Strong Lagrangian solutions of the (relativistic) Vlasov-Poisson system for non-smooth, spherically symmetric data

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Abstract

We prove a local existence and uniqueness result for the nonrelativistic and relativistic Vlasov-Poisson system for data which need not even be continuous. The corresponding solutions preserve all the standard conserved quantities and are constant along their pointwise defined characteristic flow so that these solutions are suitable for the stability analysis of not necessarily smooth steady states. They satisfy the well-known continuation criterion and are global in the nonrelativistic case. The only unwanted requirement on the data is that they be spherically symmetric.

Key words. Vlasov-Poisson system, existence and uniqueness, strong Lagrangian solutions

AMS subject classification. 35Q70, 35Q83, 85A05

1 Introduction

The Vlasov-Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{1.1}$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x| \to \infty} U(t, x) = 0, \tag{1.2}$$

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v) \, dv \tag{1.3}$$

describes a large ensemble of particles which interact only by the gravitational field which they create collectively. Here $f = f(t, x, v) \ge 0$ denotes the particle density on phase space, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$ denote time, position, and velocity, ρ is the spatial mass density induced by f, and U is the gravitational potential induced by ρ . This system is used in astrophysics for modeling galaxies or globular clusters, cf. [3]. If the Vlasov equation (1.1) is replaced by

$$\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \qquad (1.4)$$

the relativistic Vlasov-Poisson system is obtained. Here v should be thought of as momentum; $v/\sqrt{1+|v|^2}$ is then the induced velocity of a particle of unit mass.

An important feature of these systems is that they posses a plethora of steady states. One way to obtain steady states is to make an ansatz

$$f(x,v) = \phi(E(x,v)), \ E(x,v) := U(x) + \begin{cases} \frac{1}{2}|v|^2 & \text{non-relativistic case,} \\ \sqrt{1+|v|^2} & \text{relativistic case,} \end{cases}$$
(1.5)

with some ansatz function ϕ ; E is the local or particle energy in a stationary potential U = U(x). This ansatz reduces the (relativistic) Vlasov-Poisson system to a semilinear Poisson equation for U, namely (1.2) where the right hand side depends on U through the ansatz (1.5). We refer to [20] and the references there for sufficient conditions on ϕ such that this leads to physically viable steady states with finite mass and compact support. The classical example are the polytropic models where

$$\phi(E) = (E_0 - E)_+^k; \tag{1.6}$$

the subscript + denotes the positive part. Here -1 < k < 7/2 and $E_0 < 0$ is a cut-off energy. One can also take sums of such ansatz functions with different cut-off energies and/or different exponents, and if one requires Uto be spherically symmetric, the ansatz may also depend on the particle angular momentum $L := |x \times v|$. The important point for the present paper is that for these steady states f need not be smooth and not even continuous. If one wants to investigate the stability of such a steady state a natural class of perturbations are the dynamically accessible ones, where the stationary particle distribution is rearranged via a measure-preserving homeomorphism of phase space caused for example by the action of some exterior, perturbing force. The resulting, dynamically accessible data are then in general as regular or irregular as the original steady state. For example, if we pick k = 0 in (1.6), then the distribution function of both the steady state and its perturbation attains only the values 0 and 1 and is discontinuous.

It is desirable to have an existence and uniqueness result for the time dependent problem for such data, where the resulting solutions should preserve all the conserved quantities like the total energy and the so-called Casimir functionals, since these are used in the stability analysis. In addition, the characteristic flow corresponding to the Vlasov equation should exist and f should be constant along this flow; this property is more important in a stability analysis (and elsewhere) than the Vlasov equation itself. We refer to [8, 9, 10, 11, 15, 16, 21] and the references there for stability results for the (relativistic) Vlasov-Poisson system. In the present paper we provide a local existence and uniqueness result as specified above. The characteristic flow of the corresponding Vlasov equation will be defined pointwise on phase space, and f will be constant along it; we call such solutions strong Lagrangian so*lutions.* For the data we require that $\mathring{f} = f_{|t=0}$ is a non-negative, bounded, and measurable function with compact support, and in addition, that \mathring{f} is spherically symmetric; a function $q: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is spherically symmetric. if q(Ax, Av) = q(x, v) for all $x, v \in \mathbb{R}^3$ and $A \in SO(3)$. In passing we note that our result answers a question which was left open in the stability analysis [11]. The symmetry assumption is of course undesirable, and it is an open problem, how far one can relax this assumption without loosing any of the properties of the solution. It is also an open problem whether one can preserve these properties for not necessarily bounded data, such as would arise by perturbing polytropic steady states with -1 < k < 0.

