Finding Sparse Solutions for Packing and Covering Semidefinite Programs

Khaled Elbassioni* Kazuhisa Makino†

Abstract

Packing and covering semidefinite programs (SDPs) appear in natural relaxations of many combinatorial optimization problems as well as a number of other applications. Recently, several techniques were proposed, that utilize the particular structure of this class of problems, to obtain more efficient algorithms than those offered by general SDP solvers. For certain applications, such as those described in this paper, it may be desirable to obtain *sparse* dual solutions, i.e., those with support size (almost) independent of the number of primal constraints. In this paper, we give an algorithm that finds such solutions, which is an extension of a *logarithmic-potential* based algorithm of Grigoriadis, Khachiyan, Porkolab and Villavicencio (SIAM Journal of Optimization 41 (2001)) for packing/covering linear programs.

1 Introduction

1.1 Packing and Covering SDPs

We denote by $\mathbb{S}^n$ the set of all $n \times n$ real symmetric matrices and by $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ the set of all $n \times n$ positive semidefinite matrices. Consider the following pairs of *packing-covering* semidefinite programs (SDPs):

$$\begin{aligned} z_I^* = \max \quad & C \bullet X \qquad \text{(PACKING-I)} \\ \text{s.t.} \quad & A_i \bullet X \le b_i, \forall i \in [m] \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 \end{aligned}$$

$$\begin{aligned} z_I^* = \min \quad & b^T y \qquad \text{(COVERING-I)} \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i \succeq C \\ & y \in \mathbb{R}^m,\ y \ge 0 \end{aligned}$$

$$\begin{aligned} z_{II}^* = \min \quad & C \bullet X \qquad \text{(COVERING-II)} \\ \text{s.t.} \quad & A_i \bullet X \ge b_i, \forall i \in [m] \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 \end{aligned}$$

$$\begin{aligned} z_{II}^* = \max \quad & b^T y \qquad \text{(PACKING-II)} \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i \preceq C \\ & y \in \mathbb{R}^m,\ y \ge 0 \end{aligned}$$

where $C, A_1, \ldots, A_m \in \mathbb{S}^n_+$ are (non-zero) positive semidefinite matrices, and $b = (b_1, \ldots, b_n)^T \in \mathbb{R}^m_+$ is a nonnegative vector. In the above, $C \bullet X := \text{Tr}(CX) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$, and "$\succeq$" is the *Löwner order* on matrices: $A \succeq B$ if and only if $A - B$ is positive semidefinite. This type of SDPs arise in many applications, see, e.g. [20, 21] and the references therein.

We will make the following assumption throughout the paper:

(A) $b_i > 0$ and hence $b_i = 1$ for all $i \in [m]$.

It is known that, under assumption (A), *strong duality* holds for problems (PACKING-I)-(COVERING-I) (resp., (PACKING-II)-(COVERING-II)) (see Appendix B for details).

Let $\epsilon \in (0, 1]$ be a given constant. We say that (X, y) is an ϵ-*optimal* primal-dual solution for (PACKING-I)-(COVERING-I) if (X, y) is a primal-dual feasible pair such that

$$C \bullet X \ge (1 - \epsilon) b^T y \ge (1 - \epsilon) z_I^*. \tag{1}$$

Similarly, we say that (X, y) is an ϵ-optimal primal-dual solution for (PACKING-II)-(COVERING-II) if (X, y) is a primal-dual feasible pair such that

$$C \bullet X \le (1 + \epsilon) b^T y \le (1 + \epsilon) z_{II}^*. \tag{2}$$

Since in this paper we allow the number of constraints m in (PACKING-I) (resp., (COVERING-II)) to be *exponentially* (or even infinitely) large, we will assume the availability of the following *oracle*:

*Masdar Institute, Khalifa University of Science and Technology, P.O. Box 54224, Abu Dhabi, UAE; (kelbassioni@masdar.ac.ae)
†Research Institute for Mathematical Sciences (RIMS) Kyoto University, Kyoto 606-8502, Japan; (makino@kurims.kyoto-u.ac.jp)

Max(Y) **(resp., Min**(Y)**)** : Given $Y \in \mathbb{S}_+^n$, find $i \in \text{argmax}_{i\in[m]} A_i \bullet Y$ (resp., $i \in \text{argmin}_{i\in[m]} A_i \bullet Y$).

Note that an *approximation* oracle computing the maximum (resp., minimum) above within a factor of $(1-\epsilon)$ (resp., $(1+\epsilon)$) is also sufficient for our purposes.

A primal-dual solution (X, y) to (COVERING-I) (resp., (PACKING-II)) is said to be η-*sparse*, if the size of $\text{supp}(y) := \{i \in [m] : y_i > 0\}$ is at most η. Our objective in this paper is to develop *primal-dual* algorithms that find sparse ϵ-optimal solutions for (PACKING-I)-(COVERING-I) and (PACKING-II)-(COVERING-II).

1.2 Reduction to Normalized Form

When $C = I = I_n$, the identity matrix in $\mathbb{R}^{n\times n}$ and $b = \mathbf{1}$, the vector of all ones in $\mathbb{R}^m$, we say that the packing-covering SDPs are in *normalized* form:

$$\begin{aligned} z_I^* = \max \quad & I \bullet X \qquad \text{(NORM-PACKING-I)} \\ \text{s.t.} \quad & A_i \bullet X \le 1, \forall i \in [m] \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 \end{aligned}$$

$$\begin{aligned} z_I^* = \min \quad & \mathbf{1}^T y \qquad \text{(NORM-COVERING-I)} \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i \succeq I \\ & y \in \mathbb{R}^m,\ y \ge 0. \end{aligned}$$

$$\begin{aligned} z_{II}^* = \min \quad & I \bullet X \qquad \text{(NORM-COVERING-II)} \\ \text{s.t.} \quad & A_i \bullet X \ge 1, \forall i \in [m] \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 \end{aligned}$$

$$\begin{aligned} z_{II}^* = \max \quad & \mathbf{1}^T y \qquad \text{(NORM-PACKING-II)} \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i \preceq I \\ & y \in \mathbb{R}^m,\ y \ge 0. \end{aligned}$$

In the appendix, we show that, at the loss of a factor of $(1+\epsilon)$ in the objective, any pair of packing-covering SDPs of the form (PACKING-I)-(COVERING-I) can be brought in $O(n^3)$, increasing the oracle time only by $O(n^\omega)$, where ω is the exponent of matrix multiplication, to the normalized form (NORM-PACKING-I)-(NORM-COVERING-I), under the following assumption:

(B-I) There exist r matrices, say $A_1, \ldots, A_r$, such that $\bar{A} := \sum_{i=1}^r A_i \succ 0$. In particular, $\text{Tr}(X) \le \tau := \frac{r}{\lambda_{\min}(\bar{A})}$ for any optimal solution X for (PACKING-I).

Similarly, we show in the appendix (some of the results are reproduced with simplifications from [22]) that, at the loss of a factor of $(1+\epsilon)$ in the objective, any pair of packing-covering SDPs of the form (PACKING-II)-(COVERING-II) can be brought in $O(n^3)$ time, increasing the oracle time only by $O(n^\omega)$, to the normalized form (NORM-PACKING-II)-(NORM-COVERING-II). Moreover, we may assume in this normalized form that

(B-II) $\lambda_{\min}(A_i) = \Omega\big(\frac{\epsilon}{n} \cdot \min_{i'} \lambda_{\max}(A_{i'})\big)$ for all $i \in [m]$,

where, for a positive semidefinite matrix $B \in \mathbb{S}_+^{n\times n}$, we denote by $\{\lambda_j(B) :\ j = 1, \ldots, n\}$ the eigenvalues of B, and by $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ the minimum and maximum eigenvalues of B, respectively. With an additional $O(mn^2)$ time, we may also assume that:

(B-II′) $\frac{\lambda_{\max}(A_i)}{\lambda_{\min}(A_i)} = O\big(\frac{n^2}{\epsilon^2}\big)$ for all $i \in [m]$.

Thus, from now on we focus on the normalized problems.

1.3 Main Result and Related Work

Problems (PACKING-I)-(COVERING-I) and (PACKING-II)-(COVERING-II) can be solved using general SDP solvers, such as interior-point methods: for instance, the barrier method (see, e.g., [27]) can compute a solution, within an *additive* error of ϵ from the optimal, in time $O(\sqrt{n}m(n^3 + mn^2 + m^2)\log\frac{1}{\epsilon})$ (see also [1, 37]). However, due to the special nature of (PACKING-I)-(COVERING-I) and (PACKING-II)-(COVERING-II), better algorithms can be obtained. Most of the improvements are obtained by using *first order methods* [4, 5, 7, 2, 14, 20, 22, 23, 24, 30, 31, 32], or second order methods [19, 21]. In general, we can classify these algorithms according to whether they are:

(I) *width-independent*: the running time of the algorithm depends *polynomially* on the bit length of the input; for example, in the of case of (PACKING-I)-(COVERING-I), the running time is $\text{poly}(n, m, \mathcal{L}, \log\tau, \frac{1}{\epsilon})$, where $\mathcal{L}$ is the maximum bit length needed to represent any number in the input; on the other hand, the running time of a width-dependent algorithm will depend polynomially on a“width parameter” ρ, which is polynomial in $\mathcal{L}$ and τ;

(II) *parallel*: the algorithm takes $\text{polylog}(n, m, \mathcal{L}, \log\tau) \cdot \text{poly}(\frac{1}{\epsilon})$ time, on a $\text{poly}(n, m, \mathcal{L}, \log\tau, \frac{1}{\epsilon})$ number of processors;

Table 1: Different Algorithms for Packing/covering SDPs

Paper	Problem	Technique	Most Expensive Operation	# Iterations	Width-indep.	Parallel	Sparse	Oracle-based
[4, 24]	(PACKING-I) (COVERING-II)	MWU	$\max$ / $\min$ eigenvalue of a PSD matrix $\tilde{O}(\frac{n^2}{\epsilon})$	$O(\frac{\rho \log m}{\epsilon^2})$	No	No	No*	No
[7]	(PACKING-I) (COVERING-II)	Matrix MWU	Matrix exponentiation $O(n^3)$	$O(\frac{\rho^2 \tau^2 \log n}{\epsilon^2 (z_I^*)^2})$	No	No	No*	Yes
[19, 20]	(PACKING-I)	Nesterov's smoothing technique [28, 30]	Matrix exponentiation $O(n^3)$	$O(\frac{\tau \log m}{\epsilon})$	No	No	No	No
[21]	(COVERING-II)	Nesterov's smoothing technique [28, 30]	$\min$ eigenvalue of a non PSD matrix $O(n^3)$	$O(\frac{\rho^2 \log(nm)}{\epsilon})$	No	No	No	No
[22]	(PACKING-I)& (COVERING-II)	MWU technique [28, 30]	eigenvalue decomposition $O(n^3)$	$O(\frac{\log^{13} n \log m}{\epsilon^{13}})$	Yes	Yes	No	No
[31, 32]	(PACKING-II)& (COVERING-II)	Matrix MWU	Matrix exponentiation $O(n^3)$	$O(\frac{\log^3 m}{\epsilon^3})$	Yes	Yes	No	No
[2]	(PACKING-I)& (COVERING-II)	Gradient Descent + Mirror Descent	Matrix exponentiation $O(n^3)$	$O(\frac{\log^2(mn) \log \frac{1}{\epsilon}}{\epsilon^2})$	Yes	Yes	No	No
This paper Appendix A	(PACKING-II)& (COVERING-II)	Matrix MWU	Matrix exponentiation $O(n^3)$	$O(\frac{n \log n}{\epsilon^2})$	Yes	No	Yes	Yes
This paper	(PACKING-II) & (COVERING-II) (PACKING-II) & (COVERING-II)	Logarithmic potential [17]	Matrix inversion $O(n^\omega)$	$O(n \log(n\mathcal{L}\tau) + \frac{n}{\epsilon^2})$ $O(n \log(n/\epsilon) + \frac{n}{\epsilon^2})$	Yes	No	Yes	Yes

* In fact, these algorithms find sparse solutions, in the sense that the dependence of the size of the support of the dual solution on m is at most logarithmic; however, the dependence of the size of the support on the bit length $\mathcal{L}$ is not polynomial.

(III) *output sparse solutions*: the algorithm outputs an η-sparse solution to (COVERING-I) (resp., (PACKING-II)), for $\eta = \text{poly}(n, \log m, \mathcal{L}, \log \tau, \frac{1}{\epsilon})$ (resp., $\eta = \text{poly}(n, \log m, \mathcal{L}, \frac{1}{\epsilon})$), where τ is a parameter that bounds the trace of any optimal solution X (see Section 1.2 for details);

(IV) *oracle-based*: the only access of the algorithm to the matrices $A_1, \ldots, A_m$ is via the maximization/minimization oracle, and hence the running time is independent of m.

Table 1.3 below gives a summary[1] of the most relevant results together with their classifications, according to the four criteria described above. We note that almost all these algorithms for packing/covering SDP's are generalizations of similar algorithms for packing/covering linear programs (LPs), and most of them are essentially based on an *exponential potential function* in the form of *scalar exponentiation*, e.g., [4, 24], or *matrix exponentiation* [5, 7, 2, 23, 20]. For instance, several of these results use the scalar or matrix versions of the *multiplicative weights updates* (MWU) method (see, e.g., [6]), which are extensions of similar methods for packing/covering LPs [15, 16, 38, 33].

In [17], a different type of algorithm was given for covering LPs (indeed, more generally, for a class of concave covering inequalities) based on a *logarithmic* potential function. In this paper, we show that this approach can be extended to provide sparse solutions for both versions of packing and covering SDPs.

As we can see from the table, among all the algorithms, the logarithmic-potential algorithm, presented in this paper, is the only one that produces sparse solutions, in the sense described above. We also show in Appendix A that a modified version of the matrix exponential MWU algorithm [5] can yield sparse solutions for (PACKING-II)-(COVERING-II). However, the overall running time of this matrix MWU algorithm is larger by a factor of (roughly) $\Omega(n^{3-\omega})$ than that of the logarithmic-potential algorithm, where ω is the exponent of matrix multiplication. Moreover, we were not able to extend the matrix MWU algorithm to solve (PACKING-I)-(COVERING-I) (in particular, it seems tricky to bound the number of iterations).

A work that is also related to ours is the sparsification of graph Laplacians [8] and positive semidefinite sums [35]. Given matrices $A_1, \ldots, A_m \in \mathbb{S}_+^n$ and $\epsilon > 0$, it was shown in [35] that one can find, in $O\big(\frac{n}{\epsilon^2}(n^\omega + \mathcal{T})\big)$ time, a vector $y \in \mathbb{R}_+^m$ with support size $O(\frac{n}{\epsilon}^2)$, such that $B \preceq \sum_i y_i A_i \preceq (1+\epsilon)B$, where $B := \sum_i A_i$ and $\mathcal{T}$ is the time taken by a single call to the minimization oracle Min(Y) (for a not necessarily positive semidefinite matrix Y). An immediate corollary is that, given an ϵ-optimal solution y for (COVERING-I) (resp., (PACKING-I)), one can find in $O\big(\frac{n}{\epsilon^2}(n^\omega + \mathcal{T})\big)$ time an $O(\epsilon)$-optimal solution y' with support size $O(\frac{n}{\epsilon^2})$. Interestingly, the algorithm in [35] (which is an extension for the rank-one version in [8]) uses the *barrier potential function* $\Phi'(x, F) := \text{Tr}\big((H - xI)^{-1}\big)$ (resp., $\Phi'(x, H) := \text{Tr}\big((xI - H)^{-1}\big)$), while in our algorithms (generalizing the potential function in [17]) we use the logarithmic potential function $\Phi(x, H) = \ln x + \frac{\epsilon}{n} \ln \det\big(H - xI\big) = \ln x - \frac{\epsilon}{n}\int_x \Phi'(x, H)dx$ (resp., $\Phi(x, H) = \ln x - \frac{\epsilon}{n} \ln \det\big(xI - H\big) = \ln x - \frac{\epsilon}{n}\int_x \Phi'(x, H)dx$). Sparsification algorithms with better running times were recently obtained in [3, 25]. Since the sparse solutions produced

[1]We provide rough estimates of the bounds, as some of them are not stated explicitly in the corresponding paper in terms of the parameters we consider here.

by our algorithms may have support size slightly more (by polylogarithmic factors) than $O(\frac{n}{\epsilon^2})$, we may use, in a post-processing step, the sparsfication algorithms, mentioned above, to convert our solutions to ones with support size $O(\frac{n^2}{\epsilon})$, without increasing the overall asymptotic running time.

To motivate our algorithms, in Section 3, we give two applications, mainly in robust optimization, that require finding sparse solutions for a packing/covering SDP.

2 A Logarithmic Potential Algorithm

2.1 Algorithm for (PACKING-I)-(COVERING-I)

In this section we give an algorithm for finding a sparse $O(\epsilon)$-optimal primal-dual solution for (PACKING-I)-(COVERING-I).

High-level Idea of the Algorithm. The idea of the algorithm is quite intuitive. It can be easily seen that problem (NORM-COVERING-I) is equivalent to finding a convex combination of the A_i's that maximizes the minimum eigenvalue, that is, $\max_{y \in \mathbb{R}_+^m : \mathbf{1}^T y = 1} \lambda_{\min}(F(y))$, where $F(y) := \sum_{i=1}^m y_i A_i$, and $\mathbf{1}$ is the m-dimensional vector of all ones. Since $\lambda_{\min}(F(y))$ is not a *smooth* function in y, it is more convenient to work with a smooth approximation of it, which is obtained by maximizing (over x) a *logarithmic potential function* $\Phi(x, F(y))$ that captures the constraints that each eigenvalue of $F(y)$ is at least x. The unique maximizer $x = \theta^*$ of $\Phi(x, F(y))$ defines a set of "weights" (these are the eigenvalues of the primal matrix X computed in line 6 of the algorithm) such that the weighted average of the $\lambda_j(F(y))$'s is a very close approximation of $\lambda_{\min}(F(y))$. Thus, to maximize this average (which is exactly $X \bullet F(y)$), we obtain a direction (line 7) along which y is modified with an appropriate step size (line 10).

For numbers $x \in \mathbb{R}_+$ and $\delta \in (0, 1)$, a δ-(lower) approximation x_δ of x is a number such that $(1 - \delta)x \le x_\delta < x$. For $i \in [m]$, $\mathbf{1}_i$ denotes the ith unit vector of dimension m.

The algorithm is shown as Algorithm 1. The main while-loop (step 4) is embedded within a sequence of scaling phases, in which each phase starts from the vector $y(t)$ computed in the previous phase and uses double the accuracy. The algorithm stops when the scaled accuracy ε_s drops below the desired accuracy $\epsilon \in (0, 1/2)$.

1 $s \leftarrow 0$; $\varepsilon_0 \leftarrow \frac{1}{2}$; $t \leftarrow 0$; $\nu(0) \leftarrow 1$; $y(0) \leftarrow \frac{1}{r}\sum_{i=1}^{r} \mathbf{1}_i$
2 **while** $\varepsilon_s > \epsilon$ **do**
3     $\delta_s \leftarrow \frac{\varepsilon_s^3}{32n}$
4   **while** $\nu(t) > \varepsilon_s$ **do**
5         $\theta(t) \leftarrow \theta^*(t)_{\delta_s}$, where $\theta^*(t)$ is the smallest positive number root of the equation $\frac{\varepsilon_s \theta}{n}\mathrm{Tr}(F(y(t)) - \theta I)^{-1} = 1$
6         $X(t) \leftarrow \frac{\varepsilon_s \theta(t)}{n}(F(y(t)) - \theta(t) I)^{-1}$ /* Set the primal solution */
7         $i(t) \leftarrow \mathrm{argmax}_i\, A_i \bullet X(t)$ /* Call the maximization oracle */
8         $\nu(t+1) \leftarrow \dfrac{X(t) \bullet A_{i(t)} - X(t) \bullet F(y(t))}{X(t) \bullet A_{i(t)} + X(t) \bullet F(y(t))}$ /* Compute the error */
9         $\tau(t+1) \leftarrow \dfrac{\varepsilon_s \theta(t) \nu(t+1)}{4n(X(t) \bullet A_{i(t)} + X(t) \bullet F(y(t)))}$ /* Compute the step size */
10         $y(t+1) \leftarrow (1 - \tau(t+1))y(t) + \tau(t+1)\mathbf{1}_{i(t)}$ /* Update the dual solution */
11         $t \leftarrow t + 1$
12   **end**
13     $\varepsilon_{s+1} \leftarrow \frac{\varepsilon_s}{2}$
14     $s \leftarrow s + 1$
15 **end**
16 $\hat{X} \leftarrow \frac{(1-\varepsilon_{s-1})X(t-1)}{(1+\varepsilon_{s-1})^2 \theta(t-1))}$; $\hat{y} \leftarrow \frac{y(t-1)}{\theta(t-1)}$
17 **return** $(\hat{X}, \hat{y}, t)$

Algorithm 1: Logarithmic-potential Algorithm for (PACKING-I)-(COVERING-I)

2.2 Analysis

High-level Idea of the Analysis. The proof of ϵ-optimality follows easily from the stopping condition in line 4 of the algorithm, the definition of the "approximation error" ν in line 8, and the fact that $X \bullet F(y)$ is a very close approximation of $\lambda_{\min}(F(y(t)))$. The main part of the proof is to bound the number of iterations in the inner while-loop (line 4). This is done by using a *potential function argument*: we define the potential function $\Phi(t) := \Phi(\theta^*(t), F(y(t)))$ and show in Claim 14 that, in each iteration, the choice of the step size in line 9 guarantees that $\Phi(t)$ is increased substantially; on the other hand, by Claim 15, the potential difference cannot be very large, and the two claims together imply that we cannot have many iterations.

