# Kalman-Bucy filtering and minimum mean square estimator under uncertainty 

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#### Abstract

In this paper, we study a generalized Kalman-Bucy filtering problem under uncertainty. The drift uncertainty for both signal process and observation process is considered and the attitude to uncertainty is characterized by a convex operator (convex risk measure). The optimal filter or the minimum mean square estimator (MMSE) is calculated by solving the minimum mean square estimation problem under a convex operator. In the first part of this paper, this estimation problem is studied under $g$-expectation which is a special convex operator. For this case, we prove that there exists a worst-case prior $P^{\theta^{*}}$. Based on this $P^{\theta^{*}}$ we obtained the Kalman-Bucy filtering equation under $g$-expectation. In the second part of this paper, we study the minimum mean square estimation problem under general convex operators. The existence and uniqueness results of the MMSE are deduced.


Key words. Kalman-Bucy filtering; minimum mean square estimator; drift uncertainty; convex operator; minimax theorem; backward stochastic differential equation

AMS subject classifications. 62M20, 60G35, 93E11, 62F86

## 1 Introduction

It is well-known that Kalman-Bucy filtering is the foundation of modern filtering theory (see Bensoussan[3], Bian and Crisan [5], Liptser and Shiryaev 27], Xiong [34]). It lays the groundwork for further study of optimization problems under partial information in various fields. For example, Duncan and Pasik-Dunan [12], Huang, Wang and Zhang [20], Øksendal and Sulem [28], Tang [33] studied the optimal control (game) for partially observed stochastic systems; Lakner [26], Bensoussan and Keppo [4] considered the utility maximization problem under partial information in mathematical finance and so on.

Let's first recall the classic Kalman-Bucy filtering theory. The model is described as follows: under the probability measure $\mathbb{P}$,

$$
\left\{\begin{align*}
d x(t) & =(B(t) x(t)+b(t)) d t+d w(t)  \tag{1.1}\\
x(0) & =x_{0} \\
d m(t) & =(H(t) x(t)+h(t)) d t+d v(t) \\
m(0) & =0
\end{align*}\right.
$$

where $x(\cdot)$ is the signal process, $m(\cdot)$ is the observation process, $w(\cdot)$ and $v(\cdot)$ are two independent Brownian motions. The coefficients $B(t), H(t), b(t), h(t)$ are deterministic uniformly bounded functions in $t \in[0, T]$, $x_{0}$ is a given constant vector. Set $\mathcal{Z}_{t}=\sigma\{m(s) ; 0 \leq s \leq t\}$ which represents all the observable information up to time $t$. The Kalman filter $\bar{x}(t)$ of $x(t)$ is

$$
\bar{x}(t)=\mathbb{E}_{\mathbb{P}}\left[x(t) \mid \mathcal{Z}_{t}\right]
$$

where $\mathbb{E}_{\mathbb{P}}[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}$. It is well-known that the optimal estimator $\bar{x}(t)$ of the signal $x(t)$ solves the following minimum mean square estimation problem:

$$
\min _{\zeta \in L_{z_{t}}^{2}(\Omega, P)} E_{P}\|x(t)-\zeta\|^{2} .
$$

So $\bar{x}(t)$ is also called the minimum mean square estimator, or MMSE for short.
In this paper, we suppose that there exists model uncertainty for the system (1.1). In other words, we don't know the true probability $\mathbb{P}$ and only know that it falls in a set of probability measures $\mathcal{P}$ which is called the prior set. For continuous-time models, Chen and Epstein [8] first proposed one kind of model uncertainty which is usually called drift uncertainty. Later Epstein and Ji proposed more general uncertainty models (see 14] and [15] for details), Guo 19] introduced some basic scientific problems concerning the estimation, control, and games of dynamical systems with uncertainty and shared some related theoretical progress. In this paper, we introduce the following drift uncertainty model: for every $P^{\theta} \in \mathcal{P}$, consider

$$
\left\{\begin{align*}
d x(t) & =\left(B(t) x(t)+b(t)-\theta_{1}(t)\right) d t+d w^{\theta_{1}}(t)  \tag{1.2}\\
x(0) & =x_{0} \\
d m(t) & =\left(H(t) x(t)+h(t)-\theta_{2}(t)\right) d t+d v^{\theta_{2}}(t) \\
m(0) & =0
\end{align*}\right.
$$

where $w^{\theta_{1}}$ and $v^{\theta_{2}}$ are Brownian motions under $P^{\theta}$ and $\theta=\left(\theta_{1}, \theta_{2}\right) \in \Theta$ is called the uncertainty parameter. When $\theta$ changes, the distribution of the solutions $x(\cdot)$ and $m(\cdot)$ of the above equations also change. The question now is how to calculate the Kalman filter in such an uncertain environment. A natural idea is to calculate the worst-case minimum mean square estimation problem:

$$
\begin{equation*}
\min _{\zeta} \sup _{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}\left(\|x(t)-\zeta\|^{2}\right) \tag{1.3}
\end{equation*}
$$

which is to minimize the maximum expected loss over a range of possible models. Recently, Borisov [6] and [7] studied this type of estimator for finite state Markov processes with uncertainty of the transition intensity and the observation matrices. Allan and Cohen [1] investigated the Kalman-Bucy filtering with a uncertainty parameter by a control approach. Moreover, in the past decade, much research has been discussed depending on the technique of $H_{\infty}$ filter, see [9]-[10] and so on. Different from this paper, the design goal of $H_{\infty}$ filter is to guarantee that the filtering error system is asymptotically stable, while achieving a prescribed $H_{\infty}$ performance level. From another perspective, (1.3) can be rewritten as a minimum mean square estimation problem under a sublinear operator:

$$
\min _{\zeta} \mathcal{E}\left(\|x(t)-\zeta\|^{2}\right)
$$

where $\mathcal{E}(\cdot):=\sup _{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}[\cdot]$ is a sublinear operator. Recently, Ji, Kong and Sun 21] and 22] studied Kalman-Bucy filtering under sublinear operators when the drift uncertainty appears in the signal process and the observation process respectively. The related literatures about the minimum mean square estimation problems under sublinear operators include Sun, Ji 32] and Ji, Kong, Sun [23] in which they considered these problems on $L^{\infty}(\Omega, P)$ and $L^{p}(\Omega, P)$ respectively

However, when we study some problems, especially financial and risk management problems, we need to use a more general nonlinear operator: the convex operator or convex risk measure. For example, in the last decade, the concept of convex risk measure (a special convex operator) has been extensively studied in various fields (see Föllmer, Schied [17], Arai, Fukasawa [2] et al). So it is an interesting problem to solve the minimum mean square estimation problem under the convex operator. Unlike sublinear operators, the lack of positive homogeneity results in an extra penalty term in the expression of convex operators. For the convex operator $\rho(\cdot)$, that is to say, $\rho(\cdot)$ can be represented as

$$
\rho(\cdot)=\sup _{P^{\theta} \in \mathcal{P}}\left[E_{P^{\theta}}[\cdot]-\alpha\left(P^{\theta}\right)\right],
$$

where $\alpha\left(P^{\theta}\right)$ is a penalty function defined on a probability measure set. If $\rho(\cdot)$ is sublinear, the $\alpha\left(P^{\theta}\right)$ takes values in $\{0, \infty\}$. The main difference between this paper and the previous ones is how to deal with the penalty term.

In this paper, we first generalize the Kalman-Bucy filtering to accommodate drift uncertainty in both signal process and observation process and the attitude to uncertainty is characterized by a convex operator (convex risk measure). In more details, we consider system (1.2) and calculate the MMSE by solving

$$
\min _{\zeta} \sup _{P^{\theta}}\left[E_{P^{\theta}}\left[\|x(t)-\zeta\|^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right]=\min _{\zeta} \mathcal{E}_{g}\left[\|x(t)-\zeta\|^{2}\right]
$$

where

$$
\begin{equation*}
\mathcal{E}_{g}[\cdot]:=\sup _{P^{\theta}}\left[E_{P^{\theta}}\left[\|\cdot\|^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \tag{1.4}
\end{equation*}
$$

is called $g$-expectation introduced by Peng [29]. In our context, $\mathcal{E}_{g}[\cdot]$ is a special convex operator and (1.4) is it's dual representation obtained in El Karoui et al 13]. Under some mild conditions, we prove that there exists a worst-case prior $P^{\theta^{*}}$. Based on this $P^{\theta^{*}}$ we obtained the filtering equation by which the MMSE $\hat{x}$ is governed.

The convex $g$-expectation is just a special convex operator. It is worth studying the minimum mean square estimation problem under the general convex operator. In the second part of this paper, we solve the following problem (For the convenience of readers, we misused some notations in the introduction and Section (4):

$$
\min _{\zeta} \rho\left(\|x(t)-\zeta\|^{2}\right)
$$

where $\rho(\cdot)$ is a general convex operator (convex risk measure). The existence and uniqueness results of the MMSE under the general convex operator are deduced.

The paper is organized as follows. In Section 2, we give some preliminaries and formulate our filtering problem under $g$-expectations. In Section 3, the worst-case prior $P^{\theta^{*}}$ is obtained and the corresponding Kalman-Bucy filtering equation (3.8) is deduced. We study the minimum mean square estimation problem under general convex operators $L_{\mathcal{F}}^{p}(\mathbb{P})$ and obtain the existence and uniqueness results of the MMSE in Section 4.

## 2 Preliminaries and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which two independent $n$-dimensional and $m$-dimensional Brownian motions $w(\cdot)$ and $v(\cdot)$ are defined. For the sake of generality, they are not standard. The means of $w(\cdot)$ and $v(\cdot)$ are zero and the covariance matrices are $Q(\cdot)$ and $R(\cdot)$ respectively. We assume that the matrix $R(\cdot)$ is uniformly positive definite. For a fixed time $T>0$, denote by $\mathbb{F}=\left\{\mathcal{F}_{t}, 0 \leq t \leq T\right\}$ the natural filtration of $w(\cdot)$ and $v(\cdot)$ satisfying the usual conditions. We assume $\mathcal{F}=\mathcal{F}_{T}$. For any given Euclidean space $\mathbb{H}$, denote by $\langle\cdot, \cdot\rangle$ (resp. $\|\cdot\|$ ) the scalar product (resp. norm) of $\mathbb{H}$. Let $A^{\top}$ denote the transpose of a matrix $A$. For a $\mathbb{R}^{n}$-valued vector $x=\left(x_{1}, \cdots, x_{n}\right)^{\top},|x|:=\left(\left|x_{1}\right|, \cdots,\left|x_{n}\right|\right)^{\top}$; for two $\mathbb{R}^{n}$-valued vectors $x$ and $y, x \leq y$ means that $x_{i} \leq y_{i}$ for $i=1, \cdots, n$. Through out this paper, 0 denotes the matrix/vector with appropriate dimension whose all entries are zero. For $1<p<\infty$, denote by $L_{\mathbb{F}}^{p}(0, T ; \mathbb{H})$ the space of all the $\mathbb{F}$-adapted $\mathbb{H}$-valued stochastic processes on $[0, T]$ such that

$$
\mathbb{E}\left[\int_{0}^{T}\|f(r)\|^{p} d r\right]<\infty, \forall f \in L_{\mathbb{F}}^{p}(0, T ; \mathbb{H})
$$

The Kalman-Bucy filtering theory is based on a reference probability measure $\mathbb{P}$ for the system (1.1). However, if we don't know the true probability measure $\mathbb{P}$ and only know that it falls in the set $\mathcal{P}$ which is a suitably chosen space of equivalent probability measures, then it is naturally to study the worst-case minimum mean square estimators (MMSE).