There exists an extensive literature concerning the initial value problem for the Vlasov-Poisson system, and a bit less for its relativistic version, and to put the present paper into context we recall some of it. For the nonrelativistic version, Batt [2] proved local existence and uniqueness of smooth solutions together with a continuation criterion, and he used the latter to obtain global existence for smooth, spherically symmetric data. The latter result is known to be false for the relativistic version, cf. [7]. For the nonrelativistic version smooth solutions exist globally also for non-symmetric data, as was shown by Pfaffelmoser [19] and simultaneously but independently by Lions and Perthame [17], cf. also [21]. Global weak solutions for the non-relativistic system, which are neither known to be unique nor to preserve the usual conserved quantities, were obtained for example in [1, 13]. More recently, Lagrangian flows for non-smooth vector fields have been investigated and used to construct Lagrangian solutions of the Vlasov-Poisson system for L^1 data, cf. [4] and the references there. However, [4] considers the non-relativistic, repulsive case of the Vlasov-Poisson system where the sign of the right hand side in the Poisson equation (1.2) is reversed. We do not know whether these results can be extended to the attractive case stated above or to the relativistic one. The relation of the flow of ordinary differential equations with coefficients in Sobolev spaces to linear transport equations like the Vlasov equation was studied in the seminal paper [5]. It should be emphasized that the results and techniques in [4, 5] are much more far reaching and sophisticated than the present investigation and in particular do not rely on any symmetry assumption. Indeed, the main point of the present investigation is to show that for the price of assuming spherical symmetry, Lagrangian solutions with all the desired properties, in particular, with a pointwise defined characteristic flow, can be obtained by quite elementary methods for both the non-relativistic and the relativistic Vlasov-Poisson system; in passing we note that all our results hold equally well for the repulsive case mentioned above.

In the next section we state our results, and the proofs are given in Section 3. The present paper is based on the first authors master thesis [14].

2 Main results

We start by making precise our solution concept; throughout the paper integrals without an explicitly denoted domain of integration extend over \mathbb{R}^3 or \mathbb{R}^6 .

Definition 2.1. A measurable function $f: [0, T[\times \mathbb{R}^6 \to \mathbb{R} \text{ with } T > 0 \text{ is a strong Lagrangian solution of the non-relativistic or relativistic Vlasov-Poisson system iff:$

(i) The induced mass density

$$\rho_f(t,x) = \rho(t,x) := \int f(t,x,v) \, dv$$

and the induced gravitational field

$$F_f(t,x) = F(t,x) := \int \frac{x-y}{|x-y|^3} \rho(t,y) \, dy$$

exist for all $(t,x) \in [0,T[\times\mathbb{R}^3, and F \text{ is continuous and Lipschitz continuous in } x, locally uniformly in t, i.e., for every <math>0 < T' < T$ there exists L > 0 such that for all $t \in [0,T']$ and $x, x' \in \mathbb{R}^3$,

$$|F(t, x) - F(t, x')| \le L |x - x'|.$$

(ii) f is constant along its characteristics, i.e., for all $(t,z) \in [0,T[\times \mathbb{R}^6, the mapping s \mapsto f(s, Z(s,t,z))$ is constant, where $s \mapsto Z(s,t,z) = (X,V)(s,t,x,v)$ is the solution of the characteristic system

$$\dot{x} = v \text{ or } \dot{x} = \frac{v}{\sqrt{1+|v|^2}}, \qquad \dot{v} = -F(s,x)$$
 (2.1)

with Z(t, t, z) = z = (x, v).

The gravitational field F defined in part (i) is the gradient of the potential determined by (1.2), and the conditions on F guarantee the existence of the characteristic flow used in part (ii) of the definition, see also Lemma 3.1 below. Formally, the definition can be relaxed by replacing the assumptions on the field F by the properties of the induced flow, obtained in Lemma 3.1. We also note that no symmetry assumption enters in this definition.

For a measurable, bounded, and compactly supported state $g: \mathbb{R}^6 \to [0, \infty]$ we define its kinetic and potential energies as

$$E_{\rm kin}(g) := \frac{1}{2} \iint |v|^2 g(x, v) \, dv \, dx$$

or

$$E_{\mathrm{kin}}(g) := \iint \sqrt{1+|v|^2}g(x,v)\,dv\,dx,$$
$$E_{\mathrm{pot}}(g) := -\frac{1}{2} \iiint \frac{g(x,v)\,g(y,w)}{|x-y|}dv\,dw\,dx\,dy,$$

and a Casimir functional is defined as

$$\mathcal{C}(g) := \iint \Phi(g(x,v)) \, dv \, dx,$$

where $\Phi \colon \mathbb{R} \to \mathbb{R}$ is continuous with $\Phi(0) = 0$

Theorem 2.2. Let $\mathring{f}: \mathbb{R}^6 \to [0, \infty[$ be measurable, bounded, compactly supported and spherically symmetric. Then there exists a unique, spherically symmetric, strong Lagrangian solution $f: [0, T[\times \mathbb{R}^6 \to [0, \infty[$ of the (relativistic) Vlasov-Poisson system with $f(0) = \mathring{f}$. If T > 0 is chosen maximal, then in the non-relativistic case, $T = \infty$. In the relativistic case, $T = \infty$ if

 $\sup \{ |v| \mid (x, v) \in \operatorname{supp} f(t), \ 0 \le t < T \} < \infty.$

The energy and all Casimir functionals are conserved, i.e., for all $t \in [0, T]$,

$$E_{\rm kin}(f(t)) + E_{\rm pot}(f(t)) = E_{\rm kin}(\mathring{f}) + E_{\rm pot}(\mathring{f}), \quad \mathcal{C}(f(t)) = \mathcal{C}(\mathring{f}).$$

Some additional properties of the solution f which come out of the proof will be listed below.

3 Proofs

3.1 The characteristic flow and the Vlasov equation

We recall the relevant properties of the flow induced by (2.1).