2.2.1 Some Preliminaries

Up to Claim 17, we fix a particular iteration s of the outer while-loop in the algorithm. For simplicity in the following, we will sometimes write $F := F(y(t))$, $\theta := \theta(t)$, $\theta^* := \theta^*(t)$, $X := X(t)$, $\hat{F} := A_{i(t)}$, $\tau := \tau(t+1)$, $\nu := \nu(t+1)$, $F' := F(y(t+1))$, and $\theta' := \theta(t+1)$, when the meaning is clear from the context. For $H \succ 0$ and $x \in (0, \lambda_{\min}(H))$, define the *logarithmic potential function* [17, 27]:

$$\Phi(x, H) = \ln x + \frac{\varepsilon_s}{n} \ln\det\big(H - xI\big). \tag{3}$$

Note that the term $\ln\det\big(H - xI\big)$ forces the value of x to stay away from the "boundary" $\lambda_{\min}(H)$, while the term $\ln x$ pushes x towards that boundary; hence, one would expect the maximizer of $\Phi(x, H)$ to be a good approximation of $\lambda_{\min}(H)$ (see Claim 3).

Claim 1. *If $F(y(t)) \succ 0$, then $\theta^*(t) = \operatorname{argmax}_{0<x<\lambda_{\min}(F)} \Phi(x, F(y(t)))$ and $X(t) \succ 0$.*

Proof. Note that

$$\frac{d\Phi(x,F)}{dx} = \frac{1}{x} - \frac{\varepsilon_s}{n}\mathrm{Tr}\big((F - xI)^{-1}\big) \quad \text{ and } \quad \frac{d^2\Phi(x,F)}{dx^2} = -\frac{1}{x^2} - \frac{\varepsilon_s}{n}\mathrm{Tr}\big((F - xI)^{-2}\big).$$

Thus, if $F \succ 0$, then $\frac{d^2\Phi(x,F)}{dx^2} = -\frac{1}{x^2} - \frac{\varepsilon_s}{n}\sum_j \frac{1}{(\lambda_j(F)-x)^2} < 0$ for all $x \in (0, \lambda_{\min}(F))$. Thus $\Phi(x, F)$ is *strictly concave* in $x \in (0, \lambda_{\min}(F))$ and hence has a unique maximizer defined by setting $\frac{d\Phi(x,F)}{dx} = 0$, giving the definition $\theta^*(t)$ in step 5. Also, by definition of X in step 6, $\lambda_{\min}(X) = \frac{\varepsilon_s \theta}{n}(\lambda_{\min}(F) - \theta)^{-1} > 0$ (as $\theta < \theta^* < \lambda_{\min}(F)$), implying that $X \succ 0$. $\square$

For $x \in (0, \lambda_{\min}(F))$, let $g(x) := \dfrac{\varepsilon_s x}{n}\mathrm{Tr}(F - xI)^{-1}$. The following claim shows that our choice of δ_s guarantees that $g(\theta)$ is a good approximation of $g(\theta^*) = 1$.

Claim 2. $g(\theta(t)) \in (1 - \varepsilon_s, 1)$.

Proof. For $x \in (0, \lambda_{\min}(F))$, we have

$$\frac{dg(x)}{dx} = \frac{\varepsilon_s}{n}\sum_{j=1}^n \frac{1}{\lambda_j(F) - x} + \frac{\varepsilon_s x}{n}\sum_{j=1}^n \frac{1}{(\lambda_j(F)-x)^2} > 0, \tag{4}$$

$$\frac{d^2 g(x)}{dx^2} = \frac{2\varepsilon_s}{n}\sum_{j=1}^n \frac{1}{(\lambda_j(F) - x)^2} + \frac{2\varepsilon_s x}{n}\sum_{j=1}^n \frac{1}{(\lambda_j(F)-x)^3} > 0. \tag{5}$$

Thus, $g(x)$ is monotone increasing and strictly convex in x. As $\theta < \theta^*$, we have $g(\theta) < g(\theta^*) = 1$. Moreover, by convexity,

$$\begin{aligned}
g(\theta) &\geq g(\theta^*) + (\theta - \theta^*)\left.\frac{dg(x)}{dx}\right|_{x=\theta^*} \\
&\geq 1 - \delta_s \frac{\varepsilon_s\theta^*}{n}\sum_{j=1}^n \frac{1}{\lambda_j(F) - \theta^*} - \delta_s\frac{\varepsilon_s}{n}\sum_{j=1}^n\left(\frac{\theta^*}{\lambda_j(F)-\theta^*}\right)^2 && (\because (1-\delta_s)\theta^* \leq \theta) \\
&\geq 1 - \delta_s - \delta_s\frac{\varepsilon_s}{n}\left(\sum_{j=1}^n \frac{\theta^*}{\lambda_j(F)-\theta^*}\right)^2 && \text{(by defintition of } \theta^* \text{ and } \textstyle\sum_j x_j^2 \leq \big(\sum_j x_j\big)^2 \text{ for nonnengative } x_j\text{'s)} \\
&= 1 - \delta_s\left(1 + \frac{n}{\varepsilon_s}\right) > 1 - \varepsilon_s. && \text{(by defintition of } \theta^* \text{ and } \delta_s)
\end{aligned}$$

$\square$

The following two claim show that $\theta(t) \approx X(t) \bullet F(y(t))$ provides a good approximation for $\lambda_{\min}(F(y(t)))$.

Claim 3. $(1-\varepsilon_s)\lambda_{\min}(F(y(t))) < \theta(t) < \frac{\lambda_{\min}(F(y(t)))}{1+\varepsilon_s/n}$ *and* $\frac{\lambda_{\min}(F(y(t)))}{1+\varepsilon_s} \leq \theta^*(t) \leq \frac{\lambda_{\min}(F(y(t)))}{1+\varepsilon_s/n}$.

Proof. By Claim 2, we have

$$1 - \varepsilon_s < \frac{\varepsilon_s\theta(t)}{n}\sum_{j=1}^n \frac{1}{\lambda_j(F) - \theta(t)} < 1. \tag{6}$$

The middle term in (6) is at least $\frac{\varepsilon_s\theta(t)}{n}\frac{1}{\lambda_{\min}(F)-\theta(t)}$ and at most $\frac{\varepsilon_s\theta(t)}{n}\frac{n}{\lambda_{\min}(F)-\theta(t)}$, which implies the claim for $\theta(t)$. The claim for $\theta^*(t)$ follows similarly. $\square$

Claim 4. $\theta(t) < X(t) \bullet F(y(t)) < (1+\varepsilon_s)\theta(t)$.

Proof. By the definition of X, we have $(F-\theta I)X = \frac{\varepsilon_s\theta}{n}I$. It follows from Claim 2 that

$$X \bullet F = \frac{\varepsilon_s\theta}{n}\mathrm{Tr}(I) + \theta\mathrm{Tr}(X) \in \big(\varepsilon_s + (1-\varepsilon_s, 1)\big)\theta = \big(\theta, (1+\varepsilon_s)\theta\big).$$

□

Claim 5. $\mathbf{1}^T y(t) = 1$.

Proof. This is immediate from the initialization of $y(0)$ in step 1 and the update of $y(t+1)$ in step 10 of the algorithm. □

Claim 6. *For all iterations t, except possibly the last,* $\nu(t+1), \tau(t+1) \in (0,1)$.

Proof. $\nu(t+1) \geq 0$ as $X \bullet A_{i(t)} \geq X \bullet F$ by Claim 5, and except possibly for the last iteration, we have $\nu(t+1) > 0$. Also, $\nu(t+1) \leq 1$ by the non-negativity of $X \bullet A_{i(t)}$ and $X \bullet F$, while $\nu(t+1) = 1$ implies that $X \bullet F = 0$, in contradiction to Claim 4.

Note that the definition of $\nu(t+1)$ implies that

$$\tau(t+1) = \frac{\varepsilon_s\theta\nu(t+1)(1-\nu(t+1))}{8nX(t) \bullet F(y(t))},$$

and hence, $\tau(t+1) > 0$. Moreover, by Claim 4, $\tau(t+1) < \frac{\varepsilon_s}{8n} < 1$. □

Claim 7. $F(y(t)) \succ 0$.

Proof. This follows by induction on $t' = 0, 1, \ldots, t$. For $t' = 0$, the claim follows from assumption (B-I), which implies that $F(y(0)) = \frac{1}{r}\bar{A} \succ 0$. Assume now that $F = F(y(t)) \succ 0$. Then for $F' = F(y(t+1))$, we have by step 10 of the algorithm that $F' = (1-\tau)F + \tau A_{i(t)} \succ 0$. □

Claim 8. $(F - \theta^* I)^{-1} = \big(\frac{\varepsilon_s\theta}{n}I - (\theta^*-\theta)X\big)^{-1}X$.

Proof. By definition of X, we have

$$(F-\theta^* I)X = (F-\theta I)X - (\theta^*-\theta)X = \frac{\varepsilon_s\theta}{n}I - (\theta^*-\theta)X$$
$$\therefore X = (F-\theta^* I)^{-1}\left(\frac{\varepsilon_s\theta}{n}I - (\theta^*-\theta)X\right).$$

□

2.2.2 Number of Iterations

Define $B = B(t) := \frac{n}{\varepsilon_s\theta}\Big(\tau X^{1/2}(\hat{F}-F)X^{1/2} - (\theta^*-\theta)X\Big)$.

Claim 9. $F' - \theta^* I = (F-\theta I)^{1/2}(I+B)(F-\theta I)^{1/2}$.

Proof. By (the update) step 10, we have

$$\begin{aligned}
F' - \theta^* I &= (1-\tau)F + \tau\hat{F} - \theta^* I \\
&= F - \theta I + \tau(\hat{F}-F) - (\theta^*-\theta)I \\
&= (F-\theta I)^{1/2}\left(I + \frac{n\tau}{\varepsilon_s\theta}X^{1/2}(\hat{F}-F)X^{1/2} - \frac{n}{\varepsilon_s\theta}(\theta^*-\theta)X\right)(F-\theta I)^{1/2}. \qquad (\because X = \tfrac{\varepsilon_s\theta}{n}(F-\theta I)^{-1})
\end{aligned}$$

□

Claim 10. $\max_j |\lambda_j(B)| \leq \frac{1}{2}$.

Proof. By the definition of B, we have

$$\begin{aligned}
\max_j |\lambda_j(B)| &= \frac{n}{\varepsilon_s\theta}\max_j\left|\lambda_j\left(\tau X^{1/2}(\hat{F}-F)X^{1/2} - (\theta^*-\theta)X\right)\right| \\
&= \frac{n}{\varepsilon_s\theta}\max_{v:||v||=1}\left|v^T\left(\tau X^{1/2}(\hat{F}-F)X^{1/2} - (\theta^*-\theta)X\right)v\right| \\
&\leq \frac{n}{\varepsilon_s\theta}\left(\max_{v:||v||=1}\tau v^T X^{1/2}\hat{F}X^{1/2}v + \max_{v:||v||=1}\tau v^T X^{1/2}FX^{1/2}v + (\theta^*-\theta)\max_{v:||v||=1}v^T Xv\right) \\
&\leq \frac{n\tau}{\varepsilon_s\theta}\left(\mathrm{Tr}(X^{1/2}\hat{F}X^{1/2}) + \mathrm{Tr}(X^{1/2}FX^{1/2})\right) + \frac{n\delta_s}{(1-\delta_s)\varepsilon_s} \qquad (\because F, \hat{F} \succeq 0,\ |\mathrm{Tr}(X) \leq 1 \text{ and } \theta \geq (1-\delta_s)\theta^*)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n\tau}{\varepsilon_s\theta}\left(X \bullet \hat{F} + X \bullet F\right) + \frac{n\delta_s}{(1-\delta_s)\varepsilon_s} \\
&= \frac{\nu}{4} + \frac{{\varepsilon_s}^2}{32(1-{\varepsilon_s}^3/(32n))} && \text{(substituting } \tau \text{ and } \delta_s\text{)} \\
&< \frac{1}{2}. && \text{(using } \nu, \varepsilon_s \le 1\text{)}
\end{aligned}
$$

□

Claim 11. $\theta^*(t) < \lambda_{\min}(F(y(t+1)))$.

Proof. By Claim 10, $I + B \succeq I - \frac{1}{2}I = \frac{1}{2}I$, and by thus, we get by Claim 9,

$$
F' - \theta^* I \succeq \frac{1}{2}(F - \theta I) \succ 0. \qquad (\because BZB \succeq 0 \text{ for } B \in \mathbb{S}^n \text{ and } Z \in \mathbb{S}^n_+)
$$

□

Claim 12. *if $\nu > \varepsilon_s$, then Tr$(B) \ge \frac{\nu^2}{8}$.*

Proof. By the definition of B,

$$
\begin{aligned}
\mathrm{Tr}(B) &= \frac{n}{\varepsilon_s\theta}\left(\tau \mathrm{Tr}(X^{1/2}(\hat{F} - F)X^{1/2}) - (\theta^* - \theta)\mathrm{Tr}(X)\right) \\
&\ge \frac{n}{\varepsilon_s\theta}\left(\tau(X \bullet \hat{F} - X \bullet F) - (\theta^* - \theta)\right) && (\because \mathrm{Tr}(X) \le 1 \text{ by Claim 2}) \\
&\ge \frac{n}{\varepsilon_s\theta}\left(\tau(X \bullet \hat{F} - X \bullet F) - \frac{\delta_s}{1-\delta_s}\theta\right) && (\because (1-\delta_s)\theta^* \le \theta) \\
&= \frac{\nu^2}{4} - \frac{{\varepsilon_s}^2}{32(1-{\varepsilon_s}^3/32n))} && \text{(by definition of } \tau \text{ and } \delta_s\text{)} \\
&> \frac{\nu^2}{4} - \frac{\nu^2}{16} > \frac{\nu^2}{8}. && (\because \varepsilon_s < \nu \le 1)
\end{aligned}
$$

□

Claim 13. *if $\nu > \varepsilon_s$, then Tr$(B^2) < \frac{\nu^2}{10}$.*

Proof. Write $\hat{Y} = \tau X^{1/2}\hat{F}X^{1/2}$ and $Y = X^{1/2}\big(\tau F + (\theta^* - \theta)I\big)X^{1/2}$ and note that both $\hat{Y}$ and Y are in $\mathbb{S}^n_+$. It follows by the definition of B that

$$
\begin{aligned}
\mathrm{Tr}(B^2) &= \frac{n^2}{{\varepsilon_s}^2\theta^2}\mathrm{Tr}\left((\hat{Y} - Y)^2\right) \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(\hat{Y}^2) + \mathrm{Tr}(Y^2) - 2\mathrm{Tr}(\hat{Y}Y)\right) \\
&\le \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(\hat{Y}^2) + \mathrm{Tr}(Y^2) + 2\mathrm{Tr}(\hat{Y}Y)\right) && (\because \hat{Y}, Y \in \mathbb{S}^n_+) \\
&\le \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(\hat{Y}^2) + \mathrm{Tr}(Y^2) + 2\sqrt{\mathrm{Tr}(\hat{Y}^2)\mathrm{Tr}(Y^2)}\right) && \text{(by Cauchy-Schwarz Ineq.)} \\
&\le \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(\hat{Y})^2 + \mathrm{Tr}(Y)^2 + 2\mathrm{Tr}(\hat{Y})\mathrm{Tr}(Y)\right) && (\because \hat{Y}, Y \in \mathbb{S}^n_+) \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}(\mathrm{Tr}(\hat{Y}) + \mathrm{Tr}(Y))^2 \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}(\tau\mathrm{Tr}(X\hat{F}) + \tau\mathrm{Tr}(XF) + (\theta^* - \theta)\mathrm{Tr}(X))^2 \\
&\le \frac{n^2}{{\varepsilon_s}^2\theta^2}(\tau X \bullet \hat{F} + \tau X \bullet F + (\theta^* - \theta))^2 && (\because \mathrm{Tr}(X) \le 1 \text{ by Claim 2}) \\
&\le \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\tau X \bullet \hat{F} + \tau X \bullet F + \frac{\delta_s}{1-\delta_s}\theta\right)^2 && (\because (1-\delta_s)\theta^* \le \theta) \\
&= \left(\frac{\nu}{4} + \frac{{\varepsilon_s}^2}{32(1-{\varepsilon_s}^3/(32n))}\right)^2 && \text{(by definition of } \tau \text{ and } \delta_s\text{)} \\
&< \left(\frac{\nu}{4} + \frac{\nu^2}{16}\right)^2 < \frac{\nu^2}{10}. && (\because \varepsilon_s < \nu \le 1)
\end{aligned}
$$

□

Define $\Phi(t) := \Phi(\theta^*(t), F(y(t)))$.

Claim 14. $\Phi(t+1) - \Phi(t) \geq \frac{\varepsilon_s \nu(t+1)^2}{40n}$.

