gap conditions for existence

of  $N$ -dimensional IM read:

$$\lambda_{N+1} - \lambda_N > 2L, \quad (1.2)$$

where  $\{\lambda_n\}_{n=1}^{\infty}$  are the eigenvalues of the operator  $A$  enumerated in a non-decreasing order. In the present paper we are mainly interested in a more complicated version of the abstract parabolic problem, namely,

$$\partial_t A^{-\gamma} u + Au + F(u) = 0, \quad (1.3)$$

where  $\gamma \geq 0$  and  $A$  and  $F$  are the same as in (1.1). The spectral gap conditions for this equation read:

$$\frac{\lambda_{N+1}^{1+\gamma} - \lambda_N^{1+\gamma}}{\lambda_{N+1}^{\gamma} + \lambda_N^{\gamma}} > L. \quad (1.4)$$

It is known that these spectral gap conditions are sharp in the sense that if they are not satisfied one always can construct a nonlinearity  $F$  for which the corresponding IM will not exist, see [11, 38, 45, 46, 53] for more details. Thus, there is no hope to push forward the theory beyond the spectral gap conditions at least on the level of general abstract nonlinearities. However, this is possible for some partial classes of operators  $A$  and nonlinearities  $F$  (see [34] and [25] for 3D reaction-diffusion and Cahn-Hilliard equations with periodic boundary conditions, [26, 27] for 1D reaction-diffusion-advection problems, [23, 32, 19] for modified Navier-Stokes equation and [24] for the complex Ginzburg-Landau equation).

In the present paper, we are mainly interested in the so-called spatial averaging method which has been introduced in [34] in order to verify the existence of an IM for 3D scalar reaction-diffusion equation with periodic boundary conditions, see also [53] for more recent exposition of the theory and [29] for slightly different boundary conditions. Roughly speaking, the method works in the case where the derivative  $F'(u)$  contains *point-wise multiplication* and utilizes some special features of such multiplication operators which comes from harmonic analysis and number theory, see [34, 53] and Section 6 below for more details. These features allow us to replace in the analysis the multiplication on a function by the *scalar* operator of multiplication on its *spatial average* (which explains the name of the method). This trick essentially simplifies the analysis and allows us to go beyond of spectral gap conditions at least in the case of 3D problems with periodic boundary conditions. Note also that in general this method does not work for systems since we will have not a scalar operator, but matrix operator instead and this is not enough for IMs, so some further steps are necessary, see [24] for the case of complex Ginzburg-Landau equation where the spatial averaging is combined with the temporal one in order to get finally the scalar operator. But there is an important exception pointed out in [23], namely, the case of zero spatial averaging which is typical for the Navier-Stokes type nonlinearities and which allowed to treat the modified Navier-Stokes equations using the spatial averaging method, see also [19, 32].

The aim of the present paper is to give a systematic study of IMs via the spatial averaging based on the universal model (1.3) which allows to treat most part of known applications of spatial averaging technique as well as to get new ones from the unified point of view. Among the considered applications are classical reaction-diffusion equations, various types of Cahn-Hilliard (CH) equations, including the so-called fractional CH, 6th order CH, etc., and various modifications of Navier-Stokes equations including the Bardina model and Leray  $\alpha$ -model, hyperviscous Navier-Stokes and their combinations. The paper is organized as follows.

In Section 2 we discuss the analytic properties (such as existence and uniqueness of solutions, their regularity and various versions of a parabolic smoothing property) of solutions of problem (1.3) with globally Lipschitz nonlinearity  $F$ . These properties will be used throughout of the paper.

In Section 3, we recall (following mainly [25] and [53]) the strong cone property in a differential form and general theorems about existence of an IM of regularity  $C^{1+\varepsilon}$  with  $\varepsilon > 0$  adapted to the case of equation 1.3.

Verification of the strong cone property based on an abstract version of spatial averaging introduced in [25] is given in Section 4. In particular, we present here the abstract theorems on the existence and  $C^{1+\varepsilon}$ -smoothness of an IM for equation (1.3) (also in the spirit of [25]).

We note that usually most part of equations interesting from the applied point of view do not have nonlinearities which are globally Lipschitz in  $H$ , so, in order to get an IM for such equations, one usually first verify the existence of a good absorbing/attracting set in the phase space and then *truncate* the nonlinearity outside of this absorbing set to end up with globally Lipschitz nonlinearity. This truncation procedure is usually simple in the case when the spectral gap conditions are satisfied, but may be very delicate in the case of spatial averaging since the truncation should not affect much the spatial averaging property for the nonlinearity. For instance, in the original paper [34] where the spatial averaging method has been suggested, the authors have to truncate not only the nonlinearity, but also to change in a very non-trivial way the leading part  $Au$  of the equation. Analogously, the applications of spatial averaging to Navier-Stokes equations become possible due to the special truncation function  $W(u)$  which truncates the Fourier modes of the solution  $u$ , suggested in [23], see Section 5 for more details.

In Section 5, we suggest a unified truncation procedure (which somehow combines the approaches developed in [34] and [23]) which allows us to deduce the spatial averaging property for the truncated nonlinearity directly from some natural properties (Assumptions I-III, see Section 5) of the initial non-truncated nonlinearity and the extra assumption that the initial non-truncated equation possesses an absorbing set in a "good" space.

In Section 6 we restrict ourselves to the case where  $A$  is the Laplacian in a 3D domain  $(-\pi, \pi)^3$  with periodic boundary conditions and verify the spatial averaging property for all classes of nonlinearities important for our applications as well as other of Assumptions I-III. Thus, to get the existence of  $C^{1+\varepsilon}$ -smooth IM, it only remains to verify the global well-posedness of the problem and the existence of an absorbing set in the proper "good" space.

This verification is finally done in Section 7. Namely, the application of our method to the classical scalar reaction-diffusion equation:

$$\partial_t u = \Delta_x u - u + f(u) + g, \quad u|_{t=0} = u_0 \quad (1.5)$$

in a 3D domain endowed with periodic boundary conditions is given in subsection 7.1. We assume that  $g \in H = L^2(\Omega)$  and  $f$  satisfies the assumptions

$$1. f \in C^4(\mathbb{R}, \mathbb{R}), \quad 2. f(u)u \geq -C, \quad 3. f'(u) \geq -K, \quad u \in \mathbb{R}. \quad (1.6)$$

This equation formally fits to equation (1.3) with  $\gamma = 0$  and the main result is that under assumptions (1.6) this equation possesses a  $C^{1+\varepsilon}$ -smooth IM, for instance, in the phase space  $H$ . In this case our approach gives nothing new in comparison with the standard results (and it is even a bit weaker since more accurate analysis shows that  $f$  may be taken to be  $C^2$ -smooth only), but it is nevertheless presented here in order to demonstrate that this classical result is covered by our unified scheme.

Subsection 7.2 is devoted to the generalizations of the Cahn-Hilliard equations, namely,

$$\partial_t u + (-\Delta_x)^\gamma (-\Delta_x u + f(u) + g) = 0, \quad u|_{t=0} = u_0, \quad \gamma > 0 \quad (1.7)$$

in a 3D domain  $(-\pi, \pi)^3$  with periodic boundary conditions. Due to the presence of the mass conservation law it is natural to consider this equation in the spaces of functions with zero mean

$$\langle u \rangle := \frac{1}{(2\pi)^3} \int_{(-\pi, \pi)^3} u(x) dx,$$

for instance  $H = \{u \in L^2((-\pi, \pi)^3), \langle u \rangle = 0\}$ . Then the Laplacian is positive definite and this equation indeed has the form of (1.3). The choice  $\gamma = 1$  corresponds to the classical Cahn-Hilliard equation considered in [25]. The choice  $\gamma \in (0, 1)$  gives the so-called fractional Cahn-Hilliard equation (see [1]) and  $\gamma = 2$  gives us the so-called 6th order Cahn-Hilliard equation, see [36] and references therein. To the best of our knowledge the questions related with IMs for the last two equations have been not considered in the literature.

As an application of our abstract scheme, we get the existence of  $C^{1+\varepsilon}$ -smooth IM for equation (1.7) for all  $\gamma > 0$ ,  $g \in H$  and  $f$  satisfying (1.6). We also note that the natural phase space for problem (1.7) as well as for our abstract model (1.3) is  $H^{-\gamma}$ . However, due to the smoothing property for differences of solutions verified in Section 2, the statements about the existence of IM are *equivalent* in all spaces between which this smoothing property holds. By many reasons, it is more convenient to verify the existence of an IM in the space  $H^{-\gamma}$  and then to extend it to all phase spaces  $H^s$ ,  $-\gamma < s < 2$  using the above mentioned smoothing property.

Finally, the case of Navier-Stokes type nonlinearities is considered in subsection 7.3. Note that the classical 3D Navier-Stokes is out of reach of the modern theory even from the point of view of global well-posedness of solutions, so using some modified models looks unavoidable at this stage. In addition, existence of an IM even for the 2D case (where the global well-posedness is well-known) is one of the key open problems in the field, so in order to get the existence of IMs we need stronger modifications. In this paper, we consider the following combination of hyper-viscosity with Leray- $\alpha$  type regularization of the velocity vector field:

$$\begin{cases} \partial_t u + (u, \nabla_x) \bar{u} + \nabla_x p + (-\Delta_x)^{1+\gamma} u = g, & u|_{t=0} = u_0, \\ \operatorname{div} u = 0, & \bar{u} := (1 - \alpha \Delta_x)^{-\bar{\gamma}} u, \end{cases} \quad (1.8)$$

where  $u = (u_1, u_2, u_3)$ ,  $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$  and  $p(t, x)$  are unknown velocity, "filtered" velocity and pressure respectively,  $g$  is a given external forces,  $\alpha > 0$  is a given length scale parameter,  $\gamma \geq 0$  is a given hyper-viscosity exponent and a given parameter  $\bar{\gamma} \geq 0$  affects the strength of the nonlinear term.

Various regularisations of the initial Navier-Stokes equations (including (1.8)) have been intensively studied after the pioneering work of J. Leray [31] by many researches, see [2, 5, 6, 21, 30, 22, 40] and references therein. In particular, in order to guarantee the global well-posedness of problem (1.8), we need to require that

$$2\gamma + \bar{\gamma} \geq \frac{1}{2}, \quad (1.9)$$

see also subsection 7.3 for more details.

The existence of an IM for this problem in 2D case with periodic boundary conditions for  $\gamma = 0$  and  $\bar{\gamma} = 1$  has been verified in [20] using the spectral gap conditions (which hold in 2D case but fail in 3D). The spatial averaging method has been applied instead of spectral gap conditions in [23] in order to treat 3D case with the same parameters  $\gamma = 0$  and  $\bar{\gamma} = 1$ . The possibility to treat the "double-critical" case  $\gamma = 0$ ,  $\bar{\gamma} = \frac{1}{2}$  has been also outlined in [23] and then verified in details in [32]. The purely hyperviscous case  $\gamma = \frac{1}{2}$  and  $\bar{\gamma} = 0$  has studied in [19].

In the present paper, we give the existence of the  $C^{1+\varepsilon}$ -smooth IMs for all intermediate cases. Namely, as we will see below, the spatial averaging technique works if  $\gamma + \bar{\gamma} \geq \frac{1}{2}$ . The case of strict inequality is usually simpler and can be treated using the spectral gap conditions (if  $\gamma > 0$ ), so we concentrate on the critical (from the point of view of IMs) case when

$$\gamma + \bar{\gamma} = \frac{1}{2}, \quad \gamma \in [0, \frac{1}{2}]. \quad (1.10)$$

In this case, we define the basic space

$$H := \{u \in [L^2((-\pi, \pi)^3)]^3, \operatorname{div} u = 0, \langle u \rangle = 0\}$$

and the operator  $A$  as a Stokes operator (=Laplace operator restricted to the invariant subspace of divergent free vector fields with zero mean). Then, applying the operator  $A^{-\gamma}$  to both sides of (1.8), we get the equation of the form (1.3) and may apply our general theory to the obtained equation. This gives us the following result: for every  $\gamma$  and  $\bar{\gamma}$  satisfying (1.10) and every external forces  $g$  such that  $A^{-\gamma}g \in H$ , there exist a  $C^{1+\varepsilon}$ -smooth IM for the problem (1.8). For the end points, this result covers the results obtained before, but it seems new for all intermediate cases. In addition, all the previous results for this equation give only Lipschitz continuous IM and  $C^{1+\varepsilon}$ -smoothness is also a novelty.

We finally note that, analogously to the case of reaction-diffusion equations, our result is applicable and gives new results in the 2D case as well. Indeed, in the case of *square* torus  $(-\pi, \pi)^2$ , we have the spectral gaps of length  $\ln \lambda_N$  in the spectrum of the Laplace or Stokes operator, see [42], so the spectral gap condition will be satisfied and no spatial averaging is required. However, this result is not known in the case of rectangular torus  $(-\pi, \pi) \times (-\beta\pi, \beta\pi)$  if  $\beta$  is irrational. In this case, the spatial averaging works and allows us to overcome the problem and get the desired IM.

## 2. PRELIMINARIES AND AN ABSTRACT MODEL

In this section we recall some basic notations, introduce an abstract model equation which will be of our main interest throughout the paper and prove some elementary, but useful properties of its solutions. Let  $A : D(A) \rightarrow H$  be a positive definite self-adjoint operator in a Hilbert space  $H$  with compact inverse and let  $\lambda_1 \leq \lambda_2 \leq \dots$  be its eigenvalues enumerated in the non-decreasing order. The corresponding orthonormal base in  $H$  generated by its eigenvectors will be denoted by  $\{e_n\}_{n=1}^\infty$ . Then any element  $u \in H$  is presented by its Fourier series:

$$u = \sum_{n=1}^{\infty} u_n e_n, \quad u_n := (u, e_n), \quad \|u\|_H^2 = \sum_{n=1}^{\infty} u_n^2, \quad (2.1)$$

where  $(u, v)$  is the inner product in the space  $H$ .

The fractional powers  $A^s$ ,  $s \in \mathbb{R}$  of operator  $A$  are defined using the standard formula

$$A^s u := \sum_{n=1}^{\infty} \lambda_n^s (u, e_n) e_n \quad (2.2)$$

and the spaces  $H^s := D(A^{s/2})$  are defined as completions of finite linear combinations of  $\{e_n\}_{n=1}^\infty$  with respect to the norm

$$\|u\|_{H^s}^2 = \|A^{s/2} u\|_H^2 = (A^s u, u) = \sum_{n=1}^{\infty} \lambda_n^s u_n^2. \quad (2.3)$$

We consider the following abstract semi-linear parabolic problem in  $H$ :

$$\partial_t A^{-\gamma} u + Au + F(u) = g, \quad u|_{t=0} = u_0, \quad (2.4)$$

where  $\gamma \geq 0$  is a fixed exponent and  $F : H \rightarrow H$  is a given nonlinearity which is assumed to be globally bounded

$$\|F(u)\|_H \leq C \quad (2.5)$$

and is globally Lipschitz continuous with global Lipschitz constant  $L$ :

$$\|F(u_1) - F(u_2)\|_H \leq L \|u_1 - u_2\|_H, \quad u_1, u_2 \in H. \quad (2.6)$$

The external force  $g$  is time-independent and is taken from the space  $H$  ( $g \in H$ ).

The natural phase space for problem (2.4) is  $\Phi := H^{-\gamma}$  ( $u_0 \in H^{-\gamma}$ ) although as we see from the next proposition the solution  $u(t)$  becomes at least  $H^2$ -smooth at any positive time  $t > 0$ . As usual the solutions are understood in the sense of distributions, namely,  $u \in C(0, T; H^{-\gamma}) \cap$

$L^2(0, T; H^1)$  is a solution of (2.4) if for every test function  $\varphi \in C_0^\infty(0, T; H^2)$ , the following identity holds:

$$-\int_{\mathbb{R}} (u(t), A^{-\gamma} \partial_t \varphi(t)) dt + \int_{\mathbb{R}} (u(t), A \varphi(t)) dt = \int_{\mathbb{R}} (g - F(u), \varphi(t)) dt. \quad (2.7)$$

**Proposition 2.1.** *Let the nonlinearity  $F$  satisfy (2.5) and (2.6) and the external force  $g \in H$ . Then*

1. *Equation (2.4) is uniquely globally solvable for all  $u_0 \in H^{-\gamma}$  and the corresponding solution operators  $S(t) : H^{-\gamma} \rightarrow H^{-\gamma}$ ,  $t \geq 0$ , generate a dissipative semigroup in  $H^{-\gamma}$ , i.e., the following estimate holds:*

$$\|u(t)\|_{H^{-\gamma}}^2 + \|u\|_{L^2(t, t+1; H^1)}^2 \leq C e^{-\alpha t} \|u_0\|_{H^{-\gamma}}^2 + C(1 + \|g\|_H^2), \quad (2.8)$$

where  $u(t) := S(t)u_0$  and positive constants  $C$  and  $\alpha$  are independent of  $t$  and  $u_0$ .