### 2.1 Prior set and $g$-expectation

In order to characterize uncertainty, we introduce the prior set $\mathcal{P}$ and $g$-expectation which is a special convex operator.

Let $\theta(\cdot)=\left(\theta_{1}(\cdot), \theta_{2}(\cdot)\right)^{\top}$ be a $\mathbb{R}^{n+m}$-valued progressively measurable process on $[0, T]$. For a given constant $\mu$, let $\Theta$ be the set of all $\mathbb{R}^{n+m}$-valued progressively measurable processes $\theta$ with $\left|\theta_{i}(t)\right| \leq \mu, 0 \leq$ $t \leq T$. Define

$$
\begin{equation*}
\mathcal{P}=\left\{P^{\theta} \left\lvert\, \frac{d P^{\theta}}{d \mathbb{P}}=f^{\theta}(T)\right. \text { for } \theta \in \Theta\right\} \tag{2.1}
\end{equation*}
$$

where

$$
f^{\theta}(T):=\exp \left(-\int_{0}^{T} \theta_{1}^{\top}(t) d w(t)-\frac{1}{2} \int_{0}^{T}\left\|\theta_{1}(t)\right\|^{2} d t-\int_{0}^{T} \theta_{2}^{\top}(t) d v(t)-\frac{1}{2} \int_{0}^{T}\left\|\theta_{2}(t)\right\|^{2} d t\right)
$$

Due to the boundedness of $\theta$, the Novikov's condition holds (see Karatzas, Shreve 25]). Therefore, $P^{\theta}$ defined by (2.1) is a probability measure which is equivalent to probability measure $\mathbb{P}$ and the processes $w^{\theta_{1}}(t)=w(t)+\int_{0}^{t} \theta_{1}(s) d s$ and $v^{\theta_{2}}(t)=v(t)+\int_{0}^{t} \theta_{2}(s) d s$ are Brownian motions under this probability measure $P^{\theta}$ by Girsanov's theorem. The set $\Theta$ characterizes the ambiguity and $\mathcal{P}$ is usually called the prior set.

Then, we introduce $g$-expectation and it's dual representation (see [29] and [13]). In the following we will see that $g$-expectation is a powerful tool for studying uncertainty.

Definition 2.1 we call a function $g: \Omega \times[0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ a standard generator if it satisfies the following conditions:

- $\left(g\left(\omega, t, z_{1}, z_{2}\right)\right)_{t \in[0, T]}$ is an adapted process with

$$
\mathbb{E} \int_{0}^{T}\left|g\left(\omega, t, z_{1}, z_{2}\right)\right|^{2} d t<\infty
$$

for all $z_{1} \in \mathbb{R}^{n}$ and $z_{2} \in \mathbb{R}^{m}$;

- $g\left(\omega, t, z_{1}, z_{2}\right)$ is Lipschitz continuous in $z_{1}$ and $z_{2}$, uniformly in $t$ and $\omega$ : there exists constant $\mu>0$ such that for all $z_{1}, \tilde{z}_{1} \in \mathbb{R}^{n}$ and $z_{2}, \tilde{z}_{2} \in \mathbb{R}^{m}$ we have

$$
\left|g\left(\omega, t, z_{1}, z_{2}\right)-g\left(\omega, t, \tilde{z}_{1}, \tilde{z}_{2}\right)\right| \leq \mu\left(\left\|z_{1}-\tilde{z}_{1}\right\|+\left\|z_{2}-\tilde{z}_{2}\right\|\right)
$$

- $g(\omega, t, 0,0)=0$ for all $t \geq 0$ and $\omega \in \Omega$.

For a standard generator $g$, the following backward stochastic differential equation (BSDE for short)

$$
\left\{\begin{aligned}
-d Y(t) & =g\left(t, Z_{1}(t), Z_{2}(t)\right) d t-Z_{1}^{\top}(t) d w(t)-Z_{2}^{\top}(t) d v(t), t \in[0, T] \\
Y(T) & =\xi
\end{aligned}\right.
$$

with terminal condition $\xi \in L_{\mathcal{F}_{T}}^{2}(\Omega, \mathbb{P})$ has a unique square integrable solution $(Y(t)$, $\left.Z_{1}(t), Z_{2}(t)\right)_{t \in(0, T]}$ (see 29]). Peng [29] calls $Y(t):=\mathcal{E}_{g}\left(\xi \mid \mathcal{F}_{t}\right)$ the (condition) $g$-expectation of $\xi$ at time $t$.

Definition 2.2 A standard generator $g$ is called a convex generator if $g\left(\omega, t, z_{1}, z_{2}\right)$ is convex in $z_{1}$ and $z_{2}$ for $z_{1} \in \mathbb{R}^{n}$ and $z_{2} \in \mathbb{R}^{m}$. The $g$-expectation with a convex generator is called the convex $g$-expectation.

Now we give the dual representation of the convex $g$-expectation through the prior set and the concave dual function of $g$.

Let

$$
\begin{aligned}
& G\left(\omega, t, \theta_{1}, \theta_{2}\right)=\inf _{z_{1} \in \mathbb{R}^{n}, z_{2} \in \mathbb{R}^{m}}\left[g\left(\omega, t, z_{1}, z_{2}\right)+\left\langle z_{1}, \theta_{1}\right\rangle+\left\langle z_{2}, \theta_{2}\right\rangle\right], \\
& \left(\omega \in \Omega, t \in[0, T], \theta_{1} \in \mathbb{R}^{n}, \theta_{2} \in \mathbb{R}^{m}\right)
\end{aligned}
$$

be the concave dual function of $g\left(\omega, t, z_{1}, z_{2}\right)$.
EI Karoui et al. 13] (also see Delbaen et al. [11]) established the following dual representation for $g$-expectation: for a $\mathcal{F}_{s}$-measurable random variable $\xi$, the $g$-expectation at time $t$ can be represented as

$$
\begin{equation*}
\mathcal{E}_{g}\left(\xi \mid \mathcal{F}_{t}\right)=\sup _{P^{\theta} \in \mathcal{P}}\left[E_{P^{\theta}}\left[\xi \mid \mathcal{F}_{t}\right]+\alpha_{t, s}\left(P^{\theta}\right)\right] \tag{2.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{t, s}\left(P^{\theta}\right):=E_{P^{\theta}}\left[\int_{t}^{s} G\left(r, \theta_{1}(r), \theta_{2}(r)\right) d r \mid \mathcal{F}_{t}\right], 0 \leq t \leq s \leq T . \tag{2.3}
\end{equation*}
$$

Remark 2.3 It is easy to check that $\mathcal{E}_{g}\left(\cdot \mid \mathcal{F}_{t}\right)$ is a special convex operator (see 4.1)). Moreover, if we let the standard generator $g\left(t, z_{1}, z_{2}\right)=\mu\left(\left|z_{1}\right|+\left|z_{2}\right|\right)$, then the corresponding dual function of $g\left(t, z_{1}, z_{2}\right)$ and penalty term $\alpha_{t, s}\left(P^{\theta}\right)$ are simultaneously equal to 0 . Then the above convex operator $\mathcal{E}_{g}\left(\cdot \mid \mathcal{F}_{t}\right)$ degenerates to a sublinear operator.

### 2.2 Problem formulation

We formulate the Kalman-Bucy filtering problem under uncertainty. For every $\theta \in \Theta$, under the probability measure $P^{\theta} \in \mathcal{P}$

$$
\left\{\begin{align*}
d x(t) & =\left(B(t) x(t)+b(t)-\theta_{1}(t)\right) d t+d w^{\theta_{1}}(t)  \tag{2.4}\\
x(0) & =x_{0} \\
d m(t) & =\left(H(t) x(t)+h(t)-\theta_{2}(t)\right) d t+d v^{\theta_{2}}(t) \\
m(0) & =0
\end{align*}\right.
$$

where $x(\cdot) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{n}\right)$ is the signal process and $m(\cdot) \in L_{\mathbb{F}}^{2}\left(0, T ; \mathbb{R}^{m}\right)$ is the observation process. The coefficients $B(t) \in \mathbb{R}^{n \times n}, H(t) \in \mathbb{R}^{m \times n}, b(t) \in \mathbb{R}^{n}, h(t) \in \mathbb{R}^{m}$ are deterministic uniformly bounded functions in $t \in[0, T], x_{0} \in \mathbb{R}^{n}$ is a given constant vector. Set

$$
\mathcal{Z}_{t}=\sigma\{m(s) ; 0 \leq s \leq t\}
$$

which represents all the observable information up to time $t$. We want to calculate the MMSE of the signal $x(t)$ by solving the following worst-case minimum mean square estimation problem:

$$
\begin{align*}
\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)} \mathcal{E}_{g}\left(\|x(t)-\zeta(t)\|^{2}\right)= & \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)} \sup _{P^{\theta} \in \mathcal{P}}\left[E_{P^{\theta}}\left(\|x(t)-\zeta(t)\|^{2}\right)\right.  \tag{2.5}\\
& \left.+\alpha_{0, t}\left(P^{\theta}\right)\right]
\end{align*}
$$

where $L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)$ is the set of all the $\mathbb{R}^{n}$-valued $(2+\epsilon)$ integrable $\mathcal{Z}_{t}$-measurable random variables and $0<\epsilon<1$.

Definition 2.4 If $\hat{x}(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)$ satisfies

$$
\mathcal{E}_{g}\left(\|x(t)-\hat{x}(t)\|^{2}\right)=\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)} \mathcal{E}_{g}\left(\|x(t)-\zeta(t)\|^{2}\right)
$$

then we call $\hat{x}(t)$ the minimum mean square estimator (MMSE) of $x(t)$.

## 3 Kalman-Bucy filtering under $g$-expectation

In this section, we calculate the minimum mean square estimator $\hat{x}(t)$ of the problem (2.5) for $t \in[0, T]$. Without loss of generality, all the statements in this section are only proved in the one dimensional case.

Lemma 3.1 The set $\left\{\frac{d P^{\theta}}{d \mathbb{P}}: P^{\theta} \in \mathcal{P}\right\} \subset L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, \mathbb{P})$ is $\sigma\left(L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, \mathbb{P}), L^{1+\frac{\epsilon}{2}}(\Omega, \mathcal{F}, \mathbb{P})\right)$-compact and $\mathcal{P}$ is convex.

Proof. Since $\theta$ is bounded, by Theorem 5.3 in the Appendix, the set $\left\{\frac{d P^{\theta}}{d \mathbb{P}}: P^{\theta} \in \mathcal{P}\right\}$ is bounded in norm $\|\cdot\|_{1+\frac{2}{\epsilon}}$. From Theorem 4.1 of Chapter 1 in Simons 31], we know that the set $\left\{\frac{d P^{\theta}}{d \mathbb{P}}: P^{\theta} \in \mathcal{P}\right\}$ is $\sigma\left(L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, \mathbb{P}), L^{1+\frac{\epsilon}{2}}(\Omega, \mathcal{F}, \mathbb{P})\right)$-compact.