Proof. Note that Claim 11 implies that θ^* is feasible to the problem $\max\{\Phi(\xi, F') : 0 \leq \xi \leq \lambda_{\min}(F')\}$. Thus,

$$\Phi(t+1) = \Phi(\theta^*(t+1), F') \geq \ln \theta^* + \frac{\varepsilon_s}{n} \ln \det(F' - \theta^* I).$$

$$\begin{aligned}
\therefore \Phi(t+1) - \Phi(t) &\geq \frac{\varepsilon_s}{n} \left(\ln \det(F' - \theta^* I) - \ln \det(F - \theta^* I)\right) \\
&\geq \frac{\varepsilon_s}{n} \left(\ln \det\left(F' - \theta^* I\right) - \ln \det(F - \theta I)\right) && (\because \theta \leq \theta^*) \\
&= \frac{\varepsilon_s}{n} \ln \det\left(I + B\right) && \text{(by Claim 9)} \\
&= \frac{\varepsilon_s}{n} \sum_{j=1}^{n} \ln\left(1 + \lambda_j(B)\right) \\
&\geq \frac{\varepsilon_s}{n} \sum_{j=1}^{n} \left(\lambda_j(B) - \lambda_j(B)^2\right) && \text{(by Claim 10 and } \ln(1+z) \geq z - z^2, \forall z \geq -0.5) \\
&= \frac{\varepsilon_s}{n} \left(\mathrm{Tr}(B) - \mathrm{Tr}(B^2)\right) \\
&> \frac{\varepsilon_s}{8n}\nu^2 - \frac{\varepsilon_s}{10n}\nu^2 && \text{(by Claims 12 and 13)} \\
&= \frac{\varepsilon_s}{40n}\nu^2.
\end{aligned}$$

□

Claim 15. *For any* t, t',

$$\Phi(t') - \Phi(t) \leq (1 + \varepsilon_s) \ln \frac{X(t) \bullet A_{i(t)}}{(1 - \varepsilon_s) X(t) \bullet F(y(t))}.$$

Proof. Write $F = F(y(t))$, $\theta^* := \theta^*(t)$, $\theta := \theta(t)$, $X := X(t)$, $F' = F(y(t'))$, $\theta'^* := \theta^*(t')$. Then

$$\begin{aligned}
\Phi(t') - \Phi(t) &= \ln \frac{\theta'^*}{\theta^*} + \frac{\varepsilon_s}{n} \ln \det\left[(F - \theta^* I)^{-1}(F' - \theta'^* I)\right] \\
&= \ln \frac{\theta'^*}{\theta^*} + \frac{\varepsilon_s}{n} \ln \det\left[\left(\frac{\varepsilon_s \theta}{n} I - (\theta^* - \theta) X\right)^{-1} X(F' - \theta'^* I)\right] && \text{(by Claim 8)} \\
&= \ln \frac{\theta'^*}{\theta^*} + \frac{\varepsilon_s}{n} \left[\ln \det\left(\frac{\varepsilon_s \theta}{n} I - (\theta^* - \theta) X\right)^{-1} + \ln \det\left[X(F' - \theta'^* I)\right]\right] \\
&\leq \ln \frac{\theta'^*}{\theta^*} + \frac{\varepsilon_s}{n} \left[\ln\left(\frac{\varepsilon_s \theta}{n} - \frac{\delta_s \theta}{1 - \delta_s}\right)^{-n} + \ln \det\left[X(F' - \theta'^* I)\right]\right] \\
& && (\because \mathrm{Tr}(X) \leq 1 \text{ by Claim 2 and } (1 - \delta_s)\theta^* \leq \theta) \\
&\leq \ln \frac{\theta'^*}{\theta^*} + \frac{\varepsilon_s}{n} \left[\ln\left(\frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta}\right)^{n} + \ln \det X(F' - \theta'^* I)\right] && \text{(by defintion of } \delta_s) \\
&= \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta} + \frac{\varepsilon_s}{n} \ln\left[\det X(F' - \theta'^* I)\right] \\
&= \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta} + \frac{\varepsilon_s}{n} \sum_{j=1}^{n} \ln \lambda_j\left(X(F' - \theta'^* I)\right) \\
&\leq \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(\frac{1}{n} \sum_{j=1}^{n} \lambda_j\left(X(F' - \theta'^* I)\right)\right) && \text{(by the concavity of } \ln(\cdot)) \\
&= \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(\frac{1}{n} \mathrm{Tr}(XF' - \theta'^* X)\right) \\
&\leq \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{n}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(\frac{X \bullet F' - \theta'^*(1 - \varepsilon_s)}{n}\right) && (\because \mathrm{Tr}(X) \geq 1 - \varepsilon_s \text{ by Claim 2}) \\
&= \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{1}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(X \bullet F' - \theta'^*(1 - \varepsilon_s)\right) \\
&\leq \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{1}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(\max_{y \in \mathbb{R}_+^m :\, \mathbf{1}^T y = 1} X \bullet F(y) - \theta'^*(1 - \varepsilon_s)\right) && (\because \mathbf{1}^T y(t') = 1 \text{ by Claim 5}) \\
&= \ln \frac{\theta'^*}{\theta^*} + \varepsilon_s \ln \frac{1}{(1 - \varepsilon_s)\varepsilon_s \theta} + \varepsilon_s \ln\left(X \bullet A_{i(t)} - \theta'^*(1 - \varepsilon_s)\right) && \text{(by defintion of } i(t))
\end{aligned}$$

$$
\begin{aligned}
&\le \max_{0\le\xi<X\bullet A_{i(t)}} \left\{\ln\frac{\xi}{(1-\varepsilon_s)\theta^*} + \varepsilon_s\ln\frac{1}{(1-\varepsilon_s)\varepsilon_s\theta} + \varepsilon_s\ln\left(X\bullet A_{i(t)}-\xi\right)\right\} \\
&= (1+\varepsilon_s)\ln\frac{X\bullet A_{i(t)}}{(1-{\varepsilon_s}^2)\theta} + \ln\frac{\theta}{\theta^*} && (\max(\cdot)\text{ is achieved at }\xi=\tfrac{X\bullet A_{i(t)}}{1+\varepsilon_s}) \\
&\le (1+\varepsilon_s)\ln\frac{X\bullet A_{i(t)}}{(1-{\varepsilon_s}^2)\theta} && (\because \theta\le\theta^*) \\
&\le (1+\varepsilon_s)\ln\frac{X\bullet A_{i(t)}}{(1-\varepsilon_s)X\bullet F}. && (\text{by Claim 4})
\end{aligned}
$$

□

Recall by assumption (B-I) that $\bar{A} := \sum_{i=1}^r A_i \succ 0$.

Claim 16. $\frac{X(0)\bullet A_{i(0)}}{X(0)\bullet F(y(0))} \le \psi := \frac{r\cdot\lambda_{\max}(A_{i(0)})}{\lambda_{\min}(\bar{A})} \le \frac{r\cdot\max_i\lambda_{\max}(A_i)}{\lambda_{\min}(\bar{A})} \le n\tau 2^{\mathcal{L}}$.

Proof. Let $X(0)=\sum_{j=1}^n \lambda_j u_j u_j^T$ be the spectral decomposition of $X(0)$. Then,

$$
\begin{aligned}
X(0)\bullet A_{i(0)} &= \sum_{j=1}^n \lambda_j A_{i(0)}\bullet u_j u_j^T \le \sum_{j=1}^n \lambda_j\lambda_{\max}(A_{i(0)}) = \lambda_{\max}(A_{i(0)})\cdot \mathrm{Tr}(X(0)) \\
X(0)\bullet F(y(0)) &= \sum_{j=1}^n \lambda_j F(y(0))\bullet u_j u_j^T \ge \frac{1}{r}\sum_{j=1}^n \lambda_j\lambda_{\min}(\bar{A}) = \frac{1}{r}\lambda_{\min}(\bar{A})\cdot\mathrm{Tr}(X(0)).
\end{aligned}
$$

The claim follows. □

Claim 17. *The algorithm terminates in at most $O\big(n\log\psi + \frac{n}{\epsilon^2}\big)$ iterations.*

Proof. Let $t_{-1}=-1$ and, for $s=0,1,2,\ldots$, let t_s be the smallest t such that $\nu(t+1)\le 2^{-(s+1)}$ (so t_s+1 is the value of t at which the iteration $s+1$ of the outer while-loop starts). Then for $t=t_{s-1}+1,\ldots,t_s-1$, we have $\nu(t+1)>2^{-(s+1)}=\varepsilon_s$. Hence, for $s=0$,

$$
\begin{aligned}
\frac{\varepsilon_0^3 t_0}{40n} &< \Phi(t_0)-\Phi(0) && (\text{by Claim 14}) \\
&\le (1+\varepsilon_0)\ln\frac{X(0)\bullet A_{i(0)}}{(1-\varepsilon_0)X(0)\bullet F(y(0))} && (\text{by Claim 15}) \\
&\le (1+\varepsilon_0)\ln\frac{\psi}{(1-\varepsilon_0)}. && (\text{by Claim 16})
\end{aligned}
$$

Setting $\varepsilon_0=\frac{1}{2}$ in the last series of inequalities we get

$$
t_0 < 480n\ln(2\psi) = O(n\log\psi). \tag{7}
$$

Now consider $s\ge 1$:

$$
\begin{aligned}
\frac{\varepsilon_s^3(t_s-t_{s-1})}{40n} &< \Phi(t_s)-\Phi(t_{s-1}) && (\text{by Claim 14}) \\
&\le (1+\varepsilon_s)\ln\frac{X(t_{s-1})\bullet A_{i(t_{s-1})}}{(1-\varepsilon_s)X(t_{s-1})\bullet F(y(t_{s-1}))} && (\text{by Claim 15}) \\
&= (1+\varepsilon_s)\ln\frac{1+\nu(t_{s-1}+1)}{(1-\varepsilon_s)\big(1-\nu(t_{s-1}+1)\big)} && (\text{by definition of }\ \nu(t_{s-1}+1)) \\
&\le (1+\varepsilon_s)\ln\frac{1+2\varepsilon_s}{(1-\varepsilon_s)(1-2\varepsilon_s)} && (\because \nu(t_{s-1}+1)\le 2^{-s}=2\varepsilon_s) \\
&\le (1+\varepsilon_s)\ln(1+12\varepsilon_s)\le 15\varepsilon_s. && (\because \varepsilon_s\le\tfrac{1}{4})
\end{aligned}
$$

Setting $\varepsilon_s=\frac{1}{2^{s+1}}$ in the last series of inequalities we get

$$
t_s - t_{s-1} < \frac{600n}{\varepsilon_s^2} = O(n/\varepsilon_s^2). \tag{8}
$$

Summing (7), and (8) over $s=1,2,\ldots,\lceil\log\frac{1}{\epsilon}\rceil$, we get the claim. □

Remark 1. *If we do not insist on a sparse dual solution, then we can use the initialization $y(0)\leftarrow\frac{1}{m}\mathbf{1}$ in step 1 in Algorithm 1, where $\mathbf{1}$ is the m-dimensional vector of all ones, and replace ψ in Claim 16, and hence in the running time in Claim 17, by m.*

2.2.3 Primal Dual Feasibility and Approximate Optimality

Let $t_f + 1$ be the value of t when the algorithm terminates and $s_f + 1$ be the value of s at termination. For simplicity, we write $s = s_f$.

Claim 18. *(Primal feasibility).* $\hat{X} \succ 0$ *and* $\max_i A_i \bullet \hat{X} \leq 1$.

Proof. The first claim is immediate from Claim 1. To see the second claim, we use the definition of $\nu(t_f)$ and the termination condition in line 4 (which is also satisfied even if $X(t_f) \bullet A_{i(t_f)} - X(t_f) \bullet F(y(t_f)) = 0$):

$$\frac{X(t_f) \bullet A_{i(t_f)} - X(t_f) \bullet F(y(t_f))}{X(t_1) \bullet A_{i(t_f)} + X(t_f) \bullet F(y(t_f))} \leq \varepsilon_s.$$

$$\therefore (1+\varepsilon_s) X(t_f) \bullet F(y(t_f)) \geq (1-\varepsilon_s) X(t_f) \bullet A_{i(t_f)}$$

$$= (1-\varepsilon_s) \max_i X(t_f) \bullet A_i \qquad \text{(by the defintition of } i(t_f))$$

$$\therefore (1+\varepsilon_s)^2 \theta(t_f) \geq (1-\varepsilon_s) \max_i X(t_f) \bullet A_i. \qquad (\because X(t_f) \bullet F(y(t_f)) \leq (1+\varepsilon_s)\theta(t_f) \text{ by Claim 4})$$

The claim follows by the definition of $\hat{X}$ in step 16 of the algorithm. □

Claim 19. *(Dual feasibility).* $\hat{y} \geq 0$ *and* $F(\hat{y}) \succ I$.

Proof. The fact that $\hat{y} \geq 0$ follows from the initialization of $y(0)$ in step 1, Claim 6, and the update of $y(t+1)$ in step 10. For the other claim, we have

$$\lambda_{\min}\big(F(\hat{y})\big) = \frac{1}{\theta(t_f)} \lambda_{\min}\big(F(y(t_f))\big) \geq 1 + \frac{\varepsilon_s}{n}. \qquad \text{(by Claim 3)}$$

□

Claim 20. *(Approximate optimality).* $I \bullet \hat{X} \geq \left(\frac{1-\varepsilon_s}{1+\varepsilon_s}\right)^2 \mathbf{1}^T \hat{y}$.

Proof. By Claim 2, we have $\mathrm{Tr}(X(t_f)) \geq 1 - \varepsilon_s$, and by Claim 5, we have $\mathbf{1}^T y(t_f) = 1$. The claim follows by the definition of $\hat{X}$ and $\hat{y}$ in step 16. □

Remark 2. *Suppose that in step 7 of Algorithm 1, we instead define $i(t)$ to be an index $i \in [m]$ such that $A_i \bullet X(t) \geq 1 - \varepsilon_s$, and we are guaranteed that such index exists in each iteration of the algorithm. Then the dual solution $\hat{y}$ satisfies: $\mathbf{1}^T \hat{y} \leq 1 + O(\epsilon)$. Indeed, the proof of Claim 18 can be easily modified to show that $\theta(t_f) \geq \frac{(1-\varepsilon_{s_f})^2}{(1+\varepsilon_{s_f})^2}$, which combined with the definition of $\hat{y}$ in step 16 of the algorithm implies the claim.*

2.2.4 Running Time per Iteration

Computing $\theta(t)$. Given $F := F(y(t)) \succ 0$, we first compute an approximation $\widetilde{\lambda}$ of $\lambda_{\min}(F)$ using Lanczos' algorithm with a random start [26].

Lemma 21 ([26])**.** *Let $M \in \mathbb{S}^n_+$ be a positive semidefinite matrix with N non-zeros and $\gamma \in (0,1)$ be a given constant. Then there is a randomized algorithm that computes, with high (i.e., $1 - o(1)$) probability a unit vector $v \in \mathbb{R}^n$ such that $v^T M v \geq (1-\gamma)\lambda_{\max}(M)$. The algorithm takes $O\big(\frac{\log n}{\sqrt{\gamma}}\big)$ iterations, each requiring $O(N)$ arithmetic operations.*

By Claim 3, we need $\widetilde{\lambda}$ to lie in the range $[\frac{\lambda_{\min}(F)}{1+\varepsilon_s/n}, \lambda_{\min}(F)]$. To obtain $\widetilde{\lambda}$, we may apply the above lemma with $M := F^{-1}$ and $\gamma := \frac{\varepsilon_s}{2n}$. Then in $O\big(\sqrt{\frac{n}{\varepsilon_s}} \log n\big)$ iterations we get $\widetilde{\lambda} := \frac{1-\gamma}{v^T F^{-1} v}$ satisfying our requirement. However, we can save (roughly) a factor of $\sqrt{n}$ in the running time by using, instead, $M := F^{-n}$ and $\gamma := \frac{\varepsilon_s}{2}$. Let v be the vector obtained from Lemma 21, and set $\widetilde{\lambda} := \big(\frac{1-\gamma}{v^T F^{-n} v}\big)^{1/n}$. Then, as $\lambda_{\max}(M) \geq v^T M v \geq (1-\gamma)\lambda_{\max}(M)$, and $\lambda_{\min}(F) = \lambda_{\max}(F^{-n})^{-1/n}$, we get

$$\frac{\lambda_{\min}(F)}{1+\varepsilon_s/n} \leq (1-\gamma)^{1/n} \lambda_{\min}(F) \leq \widetilde{\lambda} \leq \lambda_{\min}(F). \tag{9}$$

Note that we can compute F^{-n} in $O(n^\omega \log n)$, where w is the *exponent of matrix multiplication*. Thus, the overall running time for computing $\widetilde{\lambda}$ is $O(n^\omega \log n + \frac{n^2 \log n}{\sqrt{\varepsilon_s}})$.

Given $\widetilde{\lambda}$, we know by Claim 3 and (9) that $\theta^*(t) \in [\frac{\widetilde{\lambda}}{1+\varepsilon_s}, \widetilde{\lambda}]$. Then we can apply binary search to find $\theta(t) := \theta^*(t)_{\delta_s}$ as follows. Let $\theta_k = \frac{\widetilde{\lambda}}{1+\varepsilon_s}(1+\delta_s)^k$, for $k = 0, 1, \ldots, K := \lceil \frac{2\ln(1+\varepsilon_s)}{\delta_s} \rceil$, and note that $\theta_L \geq \widetilde{\lambda}$. Then we do binary search on the exponent $k \in \{0, 1 \ldots, K\}$; each step of the search evaluates $g(\theta_k) := \dfrac{\varepsilon_s \theta_\ell}{n} \mathrm{Tr}(F - \theta_k I)^{-1}$, and depending on whether this value is less than or at least 1, the value of k is increased or decreased, respectively. The search stops when the search interval $[\ell, u]$ has $u \le \ell + 1$, in which case we set $\theta(t) = \theta_\ell$; the number of steps until this happens is $O(\log K) = O(\log \frac{1}{\delta_s}) = O(\log \frac{n}{\varepsilon_s})$. By the monotonicity of $g(x)$ (in the interval $[0, \lambda_{\min}(F)]$), and the property of binary search, we know that $\theta^* \in [\theta_\ell, \theta_u]$. Thus, by the stopping criterion,

$$\theta(t) = \theta_\ell \le \theta^*(t) \le \theta_u \le \theta_{\ell+1} = (1+\delta_s)\theta_\ell,$$

implying that $(1-\delta_s)\theta^*(t) \le \theta(t) \le \theta^*(t)$. Since evaluating $g(\theta_\ell)$ takes $O(n^\omega)$, the overall running time for the binary search procedure is $O(n^\omega \log \frac{n}{\varepsilon_s})$, and hence the total time needed for for computing $\theta(t)$ is $O(n^\omega \log \frac{n}{\epsilon} + \frac{n^2 \log n}{\sqrt{\epsilon}})$.

All other steps of the algorithm inside the inner while-loop can be done in $O(\mathcal{T} + n^2)$ time, where $\mathcal{T}$ is the time taken by a single call to the oracle $\text{Max}(X(t))$ in step 7 of the algorithm. Thus, in view of Claim 17, we obtain the following result.

Theorem 22. *For any $\epsilon > 0$, Algorithm 1 outputs an $O(n \log \psi + \frac{n}{\epsilon^2})$-sparse $O(\epsilon)$-optimal primal-dual pair in time*[2] $O\big((n \log \psi + \frac{n}{\epsilon^2})(n^\omega \log \frac{n}{\epsilon} + \frac{n^2 \log n}{\sqrt{\epsilon}} + \mathcal{T})\big) = \widetilde{O}\big(\frac{n^{\omega+1} \log \psi}{\epsilon^{2.5}} + \frac{n\mathcal{T} \log \psi}{\epsilon^2}\big)$.

2.3 Algorithm for (PACKING-II)-(COVERING-II)

In this section we give an algorithm for finding a sparse $O(\epsilon)$-optimal primal-dual solution for (PACKING-II)-(COVERING-II).

For numbers $x \in \mathbb{R}_+$ and $\delta \in (0,1)$, a δ-(upper) approximation x^δ of x is a number such that $x \le x^\delta < (1+\delta)x$.

The algorithm is shown as Algorithm 2.

$s \leftarrow 0$; $\varepsilon_0 \leftarrow \frac{1}{4}$; $t \leftarrow 0$; $\nu(0) \leftarrow 1$; $y(0) \leftarrow \mathbf{1}_i$ (for an arbitrary $i \in [m]$)
while $\varepsilon_s > \epsilon$ **do**
  $\delta_s \leftarrow \frac{\varepsilon_s^3}{32n}$
  while $\nu(t) > \varepsilon_s$ **do**
    $\theta(t) \leftarrow \theta^*(t)^{\delta_s}$, where $\theta^*(t)$ is the smallest positive number root of the equation $\frac{\varepsilon_s \theta}{n}\text{Tr}(\theta I - F(y(t)))^{-1} = 1$
    $X(t) \leftarrow \frac{\varepsilon_s \theta(t)}{n}(\theta(t) I - F(y(t)))^{-1}$ /* Set the primal solution */
    $i(t) \leftarrow \text{argmin}_i\, A_i \bullet X(t)$ /* Call the minimization oracle */
    $\nu(t+1) \leftarrow \frac{X(t) \bullet F(y(t)) - X(t) \bullet A_{i(t)}}{X(t) \bullet A_{i(t)} + X(t) \bullet F(y(t))}$ /* Compute the error */
    $\tau(t+1) \leftarrow \frac{\varepsilon_s \theta(t) \nu(t+1)}{4n(X(t) \bullet A_{i(t)} + X(t) \bullet F(y(t)))}$ /* Compute the step size */
    $y(t+1) \leftarrow (1-\tau(t+1))y(t) + \tau(t+1)\mathbf{1}_{i(t)}$ /* Update the dual solution */
    $t \leftarrow t+1$
  end
  $\varepsilon_{s+1} \leftarrow \frac{\varepsilon_s}{2}$
  $s \leftarrow s+1$
end
$\hat{X} \leftarrow \frac{(1+\varepsilon_{s-1})X(t-1)}{(1-2\varepsilon_{s-1})^2\theta(t-1))}$; $\hat{y} \leftarrow \frac{y(t-1)}{\theta(t-1)}$
return $(\hat{X}, \hat{y}, t)$

Algorithm 2: Logarithmic-potential Algorithm for (PACKING-II)-(COVERING-II)

2.4 Analysis

2.4.1 Some Preliminaries

Up to Claim 39, we fix a particular iteration s of the outer while-loop in the algorithm. For simplicity in the following, we will sometimes write $F := F(y(t))$, $\theta := \theta(t)$, $\theta^* := \theta^*(t)$, $X := X(t)$, $\hat{F} := A_{i(t)}$, $\tau := \tau(t+1)$, $\nu := \nu(t+1)$, $F' := F(y(t+1))$, and $\theta' := \theta(t+1)$, when the meaning is clear from the context. For $H \succ 0$ and $x \in (0, \lambda_{\min}(H))$, define the following logarithmic potential function:

$$\Phi(x, H) = \ln x - \frac{\varepsilon_s}{n} \ln \det \big(xI - H\big). \tag{10}$$

Claim 23. *If $\lambda_{\max}(F) > 0$, then $\theta^*(t) = \text{argmin}_{x > \lambda_{\max}(F)} \Phi(x, F(y(t)))$ and $X(t) \succ 0$.*

Proof. Note that

$$\frac{d\Phi(x,F)}{dx} = \frac{1}{x} - \frac{\varepsilon_s}{n}\text{Tr}\big((xI-F)^{-1}\big) \quad \text{and} \quad \frac{d^2\Phi(x,F)}{dx^2} = -\frac{1}{x^2} + \frac{\varepsilon_s}{n}\text{Tr}\big((xI-F)^{-2}\big).$$

[2] $\widetilde{O}(\cdot)$ hides polylogarithmic factors in n and $\frac{1}{\epsilon}$.