Lemma 3.1. Let $F: [0, T[\times \mathbb{R}^3 \to \mathbb{R}^3]$ be continuous and Lipschitz continuous with respect to x, locally uniformly in t. Then the following holds:

- (a) For every $t \in [0,T[$ and $z = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ there exists a unique solution $[0,T[\ni s \mapsto Z(s,t,z) \text{ of } (2.1) \text{ with } Z(t,t,z) = z$. The flow Z is continuous on $[0,T[\times[0,T[\times\mathbb{R}^6 \text{ and Lipschitz continuous with respect to z, locally uniformly in s and t.$
- (b) For every $s,t \in [0,T[$, the mapping $Z(s,t,\cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is measure preserving, i.e.,

$$|\det \partial_z Z(s,t,z)| = 1 \text{ for almost every } z \in \mathbb{R}^6,$$

and one-to-one and onto with inverse $Z^{-1}(s,t,\cdot) = Z(t,s,\cdot)$.

(c) For any measurable function $\Phi \colon \mathbb{R}^6 \to \mathbb{R}$, any measurable set $D \subset \mathbb{R}^6$, and any $s, t \in [0, T]$ the change-of-variables formula holds:

$$\int_{Z(s,t,D)} \Phi(z) \, dz = \int_D \Phi(Z(s,t,z)) \, dz$$

(d) If F in addition is spherically symmetric, i.e., F(t, Ax) = A F(t, x)for all $t \in [0, T[, x \in \mathbb{R}^3, and A \in SO(3), then so is Z = (X, V),$ i.e., (X, V)(s, t, Ax, Av) = (A X, A V)(s, t, x, v) for all $s, t \in [0, T[, x, v \in \mathbb{R}^3, and A \in SO(3).$

Proof. Most of parts (a) and (b) is standard ODE theory. The fact that for s and t fixed, $Z(s,t,\cdot)$ is Lipschitz implies that the derivative $\partial_z Z(s,t,z)$ exists for almost all z; the exceptional set of measure zero may depend on s and t, but this causes no problems. To prove the assertion on the functional determinant, let $J \in C_c^{\infty}(\mathbb{R}^3)$ be a smooth, compactly supported function with $\int J = 1$, i.e., a Friedrichs mollifier. For $\epsilon > 0$, we define $J_{\epsilon} := \epsilon^{-3}J(\cdot/\epsilon)$ and the smoothed field $F_{\epsilon}(t) := J_{\epsilon} * F(t)$ where $F(t) = F(t, \cdot)$ for $t \in [0, T[$. The corresponding flow Z_{ϵ} is differentiable with respect to z with

$$\det \partial_z Z_{\epsilon}(s, t, z) = 1, \ s, t \in [0, T[, \ z \in \mathbb{R}^6,$$

since the vector field generating this flow is divergence free on $\mathbb{R}^3 \times \mathbb{R}^3$, cf. for example [21, Lemma 1.2]. Moreover, $Z_{\epsilon}(s,t,z) \to Z(s,t,z)$ for $\epsilon \to 0$, uniformly in z and locally uniformly in s and t. Now let $\phi \in C_c^{\infty}(\mathbb{R}^6)$ denote any test function. Then the change-of-variables formula for Lipschitz continuous transformations—cf. [6, *263F Corollary]—and the above convergence imply that

$$\int \phi(z) |\det \partial_z Z(s,t,z)| \, dz = \int \phi(Z(t,s,z)) \, dz = \lim_{\epsilon \to 0} \int \phi(Z_\epsilon(t,s,z)) \, dz$$
$$= \lim_{\epsilon \to 0} \int \phi(z) |\det \partial_z Z_\epsilon(s,t,z)| \, dz = \int \phi(z) \, dz.$$

Hence, $|\det \partial_z Z(s, t, z)| = 1$ for almost every z. Combining this again with [6, *263F Corollary] yields part (c). Part (d) follows by uniqueness.

Given a field F as specified in the previous lemma and initial data we can solve the corresponding Vlasov equation.

Lemma 3.2. Let F be as in Lemma 3.1, Z the flow obtained there, and let $\mathring{f}: \mathbb{R}^6 \to \mathbb{R}$ be measurable, bounded, and compactly supported, and define $f(t,z) := \mathring{f}(Z(0,t,z))$ for all $t \in [0,T[$ and $z \in \mathbb{R}^6$. Then the following holds:

- (a) f is constant along solutions of (2.1), and $f(0) = \mathring{f}$.
- (b) For every $t \in [0, T[$ and $p \in [1, \infty]$, $\operatorname{supp} f(t) = Z(t, 0, \operatorname{supp} \mathring{f})$, and $\|f(t)\|_p = \|\mathring{f}\|_p$; here $\|\cdot\|_p$ is the L^p norm on \mathbb{R}^6 , and $f(t) = f(t, \cdot)$.

- (c) $f \in C([0,T[;L^1(\mathbb{R}^6))).$
- (d) If F and \mathring{f} are spherically symmetric, then so is f(t) for every $t \in [0,T]$.