2. *The constructed semigroup  $S(t)$  is globally Lipschitz continuous in  $H^{-\gamma}$ , i.e., for every two solutions  $u_1(t)$  and  $u_2(t)$  of equation (2.4), we have*

$$\|u_1(t) - u_2(t)\|_{H^{-\gamma}}^2 + \|u_1 - u_2\|_{L^2(t, t+1; H^1)}^2 \leq C \|u_1(0) - u_2(0)\|_{H^{-\gamma}}^2 e^{L_\gamma t}, \quad (2.9)$$

where the positive constants  $C$  and  $L_\gamma$  depend only on  $L$  and  $\gamma$ .

3. *The semigroup  $S(t)$  possesses an instantaneous  $H^{-\gamma}$  to  $H^2$  parabolic smoothing property, i.e.,*

$$\|u(t)\|_{H^2} \leq C t^{-1} (\|u(0)\|_{H^{-\gamma}} + \|g\|_H + 1), \quad t \in (0, 1] \quad (2.10)$$

where the positive constant  $C$  depends on  $\gamma$ ,  $A$  and  $F$  only. In addition, if we know that  $u(0) = u_0 \in H^2$ , then we have the dissipative estimate in  $H^2$  as well:

$$\|u(t)\|_{H^2}^2 \leq C \|u(0)\|_{H^2}^2 e^{-\alpha t} + C(1 + \|g\|_H^2). \quad (2.11)$$

*Proof.* Since all statements of this proposition are more or less standard and can be checked as in the linear case  $F = 0$ , we give here only the sketch of the proof and leave the details for the reader.

*Step 1. A priori estimate in  $H^{-\gamma}$ .* To this end, we multiply (take an inner product) of equation (2.4) with  $u$  (it is easy to see that all obtained terms make sense, so this multiplication is justified). This gives

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-\gamma}}^2 + \|u(t)\|_{H^1}^2 + (F(u), u) = (g, u).$$

Using the inequality  $\|u\|_{H^{-\gamma}}^2 \leq \lambda_1^{-\gamma-1} \|u\|_{H^1}^2$ , the boundedness of  $F$  and the Gronwall lemma, we get the desired dissipative estimate (2.8)

*Step 2. Existence and uniqueness.* Let  $u_1$  and  $u_2$  be two solutions and  $v(t) = u_1(t) - u_2(t)$ . Then this function solves

$$\partial_t A^{-\gamma} v(t) + A v(t) + [F(u_1(t)) - F(u_2(t))] = 0. \quad (2.12)$$

Multiplying this equation by  $v$  and using the Lipschitz continuity of  $F$ , we get

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{-\gamma}}^2 + \|v(t)\|_{H^1}^2 \leq L \|v(t)\|_H^2.$$

Using the obvious interpolation inequality

$$\|v\|_H^2 \leq \varepsilon \|v\|_{H^1}^2 + C_\varepsilon \|v\|_{H^{-\gamma}}^2$$

and the Gronwall lemma, we get the desired uniqueness and estimate (2.9).

The existence of a solution can be obtained by the standard Galerkin approximations using, e.g., the spectral base  $\{e_n\}_{n=1}^\infty$ , see e.g. [3, 51] for the details.

*Step 3. Estimates for  $\partial_t u$ .* We first note that expressing  $\partial_t u$  from equation (2.4) and using estimates (2.8), we conclude that

$$\|\partial_t u\|_{L^2(0,1;H^{-2\gamma-1})} \leq C(1 + \|g\|_H + \|u_0\|_{H^{-\gamma}}). \quad (2.13)$$

After that, formally differentiating equation (2.4) in time and denoting  $v(t) := \partial_t u(t)$ , we get the equation

$$\partial_t A^{-\gamma} v + Av + F'(u(t))v = 0. \quad (2.14)$$

Multiplying this equation by  $t^2 v(t)$ , we arrive at

$$\frac{1}{2} \frac{d}{dt} (t^2 \|v(t)\|_{H^{-\gamma}}^2) + t^2 \|v(t)\|_{H^1}^2 \leq Lt^2 \|v(t)\|_H^2 + t \|v(t)\|_{H^{-\gamma}}^2. \quad (2.15)$$

We estimate the last term in the right-hand side using the interpolation inequality:

$$t \|v(t)\|_{H^{-\gamma}}^2 \leq Ct \|v(t)\|_{H^{-2\gamma-1}} \|v(t)\|_{H^1} \leq \frac{1}{4} t^2 \|v(t)\|_{H^1}^2 + C \|v(t)\|_{H^{-2\gamma-1}}^2.$$

Integrating this inequality over  $t$  and using the obvious inequality

$$Lt^2 \|v(t)\|_H^2 \leq \frac{1}{4} t^2 \|v(t)\|_{H^1}^2 + C \|v(t)\|_{H^{-2\gamma-1}}^2,$$

we end up (using also (2.13)) with the desired inequality

$$t^2 \|\partial_t u(t)\|_{H^{-\gamma}}^2 \leq C(1 + \|u_0\|_{H^{-\gamma}}^2 + \|g\|_H^2), \quad t \in [0, 1]. \quad (2.16)$$

Being pedantic, estimate (2.16) requires justification. This justification can be done by approximating the solution  $u$  by spectral Galerkin solutions  $u_N(t)$  and on the finite-dimensional level the corresponding nonlinearity which is a priori Lipschitz can be easily approximated by smooth ones without increasing the Lipschitz constant. Since all these arguments are standard, we left the details to the reader.

*Step 4. Smoothing property for  $u(t)$ .* We rewrite equation (2.4) as a point-wise in  $t$  elliptic problem:

$$Au(t) = \tilde{g}(t) := g - A^{-\gamma} \partial_t u(t) - F(u(t)) \quad (2.17)$$

which together with the already obtained estimate for  $\partial_t u(t)$  gives the desired estimate (2.10) for  $u(t)$ . As an immediate corollary of (2.10) and (2.8) we get the desired dissipative estimate (2.11) for large enough  $t$  (say,  $t \geq 1$ ).

*Step 5.  $H^2$ -estimates for small time.* As usual for Cahn-Hilliard type equations, there is an extra small problem to get estimates of  $\|u(t)\|_{H^2}$  for finite (small) time  $t > 0$ . The above technique based on estimating  $\partial_t u(t)$  does not work well here since  $u_0 \in H^2$  is not enough to get  $\partial_t u(0) \in H^{-\gamma}$ , so we need to argue in a bit more delicate way. Namely, we will use the classical parabolic regularity stated in the following lemma.

**Lemma 2.2.** *Let  $u(t)$  solve the linear problem:*

$$\partial_t u + A^{1+\gamma} u = h(t), \quad u|_{t=0} = u_0 \in H^2, \quad h \in C^\kappa(0, 1; H^{-2\gamma}) \quad (2.18)$$

for some  $0 < \kappa \leq \frac{1}{2(\gamma+1)}$ . Then the following estimate holds:

$$\|u\|_{C^1(0,1;H^{-2\gamma}) \cap C(0,1;H^2)} + \|u\|_{C^\kappa(0,1;H^1)} \leq C(\|u_0\|_{H^2} + \|h\|_{C^\kappa(0,1;H^{-2\gamma})}). \quad (2.19)$$

*Proof.* We split  $u(t) = u_1(t) + u_2(t)$ , where

$$\partial_t u_1 + A^{1+\gamma} u_1 = h, \quad u_1|_{t=0} = 0, \quad \partial_t u_2 + A^{1+\gamma} u_2 = 0, \quad u_2|_{t=0} = u_0.$$

Then, for the first equation, using the fact that  $A^{1+\gamma}$  generates an analytic semigroup in  $H$ , we have the following maximal regularity result:

$$\|u_1\|_{C^{1+\kappa}(0,1;H^{-2\gamma}) \cap C^\kappa(0,1;H^2)} \leq C \|h\|_{C^\kappa(0,1;H^{-2\gamma})}.$$

for all  $0 < \kappa < 1$ , see e.g., [9]. For the second component  $u_2$ , we have a bit weaker estimate

$$\|u_2\|_{C^1(0,1;H^{-2\gamma}) \cap C(0,1;H^2)} \leq C\|u_0\|_{H^2},$$

see again [9]. Using now the interpolation

$$\|u_2\|_{C^\kappa(0,1;H^1)} \leq \|u_2\|_{C^1(0,1;H^{-2\gamma}) \cap C(0,1;H^2)}$$

for  $0 < \kappa \leq \frac{1}{2(\gamma+1)}$ , we get the desired result and finish the proof of the lemma.  $\square$

To apply this result to our case, we estimate the nonlinearity using the global Lipschitz continuity assumption:

$$\begin{aligned} \|F(u)\|_{C^\kappa(0,1;H)} &\leq C(1 + \|u\|_{C^\kappa(0,1;H)}) \leq \varepsilon\|u\|_{C^\kappa(0,1;H^1)} + \\ &\quad + C_\varepsilon(1 + \|u\|_{Lip(0,1;H^{-3\gamma-2})}) \leq \varepsilon\|u\|_{C^\kappa(0,1;H^1)} + \\ &\quad + C_\varepsilon(1 + \|\partial_t u\|_{L^\infty(0,1;H^{-3\gamma-2})}) \leq \varepsilon\|u\|_{C^\kappa(0,1;H^1)} + C_\varepsilon(1 + \|u_0\|_{H^2} + \|g\|_H), \end{aligned} \quad (2.20)$$

where  $\varepsilon > 0$  is arbitrary and we have used inequality (2.8) in order to estimate the  $H^{-3\gamma-2}$ -norm of  $\partial_t u$ .

Applying estimate (2.19) to equation (2.18) with  $h(t) = A^\gamma(g - F(u(t)))$  and fixing  $\varepsilon > 0$  small enough, we finally arrive at

$$\|u\|_{C(0,1;H^2)} \leq C(1 + \|u_0\|_{H^2})$$

which gives us the desired estimate (2.11) and finishes the proof of the proposition.  $\square$

In what follows we will also need smoothing estimates for differences of solutions which, in particular, will allow us to show that the IMs constructed in the phase space  $H^{-\gamma}$  will be simultaneously IMs in more regular spaces  $H^s$ .

**Proposition 2.3.** *Let the assumptions of Proposition 2.1 hold and let  $u_1(t)$  and  $u_2(t)$  be two solutions of problem (2.4). Then, for every  $\beta > 0$ , the following estimate holds:*

$$\|u_1(t) - u_2(t)\|_{H^{2-\beta}} \leq C_\beta t^{-1} \|u_1(0) - u_2(0)\|_{H^{-\gamma}}, \quad t \in (0, 1], \quad (2.21)$$

where the constant  $C_\beta$  is independent of the choice of  $u_1$  and  $u_2$ .

*Proof.* Let  $v(t) := u_1(t) - u_2(t)$  and let  $w(t) = tv(t)$ . Then, the last function solves

$$\partial_t w + A^{1+\gamma} w = \tilde{h}(t) := -tA^\gamma(F(u_1(t)) - F(u_2(t))) + v(t), \quad w|_{t=0} = 0. \quad (2.22)$$

We want to apply the analogue of estimate (2.19) with  $\kappa = 0$  to this equation. However, as well-known, the maximal regularity estimate works perfectly in Hölder spaces, but fails in  $C$ , so we need to decrease the regularity exponent (from 2 till  $2 - \beta$ ) in order to restore the validity, see, say, [9, 52] for more details. This gives us the following estimate

$$\begin{aligned} \|w\|_{C(0,1;H^{2-\beta})} &\leq C_\beta \|\tilde{h}\|_{C(0,1;H^{-2\gamma})} \leq C_\beta L \|w\|_{C(0,1;H)} + \|v\|_{C(0,1;H^{-2\gamma})} \leq \\ &\leq \varepsilon \|w\|_{C(0,1;H^{2-\beta})} + C_\varepsilon \|v\|_{C(0,1;H^{-\gamma})}. \end{aligned} \quad (2.23)$$

Fixing  $\varepsilon = \frac{1}{2}$  in this estimate and using (2.9), we get the desired estimate (2.21) and finish the proof of the proposition.  $\square$

**Remark 2.4.** The restriction that the smoothing exponent in (2.21) is restricted by  $2 - \beta < 2$  is related with the fact that  $F'(u)$  is a bounded operator from  $H$  to  $H$  only. If we know, in addition, that

$$\|F'(u)\|_{\mathcal{L}(H^{s_0}, H^{s_0})} \leq C, \quad (2.24)$$

for some  $0 < s_0 < 2$ , we may get the analogue of the smoothing property (2.21), where  $2 - \beta$  is replaced by  $2 + s_0 - \beta$  (with  $t^{-1}$  replaced by  $t^{-2}$ ). Indeed, to get this estimate we just need to make one more step. Namely, when (2.21) is already obtained, we need to return to

equation (2.22), apply the parabolic regularity theorem to it in the space  $H^{s_0-2\gamma}$  and use (2.24) to estimate the terms related with the nonlinearity.

### 3. INERTIAL MANIFOLDS AND CONE PROPERTY

The aim of this section is to recall the basic facts about the Inertial Manifolds (IMs) adapted to our model equation (2.4). We will consider here only the case where the IM is constructed over the spectral subspace  $H_N = \text{span}\{e_1, \dots, e_N\}$  generated by first  $N$  eigenvectors of the operator  $A$ . Here and below, we denote by  $P_N : H \rightarrow H_{N,+}$  the orthoprojector defined by

$$P_N u := \sum_{n=1}^N (u, e_n) e_n$$

and  $Q_N := 1 - P_N$ . It is not difficult to see that  $P_N$  and  $Q_N$  are orthoprojectors in  $H^s$ ,  $s \in \mathbb{R}$  and generate a splitting

$$H^s = H_{N,+}^s \oplus H_{N,-}^s, \quad H_{N,+}^s = H_{N,+}, \quad H_{N,-}^s = Q_N H^s$$

of the space  $H^s$  into the orthogonal sum of two spectral subspaces. Of course, the dimension of  $H_N$  is  $N$ .

**Definition 3.1.** A sub-manifold  $\mathcal{M} \subset H^{-\gamma}$  of dimension  $N$  is called an Inertial Manifold for equation (2.4) if the following conditions are satisfied:

1.  $\mathcal{M}$  is invariant with respect to the solution semigroup  $S(t)$  generated by (2.4):  $S(t)\mathcal{M} = \mathcal{M}$ ;
2.  $\mathcal{M}$  is a graph of a globally Lipschitz continuous function  $\mathbb{M} : H_{N,+}^{-\gamma} \rightarrow H_{N,-}^{-\gamma}$ :

$$\mathcal{M} = \{u_+ + u_-, \quad u_- = \mathbb{M}(u_+), \quad u_+ \in H_{N,+}^{-\gamma}\}. \quad (3.1)$$

We will say that the IM  $\mathcal{M}$  is  $C^{1+\alpha}$ -smooth if  $\mathbb{M}$  is  $C^{1+\alpha}$ -smooth.

3. The manifold  $\mathcal{M}$  possesses the exponential tracking (=asymptotic phase) property, namely, there exists a positive constant  $\theta$  such that for any  $u_0 \in H^{-\gamma}$  there exists a "trace"  $\bar{u}_0 \in \mathcal{M}$  such that

$$\|S(t)u_0 - S(t)\bar{u}_0\|_{H^{-\gamma}} \leq C e^{-\theta t} \|u_0 - \bar{u}_0\|_{H^{-\gamma}} \quad (3.2)$$

for some positive  $C$ .

**Remark 3.2.** As known, the above stated properties of IMs are closely related with *normal-hyperbolicity*. Indeed, usually the manifold  $\mathcal{M}$  is not only Lipschitz continuous, but also is  $C^{1+\alpha}$ -smooth for some small positive  $\alpha$ , so we may speak about tangential and transversal directions.