Let $\theta^{1}=\left(\theta_{1}^{1}, \theta_{2}^{1}\right)^{\top}$ and $\theta^{2}=\left(\theta_{1}^{2}, \theta_{2}^{2}\right)^{\top}$ belong to $\Theta . f^{\theta^{1}}$ and $f^{\theta^{2}}$ denote the corresponding exponential martingales: for $t \in[0, T]$,

$$
f^{\theta^{i}}(t)=\exp \left(\int_{0}^{t} \theta_{1}^{i}(s) d w(s)-\frac{1}{2} \int_{0}^{t}\left(\theta_{1}^{i}(s)\right)^{2} d s+\int_{0}^{t} \theta_{2}^{i}(s) d v(s)-\frac{1}{2} \int_{0}^{t}\left(\theta_{2}^{i}(s)\right)^{2} d s\right)
$$

which satisfies

$$
d f^{\theta^{i}}(t)=f^{\theta^{i}}(t)\left(\theta_{1}^{i}(t) d w(t)+\theta_{2}^{i}(t) d v(t)\right), i=1,2
$$

Let $\lambda_{1}$ and $\lambda_{2}$ be nonnegative constants which belong to $(0,1)$ with $\lambda_{1}+\lambda_{2}=1$. Define

$$
\left\{\begin{array}{l}
\theta_{1}^{\lambda}(t)=\frac{\lambda_{1} \theta_{1}^{1}(t) f^{\theta^{1}}(t)+\lambda_{2} \theta_{1}^{2}(t) f^{\theta^{2}}(t)}{\lambda_{1} f^{\theta^{1}}(t)+\lambda_{2} f^{\theta^{2}(t)}}, \\
\theta_{2}^{\lambda}(t)=\frac{\lambda_{1} \theta_{2}^{1}(t) f^{\theta^{1} 1}(t)+\lambda_{2} \theta_{2}^{2}(t) f^{\theta^{2}}(t)}{\lambda_{1} f^{\theta^{1}}(t)+\lambda_{2} f^{\theta^{2}(t)}} .
\end{array}\right.
$$

It is easy to verify that

$$
d\left(\lambda_{1} f^{\theta^{1}}(t)+\lambda_{2} f^{\theta^{2}}(t)\right)=\left(\lambda_{1} f^{\theta^{1}}(t)+\lambda_{2} f^{\theta^{2}}(t)\right)\left(\theta_{1}^{\lambda}(t) d w(t)+\theta_{2}^{\lambda}(t) d v(t)\right)
$$

Since $f^{\theta^{i}}(t)>0, i=1,2$, the process $\theta^{\lambda}=\left(\theta_{1}^{\lambda}, \theta_{2}^{\lambda}\right)^{\top}$ belongs to $\Theta$. Therefore, it results in that $\mathcal{P}$ is convex. This completes the proof.

Lemma 3.2 The penalty term $\alpha_{0, T}\left(P^{\theta}\right)$ is a concave functional on $\mathcal{P}$.
Proof. Let $\theta^{1}=\left(\theta_{1}^{1}, \theta_{2}^{1}\right)^{\top}$ and $\theta^{2}=\left(\theta_{1}^{2}, \theta_{2}^{2}\right)^{\top}$ belong to $\Theta$. $f^{\theta^{1}}$ and $f^{\theta^{2}}$ denote the exponential martingales respectively as in Lemma 3.1. By Lemma 3.1, the exponential martingale ( $\left.\lambda_{1} \frac{d P^{\theta^{1}}}{d \mathbb{P}^{P}}+\lambda_{2} \frac{d P^{\theta^{2}}}{d \mathbb{P}^{2}}\right)$ is generated by $\theta^{\lambda}=\left(\theta_{1}^{\lambda}, \theta_{2}^{\lambda}\right)$. It yields that

$$
\alpha_{0, T}\left(\lambda_{1} P^{\theta^{1}}+\lambda_{2} P^{\theta^{2}}\right)=\mathbb{E}\left[\left(\lambda_{1} f^{\theta^{1}}(T)+\lambda_{2} f^{\theta^{2}}(T)\right) \int_{0}^{T} G\left(t, \theta_{1}^{\lambda}(t), \theta_{2}^{\lambda}(t)\right) d t\right] .
$$

Since $G(t, \cdot, \cdot)$ is a concave function, we have

$$
\begin{aligned}
& \alpha_{0, T}\left(\lambda_{1} P^{\theta^{1}}+\lambda_{2} P^{\theta^{2}}\right) \\
& \geq \mathbb{E}\left[( \lambda _ { 1 } f ^ { \theta ^ { 1 } } ( T ) + \lambda _ { 2 } f ^ { \theta ^ { 2 } } ( T ) ) \left(\int_{0}^{T} \frac{\lambda_{1} f^{\theta^{1}}(t)}{\lambda_{1} f^{\theta^{1}(t)+\lambda_{2} f^{\theta^{2}(t)}} G\left(t, \theta_{1}^{1}(t), \theta_{2}^{1}(t)\right)}\right.\right. \\
& +\int_{0}^{T} \frac{\lambda_{2} f^{\theta^{2}}(t)}{\left.\left.\lambda_{1} f^{\theta^{1}(t)+\lambda_{2} f^{\theta^{2}(t)}} G\left(t, \theta_{1}^{2}(t), \theta_{2}^{2}(t)\right)\right) d t\right]} \\
& =\mathbb{E}\left[\left(\int_{0}^{T} \lambda_{1} f^{\theta^{1}}(t) G\left(t, \theta_{1}^{1}(t), \theta_{2}^{1}(t)\right)+\int_{0}^{T} \lambda_{2} f^{\theta^{2}}(t) G\left(t, \theta_{1}^{2}(t), \theta_{2}^{2}(t)\right)\right) d t\right] \\
& =\mathbb{E}\left[\lambda_{1} f^{\theta^{1}}(T) \int_{0}^{T} G\left(t, \theta_{1}^{1}(t), \theta_{2}^{1}(t)\right) d t\right]+\mathbb{E}\left[\lambda _ { 2 } f ^ { \theta ^ { 2 } } ( T ) \int _ { 0 } ^ { T } G \left(t, \theta_{1}^{2}(t),\right.\right. \\
& \left.\left.\theta_{2}^{2}(t)\right) d t\right]=\lambda_{1} \alpha_{0, T}\left(P^{\theta^{1}}\right)+\lambda_{2} \alpha_{0, T}\left(P^{\theta^{2}}\right) .
\end{aligned}
$$

Therefore, the penalty term $\alpha\left(P^{\theta}\right)$ is a concave functional on $\mathcal{P}$. This completes the proof.
Remark 3.3 It is easy to check that for any $t \in[0, T], \alpha_{0, t}\left(P^{\theta}\right)$ is a concave functional on $\mathcal{P}$ and

$$
\alpha_{0, t}\left(P^{\theta}\right)=\mathbb{E}\left[f^{\theta}(T) \cdot \int_{0}^{t} G\left(s, \theta_{1}(s), \theta_{2}(s)\right) d s\right]=\mathbb{E}\left[f^{\theta}(t) \cdot \int_{0}^{t} G\left(s, \theta_{1}(s), \theta_{2}(s)\right) d s\right] .
$$

Lemma 3.4 Suppose that the stochastic processes $\left(g_{m}(t)\right)_{t \in[0, T]}, m=1,2, \ldots$ and $\left(f^{*}(t)\right)_{t \in[0, T]}$ are exponential martingales respect to the filtration $\mathbb{F}$ and $\left(g_{m}(T)-f^{*}(T)\right) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})} 0$. Then for any $0 \leq t \leq T$, we have

$$
\left(\theta_{i}^{m}(t)-\theta_{i}^{*}(t)\right) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})} 0, \quad i=1,2,
$$

where $\theta^{m}(t)=\left(\theta_{1}^{m}(t), \theta_{2}^{m}(t)\right) \in \Theta$ and $\theta^{*}(t)=\left(\theta_{1}^{*}(t), \theta_{2}^{*}(t)\right) \in \Theta$ are respectively generators of $\left(g_{m}(t)\right)_{t \in[0, T]}, m=$ $1,2, \ldots$ and $\left(f^{*}(t)\right)_{t \in[0, T]}$.

Proof. Denote the generator of $g_{m}(\cdot)$ by $\theta^{m}=\left(\theta_{1}^{m}, \theta_{2}^{m}\right)$, i.e., for $0 \leq t \leq T$,

$$
g_{m}(t)=\exp \left(\int_{0}^{t} \theta_{1}^{m}(s) d w(s)-\frac{1}{2} \int_{0}^{t}\left(\theta_{1}^{m}(s)\right)^{2} d s+\int_{0}^{t} \theta_{2}^{m}(s) d v(s)-\frac{1}{2} \int_{0}^{t}\left(\theta_{2}^{m}(s)\right)^{2} d s\right)
$$

We want to prove that $\left(\theta^{m}\right)$ converges to $\theta^{*}$. Since $g_{m}(\cdot)$ and $f^{*}(\cdot)$ are martingales and $g_{m}(T) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})}$ $f^{*}(T)$, it is easy to verify that $g_{m}(t) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})} f^{*}(t)$ for any $t \in[0, T]$. Applying Itô's formula to $\left(g_{m}(t)-\right.$ $\left.f^{*}(t)\right)^{2}$, we have

$$
\begin{aligned}
& d\left(g_{m}(t)-f^{*}(t)\right)^{2} \\
& =2\left(g_{m}(t)-f^{*}(t)\right)\left[\left(g_{m}(t) \theta_{1}^{m}(t)-f^{*}(t) \theta_{1}^{*}(t)\right) d w(t)+\left(g_{m}(t) \theta_{2}^{m}(t)\right.\right. \\
& \left.\left.-f^{*}(t) \theta_{2}^{*}(t)\right) d v(t)\right]+\left(g_{m}(t) \theta_{1}^{m}(t)-f^{*}(t) \theta_{1}^{*}(t)\right)^{2} d t+\left(g_{m}(t) \theta_{2}^{m}(t)\right. \\
& \left.-f^{*}(t) \theta_{2}^{*}(t)\right)^{2} d t .
\end{aligned}
$$

Taking expectation on both sides,

$$
\begin{align*}
\mathbb{E}\left[\left(g_{m}(T)-f^{*}(T)\right)^{2}\right]= & \mathbb{E}\left[\int_{0}^{T}\left(g_{m}(t) \theta_{1}^{m}(t)-f^{*}(t) \theta_{1}^{*}(t)\right)^{2} d t\right]  \tag{3.1}\\
& +\mathbb{E}\left[\int_{0}^{T}\left(g_{m}(t) \theta_{2}^{m}(t)-f^{*}(t) \theta_{2}^{*}(t)\right)^{2}\right]
\end{align*}
$$

Since $\lim _{m \rightarrow \infty} \mathbb{E}\left[\left(g_{m}(T)-f^{*}(T)\right)^{2}\right]=0$, it yields that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbb{E}\left[\int_{0}^{T}\left(g_{m}(t) \theta_{i}^{m}(t)-f^{*}(t) \theta_{i}^{*}(t)\right)^{2} d t\right]=0, i=1,2 . \tag{3.2}
\end{equation*}
$$

Note that

$$
\begin{aligned}
& \mathbb{E}\left[\int_{0}^{T}\left(g_{m}(t) \theta_{1}^{m}(t)-f^{*}(t) \theta_{1}^{*}(t)\right)^{2} d t\right] \\
& =\mathbb{E} \int_{0}^{T}\left[\left(f^{*}(t)-g_{m}(t)\right)^{2}\left(\theta_{1}^{*}(t)\right)^{2}+\left(g_{m}(t)\right)^{2}\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)^{2}\right. \\
& \left.+2\left(f^{*}(t)-g_{m}(t)\right) g_{m}(t) \theta_{1}^{*}(t)\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)\right] d t .
\end{aligned}
$$

Because $g_{m}(t) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})} f^{*}(t)$ and $\theta$ is bounded, we have

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \mathbb{E}\left[\left(f^{*}(t)-g_{m}(t)\right)^{2}\left(\theta_{1}^{*}(t)\right)^{2}\right] & =0 \\
\lim _{m \rightarrow \infty} \mathbb{E}\left[\left(f^{*}(t)-g_{m}(t)\right) g_{m}(t) \theta_{1}^{*}(t)\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)\right] & =0
\end{aligned}
$$

Therefore, $\lim _{m \rightarrow \infty} \mathbb{E}\left[\left(g_{m}(t)\right)^{2}\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)^{2}\right]=0$. It results in that $\left(g_{m}(t)\right)^{2}\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)^{2} \xrightarrow{\mathbb{P}} 0$. Since $g_{m}(t) \xrightarrow{\mathbb{P}} f^{*}(t)$, we have $\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right)^{2} \xrightarrow{\mathbb{P}} 0$. Due to the boundedness of $\theta$, we obtain $\left(\theta_{1}^{*}(t)-\theta_{1}^{m}(t)\right) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})}$ 0 . Similarly, we can obtain $\left(\theta_{2}^{*}(t)-\theta_{2}^{m}(t)\right) \xrightarrow{L^{2}(\Omega, \mathcal{F}, \mathbb{P})} 0$. This completes the proof.