Proof. With the possible exception of part (c) all of this is quite obvious by the definition of f and Lemma 3.1. As to part (c), we first notice that f(t) is measurable, since \mathring{f} is measurable and $Z(t, 0, \cdot)$ is one-to-one and Lipschitz. Now let $\epsilon > 0$ be arbitrary and choose $g \in C_c^{\infty}(\mathbb{R}^6)$ such that $\|\mathring{f} - g\|_1 < \epsilon$. Then for any $t, t' \in [0, T]$,

$$\begin{split} \|f(t) - f(t')\|_1 &\leq \int |\mathring{f}(Z(0,t,z)) - g(Z(0,t,z))| \, dz \\ &+ \int |\mathring{f}(Z(0,t',z)) - g(Z(0,t',z))| \, dz \\ &+ \int |g(Z(0,t,z)) - g(Z(0,t',z))| \, dz \\ &\leq 2\epsilon + \int_{B_R} |g(Z(0,t,z)) - g(Z(0,t',z))| \, dz, \end{split}$$

where $B_R \subset \mathbb{R}^6$ is a sufficiently large ball about the origin, and the assertion follows by continuity of g and Z.

3.2 Local existence

In this section we prove the local existence part of Theorem 2.2. To this end we consider the following iteration scheme which is essentially the same as in [21, Thm. 1.1].

We define the 0th iterate of the field as $F_0(t, x) = 0$ for all $t \in [0, \infty[$ and $x \in \mathbb{R}^3$. Assume that for some $n \in \mathbb{N}_0$ a field $F_n: [0, \infty[\times \mathbb{R}^3 \to \mathbb{R}^3]$ is already defined which has the following properties.

Field properties: F_n is continuous in t and x, Lipschitz continuous in x, locally uniformly in t, bounded on $[0, T'] \times \mathbb{R}^3$ for any T' > 0, and spherically symmetric.

The field F_0 clearly has these properties. Lemma 3.1 yields a corresponding flow Z_n , and Lemma 3.2 yields the *n*-th iterate f_n . We complete one iteration step by defining $\rho_n := \rho_{f_n}$ and $F_{n+1} := F_{f_n}$, cf. Definition 2.1 (i). Local existence now follows in three steps.

Step 1. In this step we prove that the iteration is well defined. Let

$$P_n(t) := \sup\{|V_n(s, 0, z)| \mid z \in \operatorname{supp} f, \ 0 \le s \le t\},\$$

and pick $\mathring{R}, \mathring{P} > 0$ such that $\operatorname{supp} \mathring{f} \subset B_{\mathring{R}} \times B_{\mathring{P}}$; the latter balls are now in \mathbb{R}^3 . Then

$$f_n(t, x, v) = 0 \text{ for } |v| \ge P_n(t) \text{ or } |x| \ge \mathring{R} + \int_0^t P_n(s) \, ds,$$
$$\rho_n(t, x) = 0 \text{ for } |x| \ge \mathring{R} + \int_0^t P_n(s) \, ds,$$

and

$$\|\rho_n(t)\|_{\infty} \le \frac{4\pi}{3} \|\mathring{f}\|_{\infty} P_n(t)^3.$$

Now we recall that for any $\rho \in L^1 \cap L^\infty(\mathbb{R}^3)$ the field generated by ρ satisfies the estimate

$$\|F_{\rho}\|_{\infty} \le 3(2\pi)^{2/3} \|\rho\|_{1}^{1/3} \|\rho\|_{\infty}^{2/3}, \tag{3.1}$$

cf. for example [21, Lemma P1]. Since $f_n \in C([0, \infty[; L^1(\mathbb{R}^6)))$ and hence $\rho_n \in C([0, \infty[; L^1(\mathbb{R}^3)), (3.1))$ implies that F_{n+1} is continuous in t. Moreover, for all $t \geq 0$,

$$||F_{n+1}(t)||_{\infty} \le 3(2\pi)^{2/3} ||\rho_n(t)||_1^{1/3} ||\rho_n(t)||_{\infty}^{2/3} \le C_{\mathring{f}} P_n(t)^2, \qquad (3.2)$$

where

$$C_{\mathring{f}} := 4 \cdot 3^{1/3} \pi^{4/3} \|\mathring{f}\|_1^{1/3} \|\mathring{f}\|_{\infty}^{2/3}, \qquad (3.3)$$

in particular, the field F_{n+1} is bounded, locally uniformly in t. The spherical symmetry is inherited by f_n , and hence by ρ_n and F_{n+1} , and to see that F_{n+1} has the field properties formulated above, it remains to show its Lipschitz property; this is the first instance where we need to exploit the symmetry assumption. Because of the latter,

$$F_{n+1}(t,x) = G_{n+1}(t,r)\frac{x}{r}, \text{ where } G_{n+1}(t,r) := \frac{4\pi}{r^2} \int_0^r \rho_n(t,s) \, s^2 ds; \quad (3.4)$$

here r = |x|, and we identify $\rho_n(t, x)$ and $\rho_n(t, r)$. For any $t \ge 0$ and 0 < u < r,

$$|G_{n+1}(t,r) - G_{n+1}(t,u)| \leq \frac{4\pi}{r^2} \int_u^r \rho_n(t,s) \, s^2 ds + 4\pi \Big| \frac{1}{r^2} - \frac{1}{u^2} \Big| \int_0^u \rho_n(t,s) \, s^2 ds \leq \frac{20\pi}{3} \|\rho_n(t)\|_{\infty} |r-u|.$$
(3.5)

The required Lipschitz property of F_{n+1} follows from (3.5).