Then, as a rule the exponent of attraction in directions transversal to the manifold ( $\theta$ ) is not only positive, but also *larger* than the Lyapunov exponents in the tangential directions. This, in particular, gives us the robustness of the IM with respect to perturbations, see [14, 34, 47, 25, 53] for more details.

The existence of an IM is usually verified by checking the so-called invariant *cone* property. To state it in our situation we introduce the following quadratic form:

$$V(\xi) = V_N(\xi) := \|Q_N \xi\|_{H^{-\gamma}}^2 - \|P_N \xi\|_{H^{-\gamma}}^2, \quad \xi \in H^{-\gamma} \quad (3.3)$$

and define the associated cone in the phase space  $H^{-\gamma}$ :

$$K^+ := \left\{ \xi \in H^{-\gamma}, V(\xi) \leq 0 \right\}. \quad (3.4)$$

**Definition 3.3.** Let the above assumptions hold. We say that the solution semigroup  $S(t)$  generated by equation (2.4) possesses the cone property (invariance of the cone  $K^+$ ) if

$$\xi_1 - \xi_2 \in K^+ \Rightarrow S(t)\xi_1 - S(t)\xi_2 \in K^+, \quad \text{for all } t \geq 0, \quad (3.5)$$

where  $\xi_1, \xi_2 \in H^{-\gamma}$ .

Analogously, we say that  $S(t)$  possesses the squeezing property if there exist positive  $\theta$  and  $C$  such that

$$S(T)\xi_1 - S(T)\xi_2 \notin K^+ \Rightarrow \|S(t)\xi_1 - S(t)\xi_2\|_{H^{-\gamma}} \leq Ce^{-\theta t} \|\xi_1 - \xi_2\|_{H^{-\gamma}}, \quad t \in [0, T]. \quad (3.6)$$

The key result of the theory of invariant manifolds is that (at least on the level of abstract semi-linear parabolic equations) the cone and squeezing properties imply the existence of an IM.

**Theorem 3.4.** *Let the solution semigroup  $S(t)$  of problem (2.4) possess the cone and squeezing properties. Then there exists a Lipschitz IM for this problem in the phase space  $H^{-\gamma}$ .*

The proof of this theorem can be found, e.g., in [34, 53].

We just mention that the desired Lipschitz function  $\mathbb{M} : H_{N,+}^{-\gamma} \rightarrow H_{N,-}^{-\gamma}$  can be obtained as follows: for a given  $u_+ \in H_{N,+}^{-\gamma}$  and  $T > 0$ , one finds a unique solution  $u = u_{T,u_+}(t)$  of the boundary value problem

$$\partial_t A^{-\gamma} u + Au + F(u) = g, \quad P_N u|_{t=0} = u_+, \quad Q_N u|_{t=-T} = 0. \quad (3.7)$$

Then, at the next step one passes to the limit  $T \rightarrow \infty$  and find a backward trajectory  $u_{u_+}(t)$ ,  $t \leq 0$ :

$$u_{u_+}(t) := \lim_{T \rightarrow \infty} u_{T,u_+}(t). \quad (3.8)$$

The existence of this limit is guaranteed by the squeezing property, see [53] for details. Finally we define

$$\mathbb{M}(u_+) := Q_N u_{u_+}(0). \quad (3.9)$$

Then the cone property guarantees us the Lipschitz continuity of  $\mathbb{M}$  and the squeezing property implies in a standard way the exponential tracking property, see [53] for more details. We also mention that the semigroup  $S(t)$  restricted to the IM  $\mathcal{M}$  can be extended to a globally Lipschitz continuous group

$$\|S(-t)\xi_1 - S(-t)\xi_2\|_{H^{-\gamma}} \leq Ce^{K|t|} \|\xi_1 - \xi_2\|_{H^{-\gamma}}, \quad \xi_1, \xi_2 \in \mathcal{M}. \quad (3.10)$$

This estimate follows from the fact that any trajectory  $u(t) \in \mathcal{M}$  has a structure  $u(t) = u_+(t) + \mathbb{M}(u_+(t))$ , where the function  $u_+(t) \in H_{N,+}$  solves a system of ODEs

$$\partial_t u_+ + A^{1+\gamma} u_+ + A^\gamma P_N F(u_+ + \mathbb{M}(u_+)) = P_N A^\gamma g \quad (3.11)$$

with globally Lipschitz continuous nonlinearity. This system of ODEs is usually referred as an *Inertial Form* (IF) associated with equation (2.4) and gives us the desired finite-dimensional reduction constructed via IMs.

**Corollary 3.5.** *Let the solution semigroup  $S(t)$  of equation (2.4) satisfy the cone and squeezing properties in the phase space  $H^{-\gamma}$ . Then the IM  $\mathcal{M}$  in the space  $H^{-\gamma}$  constructed in Theorem 3.4 is simultaneously an IM for equation (2.4) in any phase space  $H^s$ ,  $-\gamma \leq s < 2$ .*

Indeed, this statement is an immediate corollary of the construction of an IM for  $H^{-\gamma}$  described in Theorem 3.4 and the smoothing property (2.21).

**Remark 3.6.** The result of Corollary 3.5 shows that the choice of the phase space where to verify the cone and squeezing properties is in our disposal and it is natural to fix this phase space in the way which simplifies calculations. In particular, there are no connections between the initial problem *before* the cut-off of the nonlinearities making them globally Lipschitz and the technical choice of the phase space for proving the IM existence. Since the most delicate procedure in our proof is related with spatial averaging, we fix the  $H^{-\gamma}$  as a phase space just in order to be able to treat the spatial averaging in the most convenient space  $H$ .

We now discuss the ways to verify the above introduced cone and squeezing properties for equation (2.4). To this end we introduce, following [25] the so called strong cone property in a differential form which allows us to verify cone and squeezing properties simultaneously and also gives normal hyperbolicity of the IM and its extra smoothness if  $F(u)$  is smooth enough.

**Definition 3.7.** Assume in addition that the function  $F : H \rightarrow H$  is Gateaux differentiable at every point  $u \in H$  and its Gateaux derivative  $F'(u)$  is a linear continuous operator in  $H$ . Then, the equation of variations

$$\partial_t A^{-\gamma} v + Av + l(t)v = 0, \quad l(t) := F'(u(t)), \quad (3.12)$$

where  $u(t) := S(t)u_0$  which corresponds to equation (2.4) is well-defined. Clearly,

$$\|F'(u)\|_{\mathcal{L}(H,H)} \leq L \quad (3.13)$$

We say that equation (2.4) satisfies the strong cone property in a differential form if there are Borel measurable bounded function  $\alpha : H \rightarrow \mathbb{R}$  and a positive constant  $\mu$  such that

$$0 < \alpha_1 \leq \alpha(u) \leq \alpha_2$$

and

$$\frac{1}{2} \frac{d}{dt} V(v(t)) + \alpha(u(t))V(v(t)) \leq -\mu \|v(t)\|_H^2 \quad (3.14)$$

for any  $u_0 \in H^{-\gamma}$  and any solution  $v(t)$  of problem (3.12) starting from  $v_0 \in H^{-\gamma}$ .

The next theorem is a key point in our method of constructing the IMs.

**Theorem 3.8.** *Let the assumptions of Proposition 2.1 be satisfied and let, in addition, equation (2.4) possess a strong cone property in a differential form for some  $N \in \mathbb{N}$ . Then, the solution semigroup  $S(t)$  possesses a cone and squeezing properties and, according to Theorem 3.4 also possesses an IM with the base  $H_{N,+} = H_{N,+}^{-\gamma}$ .*

The proof of this result is given in [25].

Thus, in order to prove the existence of an IM for our equation (2.4), it is sufficient to verify only estimate (3.14) for the linearized equation (3.12).

The next result gives the extra smoothness of the constructed IM.

**Theorem 3.9.** *Let the assumptions of Theorem 3.8 hold and let, in addition, the nonlinearity  $F$  satisfy*

$$\|F(u_1) - F(u_2) - F'(u_1)(u_1 - u_2)\|_H \leq C \|u_1 - u_2\|_H \|u_1 - u_2\|_{H^{2-\kappa}}^\delta, \quad u_1, u_2 \in H^{2-\kappa} \quad (3.15)$$

for some small positive constants  $\delta$  and  $\kappa$ . Then, the associated IM is  $C^{1+\delta}$ -smooth.

The proof of this theorem is given in [25] for the case  $\gamma = 1$ , but the case of general  $\gamma$  is completely analogous.

**Remark 3.10.** We emphasize that the theorem gives  $C^{1+\delta}$ -smoothness of the IM for *small* positive  $\delta$  only no matter how smooth the nonlinearity  $F$  is. The space  $H^{2-\kappa}$  in (3.15) is related only with the fact that in general we have parabolic smoothing property (2.21) for the exponents less than 2. If we somehow know, in addition, that this smoothing property holds for the space  $H^s$  with  $s > 2$ , then  $H^{2-\kappa}$  in (3.15) can be replaced by  $H^s$ . For instance, if (2.24) is satisfied,  $H^{2-\kappa}$  can be replaced by  $H^s$  with  $s < s_0 + 2$ . We also mention that estimate (3.15) is actually used only for  $u_1, u_2 \in \mathcal{M}$ , so we may check it only under the extra assumption that

$$\|Q_N u_1\|_{H^{2-\kappa}} + \|Q_N u_2\|_{H^{2-\kappa}} \leq C$$

for some  $\kappa > 0$  and sufficiently large  $C$ . Moreover, the estimate (2.24) should also be checked for  $\|u\|_{H^{2-\kappa}} \leq C$  only.

## 4. VERIFICATION OF THE CONE PROPERTY VIA SPATIAL AVERAGING

This section is devoted to verifying the strong cone condition (3.14) for the solutions  $v(t)$  of (3.12). We start with the simplest case where the so-called spectral gap conditions are satisfied.

**Proposition 4.1.** *Let  $N \in \mathbb{N}$  be such that*

$$\frac{\lambda_{N+1}^{1+\gamma} - \lambda_N^{1+\gamma}}{\lambda_{N+1}^\gamma + \lambda_N^\gamma} > L. \quad (4.1)$$

*Then the corresponding equation (3.12) possesses the strong cone property (3.14) with*

$$\alpha := \lambda_N^{1+\gamma} \frac{\lambda_{N+1}^\gamma}{\lambda_N^\gamma + \lambda_{N+1}^\gamma} + \lambda_{N+1}^{1+\gamma} \frac{\lambda_N^\gamma}{\lambda_N^\gamma + \lambda_{N+1}^\gamma}, \quad \mu := \frac{\lambda_{N+1}^{1+\gamma} - \lambda_N^{1+\gamma}}{\lambda_{N+1}^\gamma + \lambda_N^\gamma} - L. \quad (4.2)$$

*Proof.* Multiplying equation (3.12) by  $Q_N v - P_N v$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} V(v(t)) + \alpha V(v(t)) + ((\alpha A^{-\gamma} - A)P_N v, P_N v) + \\ + ((A - \alpha A^{-\gamma})Q_N v, Q_N v) = -(l(t)v, Q_N v - P_N v). \end{aligned} \quad (4.3)$$

Using the fact that the function  $x \rightarrow x - \alpha x^{-\gamma}$  is monotone increasing, we can estimate

$$((\alpha A^{-\gamma} - A)P_N v, P_N v) = \sum_{n=1}^N (\alpha \lambda_n^{-\gamma} - \lambda_n) |v_n|^2 \geq \sum_{n=1}^N (\alpha \lambda_N^{-\gamma} - \lambda_N) |v_n|^2 = (\alpha \lambda_N^{-\gamma} - \lambda_N) \|P_N v\|_H^2$$

and, analogously,

$$((A - \alpha A^{-\gamma})Q_N v, Q_N v) = \sum_{n=N+1}^{\infty} (\lambda_n - \alpha \lambda_n^{-\gamma}) |v_n|^2 \geq (\lambda_{N+1} - \alpha \lambda_{N+1}^{-\gamma}) \|Q_N v\|_H^2.$$

Since, by our choice the exponent  $\alpha$  solves

$$\lambda_{N+1} - \alpha \lambda_{N+1}^{-\gamma} = \alpha \lambda_N^{-\gamma} - \lambda_N,$$

and elementary calculation shows that

$$((\alpha A^{-\gamma} - A)P_N v, P_N v) + ((A - \alpha A^{-\gamma})Q_N v, Q_N v) \geq \frac{\lambda_{N+1}^{1+\gamma} - \lambda_N^{1+\gamma}}{\lambda_{N+1}^\gamma + \lambda_N^\gamma} \|v\|_H^2.$$

Finally, the Cauchy-Schwarz inequality together with assumption (3.13) gives

$$|(l(t)v, Q_N v - P_N v)| \leq L \|v\|_H^2$$

and inserting the obtained estimates to (4.3) we arrive at (3.14) and finish the proof of the proposition.  $\square$

The rest of this section is devoted to the case when the spectral gap condition (4.1) is not satisfied, but instead the nonlinearity satisfies the so-called spatial averaging principle. To state this principle, we introduce for every  $k \in \mathbb{N}$  the following orthoprojectors:

$$\mathcal{P}_{k,N} u := \sum_{j: \lambda_j < \lambda_N - k} (u, e_j) e_j, \quad \mathcal{Q}_{k,N} u := \sum_{j: \lambda_j > \lambda_N + k} (u, e_j) e_j, \quad \mathcal{I}_{k,N} := 1 - \mathcal{P}_{k,N} - \mathcal{Q}_{k,N}$$

Thus, instead of splitting  $v = v_+ + v_-$  on lower ( $v_+ := P_N v$ ) and higher ( $v_- = Q_N v$ ) modes, we now use the splitting

$$v = v_{++} + v_I + v_{--}, \quad v_{++} := \mathcal{P}_{k,N} v, \quad v_I := \mathcal{I}_{k,N} v, \quad v_{--} := \mathcal{Q}_{k,N} v$$

on essentially lower, essentially higher and *intermediate* modes. The key assumption in the spatial averaging method is that the operator  $F'(u(t))$  restricted to the intermediate modes is close to the scalar operator. Then, we say that  $F$  satisfies the spatial averaging principle if there

exists  $\theta > 0$  such that, for every positive  $\delta < L$  and natural number  $k$  there exist infinitely many values of  $N \in \mathbb{N}$  such that

$$\|\mathcal{I}_{k,N}F'(u)\mathcal{I}_{k,N} - a(u)\mathcal{I}_{k,N}\|_{\mathcal{L}(H,H)} \leq \delta \quad (4.4)$$

uniformly with respect to  $u \in H$  and  $\lambda_{N+1} - \lambda_N \geq \theta$ . Here  $a : H \rightarrow \mathbb{R}$  may depend on  $\delta$  and  $N$ .

We are now ready to state and prove the main result of this section.

**Theorem 4.2.** *Let the nonlinearity satisfy the spatial averaging principle and let the involving constants  $\theta$ ,  $k$ ,  $L$  and  $N$  satisfy*

$$\frac{\theta}{8} - \delta - \gamma 2^{\gamma+1} L \frac{k}{\lambda_N - k} > 0, \quad \frac{1}{2}k - \frac{8L^2}{\theta} - 2L \geq 0 \quad (4.5)$$

and, in addition,  $\lambda_N > L$  and  $k \leq \lambda_N/2$ .

Then equation (2.4) possesses a strong cone property in the form of (3.14).