Note that $\Phi(x,F)$ is not convex in $x \in (\lambda_{\max}(F), +\infty)$, but has a unique minimizer in this interval, defined by setting $\frac{d\Phi(x,F)}{dx} = 0$, giving the definition $\theta^*(t)$ in step 5 of Algorithm 2. (Indeed, $\frac{d\Phi(x,F)}{dx} < 0$ for $\lambda_{\max}(F) < x < \theta^*(t)$, while $\frac{d\Phi(x,F)}{dx} > 0$ for $x > \theta^*(t)$.) Also, by definition of X in step 6, $\lambda_{\min}(X) = \frac{\varepsilon_s\theta}{n}(\theta - \lambda_{\min}(F))^{-1} > 0$ (as $\theta \geq \theta^* > \lambda_{\max}(F) \geq \lambda_{\min}(F)$), implying that $X \succ 0$. ☐

For $x \in (\lambda_{\max}(H), +\infty)$, let $g(x) := \dfrac{\varepsilon_s x}{n}\mathrm{Tr}(xI - H)^{-1}$. The following claim shows that our choice of δ_s guarantees that $g(\theta)$ is a good approximation of $g(\theta^*) = 1$.

Claim 24. $g(\theta(t)) \in (1-\varepsilon_s, 1]$.

Proof. For $x \in (\lambda_{\max}(H), +\infty)$, we have

$$\frac{dg(x)}{dx} = \frac{\varepsilon_s}{n}\sum_{j=1}^{n}\frac{1}{x-\lambda_j(F)} - \frac{\varepsilon_s x}{n}\sum_{j=1}^{n}\frac{1}{(x-\lambda_j(F))^2} = -\frac{\varepsilon_s}{n}\sum_{j=1}^{n}\frac{\lambda_j(F)}{(x-\lambda_j(F))^2} < 0, \tag{11}$$

$$\frac{d^2g(x)}{dx^2} = -\frac{2\varepsilon_s}{n}\sum_{j=1}^{n}\frac{1}{(\lambda_j(F)-x)^2} + \frac{2\varepsilon_s x}{n}\sum_{j=1}^{n}\frac{1}{(\lambda_j(F)-x)^3} = \frac{2\varepsilon_s}{n}\sum_{j=1}^{n}\frac{\lambda_j(F)}{(x-\lambda_j(F))^3} > 0. \tag{12}$$

Thus, $g(x)$ is monotone decreasing and strictly convex in x. As $\theta \geq \theta^*$, we have $g(\theta) \leq g(\theta^*) = 1$. Moreover, by convexity,

$$\begin{aligned}
g(\theta) &\geq g(\theta^*) + (\theta-\theta^*)\left.\frac{dg(x)}{dx}\right|_{x=\theta^*} \\
&\geq 1 + \delta_s\frac{\varepsilon_s\theta^*}{n}\sum_{j=1}^{n}\frac{1}{\theta^*-\lambda_j(F)} - \delta_s\frac{\varepsilon_s}{n}\sum_{j=1}^{n}\left(\frac{\theta^*}{\theta^*-\lambda_j(F)}\right)^2 && (\because \theta < (1+\delta_s)\theta^* \text{ and } \left.\tfrac{dg(x)}{dx}\right|_{x=\theta^*} < 0) \\
&\geq 1 + \delta_s - \delta_s\frac{\varepsilon_s}{n}\left(\sum_{j=1}^{n}\frac{\theta^*}{\theta^*-\lambda_j(F)}\right)^2 && (\text{by defintition of } \theta^* \text{ and } \textstyle\sum_j x_j^2 \leq \left(\sum_j x_j\right)^2 \text{ for nonnengative } x_j\text{'s}) \\
&= 1 + \delta_s - \delta_s\frac{n}{\varepsilon_s} > 1-\varepsilon_s. && (\text{by defintition of } \theta^* \text{ and } \delta_s)
\end{aligned}$$

☐

The following claim shows that $\theta(t)$ provides a good approximation for $\lambda_{\max}(F(y(t)))$.

Claim 25. $\frac{\lambda_{\max}(F(y(t)))}{1-\varepsilon_s/n} < \theta(t) \leq \frac{(1-\varepsilon_s)\lambda_{\max}(F(y(t)))}{1-2\varepsilon_s}$ *and* $\frac{\lambda_{\max}(F(y(t)))}{1-\varepsilon_s/n} \leq \theta^*(t) \leq \frac{\lambda_{\max}(F(y(t)))}{1-\varepsilon_s}$.

Proof. By Claim 24, we have

$$1-\varepsilon_s < \frac{\varepsilon_s\theta(t)}{n}\sum_{j=1}^{n}\frac{1}{\theta(t)-\lambda_j(F)} \leq 1. \tag{13}$$

The middle term in (13) is at least $\frac{\varepsilon_s\theta(t)}{n}\frac{1}{\theta(t)-\lambda_{\max}(F)}$ and at most $\frac{\varepsilon_s\theta(t)}{n}\frac{n}{\theta(t)-\lambda_{\max}(F)}$, which implies the claim for $\theta(t)$. The claim for $\theta^*(t)$ follows similarly. ☐

Claim 26. $(1-2\varepsilon_s)\theta(t) < X(t) \bullet F(y(t)) \leq (1-\varepsilon_s)\theta(t)$.

Proof. By the definition of X, we have $(\theta I - F)X = \frac{\varepsilon_s\theta}{n}I$. It follows from Claim 24 that

$$X \bullet F = \theta\mathrm{Tr}(X) - \frac{\varepsilon_s\theta}{n}\mathrm{Tr}(I) \in \big((1-\varepsilon_s, 1] - \varepsilon_s\big)\theta = \big((1-2\varepsilon_s)\theta, (1-\varepsilon_s)\theta\big].$$

☐

Claim 27. $\mathbf{1}^T y(t) = 1$.

Proof. This is immediate from the initialization of $y(0)$ in step 1 and the update of $y(t+1)$ in step 10 of the algorithm. ☐

Claim 28. *For all iterations t in the while-loop, except possibly the last,* $\nu(t+1), \tau(t+1) \in (0,1)$.

Proof. $\nu(t+1) \geq 0$ as $X \bullet A_{i(t)} \leq X \bullet F$ by Claim 27, and except possibly for the last iteration, we have $\nu(t+1) > 0$. Also, $\nu(t+1) \leq 1$ by the non-negativity of $X \bullet A_{i(t)}$ and $X \bullet F$, while $\nu(t+1) = 1$ implies that $X \bullet A_{i(t)} = 0$, in contradiction to the assumption that $A_{i(t)} \neq 0$ (as $X \succ 0$ by Claim 23).

Note that the definition of $\nu(t+1)$ implies that

$$\tau(t+1) = \frac{\varepsilon_s\theta\nu(t+1)(1+\nu(t+1))}{8nX(t) \bullet F(y(t))},$$

and hence, $\tau(t+1) > 0$. Moreover, by Claim 26, $\tau(t+1) < \frac{\varepsilon_s}{2n} < 1$. ☐

Claim 29. $\lambda_{\max}(F(y(t))) > 0$.

Proof. This follows by induction on $t' = 0, 1, \ldots, t$. For $t' = 0$, the claim follows from the assumption that $A_i \neq 0$ for all i. Assume now that $F = F(y(t)) \neq 0$. Then for $F' = F(y(t+1))$, we have by step 10 of the algorithm that $F' = (1-\tau)F + \tau A_{i(t)} \neq 0$. As $F' \succeq 0$, we get $\lambda_{\max}(F') > 0$. □

Claim 30. $(\theta^* I - F)^{-1} = \left(\frac{\varepsilon_s \theta}{n} I - (\theta - \theta^*)X\right)^{-1} X$.

Proof. By definition of X, we have

$$(\theta^* I - F)X = (\theta I - F)X - (\theta - \theta^*)X = \frac{\varepsilon_s \theta}{n} I - (\theta - \theta^*)X$$
$$\therefore X = (\theta^* I - F)^{-1} \left(\frac{\varepsilon_s \theta}{n} I - (\theta - \theta^*)X\right).$$

□

2.4.2 Number of Iterations

Define $B = B(t) := \frac{n}{\varepsilon_s \theta}\left(\tau X^{1/2}(F - \hat{F})X^{1/2} - (\theta - \theta^*)X\right)$.

Claim 31. $\theta^* I - F' = (\theta I - F)^{1/2}(I + B)(\theta I - F)^{1/2}$.

Proof. By (the update) step 10, we have

$$\begin{aligned}
\theta^* I - F' &= \theta^* I - (1-\tau)F - \tau \hat{F} \\
&= \theta I - F + \tau(F - \hat{F}) - (\theta - \theta^*)I \\
&= (\theta I - F)^{1/2}\left(I + \frac{n\tau}{\varepsilon_s \theta} X^{1/2}(F - \hat{F})X^{1/2} - \frac{n}{\varepsilon_s \theta}(\theta - \theta^*)X\right)(\theta I - F)^{1/2}. && (\because X = \tfrac{\varepsilon_s \theta}{n}(\theta I - F)^{-1})
\end{aligned}$$

□

Claim 32. $\max_j |\lambda_j(B)| \leq \frac{1}{2}$.

Proof. By the definition of B, we have

$$\begin{aligned}
\max_j |\lambda_j(B)| &= \frac{n}{\varepsilon_s \theta} \max_j \left|\lambda_j\left(\tau X^{1/2}(F - \hat{F})X^{1/2} - (\theta - \theta^*)X\right)\right| \\
&= \frac{n}{\varepsilon_s \theta} \max_{v:||v||=1} \left|v^T\left(\tau X^{1/2}(F - \hat{F})X^{1/2} - (\theta - \theta^*)X\right)v\right| \\
&\leq \frac{n}{\varepsilon_s \theta}\left(\max_{v:||v||=1} \tau v^T X^{1/2} F X^{1/2} v + \max_{v:||v||=1} \tau v^T X^{1/2} \hat{F} X^{1/2} v + (\theta - \theta^*) \max_{v:||v||=1} v^T X v\right) \\
&< \frac{n\tau}{\varepsilon_s \theta}\left(\mathrm{Tr}(X^{1/2} F X^{1/2}) + \mathrm{Tr}(X^{1/2} \hat{F} X^{1/2})\right) + \frac{n\delta_s}{\varepsilon_s} && (\because F, \hat{F} \succeq 0,\ |\mathrm{Tr}(X) \leq 1 \text{ and } \theta^* \leq \theta < (1+\delta_s)\theta^*) \\
&= \frac{n\tau}{\varepsilon_s \theta}\left(X \bullet \hat{F} + X \bullet F\right) + \frac{n\delta_s}{\varepsilon_s} \\
&= \frac{\nu}{4} + \frac{{\varepsilon_s}^2}{32} && \text{(substituting } \tau \text{ and } \delta_s) \\
&< \frac{1}{2}. && \text{(using } \nu, \varepsilon_s \leq 1)
\end{aligned}$$

□

Claim 33. $\theta^*(t) > \lambda_{\max}(F(y(t+1)))$.

Proof. By Claim 32, $I + B \succeq I - \frac{1}{2}I = \frac{1}{2}I$, and by thus, we get by Claim 31,

$$\theta^* I - F' \succeq \frac{1}{2}(\theta I - F) \succ 0. \qquad (\because BZB \succeq 0 \text{ for } B \in \mathbb{S}^n \text{ and } Z \in \mathbb{S}^n_+) \tag{14}$$

The claim follows. □

Claim 34. *if $\nu > \varepsilon_s$, then $Tr(B) \geq \frac{\nu^2}{8}$.*

Proof. By the definition of B,

$$\begin{aligned}
\mathrm{Tr}(B) &= \frac{n}{\varepsilon_s\theta}\left(\tau\mathrm{Tr}(X^{1/2}(F-\hat{F})X^{1/2}) - (\theta-\theta^*)\mathrm{Tr}(X)\right) \\
&\geq \frac{n}{\varepsilon_s\theta}\left(\tau(X\bullet F - X\bullet\hat{F}) - (\theta-\theta^*)\right) && (\because \mathrm{Tr}(X)\leq 1 \text{ by Claim 24}) \\
&> \frac{n}{\varepsilon_s\theta}\left(\tau(X\bullet F - X\bullet\hat{F}) - \delta_s\theta\right) && (\because \theta^*\leq\theta<(1+\delta_s)\theta^*) \\
&= \frac{\nu^2}{4} - \frac{{\varepsilon_s}^2}{32} && (\text{by definition of } \tau \text{ and } \delta_s) \\
&> \frac{\nu^2}{4} - \frac{\nu^2}{32} > \frac{\nu^2}{8}. && (\because \varepsilon_s<\nu\leq 1)
\end{aligned}$$

□

Claim 35. *if $\nu > \varepsilon_s$, then $Tr(B^2) < \frac{\nu^2}{10}$.*

Proof. Write $Y = \tau X^{1/2}FX^{1/2}$ and $\hat{Y} = X^{1/2}\big(\tau\hat{F} + (\theta-\theta^*)I\big)X^{1/2}$ and note that both $\hat{Y}$ and Y are in $\mathbb{S}^n_+$. It follows by the definition of B that

$$\begin{aligned}
\mathrm{Tr}(B^2) &= \frac{n^2}{{\varepsilon_s}^2\theta^2}\mathrm{Tr}\left((Y-\hat{Y})^2\right) \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(Y^2)+\mathrm{Tr}(\hat{Y}^2)-2\mathrm{Tr}(Y\hat{Y})\right) \\
&\leq \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(Y^2)+\mathrm{Tr}(\hat{Y}^2)+2\mathrm{Tr}(Y\hat{Y})\right) && (\because \hat{Y}, Y\in\mathbb{S}^n_+) \\
&\leq \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(Y^2)+\mathrm{Tr}(\hat{Y}^2)+2\sqrt{\mathrm{Tr}(Y^2)\mathrm{Tr}(\hat{Y}^2)}\right) && (\text{by Cauchy-Schwarz Ineq.}) \\
&\leq \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\mathrm{Tr}(Y)^2+\mathrm{Tr}(\hat{Y})^2+2\mathrm{Tr}(\hat{Y})\mathrm{Tr}(Y)\right) && (\because \hat{Y}, Y\in\mathbb{S}^n_+) \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}(\mathrm{Tr}(Y)+\mathrm{Tr}(\hat{Y}))^2 \\
&= \frac{n^2}{{\varepsilon_s}^2\theta^2}(\tau\mathrm{Tr}(XF)+\tau\mathrm{Tr}(X\hat{F})+(\theta-\theta^*)\mathrm{Tr}(X))^2 \\
&\leq \frac{n^2}{{\varepsilon_s}^2\theta^2}(\tau X\bullet F+\tau X\bullet\hat{F}+(\theta-\theta^*))^2 && (\because \mathrm{Tr}(X)\leq 1 \text{ by Claim 24}) \\
&< \frac{n^2}{{\varepsilon_s}^2\theta^2}\left(\tau X\bullet F+\tau X\bullet\hat{F}+\delta_s\theta\right)^2 && (\because \theta^*\leq\theta<(1+\delta_s)\theta^*) \\
&= \left(\frac{\nu}{4}+\frac{{\varepsilon_s}^2}{32}\right)^2 && (\text{by definition of } \tau \text{ and } \delta_s) \\
&< \left(\frac{\nu}{4}+\frac{\nu^2}{32}\right)^2 < \frac{\nu^2}{10}. && (\because \varepsilon_s<\nu\leq 1)
\end{aligned}$$

□

Define $\Phi(t) := \Phi(\theta^*(t), F(y(t)))$.

Claim 36. $\Phi(t+1)-\Phi(t) \leq -\frac{\varepsilon_s\nu(t+1)^2}{40n}$.

Proof. Note that Claim 33 implies that θ^* is feasible to the problem $\min\{\Phi(\xi,F') : \xi\geq\lambda_{\max}(F')\}$. Thus,

$$\Phi(t+1) = \Phi(\theta^*(t+1),F') \leq \ln\theta^* - \frac{\varepsilon_s}{n}\ln\det(\theta^*I-F').$$

$$\begin{aligned}
\therefore \Phi(t+1)-\Phi(t) &\leq -\frac{\varepsilon_s}{n}\left(\ln\det(\theta^*I-F')-\ln\det(\theta^*I-F)\right) \\
&\leq -\frac{\varepsilon_s}{n}\left(\ln\det\left(\theta^*I-F'\right)-\ln\det(\theta I-F)\right) && (\because \theta^*\leq\theta) \\
&= -\frac{\varepsilon_s}{n}\ln\det\left(I+B\right) && (\text{by Claim 31}) \\
&= -\frac{\varepsilon_s}{n}\sum_{j=1}^{n}\ln\left(1+\lambda_j(B)\right) \\
&\leq -\frac{\varepsilon_s}{n}\sum_{j=1}^{n}\left(\lambda_j(B)-\lambda_j(B)^2\right) && (\text{by Claim 10 and } \ln(1+z)\geq z-z^2, \forall z\geq -0.5) \\
&= -\frac{\varepsilon_s}{n}\left(\mathrm{Tr}(B)-\mathrm{Tr}(B^2)\right)
\end{aligned}$$

$$
\begin{aligned}
&< -\frac{\varepsilon_s}{8n}\nu^2 + \frac{\varepsilon_s}{10n}\nu^2 && \text{(by Claims 34 and 35)}\\
&= -\frac{\varepsilon_s}{40n}\nu^2.
\end{aligned}
$$

□

Claim 37. *For any t, t',*

$$\Phi(t') - \Phi(t) \geq (1-\varepsilon_s)\ln\frac{(1-2\varepsilon_s)X \bullet A_{i(t)}}{(1-\varepsilon_s)^2 X \bullet F} + \ln(1-\varepsilon_s).$$