Step 2: We establish bounds that are uniform in n. The definition of P_n and (3.2) imply that for $n \in \mathbb{N}_0$ and $t \ge 0$,

$$P_{n+1}(t) \le \mathring{P} + \int_0^t \|F_{n+1}(s)\|_{\infty} ds \le \mathring{P} + C_{\mathring{f}} \int_0^t P_n(s)^2 ds.$$

If we drop the subscripts of P and replace \leq by =, we obtain an integral equation the unique, maximal solution of which is

$$Q: [0, (\mathring{P}C_{\mathring{f}})^{-1}[\to [0, \infty[, t \mapsto \frac{\mathring{P}}{1 - \mathring{P}C_{\mathring{f}}t},$$
(3.6)

and a straight forward induction argument shows that $P_n(t) \leq Q(t)$ for all $n \in \mathbb{N}_0$ and $t \in [0, \delta_0[$, where $\delta_0 := (\mathring{P}C_{\mathring{f}})^{-1}$.

Step 3: We show that on any compact subinterval $[0, \delta] \subset [0, \delta_0[$ the iteration sequence converges in a suitable sense, and its limit is a strong Lagrangian solution.

Using the characteristic system and observing that in view of the uniform bounds and (3.5) the fields F_n are Lipschitz in x uniformly on $[0, \delta]$ and uniformly in n we find that

$$|Z_{n+1}(t,0,z) - Z_n(t,0,z)| \le C \int_0^t |Z_{n+1}(s,0,z) - Z_n(s,0,z)| \, ds$$
$$+ \int_0^t ||G_{n+1}(s) - G_n(s)||_{\infty} ds;$$

here and in what follows C denotes a positive constant which may only depend on \mathring{f} and δ_0 and which may change its value from line to line. For the relativistic case we note that the map $v \mapsto v/\sqrt{1+|v|^2}$ is Lipschitz continuous. By Gronwall,

$$|Z_{n+1}(t,0,z) - Z_n(t,0,z)| \le C \int_0^t ||G_{n+1}(s) - G_n(s)||_{\infty} ds.$$
(3.7)

The crucial point is to estimate the latter difference, and this is also the point where the symmetry assumption enters most strongly, cf. (3.4). For $t \in [0, \delta], r \ge 0$ and $n \in \mathbb{N}$ we first note that by the uniform estimate on ρ_n ,

$$DG_n(t,r) := |G_{n+1}(t,r) - G_n(t,r)|$$

= $\frac{4\pi}{r^2} \left| \int_0^r \left(\rho_n(t,s) - \rho_{n-1}(t,s) \right) s^2 ds \right| \le C r.$ (3.8)

On the other hand, denoting $z = (y, v) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$\{Z_n(0,t,z) \mid |y| \le r\} = \{\tilde{z} \in \mathbb{R}^6 \mid \exists z \in B_r \times \mathbb{R}^3 \text{ such that } Z_n(t,0,\tilde{z}) = z\}$$
$$= \{\tilde{z} \in \mathbb{R}^6 \mid |X_n(t,0,\tilde{z})| \le r\}.$$

Hence we can rewrite the modulus of the field as

$$G_{n+1}(t,r) = \frac{1}{r^2} \int_{|y| \le r} \rho_n(t,y) \, dy = \frac{1}{r^2} \int_{\{z \in \mathbb{R}^6 | |y| \le r\}} \mathring{f}(Z_n(0,t,z)) \, dz$$

$$= \frac{1}{r^2} \int_{Z_n(0,t,B_r \times \mathbb{R}^3)} \mathring{f}(z) \, dz = \frac{1}{r^2} \int_{\{z \in \mathbb{R}^6 | |X_n(t,0,z)| \le r\}} \mathring{f}(z) \, dz,$$

(3.9)

where we have used the change-of-variables formula from Lemma 3.1 (c). This implies that

$$DG_{n}(t,r) \leq \frac{1}{r^{2}} \Big| \int_{\{z \in \mathbb{R}^{6} ||X_{n}(t,0,z)| \leq r\}} \mathring{f}(z) \, dz - \int_{\{z \in \mathbb{R}^{6} ||X_{n-1}(t,0,z)| \leq r\}} \mathring{f}(z) \, dz \Big|$$

$$\leq \frac{1}{r^{2}} \|\mathring{f}\|_{\infty} \lambda(D_{n}), \qquad (3.10)$$

where λ denotes the Lebesgue measure, and

$$D_n := \left\{ z \in \operatorname{supp} \mathring{f} \mid |X_n(t,0,z)| \le r < |X_{n-1}(t,0,z)| \\ \lor |X_{n-1}(t,0,z)| \le r < |X_n(t,0,z)| \right\}.$$

Defining

$$d_n := \sup_{z \in \text{supp } \mathring{f}} |X_n(t, 0, z) - X_{n-1}(t, 0, z)|,$$

we observe that

$$\begin{split} \lambda(D_n) &\leq \lambda \Big(\Big\{ z \in \operatorname{supp} \mathring{f} \mid |X_n(t,0,z)| \leq r < d_n + |X_n(t,0,z)| \\ & \vee |X_{n-1}(t,0,z)| \leq r < d_n + |X_{n-1}(t,0,z)| \Big\} \Big) \\ &\leq \lambda \Big(\Big\{ z \in \operatorname{supp} \mathring{f} \mid |X_n(t,0,z)| \leq r < d_n + |X_n(t,0,z)| \Big\} \Big) \\ & + \lambda \Big(\Big\{ z \in \operatorname{supp} \mathring{f} \mid |X_{n-1}(t,0,z)| \leq r < d_n + |X_{n-1}(t,0,z)| \Big\} \Big) \\ &=: \lambda(D_n^1) + \lambda(D_n^2). \end{split}$$