*Proof.* We just need to estimate the terms in (4.3) in a more accurate way. Namely, for lower modes  $\mathcal{P}_{k,N}v$ , we will have

$$\begin{aligned} ((\alpha A^{-\gamma} - A)\mathcal{P}_{k,N}v, \mathcal{P}_{k,N}v) &= \sum_{n:\lambda_n < \lambda_N - k} \left( \frac{\lambda_N^\gamma}{\lambda_n^\gamma} \cdot \frac{\lambda_{N+1}^\gamma(\lambda_{N+1} + \lambda_N)}{\lambda_N^\gamma + \lambda_{N+1}^\gamma} - \lambda_n \right) |v_n|^2 \geq \\ &\geq \sum_{n:\lambda_n < \lambda_N - k} \left( \frac{\lambda_{N+1}^\gamma(\lambda_{N+1} + \lambda_N)}{\lambda_N^\gamma + \lambda_{N+1}^\gamma} - \lambda_N + k \right) |v_n|^2 = (\bar{\mu} + k) \|\mathcal{P}_{k,N}v\|_H^2, \end{aligned} \quad (4.6)$$

where  $\bar{\mu} := \frac{\lambda_{N+1}^{1+\gamma} - \lambda_N^{1+\gamma}}{\lambda_{N+1}^\gamma + \lambda_N^\gamma}$ . Arguing analogously, we also get

$$((A - \alpha A^{-\gamma})\mathcal{Q}_{k,N}v, \mathcal{Q}_{k,N}v) \geq (\bar{\mu} + k) \|\mathcal{Q}_{k,N}v\|_H^2.$$

In addition, we need the analogue of (4.6) for the  $H^{-\gamma}$ -norm. Namely,

$$\begin{aligned} ((\alpha A^{-\gamma} - A)\mathcal{P}_{k,N}v, \mathcal{P}_{k,N}v) &= \sum_{n:\lambda_n < \lambda_N - k} \left( \lambda_N^\gamma \frac{\lambda_{N+1}^\gamma(\lambda_{N+1} + \lambda_N)}{\lambda_N^\gamma + \lambda_{N+1}^\gamma} - \lambda_n \lambda_n^\gamma \right) (\lambda_n^{-\gamma} |v_n|^2) \geq \\ &\geq \sum_{n:\lambda_n < \lambda_N - k} \lambda_N^\gamma \left( \frac{\lambda_{N+1}^\gamma(\lambda_{N+1} + \lambda_N)}{\lambda_N^\gamma + \lambda_{N+1}^\gamma} - \lambda_N + k \right) \lambda_n^{-\gamma} |v_n|^2 = \lambda_N^\gamma (\bar{\mu} + k) \|\mathcal{P}_{k,N}v\|_{H^{-\gamma}}^2. \end{aligned} \quad (4.7)$$

Moreover, estimating lower-intermediate and higher-intermediate modes exactly as in Proposition 4.1 and using that

$$((\alpha A^{-\gamma} - A)P_N v, P_N v) = ((\alpha A^{-\gamma} - A)\mathcal{P}_{k,N}v, \mathcal{P}_{k,N}v) + ((\alpha A^{-\gamma} - A)P_N \mathcal{I}_{k,N}v, P_N \mathcal{I}_{k,N}v)$$

and the analogous expression for  $Q_N v$  component, we transform (4.3) to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} V(v(t)) + \alpha V(v(t)) + \frac{1}{2} \bar{\mu} \|v(t)\|_H^2 + \\ + \frac{1}{2} k (\|\mathcal{P}_{k,N}v\|_H^2 + \|\mathcal{Q}_{k,N}v\|_H^2) + \frac{1}{2} \lambda_N^\gamma (\bar{\mu} + k) \|\mathcal{P}_{k,N}v\|_{H^{-\gamma}}^2 \leq -(l(t)v, Q_N v - P_N v). \end{aligned} \quad (4.8)$$

To estimate the right-hand side we use that  $1 = \mathcal{P}_{k,N} + \mathcal{I}_{k,N} + \mathcal{Q}_{k,N}$ :

$$\begin{aligned} -(l(t)v, Q_N v - P_N v) &= (l(t)v, \mathcal{P}_{k,N}v) - (l(t)v, \mathcal{Q}_{k,N}v) - (l(t)v, Q_N \mathcal{I}_{k,N}v - P_N \mathcal{I}_{k,N}v) \leq \\ &\leq L \|v\|_H (\|\mathcal{P}_{k,N}v\|_H + \|\mathcal{Q}_{k,N}v\|_H) - (l(t)v, Q_N \mathcal{I}_{k,N}v - P_N \mathcal{I}_{k,N}v) \leq \\ &\leq \frac{\bar{\mu}}{8} \|v\|_H^2 + \frac{2L^2}{\bar{\mu}} (\|\mathcal{P}_{k,N}v\|_H^2 + \|\mathcal{Q}_{k,N}v\|_H^2) - (l(t)v, Q_N \mathcal{I}_{k,N}v - P_N \mathcal{I}_{k,N}v). \end{aligned} \quad (4.9)$$

We may continue this estimate as follows

$$\begin{aligned}
& - (l(t)v, Q_N \mathcal{I}_{k,N} v - P_N \mathcal{I}_{k,N} v) = - (l(t) \mathcal{P}_{k,N} v, Q_N \mathcal{I}_{k,N} v - P_N \mathcal{I}_{k,N} v) - \\
& \quad - (l(t) \mathcal{Q}_{k,N} v, Q_N \mathcal{I}_{k,N} v - P_N \mathcal{I}_{k,N} v) - (l(t) \mathcal{I}_{k,N} v, Q_N \mathcal{I}_{k,N} v - P_N \mathcal{I}_{k,N} v) \leq \\
& \quad \leq L \|v\|_H (\|\mathcal{P}_{k,N} v\|_H + \|\mathcal{Q}_{k,N} v\|_H) - (\mathcal{I}_{k,N} l(t) \mathcal{I}_{k,N} v, Q_N v - P_N v) \leq \\
& \quad \leq \frac{\bar{\mu}}{8} \|v\|_H^2 + \frac{2L^2}{\bar{\mu}} (\|\mathcal{P}_{k,N} v\|_H^2 + \|\mathcal{Q}_{k,N} v\|_H^2) - (\mathcal{I}_{k,N} l(t) \mathcal{I}_{k,N} v, Q_N v - P_N v). \quad (4.10)
\end{aligned}$$

Using now (4.4), we get

$$- (\mathcal{I}_{k,N} l(t) \mathcal{I}_{k,N} v, Q_N v - P_N v) \leq -a(u(t)) (\|Q_N \mathcal{I}_{k,N} v\|_H^2 - \|P_N \mathcal{I}_{k,N} v\|_H^2) + \delta \|v\|_H^2. \quad (4.11)$$

To transform the right-hand side of this inequality, we need the following straightforward estimates

$$\begin{aligned}
& \left| \lambda_N^\gamma \|P_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2 - \|P_N \mathcal{I}_{k,N} v\|_H^2 \right| \leq \\
& \leq \sum_{n: \lambda_N - k \leq \lambda_n \leq \lambda_N} |\lambda_N^\gamma - \lambda_n^\gamma| \lambda_n^{-\gamma} |v_n|^2 \leq \frac{(\lambda_N^\gamma - (\lambda_N - k)^\gamma)}{(\lambda_N - k)^\gamma} \|P_N \mathcal{I}_{k,N} v\|_H^2 \quad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
& \left| \lambda_N^\gamma \|Q_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2 - \|Q_N \mathcal{I}_{k,N} v\|_H^2 \right| \leq \\
& \leq \sum_{n: \lambda_{N+1} \leq \lambda_n \leq \lambda_{N+k}} |\lambda_N^\gamma - \lambda_n^\gamma| \lambda_n^{-\gamma} |v_n|^2 \leq \frac{(\lambda_N + k)^\gamma - \lambda_N^\gamma}{(\lambda_N + k)^\gamma} \|Q_N \mathcal{I}_{k,N} v\|_H^2. \quad (4.13)
\end{aligned}$$

Moreover, as not difficult to check,

$$\frac{(a+x)^\gamma - a^\gamma}{(a+x)^\gamma} \leq \frac{a^\gamma - (a-x)^\gamma}{(a-x)^\gamma}, \quad 0 < x \leq a.$$

Therefore

$$\begin{aligned}
& -a(u(t)) (\|Q_N \mathcal{I}_{k,N} v\|_H^2 - \|P_N \mathcal{I}_{k,N} v\|_H^2) \leq \\
& \leq -a(u(t)) \lambda_N^\gamma (\|Q_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2 - \|P_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2) + 2L \frac{\lambda_N^\gamma - (\lambda_N - k)^\gamma}{(\lambda_N - k)^\gamma} \|v\|_H^2. \quad (4.14)
\end{aligned}$$

Finally, we estimate the first-term in the right-hand side through the function  $V(v(t))$  as follows:

$$\begin{aligned}
& -a(u(t)) \lambda_N^\gamma (\|Q_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2 - \|P_N \mathcal{I}_{k,N} v\|_{H^{-\gamma}}^2) = \\
& = -a(u(t)) \lambda_N^\gamma V(v(t)) + a(u(t)) \lambda_N^\gamma (\|\mathcal{P}_{k,N} v\|_{H^{-\gamma}}^2 - \|\mathcal{Q}_{k,N} v\|_{H^{-\gamma}}^2) \leq \\
& \leq -a(u(t)) \lambda_N^\gamma V(v(t)) + 2L \lambda_N^\gamma \|\mathcal{P}_{k,N} v\|_{H^{-\gamma}}^2 + 2L \|\mathcal{Q}_{k,N} v\|_H^2. \quad (4.15)
\end{aligned}$$

Combining the obtained estimates, we get

$$\begin{aligned}
& - (l(t)v, Q_N v - P_N v) \leq -a(u(t)) \lambda_N^\gamma V(v(t)) + \left( \frac{\bar{\mu}}{4} + \delta + 2L \left( \left( 1 + \frac{k}{\lambda_N - k} \right)^\gamma - 1 \right) \right) \|v\|_H^2 + \\
& \quad + \left( \frac{4L^2}{\bar{\mu}} + 2L \right) (\|\mathcal{P}_{k,N} v\|_H^2 + \|\mathcal{Q}_{k,N} v\|_H^2) + 2L \lambda_N^\gamma \|\mathcal{P}_{k,N} v\|_{H^{-\gamma}}^2. \quad (4.16)
\end{aligned}$$

Using the elementary inequality

$$(1+x)^\gamma - 1 \leq \gamma 2^\gamma x, \quad x \in (0, 1)$$

and inserting (4.16) into (4.8), we get the desired inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} V(v(t)) + (\alpha + a(u(t))\lambda_N^\gamma) V(v(t)) + \\ + \left( \frac{\bar{\mu}}{4} - \delta - \gamma 2^{\gamma+1} L \frac{k}{\lambda_N - k} \right) \|v\|_H^2 + \\ + \left( \frac{1}{2} k - \frac{4L^2}{\bar{\mu}} - 2L \right) (\|\mathcal{P}_{k,N} v\|_H^2 + \|\mathcal{Q}_{k,N} v\|_H^2) + \lambda_N^\gamma \left( \frac{k}{2} - 2L \right) \|\mathcal{P}_{k,N} v\|_{H^{-\gamma}}^2 \leq 0. \end{aligned} \quad (4.17)$$

Using the obvious inequality

$$\frac{|x^{1+\gamma} - y^{1+\gamma}|}{x^\gamma + y^\gamma} \geq \frac{1}{2} |x - y|, \quad x, y \geq 0$$

and the assumptions of the theorem, we see that (4.17) implies the desired cone property and finishes the proof of the theorem.  $\square$

## 5. THE TRUNCATION PROCEDURE

Note that in the previous sections, we have assumed that the nonlinearity  $F(u)$  is *globally* Lipschitz continuous and satisfies the spatial averaging principle also uniformly with respect to  $u \in H$ . These assumptions look very restrictive since in applications we usually have *growing* nonlinearities. The standard strategy here is to verify first the existence of an absorbing ball in some higher order space  $H^s$  and then cut-off the nonlinearity outside of this ball, see [18, 34, 53]. However this truncation is rather delicate when the spatial averaging is involved since we should preserve spatial averaging structure under this truncation. For the case of scalar reaction-diffusion equation, the proper cut-off procedure has been suggested in [34] and alternative construction which is well-adapted for the case when the average  $a(u)$  of the nonlinearity  $f'(u)$  is identically zero has been introduced in [23]. In this section, we present a combination of two above mentioned methods which will allow us to treat both cases from the unified point of view.

Let  $\phi \in C^\infty(\mathbb{R})$  be such that  $\phi(z) = z$  for  $|z| \leq 1$  and  $\phi(z) = 2$  for  $|z| \geq 2$ . Then for a given positive constant  $C_*$  and sufficiently large exponent  $s$ , we define the function  $W : H \rightarrow H$  via

$$W(u) = \sum_{n=1}^{\infty} C_* \lambda_n^{-s/2} \phi \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) e_n. \quad (5.1)$$

The elementary properties of this truncation function are collected in the following proposition.

**Proposition 5.1.** *Let the function  $W$  be defined via (5.1). Then,*

1. *The map  $W$  is bounded and continuous as a map from  $H$  to  $H^{s_0}$ , where  $s_0 > 0$  is such that*

$$\sum_{n=1}^{\infty} \lambda_n^{s_0-s} < \infty. \quad (5.2)$$

2.  *$W(u) \equiv u$  if  $u \in H^s$  and  $\|u\|_{H^s} \leq C_*$ .*

3. *The function  $W$  is Hadamard differentiable as a map from  $H$  to  $H$  and the derivative is given by*

$$W'(u)v = \sum_{n=1}^{\infty} \phi' \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) (v, e_n) e_n. \quad (5.3)$$

4. *There exists a positive constant  $C$  such that, for every  $\kappa \in \mathbb{R}$*

$$\|W'(u)\|_{\mathcal{L}(H^\kappa, H^\kappa)} \leq C, \quad \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H^\kappa, H^\kappa)} \leq C \|u_1 - u_2\|_{H^s} \quad (5.4)$$

for all  $u, u_1, u_2 \in H^s$ .

*Proof.* The first statement is straightforward. Indeed, let  $u, v \in H$ . Then, due to (5.2) and boundedness of  $\phi$ , for every  $\varepsilon > 0$ , there exists  $M = M(\varepsilon)$  such that

$$\|W(u+v) - W(u)\|_{H^{s_0}}^2 \leq \frac{\varepsilon^2}{2} + \sum_{n=1}^M C_*^2 \lambda_n^{s_0-s} \left( \phi \left( \frac{\lambda_n^{s/2}(u+v, e_n)}{C_*} \right) - \phi \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) \right)^2.$$

Since the sum in the RHS has now only finitely many terms and  $\phi$  is continuous, we may make the sum less than  $\frac{\varepsilon^2}{2}$  by taking the  $H$ -norm of  $v$  small enough. This proves the continuity.

To verify the second property, let us take  $u \in H^s$  such that

$$\|u\|_{H^s}^2 := \sum_{n=1}^{\infty} \lambda_n^s (u, e_n)^2 \leq C_*^2,$$

then  $|(u, e_n)| \leq C_* \lambda_n^{-s/2}$  and therefore  $\phi \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) = \frac{\lambda_n^{s/2}(u, e_n)}{C_*}$  and  $W(u) = u$ .

Let us verify the differentiability. To this end, we need to estimate

$$\|W(u+th) - W(u) - tW'(u)h\|_H^2 = \sum_{n=1}^{\infty} \left( \int_0^1 \phi' \left( \frac{\lambda_n^{s/2}(u+tlh, e_n)}{C_*} \right) - \phi' \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) dl \right)^2 t^2 (h, e_n)^2.$$

To check the Hadamard differentiability, we need to take  $h \in \mathcal{K}$  where  $\mathcal{K}$  is a compact set in  $H$ , so we have uniform smallness of the tails  $\sum_{n=M}^{\infty} (h, e_n)^2$ . Using also that  $\phi'$  is bounded, for every  $\varepsilon > 0$ , we may find  $M = M(\varepsilon)$  such that

$$\sum_{n=M}^{\infty} \left( \int_0^1 \phi' \left( \frac{\lambda_n^{s/2}(u+tlh, e_n)}{C_*} \right) - \phi' \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) dl \right)^2 t^2 (h, e_n)^2 \leq \frac{\varepsilon}{2} t^2$$

for all  $t \in [0, 1]$  and all  $h \in \mathcal{K}$ . Passing to the limit  $t \rightarrow 0$  in the remaining *finite* sum

$$\sum_{n=1}^M \left( \int_0^1 \phi' \left( \frac{\lambda_n^{s/2}(u+tlh, e_n)}{C_*} \right) - \phi' \left( \frac{\lambda_n^{s/2}(u, e_n)}{C_*} \right) dl \right)^2 t^2 (h, e_n)^2$$

is immediate since  $\phi'$  is smooth, so we may make it less than  $\frac{\varepsilon}{2} t^2$  by taking  $t$  small enough. This proves the differentiability.