Proof. Write $F = F(y(t))$, $\theta^* := \theta^*(t)$, $\theta := \theta(t)$, $X := X(t)$, $F' = F(y(t'))$, $\theta'^* := \theta^*(t')$. Then

$$
\begin{aligned}
\Phi(t') - \Phi(t) &= \ln\frac{\theta'^*}{\theta^*} - \frac{\varepsilon_s}{n}\ln\det\left[(\theta^* I - F)^{-1}(\theta'^* I - F')\right]\\
&= \ln\frac{\theta'^*}{\theta^*} - \frac{\varepsilon_s}{n}\ln\det\left[\left(\frac{\varepsilon_s\theta}{n}I - (\theta-\theta^*)X\right)^{-1} X(\theta'^* I - F')\right] && \text{(by Claim 30)}\\
&= \ln\frac{\theta'^*}{\theta^*} - \frac{\varepsilon_s}{n}\left[\ln\det\left(\frac{\varepsilon_s\theta}{n}I - (\theta-\theta^*)X\right)^{-1} + \ln\det\left[X(\theta'^* I - F')\right]\right]\\
&\geq \ln\frac{\theta'^*}{\theta^*} - \frac{\varepsilon_s}{n}\left[\ln\left(\frac{\varepsilon_s\theta}{n} - \delta_s\theta\right)^{-n} + \ln\det\left[X(\theta'^* I - F')\right]\right]\\
& && (\because \text{Tr}(X) \leq 1 \text{ by Claim 24 and } \theta^* \leq \theta \leq (1+\delta_s)\theta^*)\\
&\geq \ln\frac{\theta'^*}{\theta^*} - \frac{\varepsilon_s}{n}\left[\ln\left(\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta}\right)^{n} + \ln\det X(\theta'^* I - F')\right] && \text{(by defintion of } \delta_s)\\
&= \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta} - \frac{\varepsilon_s}{n}\ln\left[\det X(\theta'^* I - F')\right]\\
&= \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta} - \frac{\varepsilon_s}{n}\sum_{j=1}^{n}\ln\lambda_j\left(X(\theta'^* I - F')\right)\\
&\geq \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j\left(X(\theta'^* I - F')\right)\right) && \text{(by the concavity of } \ln(\cdot))\\
&= \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\frac{1}{n}\text{Tr}(\theta'^* X - XF')\right)\\
&\geq \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{n}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\frac{\theta'^* - X\bullet F'}{n}\right) && (\because \text{Tr}(X) \leq 1 \text{ by Claim 24})\\
&= \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{1}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\theta'^* - X\bullet F'\right)\\
&\geq \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{1}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\theta'^* - \min_{y\in\mathbb{R}_+^m:\ \mathbf{1}^T y = 1} X\bullet F(y)\right) && (\because \mathbf{1}^T y(t') = 1 \text{ by Claim 27})\\
&= \ln\frac{\theta'^*}{\theta^*} - \varepsilon_s\ln\frac{1}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\theta'^* - X\bullet A_{i(t)}\right) && \text{(by defintion of } i(t))\\
&\geq \min_{\xi > X\bullet A_{i(t)}}\left\{\ln\frac{\xi}{\theta^*} - \varepsilon_s\ln\frac{1}{(1-\varepsilon_s)\varepsilon_s\theta} - \varepsilon_s\ln\left(\xi - X\bullet A_{i(t)}\right)\right\}\\
&= (1-\varepsilon_s)\ln\frac{X\bullet A_{i(t)}}{(1-\varepsilon_s)^2\theta} + \ln\frac{(1-\varepsilon_s)\theta}{\theta^*} && \left(\min(\cdot) \text{ is achieved at } \xi = \tfrac{X\bullet A_{i(t)}}{1-\varepsilon_s}\right)\\
&\geq (1-\varepsilon_s)\ln\frac{X\bullet A_{i(t)}}{(1-\varepsilon_s)^2\theta} + \ln(1-\varepsilon_s) && (\because \theta \geq \theta^*)\\
&\geq (1-\varepsilon_s)\ln\frac{(1-2\varepsilon_s)X\bullet A_{i(t)}}{(1-\varepsilon_s)^2 X\bullet F} + \ln(1-\varepsilon_s). && \text{(by Claim 26)}
\end{aligned}
$$

□

Claim 38. $\frac{X(0)\bullet A_{i(0)}}{X(0)\bullet F(y(0))} \geq \frac{1}{\psi} := \frac{\lambda_{\min}(A_{i(0)})}{\lambda_{\max}(A_{i'})} \geq \frac{\min_i \lambda_{\min}(A_i)}{\max_i \lambda_{\max}(A_i)} \geq \frac{\epsilon}{n^2 2^{2\mathcal{L}}}$*, where i' is the index such that $y(0) = \mathbf{1}_{i'}$.*

Proof. Let $X(0) = \sum_{j=1}^n \lambda_j u_j u_j^T$ be the spectral decomposition of $X(0)$. Then,

$$X(0)\bullet A_{i(0)} = \sum_{j=1}^n \lambda_j A_{i(0)}\bullet u_j u_j^T \geq \sum_{j=1}^n \lambda_j \lambda_{\min}(A_{i(0)}) = \lambda_{\min}(A_{i(0)})\cdot \text{Tr}(X(0))$$

$$X(0) \bullet F(y(0)) = \sum_{j=1}^{n} \lambda_j A_{i'} \bullet u_j u_j^T \le \sum_{j=1}^{n} \lambda_j \lambda_{\max}(A_{i'}) = \lambda_{\max}(A_{i'}) \cdot \mathrm{Tr}(X(0)).$$

Note that $\psi \le \frac{n^2 2^{2\mathcal{L}}}{\epsilon}$ by Assumption (B-II). The claim follows. $\square$

Claim 39. *The algorithm terminates in at most $O(n \log \psi + \frac{n}{\epsilon^2})$ iterations.*

Proof. Let $t_{-1} = -1$ and, for $s = 0, 1, 2, \ldots$, let t_s be the smallest t such that $\nu(t+1) \le 2^{-(s+1)}$ (so $t_s + 1$ is the value of t at which the iteration $s+1$ of the outer while-loop starts). Then for $t = t_{s-1}+1, \ldots, t_s - 1$, we have $\nu(t+1) > 2^{-(s+1)} = 2\varepsilon_s$. Hence, for $s = 0$,

$$\begin{aligned}
-\frac{\varepsilon_0^3 t_0}{40n} &> \Phi(t_0) - \Phi(0) && \text{(by Claim 36)}\\
&\ge (1-\varepsilon_0) \ln \frac{(1-2\varepsilon_0) X(0) \bullet A_{i(0)}}{(1-\varepsilon_0)^2 X(0) \bullet F(y(0))} + \ln(1-\varepsilon_0) && \text{(by Claim 37)}\\
&\ge (1-\varepsilon_0) \ln \frac{\psi(1-2\varepsilon_0)}{(1-\varepsilon_0)^2} + \ln(1-\varepsilon_0). && \text{(by Claim 38)}
\end{aligned}$$

Setting $\varepsilon_0 = \frac{1}{4}$ in the last series of inequalities we get

$$t_0 < 1920 n \ln\big(\frac{9\psi}{8}\big) + \ln\frac{4}{3} = O(n \log \psi). \tag{15}$$

Now consider $s \ge 1$:

$$\begin{aligned}
-\frac{\varepsilon_s^3 (t_s - t_{s-1})}{40n} &> \Phi(t_s) - \Phi(t_{s-1}) && \text{(by Claim 36)}\\
&\ge (1-\varepsilon_s) \ln \frac{(1-2\varepsilon_s) X(t_{s-1}) \bullet A_{i(t_{s-1})}}{(1-\varepsilon_s)^2 X(t_{s-1}) \bullet F(y(t_{s-1}))} + \ln(1-\varepsilon_s) && \text{(by Claim 37)}\\
&= (1-\varepsilon_s) \ln \frac{(1-2\varepsilon_s)\big(1-\nu(t_{s-1}+1)\big)}{(1-\varepsilon_s)^2\big(1+\nu(t_{s-1}+1)\big)} + \ln(1-\varepsilon_s) && \text{(by definition of } \nu(t_{s-1}+1))\\
&\ge (1-\varepsilon_s) \ln \frac{(1-2\varepsilon_s)(1-4\varepsilon_s)}{(1-\varepsilon_s)^2(1+4\varepsilon_s)} + \ln(1-\varepsilon_s) && (\because \nu(t_{s-1}+1) \le 2^{-s} = 4\varepsilon_s)\\
&\ge -(1-\varepsilon_s) \ln(1+32\varepsilon_s) - \ln(1+3\varepsilon_s) > -35\varepsilon_s. && (\because \varepsilon_s \le \tfrac{1}{8})
\end{aligned}$$

Setting $\varepsilon_s = \frac{1}{2^{s+2}}$ in the last series of inequalities we get

$$t_s - t_{s-1} < \frac{1400n}{\varepsilon_s^2} = O(n/\varepsilon_s^2). \tag{16}$$

Summing (15), and (16) over $s = 1, 2, \ldots, \lceil \log \frac{1}{\epsilon} \rceil$, we get the claim. $\square$

2.4.3 Primal Dual Feasibility and Approximate Optimality

Let $t_f + 1$ be the value of t when the algorithm terminates and $s_f + 1$ be the value of s at termination. For simplicity, we write $s = s_f$.

Claim 40. *(Primal feasibility). $\hat{X} \succ 0$ and $\min_i A_i \bullet \hat{X} \ge 1$.*

Proof. The first claim is immediate from Claim 23. To see the second claim, we use the definition of $\nu(t_f)$ and the termination condition in line 4 (which is also satisfied even if $X(t_f) \bullet F(y(t_f)) - X(t_f) \bullet A_{i(t_f)} = 0$):

$$\begin{aligned}
&\frac{X(t_f) \bullet F(y(t_f)) - X(t_f) \bullet A_{i(t_f)}}{X(t_1) \bullet A_{i(t_f)} + X(t_f) \bullet F(y(t_f))} \le \varepsilon_s.\\
&\quad\therefore (1-\varepsilon_s) X(t_f) \bullet F(y(t_f)) \le (1+\varepsilon_s) X(t_f) \bullet A_{i(t_f)}\\
&\qquad\qquad = (1-\varepsilon_s) \min_i X(t_f) \bullet A_i && \text{(by the defintition of } i(t_f))\\
&\quad\therefore (1-\varepsilon_s)(1-2\varepsilon_s)\theta(t_f) < (1+\varepsilon_s) \min_i X(t_f) \bullet A_i. && (\because X(t_f) \bullet F(y(t_f)) > (1-2\varepsilon_s)\theta(t_f) \text{ by Claim 26})
\end{aligned}$$

The claim follows by the definition of $\hat{X}$ in step 16 of the algorithm. $\square$

Claim 41. *(Dual feasibility). $\hat{y} \ge 0$ and $F(\hat{y}) \prec I$.*

Proof. The fact that $\hat{y} \geq 0$ follows from the initialization of $y(0)$ in step 1, Claim 6, and the update of $y(t+1)$ in step 10. For the other claim, we have

$$\lambda_{\max}\big(F(\hat{y})\big) = \frac{1}{\theta(t_f)}\lambda_{\max}\big(F(y(t_f))\big) \leq 1 - \frac{\varepsilon_s}{n}. \qquad \text{(by Claim 25)}$$

$\square$

Claim 42. *(Approximate optimality).* $I \bullet \hat{X} \leq \frac{1+\varepsilon_s}{(1-2\varepsilon_s)^2}\mathbf{1}^T\hat{y}$.

Proof. By Claim 24, we have $\text{Tr}(X(t_f) \leq 1$, and by Claim 27, we have $\mathbf{1}^T y(t_f) = 1$. The claim follows by the definition of $\hat{X}$ and $\hat{y}$ in step 16. $\square$

Remark 3. *Similar to the packing case, suppose that in step 7 of Algorithm 2, we instead define $i(t)$ to be an index $i \in [m]$ such that $A_i \bullet X(t) \leq 1+\varepsilon_s$, and we are guaranteed that such index exists in each iteration of the algorithm. Then the dual solution $\hat{y}$ satisfies: $\mathbf{1}^T\hat{y} \geq 1 - O(\epsilon)$. Indeed, the proof of Claim 40 can be easily modified to show that $\theta(t_f) \leq \frac{(1+\varepsilon_{s_f})^2}{(1-2\varepsilon_{s_f})^2}$, which combined with the definition of $\hat{y}$ in step 16 of the algorithm implies the claim.*

2.4.4 Running Time per Iteration

Computing $\theta(t)$**.** Given $F := F(y(t)) \succeq 0$, we first compute an approximation $\widetilde{\lambda}$ of $\lambda_{\max}(F)$ using Lanczos' algorithm with a random start. By Claim 25, we need $\widetilde{\lambda}$ to lie in the range $[\lambda_{\max}(F), \frac{\lambda_{\max}(F)}{1-\varepsilon_s/n}]$. To obtain $\widetilde{\lambda}$, we apply Lemma 21 with $M := F^n$ and $\gamma := \frac{\varepsilon_s}{2}$. Then in $O(\frac{\log n}{\sqrt{\varepsilon_s}})$ iterations we get $\widetilde{\lambda} := \left(\frac{v^T F^n v}{1-\gamma}\right)^{1/n}$(where v be the vector obtained from Lemma 21) satisfying

$$\lambda_{\max}(F) \leq \widetilde{\lambda} \leq \frac{\lambda_{\max}(F)}{(1-\gamma)^{1/n}} \leq (1+\varepsilon_s)^{1/n}\lambda_{\max}(F) \leq \frac{\lambda_{\max}(F)}{1-\varepsilon_s/n}. \tag{17}$$

Thus, the overall running time for computing $\widetilde{\lambda}$ is $O(n^\omega \log n + \frac{n^2 \log n}{\sqrt{\varepsilon_s}})$. Given $\widetilde{\lambda}$, we know by Claim 25 and (17) that $\theta^*(t) \in [\widetilde{\lambda}, \frac{\widetilde{\lambda}}{1-\varepsilon_s}]$. Then we can apply binary search to find $\theta(t) := \theta^*(t)^{\delta_s}$ as follows. Let $\theta_k = \widetilde{\lambda}(1+\delta_s)^k$, for $k = 0, 1, \ldots, K := \lceil\frac{-2\ln(1-\varepsilon_s)}{\delta_s}\rceil$, and note that $\theta_K \geq \widetilde{\lambda}$. Then we do binary search on the exponent $k \in \{0, 1 \ldots, K\}$; each step of the search evaluates $g(\theta_k) := \dfrac{\varepsilon_s\theta_k}{n}\text{Tr}(\theta_k I - F)^{-1}$, and depending on whether this value is less than or at least 1, the value of k is decreased or increased, respectively. The search stops when the search interval $[\ell, u]$ has $u \leq \ell + 1$, in which case we set $\theta(t) = \theta_u$; the number of steps until this happens is $O(\log K) = O(\log \frac{1}{\delta_s}) = O(\log \frac{n}{\varepsilon_s})$. By the monotonicity of $g(x)$ (in the interval $[\lambda_{\max}(F), +\infty)$), and the property of binary search, we know that $\theta^* \in [\theta_\ell, \theta_u]$. Thus, by the stopping criterion,

$$\theta_\ell \leq \theta^*(t) \leq \theta(t) = \theta_u \leq \theta_{\ell+1} = (1+\delta_s)\theta_\ell,$$

implying that $\theta^*(t) \leq \theta(t) \leq (1+\delta_s)\theta^*(t)$. Since evaluating $g(\theta_k)$ takes $O(n^\omega)$, the overall running time for the binary search procedure is $O(n^\omega \log \frac{n}{\varepsilon_s})$, and hence the total time needed for for computing $\theta(t)$ is $O(n^\omega \log \frac{n}{\epsilon} + \frac{n^2 \log n}{\sqrt{\epsilon}})$.

As all other steps of the algorithm inside the inner while-loop can be done in $O(\mathcal{T} + n^2)$ time, where $\mathcal{T}$ is the time taken by a single call to the minimization oracle in step 7, in view of Claim 17, we obtain the following result.

Theorem 43. *For any $\epsilon > 0$, Algorithm 2 outputs an $O(n\log\psi + \frac{n}{\epsilon^2})$-sparse $O(\epsilon)$-optimal primal-dual pair in time $O((n\log\psi + \frac{n}{\epsilon^2})(n^\omega \log \frac{n}{\epsilon} + \frac{n^2\log n}{\sqrt{\epsilon}} + \mathcal{T})) = \tilde{O}(\frac{n^{\omega+1}\log\psi}{\epsilon^{2.5}} + \frac{n\mathcal{T}\log n}{\epsilon^2})$.*

3 Applications

3.1 Robust Packing and Covering SDPs

Consider a packing-covering pair of the form (PACKING-I)-(COVERING-I) or (PACKING-II)-(COVERING-II). In the framework of *robust optimization* (see, e.g. [9, 10]), we assume that each constraint matrix A_i is not known exactly; instead, it is given by a convex uncertainty set $\mathcal{A}_i \subseteq \mathbb{S}_+^n$. It is required to find a (near)-optimal solution for the packing-covering pair under the *worst-case* choice $A_i \in \mathcal{A}_i$ of the constraints in each uncertainty set. A typical example of a *convex* uncertainty set is given by an *affine perturbation* around a nominal matrix $A_i^0 \in \mathbb{S}_+^n$:

$$\mathcal{A}_i = \left\{A_i := A_i^0 + \sum_{r=1}^k \delta_r A_i^r :\ \delta = (\delta_1, \ldots, \delta_k) \in \mathcal{D}\right\}, \tag{18}$$

where $A_i^1, \ldots, A_i^k \in \mathbb{S}_+^n$, and $\mathcal{D} \subseteq \mathbb{R}_+^k$ can take, for example, one of the following forms:

- *Ellipsoidal uncertainty*: $\mathcal{D} = E(\delta_0, D) := \{\delta \in \mathbb{R}^k_+ : (\delta - \delta_0)^T D^{-1}(\delta - \delta_0) \le 1\}$, for given positive definite matrix $D \in \mathbb{S}^k_+$ and vector $\delta_0 \in \mathbb{R}^k_+$ such that $E(\delta_0, D) \subseteq \mathbb{R}^k_+$;
- *Box uncertainty*: $\mathcal{D} = B(\delta_0, \rho) := \{\delta \in \mathbb{R}^k_+ : \|\delta - \delta_0\|_1 \le \rho\}$, for given positive number $\rho \in \mathbb{R}_+$ and vector $\delta_0 \in \mathbb{R}^k_+$ such that $B(\delta_0, \rho) \subseteq \mathbb{R}^k_+$;
- *Polyhedral uncertainty*: $\mathcal{D} := \{\delta \in \mathbb{R}^k_+ : D\delta \le w\}$, for given matrix $D \in \mathbb{R}^{h\times k}$ and vector $w \in \mathbb{R}^h$.

Without loss of generality, we consider the robust version of (NORM-PACKING-I)-(NORM-COVERING-I), where A_i, for $i \in [m]$, belongs to a convex uncertainty set $\mathcal{A}_i$. Then the robust optimization problem and its dual can be written as follows:

$$\begin{aligned} z_P^* = \max \quad & I \bullet X && \text{(RBST-PACKING-I)} \\ \text{s.t.} \quad & A_i \bullet X \le 1, \quad \forall A_i \in \mathcal{A}_i \quad \forall i \in [m] \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 \end{aligned}$$

$$\begin{aligned} z_D^* = \inf \quad & \sum_{i=1}^m \int_{\mathcal{A}_i} y^i_{A_i} dA_i && \text{(RBST-COVERING-I)} \\ \text{s.t.} \quad & \sum_{i=1}^m \int_{\mathcal{A}_i} y^i_{A_i} A_i dA_i \succeq I \\ & y^i \text{ is a } \textit{discrete measure} \text{ on } \mathcal{A}_i, \quad \forall i \in [m]. \end{aligned}$$

As before, we assume (B-I), where $A_1, \ldots, A_r \in \bigcup_{i\in[m]} \mathcal{A}_i$. We call a pair of solutions (X, y) to be ϵ-optimal for (RBST-PACKING-I)-(RBST-COVERING-I), if

$$z_P^* \ge I \bullet X \ge (1-\epsilon)\sum_{i=1}^m \int_{\mathcal{A}_i} y^i_{A_i} dA_i \ge (1-\epsilon) z_D^*.$$

As a corollary of Theorem 22, we obtain the following result.

Theorem 44. *For any $\epsilon > 0$, Algorithm 1 outputs an $O(\epsilon)$-optimal primal-dual pair for (*RBST-PACKING-*I)-(*RBST-COVERING-*I) in time $\tilde{O}\big(\frac{n^{\omega+1}\log\psi}{\epsilon^{2.5}} + \frac{n\mathcal{T}\log\psi}{\epsilon^2}\big)$, where $\psi := \frac{r\cdot\max_{i\in[m],A_i\in\mathcal{A}_i}\lambda_{\max}(A_i)}{\lambda_{\min}(A)}$ and $\mathcal{T}$ is the time to compute, for a given $Y \in \mathbb{S}^n_+$, a pair (i, A_i) such that*

$$(i, A_i) \in \operatorname{argmax}_{i\in[m],\ A_i\in\mathcal{A}_i} A_i \bullet Y. \tag{19}$$

Note that (19) amounts to solving a linear optimization problem over a convex set. Moreover, for simple uncertainty sets, such as balls or ellipsoids, such computation can be done very efficiently.