In the following, we prove that the worst-case prior $P^{\theta^{*}}$ exists.
Theorem 3.5 For a given $t \in[0, T]$, there exists a $\theta^{*} \in \Theta$ such that

$$
\begin{align*}
& \inf _{\zeta(t) \in L_{z_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})} \mathcal{E}_{g}\left[(x(t)-\zeta(t))^{2}\right] \\
& =\sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right]  \tag{3.3}\\
& =\inf _{\zeta(t) \in L_{z_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{*}}\right)\right] .
\end{align*}
$$

Proof. Firstly, we prove the first equality. According to Lemmas 3.1 and 3.2, the original robust estimation problem (2.5) satisfies minimax theorem 5.1. Therefore, the first equality is verified.

Secondly, we prove the second equality. Choose a sequence $\left\{\theta^{n}\right\}, n=1,2, \cdots$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{n}}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{n}}\right)\right]  \tag{3.4}\\
& =\sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\alpha_{0, t}\left(P^{\theta}\right)+E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]\right] .
\end{align*}
$$

Set $f^{\theta^{n}}(T)=\frac{d P^{\theta^{n}}}{d \mathbb{P}}$. By Komlós theorem A.3.4 in 30], there exists a subsequence $\left\{f^{\theta^{n_{k}}}(T)\right\}_{k \geq 1}$ of $\left\{f^{\theta^{n}}(T)\right\}_{n \geq 1}$ and a $f^{*}(T) \in L^{1}(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^{m} f^{\theta^{n_{k}}}(T)=f^{*}(T), \mathbb{P}-\text { a.s.. } \tag{3.5}
\end{equation*}
$$

Let $g_{m}(T)=\frac{1}{m} \sum_{k=1}^{m} f^{\theta^{n_{k}}}(T)$. We have $g_{m}(T) \xrightarrow{\mathbb{P}-a . s .} f^{*}(T)$. By Theorem 5.3 in the Appendix, for any given constant $p>1$ and $m$, we have $\mathbb{E}\left(g_{m}(T)\right)^{K} \leq M$ where $K=\left(1+\frac{2}{\epsilon}\right) p$ and $M=\exp \left(\left(K^{2}-K\right) \mu^{2} T\right)$. Then, we have $\left\{\left|g_{m}(T)\right|^{1+\frac{2}{\varepsilon}}: m=1,2, \cdots\right\}$ is uniformly integrable. Therefore, it results in that $g_{m}(T) \xrightarrow{L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, \mathbb{P})}$ $f^{*}(T)$ and $f^{*}(T) \in L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, \mathbb{P})$. According to the convexity and weak compactness of the set $\left\{\frac{d P^{\theta}}{d \mathbb{P}^{P}}: P^{\theta} \in\right.$ $\mathcal{P}\}$, there exists a $\theta^{*}$ such that $\frac{d P^{\theta^{*}}}{d \mathbb{P}}=f^{*}(T)$.

Then we prove that the probability measure $P^{\theta^{*}}$ with respect to obtained generator $\theta^{*}$ satisfies (3.3).

Based on (3.4) and (3.5), we have

$$
\begin{align*}
& \sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \\
= & \lim _{n \rightarrow \infty} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[f^{P^{\theta_{n}}}(T)(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta_{n}}\right)\right] \\
= & \lim _{k \rightarrow \infty} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[f^{P^{\theta_{n_{k}}}}(T)(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta_{n_{k}}}\right)\right] \\
= & \lim _{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^{m} \inf _{\zeta(t) \in L^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[f^{P^{\theta_{n}}}(T)(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta_{n_{k}}}\right)\right]  \tag{3.6}\\
\leq & \liminf _{m \rightarrow \infty} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})} \frac{1}{m} \sum_{k=1}^{m}\left[\mathbb{E}\left[f^{P^{\theta_{n}}}(T)(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta_{n_{k}}}\right)\right] \\
\leq & \liminf _{m \rightarrow \infty} \inf _{\zeta(t) \in L_{z_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[g_{m}(T)(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{m}}\right)\right]
\end{align*}
$$

where the last inequality is due to the concavity of $\alpha(\cdot)$. By (3.6) and Lemma 3.4, it results in that

$$
\begin{align*}
& \sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \\
& \geq \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{*}}\right)\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[\lim _{m \rightarrow \infty} g_{m}(T)(x(t)-\zeta(t))^{2}\right]\right. \\
& \left.+\mathbb{E}\left[f^{*}(T) \int_{0}^{t} G\left(r, \theta_{1}^{*}(r), \theta_{2}^{*}(r)\right) d r\right]\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[\lim _{m \rightarrow \infty} g_{m}(T)(x(t)-\zeta(t))^{2}\right]\right.  \tag{3.7}\\
& \left.+\mathbb{E}\left[\lim _{m \rightarrow \infty}\left(g_{m}(T) \int_{0}^{t} G\left(r, \theta_{1}^{m}(r), \theta_{2}^{m}(r)\right) d r\right)\right]\right] \\
& \geq \limsup _{m \rightarrow \infty} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}+\epsilon}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[\mathbb{E}\left[g_{m}(T)(x(t)-\zeta(t))^{2}\right]\right. \\
& \left.+\mathbb{E}\left[g_{m}(T) \int_{0}^{t} G\left(r, \theta_{1}^{m}(r), \theta_{2}^{m}(r)\right) d r\right]\right] \\
& \geq \sup _{P^{\theta} \in \mathcal{P}} \inf ^{\inf }(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R}) \\
& {\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right]}
\end{align*}
$$

where the second inequality is based on the upper semi-continuous property. Therefore,

$$
\begin{aligned}
& \sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{*}}\right)\right] .
\end{aligned}
$$

By minimax theorem (Theorem5.1 in the Appendix), we obtain

$$
\begin{aligned}
& \sup _{P^{\theta} \in \mathcal{P}} \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})} \sup _{P^{\theta} \in \mathcal{P}}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right]
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})} \mathcal{E}_{g}\left[(x(t)-\zeta(t))^{2}\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})} \sup _{P^{\theta} \in \mathcal{P}}\left[E_{P^{\theta}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta}\right)\right] \\
& =\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{*}}\right)\right] .
\end{aligned}
$$

This completes the proof.
For the obtained $\theta^{*}(t)=\left(\theta_{1}^{*}(t), \theta_{2}^{*}(t)\right)$ in Theorem 3.5] set $\widehat{\theta_{i}^{*}(t)}=E_{P^{\theta^{*}}}\left[\theta_{i}^{*}(t) \mid \mathcal{Z}_{t}\right], i=1,2$.
Theorem 3.6 The MMSE $\hat{x}(t)$ of problem (2.5) equals $E_{P^{\theta^{*}}}\left[x(t) \mid \mathcal{Z}_{t}\right]$ and satisfies the following equation:

$$
\left\{\begin{align*}
d \hat{x}(t) \quad & =\left(B(t) \hat{x}(t)+b(t)-\widehat{\theta_{1}^{*}(t)}\right) d t+\left(P(t) H(t)-x \widehat{x(t) \theta_{2}^{*}(t)}\right.  \tag{3.8}\\
& \left.+\hat{x}(t) \widehat{\theta_{2}^{*}(t)}\right) R(t)^{-1} d \hat{I}(t) \\
\hat{x}(0) & =x_{0}
\end{align*}\right.
$$

where $\theta^{*}$ is obtained in Theorem [3.5, $x \widehat{x(t) \theta_{2}^{*}}(t):=E_{P^{*}}\left[x(t) \theta_{2}^{*}(t) \mid \mathcal{Z}_{t}\right]$ and the so called innovation process $\hat{I}(t):=m(t)-\int_{0}^{t}\left(H(s) \hat{x}(s)+g(s)-\widehat{\theta_{2}^{*}(s)}\right) d s, 0 \leq t \leq T$ is a $\mathcal{Z}_{t}$-measurable Brownian motion. The variance of the estimation error $P(t)=E_{P^{\theta^{*}}}\left[(x(t)-\hat{x}(t))^{2}\right]$ satisfies the following equation:

$$
\left\{\begin{align*}
\frac{d P(t)}{d t}= & -E_{P^{\theta^{*}}}\left[( P ( t ) H ( t ) - x \widehat { ( t ) \theta _ { 2 } ^ { * } ( t ) } + \hat { x } ( t ) \widehat { \theta _ { 2 } ^ { * } ( t ) } ) R ^ { - 1 } ( t ) \left(H(t) P(t)-\widehat{\theta_{2}^{*}(t) x(t)}\right.\right.  \tag{3.9}\\
& \left.\left.+\widehat{\theta_{2}^{*}(t)} \hat{x}(t)\right)\right]+2 E_{P^{\theta^{*}}}\left[-x \widehat{(t) \theta_{1}^{*}}(t)+\hat{x}(t) \widehat{\theta_{1}^{*}(t)}\right]+2 B(t) P(t)+Q(t), \\
P(0)= & 0 .
\end{align*}\right.
$$

Proof. For the obtained optimal $\theta^{*}(t)=\left(\theta_{1}^{*}(t), \theta_{2}^{*}(t)\right)$ in Theorem 3.5 the system (2.4) and problem (2.5) can be reformulated correspondingly under $P^{\theta^{*}}$. In more detail, on the filtered probability space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{0 \leq t \leq T}, P^{\theta^{*}}\right)$, the processes $x(\cdot)$ and $m(\cdot)$ satisfy the following equations:

$$
\left\{\begin{align*}
d x(t) & =\left(B(t) x(t)+b(t)-\theta_{1}^{*}(t)\right) d t+d w^{\theta_{1}^{*}}(t)  \tag{3.10}\\
x(0) & =x_{0} \\
d m(t) & =\left(H(t) x(t)+h(t)-\theta_{2}^{*}(t)\right) d t+d v^{\theta_{2}^{*}}(t) \\
m(0) & =0
\end{align*}\right.
$$

We solve the minimum mean square estimation problem

$$
\begin{equation*}
\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right]+\alpha_{0, t}\left(P^{\theta^{*}}\right)\right] . \tag{3.11}
\end{equation*}
$$

Since $\alpha_{0, t}\left(P^{\theta^{*}}\right)$ is a constant, we only need to consider the following optimization problem:

$$
\begin{equation*}
\inf _{\zeta(t) \in L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})}\left[E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right] .\right. \tag{3.12}
\end{equation*}
$$

In [27], Liptser and Shiryaev studied the optimal estimator of the following problem:

$$
\begin{equation*}
\inf _{\zeta(t) \in L_{z_{t}}^{2}\left(\Omega, P^{\theta^{*}}, \mathbb{R}\right)} E_{P^{\theta^{*}}}\left[(x(t)-\zeta(t))^{2}\right] . \tag{3.13}
\end{equation*}
$$

By Theorem 8.1 in [27], the optimal estimator $\hat{x}(t)=E_{P^{\theta^{*}}}\left[x(t) \mid \mathcal{Z}_{t}\right]$ satisfies (3.8). Since $B(t), H(t), b(t)$ and $h(t)$ are uniformly bounded, deterministic functions and $\theta^{*}$ is bounded, by Theorem 6.3 (see Chapter 1 in [35]), the solution $\hat{x}(t)$ to (3.8) also belongs to $L_{\mathcal{Z}_{t}}^{2+\epsilon}(\Omega, \mathbb{P}, \mathbb{R})$. It yields that $\hat{x}(t)$ is the optimal solution of problem (3.12) at time $t \in[0, T]$. This completes the proof.

