# A POLYNOMIAL TIME ALGORITHM FOR SOLVING THE CLOSEST VECTOR PROBLEM IN ZONOTOPAL LATTICES 

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#### Abstract

In this note we give a polynomial time algorithm for solving the closest vector problem in the class of zonotopal lattices. The Voronoi cell of a zonotopal lattice is a zonotope, i.e. a projection of a regular cube. Examples of zonotopal lattices include lattices of Voronoi's first kind and tensor products of root lattices of type A. The combinatorial structure of zonotopal lattices can be described by regular matroids/totally unimodular matrices. We observe that a linear algebra version of the minimum mean cycle canceling method can be applied for efficiently solving the closest vector problem in a zonotopal lattice if the lattice is given as the integral kernel of a totally unimodular matrix.


## 1. Introduction

A lattice $L$ of rank $r$ is a discrete subgroup of $\left(\mathbb{R}^{m},+\right)$ which spans a linear subspace of dimension $r$. One can specify a lattice by a lattice basis; these are $r$ linearly independent vectors $b_{1}, \ldots, b_{r} \in L$ so that $L$ is given by all their integral linear combinations:

$$
L=\left\{\sum_{i=1}^{r} \alpha_{i} b_{i}: \alpha_{1}, \ldots, \alpha_{r} \in \mathbb{Z}\right\} .
$$

The central computational problems for lattices are the shortest vector problem (SVP) and the closest vector problem (CVP). They have many applications in mathematics, computer science, and engineering, in particular in complexity theory, cryptography, information theory, mathematical optimization, and the geometry of numbers; see for instance [28].

Solving the shortest vector problem amounts to finding a shortest nonzero vector in a given lattice. In this paper we are concerned with the closest vector problem (CVP): Given a lattice basis of $b_{1}, \ldots, b_{r}$ of $L$ and given a target vector $t \in \mathbb{R}^{m}$ find a lattice vector $u \in L$ which is closest to $t$, i.e.

$$
\text { determine } u \in L \text { with }\|u-t\|=\min _{v \in L}\|v-t\| \text {, }
$$

where $\|x\|$ denotes the standard Euclidean norm of a vector $x \in \mathbb{R}^{m}$. Without loss of generality, after performing an orthogonal projection, we may assume that the target vector $t$ lies in the span of $L$, which we denote by $\mathcal{L}$.

[^0]One can interpret CVP geometrically via the Voronoi cell of the lattice $L$ which is defined as

$$
\mathcal{V}(L)=\{x \in \mathcal{L}:\|x\| \leq\|v-x\| \text { for all } v \in L\}
$$

The Voronoi cell of $L$ is a polytope which tessellates the space $\mathcal{L}$ by lattice translates $v+\mathcal{V}(L)$ for $v \in L$. Now the CVP asks for a lattice vector $u$ so that the target vector $t$ lies in $u+\mathcal{V}(L)$.

In the past the closest vector problem has been studied intensively. Here we only discuss results on algorithms and complexity which are most relevant for us. We refer to [1] for an up-to-date discussion of the computational complexity of CVP.

Van Emde Boas [14] established the NP-hardness of exactly solving CVP. Dinur, Kindler, Raz, Safra [11] showed that approximating CVP within a factor of $r^{c / \log \log r}$, for some positive constant $c$, is NP-hard as well. Aharonov and Regev [2] showed that approximating CVP within a factor of $\sqrt{r}$ lies in NP $\cap$ coNP.

On the algorithmic side, Micciancio and Voulgaris [27] developed a deterministic algorithm for exactly solving CVP which runs in $\tilde{O}\left(2^{2 r}\right)$ time and needs $\tilde{O}\left(2^{r}\right)$ space. This was improved by Aggarwal, Dadush, and Stephens-Davidowitz [1] who achieved a $2^{r+o(r)}$-time and space randomized algorithm. Hunkenschröder, Reuland, Schymura [20] considered the possibility to improve the space complexity of the algorithm by Micciancio and Voulgaris if one has a compact representation of the lattice' Voronoi cell.

In this note we are concerned with the polynomial time solvability of the closest vector problem restricted to a special class of lattices.

That CVP can be solved in polynomial time for special classes of lattices has been proved in the case of lattices of Voronoi's first kind by McKilliam, Grant, and Clarkson [25] and in the case of tensor products $A_{m} \otimes A_{n}$ of root lattices of type $A$ by Ducas and van Woerden [12].