We use the fact that the characteristic flow is measure preserving to eliminate the X_n -terms:

$$\begin{split} \lambda(D_n^1) &= \lambda(Z_n(t, 0, D_n^1)) \\ &= \lambda\Big(\Big\{Z_n(t, 0, z) \mid z \in \operatorname{supp} \mathring{f} \land |X_n(t, 0, z)| \le r < d_n + |X_n(t, 0, z)|\Big\}\Big) \\ &= \lambda\Big(\Big\{(y, v) \in Z_n(t, 0, \operatorname{supp} \mathring{f}) \mid |y| \le r < d_n + |y|\Big\}\Big) \\ &\le \lambda\Big(\{y \in B_R \mid r - d_n < |y| \le r\} \times B_R\Big) \\ &\le C\left(r^3 - (r - d_n)^3_+\right), \end{split}$$

where we recall that by our uniform estimates, $Z_n(t, 0, \operatorname{supp} \mathring{f}) \subset B_R \times B_R$ with some radius R > 0 which is uniform in $n \in \mathbb{N}_0$ and $t \in [0, \delta]$. The same result holds for D_n^2 ; we just have to replace Z_n by Z_{n-1} . Hence

$$\lambda(D_n) \le C \left(r^3 - (r - d_n)_+^3 \right). \tag{3.11}$$

If $r \leq d_n$, we use (3.8) to find that

$$DG_n(t,r) \le Cr \le Cd_n.$$

If $r > d_n$, (3.10) and (3.11) imply that

$$DG_n(t,r) \le \frac{C}{r^2}(d_n^3 + 3r^2d_n) \le C d_n.$$

Combining both results, we see that

$$\begin{aligned} \|G_{n+1}(t) - G_n(t)\|_{\infty} &\leq C \sup_{z \in \operatorname{supp} \mathring{f}} |X_n(t, 0, z) - X_{n-1}(t, 0, z)| \\ &\leq C \sup_{z \in \operatorname{supp} \mathring{f}} |Z_n(t, 0, z) - Z_{n-1}(t, 0, z)|, \end{aligned}$$

and together with (3.7) we finally arrive at the estimate

$$||F_{n+1}(t) - F_n(t)||_{\infty} \le C \int_0^t ||F_n(s) - F_{n-1}(s)||_{\infty} ds, \qquad (3.12)$$

which holds for all $n \in \mathbb{N}$ and $t \in [0, \delta]$. This implies that (F_n) is a Cauchy sequence in the space $C([0, \delta]; L^{\infty}(\mathbb{R}^3))$. Thus there exists a limiting field $F: [0, \delta] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $F_n \to F$ uniformly on $[0, \delta] \times \mathbb{R}^3$. The field is bounded and continuous. By the previous two steps, F_n is Lipschitz continuous in x, uniformly in $t \in [0, \delta]$ and in $n \in \mathbb{N}$. Hence F is Lipschitz continuous in x, uniformly in $t \in [0, \delta]$. Since $\delta < \delta_0$ is arbitrary, the field F exists and has the desired properties on $[0, \delta_0[$. Lemma 3.1 yields the corresponding flow Z, and $Z_n \to Z$ uniformly on $[0, \delta] \times [0, \delta] \times \mathbb{R}^6$ for all $\delta < \delta_0$. If we define f(t, z) = f(Z(0, t, z)) according to Lemma 3.2 it remains to show that F is indeed the field induced by f.

By Lebesgue's dominated convergence theorem and (3.9),

$$\begin{aligned} G(t,r) &= \lim_{n \to \infty} G_n(t,r) = \frac{1}{r^2} \lim_{n \to \infty} \int_{\{z \in \mathbb{R}^6 \mid \mid X_{n-1}(t,0,z) \mid \le r\}} \mathring{f}(z) \, dz \\ &= \frac{1}{r^2} \int_{\{z \in \mathbb{R}^6 \mid \mid X(t,0,z) \mid \le r\}} \mathring{f}(z) \, dz = \frac{1}{r^2} \int_{\{z \in \mathbb{R}^6 \mid \mid y \mid \le r\}} \mathring{f}(Z(0,t,z)) \, dz, \end{aligned}$$

which implies that

$$F(t,x) = G(t,r)\frac{x}{r} = \iint \frac{x-y}{|x-y|^3} f(t,y,v) \, dv \, dy.$$

Hence f has all the properties of a strong Lagrangian solution on $[0, \delta]$, and of course $f(0) = \mathring{f}$.

3.3 Uniqueness

Assume that we have two spherically symmetric, strong Lagrangian solutions to the same initial data with fields F and \tilde{F} . Then on any time interval $[0, \delta]$ where both are defined we can treat the difference $F-\tilde{F}$ exactly as we treated $F_{n+1} - F_n$ above, in particular $F - \tilde{F}$ must satisfy the analogue of (3.12) which implies that the two fields and hence the two solutions are equal on $[0, \delta]$.