3.2 Carr-Vempala Type Decomposition

Consider a maximization (resp., minimization) problem over a discrete set $\mathcal{S} \subseteq \mathbb{Z}^n$ and a corresponding SDP-relaxation over $\mathcal{Q} \subseteq \mathbb{S}^n_+$:

$$\begin{aligned} z_{CO}^* = \begin{Bmatrix} \max \\ \min \end{Bmatrix} \quad & C \bullet qq^T && \text{(COP)} \\ \text{s.t.} \quad & q \in \mathcal{S} \end{aligned}$$

$$\begin{aligned} z_{SDP}^* = \begin{Bmatrix} \max \\ \min \end{Bmatrix} \quad & C \bullet Q && \text{(SDP-RLX)} \\ \text{s.t.} \quad & Q \in \mathcal{Q}, \end{aligned}$$

where $C \in \mathbb{S}^n_+$.

Definition 1. *For $\alpha \in (0, 1]$ (resp., $\alpha \ge 1$), an α-integrality gap verifier $\mathcal{A}$ for (SDP-RLX) is a polytime algorithm that, given any $C \in \mathbb{S}^n_+$ and any $Q \in \mathcal{Q}$ returns a $q \in \mathcal{S}$ such that $B \bullet qq^T \ge \alpha B \bullet Q$ (resp., $C \bullet qq^T \le \alpha C \bullet Q$).*

For instance, if $\mathcal{S} = \{-1, 1\}^n$ and $Q = \{X \in \mathbb{S}^n_+ : X_{ii} = 1 \ \ \forall i \in [n]\}$, then a $\frac{2}{\pi}$-integrality gap verifier for the maximization version of (SDP-RLX) is known [29].

Carr and Vempala [12] gave a decomposition theorem that allows one to use an α-integrality gap verifier for a given LP-relaxation of a combinatorial maximization (resp., minimization) problem, to decompose a given fractional solution to the LP into a convex combination of integer solutions that is dominated by (resp., dominates) α times the fractional solution. In [13], we prove a similar result for SDP relaxations:

Theorem 45. *Consider a combinatorial maximization (resp., minimization) problem (COP) and its SDP relaxation (SDP-RLX), admitting an α-integrality gap verifier $\mathcal{A}$. Assume the set $\mathcal{S}$ is full-dimensional. Then there is a polytime algorithm that, for any given $Q \in \mathcal{Q}$, finds a set $\mathcal{X} \subseteq \mathcal{S}$, of polynomial size, and a set of convex multipliers $\{\lambda_q \in \mathbb{R}_+ :\ q \in \mathcal{X}\}$, $\sum_{q\in\mathcal{X}} \lambda_q = 1$, such that*

$$\alpha Q \preceq \sum_{q\in\mathcal{X}} \lambda_q qq^T \qquad \textit{(resp., } \alpha Q \succeq \sum_{q\in\mathcal{X}} \lambda_q qq^T\textit{)}.$$

The proof of Theorem 45 is obtained by considering the following pairs of packing and covering SDPs (of types I and II, respectively):

$$
\begin{aligned}
z_I^* = \min \quad & \sum_{q\in\mathcal{S}} \lambda_q && \text{(CVX-I)}\\
\text{s.t.} \quad & \sum_{q\in\mathcal{S}} \lambda_q qq^T \succeq \alpha Q && (20)\\
& \sum_{q\in\mathcal{S}} \lambda_q \geq 1 && (21)\\
& \lambda\in\mathbb{R}^{\mathcal{S}},\ \lambda\geq 0
\end{aligned}
\qquad
\begin{aligned}
z_I^* = \max \quad & \alpha Q \bullet Y + u && \text{(CVX-dual-I)}\\
\text{s.t.} \quad & qq^T \bullet Y + u \leq 1, \forall q\in\mathcal{S} && (22)\\
& Y\in\mathbb{S}_+^n,\ u\geq 0.
\end{aligned}
$$

$$
\begin{aligned}
z_{II}^* = \max \quad & \sum_{q\in\mathcal{S}} \lambda_q && \text{(CVX-II)}\\
\text{s.t.} \quad & \sum_{q\in\mathcal{S}} \lambda_q qq^T \preceq \alpha Q && (23)\\
& \sum_{x\in\mathcal{S}} \lambda_q \leq 1 && (24)\\
& \lambda\in\mathbb{R}^{\mathcal{S}},\ \lambda\geq 0
\end{aligned}
\qquad
\begin{aligned}
z_{II}^* = \min \quad & \alpha Q \bullet Y + u && \text{(CVX-dual-II)}\\
\text{s.t.} \quad & qq^T \bullet Y + u \geq 1, \forall q\in\mathcal{S} && (25)\\
& Y\in\mathbb{S}_+^n,\ u\geq 0.
\end{aligned}
$$

It can be shown, using the fact that the SDP relaxation admits an α-integrality gap verifier, that $z_I^* = z_{II}^* = 1$, and that the two primal-dual pairs can be solved in polynomial time using the Ellipsoid method. Here, we derive a more efficient but approximate version of Theorem 45.

Theorem 46. *Consider a combinatorial maximization (resp., minimization) problem (COP) and its SDP relaxation (SDP-RLX), admitting an α-integrality gap verifier $\mathcal{A}$. Assume the set $\mathcal{S}$ is full-dimensional and let $\epsilon > 0$ be a given constant. Then there is a polytime algorithm that, for any given $Q \in \mathcal{Q}$, finds a set $\mathcal{X} \subseteq \mathcal{S}$ of size $|\mathcal{X}| = O(\frac{n^3}{\epsilon^2}\log(nW))$ (resp., of size $|\mathcal{X}| = O(n\log\frac{n}{\epsilon} + \frac{n}{\epsilon^2})$), where $W := \max_{q\in\mathcal{S},\ i\in[n]} |q_i|$, and a set of convex multipliers $\{\lambda_q \in \mathbb{R}_+ :\ q \in \mathcal{X}\}$, $\sum_{q\in\mathcal{X}}\lambda_q = 1$, such that*

$$(1-O(\epsilon))\alpha Q \preceq \sum_{q\in\mathcal{X}} \lambda_q qq^T \qquad \textit{(resp., } (1+O(\epsilon))\alpha Q \succeq \sum_{q\in\mathcal{X}} \lambda_q qq^T\textit{)}. \tag{26}$$

Proof. Let us first consider the maximization problem and the corresponding covering SDP (CVX-I). We can write (CVX-I)-(CVX-dual-I) in the form of (COVERING-I)-(PACKING-I), where the set of constraints $[m]$ corresponds to $\mathcal{S}$, by setting

$$A_q := \begin{bmatrix} qq^T & 0\\ 0 & 1\end{bmatrix}, \qquad C := \begin{bmatrix} \alpha Q & 0\\ 0 & 1\end{bmatrix}, \qquad X := \begin{bmatrix} Y & 0\\ 0 & u\end{bmatrix}. \tag{27}$$

Let us fix any linearly independent subset $\mathcal{S}' \subseteq \mathcal{S}$ of $\mathcal{S}$ of size n. Write $\bar{A} := \sum_{q\in\mathcal{S}'} qq^T$. Then for any $Y \succeq 0$, feasible for (CVX-dual-I), we have $I \bullet Y \leq \frac{\bar{A}\bullet Y}{\lambda_{\min}(B)} \leq \frac{n}{\lambda_{\min}(A)}$. To arrive at a bound τ as in Assumption (B-I), we need to lower-bound $\lambda_{\min}(\bar{A})$. Let $\mathcal{L}'$ be the total bit length needed to describe $\mathcal{S}'$. Then we have the following bound.

Claim 47. $\lambda_{\min}(\bar{A}) \geq \gamma := 2^{-2\mathcal{L}'-1}$.

Proof. Equivalently, we need to show that $\sum_{q\in\mathcal{S}'}(q^Tv)^2 + v_0^2 \geq \gamma$, for any unit vector $(v, v_0) \in \mathbb{R}^{n+1}$. Suppose for the sake of contradiction that $|v_0| < \sqrt{\gamma}$ and $|q^Tv| < \sqrt{\gamma}$ for all $q \in \mathcal{S}$. Let $H \in \mathbb{R}^{n\times n}$ be the matrix whose columns are the vectors $q \in \mathcal{S}'$ and $h \in \mathbb{R}^n$ be a vector with component q^Tv in the position corresponding to $q \in \mathcal{S}'$. Then the linear system $Hx = h$ has a unique solution $x = v = H^{-1}h$ such that each component is bounded in absolute value from above by $2^{\mathcal{L}'}\sqrt{\gamma}$ (see. e.g., [18, chapter 1]). Since (v, v_0) is a unit vector, it follows that

$$1 = \|v\|^2 + v_0^2 < 2^{2\mathcal{L}'}\gamma + \gamma < 1,$$

a contradiction. $\square$

From Claim 47, we know that assumption (B-I) is satisfied with $\tau := 2^{2\mathcal{L}'+1}n + 1$, where $\mathcal{L}' \leq n^2\log(W+1)$. Let $\alpha Q = L^TDL$ be the LDL-decomposition of αQ and write $U := L^{-1}$. By the reduction in Appendix B.1, we can use $\alpha Q(\delta) = L^TD(\delta)L = \alpha Q + \delta L^T\bar{I}L$, where $D(\delta) = D + \delta L^T\bar{I}L$ and $\delta \leq \frac{\epsilon}{\tau I\bullet L^TL}$ (as $z_I^* = 1$), instead of αQ without changing the objective value by a factor more than $(1+\epsilon)$ (if Q is nonsingular, then we set $\delta = 0$). (Recall that $\bar{I}$ is the $0/1$-diagonal matrix with ones only in the entries corresponding to the *zero* diagonal entries of the diagonal matrix D, and note that the matrix L is *independent* of δ.) For $q \in \mathcal{S}$, let $p(q) := D(\delta)^{-1/2}U^Tq$. Using the transformation of variables

$Y' := D(\delta)^{1/2}LYL^TD(\delta)^{1/2}$, we get $\alpha Q(\delta)\bullet Y = I\bullet Y'$ and $qq^T\bullet Y = p(q)p(q)^T\bullet Y'$. Hence, we obtain a normalized form of (an approximate version of) (CVX-I)-(CVX-dual-I), where $q\in\mathcal{S}$ is replaced by $p(q)$. In view of Remark 2, it is enough to show that in each iteration t of Algorithm 1, we can find efficiently a $q\in\mathcal{S}$ such that $p(q)p(q)^T\bullet Y'+u\geq 1-O(\varepsilon_s)$ for given $Y'=Y'(t)\succeq 0$ and $u=u(t)\geq 0$ such that $\mathrm{Tr}(Y')+u\in(1-\varepsilon_s,1)$ (by Claim 2, where $X(t) := \begin{pmatrix} Y' & 0\\ 0 & u\end{pmatrix}$ in step 6 of the algorithm). To do this, let $Y := UD(\delta)^{-1/2}Y'D(\delta)^{-1/2}U^T$ and call the integrality gap verifier $\mathcal{A}$ on (Y,Q) to get a vector $q\in\mathcal{S}$ such that $qq^T\bullet Y\geq \alpha Q\bullet Y$. Then

$$p(q)p(q)^T\bullet Y'+u = qq^T\bullet Y+u\geq \alpha Q\bullet Y+u = \alpha Q(\delta)\bullet Y+u-\delta L^T\bar{I}L\bullet Y = I\bullet Y'+u-\delta L^T\bar{I}L\bullet Y. \tag{28}$$

We bound the "error term" $\delta L^T\bar{I}L\bullet Y$ using the definition of $Y'=Y'(t) := \frac{\varepsilon_s\theta(t)}{n+1}\big(\sum_{q\in\mathcal{S}}\lambda_q(t)p(q)p(q)^T-\theta(t)I\big)^{-1}$ in step 6 of the algorithm as follows:

$$\begin{aligned}\delta L^T\bar{I}L\bullet Y &= \delta L^T\bar{I}L\bullet UD(\delta)^{-1/2}Y'D(\delta)^{-1/2}U^T = \delta\bar{I}\bullet D(\delta)^{-1/2}Y'D(\delta)^{-1/2} = \delta D(\delta)^{-1/2}\bar{I}D(\delta)^{-1/2}\bullet Y' = \bar{I}\bullet Y'\\ &= \frac{\varepsilon_s\theta(t)}{n+1}\bar{I}\bullet\big(\sum_{q\in\mathcal{S}}\lambda_q(t)p(q)p(q)^T-\theta(t)I\big)^{-1} = \frac{\varepsilon_s\theta(t)}{n+1}\bar{I}\bullet\big(D(\delta)^{-1/2}H(t)D(\delta)^{-1/2}-\theta(t)I\big)^{-1},\end{aligned} \tag{29}$$

where, for brevity, we write $H=H(t) := \sum_{q\in\mathcal{S}}\lambda_q(t)U^Tqq^TU$. To bound (29), we need to compute the submatrix of $\big(D(\delta)^{-1/2}H(t)D(\delta)^{-1/2}-\theta(t)I\big)^{-1}$ corresponding to the non-zeros of $\bar{I}$. Let the corresponding decompositions of the matrices $D(\delta)$ and $G(t) := D(\delta)^{-1/2}H(t)D(\delta)^{-1/2}-\theta(t)I$ be as follows:

$$D(\delta)=\begin{pmatrix} D' & 0\\ 0 & \delta\bar{I}\end{pmatrix},\qquad H(t)=\begin{pmatrix} H_1 & H_2\\ H_2^T & H_3\end{pmatrix},\qquad G(t)=\begin{pmatrix} G_1 & G_2\\ G_2^T & G_3\end{pmatrix}=\begin{pmatrix} (D')^{-1/2}H_1(D')^{-1/2}-\theta(t)I & \frac{1}{\sqrt{\delta}}(D')^{-1/2}H_2\\ \frac{1}{\sqrt{\delta}}H_2^T(D')^{-1/2} & \frac{1}{\delta}H_3-\theta(t)I\end{pmatrix}, \tag{30}$$

where, for simplicity, we use I to denote the identity matrix of the proper dimension, according to the context. As $G(t)\succ 0$, we have

$$\theta(t)\leq\lambda_{\min}\big((D')^{-1/2}H_1(D')^{-1/2}\big),\ \text{and}\ M := H_3-H_2^T(D')^{-1}H_1(D')^{-1}H_2\succ 0. \tag{31}$$

Using the block inversion formula:

$$\begin{aligned}\bar{I}\bullet G(t)^{-1} &= I\bullet\big(G_3-G_2^TG_1G_2\big)^{-1} = I\bullet\Big(\frac{1}{\delta}H_3-\theta(t)I-\frac{1}{\delta}H_2^T(D')^{-1/2}\big((D')^{-1/2}H_1(D')^{-1/2}-\theta(t)I\big)(D')^{-1/2}H_2\Big)^{-1}\\ &= \delta I\bullet\Big(H_3-\delta\theta(t)I-H_2^T(D')^{-1/2}\big((D')^{-1/2}H_1(D')^{-1/2}-\theta(t)I\big)(D')^{-1/2}H_2\Big)^{-1},\end{aligned} \tag{32}$$

and writing $\bar{M} := H_3-H_2^T(D')^{-1/2}\big((D')^{-1/2}H_1(D')^{-1/2}-\theta(t)I\big)(D')^{-1/2}H_2$, we get by (32),

$$\bar{I}\bullet G(t)^{-1}=\sum_{j=1}^n\frac{\delta}{\lambda_j(\bar{M})-\delta\theta(t)}\leq\frac{\delta n}{\lambda_{\min}(\bar{M})-\delta\theta(t)}. \tag{33}$$

Note that $\bar{M}=M+\theta(t)H_2^T(D')^{-1}H_1(D')^{-1}H_2\succeq M\succ 0$ by (31), and that M, D' and H_1 are independent of δ. It follows that, if we set

$$\delta := \min\left\{\frac{\epsilon}{\tau I\bullet L^TL},\frac{\lambda_{\min}(M)}{2\lambda_{\min}((D')^{-1/2}H_1(D')^{-1/2})}\right\}, \tag{34}$$

then by (29), (31) and (33),

$$\delta L^T\bar{I}L\bullet Y\leq\frac{\varepsilon_s\theta(t)}{n+1}\bar{I}\bullet G(t)^{-1}\leq\frac{\varepsilon_s\theta(t)}{n+1}\cdot\frac{\delta n}{\lambda_{\min}(M)-\delta\theta(t)}\leq\frac{\varepsilon_s n}{n+1}<\varepsilon_s. \tag{35}$$

Using (29), (35) and $I\bullet Y'+u\geq 1-\varepsilon_s$, we get the desired inequality. Let $\mathcal{X}\subseteq\mathcal{S}$ be the set of vectors $q\in\mathcal{S}'$ such that $\lambda_q>0$ when the algorithm terminates. Since each iteration of the algorithm adds at most one element to $\mathcal{X}$, we have by Claim 39 that $|\mathcal{X}|=O\big(n\log\psi+\frac{n}{\epsilon^2}\big)$, where we set $r=n$, $\bar{A} := \sum_{i=1}^r A_i\succ 0$, and use the set of matrices $\{p(q)p(q)^T : q\in\mathcal{S}'\}$ for $A_1,\ldots,A_r$ in assumption (B-I), where $\mathcal{S}'\subseteq\mathcal{S}$ is a linearly independent subset of $\mathcal{S}$. We bound ψ in the same way as in the proof of Claim 17:

$$\psi\leq\frac{\max_{q\in\mathcal{S}}Y'(0)\bullet p(q)p(q)^T+u(0)}{Y'(0)\bullet\frac{1}{n}\sum_{q\in\mathcal{S}'}p(q)p(q)^T+u(0)}=\frac{\max_{q\in\mathcal{S}}Y(0)\bullet qq^T+u(0)}{Y(0)\bullet\frac{1}{n}\sum_{q\in\mathcal{S}'}qq^T+u(0)}\leq\frac{n\cdot\max_{q\in\mathcal{S}}\|q\|^2}{\lambda_{\min}(\sum_{q\in\mathcal{S}'}qq^T)}\leq n^2W^2\big(2^{2\mathcal{L}'+1}n+1\big)=n^3W^{O(n^2)}.$$

It follows that $|\mathcal{X}|=O(\frac{n^3}{\epsilon^2}\log(nW))$ (which is also a bound on the number of iterations of the algorithm). Moreover, by Remark 2, we have $\sum_{q\in\mathcal{X}}\lambda_q\leq 1+O(\epsilon)$. Thus scaling each λ_q by $\sum_{q'\in\mathcal{X}}\lambda_{q'}$ yields the sought convex combination satisfying the first inequality in (26).