Corollary 3.7 If $\theta^{*}(t)$ is adapted to $\mathcal{Z}_{t}$, then $\hat{x}(t)$ satisfies the following equation:

$$
\left\{\begin{align*}
d \hat{x}(t) & =\left(B(t) \hat{x}(t)+b(t)-\theta_{1}^{*}(t)\right) d t+P(t) H(t) R(t)^{-1} d \hat{I}(t)  \tag{3.14}\\
\hat{x}(0) & =x_{0}
\end{align*}\right.
$$

where $P(t)$ satisfies the following Riccati equation:

$$
\left\{\begin{align*}
\frac{d P(t)}{d t} & =B(t) P(t)+P(t) B(t)^{\top}-P(t) H(t)^{\top} R(t)^{-1} H(t) P(t)+Q(t)  \tag{3.15}\\
P(0) & =0
\end{align*}\right.
$$

Define

$$
A(t, s)=\exp ^{\int_{s}^{t}\left(B(r)-P(r) H(r)^{2} R^{-1}(r)\right) d r}
$$

$\bar{x}(t)$ is governed by

$$
\left\{\begin{align*}
d \bar{x}(t) & =(B(t) \bar{x}(t)+b(t)) d t+P(t) H(t)^{\top} R(t)^{-1} d I(t)  \tag{3.16}\\
\bar{x}(0) & =x_{0}
\end{align*}\right.
$$

where

$$
I(t)=m(t)-\int_{0}^{t}(H(s) \bar{x}(s)+h(s)) d s
$$

Corollary 3.8 If the optimal $\theta^{*}(t)$ adapted to subfiltration $\mathcal{Z}_{t}$, with equations (3.16) and (3.8), then the optimal estimator $\hat{x}(t)$ for any time $t \in[0 . T]$ can be expressed as

$$
\begin{equation*}
\hat{x}(t)=\bar{x}(t)+\int_{0}^{t}\left(P(s) H(s) R^{-1}(s) \theta_{2}^{*}(s)-\theta_{1}^{*}(s)\right) A(t, s) d s \tag{3.17}
\end{equation*}
$$

where $\bar{x}(t)$ is defined by equation (3.16).
Remark 3.9 So far, we have only proved the existence of the optimal $\theta^{*}$ from the mathematical theory. Since the complexity of the problem is considered in this paper, it is still a problem to be solved how to calculate the optimal $\theta^{*}$. In the future, we plan to study the numerical solutions to the robust estimation (2.5).

## 4 MMSE under general convex operators on $L_{\mathcal{F}}^{p}(\mathbb{P})$

In section 3, we boil down the calculation of the Kalman-Bucy filter under uncertainty to solving a minimum mean square estimation problem under the convex $g$-expectation. The worst-case prior $P^{\theta^{*}}$ is obtained and the corresponding filtering equation (3.8) is deduced.

It is an interesting question whether there are similar results for general convex operators. So in this section, we investigate the minimum mean square estimation problem under general convex operators on $L_{\mathcal{F}}^{p}(\mathbb{P})$ and obtain the existence and uniqueness results of the MMSE.

### 4.1 General convex operators on $L_{\mathcal{F}}^{p}(\mathbb{P})$

For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote the set of all $\mathcal{F}$-measurable $p$-th power integrable random variables by $L^{p}(\Omega, \mathcal{F}, \mathbb{P})$. Sometimes we use $L_{\mathcal{F}}^{p}(\mathbb{P})$ for short. Let $\mathcal{C}$ be a sub $\sigma$-algebra of $\mathcal{F} . L_{\mathcal{C}}^{p}(\mathbb{P})$ denotes the set of all the $p$-th power integrable $\mathcal{C}$-measurable random variables. In this paper, we only consider the case that $1<p \leq 2$.

Let $\mathcal{M}$ denote the set of probability measures absolutely continuous with respect to $\mathbb{P}$. For $P \in \mathcal{M}$, we will use $f^{P}$ to denote the Radon-Nikodym derivative $\frac{d P}{d \mathbb{P}}$ and $E_{P}[\cdot]$ to denote the expectation under $P$. Especially, the expectation under $\mathbb{P}$ is denoted as $\mathbb{E}[\cdot]$. For a sub $\sigma$-algebra $\mathcal{C}$ of $\mathcal{F}$ and $P \in \mathcal{M}$, define $f_{\mathcal{C}}^{P}=\mathbb{E}\left[f^{P} \mid \mathcal{C}\right]$.

Definition 4.1 $A$ convex operator is an operator $\rho(\cdot): L_{\mathcal{F}}^{p}(\mathbb{P}) \mapsto \mathbb{R}$ satisfying
(i) Monotonicity: for any $\xi_{1}, \xi_{2} \in L_{\mathcal{F}}^{p}(\mathbb{P}), \rho\left(\xi_{1}\right) \geq \rho\left(\xi_{2}\right)$ if $\xi_{1} \geq \xi_{2}$;
(ii) Constant invariance: $\rho(\xi+c)=\rho(\xi)+c$ for any $\xi \in L_{\mathcal{F}}^{p}(\mathbb{P})$ and $c \in \mathbb{R}$;
(iii) Convexity: for any $\xi_{1}, \xi_{2} \in L_{\mathcal{F}}^{p}(\mathbb{P})$ and $\lambda \in[0,1], \rho\left(\lambda \xi_{1}+(1-\lambda) \xi_{2}\right) \leq \lambda \rho\left(\xi_{1}\right)+(1-\lambda) \rho\left(\xi_{2}\right)$.

Definition 4.2 A convex operator $\rho(\cdot)$ is called normalized if $\rho(0)=0$.
Remark 4.3 In this paper, we will always assume the convex operator is normalized. Moreover, if we define $\rho^{\prime}(\xi)=\rho(-\xi)$, then $\rho^{\prime}(\cdot)$ is a convex risk measure on $L_{\mathcal{F}}^{p}(\mathbb{P})$.

If $\rho(\cdot)$ is a convex operator, then by Proposition 2.10 and Theorem 2.11 in [24], for any random variable $\xi \in L_{\mathcal{F}}^{p}(\mathbb{P})$, there exists a set $\mathcal{P}$ such that $\rho(\cdot)$ can be represented as

$$
\rho(\xi)=\sup _{P \in \mathcal{P}}\left[E_{P}[\xi]-\alpha(P)\right],
$$

where $\alpha(P):=\sup _{\zeta \in \mathcal{A}_{\rho}} E_{P}[\zeta], \mathcal{A}_{\rho}:=\left\{\zeta \in L_{\mathcal{F}}^{p}(\mathbb{P}) ; \rho(\zeta) \leq 0\right\}$ called acceptance set, $\mathcal{P}:=\{P \in \mathcal{M}$; $\left.f^{P} \in L_{\mathcal{F}}^{q}(\mathbb{P}), \alpha(P)<\infty\right\}$. Moreover, $\mathcal{D}:=\left\{f^{P} ; P \in \mathcal{P}\right\}$ is norm-bounded in $L_{\mathcal{F}}^{q}(\mathbb{P})$ and $\sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)$ compact, where $\sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)$ denotes the weak topology defined on $L_{\mathcal{F}}^{q}(\mathbb{P})$ and $\frac{1}{p}+\frac{1}{q}=1$. The set $\mathcal{P}$ is called the representation set of $\rho(\cdot)$. Since $\alpha(\cdot)$ is a convex function defined on $\mathcal{M}, \mathcal{P}$ is a convex set.

Remark 4.4 Note that $\alpha(P)=\sup _{\zeta \in \mathcal{A}_{\rho}} E_{P}[\zeta]=\sup _{\zeta \in \mathcal{A}_{\rho}} \mathbb{E}\left[f^{P} \zeta\right]$. By abuse of notation, we sometimes write $\alpha\left(f^{P}\right)$ instead of $\alpha(P)$.
Definition 4.5 The set $\mathcal{P}$ is called stable if for any element $P \in \mathcal{P}$ and any sub $\sigma$-algebra $\mathcal{C}$ of $\mathcal{F}$, $\frac{f^{P}}{f_{\mathcal{C}}^{P}}$ still lies in the set $\mathcal{D}$.

Definition 4.6 A convex operator $\rho(\cdot)$ is called stable, if its representation set $\mathcal{P}$ is stable.
Definition 4.7 $A$ convex operator $\rho(\cdot)$ is called proper if all the elements in its representation set $\mathcal{P}$ are equivalent to $\mathbb{P}$.

For a given $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$, when we only know the information $\mathcal{C}$, we want to find the minimum mean square estimator of $\xi$ under the convex operator $\rho(\cdot)$. In more detail, we will solve the following optimization problem:

Problem: For a given $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$, find a $\hat{\eta} \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$ such that

$$
\begin{equation*}
\rho(\xi-\hat{\eta})^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi-\eta)^{2} \tag{4.1}
\end{equation*}
$$

The optimal solution $\hat{\eta}$ of (4.1) is called the minimum mean square estimator and we will denote it by $\rho(\xi \mid \mathcal{C})$.

Remark 4.8 If we set $\mathcal{C}=\mathcal{Z}_{t}$ and $p=1+\frac{\epsilon}{2}$ with $\epsilon \in(0,1)$, then $L_{\mathcal{C}}^{2 p}(\mathbb{P})$ is just the space $L_{\mathcal{Z}_{t}}^{2+\epsilon}\left(\Omega, \mathbb{P}, \mathbb{R}^{n}\right)$ in subsection 2.2.

### 4.2 Existence and uniqueness results

In this section, we study the existence and uniqueness of the minimum mean square estimator for problem 4.1 We first give the following assumption.

Assumption 4.9 The convex operator $\rho(\cdot)$ is stable and proper.

### 4.2.1 Existence

Lemma 4.10 For any given real number $\gamma \geq 2$, if $\xi \in L_{\mathcal{F}}^{\gamma p}(\mathbb{P})$, then we have $\sup _{P \in \mathcal{P}} E_{P}\left[\xi^{\frac{\gamma p}{2}}\right]<\infty$.
Proof. Since $\left\{f^{P} ; P \in \mathcal{P}\right\}$ is normed bounded in $L_{\mathcal{F}}^{q}(\mathbb{P})$ and $1<p \leq 2$, we have

$$
\sup _{P \in \mathcal{P}} E_{P}\left[\xi^{\frac{\gamma p}{2}}\right]=\sup _{P \in \mathcal{P}} \mathbb{E}\left[f^{P} \xi^{\frac{\gamma p}{2}}\right] \leq \sup _{P \in \mathcal{P}}\left\|f^{P}\right\|_{L^{q}}\left\|\xi^{\frac{\gamma p}{2}}\right\|_{L^{p}} \leq \sup _{P \in \mathcal{P}}\left\|f^{P}\right\|_{L^{q}}\left(\|\xi\|_{L^{\gamma p}}\right)^{\gamma}<\infty
$$

This completes the proof.
Lemma 4.11 Suppose that Assumption 4.9 holds. Then for any $P \in \mathcal{P}, \xi \in L_{\mathcal{F}}^{p}(\mathbb{P})$ and sub $\sigma$-algebra $\mathcal{C}$ of $\mathcal{F}$, there exists a $\bar{P} \in \mathcal{P}$ such that $E_{\bar{P}}[\xi]=\mathbb{E}\left[E_{P}[\xi \mid \mathcal{C}]\right]$.