Remark. The argument above yields uniqueness only within the class of spherically symmetric, strong Lagrangian solutions, which is sufficient for what follows below. However, uniqueness holds also within the class of (not necessarily symmetric) strong Lagrangian solutions for data \mathring{f} which are measurable, bounded, and compactly supported. To see this, we observe that for such solutions the induced spatial density is again bounded and compactly supported with respect to x, locally uniformly in t. Moreover, such solutions are easily seen to be weak solutions, and the uniqueness results in [18, 22] apply, at least in the non-relativistic case; it seems reasonable to expect this to remain true also in the relativistic case.

3.4 Continuation and global existence

We can extend the unique, strong Lagrangian solution to its maximal interval of existence [0, T]. Assume that

$$P^* := \sup \left\{ |v| \mid (x, v) \in \operatorname{supp} f(t), \ 0 \le t < T \right\} < \infty.$$
(3.13)

For any $t_0 \in [0, T]$,

$$||f(t_0)||_{\infty} = ||\mathring{f}||_{\infty}, \quad ||f(t_0)||_1 = ||\mathring{f}||_1,$$

and hence $C_{f(t_0)} = 4 \cdot 3^{1/3} \pi^{4/3} ||f(t_0)||_1^{1/3} ||f(t_0)||_{\infty}^{2/3} = C_{\hat{f}}$, cf. (3.3). We define $\delta_0^* := (P^* C_{f(t_0)})^{-1} = (P^* C_{\hat{f}})^{-1}$. Arguing exactly as before we obtain a strong Lagrangian solution on the interval $[t_0, t_0 + \delta_0^*]$ to the initial data $f(t_0)$ prescribed at t_0 . By uniqueness, this solution must coincide with f as long as both exist. If T were finite, this would extend the maximal solution beyond T, provided we choose t_0 close enough to T; note that δ_0^* is independent of t_0 .

The continuation criterion which is now established applies to both the non-relativistic and the relativistic case, and in the former we can verify that (3.13) indeed holds and hence $T = \infty$. To this end we observe that the corresponding argument of Horst [12] applies to strong Lagrangian solutions, see also [21, Thm. 1.4].

3.5 Conservation laws

The conservation of Casimir functionals is a direct consequence of the change-of-variables formula in Lemma 3.1 (c) and the definition of a strong Lagrangian solution.

Next we prove conservation of energy for the non-relativistic case, the relativistic case being completely analogous. We use the fact that the flow is measure preserving, cf. Lemma 3.1 (b), together with the fact that f is constant along the flow and the fundamental theorem of calculus, and we recall the notation $z = (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and analogously, $\tilde{z} = (\tilde{x}, \tilde{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Then

$$\begin{split} &2E_{\rm kin}(f(t)) + 2E_{\rm pot}(f(t)) \\ &= \int |v|^2 f(t,z) \, dz - \iint \frac{f(t,z) \, f(t,\tilde{z})}{|x-\tilde{x}|} dz \, d\tilde{z} \\ &= \int |V(t,0,z)|^2 f(t,Z(t,0,z)) \, dz - \iint \frac{f(t,Z(t,0,z)) \, f(t,Z(t,0,\tilde{z}))}{|X(t,0,z) - X(t,0,\tilde{z})|} dz \, d\tilde{z} \\ &= \iint_0^t \frac{d}{ds} \Big(|V(s,0,z)|^2 f(s,Z(s,0,z)) \Big) \, ds \, dz + \int |v|^2 \mathring{f}(z) \, dz \\ &- \iint_0^t \frac{d}{ds} \bigg(\frac{f(s,Z(s,0,z)) \, f(s,Z(s,0,\tilde{z}))}{|X(s,0,z) - X(s,0,\tilde{z})|} \bigg) \, ds \, dz \, d\tilde{z} \\ &- \iint \frac{\mathring{f}(z) \, \mathring{f}(\tilde{z})}{|x-\tilde{x}|} dz \, d\tilde{z} \\ &= 2E_{\rm kin}(\mathring{f}) + 2E_{\rm pot}(\mathring{f}) \\ &- 2 \iint_0^t \frac{X(s,0,z) \cdot F(s,X(s,0,z)) \, f(s,Z(s,0,z)) \, ds \, dz \\ &+ \iiint_0^t \frac{X(s,0,z) - X(s,0,\tilde{z})}{|X(s,0,z) - X(s,0,\tilde{z})|^3} \cdot \big(V(s,0,z) - V(s,0,\tilde{z}) \big) \\ &\quad f(s,Z(s,0,z)) \, f(s,Z(s,0,\tilde{z})) \, ds \, dz \, d\tilde{z}; \end{split}$$

in the last step we used that f is a strong Lagrangian solution so that

$$\frac{d}{ds}f(s,Z(s,0,z)) = 0, \quad s \in [0,T[.$$

Using Fubini's theorem and reversing the change of variables via $z \mapsto Z(0,s,z)$ and $\tilde{z} \mapsto Z(0,s,\tilde{z})$, we obtain

$$E_{\rm kin}(f(t)) + E_{\rm pot}(f(t)) = E_{\rm kin}(\mathring{f}) + E_{\rm pot}(\mathring{f}) - \int_0^t \int v \cdot F(s, x) f(s, z) \, dz \, ds + \frac{1}{2} \int_0^t \iint (v - \tilde{v}) \cdot \frac{x - \tilde{x}}{|x - \tilde{x}|^3} f(s, \tilde{z}) \, f(s, z) \, d\tilde{z} \, dz \, ds.$$
(3.14)