Now consider the minimization problem. (In in this part of the proof, we do not require $\mathcal{S}$ to be full-dimensional.) We can write (CVX-II)-(CVX-dual-II) in the form of (PACKING-II)-(COVERING-II), where the set of constraints $[m]$ corresponds to $\mathcal{S}$, and where A_q, C and X are given by (27). By the reduction in Appendix B.2, we can reduce (CVX-II)-(CVX-dual-II) to normalized form without changing the value of the objective, but we need to show that each step of this reduction can be implemented in polynomial time. Consider assumption (C-II). Suppose that this assumption does not hold. Then there is an $x \in \mathbb{R}^n$ such that $Qx = 0$ and $q^T x \neq 0$ for some $q \in \mathcal{S}$, implying that $q \notin \mathrm{image}(Q) := \{Qv \;:\; v \in \mathbb{R}^n\}$. Conversely, if $q \notin \mathrm{image}(Q)$, then (by Farkas' Lemma) there exists an $x \in \mathbb{R}^n$ such that $Qx = 0$ and $q^T x \neq 0$. We conclude (by the same argument following assumption (C-II) in Appendix B.2) that for $q \in \mathcal{S} \setminus \mathrm{image}(Q)$, the primal variable $\lambda_q = 0$, and hence, we may replace $\mathcal{S}$ by $\mathcal{S}' := \mathcal{S} \cap \mathrm{image}(Q)$ in (CVX-dual-II). Let $\alpha Q = L^T DL$ be the LDL-decomposition of αQ, and write $U = [U' \mid U''] := L^{-1}$, where U' is the submatrix of U whose columns correspond to the columns of the submatrix D' containing the *positive* diagonal entries of the diagonal matrix D. Let $p(q) := (D')^{-1/2}(U')^T q$, for $q \in \mathcal{S}'$. Then (23) becomes equivalent to $\sum_{q\in\mathcal{S}'} \lambda_q p(q)p(q)^T \preceq I$. Next, we need to show that Assumption (B-II) can be made to hold in polynomial time. For our purposes, it is enough to show a weaker version of this assumption, as we shall see below. We begin by (implicitly) perturbing $p(q)p(q)^T$ into $\tilde{A}_q := p(q)p(q)^T + \frac{\epsilon}{n}I$, for $q \in \mathcal{S}'$. By the argument following Assumption (B-II) in Appendix B.2, $\frac{1}{\beta} \le z^*_{II} = 1 \le \frac{n}{\beta}$, where $\beta := \min_{q\in\mathcal{S}'} \|p(q)\|^2$, from which we obtain that $1 \le \beta \le n$. Furthermore, by the same argument, the optimal value $\tilde{z}_{II}$ of the perturbed problem satisfies $1 - 2\epsilon \le \tilde{z}_{II} \le 1$. Then, in view of Remark 3, it is enough to show that in each iteration t of Algorithm 2, we can find efficiently a $q \in \mathcal{S}'$ such that $\tilde{A}_q \bullet Y' + u \le 1 + O(\varepsilon_s)$ for given $Y' = Y'(t) \succeq 0$ and $u = u(t) \ge 0$ such that $\mathrm{Tr}(Y') + u \in (1 - \varepsilon_s, 1)$ (by Claim 24, where $X(t) := \begin{pmatrix} Y' & 0 \\ 0 & u \end{pmatrix}$ in step 6 of the algorithm). To do this, let $\mathcal{L}'$ be the total bit length needed to describe Q and $\{v_1, \ldots, v_k\}$ be a basis of $\mathrm{null}(Q) := \{x \in \mathbb{R}^n : Qx = 0\}$. Note that, for each $i \in [k]$, each nonzero component of v_i is bounded in absolute value from below by $2^{-\mathcal{L}'}$ (see. e.g., [18, chapter 1]). Given $Y' \succeq 0$ and $u \ge 0$, we apply $\mathcal{A}$ to (Y, Q), where $Y := U'(D')^{-\frac{1}{2}}Y'(D')^{-\frac{1}{2}}(U')^T + \gamma\sum_{i=1}^k v_i v_i^T$ and $\gamma := 2^{2\mathcal{L}'}\alpha Q \bullet Y + 1 = 2^{2\mathcal{L}'}\alpha Q \bullet U'(D')^{-\frac{1}{2}}Y'(D')^{-\frac{1}{2}}(U')^T + 1$, to get a $q \in \mathcal{S}$ such that $qq^T \bullet Y \le \alpha Q \bullet Y$. We claim that $q \in \mathcal{S}'$. For this, it is enough to show that $q^T v_i = 0$, for all $i \in [k]$. Suppose $v_i^T q \neq 0$ for some $i \in [k]$. Then $|v_i^T q| \ge 2^{-L}$, implying that

$$qq^T \bullet Y = qq^T \bullet U'(D')^{-\frac{1}{2}}Y'(D')^{-\frac{1}{2}}(U')^T + \gamma\sum_{i=1}^k (q^T v_i)^2 \ge (2^{2\mathcal{L}'}\alpha Q \bullet Y + 1)2^{-2\mathcal{L}'} > \alpha Q \bullet Y,$$

a contradiction. We conclude that $q \in \mathcal{S}'$, and moreover that $p(q)p(q)^T \bullet Y' = qq^T \bullet Y \le \alpha Q \bullet Y = (L')^T D'L' \bullet Y = I \bullet Y' \le 1 - u$. Then, $\tilde{A}_q \bullet Y' + u \le 1 + \frac{\epsilon}{n} I \bullet Y' < 1 + \varepsilon_s$, as required. To bound the number of iterations of the algorithm, we need to specify which q' is used initially. This is done as follows. We start the algorithm by setting $Y' = I$ and applying the integrality gap verifier $\mathcal{A}$ to (Y, Q), as above, to obtain a $q' \in \mathcal{S}'$ such that

$$\begin{aligned} \|p(q')\|^2 &= p(q')p(q')^T \bullet Y' = q'q'^T \bullet Y \le \alpha Q \bullet Y = \alpha Q \bullet U'(D')^{-1}(U')^T \\ &= L^T \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} L \bullet U \begin{bmatrix} (D')^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T = \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} (D')^{-1} & 0 \\ 0 & 0 \end{bmatrix} \le n. \end{aligned} \tag{36}$$

Let $\mathcal{X} \subseteq \mathcal{S}$ be the set of vectors $q \in \mathcal{S}$ such that $\lambda_q > 0$ when the algorithm terminates. Since each iteration of the algorithm adds at most one element to $\mathcal{X}$, we have by Claim 17 that $|\mathcal{X}| = O\big(n \log \frac{1}{\psi} + \frac{n}{\epsilon^2}\big)$, where

$$\psi = \frac{\lambda_{\min}(\widetilde{A}_{q(0)})}{\lambda_{\max}(\widetilde{A}_{q'})} \ge \frac{\epsilon/n}{n + \epsilon/n} \ge \frac{\epsilon}{2n^2}.$$

It follows that $|\mathcal{X}| = O(n \log \frac{n}{\epsilon} + \frac{n}{\epsilon^2})$. Moreover, by Remark 3, we have $\sum_{q\in\mathcal{X}} \lambda_q \ge 1 - O(\epsilon)$. Thus scaling each λ_q by $\sum_{q'\in\mathcal{X}} \lambda_{q'}$ yields the sought convex combination satisfying the second inequality in (26). ☐

Note that, once we have a set $\mathcal{X}$ as in Theorem 46, its support can be reduced to $O(\frac{n^2}{\epsilon})$ using the sparsification techniques of [8, 35]. Applications of the Carr-Vempala type decomposition for SDPs in robust *discrete* optimization can be found in [13].

A A Matrix MWU Algorithm for (PACKING-II)-(COVERING-II)

Given positive semidefinite matrices $A_1, \ldots, A_m \in \mathbb{S}^n_+$, we consider the dual packing-covering pair (NORM-PACKING-II)-(NORM-COVERING-II). Here is a matrix MWU algorithm.

```
1  t ← 0; y(0) ← 0; X(0) ← 0; M(0) ← 0; T ← ε^{-2} ln n
2  while M(t) < T do
3      P(t) = (1 + ε)^{Σ_{i=1}^m y_i(t) A_i} /* Update the weight matrix by exponentiation */
4      i(t) ← argmin_i A_i • X(t)
5      δ(t) ← 1/λ_max(A_{i(t)}) /* Define the update step size */
6      X(t + 1) ← X(t) + δ(t)P(t)/(I•X(t)); y(t + 1) ← y(t) + δ(t)1_{i(t)} /* Update the primal-dual solution */
7      M(t + 1) ← λ_max(Σ_i y_i(t) A_i) /* Compute the largest eigenvalue of LHS of dual */
8      t ← t + 1
9  end
10 L(t) ← min_i A_i • X(t)
11 Output (X̂, ŷ) = (X(t)/L(t), y(t)/M(t))
```

Algorithm 3: Matrix MWU Algorithm for (PACKING-II)-(COVERING-II)

A.1 Analysis

Let $F(t) := \sum_{i=1}^m y_i(t) A_i$.

A.1.1 Number of Iterations

Claim 48. *The algorithm terminates in at most nT iterations. Note that by Claim 56 below, $L(t_f) > 0$.*

Proof. Note that $\sum_{j=1}^n \lambda_j(F(t)) = I \bullet F(t)$. Then

$$\sum_{j=1}^n \lambda_j(F(t+1)) - \sum_{j=1}^n \lambda_j(F(t)) = I \bullet F(t+1) - I \bullet F(t) = \delta(t) I \bullet A_{i(t)}$$

$$= I \bullet \frac{A_{i(t)}}{\lambda_{\max}(A_{i(t)})} = \frac{\text{Tr}(A_{i(t)})}{\lambda_{\max}(A_{i(t)})} = \frac{\sum_j \lambda_j(A_{i(t)})}{\lambda_{\max}(A_{i(t)})} \geq 1.$$

It follows that $\sum_{j=1}^n \lambda_j(F(nT)) \geq nT$ and thus

$$\lambda_{\max}(F(nT)) \geq \frac{\sum_{j=1}^n \lambda_j(F(nT))}{n} \geq T.$$

The claim follows by the termination condition in step 2. ☐

Let t_f be the value of t when the algorithm terminates.

A.1.2 Primal Dual Feasibility and Approximate Optimality

Claim 49. *(Primal and dual feasibility:)* $A_i \bullet \hat{X} \geq 1\ \forall i \in [m]$, $\hat{X} \succeq 0$, *and* $\sum_{i=1}^m \hat{y}_i A_i \preceq I$.

Proof. For any $i \in [m]$, we have

$$A_i \bullet \hat{X} = \frac{A_i \bullet X(t_f)}{L(t_f)} = \frac{A_i \bullet X(t_f)}{\min_i A_i \bullet X(t_f)} \geq 1.$$

Also, $\hat{X}(t) = \frac{1}{L(t_f)} \sum_{t=0}^{t_f - 1} \frac{\delta(t) X(t)}{I \bullet X(t)} \succeq 0$, since $X(t) \succeq 0$. Thus the primal is feasible. To see dual-feasibility, note that

$$\sum_{i=1}^m \hat{y}_i A_i = \frac{F(t_f)}{M(t_f)} = \frac{F(t_f)}{\lambda_{\max}(F(t_f))}.$$

Thus, $\lambda_{\max}(F(t_f)) = 1$, implying that $\sum_{i=1}^m \hat{y}_i A_i \preceq I$.

☐

Claim 50. $L(t) \geq \sum_{t'=0}^{t-1} \frac{\delta(t') P(t') \bullet A_{i(t')}}{I \bullet P(t')}$.

Proof. For any $i \in [m]$, we have for all t'

$$A_i \bullet X(t') = A_i \bullet \frac{\delta(t')P(t')}{I \bullet P(t')} = \frac{\delta(t')A_i \bullet P(t')}{I \bullet P(t')} \geq \frac{\delta(t')A_{i(t')} \bullet P(t')}{I \bullet P(t')} \qquad \text{(by the definition of } i(t) \text{ in step 4.)}$$

Summing the above inequality over all $t' < t$, we get the claim. ☐

Claim 51.

$$I \bullet P(t+1) \leq I \bullet (1+\epsilon)^{F(t)}(1+\epsilon)^{\delta(t)A_{i(t)}}.$$

Proof. We will use the Golden-Thompson inequality (see, e.g., [36]): for any two symmetric matrices B and C:

$$\mathrm{Tr}(e^{B+C}) \leq \mathrm{Tr}(e^B e^C).$$

Now,

$$\begin{aligned} I \bullet P(t+1) &= \mathrm{Tr}\left(e^{\ln(1+\epsilon)(F(t)+\delta(t)A_{i(t)})}\right) \leq \mathrm{Tr}\left((1+\epsilon)^{F(t)}(1+\epsilon)^{\delta(t)A_{i(t)}}\right) \qquad \text{(by the Golden-Thompson inequality)} \\ &= I \bullet (1+\epsilon)^{F(t)}(1+\epsilon)^{\delta(t)A_{i(t)}}. \end{aligned}$$

☐

Fact 1. *For* $0 \preceq B \preceq I$ *and* $\epsilon > 0$,

$$(1+\epsilon)^B \preceq I + \epsilon B.$$

Proof. Let $B = U^T \Lambda U$ be the eigen decomposition of B, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$\begin{aligned} (1+\epsilon)^B - (I+\epsilon B) &= U^T \,\mathrm{diag}\left((1+\epsilon)^{\lambda_1}, \ldots, (1+\epsilon)^{\lambda_n}\right) U - U^T \,\mathrm{diag}\left(1+\epsilon\lambda_1, \ldots, 1+\epsilon\lambda_n\right) U \\ &= U^T \,\mathrm{diag}\left((1+\epsilon)^{\lambda_1} - (1+\epsilon\lambda_1), \ldots, (1+\epsilon)^{\lambda_n} - (1+\epsilon\lambda_n)\right) U. \end{aligned} \tag{37}$$

Using the inequality: $(1+\epsilon)^x \leq 1 + \epsilon x$, valid for for $x \in [0,1]$ and $\epsilon > 0$, we obtain that $(1+\epsilon)^{\lambda_j} - (1+\epsilon\lambda_j) \leq 0$ and the claim follows from (37). ☐

Claim 52.

$$(1+\epsilon)^{\delta(t)A_{i(t)}} \preceq I + \epsilon\delta(t)A_{i(t)}.$$

Proof. The claim follows from Fact 1, applied with $B := \delta(t)A_{i(t)}$, which satisfies $0 \preceq B \preceq I$ by the definition of $\delta(t)$ in step 5 of the algorithm. ☐

Fact 2. *For three symmetric matrices* $B, C, D \in \mathbb{R}^{n\times n}$ *if* $B \succeq 0$ *and* $C \preceq D$ *then*

$$B \bullet C \leq B \bullet D.$$

Proof. Immediate from $B \bullet (D - C) \geq 0$ which holds by the positive semidefiniteness of B and $D - C$. ☐

Claim 53.

$$I \bullet P(t+1) \leq I \bullet P(t)\left(1 + \frac{\epsilon\delta(t)P(t) \bullet A_{i(t)}}{I \bullet P(t)}\right).$$

Proof. We conclude from Claims 51 and 52, and Fact 2 applied with $B := (1+\epsilon)^{F(t)}$, $C := (1+\epsilon)^{\delta(t)A_{i(t)}}$ and $D := I + \epsilon\delta(t)A_{i(t)}$, that

$$\begin{aligned} I \bullet P(t+1) &\leq (1+\epsilon)^{F(t)} \bullet (1+\epsilon)^{\delta(t)A_{i(t)}} \leq (1+\epsilon)^{F(t)} \bullet \left(I + \epsilon\delta(t)A_{i(t)}\right) \\ &= I \bullet P(t)\left(1 + \frac{\epsilon\delta(t)P(t) \bullet A_{i(t)}}{I \bullet P(t)}\right). \end{aligned}$$

☐

Claim 54. $I \bullet X(t) \leq I \bullet P(0) e^{\epsilon \sum_{t'=0}^{t-1} \frac{\delta(t')P(t') \bullet A_{i(t')}}{I \bullet P(t')}}$.

Proof. Using the inequality $1 + x \leq e^x$, valid for all $x \in \mathbb{R}$, we get from Claim 54,

$$I \bullet P(t'+1) \leq e^{\frac{\epsilon\delta(t')P(t') \bullet A_{i(t')}}{I \bullet P(t')}}. \tag{38}$$

Iterating (38) for $t' = 0, 1, \ldots, t-1$, we arrive at the claim. ☐

Claim 55. $M(t)\ln(1+\epsilon) \le \ln n + \epsilon \sum_{t'=0}^{t-1} \frac{\delta(t')P(t')\bullet A_{i(t')}}{I\bullet P(t')}$.

Proof. Taking logs in Claim 54, and using that $I \bullet P(0) = n$ and

$$I \bullet X(t) = \sum_{j=1}^{n} \lambda_j((1+\epsilon)^{F(t)}) = \sum_{j=1}^{n} (1+\epsilon)^{\lambda_j(F(t))} \ge (1+\epsilon)^{\lambda_{\max}(F(t))},$$

we get

$$M(t)\ln(1+\epsilon) = \lambda_{\max}(F(t))\ln(1+\epsilon) \le \ln n + \epsilon \sum_{t'=0}^{t-1} \frac{\delta(t')P(t')\bullet A_{i(t')}}{I\bullet P(t')}.$$

□

Claim 56. $\frac{L(t_f)}{M(t_f)} \ge \frac{\ln(1+\epsilon)}{\epsilon} - \epsilon \ge 1 - 1.5\epsilon$ *for* $\epsilon \in (0, 0.5]$.

Proof. Using Claims 50 and 55, we obtain after rearranging terms

$$\begin{aligned}
\frac{L(t_f)}{M(t_f)} \ge \frac{\ln(1+\epsilon)}{\epsilon} - \frac{\ln n}{\epsilon M(t_f)} & \\
\ge \frac{\ln(1+\epsilon)}{\epsilon} - \frac{\ln n}{\epsilon T} & \quad \text{(by the termination condition)} \\
= \frac{\ln(1+\epsilon)}{\epsilon} - \epsilon & \quad \text{(by the definition of } T\text{)} \\
\ge 1 - 1.5\epsilon & \quad \left(\because \frac{\ln(1+\epsilon)}{\epsilon} - \epsilon \ge 1 - 1.5\epsilon \text{ for } \epsilon \in (0, 0.5].\right)
\end{aligned}$$

□

Claim 57. $I \bullet X(t) = \mathbf{1}^T y(t) = \sum_{t'=0}^{t-1} \delta(t')$. *Thus, the following (approximate strong duality) holds*

$$(1-1.5\epsilon) I \bullet \hat{X} \le \mathbf{1}^T \hat{y}.$$

Proof. The first claim follows by

$$I \bullet \Delta X(t) = \frac{I \bullet \delta(t)P(t)}{I \bullet P(t)} = \delta(t) = \mathbf{1}^T y(t).$$

From this we get from Claim 56

$$\frac{\mathbf{1}^T \hat{y}}{I \bullet \hat{X}} = \frac{\mathbf{1}^T y(t_f)}{M(t_f)} \Big/ \frac{I \bullet X(t_f)}{L(t_f)} = \frac{L(t_f)}{M(t_f)} \ge 1 - 1.5\epsilon,$$

from which the second claim follows. □

A.1.3 Running Time per Iteration

The most expensive step is the matrix exponentiation. It can be done (approximately) in time $O(n^3)$, by computing the eigenvalue decomposition of $F(t)$ (more efficient algorithms are available if $F(t)$ is sparse, see, e.g. [20]).

Theorem 58. *For any* $\epsilon > 0$*, Algorithm 3 outputs an* $O(\frac{n \log n}{\epsilon^2})$*-sparse* $O(\epsilon)$*-optimal primal-dual pair in time* $O(\frac{n^4 \log n}{\epsilon^2} + \frac{n\mathcal{T} \log n}{\epsilon^2})$*, where* $\mathcal{T}$ *is the time taken by a single call to the minimization oracle in step 4.*

B Reduction to Normalized Form

When $C = I = I_n$, the identity matrix in $\mathbb{R}^{n\times n}$ and $b = \mathbf{1}$, the vector of all ones in $\mathbb{R}^m$, we say that the packing-covering SDPs (PACKING-I)-(COVERING-I) and (PACKING-II)-(COVERING-II) are in *normalized* form. We recall below how a general packing covering pair of SDPs can be reduced to normalized form (see e.g., [22]). We denote by I the identity matrix of appropriate dimension.

We first note that under assumption (A), strong duality holds for both pairs (PACKING-I)-(COVERING-I) and (PACKING-II)-(COVERING-II). Indeed, it is enough for this (see, e.g.,[27]) to show the strict feasibility of the primal (the so-called the Slater's condition). For (PACKING-I) (resp., (COVERING-II)), a strict primal feasible solution is given by $X := \delta I$, where $\delta := \frac{1}{2\max_i I\bullet A_i}$ (resp., $\delta := \frac{2}{\min_i I\bullet A_i}$).

B.1 Reduction to Normalized Form for (PACKING-I)-(COVERING-I)

Under assumption (B-I), we may further assume that

(C-I) $C \succ 0$ and hence $C = I$.

If this is not the case, we slightly perturb the matrix C to make it positive definite without changing the objective value by much[3]. Let $C = L^T DL$ be the LDL-decomposition of C and $\bar{I}$ be the $0/1$-diagonal matrix with ones only in the entries corresponding to the *zero* diagonal entries of the diagonal matrix D. For $\delta > 0$, define $D(\delta) := D + \delta\bar{I} \succ 0$, $C(\delta) := L^T D(\delta)L = C + \delta L^T \bar{I} L$, and let $z_I^*(\delta)$ be the common optimum value of (PACKING-I)-(COVERING-I) when C is replaced by $C(\delta)$, and $X^*(\delta)$ and $y^*(\delta)$ be corresponding optimal primal and dual solutions, respectively.