Proof. It is obvious that

$$
\mathbb{E}\left[E_{P}[\xi \mid \mathcal{C}]\right]=\mathbb{E}\left[\frac{\mathbb{E}\left[\xi f^{P} \mid \mathcal{C}\right]}{\mathbb{E}\left[f^{P} \mid \mathcal{C}\right]}\right]=\mathbb{E}\left[\mathbb{E}\left[\left.\xi \frac{f^{P}}{f_{\mathcal{C}}^{P}} \right\rvert\, \mathcal{C}\right]\right]=\mathbb{E}\left[\xi \frac{f^{P}}{f_{\mathcal{C}}^{P}}\right]
$$

By Definition 4.5, there exists a $\bar{P} \in \mathcal{P}$ such that $\frac{d \bar{P}}{d \mathbb{P}}=\frac{f^{P}}{f_{\mathcal{C}}^{P}}$ which implies that $E_{\bar{P}}[\xi]=\mathbb{E}\left[E_{P}[\xi \mid \mathcal{C}]\right]$. This completes the proof.

Proposition 4.12 Suppose that Assumption 4.9 holds. If $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$, then there exists a constant $M$ such that for any probability measure $P \in \mathcal{P}$,

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p_{p}(\mathbb{P})}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
$$

where $L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$ denotes all the elements in $L_{\mathcal{C}}^{2 p}(\mathbb{P})$ which are norm-bounded by the constant $M$.
Proof. Set $\mathbb{G}=\left\{E_{P}[\xi \mid \mathcal{C}] ; P \in \mathcal{P}\right\}$. For any $P \in \mathcal{P}$, we have $\mathbb{E}\left[\left(E_{P}[\xi \mid \mathcal{C}]\right)^{2 p}\right] \leq \mathbb{E}\left[E_{P}\left[\xi^{2 p} \mid \mathcal{C}\right]\right]$. By Lemma 4.11, there exists a $\bar{P} \in \mathcal{P}$ such that $E_{\bar{P}}\left[\xi^{2 p}\right]=\mathbb{E}\left[E_{P}\left[\xi^{2 p} \mid \mathcal{C}\right]\right]$. By Lemma 4.10, there exists a constant $M_{1}$ such that $\sup _{P \in \mathcal{P}} E_{P}\left[\xi^{2 p}\right] \leq M_{1}$. Then $\mathbb{G} \subset L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$ where $M=M_{1}^{\frac{1}{2 p}}$. Since $1<p \leq 2$, it is obvious that

$$
\mathbb{G} \subset L_{\mathcal{C}}^{2 p}(\mathbb{P}) \subset\left(\bigcup_{0<\epsilon \leq 2} L_{\mathcal{C}}^{2+\epsilon}(\mathbb{P})\right)
$$

By the project property of conditional expectations, for any $P \in \mathcal{P}$ and $\eta \in L_{\mathcal{C}}^{2+\epsilon}(\mathbb{P})$ with $\epsilon \in(0,2]$, we have that

$$
E_{P}\left[\left(\xi-E_{P}[\xi \mid \mathcal{C}]\right)^{2}\right] \leq E_{P}\left[(\xi-\eta)^{2}\right]
$$

which leads to

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] \geq \inf _{\eta^{\prime} \in \mathbb{G}}\left[E_{P}\left[\left(\xi-\eta^{\prime}\right)^{2}\right]-\alpha(P)\right] .
$$

On the other hand, the inverse inequality is obviously true. Then the following equality holds for any $P \in \mathcal{P}$ :

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\inf _{\eta \in \mathbb{G}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
$$

Since $\mathbb{G} \subset L_{\mathcal{C}}^{2 p, M}(\mathbb{P}) \subset L_{\mathcal{C}}^{2 p}(\mathbb{P})$, it follows that

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] .
$$

This completes the proof.
By Proposition 4.12 it is easy to see that

$$
\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
$$

Lemma $4.13 \alpha(\cdot)$ is a lower semi-continuous (l.s.c.) function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right)$.
Proof. For any fixed random variable $\zeta \in \mathcal{A}_{\rho}$, define $\varphi\left(\zeta, f^{P}\right)=\mathbb{E}\left[f^{P} \zeta\right]$ where $f^{P}$ belongs to $\mathcal{D}$. Then $\varphi(\zeta, \cdot)$ is a continuous function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P})\right.\right.$
, $\left.L_{\mathcal{F}}^{p}(\mathbb{P})\right)$ ). Since $\alpha\left(f^{P}\right)=\sup _{\zeta \in \mathcal{A}_{\rho}} \varphi\left(\zeta, f^{P}\right)$, based on lower-semicontinuous definition B.1.1 in Pham 30], then $\alpha(P)$ is a l.s.c. function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right)$. This completes the proof.

For $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P}), \eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$ and $P \in \mathcal{P}$, define

$$
l\left(\xi, \eta, f^{P}\right)=\mathbb{E}\left[f^{P}(\xi-\eta)^{2}\right]-\alpha\left(f^{P}\right)
$$

Lemma 4.14 For any random variables $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$ and $\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P}), l(\xi, \eta, \cdot)$ is an upper semi-continuous (u.s.c.) function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right)$.

Proof. Since $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$ and $\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$, then $(\xi-\eta)^{2} \in L_{\mathcal{F}}^{p}(\mathbb{P})$ which implies that $\mathbb{E}\left[f^{P}(\xi-\eta)^{2}\right]$ is a continuous function with respect to $f^{P}$ on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right.$ ). By Lemma 4.13, $\alpha(\cdot)$ is a l.s.c. function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right)$. Thus, $l(\xi, \eta, \cdot)$ is an u.s.c. function on the topology space $\left(\mathcal{D}, \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)\right)$. This completes the proof.

Proposition 4.15 Suppose that Assumption 4.9holds. Then for a given $\xi \in L_{\mathcal{F}}^{4 p}(\Omega, \mathbb{P})$, there exists a $\hat{P} \in \mathcal{P}$ such that

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{\hat{P}}\left[(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right]=\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
$$

where $M$ is the constant given in Proposition 4.12.
Proof. Define

$$
\beta=\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\sup _{f^{P} \in \mathcal{D}} \inf _{\eta \in L_{\mathcal{C}}^{2, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{P}(\xi-\eta)^{2}\right]-\alpha\left(f^{P}\right)\right]
$$

Take a sequence $\left\{f^{P_{n}} ; P_{n} \in \mathcal{P}\right\}_{n \geq 1}$ such that

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{P_{n}}(\xi-\eta)^{2}\right]-\alpha\left(f^{P_{n}}\right)\right] \geq \beta-\frac{1}{2^{n}}
$$

Since $\mathcal{D}$ is a weakly compact set, we can take a subsequence $\left\{f^{P_{n_{i}}}\right\}_{i \geq 1}$ which weakly converges to some $f^{\hat{P}} \in L_{\mathcal{F}}^{q}(\mathbb{P})$. Therefore, $\hat{P} \in \mathcal{P}$ and there exists a sequence $\left\{f^{\tilde{P}_{i}} \in \operatorname{conv}\left(f^{P_{n_{i}}}, f^{P_{n_{i+1}}}, \ldots\right)\right\}_{i \geq 1}$ such that $f^{\tilde{P}_{i}}$ converges to $f^{\hat{P}}$ in $L_{\mathcal{F}}^{q}(\mathbb{P})$-norm by Theorem 5.4 in the Appendix.

For any $\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$ and $i \in \mathbb{N}$,

$$
\lim _{i \rightarrow \infty} \mathbb{E}\left|f^{\tilde{P}_{i}}(\xi-\eta)^{2}-f^{\hat{P}}(\xi-\eta)^{2}\right| \leq \lim _{i \rightarrow \infty}\left\|\left(f^{\tilde{P}_{i}}-f^{\hat{P}}\right)\right\|_{L^{q}(\mathbb{P})}\left\|(\xi-\eta)^{2}\right\|_{L^{p}(\mathbb{P})}=0
$$

which leads to

$$
\lim _{i \rightarrow \infty} \mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\eta)^{2}\right]=\mathbb{E}\left[f^{\hat{P}}(\xi-\eta)^{2}\right]
$$

On the other hand,

$$
\begin{aligned}
\left|\alpha\left(f^{\hat{P}}\right)-\alpha\left(f^{\tilde{P}_{i}}\right)\right| & =\left|\sup _{\zeta \in \mathcal{A}_{\rho}} \mathbb{E}\left[f^{\hat{P}} \zeta\right]-\sup _{\zeta \in \mathcal{A}_{\rho}} \mathbb{E}\left[f^{\tilde{P}_{i}} \zeta\right]\right| \leq \sup _{\zeta \in \mathcal{A}_{\rho}} \mathbb{E}\left[\left|\left(f^{\hat{P}}-f^{\tilde{P}_{i}}\right) \zeta\right|\right] \\
& \leq \sup _{\zeta \in \mathcal{A}_{\rho}}\left\|\left(f^{\tilde{P}_{i}}-f^{\hat{P}}\right)\right\|_{L^{q}(\mathbb{P})}\|\zeta\|_{L^{p}(\mathbb{P})} .
\end{aligned}
$$

Then,

$$
\lim _{i \rightarrow \infty}\left[\mathbb{E}\left[\left[f^{\tilde{P}_{i}}(\xi-\eta)^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right]=\mathbb{E}\left[f^{\hat{P}}(\xi-\eta)^{2}\right]-\alpha\left(f^{\hat{P}}\right)\right]
$$

Since

$$
\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\eta)^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right] \geq \inf _{\tilde{\eta} \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\tilde{\eta})^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right]
$$

for any $\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$, we have that

$$
\lim _{i \rightarrow \infty}\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\eta)^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right] \geq \limsup _{i \rightarrow \infty} \inf _{\tilde{\eta} \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\tilde{\eta})^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right]
$$

It yields that

$$
\begin{align*}
& \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{\hat{P}}(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right] \\
= & \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})} \lim _{i \rightarrow \infty}\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\eta)^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right]  \tag{4.2}\\
\geq & \limsup _{i \rightarrow \infty} \inf _{\tilde{\eta} \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{\tilde{P_{i}}}(\xi-\tilde{\eta})^{2}\right]-\alpha\left(f^{\tilde{P}}\right)\right] .
\end{align*}
$$

As $\alpha(\cdot)$ is a convex function and $f^{\tilde{P}_{i}} \in \operatorname{conv}\left(f^{P_{n_{i}}}, f^{P_{n_{i+1}}}, \ldots\right)$, we have

$$
\begin{equation*}
\limsup _{i \rightarrow \infty} \inf _{\tilde{\eta} \in L_{\mathcal{C}}^{2 \mathcal{P}, M}(\mathbb{P})}\left[\mathbb{E}\left[f^{\tilde{P}_{i}}(\xi-\tilde{\eta})^{2}\right]-\alpha\left(f^{\tilde{P}_{i}}\right)\right] \geq \beta \tag{4.3}
\end{equation*}
$$

Combining (4.2) and (4.3), we obtain the result.
Corollary 4.16 Suppose that Assumption 4.9 holds. Then for a given $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$, there exists a $\hat{P} \in \mathcal{P}$ such that

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{\hat{P}}\left[(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right]=\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] .
$$

Proof. Choose $\hat{P}$ as in Proposition 4.15, By Propositions 4.12 and 4.15, the following relations hold

$$
\begin{aligned}
& \sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\sup _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] \\
& =\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{\hat{P}}\left[(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right]=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{\hat{P}}\left[(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right] .
\end{aligned}
$$

This completes the proof.
Theorem 4.17 (Existence theorem) Suppose that Assumption 4.9 holds. Then there exists a $\hat{\eta} \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$ which solves problem (4.1).