Finally, the 4th property is an immediate corollary of the estimate

$$\left| \phi' \left( \frac{\lambda_n^{s/2}(u_1, e_n)}{C_*} \right) - \phi' \left( \frac{\lambda_n^{s/2}(u_2, e_n)}{C_*} \right) \right|^2 \leq C \lambda_n^s |(u_1 - u_2, e_n)|^2 \leq C \|u_1 - u_2\|_{H^s}^2$$

and the fact that  $\phi'$  is uniformly bounded. Thus, the proposition is proved.  $\square$

We now turn to more general semi-linear parabolic equation

$$A^{-\gamma} \partial_t u + Au + f(u) = g, \quad u|_{t=0} = u_0, \quad (5.5)$$

where  $g \in H$  and  $f$  is a given nonlinearity which is no more assumed to be globally bounded or/and globally Lipschitz. Instead, we assume that this problem is well-posed in a phase space  $H^{s'}$  for some  $s' \in \mathbb{R}$  and generates a dissipative semigroup  $\bar{S}(t) : H^{s'} \rightarrow H^{s'}$  there. We also assume that the ball  $\mathcal{B}$  of radius  $C_*$  in the space  $H^s$ , for some  $s > \max\{s', -\gamma\}$  is a (semi)invariant absorbing ball for the semigroup  $\bar{S}(t)$ . The latter means that

1.  $\bar{S}(t)\mathcal{B} \subset \mathcal{B}$ ;
2. For every bounded set  $B \subset H^{s'}$ , there exists  $T = T(B)$  such that

$$\bar{S}(t)B \subset \mathcal{B}, \quad \text{if } t \geq T.$$

Roughly speaking, the idea is to define the truncated nonlinearity as  $F(u) := f(W(u))$ . Then, we will have

$$F(u) = f(u), \quad u \in \mathcal{B},$$

but in order to verify that  $F$  satisfies the conditions of Theorem 4.2, we need some restrictions on the map  $f$ . Namely,

**Assumption I.** The map  $f : H^{s_0} \rightarrow H$  is continuous and is locally bounded. Here  $s_0 = s_0(s)$  is the same as in Proposition 5.1. The map  $a : H^{s_0} \rightarrow \mathbb{R}$  is also continuous and locally bounded.

**Assumption II.**

**a)** The map  $f : H^{s_0} \rightarrow H$  is Gateaux differentiable at any point  $u \in H^{s_0}$  and its derivative  $f'(u)$  is linear and can be extended to the linear continuous operator in  $H$ :  $f'(u) \in \mathcal{L}(H, H)$  for any  $u \in H^{s_0}$ . Moreover,

$$f' \in C^\varepsilon(H^{s_0}, \mathcal{L}(H, H)) \quad (5.6)$$

for some  $\varepsilon > 0$ .

**b)** The map  $u \rightarrow a(u)$  is Gateaux differentiable as a map from  $H^{s_0}$  to  $\mathbb{R}$  and its derivative has a form  $a'(u)v = (a'(u), v)$  where  $a'(u) \in H$ . Moreover,

$$a' \in C^\varepsilon(H^{s_0}, H)$$

for some  $\varepsilon > 0$ .

**c)** The map  $f'$  is well-defined and is locally bounded as a map from  $H^{s_0}$  to  $\mathcal{L}(H^{s_0}, H^{s_0})$ .

**Assumption III.** The following version of spatial averaging principle is satisfied: there exists a function  $a : H^{s_0} \rightarrow \mathbb{R}$  such that, for every bounded set  $B \subset H^{s_0}$  and every  $\delta > 0$  and  $k > 0$ , there exists an infinite sequence of  $N \in \mathbb{N}$  such that

$$\sup_{u \in B} \|\mathcal{I}_{k,N} f'(u) \mathcal{I}_{k,N} v - a(u) \mathcal{I}_{k,N} v\|_H \leq \delta \|v\|_H, \quad \forall v \in H, \quad (5.7)$$

compare with (4.4).

We start with the simplest case where the spatial average  $a(u)$  vanishes identically.

**Theorem 5.2.** *Let the nonlinearity  $f$  satisfy Assumptions I, II and III and let  $a(u) \equiv 0$ . Then, the truncated nonlinearity*

$$F(u) := f(W(u)) \quad (5.8)$$

*satisfies the assumptions of Theorem 4.2 and therefore, the associated truncated equation (2.4) possesses a family of IMs  $\mathcal{M} = \mathcal{M}_N$  for infinitely many values of  $N$ .*

*Proof.* Indeed, according to Assumption I and the first statement of Proposition 5.1, the map  $F : H \rightarrow H$  is globally bounded and continuous. From Assumption II and the third statement of Proposition 5.1, we conclude that the map  $F$  is Gateaux (and even Hadamard) differentiable and the following chain rule formula holds:

$$F'(u) = f'(W(u))W'(u). \quad (5.9)$$

Indeed, let  $u, v \in H$  and  $t \geq 0$ . Then

$$\begin{aligned} & \|f(W(u+tv)) - f(W(u)) - tf'(W(u))W'(u)v\|_H \leq \\ & \leq \left\| \int_0^1 [f'(W(u) + \kappa(W(u+tv) - W(u))) - f'(W(u))] d\kappa(W(u+tv) - W(u)) \right\|_H + \\ & + \|f'(W(u))(W(u+tv) - W(u) - tW'(u)v)\|_H \leq C\|W(u+tv) - W(u) - tW'(u)v\|_H + \\ & + C\|W(u+tv) - W(u)\|_{H^{s_0}}^\varepsilon \|W(u+tv) - W(u)\|_H \quad (5.10) \end{aligned}$$

and, since  $W$  is Gateaux differentiable as a map from  $H$  to  $H$  and is continuous as a map from  $H$  to  $H^{s_0}$ , we see that the right-hand side is of order  $o(t)$ . This proves the differentiability and verifies (5.9).

In particular, (5.9) shows that  $F'(u)$  is globally bounded in  $\mathcal{L}(H, H)$ , so (3.13) is satisfied for the properly chosen constant  $L$ .

Finally, inserting  $W'(u)v$  instead of  $v$  in (5.7) and using that  $a \equiv 0$  and that the operator  $W'(u)$  is diagonal in the base of eigenvectors of  $A$  (and consequently  $\mathcal{I}_{k,N}W' = W'\mathcal{I}_{k,N}$ ), together with the boundedness of  $W'$ , we get that the spatial averaging condition (4.4) is also satisfied for infinitely many values of  $N$ s. This finishes the proof of the theorem.  $\square$

We now return to the non-truncated equation (5.5) and give (following [26]) the natural definition of the IM for non-scaled case.

**Definition 5.3.** Let  $\bar{S}(t) : \Phi \rightarrow \Phi$  be a semigroup acting in a Banach space  $\Phi$  and possessing the invariant bounded absorbing set  $\mathcal{B}$  in it. Assume that

- 1) There exists another Banach space  $\Psi$  and a dissipative semigroup  $S(t)$  in  $\Psi$ .
- 2) There exists a bi-Lipschitz embedding  $E : \mathcal{B} \rightarrow \Psi$  such that

$$S(t) = E \circ \bar{S}(t) \circ E^{-1}$$

on  $E(\mathcal{B}) \subset \Psi$ .

- 3) The dynamical system  $S(t)$  possesses an IM  $\mathcal{M}$  in the phase space  $\Psi$ .

Then  $\mathcal{M}$  is referred as a (generalized) inertial manifold for the semigroup  $\bar{S}(t)$  in the sense of Definition 3.1. This manifold is called  $C^{1+\varepsilon}$ -smooth if both  $E$  and  $\mathcal{M}$  are  $C^{1+\varepsilon}$ -smooth.

In our particular case  $\Phi = H^{s'}$ ,  $\Psi = H^{-\gamma}$ ,  $\mathcal{B} \subset \Phi \cap \Psi$  and the semigroups  $\bar{S}(t)$  and  $S(t)$  are the solution operators for equations (5.5) and (2.4) respectively and  $E = \text{Id}$ . So, we have proved the following result.

**Corollary 5.4.** *Let the assumptions of Theorem 5.2 hold and let, in addition, the solution semigroup  $\bar{S}(t)$  possess an invariant absorbing ball  $\mathcal{B}$  in  $H^s$ . Then, there are infinitely many  $N$ s such that equation (5.5) possesses an IM of dimension  $N$  in the sense of Definition 5.3 and the associated truncated semigroup  $S(t)$  is defined by equation (2.4).*

**Remark 5.5.** The IM for the equation (2.4) with already truncated nonlinearity is usually unique if the dimension  $N$  is fixed. However, the non-uniqueness of the IM for the initial non-truncated equations appears since there are many ways to make the cut-off procedure. Note also that the IM  $\mathcal{M}$  is strictly invariant for the truncated semigroup  $S(t)$  only, and may be not invariant for the initial semigroup  $\bar{S}(t)$ . On the other hand, the manifold  $\mathcal{M}$  always contains the image  $E(\mathcal{A})$  of a global attractor  $\mathcal{A}$  of the initial equation, so it always generates an Inertial Form for the initial dynamics on the global attractor, see [18, 51, 34, 26, 27, 53] for more details.

We now return to the general case  $a(u) \neq 0$ . In this case, the naive choice (5.8) is no longer working (since for truncated nonlinearity we then have  $a(W(u))W'(u)$  which is no more a scalar operator and everything crushes). So, we need to proceed in a more delicate way.

Namely, following [34], we assume that the absorbing ball  $\mathcal{B}$  is bounded in  $H^2$  by the constant  $R$  and introduce a cut-off function  $\varphi(z)$  which equals to 0 for  $z \leq R^2$  and equals to  $-1/2$  for  $z \geq R_1^2$  for some  $R_1 > R$ . Then, we define the map  $T = T_N : H \rightarrow H$  via

$$T(u) := \varphi(\|AP_N u\|_H^2) AP_N u. \quad (5.11)$$

The key property of this map is stated in the following lemma.

**Lemma 5.6.** *It is possible to fix the cut-off function  $\varphi$  in such a way that*

$$(T'(u)v, v) \leq 0, \quad v \in H \quad (5.12)$$

and  $(T'(u)v, v) = -\frac{1}{2}\|P_N v\|_{H^1}^2$  if  $\|P_N u\|_{H^2} \geq R_1$ .

The proof of this lemma is given in [34] (see also [53]).

We fix one more smooth cut-off function  $\theta(z)$  which equals to one if  $z \leq \bar{R}^2$  and zero if  $z > 4\bar{R}^2$ , where the parameter  $\bar{R}$  is chosen in such a way that  $\|\mathcal{B}\|_H \leq \bar{R}$  and define

$$F(u) := f(W(u)) - a(W(u))W(u) + \theta(\|u\|_H^2)a(W(u))u + T_N(u). \quad (5.13)$$

Then, exactly as in the case,  $a = 0$ , the function  $F$  will be bounded and continuous as the map from  $H$  to  $H$  and its Gateaux derivative will have the form

$$\begin{aligned} F'(u)v &= [f'(W(u))W'(u)v - a(W(u))W'(u)v] + \theta(\|u\|_H^2)a(W(u))v - \\ &\quad - (a'(W(u)), W'(u)v)W(u) + [2\theta'(\|u\|_H^2)(u, v)a(W(u)) + \\ &\quad + \theta(\|u\|_H^2)(a'(W(u)), W'(u)v)]u + T'_N(u)v = \\ &= l_1(u)v + l_2(u)v + l_3(u)v + l_4(u)v + T'_N(u)v. \end{aligned} \quad (5.14)$$

Indeed, the verification of (5.14) is straightforward and is similar to what we did to check (5.9), so we left the details to the reader.

Note that only the term  $T_N(u)$  depends explicitly on  $N$  now, so the norms of all other terms are independent of  $N$ . In particular, since  $Q_N T_N(u) \equiv 0$ , we have that

$$\|Q_N F(u)\|_H \leq C, \quad u \in H, \quad (5.15)$$

where  $C$  is independent of  $N$ .

**Lemma 5.7.** *Let the estimate (5.15) hold. Then, for any  $\kappa > 0$ , the  $Q_N$ -component of the solution  $u(t)$  of problem (2.4) possesses the following estimate:*

$$\|Q_N u(t)\|_{H^{2-\kappa}} \leq C_1 \frac{1+t^M}{t^M} e^{-\beta t} \|Q_N u(0)\|_{H^{-\gamma}} + C_2(1 + \|g\|_H), \quad (5.16)$$

where the constants  $M, \beta > 0$  and  $C_1, C_2$  are independent of  $N$  and  $u$  (but may depend on  $\kappa$ ). Moreover, for the existence of an IM, the strong cone property (3.14) can be verified for the trajectories  $u(t)$  satisfying

$$\|Q_N u(t)\|_{H^{2-\kappa}} \leq C_2, \quad t \geq 0 \quad (5.17)$$

only.

Indeed, estimate (5.16) follows from (5.15) and the parabolic regularity estimates applied to the equation

$$A^{-\gamma} \partial_t Q_N u + A Q_N u = Q_N g - Q_N F(u),$$

see the proof of Proposition 2.1. The second statement is also standard and follows from more detailed analysis of the proof of Theorem 3.4, namely, from the fact that the cone property is actually used for the solutions  $u(t)$  with the control of  $Q_N$ -component (see, e.g., (3.7)). More details can be found in [34, 25, 53].

Note that in contrast to the  $Q_N$ -component of  $u(t)$ , the  $P_N$ -component is typically unbounded on the IM, so we cannot assume any uniform bounds for it. Instead, we will use the extra map  $T$  and Lemma 5.6 in order to control it.

The next theorem can be considered as the main result of this section.

**Theorem 5.8.** *Let the nonlinearity  $f$  and its spatial average  $a$  satisfy Assumptions I, II and III and let the truncated nonlinearity  $F(u)$  be constructed via (5.13) (for the properly chosen function  $W$  as explained above). Then, there are infinitely many  $N$ s for which equation (2.4) satisfies the strong cone condition and, therefore, also possesses a Lipschitz IM of dimension  $N$ .*

*Proof.* We need to check, following Theorem 4.2, that the strong cone property (3.14) is satisfied for all solutions  $v$  of (3.12) with an extra condition (5.17).

We have already verified that exactly as in Theorem 5.2, the map  $F$  is uniformly bounded in  $H$  and its Gateaux (Hadamard) derivative is also bounded. The small change here is the

fact that now these bounds are depending on  $N$  through the term  $T_N(u)$ , but this term is not dangerous for the cone property. Indeed, due to Lemma 5.6, we have

$$(T'(u)v, P_N v - Q_N v) = (T'(u)P_N v, P_N v) \leq 0, \quad (5.18)$$

so we need not any extra assumptions to control it. All other terms are independent of  $N$ .

Let us analyze the impact of every term of the derivative (5.14) to the key estimate (4.3) for the cone inequality. We first note that due to the fact that all involving operators except of  $T'(u)$  are bounded by some constant  $L$  which is independent of  $N$  and the term  $T'(u)$  is not dangerous, we only need to analyze the intermediate modes.

The term  $l_1(u)v$  has zero spatial average, so its intermediate modes are estimated based on (5.7) exactly as in the proof of Theorem 4.2. The term  $l_2(u)v$  is a scalar operator and it gives the truncated analogue of spatial averaging for the function  $F$ .

The spatial averaging of the term  $l_3(t)v$  also vanishes. Indeed,

$$\|\mathcal{I}_{k,N} l_3(u) \mathcal{I}_{k,N} v\|_H \leq C \|\mathcal{I}_{k,N} W(u)\|_H \|v\|_H \leq C(\lambda_N - k)^{-s_0/2} \|v\|_H \quad (5.19)$$

and the right-hand side of it can be made arbitrarily small by chosen  $N$  large enough (since  $s_0 > 0$ ).