By assumption (B-I), for any feasible solution X to (PACKING-I), we have $I \bullet X \le \tau$. Also, since $X = \frac{1}{\max_i A_i \bullet I} I$ is feasible for (PACKING-I), $z_I^* \ge \zeta := \frac{C \bullet I}{\max_i A_i \bullet I}$. Thus, for any desired accuracy $\epsilon > 0$, selecting $\delta = \frac{\epsilon\zeta}{\tau L \bullet L^T}$ gives

$$
\begin{aligned}
z_I^* \le z_I^*(\delta) &= C \bullet X^*(\delta) + \delta L^T \bar{I} L \bullet X^*(\delta) = C \bullet X^*(\delta) + \delta\bar{I} \bullet LX^*(\delta)L^T \le C \bullet X^*(\delta) + \delta I \bullet LX^*(\delta)L^T \\
&= C \bullet X^*(\delta) + \delta L^T L \bullet X^*(\delta) \le C \bullet X^*(\delta) + \delta\lambda_{\max}(L^T L) I \bullet X^*(\delta) \le C \bullet X^*(\delta) + \delta I \bullet L^T L \cdot I \bullet X^*(\delta) \\
&\le C \bullet X^*(\delta) + \delta\tau L \bullet L^T = C \bullet X^*(\delta) + \epsilon\zeta \le C \bullet X^*(\delta) + \epsilon z_I^* \le (1+\epsilon) z_I^*.
\end{aligned}
$$

It follows that $X^*(\delta)$ is feasible solution to (PACKING-I) with objective value $C \bullet X^*(\delta) \ge (1-\epsilon) z_I^*$. It follows also that $y^*(\delta)$ is feasible for (COVERING-I) (as $\sum_i y^*(\delta) A_i \succeq C(\delta) \succ C$) with objective value $z_I^*(\delta) \le (1+\epsilon) z_I^*$.

Finally, writing $U := L^{-1}$, and replacing X by $X' := D(\delta)^{\frac{1}{2}} LXL^T D(\delta)^{\frac{1}{2}}$, A_i by $A_i' := D(\delta)^{-\frac{1}{2}} U^T A_i U D(\delta)^{-\frac{1}{2}}$ and $C(\delta)$ by $C' = I$, we obtain an equivalent version of the perturbed (PACKING-I)-(COVERING-I) in normalized form. Given an optimal primal solution X' for the normalized problem we get a feasible solution $X = UD(\delta)^{-\frac{1}{2}} X' D(\delta)^{-\frac{1}{2}} U^T$ to the perturbed (PACKING-I) with the same objective value. Similarly an optimal dual solution for the normalized problem is an optimal solution for the perturbed (COVERING-I) as $\sum_i y_i A_i' \succeq I \Leftrightarrow \sum_i y_i A_i \succeq C(\delta)$. Note that this reduction can be implemented in $O(n^3 + n^\omega m)$ time. Moreover, given a maximization oracle Max$(\cdot)$ for (PACKING-I)-(COVERING-I), we obtain a maximization oracle for the normalized problem as follows: given $Y \in \mathbb{S}_+^n$, we return Max(Y') with $Y' := UD(\delta)^{-\frac{1}{2}} Y D(\delta)^{-\frac{1}{2}} U^T$. (For simplicity we ignore roundoff errors resulting from computing square roots, which can be dealt with using standard techniques)

B.2 Reduction to Normalized Form for (PACKING-II)-(COVERING-II)

For a matrix $B \in \mathbb{R}^{n\times n}$, define $\mathrm{supp}(B) := \{x \in \mathbb{R}^n : Bx \ne 0\}$.

We may assume that

(C-II) $\mathrm{supp}(C) \supseteq \bigcup_i \mathrm{supp}(A_i)$.

If this is not the case, that is, there is an $i \in [m]$ such that $\mathrm{supp}(A_i) \not\subseteq \mathrm{supp}(C)$ then $y_i = 0$ for any feasible solution y to (PACKING-II). Indeed, the existence of an $x \in \mathbb{R}^n$ such that $A_i x \ne 0$ and $Cx = 0$ implies that $y_i x^T A_i x \le y_i x^T A_i x + \sum_{j\ne i} y_j x^T A_j x \le x^T C x = 0$, giving that $y_i = 0$. Furthermore, the existence of such x allows us to remove the ith inequality from (COVERING-II); given an optimal solution X to the reduced covering system, we obtain a feasible solution with the same objective value (and hence optimal) to the original system by setting $X = X' + \frac{xx^T}{A_i \bullet xx^T}$. Note that (by Farkas' Lemma [34, Chapter 7]) we can check if (C-II) holds by solving m linear systems of equations $C\Gamma = A_i$, for $i = 1, \ldots, m$. This can be done in $O(n^3 + n^\omega m)$ time, where ω is the exponent of matrix multiplication, by computing the LDL-decomposition of C.

We may assume next that

(D-II) $\mathrm{supp}(C) = \mathbb{R}^n \setminus \{0\}$ and hence $C = I$.

Suppose that (D-II) does not hold. Let $C = L^T DL$ be the LDL-decomposition of C and write $U = [U' \mid U''] := L^{-1}$, where U' is the submatrix of U whose columns correspond to the columns of the submatrix D' containing the *positive* diagonal entries of the diagonal matrix D. Then $U^T CU = D$ implies that $(U'')^T CU'' = 0$, which in turn implies that $CU'' = 0$ (since $C \succeq 0$). The latter condition gives by (C-II) that $A_i U'' = 0$ for all i, and consequently,

$$
U^T A_i U = \left[\begin{array}{c} (U')^T \\ \hline (U'')^T \end{array}\right] A_i \left[\begin{array}{c|c} U' & U'' \end{array}\right] = \left[\begin{array}{c|c} (U')^T A_i U' & (U')^T A_i U'' \\ \hline (U'')^T A_i U' & (U'')^T A_i U'' \end{array}\right] = \left[\begin{array}{c|c} (U')^T A_i U' & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{39}
$$

[3] such a reduction has been used, e.g., in [11]

It follows that

$$\sum_i y_i A_i \preceq C \iff \sum_i y_i U^T A_i U \preceq U^T C U = D = \left[\begin{array}{c|c} D' & 0 \\ \hline 0 & 0 \end{array}\right]$$
$$\iff \sum_i y_i (U')^T A_i U' \preceq D'$$
$$\iff \sum_i y_i (D')^{-\frac{1}{2}} (U')^T A_i U' (D')^{-\frac{1}{2}} \preceq I. \tag{40}$$

Thus, replacing A_i by $A'_i := (D')^{-\frac{1}{2}}(U')^T A_i U'(D')^{-\frac{1}{2}}$ and C by I, we obtain an equivalent dual problem in normalized form whose optimal solution y is optimal for (PACKING-II). Also, a feasible primal solution X' to the corresponding normalized primal problem can be transformed to a feasible solution $X = U'(D')^{-\frac{1}{2}}X'(D')^{-\frac{1}{2}}(U')^T$ to (COVERING-II) with the same objective value, as $C \bullet X = C \bullet U'(D')^{-\frac{1}{2}}X'(D')^{-\frac{1}{2}}(U')^T = (D')^{-\frac{1}{2}}(U')^T C U'(D')^{-\frac{1}{2}} \bullet X' = I \bullet X'$, and $A_i \bullet X = A_i \bullet U'(D')^{-\frac{1}{2}}X'(D')^{-\frac{1}{2}}(U')^T = (D')^{-\frac{1}{2}}(U')^T A_i U'(D')^{-\frac{1}{2}} \bullet X' = A'_i \bullet X'$. Conversely, write $L^T = [(L')^T|(L'')^T]$, where L' is the submatrix of L whose rows correspond to the rows of the submatrix D', and note by definition that $U'L' + U''L'' = I$. Then, given any feasible solution X to (COVERING-II), a feasible solution to the normalized primal problem with the same objective value is given by $X' := (D')^{\frac{1}{2}}L'X(L')^T(D')^{\frac{1}{2}}$ since

$$A'_i \bullet X' = (L')^T(U')^T A_i U' L' \bullet X = (I - U''L'')^T A_i (I - U''L'') \bullet X = A_i \bullet X,$$

and similarly $I \bullet X' = I \bullet (D')^{\frac{1}{2}}L'X(L')^T(D')^{\frac{1}{2}} = (L')^T D' L' \bullet X = C \bullet X$. This step takes $O(n^3 + n^\omega m)$ time. Moreover, given a minimization oracle Min$(\cdot)$ for (PACKING-II)-(COVERING-II), we obtain a minimization oracle for the normalized problem as follows: given $Y \in \mathbb{S}^n_+$, we return Max(Y') with $Y' := U'(D')^{-\frac{1}{2}}Y(D')^{-\frac{1}{2}}(U')^T$.

We may next make the following further assumption on (NORM-PACKING-II)-(NORM-COVERING-II):

(B-II′) $\frac{\epsilon\beta}{2n} \le \lambda_{\min}(A_i) \le \lambda_{\max}(A_i) \le \frac{3n\beta}{\epsilon}$, for all $i \in [m]$, where $\beta := \min_i \lambda_{\max}(A_i)$.

(The same argument shows (B-II).) Indeed, let $J := \{i \in [m] : \lambda'_{\max}(A_i) \le \frac{n\beta'}{\epsilon}\}$, and for $i \in J$, let $\widetilde{A}_i := A_i + \frac{\epsilon\beta'}{n}I$, where $\lambda'_{\max}(A_i)$ is a $\frac{1}{2}$-approximation $\lambda'_{\max}(A_i)$ of $\lambda_{\max}(A_i)$ and $\beta' := \min_i \lambda'_{\max}(A_i)$. Consider the following pair of packing-covering SDP's:

$$\begin{array}{llll} & \tilde{z}_{II} = \min & I \bullet X & (\text{NORM-COVERING-}\widetilde{II}) \\ \text{s.t.} & \widetilde{A}_i \bullet X \ge 1, \forall i \in J & & \\ & X \in \mathbb{R}^{n\times n},\ X \succeq 0 & & \end{array}$$

$$\begin{array}{lll} & \tilde{z}_{II} = \max \quad \sum_{i\in J} y_i & (\text{NORM-PACKING-}\widetilde{II}) \\ \text{s.t.} & \sum_{i\in J} y_i \widetilde{A}_i \preceq I & \\ & y \in \mathbb{R}^m,\ y \ge 0. & \end{array}$$

Note that $\lambda_{\min}(\widetilde{A}_i) \ge \frac{\epsilon\beta'}{n} \ge \frac{\epsilon\beta}{2n}$, while for $i \in J$, $\lambda_{\max}(\widetilde{A}_i) = \lambda_{\max}(A_i) + \frac{\epsilon\beta'}{n} \le \frac{2n\beta}{\epsilon} + \frac{\epsilon\beta}{n} \le \frac{3n\beta}{\epsilon}$.

Let us also note that $\frac{1}{\beta} \le z^*_{II} \le \frac{n}{\beta}$ (see.e.g., [22]), as $\frac{1}{\beta}I$ is feasible for (NORM-COVERING-II), and if X^* is optimal for (NORM-COVERING-II), then for $i \in \operatorname{argmin}_{i'} \lambda_{\max}(A_{i'})$, we have $I \bullet X^* \ge \frac{A_i \bullet X^*}{\lambda_{\max}(A_i)} = \frac{A_i \bullet X^*}{\beta} \ge \frac{1}{\beta}$.

Let us note next that if X is feasible for (NORM-COVERING-II), then it is also feasible for (NORM-COVERING-$\widetilde{II}$). Hence, $\tilde{z}_{II} \le z^*_{II}$. On the other hand, suppose $\widetilde{X}$ is optimal for (NORM-COVERING-$\widetilde{II}$). Then, $X := \frac{1}{1-\epsilon}\left(\widetilde{X} + \frac{\epsilon}{\beta' n}I\right)$ is feasible for (NORM-COVERING-II), since for $i \in J$,

$$\begin{aligned} A_i \bullet X &= \frac{1}{1-\epsilon}\left(\widetilde{A}_i - \frac{\epsilon\beta'}{n}I\right) \bullet \left(\widetilde{X} + \frac{\epsilon}{\beta' n}I\right) \\ &= \frac{1}{1-\epsilon}\left(\widetilde{A}_i \bullet \widetilde{X} - \frac{\epsilon\beta'}{n}I \bullet \widetilde{X} + \frac{\epsilon}{\beta' n}I \bullet \widetilde{A}_i - \frac{\epsilon^2}{n^2}I \bullet I\right) \\ &\ge \frac{1}{1-\epsilon}\left(1 - \epsilon + \frac{\epsilon}{n} - \frac{\epsilon^2}{n}\right) > 1, \qquad (\because I \bullet \widetilde{X} = \tilde{z}_{II} \le z^*_{II} \le \tfrac{n}{\beta} \le \tfrac{n}{\beta'} \text{ and } I \bullet \widetilde{A}_i \ge I \bullet A_i \ge \beta \ge \beta') \end{aligned}$$

while for $i \notin J$, we have $A_i \bullet X \ge A_i \bullet \frac{\epsilon}{\beta' n}I \ge \frac{\epsilon}{\beta' n}\lambda_{\max}(A_i) \ge \frac{\epsilon}{\beta' n}\lambda'_{\max}(A_i) > 1$. Moreover, $I \bullet X = \frac{1}{1-\epsilon}\left(I \bullet \widetilde{X} + \frac{\epsilon}{\beta'}\right) \le \frac{1}{1-\epsilon}\left(I \bullet \widetilde{X} + \frac{2\epsilon}{\beta}\right) \le \frac{1}{1-\epsilon}(\tilde{z}_{II} + 2\epsilon z^*_{II}) \le \frac{1+2\epsilon}{1-\epsilon}z^*_{II}$. Obviously, given a feasible solution $\tilde{y}$ to (NORM-PACKING-$\widetilde{II}$), it can be extended to a feasible solution y to (NORM-PACKING-II), with the same objective value, by setting $y_i = \tilde{y}_i$ for $i \in J$ and $y_i = 0$ for $i \notin J$. To implement this step of the reduction, we can use Lanczos' algorithm to compute $\lambda'_{\max}(A_i)$, for $i \in [m]$. The running time is $O(mn^2)$.

Acknowledgment. We thank Waleed Najy for his help in the proofs in Section 2.2, and for many helpful discussions.

References

[1] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM Journal on Optimization*, 5(1):13–51, 1995.

[2] Zeyuan Allen-Zhu, Yin Tat Lee, and Lorenzo Orecchia. Using optimization to obtain a width-independent, parallel, simpler, and faster positive sdp solver. In *Proceedings of the Twenty-seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '16, pages 1824–1831, Philadelphia, PA, USA, 2016. Society for Industrial and Applied Mathematics.

[3] Zeyuan Allen-Zhu, Zhenyu Liao, and Lorenzo Orecchia. Spectral sparsification and regret minimization beyond matrix multiplicative updates. In *Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing*, STOC '15, pages 237–245, New York, NY, USA, 2015. ACM.

[4] S. Arora, E. Hazan, and S. Kale. Fast algorithms for approximate semidefinite programming using the multiplicative weights update method. In *Proc. 46th Symp. Foundations of Computer Science (FOCS)*, pages 339–348, 2005.

[5] S. Arora and S. Kale. A combinatorial, primal-dual approach to semidefinite programs. In *Proc. 39th Symp. Theory of Computing (STOC)*, pages 227–236, 2007.

[6] Sanjeev Arora, Elad Hazan, and Satyen Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing*, 8(1):121–164, 2012.

[7] Sanjeev Arora and Satyen Kale. A combinatorial, primal-dual approach to semidefinite programs. *J. ACM*, 63(2):12:1–12:35, May 2016.

[8] J. Batson, D. Spielman, and N. Srivastava. Twice-ramanujan sparsifiers. *SIAM Review*, 56(2):315–334, 2014.

[9] A. Ben-Tal, L. El Ghaoui, and A.S. Nemirovski. *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press, October 2009.

[10] Aharon Ben-Tal and Arkadi Nemirovski. Robust optimization - methodology and applications. *Math. Program.*, 92(3):453–480, 2002.

[11] S. Bose, D. F. Gayme, K. M. Chandy, and S. H. Low. Quadratically constrained quadratic programs on acyclic graphs with application to power flow. *IEEE Transactions on Control of Network Systems*, 2(3):278–287, Sept 2015.

[12] Robert D. Carr and Santosh Vempala. Randomized metarounding. *Random Struct. Algorithms*, 20(3):343–352, 2002.

[13] Khaled Elbassioni, Kazuhisa Makino, and Kazuki Ohara. A black-box reduction for robust combinatorial optimization problems.*Manuscript*, 2018.

[14] Dan Garber and Elad Hazan. Sublinear time algorithms for approximate semidefinite programming. *Math. Program.*, 158(1-2):329–361, July 2016.

[15] N. Garg and J. Könemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. *SIAM Journal on Computing*, 37(2):630–652, 2007.

[16] M. D. Grigoriadis and L. G. Khachiyan. A sublinear-time randomized approximation algorithm for matrix games. *Operations Research Letters*, 18(2):53–58, 1995.

[17] M. D. Grigoriadis, L. G. Khachiyan, L. Porkolab, and J. Villavicencio. Approximate max-min resource sharing for structured concave optimization. *SIAM Journal of Optimization*, 41:1081–1091, 2001.

[18] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, New York, 1988.

[19] G. Iyengar, D. J. Phillips, and C. Stein. Approximation algorithms for semidefinite packing problems with applications to maxcut and graph coloring. In Michael Jünger and Volker Kaibel, editors, *Integer Programming and Combinatorial Optimization (IPCO)*, pages 152–166, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.

[20] G. Iyengar, D. J. Phillips, and C. Stein. Approximating semidefinite packing programs. *SIAM J. on Optimization*, 21(1):231–268, January 2011.

[21] Garud Iyengar, David J. Phillips, and Cliff Stein. Feasible and accurate algorithms for covering semidefinite programs. In Haim Kaplan, editor, *Algorithm Theory - SWAT 2010*, pages 150–162, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

[22] Rahul Jain and Penghui Yao. A parallel approximation algorithm for positive semidefinite programming. In *IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS 2011, Palm Springs, CA, USA, October 22-25, 2011*, pages 463–471, 2011.

[23] Rahul Jain and Penghui Yao. A parallel approximation algorithm for mixed packing and covering semidefinite programs. *CoRR*, abs/1201.6090, 2012. URL: `http://arxiv.org/abs/1201.6090`.

[24] Philip Klein and Hsueh-I Lu. Efficient approximation algorithms for semidefinite programs arising from max cut and coloring. In *Proceedings of the Twenty-eighth Annual ACM Symposium on Theory of Computing*, STOC '96, pages 338–347, New York, NY, USA, 1996. ACM.

[25] Yin Tat Lee and He Sun. An sdp-based algorithm for linear-sized spectral sparsification. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, pages 678–687, New York, NY, USA, 2017. ACM.

[26] Zbigniew Leyk and Henryk Woniakowski. Estimating a largest eigenvector by lanczos and polynomial algorithms with a random start. *Numerical Linear Algebra with Applications*, 5(3):147–164, 1999.

[27] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics, 1994.

[28] Yu. Nesterov. Smooth minimization of non-smooth functions. *Mathematical Programming*, 103(1):127–152, May 2005.

[29] Yurii Nesterov. Quality of semidefinite relaxation for nonconvex quadratic optimization. CORE Discussion Papers 1997019, Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), March 1997.

[30] Yurii Nesterov. Smoothing technique and its applications in semidefinite optimization. *Mathematical Programming*, 110(2):245–259, Jul 2007.

[31] Richard Peng and Kanat Tangwongsan. Faster and simpler width-independent parallel algorithms for positive semidefinite programming. In *Proceedings of the Twenty-fourth Annual ACM Symposium on Parallelism in Algorithms and Architectures*, SPAA '12, pages 101–108, New York, NY, USA, 2012. ACM.

[32] Richard Peng, Kanat Tangwongsan, and Peng Zhang. Faster and simpler width-independent parallel algorithms for positive semidefinite programming. *CoRR*, abs/1201.5135, 2016. URL: `http://arxiv.org/abs/1201.5135`.

[33] S.A. Plotkin, D.B. Shmoys, and É. Tardos. Fast approximation algorithms for fractional packing and covering problems. In *Proc. 32nd Symp. Foundations of Computer Science (FOCS)*, pages 495–504, 1991.

[34] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley, New York, 1986.

[35] Marcel K. De Carli Silva, Nicholas J. A. Harvey, and Cristiane M. Sato. Sparse sums of positive semidefinite matrices. *ACM Trans. Algorithms*, 12(1):9:1–9:17, December 2015.

[36] Joel A. Tropp. An introduction to matrix concentration inequalities. *Found. Trends Mach. Learn.*, 8(1-2):1–230, May 2015.

[37] Lieven Vandenberghe and Stephen Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996. URL: `http://www.jstor.org/stable/2132974`.

[38] N.E. Young. Sequential and parallel algorithms for mixed packing and covering. In *Proc. 42nd Symp. Foundations of Computer Science (FOCS)*, pages 538–546, 2001.