Proof. For given $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P}), \eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$ and $P \in \mathcal{P}$, it is easy to check that $l\left(\xi, \cdot, f^{P}\right)$ is convex on $L_{\mathcal{C}}^{2 p}(\mathbb{P})$ and $l(\xi, \eta, \cdot)$ is concave on $L_{\mathcal{F}}^{q}(\mathbb{P})$. As $\mathcal{D}$ is $\sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P}), L_{\mathcal{F}}^{p}(\mathbb{P})\right)$-compact and $l(\xi, \eta, \cdot)$ is u.s.c on the topology space $\left(L_{\mathcal{F}}^{q}(\mathbb{P}), \sigma\left(L_{\mathcal{F}}^{q}(\mathbb{P})\right.\right.$
, $\left.L_{\mathcal{F}}^{p}(\mathbb{P})\right)$ ) by Lemma 4.14, we have

$$
\begin{aligned}
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \max _{P \in \mathcal{P}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] & =\max _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] ; \\
\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})} \max _{P \in \mathcal{P}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right] & =\max _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
\end{aligned}
$$

by Proposition 4.15, Corollary 4.16 and Theorem 5.1 in the Appendix. With the help of Proportion 4.12,

$$
\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \max _{P \in \mathcal{P}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]=\inf _{\eta \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})} \max _{P \in \mathcal{P}}\left[E_{P}\left[(\xi-\eta)^{2}\right]-\alpha(P)\right]
$$

Therefore, we can take a sequence $\left\{\eta_{n} ; n \in \mathbb{N}\right\} \subset L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$ such that

$$
\rho\left(\xi-\eta_{n}\right)^{2}<\beta+\frac{1}{2^{n}}
$$

where $\beta:=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi-\eta)^{2}$. Since $L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$ is a weakly compact set, we can take a subsequence $\left\{\eta_{n_{i}}\right\}_{i \in \mathbb{N}}$ of $\left\{\eta_{n}\right\}_{n \in \mathbb{N}}$ which weakly converges to some $\hat{\eta} \in L_{\mathcal{C}}^{2 p, M}(\mathbb{P})$. By theorem 5.4 in the Appendix, there exists a sequence $\left\{\tilde{\eta}_{i} \in \operatorname{conv}\left(\eta_{n_{i}}, \eta_{n_{i+1}}, \cdots\right)\right\}_{i \in \mathbb{N}}$ such that $\tilde{\eta}_{i}$ converges to $\hat{\eta}$ in $L_{\mathcal{C}}^{2 p}(\mathbb{P})$-norm. Then

$$
\begin{align*}
\rho(\xi-\hat{\eta})^{2} & =\rho\left(\xi-\tilde{\eta}_{i}+\tilde{\eta}_{i}-\hat{\eta}\right)^{2} \\
& =\sup _{P \in \mathcal{P}}\left[E_{P}\left[\left(\xi-\tilde{\eta}_{i}\right)^{2}+\left(\tilde{\eta}_{i}-\hat{\eta}\right)^{2}+2\left(\xi-\tilde{\eta}_{i}\right)\left(\tilde{\eta}_{i}-\hat{\eta}\right)\right]-\alpha(P)\right] \\
& \leq \sup _{P \in \mathcal{P}}\left[E_{P}\left[\left(\xi-\tilde{\eta}_{i}\right)^{2}\right]-\alpha(P)\right]+\sup _{P \in \mathcal{P}} E_{P}\left[\left(\tilde{\eta}_{i}-\hat{\eta}\right)^{2}\right.  \tag{4.4}\\
& \left.+2\left(\xi-\tilde{\eta}_{i}\right)\left(\tilde{\eta}_{i}-\hat{\eta}\right)\right] \\
& =\rho\left(\xi-\tilde{\eta}_{i}\right)^{2}+\sup _{P \in \mathcal{P}} E_{P}\left[-\left(\tilde{\eta}_{i}-\hat{\eta}\right)^{2}+2(\xi-\hat{\eta})\left(\tilde{\eta}_{i}-\hat{\eta}\right)\right] \\
& \leq \beta+\frac{1}{2^{i-1}}+2 \sup _{P \in \mathcal{P}}\left\|f^{P}\right\|_{L^{q}}\left\|\tilde{\eta}_{i}-\hat{\eta}\right\|_{L^{2 p}}\left(1+\|\xi-\hat{\eta}\|_{L^{2 p}}\right) .
\end{align*}
$$

As (4.4) holds for any $i \geq 1$, we have that $\rho(\xi-\hat{\eta})^{2}=\beta$.

### 4.2.2 Uniqueness

In this subsection, we prove that the optimal solution of problem (4.1) is unique.
Proposition 4.18 Suppose that Assumption 4.9 holds. If $\hat{\eta}$ is an optimal solution of problem (4.1), then there exists a $\hat{P} \in \mathcal{P}$ such that $\hat{\eta}=E_{\hat{P}}[\xi \mid \mathcal{C}]$.

Proof. If $\hat{\eta}$ is an optimal solution of problem (4.1), then there exists a $\hat{P} \in \mathcal{P}$ such that

$$
\begin{aligned}
& \sup _{P \in \mathcal{P}}\left[\mathbb{E}\left[f_{P}(\xi-\hat{\eta})^{2}\right]-\alpha(P)\right] \\
& =\max _{P \in \mathcal{P}}\left[\mathbb{E}\left[f_{P}(\xi-\hat{\eta})^{2}\right]-\alpha(P)\right] \\
& =\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \max _{P \in \mathcal{P}}\left[\mathbb{E}\left[f_{P}(\xi-\eta)^{2}\right]-\alpha(P)\right] \\
& =\max _{P \in \mathcal{P}} \inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[\mathbb{E}\left[f_{P}(\xi-\eta)^{2}\right]-\alpha(P)\right] \\
& =\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})}\left[\mathbb{E}\left[f^{\hat{P}}(\xi-\eta)^{2}\right]-\alpha(\hat{P})\right]
\end{aligned}
$$

by Corollary 4.16. Theorem 4.17 and Theorem 5.1 in the Appendix. Thus, by Theorem 5.2 in the Appendix, $(\hat{\eta}, \hat{P})$ is a saddle point, i.e., for $\forall P \in \mathcal{P}, \eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$, we have

$$
\mathbb{E}\left[f^{P}(\xi-\hat{\eta})^{2}\right]-\alpha(P) \leq \mathbb{E}\left[f^{\hat{P}}(\xi-\hat{\eta})^{2}\right]-\alpha(\hat{P}) \leq \mathbb{E}\left[f^{\hat{P}}(\xi-\eta)^{2}\right]-\alpha(\hat{P})
$$

This shows that if $\hat{\eta}$ is an optimal solution, then there exists a $\hat{P} \in \mathcal{P}$ such that $\hat{\eta}=E_{\hat{P}}[\xi \mid \mathcal{C}]$ by the project property of conditional expectations.

Theorem 4.19 (Uniqueness theorem) Suppose that Assumption 4.9 holds.
Then, the optimal solution of problem (4.1) is unique.

Proof. Suppose that there exist two optimal solutions $\hat{\eta}_{1}$ and $\hat{\eta}_{2}$. Denote the corresponding probabilities in Proposition 4.18 by $\hat{P}_{1}$ and $\hat{P}_{2}$ respectively. Then $\hat{\eta}_{1}=E_{\hat{P}_{1}}[\xi \mid \mathcal{C}]$ and $\hat{\eta}_{2}=E_{\hat{P}_{2}}[\xi \mid \mathcal{C}]$. For $\lambda \in(0,1)$, set

$$
\begin{aligned}
P^{\lambda} & =\lambda \hat{P}_{1}+(1-\lambda) \hat{P}_{2}, \\
\lambda_{\hat{P}_{1}} & =\lambda E_{P^{\lambda}}\left[\left.\frac{d \hat{P}_{1}}{d P^{\lambda}} \right\rvert\, \mathcal{C}\right], \\
\lambda_{\hat{P}_{2}} & =(1-\lambda) E_{P^{\lambda}}\left[\left.\frac{d \hat{P}_{2}}{d P^{\lambda}} \right\rvert\, \mathcal{C}\right] .
\end{aligned}
$$

It is easy to verify that $\lambda_{\hat{P}_{1}}+\lambda_{\hat{P}_{2}}=1$ and $E_{P^{\lambda}}[\xi \mid \mathcal{C}]=\lambda_{\hat{P}_{1}} \hat{\eta}_{1}+\lambda_{\hat{P}_{2}} \hat{\eta}_{2}$. Noticing that $E_{\hat{P}_{i}}\left[\xi-\hat{\eta}_{i} \mid \mathcal{C}\right]=0$, $i=1,2$, then we have the following inequality (Details of the calculation can be found in Lemma 5.5 in the Appendix):

$$
\begin{aligned}
& E_{P^{\lambda}}\left[\left(\xi-E_{P^{\lambda}}[\xi \mid \mathcal{C}]\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\left(\xi-\lambda_{\hat{P}_{1}} \hat{\eta}_{1}-\lambda_{\hat{P}_{2}} \hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\left(\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)+\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)^{2}+\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}-\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & \lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}-\lambda_{\hat{P}_{2}}\left(\left(\xi-\hat{\eta}_{1}\right)^{2}-\left(\xi-\hat{\eta}_{2}\right)^{2}+\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right)+\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right] \\
& +(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}\left(\left(\xi-\hat{\eta}_{1}\right)^{2}-\left(\xi-\hat{\eta}_{2}\right)^{2}-\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right)+\left(\xi-\hat{\eta}_{2}\right)^{2}+\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right] \\
& -\alpha\left(P^{\lambda}\right) \\
= & \left.\lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]\right)+(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
& +(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right] .
\end{aligned}
$$

Set $\beta=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi-\eta)^{2}$. By the above equation and the convexity of $\alpha(\cdot)$,

$$
\begin{aligned}
& E_{P^{\lambda}}\left[\left(\xi-E_{P^{\lambda}}[\xi \mid \mathcal{C}]\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
\geq & \lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right]-\left[\lambda \alpha\left(\hat{P}_{1}\right)+(1-\lambda) \alpha\left(\hat{P}_{2}\right)\right] \\
& +\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right] \\
= & \beta+\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]
\end{aligned}
$$

$$
\geq \beta
$$

On the other hand, since $\left(\hat{\eta}_{1}, \hat{P}_{1}\right)$ is a saddle point, we have

$$
E_{P^{\lambda}}\left[\left(\xi-E_{P^{\lambda}}[\xi \mid \mathcal{C}]\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \leq E_{P^{\lambda}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \leq E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]-\alpha\left(\hat{P}_{1}\right)=\beta .
$$

It yields that $E_{P^{\lambda}}\left[\left(\xi-E_{P^{\lambda}}[\xi \mid \mathcal{C}]\right)^{2}\right]-\alpha\left(P^{\lambda}\right)=\beta$. By (4.5), we deduce that $\hat{\eta}_{1}=\hat{\eta}_{2} \mathbb{P}$-a.s..