Finally, the term  $l_4(u)v$  possesses the analogous estimate

$$\|\mathcal{I}_{k,N} l_4(u) \mathcal{I}_{k,N} v\|_H \leq C \|\mathcal{I}_{k,N} P_N u\|_H \|v\|_H + C \|\mathcal{I}_{k,N} Q_N u\|_H \|v\|_H, \quad (5.20)$$

but in contrast to  $W(u)$  the function  $u$  is not uniformly bounded in the higher energy space  $H^{s_0}$ . So, we need to argue in a more accurate way. To estimate the term containing  $\|\mathcal{I}_{k,N} Q_N u\|_H$  is easy due to Lemma 5.7:

$$\|\mathcal{I}_{k,N} Q_N u\|_H \leq C \lambda_N^{\frac{\kappa-2}{2}} \|Q_N u\|_{H^{2-\kappa}} \leq C_1 \lambda_N^{\frac{\kappa-2}{2}}.$$

Thus, it only remains to estimate the  $P_N$ -component. We consider two cases: the first case is when the estimate  $\|P_N u\|_{H^2} \leq R_1$  holds. In this case everything is also easy:

$$\|\mathcal{I}_{k,N} P_N u\|_H \leq R_1 (\lambda_N - k)^{-1}.$$

The alternative case  $\|P_N u\|_{H^2} \geq R_1$  is slightly more delicate and exactly for estimating it we have introduced the auxiliary operator  $T$ . Indeed, due to this operator we now have for free the extra good term  $-\|P_N v\|_{H^1}^2$  which is crucial for our estimate. Namely, using the fact that  $\theta(z)$  vanishes if  $z$  is large enough, we get

$$\begin{aligned} |(P_N \mathcal{I}_{k,N} l_4(u) \mathcal{I}_{k,N} v, P_N \mathcal{I}_{k,N} v)| &\leq C \theta(\|u\|_H^2) \|u\|_H \|v\|_H \|P_N \mathcal{I}_{k,N} v\|_H \leq \\ &\leq C_1 \|v\|_H \|P_N v\|_H \leq \varepsilon \|v\|_H^2 + C_\varepsilon (\lambda_N - k)^{-1} \|P_N v\|_{H^1}^2, \end{aligned} \quad (5.21)$$

for every  $\varepsilon > 0$ . Since we have the term  $-\mu \|v\|_H^2$  in the cone inequality with  $\mu$  independent of  $N$ , fixing  $\varepsilon > 0$  small enough and  $N$  big enough gives the desired estimate in the second case as well and finishes the proof of the theorem.  $\square$

**Corollary 5.9.** *Let the assumptions of Theorem 5.8 hold and let, in addition, the solution semigroup  $\bar{S}(t)$  associated with equation (5.5) possess an absorbing set  $\mathcal{B}$  which is a bounded set of  $H^s$  with  $s > 2$ . Assume also that the constants  $C_*$ ,  $R$  and  $\bar{R}$  are fixed in such a way that*

$$\|u\|_{H^s} \leq C_*, \quad \|u\|_{H^2} \leq R, \quad \|u\|_H \leq \bar{R}, \quad \forall u \in \mathcal{B}.$$

*Then, there exist infinitely many  $N$ s, such that equation (5.5) possesses an IM of dimension  $N$  in the sense of Definition 5.3. The truncated semigroup  $S(t)$  is defined as a solution semigroup of equation (2.4) with the nonlinearity  $F$  defined via (5.13).*

Indeed, this is an immediate corollary of Theorem 5.8 and the fact that, by the construction,  $F(u) = f(u)$  for all  $u \in \mathcal{B}$ .

To conclude this section, we discuss the  $C^{1+\varepsilon}$ -smoothness of the obtained IMs.

**Lemma 5.10.** *Let the nonlinearity  $f$  satisfy Assumptions I-II and let  $F$  be constructed via (5.13). Then,*

$$\|F(u_1) - F(u_2) - F'(u_1)(u_1 - u_2)\|_H \leq C \|u_1 - u_2\|_{H^s}^\varepsilon \|u_1 - u_2\|_H, \quad u_1, u_2 \in H^s, \quad (5.22)$$

where the constant  $C$  may depend on  $N$  but is independent of  $u_1, u_2$ . Moreover, if  $\kappa > 0$  is such that  $s_0 \leq 2 - \kappa$  and  $u \in H^s$ ,  $s \geq 2 - \kappa$  satisfies

$$\|Q_N u\|_{H^{2-\kappa}} \leq \bar{C} \quad (5.23)$$

for some positive constant  $\bar{C}$ , then the following estimate holds:

$$\|F'(u)v\|_{H^{s_0}} \leq C \|v\|_{H^{s_0}}, \quad (5.24)$$

where the constant  $C$  is independent of  $u \in H^s$ , but depends on  $\bar{C}$ .

*Proof.* Let us first check estimate (5.22). Analogously to (5.10), it is sufficient to verify that  $F' \in C^\varepsilon(H^s, \mathcal{L}(H, H))$ . Let us verify this property for every term in (5.14) separately. For the first term, due to Assumption II and Proposition 5.1, we have

$$\begin{aligned} & \|f'(W(u_1))W'(u_1)v - f'(W(u_2))W'(u_2)v\|_H \leq \\ & \leq \|f'(W(u_1)) - f'(W(u_2))\|_{\mathcal{L}(H, H)} \|W'(u_1)\|_{\mathcal{L}(H, H)} \|v\|_H + \\ & \quad + \|f'(W(u_2))\|_{\mathcal{L}(H, H)} \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H, H)} \|v\|_H \leq \\ & \leq C \|W(u_1) - W(u_2)\|_{H^{s_0}}^\varepsilon \|v\|_H + C \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H, H)} \|v\|_H. \end{aligned} \quad (5.25)$$

Using the Proposition 5.1 again, we infer

$$\begin{aligned} & \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H, H)} = \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H, H)}^\varepsilon \|W'(u_1) - W'(u_2)\|_{\mathcal{L}(H, H)}^{1-\varepsilon} \leq \\ & \leq C \|u_1 - u_2\|_{H^s}^\varepsilon (\|W'(u_1)\|_{\mathcal{L}(H, H)} + \|W'(u_2)\|_{\mathcal{L}(H, H)})^{1-\varepsilon} \leq C \|u_1 - u_2\|_{H^s}^\varepsilon. \end{aligned} \quad (5.26)$$

Thus, since  $s_0 < s$ , we have checked that

$$\|f'(W(u_1))W'(u_1) - f'(W(u_2))W'(u_2)\|_{\mathcal{L}(H, H)} \leq C \|u_1 - u_2\|_{H^{s_0}}^\varepsilon + C \|u_1 - u_2\|_{H^s}^\varepsilon \leq C \|u_1 - u_2\|_{H^s}^\varepsilon,$$

where the constant  $C$  is independent of  $u_1, u_2 \in H^s$ .

The Hölder continuity of the terms  $a(W(u))W'(u)$  and  $\theta(\|u\|_H^2)a(W(u))$  can be established analogously using Assumption II, and the term  $T'_N(u)$  is also straightforward since it is finite-dimensional. So, it only remains to estimate  $l_4(u)$ . To estimate these terms, we actually only need to verify the uniform Hölder continuity of maps  $\Psi_1 : u \rightarrow \theta(\|u\|_H^2)u$  and  $\Psi_2 : u \rightarrow 2\theta'(\|u\|_H^2)u(u, \cdot)$  as maps from  $H$  to  $H$  and  $\mathcal{L}(H, H)$  and respectively. Let us start with the first map.

Since the function  $\theta$  is smooth and has a finite support, the map  $\Psi_1(u)$  is at least Gateaux differentiable and its derivatives is given by

$$\Psi'_1(u)v := \theta(\|u\|_H^2)v + 2\theta'(\|u\|_H^2)(u, v)u$$

and, therefore,

$$\|\Psi'_1(u)\|_{\mathcal{L}(H, H)} \leq \max_{z \in \mathbb{R}_+} \{|\theta(z)|\} + 2 \max_{z \in \mathbb{R}_+} \{z|\theta'(z)|\} \leq C.$$

Since  $\|\Psi_1(u)\|_H \leq C$ , we end up with

$$\|\Psi_1(u_1) - \Psi_1(u_2)\|_H \leq C \|u_1 - u_2\|_H^\varepsilon \leq C \|u_1 - u_2\|_{H^s}^\varepsilon,$$

where the constant  $C$  is independent of  $u_1$  and  $u_2$ . Let us now look at the second map  $\Psi_2$ . Analogously, its Gateaux derivative reads

$$\Psi'_2(u)[w, v] = 2\theta'(\|u\|_H^2)u(w, v) + 2\theta''(\|u\|_H^2)w(u, v) + 4\theta''(\|u\|_H^2)(w, u)(u, v)u,$$

and using the fact that  $\theta$  has a finite support, we get

$$\|\Psi'_2(u)[w, v]\|_H \leq C \|w\|_H \|v\|_H,$$

where  $C$  is independent of  $u, v, w \in H$ . Since  $\Psi_2$  is bounded as a map from  $H$  to  $\mathcal{L}(H, H)$ , we infer from here that

$$\|\Psi_2(u_1) - \Psi_2(u_2)\|_{\mathcal{L}(H, H)} \leq C \|u_1 - u_2\|_{H^s}^\varepsilon.$$

The obtained estimates, together with Assumption II and Proposition 5.1, give

$$\|l_4(u_1) - l_4(u_2)\|_{\mathcal{L}(H, H)} \leq C \|u_1 - u_2\|_{H^s}^\varepsilon$$

and finish the proof of estimate (5.22).

Let us verify estimate (5.24). This estimate is an almost immediate corollary of Assumption II c) and Proposition 5.1. The only problem which appears here is related with the term  $l_4(u)$ . Indeed, arguing as before, we get

$$\|l_4(u)\|_{\mathcal{L}(H^{s_0}, H^{s_0})} \leq C (\theta(\|u\|_H^2) \|u\|_H + C |\theta'(\|u\|_H^2)|) \|u\|_{H^{s_0}},$$

where  $C$  is independent of  $u$ . To handle this term, we write

$$\|u\|_{H^{s_0}} \leq \|P_N u\|_{H^{s_0}} + \|Q_N u\|_{H^{s_0}} \leq C_N \|u\|_H + \|Q_N u\|_{H^{s_0}}.$$

Therefore, since  $\theta$  has a finite support, we end up with

$$\|l_4(u)\|_{\mathcal{L}(H^{s_0}, H^{s_0})} \leq C_N (1 + \|Q_N u\|_{H^{s_0}}).$$

Since  $s_0 \leq 2 - \kappa$ , the right-hand side is bounded due to condition (5.23). This finishes the proof of estimate (5.24) as well as the lemma.  $\square$

**Corollary 5.11.** *Let the assumptions of Theorem 5.8 and Lemma 5.10 hold and let, in addition  $s_0 < 2$  and  $s_0 < s < s_0 + 2$ . Then there exists an infinite sequence of  $N$ s such that problem (5.5) possesses an  $N$ -dimensional IM which is  $C^{1+\varepsilon_N}$ -smooth for some  $\varepsilon_N > 0$ .*

Indeed, this result follows from Lemma 5.10, Theorem 3.9 and Remark 3.10.

## 6. SPATIAL AVERAGING IN THE CASE OF PERIODIC BOUNDARY CONDITIONS

In this section, we discuss the spatial averaging Assumption III in the most usual (from the point of view of applications) case where  $A$  is the Laplacian  $A = -\Delta_x$  in the 3D domain  $\Omega = (-\pi, \pi)^3$  endowed with periodic boundary conditions. In this case the eigenvalues  $\lambda_n$  of  $A$  are naturally parameterised by triples  $\vec{n} := (q, l, m)$  of integer numbers:

$$\lambda_{\vec{n}} = q^2 + l^2 + m^2, \quad e_{\vec{n}} := e^{ix_1 \vec{n}} = e^{ix_1 q + ix_2 l + ix_3 m}. \quad (6.1)$$

Then, the Fourier series (2.1) become the classical Fourier expansions. We will use the notation  $\{\lambda_{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^3}$  for this parametrization keeping the notation  $\{\lambda_n\}_{n \in \mathbb{N}}$  for the parametrization in the non-decreasing order used in previous sections.

Note also that all  $\lambda_n$  are integer, so the distance from non-identical eigenvalues is at least one and due to the Gauss theorem about sums of squares there are no spectral gaps of size more than 3, see [34, 53] for more details. Thus, in general, the spectral gap conditions are *not satisfied* in all of examples considered below.

We also recall that due to the Weyl asymptotic  $\lambda_n \sim Cn^{2/3}$ , so the key condition (5.2) is satisfied if and only if

$$s > s_0 + \frac{3}{2}. \quad (6.2)$$

There is also a small problem here related with possible zero eigenvalue which corresponds to  $\vec{n} = (0, 0, 0)$ . This can be overcome in two alternative ways. First, we may consider  $A = 1 - \Delta_x$  instead of  $A = -\Delta_x$  which removes the problem up to the nonessential shift of the spectrum. This is typically done, say, for reaction-diffusion equations. Alternatively, we may work in spaces with zero mean which is natural for Navier-Stokes or Cahn-Hilliard type problems. In this case the problem does not arise at all.

The spatial averaging method takes its origin in the following number theoretic result which claims that the sums of 3 squares of integers are distributed irregularly enough.

**Proposition 6.1.** *Let*

$$\mathcal{C}_N^k := \{\vec{l} \in \mathbb{Z}^3 : N - k \leq |\vec{l}|^2 \leq N + k\}, \quad \mathcal{B}_r := \{\vec{l} \in \mathbb{Z}^3 : |\vec{l}| \leq r\}. \quad (6.3)$$

*Then, for every  $k > 0$  and  $r > 0$ , there exist infinitely many  $N \in \mathbb{N}$  such that*

$$\left(\mathcal{C}_N^k - \mathcal{C}_N^k\right) \cap \mathcal{B}_r = \{0\}. \quad (6.4)$$

The proof of this proposition is given in [34].

With a slight abuse of notations, we redefine the projector  $\mathcal{I}_{k,N}$  as follows:

$$\mathcal{I}_{k,N}v := \sum_{\vec{n} \in \mathcal{C}_N^k} (v, e_{\vec{n}}) e_{\vec{n}}. \quad (6.5)$$

Being pedantic, we should write  $\mathcal{I}_{k,N'}$  in the left-hand side of this formula, where  $N' = N'(N)$  is defined by

$$N' := \max\{M \in \mathbb{N}, \lambda_M \leq N\}.$$

However, the difference between  $N$  and  $N'$  is not essential for us and to simplify the notations, we ignore it. The next proposition is also crucial for the spatial averaging machinery.

**Proposition 6.2.** *Let  $\mathcal{N}_\psi : L^2(\Omega) \rightarrow L^2(\Omega)$  be the operator of point-wise multiplication on a function  $\psi \in H^{s_0}$  for some  $s_0 > \frac{3}{2}$ :*

$$(\mathcal{N}_\psi v)(x) := \psi(x)v(x). \quad (6.6)$$

*Then, the operator  $\mathcal{N}_\psi$  satisfies the spatial averaging property in the following sense: for every  $k > 0$  and every  $\delta > 0$  there exists an infinitely many  $N \in \mathbb{N}$  such that*

$$\|\mathcal{I}_{k,N}\mathcal{N}_\psi\mathcal{I}_{k,N}v - a\mathcal{I}_{k,N}v\|_{L^2} \leq \delta\|v\|_{L^2}, \quad v \in L^2, \quad (6.7)$$

where  $a = \langle \psi \rangle := \frac{1}{2\pi^3} \int_{(-\pi,\pi)^3} \psi(x) dx$ .

*Proof.* Although this result is well-known, its proof is crucial for understanding the spatial averaging technique, so we give some details below following mainly [53].

The multiplication  $\psi(x)v(x)$  is a convolution in Fourier modes, so the corresponding Fourier coefficients  $[\psi v]_{\vec{m}}$ ,  $\vec{m} \in \mathbb{Z}^3$  satisfy

$$[\psi v]_{\vec{m}} = \sum_{\vec{l} \in \mathbb{Z}^3} \psi_{\vec{m}-\vec{l}} v_{\vec{l}} \quad (6.8)$$

and, due to condition (6.4),

$$\mathcal{I}_{k,N}((\psi - \langle \psi \rangle)\mathcal{I}_{k,N}v) = \mathcal{I}_{k,N}(\psi_{>r}\mathcal{I}_{k,N}v), \quad (6.9)$$

where  $\psi_{>r}(x) := \sum_{\vec{l} \notin \mathcal{B}_r} \psi_{\vec{l}} e^{i\vec{l} \cdot x}$ . Therefore,

$$\|\mathcal{I}_{k,N}((\psi - \langle \psi \rangle)\mathcal{I}_{k,N}v)\|_H \leq \|(\psi_{>r}\mathcal{I}_{k,N}v)\|_H \leq \|\psi_{>r}\|_{L^\infty} \|v\|_H. \quad (6.10)$$

Thus, we only need to check that

$$\lim_{r \rightarrow \infty} \|\psi_{>r}\|_{L^\infty} = 0. \quad (6.11)$$

To verify this property, we use the interpolation inequality

$$\|\psi_{>r}\|_{L^\infty} \leq C \|\psi_{>r}\|_{L^2}^\kappa \|\psi_{>r}\|_{H^{s_0}}^{1-\kappa}$$

for the properly chosen  $\kappa = \kappa(\alpha) \in (0, 1)$  (here we have used that  $s_0 > \frac{3}{2}$ ), together with the standard inequality  $\|\psi_{>r}\|_{H^{s_0}} \leq C\|\psi\|_{H^{s_0}}$ . Thus, we have

$$\|\psi_{>r}\|_{L^\infty} \leq C \|\psi_{>r}\|_{L^2}^\kappa \|\psi\|_{H^{s_0}}^{1-\kappa} \leq Cr^{-\kappa s_0} \|\psi_{>r}\|_{H^{s_0}}^\kappa \|\psi\|_{H^{s_0}}^{1-\kappa} \leq Cr^{-\kappa s_0} \|\psi\|_{H^{s_0}}$$

and the proposition is proved.  $\square$

At the next step, we consider the particular case where the function  $\psi$  has zero mean. Then the class of operators with spatial averaging property can be essentially extended.