The main result of this paper unifies and extends these two cases. For this we consider lattices whose Voronoi cell is a zonotope. Zonotopes are defined as projections of cubes; all of their faces (of any dimension) are centrally symmetric. All lattices up to dimension three have a zonotope as Voronoi cell, but starting from dimension four on, there are lattices which do not have this property, for example the root lattice $\mathrm{D}_{4}$ whose Voronoi cell is the 24 -cell. Indeed, the threedimensional facets of the 24 -cell are regular octahedra and their two-dimensional faces are regular triangles and thus are not centrally symmetric.

We show that one can exactly solve CVP for zonotopal lattices in polynomial time using an algorithm of Karzanov and McCormick [23]. Their algorithm can be seen as a linear algebra version of the minimum mean cycle canceling algorithm of Goldberg and Tarjan [18] for finding a minimum-cost circulation in a network.

The set up is a follows: A totally unimodular matrix $M \in\{-1,0,+1\}^{n \times m}$, i.e. every minor of $M$ is either equal to $-1,0$, or 1 , is given. We consider the lattice $L$ of all integer points lying in the kernel of $M$; the matrix $M$ will be part of the algorithm's input. Furthermore, a separable convex objective function $w: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is given. Separability means that for every $i \in[m]$ we have a convex function $w_{i}: \mathbb{R} \rightarrow \mathbb{R}$ so that

$$
w(x)=\sum_{i=1}^{m} w_{i}\left(x_{i}\right) \quad \text { for } \quad\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}
$$

Then, under some technical conditions on the separable convex objective function $w$, one can compute in polynomial time a lattice vector $v \in L$ so that $w(v)$ is as small as possible.

Since the work of Coxeter [10], Shephard [33] and McMullen [26] it is known that the combinatorial structure of zonotopes which tile space by translations is determined by a regular matroid and thus it is related to totally unimodular matrices.

Zonotopal lattices are defined in Section 2 and the relation to regular matroids is reviewed. We develop the theory in such a way that the separability of the objective function which solves the CVP in this setting becomes apparent. We show that lattices of Voronoi's first kind and that tensor product lattices $\mathrm{A}_{n} \otimes \mathrm{~A}_{m}$ are zonotopal lattices.

In Section 3 we discuss the algorithm of Karzanov and McCormick. We cast the CVP for zonotopal lattices into a separable convex optimization problem and verify that the technical conditions on the separable convex objective function are fulfilled to ensure the polynomial time solvability.

## 2. Zonotopal lattices

In this section we collect basic definitions and facts about zonotopal lattices. Zonotopal lattices were first defined by Gerritzen [17] when he gave a metric to Tutte's regular chain groups (see for example Tutte [37]). The theory of zonotopal lattices was further developed by Loesch [24] and Vallentin [38], [39], [40].

Space tiling zonotopes have been thoroughly investigated in the literature: Main examples of zonotopal lattices are the lattice of integral flows and the lattice of integral cuts on a finite graph which were considered by Bacher, de la Harpe, Nagnibeda [4]. Lattices whose Voronoi cell are zonotopes can be dually interpreted by Delone subdivisions and hyperplane arrangements; this has been done by Erdahl and Ryshkov [15] who developed the theory of lattice dicings for this. Zonotopes which tile space by translations were studied by Coxeter [10], Jaeger [21], Shephard [33], and McMullen [26], see also [5].
2.1. Combinatorics: Regular chain groups, regular matroids, totally unimodular matrices. We start by briefly recalling fundamental definitions and results of Tutte's theory of regular chain groups. Chain groups are defined over general integral domains $R$ (commutative rings with a unit element and no divisors of zero). In this paper we only need $R=\mathbb{R}$ or $R=\mathbb{Z}$. So we sometimes simplify Tutte's original notation. Regular chain groups are closely related to regular matroids and totally unimodular matrices. We refer, for example, to Camion [6], Oxley [29], Schrijver [31], Tutte [36], [37], Welsh [41] for proofs and more details.

Let $\mathcal{L}$ be a subspace of $\mathbb{R}^{m}$. The support of a vector $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{L}$ is given by

$$
\operatorname{supp} x=\left\{i \in[m]: x_{i} \neq 0\right\} \quad \text { with } \quad[m]=\{1, \ldots, m\}
$$

A non-zero vector $x \in \mathcal{L}$ is called an elementary chain if it has minimal (inclusionwise) support among all non-zero vectors in $\mathcal{L}$. An elementary chain $x$ is called a primitive chain if $x_{i} \in\{-1,0,+1\}$ for all $i \in[m]$. A subspace $\mathcal{L}$ is called regular if every elementary chain is a multiple of a primitive chain.

The set of supports of elementary chains in a regular subspace forms the circuits of a regular matroid, a matroid which is representable over every field. If a matrix
$M$ is totally unimodular, then the kernel of $M$ is a regular subspace. Conversely, every regular subspace can be represented as kernel of a totally unimodular matrix.