### 4.2.3 Properties of the minimum mean square estimator

Finally, in this subsection, we will list some properties of the MMSE $\rho(\xi \mid \mathcal{C})$.
Proposition 4.20 If a convex operator $\rho(\cdot)$ is stable and proper, then for any $\xi \in L_{\mathcal{F}}^{4 p}(\mathbb{P})$, we have:
i) If $C_{1} \leq \xi(\omega) \leq C_{2}$ for two constants $C_{1}$ and $C_{2}$, then $C_{1} \leq \rho(\xi \mid \mathcal{C}) \leq C_{2}$;
ii) $\rho(-\xi \mid \mathcal{C})=-\rho(\xi \mid \mathcal{C})$;
iii) For any given $\eta_{0} \in L_{\mathcal{C}}^{2 p}(\mathbb{P})$, we have $\rho\left(\xi+\eta_{0} \mid \mathcal{C}\right)=\rho(\xi \mid \mathcal{C})+\eta_{0}$;
iv) If $\xi$ is independent of the sub $\sigma$-algebra $\mathcal{C}$ under every probability measure $P \in \mathcal{P}$, then $\rho(\xi \mid \mathcal{C})$ is a constant.

Proof. i) If $C_{1} \leq \xi(\omega) \leq C_{2}$, then for any $P \in \mathcal{P}, C_{1} \leq E_{P}[\xi \mid \mathcal{C}] \leq C_{2}$. According to the proof of Theorem 4.19, $\rho(\xi \mid \mathcal{C}) \in\left\{E_{P}[\xi \mid \mathcal{C}] ; P \in \mathcal{P}\right\}$ which leads to $C_{1} \leq \rho(\xi \mid \mathcal{C}) \leq C_{2}$.
ii) Since

$$
\rho(\xi-\rho(\xi \mid \mathcal{C}))^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi-\eta)^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi+\eta)^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(-\xi-\eta)^{2},
$$

we have

$$
\rho(-\xi-(-\rho(\xi \mid \mathcal{C})))^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(-\xi-\eta)^{2} .
$$

By Theorem 4.19, $-\rho(\xi \mid \mathcal{C})=\rho(-\xi \mid \mathcal{C})$.
iii) Note that

$$
\rho\left(\xi+\eta_{0}-\left(\eta_{0}+\rho(\xi \mid \mathcal{C})\right)\right)^{2}=\rho(\xi-\rho(\xi \mid \mathcal{C}))^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho(\xi-\eta)^{2}=\inf _{\eta \in L_{\mathcal{C}}^{2 p}(\mathbb{P})} \rho\left(\xi+\eta_{0}-\eta\right)^{2}
$$

By Theorem 4.19, we have $\eta_{0}+\rho(\xi \mid \mathcal{C})=\rho\left(\xi+\eta_{0} \mid \mathcal{C}\right)$.
iv) If $\xi$ is independent of the sub $\sigma$-algebra $\mathcal{C}$ under every $P \in \mathcal{P}$, then $E_{P}[\xi \mid \mathcal{C}]$ is a constant for any $P \in \mathcal{P}$. Since $\rho(\xi \mid \mathcal{C}) \in\left\{E_{P}[\xi \mid \mathcal{C}] ; P \in \mathcal{P}\right\}$, we know that $\rho(\xi \mid \mathcal{C})$ is a constant. This completes the proof.

## 5 Appendix

For the convenience of the reader, we list the main theorems used in our proofs.
Theorem 5.1 (Fan [16] Theorem 2) Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{Y}$ be an arbitrary set. Let $F$ be a real valued function defined on $\mathcal{X} \times \mathcal{Y}$ such that, for every $y \in \mathcal{Y}, F(x, y)$ is a l.s.c(lowersemicontinuous) on $\mathcal{X}$. If $F$ is convex on $\mathcal{X}$ and concave on $\mathcal{Y}$, then

$$
\min _{x \in \mathcal{X}} \sup _{y \in \mathcal{Y}} F(x, y)=\sup _{y \in \mathcal{Y}} \min _{x \in \mathcal{X}} F(x, y)
$$

Proof. Refer to Theorem 2 in 16].
Theorem 5.2 (Zălinescu [37] Theorem 2.10.1) Let $A$ and $B$ be two nonempty sets and from $A \times B$ to $\mathbb{R} \bigcup\{\infty\}$. Then $f$ has saddle points, i.e., there exists $(\bar{x}, \bar{y}) \in A \times B$ such that

$$
\forall x \in A, \forall y \in B: \quad f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y)
$$

if and only if

$$
\inf _{y \in B} f(\bar{x}, y)=\max _{x \in A} \inf _{y \in B} f(x, y)=\min _{y \in B} \sup _{x \in A} f(x, y)=\sup _{x \in A} f(x, \bar{y}) .
$$

Theorem 5.3 (Girsanov [18]) We suppose that $\phi(t, \omega)$ satisfies the following conditions:
(1) $\phi(\cdot, \cdot)$ are measurable in both variables;
(2) $\phi(t, \cdot)$ is $\mathcal{F}_{t}$-measurable for fixed $t$;
(3) $\int_{0}^{T}|\phi(t, \omega)|^{2} d t<\infty$ almost everywhere; and $0<c_{1} \leq|\phi(t, \omega)| \leq c_{2}$ for almost all $(t, \omega)$, then $\exp \left[\alpha \zeta_{s}^{t}(\phi)\right]$ is integrable and for $\alpha>1$

$$
\begin{equation*}
\exp \left[\frac{\left(\alpha^{2}-\alpha\right)}{2}(t-s) c_{1}^{2}\right] \leq \mathbb{E}\left[\exp \left[\alpha \zeta_{s}^{t}(\phi)\right]\right] \leq \exp \left[\frac{\left(\alpha^{2}-\alpha\right)}{2}(t-s) c_{2}^{2}\right] \tag{5.1}
\end{equation*}
$$

where $\zeta_{s}^{t}(\phi)=\int_{s}^{t} \phi(u, \omega) d w_{u}-\frac{1}{2} \int_{s}^{t} \phi^{2}(u, \omega) d u$.
Theorem 5.4 (Kôsaku Yosida [36]) Let $(X,\|\cdot\|)$ be a Banach space and $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ be a sequence in $X$ that converges weakly to some $x \in X$. Then there exists, for any $\epsilon>0$, a convex combination $\sum_{j=1}^{n} \alpha_{j} x_{j},\left(\alpha_{j} \geq\right.$ $\left.0, \sum_{j=1}^{n} \alpha_{j}=1\right)$ such that $\left\|x-\sum_{j=1}^{n} \alpha_{j} x_{j}\right\| \leq \epsilon$.

Lemma 5.5 Let $\hat{\eta}_{1}=E_{\hat{P}_{1}}[\xi \mid \mathcal{C}], \hat{\eta}_{2}=E_{\hat{P}_{2}}[\xi \mid \mathcal{C}], P^{\lambda}=\lambda \hat{P}_{1}+(1-\lambda) \hat{P}_{2}, \lambda_{\hat{P}_{1}}=\lambda E_{P^{\lambda}}\left[\left.\frac{d \hat{P}_{1}}{d P^{\lambda}} \right\rvert\, \mathcal{C}\right], \lambda_{\hat{P}_{2}}=$ $(1-\lambda) E_{P^{\lambda}}\left[\left.\frac{d \hat{P}_{2}}{d P^{\lambda}} \right\rvert\, \mathcal{C}\right]$. Then we have

$$
\begin{aligned}
& E_{P^{\lambda}}\left[\left(\xi-\lambda_{\hat{P}_{1}} \hat{\eta}_{1}-\lambda_{\hat{P}_{2}} \hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & \lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right] \\
& +\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) .
\end{aligned}
$$

## Proof.

$$
\begin{align*}
& E_{P^{\lambda}}\left[\left(\xi-\lambda_{\hat{P}_{1}} \hat{\eta}_{1}-\lambda_{\hat{P}_{2}} \hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\left(\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)+\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\xi-\hat{\eta}_{1}\right)^{2}+\lambda_{\hat{P}_{2}}^{2}\left(\xi-\hat{\eta}_{2}\right)^{2}+2 \lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{1}\right)\left(\xi-\hat{\eta}_{1}\right)\right]-\alpha\left(P^{\lambda}\right) \\
= & E_{P^{\lambda}}\left[\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)^{2}+\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}-\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & \lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}\right] \\
& +(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}\right]-\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]  \tag{5.2}\\
& -(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & \lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]-(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{1}\right)^{2}\right] \\
& \left.-\lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right]-\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}\right]+\left(1-\hat{\eta}_{1}\right)^{2}\right]-(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right]\right. \\
& -(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) .
\end{align*}
$$

Since

$$
(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]=(1-\lambda) E_{\hat{P}_{2}}\left[\left(\lambda_{\hat{P}_{1}}+\lambda_{\hat{P}_{2}}\right)\left(\xi-\hat{\eta}_{1}\right)^{2}\right]
$$

and

$$
\lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right]=\lambda E_{\hat{P}_{1}}\left[\left(\lambda_{\hat{P}_{1}}+\lambda_{\hat{P}_{2}}\right)\left(\xi-\hat{\eta}_{2}\right)^{2}\right]
$$

it results in that

$$
\text { (5.2) } \begin{aligned}
& =\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{2}\right)^{2}-\lambda_{\hat{P}_{2}}\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{1}\right)^{2}-\lambda_{\hat{P}_{1}}\left(\xi-\hat{\eta}_{2}\right)^{2}\right] \\
& \left.-\lambda E_{\hat{P}_{1}} \lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]+\lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right] \\
& +(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right] .
\end{aligned}
$$

Firstly, we calculate the items with respect to the expectation $\lambda E_{\hat{P}_{1}}[\cdot]$, the following relations hold:

$$
\begin{aligned}
& \lambda_{\hat{P}_{2}}\left(\xi^{2}+\hat{\eta}_{2}^{2}-2 \xi \hat{\eta}_{2}\right)-\lambda_{\hat{P}_{2}}\left(\xi^{2}+\hat{\eta}_{1}^{2}-2 \xi \hat{\eta}_{1}\right)-\lambda_{\hat{P}_{1}} \lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2} \\
= & \lambda_{\hat{P}_{2}}\left[2 \hat{\eta}_{1}\left(\hat{\eta}_{2}-\hat{\eta}_{1}\right)+2 \xi\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)\right]+\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2} \\
= & \lambda_{\hat{P}_{2}}\left[2\left(\xi-\hat{\eta}_{1}\right)\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)\right]+\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2} .
\end{aligned}
$$

Since $\lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)$ is $\mathcal{C}$-measurable and $\left(\xi-\hat{\eta}_{1}\right)$ is orthogonal with $\sigma$-algebra $\mathcal{C}$ under probability measure $\hat{P}_{1}$, it results that

$$
\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}} 2\left(\xi-\hat{\eta}_{1}\right)\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)\right]=\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)\right] E_{\hat{P}_{1}}\left[2\left(\xi-\hat{\eta}_{1}\right)\right]=0 .
$$

Secondly, we can also similarly calculate the items with respect to the expectation $(1-\lambda) E_{\hat{P}_{2}}[\cdot]$. Finally, the equation (5.2) can be expressed as

$$
\begin{aligned}
& E_{P^{\lambda}}\left[\left(\xi-\lambda_{\hat{P}_{1}} \hat{\eta}_{1}-\lambda_{\hat{P}_{2}} \hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) \\
= & \lambda E_{\hat{P}_{1}}\left[\left(\xi-\hat{\eta}_{1}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\left(\xi-\hat{\eta}_{2}\right)^{2}\right] \\
& +\lambda E_{\hat{P}_{1}}\left[\lambda_{\hat{P}_{2}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]+(1-\lambda) E_{\hat{P}_{2}}\left[\lambda_{\hat{P}_{1}}^{2}\left(\hat{\eta}_{1}-\hat{\eta}_{2}\right)^{2}\right]-\alpha\left(P^{\lambda}\right) .
\end{aligned}
$$

This completes the proof.

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