**Proposition 6.3.** *Let  $A_1, A_2$  be two linear operators which commute with the operator  $A := -\Delta_x$  with periodic boundary conditions (more precisely, we need the commutation of them with spectral projectors  $\mathcal{I}_{k,N}$ ) and let*

$$A_1 \in \mathcal{L}(H^{-\beta}, H), \quad A_2 \in \mathcal{L}(H, H^{-\beta}), \quad (6.12)$$

for some  $\beta \in [0, 1]$ . Assume also that  $\psi \in H^{s_0}$  for some  $s_0 > \frac{3}{2}$  and  $\langle \psi \rangle = 0$ . Then the operators

$$\mathcal{N}_{A_1, \psi, A_2} := A_1 \circ \mathcal{N}_\psi \circ A_2 \quad \text{and} \quad \mathcal{N}_{A_1, A_2 \psi} := A_1 \circ \mathcal{N}_{A_2 \psi}$$

satisfy the spatial averaging property (6.7) with  $a = 0$ .

*Proof.* Let us start with the first operator. Arguing as in the proof of Proposition 6.2 and using that  $\mathcal{I}_{k,N}$  commute with  $A_1$  and  $A_2$ , we see that it is sufficient to show that

$$\lim_{r \rightarrow \infty} \|\mathcal{N}_{A_1, \psi_{>r}, A_2}\|_{\mathcal{L}(H, H)} = 0.$$

In turn, to this end, we only need to show that

$$\lim_{r \rightarrow \infty} \|\mathcal{N}_{\psi_{>r}}\|_{\mathcal{L}(H^{-\beta}, H^{-\beta})} = 0.$$

To check this property, we will use the following version of the Kato-Ponce inequality, see [4, 28]:

$$\|\psi w\|_{H^\beta} \leq C \|\psi\|_{L^\infty} \|w\|_{H^\beta} + C \|\psi\|_{H^{\beta, q}} \|w\|_{L^p},$$

where  $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$ . We fix the exponents  $\frac{1}{p} = \frac{1}{2} - \frac{\beta}{3}$ ,  $\frac{1}{q} = \frac{\beta}{3}$  in order to have the Sobolev embeddings  $H^\beta \subset L^p$  and  $H^{s'} \subset H^{\beta, q}$  for all  $s' > \frac{3}{2}$ . This gives us the estimate

$$\|\psi w\|_{H^\beta} \leq C \|\psi\|_{H^{s'}} \|w\|_{H^\beta} \quad (6.13)$$

and, therefore, taking  $s' \in (3/2, s_0)$  and using the standard trick with adjoint operator, we have

$$\|\mathcal{N}_{\psi_{>r}}\|_{\mathcal{L}(H^{-\beta}, H^{-\beta})} \leq C \|\psi_{>r}\|_{H^{s'}} \leq C r^{s' - s_0} \|\psi\|_{H^{s_0}}$$

which finishes the proof of the proposition for the operator  $\mathcal{N}_{A_1, \psi, A_2}$ .

Let us now study the second operator  $\mathcal{N}_{A_1, A_2 \psi}$ . Using again that  $A_1, A_2$  commute with  $\mathcal{I}_{k,N}$  and arguing as in the proof of Proposition 6.2, we see that, we only need to verify that

$$\lim_{r \rightarrow \infty} \|\mathcal{N}_{A_2(\psi_{>r})}\|_{\mathcal{L}(H, H^{-\beta})} = 0.$$

To verify this, we use the Sobolev embedding  $L^q \subset H^{-\beta}$  for  $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{3}$ ,  $H^\mu \subset L^p$  for  $\frac{1}{p} = \frac{1}{2} - \frac{\mu}{3}$ , where  $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$ , i.e.,  $\mu = \frac{3}{2} - \beta$ , together with Hölder's inequality. This gives

$$\begin{aligned} \|A_2(\psi_{>r})v\|_{H^{-\beta}} &\leq C \|A_2(\psi_{>r})v\|_{L^q} \leq C \|A_2\psi_{>r}\|_{L^p} \|v\|_H \leq C \|A_2\psi_{>r}\|_{H^{3/2-\beta}} \|v\|_{L^2} \leq \\ &\leq C \|\psi_{>r}\|_{H^{3/2}} \|v\|_H \leq C r^{3/2-s_0} \|\psi\|_{H^{s_0}} \|v\|_H \end{aligned} \quad (6.14)$$

and the proposition is proved.  $\square$

We conclude this section by verifying the spatial averaging property as well as other properties stated in Assumptions I-II for a number of concrete nonlinearities related with our applications to reaction-diffusion, Cahn-Hilliard and Navier-Stokes equations.

**Example 6.4.** Let us consider the local scalar nonlinearity  $f(u)$  for some  $f \in C^4(\mathbb{R}, \mathbb{R})$ . This will correspond to the case of reaction-diffusion equation (5.5) with  $\gamma = 0$ . In this case the derivative  $f'(u)v$  is a multiplication operator on a function  $\psi = f'(u)$ . Thus, according to Proposition 6.2, the spatial averaging assumption (Assumption III) will be satisfied with  $a := \langle f'(u) \rangle$  if we take

$\frac{3}{2} < s_0 < 2$ . Using the fact that  $H^{s_0}$  is an algebra if  $s_0 > \frac{3}{2}$ , we see that the other regularity assumptions (Assumptions I-II) are also automatically satisfied if

$$\frac{3}{2} < s_0 < 2, \quad s_0 + \frac{3}{2} < s < s_0 + 2 < 4. \quad (6.15)$$

In this case, we take  $A = 1 - \Delta_x$  in order to remove zero eigenvalue.

Thus, to verify the existence of an IM of smoothness  $C^{1+\varepsilon}$  for this type of the nonlinearity (according to Corollary 5.11) it is enough to verify that the corresponding equation (5.5) possesses an absorbing ball in the space  $H^s$  where  $s$  satisfies (6.15). This will be discussed in the next section.

**Example 6.5.** Let us modify slightly the previous example to adapt it to the case of Cahn-Hilliard type equations. Namely, we will consider the space  $H = \{u \in L^2, \langle u \rangle = 0\}$  and consider the nonlinearity

$$f(u) - \langle f(u) \rangle. \quad (6.16)$$

The extra non-local term  $\langle f(u) \rangle$  has 1 dimensional range and does not affect the spectral averaging property as well as other regularity properties of the nonlinearity. Thus, to get the existence of  $C^{1+\varepsilon}$ -smooth IMs, we only need to get the absorbing set in  $H^s$  satisfying (6.15).

**Example 6.6.** We now consider the Navier-Stokes type nonlinearities. We assume that

$$H := \{u \in [L^2(\Omega)]^3, \langle u \rangle = 0, \operatorname{div} u = 0\} \quad (6.17)$$

and denote by  $P$  the Leray orthoprojector from  $[L^2(\Omega)]^3$  to  $H$ . It is well-known that in the case of periodic boundary conditions, the Leray projector  $P$  commutes with the Laplacian and, therefore, the Stokes operator  $A = -P\Delta_x$  is just a restriction of the Laplacian to the space  $H$  of divergence-free vector fields. Thus, the results of this section on spatial averaging and, in particular, Proposition 6.3 remain valid for the Stokes operator as well.

Let us now consider a special class of modified Navier-Stokes nonlinearities. First, in the spirit of Leray- $\alpha$  model, we define

$$\bar{u} := (1 - \alpha\Delta_x)^{-\bar{\gamma}}u$$

for some  $\bar{\gamma} \geq 0$  and then we consider the nonlinearity

$$f(u) := P(-\Delta_x)^{-\gamma}[(u \cdot \nabla_x)\bar{u}] = P(-\Delta_x)^{-\gamma}[(u \cdot \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}}u] \quad (6.18)$$

for the corresponding  $\bar{\gamma}, \gamma \geq 0$ . In order to have zero order nonlinearity, we need to add extra condition  $\gamma + \bar{\gamma} \geq \frac{1}{2}$ . Since the situation where this inequality is strict can be only simpler, we will assume from now on that

$$\gamma + \bar{\gamma} = \frac{1}{2}. \quad (6.19)$$

As we will see below, the limit case  $\bar{\gamma} = 0, \gamma = \frac{1}{2}$  corresponds to hyperviscous Navier-Stokes equation and another limit case  $\gamma = 0, \bar{\gamma} = \frac{1}{2}$  gives us the so-called Leray- $\alpha$ -Bardina model.

Note that the gradient also commutes with the Leray projector and with the Laplacian, so the derivative

$$\begin{aligned} f'(u)v &= P(-\Delta_x)^{-\gamma}[(u \cdot \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}}v] + \\ &\quad + P(-\Delta_x)^{-\gamma}[(v \cdot \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}}u] := B(u, v) + B(v, u) \end{aligned} \quad (6.20)$$

can be written as a sum of operators considered in Proposition 6.3 with  $\beta = 2\gamma$ . Moreover, the spatial averaging of every such a term is equal to zero due to the assumption that  $H$  consists of functions with zero mean. Thus, spatial averaging Assumption III is satisfied if  $s_0 > \frac{3}{2}$ .

Let us verify the regularity assumptions (Assumptions I-II) for the Navier-Stokes nonlinearity  $f$  given by (6.18). To this end, we recall that, we have actually proved in Proposition 6.3 that

$$\|f'(u)v\|_H \leq C\|u\|_{H^{s_0}}\|v\|_H. \quad (6.21)$$

Moreover, since the map  $u \rightarrow f'(u)$  is a *linear* continuous map from  $H^{s_0}$  to  $\mathcal{L}(H, H)$ , its Hölder continuity (as well as even  $C^\infty$ -smoothness) is also an immediate corollary of (6.21). Thus, we have verified properties a) and b) of Assumption II. To verify Assumption I, it is enough to note that  $f$  is a homogeneous quadratic form, so by Euler identity,

$$f(u) = \frac{1}{2}f'(u)u$$

and Assumption I also follows from (6.21). Thus, we only need to verify property c) of Assumption II, namely, that  $f'(u)$  is a bounded operator from  $H^{s_0}$  to  $H^{s_0}$ . To this end, we will use again the Kato-Ponce formula together with the proper Sobolev embeddings. Namely,

$$\begin{aligned} \|B(u, v)\|_{H^{s_0}} &\leq C\|(u \cdot \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}}v\|_{H^{s_0-2\bar{\gamma}}} \leq \\ &\leq C\|u\|_{L^\infty}\|v\|_{H^{s_0-2\bar{\gamma}-2\bar{\gamma}+1}} + \|u\|_{H^{s_0-2\bar{\gamma}, p}}\|v\|_{H^{1-2\bar{\gamma}, q}}, \end{aligned} \quad (6.22)$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ . The first term in the right-hand side of (6.22) is under control due to embedding  $H^{s_0} \subset L^\infty$  for  $s_0 > \frac{3}{2}$ . To estimate the second one, we fix  $\frac{1}{p} = \frac{1}{2} - \frac{2\bar{\gamma}}{3}$ . Then,  $H^{s_0} \subset H^{s_0-2\bar{\gamma}, p}$  and  $\frac{1}{q} = \frac{2\bar{\gamma}}{3}$ . Therefore,  $s_0 > \frac{3}{2}$  implies that

$$\frac{1}{q} > \frac{1}{2} - \frac{s_0 - (1 - 2\bar{\gamma})}{3}$$

and  $H^{s_0} \subset H^{1-2\bar{\gamma}, q}$ . Finally, (6.22) implies that

$$\|B(u, v)\|_{H^{s_0}} \leq C\|u\|_{H^{s_0}}\|v\|_{H^{s_0}}$$

and property c) of Assumption II is also verified.

Thus, for the existence of  $C^{1+\varepsilon}$ -smooth IM for such nonlinearities it is sufficient to verify the existence of an absorbing ball in the space  $H^s$ . This will be discussed in the next section.

## 7. APPLICATIONS

In this section we consider the applications of our general theory to several classes of equations. Note that the regularity and spatial averaging assumptions for the nonlinearities considered are already verified in the previous section, so it only remains to check the dissipativity in the proper spaces.

**7.1. Scalar Reaction-Diffusion equation.** Let us consider the equation

$$\partial_t u = \Delta_x u - u - f(u) + g, \quad u|_{t=0} = u_0, \quad g \in L^2((-\pi, \pi)^3) \quad (7.1)$$

endowed with the periodic boundary conditions. This equation is of the form (5.5) with  $\gamma = 0$  and  $Au := -\Delta_x u + u$ . Let us pose the following conditions on the scalar function  $f$ :

$$\begin{cases} 1. f \in C^4(\mathbb{R}, \mathbb{R}), \\ 2. f(u)u \geq -C, \quad u \in \mathbb{R}, \\ 3. f'(u) \geq -K, \quad u \in \mathbb{R}. \end{cases} \quad (7.2)$$

Then, as well-known (see, e.g., [3, 49, 51]), problem (7.1) is well-posed in the phase space  $H = L^2((-\pi, \pi)^2)$  and generates a dissipative semigroup  $\bar{S}(t) : H \rightarrow H$ :

$$\|u(t)\|_H \leq C\|u(0)\|_H e^{-\kappa t} + C(\|g\|_H + 1). \quad (7.3)$$

Moreover, the following smoothing property holds:

$$\|u(t)\|_{H^2} \leq C \frac{t+1}{t} (e^{-\kappa t}\|u_0\|_H + \|g\|_H + 1) \quad (7.4)$$

and therefore the invariant bounded absorbing set  $\mathcal{B} \subset H^2$  exists.

However, we need a bit more regularity, namely, the absorbing ball in the space  $H^s$  with  $3 < s < 4$ . To get this, we either need to require  $g \in H^2$  which looks a bit restrictive or to use the standard trick with introducing the auxiliary function  $G \in H^2$  as a solution of the following elliptic problem

$$\Delta_x G - G + g = 0. \quad (7.5)$$

Obviously, the solution of this problem exists and introducing the new variable  $v = u - G$ , we get the equivalent equation for  $v$ :

$$\partial_t v = \Delta_x v - v - f(v + G). \quad (7.6)$$

The advantage of this equation is that, due to the fact that  $H^2$  is algebra, we have the control of the  $H^2$ -norm of  $f(v + G)$  on the  $H^2$ -absorbing ball and then from linear parabolic smoothing property, we get the desired absorbing ball for  $v$  in the space  $H^{4-\varepsilon}$  for all  $\varepsilon > 0$  which is enough for our purposes. Since the shift  $u \rightarrow u + G$  does not affect the Assumptions I-III for the nonlinearity  $f$  in Example 6.4, we end up with the following result.

**Corollary 7.1.** *Let  $g \in H$  and the nonlinearity  $f$  satisfy assumptions (7.2). Then, there are infinitely many values of  $N$ , such that the reaction-diffusion equation (7.1) possesses  $C^{1+\varepsilon_N}$ -smooth IM in the sense of Definition 5.3. The truncated nonlinearity can be chosen in the form of (5.13) with exponents  $s_0$  and  $s$  satisfying (6.15).*

**Remark 7.2.** Of course, this result is well-known and has been first obtained in the pioneering paper [34], see also [25, 53] for the smoothness of the manifold. In fact,  $C^4$ -smoothness here also can be relaxed, but we start from this example just in order to show that our general theory covers this classical result.

**7.2. Cahn-Hilliard type equations.** We now turn to more interesting problem related with generalizations of the Cahn-Hilliard equation:

$$\partial_t u + (-\Delta_x)^\gamma (-\Delta_x u + f(u) + g) = 0, \quad u|_{t=0} = u_0 \quad (7.7)$$

in  $\Omega = (-\pi, \pi)^3$  endowed with periodic boundary conditions. The case  $\gamma = 1$  corresponds to the classical Cahn-Hilliard equation, see [51, 12, 39] and references therein for more details. The case  $0 < \gamma < 1$  is the so-called fractional Cahn-Hilliard equation which is of a big current interest, see [1] and references therein. The other choices of  $\gamma > 0$  are also interesting, for instance,  $\gamma = 2$  corresponds to the so-called 6th order Cahn-Hilliard equation, see [36] and references therein.