Since f is a strong Lagrangian solution,

$$F(s,x) = \int \frac{x - \tilde{x}}{|x - \tilde{x}|^3} f(s, \tilde{z}) d\tilde{z}$$

for all $(s,x) \in [0,T[\times \mathbb{R}^3 \text{ so that the two integrals in (3.14) cancel, and the proof of the conservation laws and of Theorem 2.2 is complete.$

3.6 Further solution properties and remarks

- (a) The above proof shows that the conservation laws hold for any strong Lagrangian solution; the symmetry assumption on the data did not enter in that argument.
- (b) In none of the preceding arguments did we use the attractive nature of the force field so that our results hold for the plasma physics case as well, where the sign in the right hand side of (1.2) is reversed. In the relativistic, plasma physics case the estimates in [7], which equally well apply to strong Lagrangian solutions, imply that $T = \infty$, i.e., the solutions are global (for spherically symmetric data).
- (c) The proof of Theorem 2.2 implies that for all $t \in [0, T]$ the functions f(t) and $\rho(t)$ are bounded, measurable functions with compact support, and the control on the support is locally uniform in t. In the non-relativistic case the bound on the velocity support of f(t) is globally uniform in t, which follows from the estimates in Horst [12], cf. [21, Thm. 1.4].
- (d) Our existence proof is completely constructive and does not rely on compactness arguments, which typically are used for obtaining weak solutions, and it covers both the non-relativistic and the relativistic case. The price to pay for this is the unwanted symmetry assumption on the initial data.

References

- A. A. Arsen'ev, Global existence of a weak solution of Vlasov's system of equations, *Comput. Math. Math. Phys.* 15, 131–141 (1975).
- [2] J. Batt, Global symmetric solutions of the initial value problem in stellar dynamics, J. Differential Equations 25, 342–364 (1977).
- [3] J. Binney, S. Tremaine, *Galactic Dynamics*, Princeton University Press 1987.
- [4] A. Bohun, F. Bouchut, G. Crippa, Lagrangian solutions to the Vlasov-Poisson system with L¹ density, J. Differential Equations 260, 3576–3597 (2016).
- [5] R. J. DiPerna, P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* 98, 511–547 (1989).

- [6] D. H. Fremlin, Measure Theory, Vol. 2: Broad Foundations, Torres Fremlin, Colchester 2001.
- [7] R. T. Glassey, J. Schaeffer, On symmetric solutions of the relativistic Vlasov-Poisson system, *Commun. Math. Phys.* **101**, 459–473 (1985).
- [8] Y. Guo, G. Rein, Isotropic steady states in galactic dynamics, Commun. Math. Phys. 219, 607–629 (2001).
- [9] Y. Guo, G. Rein, A non-variational approach to nonlinear stability in stellar dynamics applied to the King model, *Commun. Math. Phys.* 271, 489–509 (2007).
- [10] Y. Guo, Z. Lin, Unstable and Stable Galaxy Models, Commun. Math. Phys., 279, 789–813 (2008).
- [11] M. Hadžić, G. Rein, Global existence and nonlinear stability for the relativistic Vlasov-Poisson system in the gravitational case, *Indiana Uni*versity Math. J. 56, 2453–2488 (2007).
- [12] E. Horst, On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I, Math. Methods Appl. Sci. 3, 229–248 (1981).
- [13] E. Horst, R. Hunze, Weak solutions of the initial value problem for the unmodified non-linear Vlasov equation, *Math. Methods Appl. Sci.* 6, 262–279 (1984).
- [14] J. Körner, The spherically symmetric Vlasov-Poisson system, Master Thesis, Bayreuth 2020.
- [15] M. Lemou, F. Mehats, P. Raphaël, A new variational approach to the stability of gravitational systems, *Commun. Math. Phys.* **302**, 161–224 (2011).
- [16] M. Lemou, F. Mehats, P. Raphaël, Orbital stability of spherical systems, *Invent. Math.* 187, 145–194 (2012).
- [17] P.-L. Lions, B. Perthame, Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system, *Invent. Math.* **105**, 415–430 (1991).
- [18] G. Loeper, Uniqueness of the solution to the Vlasov-Poisson system with bounded density, J. Math. Pures Appl. 86, 68–79 (2006).

- [19] K. Pfaffelmoser, Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, J. Differential Equations 95, 281–303 (1992).
- [20] T. Ramming, G. Rein, Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic and relativistic case—A simple proof for finite extension, *SIAM J. Math. Anal.* 45, 900– 914 (2013).
- [21] G. Rein, Collisionless Kinetic Equations from Astrophysics—The Vlasov-Poisson System, Handbook of Differential Equations, Evolutionary Equations, Eds. C. M. Dafermos and E. Feireisl, Elsevier, 3 (2007).
- [22] R. Robert, Unicité de la solution faible á support compact de l'équation de Vlasov-Poisson, C. R. Acad. Sci. Paris 324, 873–877 (1997).