This equation has a natural (mass) conservation law:

$$\frac{d}{dt} \langle u(t) \rangle = 0,$$

so, without loss of generality, we may assume that  $\langle u \rangle = 0$  and consider  $A = -\Delta_x$  in the space  $H = \{u \in L^2((-\pi, \pi)^3), \langle u \rangle = 0\}$ . Then zero eigenvalue disappears and the operator  $A$  becomes positive definite. We also assume that  $g \in H$ . Then equation (7.7) is equivalent to the following one:

$$\partial_t (-\Delta_x)^{-\gamma} u - \Delta_x u + f(u) - \langle f(u) \rangle + g = 0, \quad (7.8)$$

so the equation is indeed in the form of (5.5). We pose exactly the same conditions (7.2) to the nonlinearity  $f$ . Then, verification of Assumptions I-III is also exactly the same as for the case of the reaction-diffusion equation (with the same values of  $s_0$  and  $s$ ) since the presence of the extra one-dimensional term  $\langle f(u) \rangle$  changes nothing. Thus, we only need to check the existence of the absorbing ball in  $H^s$ . Moreover, we only need this absorbing ball in  $H^2$  since the further regularity can be obtained in a straightforward way using the linear parabolic regularity estimates (similarly to the case of a reaction-diffusion equation). So, we will briefly discuss below only the well-posedness and  $H^2$ -regularity and dissipativity of solutions of (7.8) in  $H^2$ . This is a straightforward generalization of the standard Cahn-Hilliard theory (for  $\gamma = 1$ ), see [51, 12, 39] for more details.

Similarly to the classical CH-equation, the natural phase space for problem (7.8) is  $H^{-\gamma}$  since exactly in this case we may utilize the monotonicity of the nonlinearity  $f$  and get nice estimates in the same way as in Proposition 2.1. Indeed, multiplying equation (7.8) on  $u$ , integrating over  $x$  and using that  $f(u) \cdot u \geq -C$ , we get the following analogue of dissipative estimate (2.8):

$$\|u(t)\|_{H^{-\gamma}}^2 + \int_t^{t+1} \|u(s)\|_{H^1}^2 + |f(u(s)) \cdot u(s)| ds \leq C e^{-\kappa t} \|u_0\|_{H^{-\gamma}}^2 + C(1 + \|g\|_H^2) \quad (7.9)$$

for some positive constants  $C$  and  $\kappa$ .

Moreover, writing the equation for differences  $v(t) = u_1(t) - u_2(t)$  of two solutions, multiplying it by  $v(t)$ , using that  $f'(u) \geq -K$ , and arguing exactly as in the proof of Proposition 2.1, we get the global Lipschitz continuity (2.9). The existence of a solution can be obtained in a standard way using, e.g., the Galerkin approximations, see [3, 51]. Thus, we have verified the global well-posedness and dissipativity of the solution semigroup  $\bar{S}(t) : H^{-\gamma} \rightarrow H^{-\gamma}$  associated with the equation (7.7).

Let us now discuss the smoothing property. First, multiplying equation (7.8) by  $t\partial_t u$ , we get

$$\frac{d}{dt} \left( \frac{1}{2} t \|u\|_{H^1}^2 + t(\Phi(u), 1) + t(g, u) \right) + t \|\partial_t u(t)\|_{H^{-\gamma}}^2 = \frac{1}{2} \|u\|_{H^1}^2 + (\Phi(u), 1) + (g, u), \quad (7.10)$$

where  $\Phi(z) := \int_0^z f(s) ds$ . Using the elementary inequality

$$\Phi(u) \leq f(u) \cdot u + \frac{K}{2} |u|^2$$

and (7.9) for estimating the terms in the right-hand side of (7.10), we end up with

$$\|u(t)\|_{H^1}^2 + \int_t^{t+1} \|\partial_t u(s)\|_{H^{-\gamma}}^2 ds \leq C \frac{t+1}{t} (e^{-\kappa t} \|u_0\|_{H^{-\gamma}}^2 + 1 + \|g\|_H^2) \quad (7.11)$$

for some positive  $C$  and  $\kappa$ . This estimate gives us the absorbing ball for the semigroup  $\bar{S}(t)$  in  $H^1$ , but to get the desired  $H^2$ -smoothing, we need more steps.

At the next step, we differentiate equation (7.8) by  $t$  and denote  $\theta(t) := \partial_t u(t)$ . Then multiplying the result by  $t^2\theta(t)$ , using the fact that  $f'(u) \geq -K$  and the estimate for the integral norm of  $\theta(t)$  obtained in (7.11) analogously to (2.15), we get

$$\|\theta(t)\|_{H^{-\gamma}}^2 \leq C \frac{t^2 + 1}{t^2} (e^{-\kappa t} \|u(0)\|_{H^{-\gamma}}^2 + 1 + \|g\|_H^2) \quad (7.12)$$

for some positive  $C$  and  $\kappa$ .

Finally, analogously to (2.17), we write our problem as an elliptic problem

$$\Delta_x u(t) - f(u(t)) + \langle f(u(t)) \rangle = A^{-\gamma} \theta(t) + g$$

and multiply this equation by  $\Delta_x u(t)$  followed by integration over  $x$ . Then, using the obtained estimates (7.11) and (7.12) for  $\theta(t)$  and the assumption  $f'(u) \geq -K$ , we arrive at

$$\|u(t)\|_{H^2}^2 \leq C \frac{t^2 + 1}{t^2} (e^{-\kappa t} \|u(0)\|_{H^{-\gamma}}^2 + 1 + \|g\|_H^2) \quad (7.13)$$

which gives us the desired existence of an absorbing ball in the space  $H^2$ . Since  $H^2((-\pi, \pi)^3)$  is an algebra the further smoothing estimates are straightforward and we have proved the following result.

**Corollary 7.3.** *Let  $g \in H$  and the nonlinearity  $f$  satisfy assumptions (7.2). Then, there are infinitely many values of  $N$ , such that the Cahn-Hilliard type equation (7.7) possesses  $C^{1+\varepsilon_N}$ -smooth IM in the sense of Definition 5.3. The truncated nonlinearity can be chosen in the form of (5.13) with an extra term  $\langle f(W(u)) \rangle$  and exponents  $s_0$  and  $s$  satisfying (6.15).*

**Remark 7.4.** For the case  $\gamma = 1$  which corresponds to the classical Cahn-Hilliard equation, this result has been established in [25]. However, for other values of  $\gamma > 0$  this result seems new. One of the main achievements of our approach is that we can treat all the cases  $\gamma > 0$  as well as  $\gamma = 0$  from the unified point of view.

**7.3. Modified Navier-Stokes equations.** We now turn to the other class of examples related with Navier-Stokes equations which also fits our general theory. Namely, we will consider the following class of modified 3D Navier-Stokes equations:

$$\begin{cases} \partial_t u + (u, \nabla_x \bar{u}) + (-\Delta_x)^{1+\gamma} u + \nabla_x p = g, & u|_{t=0} = u_0, \\ \operatorname{div} u = 0, & \bar{u} = (1 - \alpha \Delta_x)^{-\bar{\gamma}} u. \end{cases} \quad (7.14)$$

The case  $\gamma = \bar{\gamma} = 0$  corresponds to the classical 3D Navier-Stokes problem. However, the global well-posedness of this problem is out of reach of the modern theory and is actually one of the Millennium problems, see [13] and references therein, so some modifications/regularisations look unavoidable. Introducing the truncated variable  $\bar{u}$  is in the spirit of Leray  $\alpha$ -regularization or the so-called Bardina model, see [2, 5, 6] and references therein. The term  $(-\Delta_x)^{1+\gamma} u$  with  $\gamma > 0$  gives an alternative popular type of regularization - the so-called hyperviscous regularization of the Navier-Stokes problem, see [21, 30, 40].

The IM theory requires extra assumptions on the exponents  $\gamma$  and  $\bar{\gamma}$  in comparison with well-posedness. For instance, for the 2D case the classical Navier-Stokes equations are globally well-posed, but for the existence of an IM for periodic boundary conditions, we still need  $\bar{\gamma} \geq \frac{1}{2}$ , see [20], and the existence of an IM for  $\bar{\gamma} < \frac{1}{2}$  is still an open problem.

For problem (7.14) the borderline for the IM theory is given by the condition

$$\gamma + \bar{\gamma} = \frac{1}{2}, \quad \gamma \in [0, \frac{1}{2}]. \quad (7.15)$$

As we will see, under this assumption, equation (7.14) can be reduced to our abstract equation (5.5) and the existence of an IM follows from the general theory. By this reason, in order to avoid technicalities, we restrict ourselves to consider the case of equality (7.15) only. The case when  $\gamma + \bar{\gamma} > \frac{1}{2}$  is similar (but simpler since we have the extra regularity for the nonlinearity) and also fits our theory, but the case  $\gamma + \bar{\gamma} < \frac{1}{2}$  is out of reach of the theory and remains an open problem.

To embed this problem into a general theory developed above, we take as in Example 6.6 the space  $H$  as the space of divergent free vector fields defined by (6.17) and rewrite (7.14) in the equivalent form

$$A^{-\gamma} \partial_t u + Au + A^{-\gamma} P[(u, \nabla_x)(1 - \alpha \Delta_x)^{-\bar{\gamma}} u] = A^{-\gamma} g, \quad u|_{t=0} = u_0, \quad (7.16)$$

where  $P$  is a Leray projector to the divergent free vector fields and  $A$  is a Stokes operator which coincides in the case of periodic boundary conditions with the restriction of the minus Laplacian to the space  $H$ . We also assume that  $g \in H$ .

Equation (7.16) has the form of (5.5) with the nonlinearity (6.18) considered in Example 6.6. As we have established there, this nonlinearity satisfies our Assumptions I-III for the IM-existence theorem with exponents  $s$  and  $s_0$  satisfying (6.15). Thus, in order to get the desired existence of  $C^{1+\varepsilon}$  IM for problem (7.14), we only need to verify the well-posedness in the proper phase space (which in general need not to coincide with  $H^{-\gamma}$ ) and the existence of an absorbing set, bounded in  $H^s$  for some  $3 < s < 4$ .

The well-posedness and regularity theory for the Navier-Stokes type equations of the form (7.14) is also well-understood nowadays, so we will restrict ourselves only to a brief exposition indicating the key features, see [21, 22, 30, 40] and references therein for more details.

The natural phase space for problem (7.14) is  $H^{-\bar{\gamma}}$ . This is related with the fact that the analogue of the energy estimate holds exactly in this space. Indeed, as known

$$((u, \nabla_x v), v) \equiv 0, \quad u, v \in H^1,$$

so the multiplication of equation (7.14) by  $\bar{u} = (1 - \alpha\Delta_x)^{-\bar{\gamma}}u$  gives

$$\frac{1}{2} \frac{d}{dt} \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} u\|_H^2 + \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} u\|_{H^{1+\gamma}}^2 = (g, (1 - \alpha\Delta_x)^{-\bar{\gamma}} u) \quad (7.17)$$

and applying the Gronwall inequality to this relation, we end up with the desired dissipative estimate in  $H^{-\bar{\gamma}}$ :

$$\|u(t)\|_{H^{-\bar{\gamma}}}^2 + \int_t^{t+1} \|u(s)\|_{H^{1+\gamma-\bar{\gamma}}}^2 ds \leq C e^{-\kappa t} \|u_0\|_{H^{-\bar{\gamma}}}^2 + C(1 + \|g\|_H^2) \quad (7.18)$$

for some positive  $\kappa$  and  $C$ . We gave only formal derivation of this estimate, but it can be easily justified, say, by the Galerkin method.

The restriction for the uniqueness of a solution and further regularity reads

$$2\gamma + \bar{\gamma} \geq \frac{1}{2}. \quad (7.19)$$

Indeed, let us indicate how get the uniqueness under this assumption. Let  $u_1(t)$  and  $u_2(t)$  be two solutions of (7.14) and let  $v(t) = u_1(t) - u_2(t)$ . Then, writing the equation on  $v(t)$  and multiplying it by  $\bar{v} := (1 - \alpha\Delta_x)^{-\bar{\gamma}}v$ , we end up with

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} v\|_H^2 + \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} v\|_{H^{1+\gamma}}^2 = \\ = -((v, \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}} u_1, (1 - \alpha\Delta_x)^{-\bar{\gamma}} v). \end{aligned} \quad (7.20)$$

As an elementary exercise on Sobolev's embeddings and Hölder inequality, one gets

$$|((v, \nabla_x)(1 - \alpha\Delta_x)^{-\bar{\gamma}} u_1, (1 - \alpha\Delta_x)^{-\bar{\gamma}} v)| \leq C \|v\|_{H^{1+\gamma-\bar{\gamma}}} \|u_1\|_{H^{1+\gamma-\bar{\gamma}}} \|v\|_{H^{-\bar{\gamma}}}$$

if the criticality assumption (7.19) is satisfied (we left the details of this exercise to the reader). Inserting this estimate to (7.20), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} v\|_H^2 \leq C \|u_1(t)\|_{H^{1+\gamma-\bar{\gamma}}}^2 \|(1 - \alpha\Delta_x)^{-\bar{\gamma}/2} v\|_H^2 \quad (7.21)$$

and the Gronwall inequality applied to this relation together with the control of the proper norm of  $u_1$  obtained in (7.18) gives the desired uniqueness. Note that condition (7.19) is weaker than our assumption (7.15), so we have verified that equation (7.14) generates a dissipative semigroup  $\bar{S}(t) : H^{-\bar{\gamma}} \rightarrow H^{-\bar{\gamma}}$ .

Moreover, we see that under the condition (7.15), the considered equation has a critical nonlinearity only in the case  $\gamma = 0$  and  $\bar{\gamma} = \frac{1}{2}$ . The parabolic smoothing property for this case is discussed in details in [32] (see also [19] for other end-point case  $\gamma = \frac{1}{2}$ ,  $\bar{\gamma} = 0$ ), so we will not present this analysis here. The case  $\gamma > 0$  is *sub-critical* and the further regularity and the existence of absorbing balls in smoother spaces are standard corollaries of the *linear* parabolic smoothing estimates and bootstrapping arguments. So, the actual smoothness of the solution is restricted by the smoothness of the nonlinearity and external forces only. Under our assumption  $g \in H$ , we may guarantee the existence of the absorbing ball in  $H^2$  only, but the trick with subtraction of an equilibrium described in subsection 7.1 allows us to get the desired absorbing ball in  $H^s$  with  $s < 4$ . Thus, we have proved the following result.

**Corollary 7.5.** *Let  $g \in H$  and the exponents  $\gamma$  and  $\bar{\gamma}$  satisfy (7.15). Then, there are infinitely many values of  $N$ , such that the Navier-Stokes type equation (7.14) possesses  $C^{1+\varepsilon_N}$ -smooth IM in the sense of Definition 5.3. The truncated nonlinearity can be chosen in the form of (5.8) and (6.18) and exponents  $s_0$  and  $s$  satisfying (6.15).*

**Remark 7.6.** The existence of a Lipschitz IM for (7.14) in 2D-case with  $\gamma = 0$ ,  $\bar{\gamma} = 1$  has been obtained in [20] based on verifying the spectral gap conditions. The IM in the 3D case with  $\gamma = 0$  and  $\bar{\gamma} = 1$  has been constructed in [23] based on a novel idea to use spatial averaging technique for Navier-Stokes type equations (in particular, the special form of the cut off function  $W(u)$  which is crucial for this approach has been also suggested there). The end points  $\gamma = 0$ ,  $\bar{\gamma} = \frac{1}{2}$  and  $\gamma = \frac{1}{2}$ ,  $\bar{\gamma} = 0$  have been treated in [32] and [19] respectively. However, in the intermediate case  $0 < \gamma < \frac{1}{2}$  our result seems new. In addition, to the best of our knowledge, the question about  $C^{1+\varepsilon}$ -smoothness of the IMs for the modified Navier-Stokes equations has been never considered before. We also emphasize that all the previous partial results for IMS related with equation (7.14) as well as the new ones are now obtained in a unified way as corollaries of a general theorem.

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