The orthogonal complement of a regular subspace $\mathcal{L}$ which is defined by

$$
\mathcal{L}^{\perp}=\left\{y \in \mathbb{R}^{m}: \sum_{i=1}^{m} x_{i} y_{i}=0 \text { for all } x \in \mathcal{L}\right\}
$$

is again regular.
Let $S$ be a subset of $[m]$. We define the deletion $\mathcal{L} \backslash S$ by

$$
\mathcal{L} \backslash S=\left\{\left(x_{i}\right)_{i \notin S}: x=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{L}, \operatorname{supp} x \cap S=\emptyset\right\}
$$

and the contraction $\mathcal{L} / S$ by

$$
\mathcal{L} / S=\left\{\left(x_{i}\right)_{i \notin S}: x=\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{L}\right\} .
$$

Both operations preserve regularity. We say that a subspace is a minor of $\mathcal{L}$ if it is obtained from $\mathcal{L}$ by a sequence of deletions and contractions.

Two main examples of regular subspaces come from directed graphs. Let $D=$ $(V, A)$ be an acyclic, directed graph with vertex set $V$ and arc set $A$. By $M(D) \in$ $\{-1,0,+1\}^{V \times A}$ we denote the vertex-arc incidence matrix of $D$ which is a totally unimodular matrix. Define the regular subspace $\mathcal{L}(D)$ as the kernel of $M$ :

$$
\mathcal{L}(D)=\left\{x \in \mathbb{R}^{A}: M(D) x=0\right\}
$$

The primitive chains of $\mathcal{L}(D)$ correspond to the simple circuits/cycles (forward and backward arcs are allowed) of the directed graph $D$. Regular subspaces which can be realized by this construction are called graphic. The primitive chains of the orthogonal complement $\mathcal{L}(D)^{\perp}$ correspond to the simple cuts/bonds (forward and backward arcs are allowed) of $D$. Such a regular subspace is called cographic. Minors of graphic (resp. cographic) subspaces are graphic (resp. cographic). The dimension of $\mathcal{L}(D)$ equals $|A|-|V|+k$ where $k$ is the number of connected components of the underlying undirected graph and the dimension of $\mathcal{L}(D)^{\perp}$ is $|V|-k$.

Tutte [35] gave a characterization of graphic and cographic subspaces in terms of forbidden minors. For this let $\mathrm{K}_{m}$ be the complete graph on $m$ vertices and let $\mathrm{K}_{m, n}$ the complete bipartite graph where one partition has $m$ vertices and the other one has $n$ vertices. Tutte showed that a regular subspace is graphic if and only if it contains neither $\mathcal{L}\left(\mathrm{K}_{5}\right)^{\perp}$ nor $\mathcal{L}\left(\mathrm{K}_{3,3}\right)^{\perp}$ as minors. Dually, a regular subspace is cographic if and only if it contains neither $\mathcal{L}\left(\mathrm{K}_{5}\right)$ nor $\mathcal{L}\left(\mathrm{K}_{3,3}\right)$ as minors. The central structure theorem about regular subspaces is Seymour's decomposition theorem [32]: One may construct every regular subspace as 1-, 2-, and 3-sums of regular subspaces starting from graphic, or cographic subspaces, or the special regular subspace called $\mathrm{R}_{10} \subseteq \mathbb{R}^{10}$; see also Truemper [34].
2.2. Geometry: Strict Voronoi vectors, Voronoi cells. A regular subspace $\mathcal{L}$ comes together with a regular lattice $L=\mathcal{L} \cap \mathbb{Z}^{m}$. One can show, see [37, Chapter 1.2], that in a regular lattice every vector $v \in L$ is a conformal sum of primitive chains $w_{1}, \ldots, w_{s} \in L$ :

$$
\begin{equation*}
v=w_{1}+\cdots+w_{s} \text { with }\left(w_{j}\right)_{i}\left(w_{k}\right)_{i} \geq 0 \text { for all } i \in[m] \text { and } j, k \in[s] . \tag{1}
\end{equation*}
$$

When $\mathcal{L} \subseteq \mathbb{R}^{m}$ is a graphic (cographic) subspace we call the associated lattice $L=\mathcal{L} \cap \mathbb{Z}^{m}$ graphic (cographic) as well. The graphic lattices are the lattices
of integral flows and the cographic lattices are the lattices of integral cuts in the framework of [4].

We equip the space $\mathbb{R}^{m}$ with an inner product which is defined by giving positive weights on the set $[m]$ : For a positive vector $g \in \mathbb{R}_{>0}^{m}$ define the inner product

$$
(x, y)_{g}=\sum_{i=1}^{m} g_{i} x_{i} y_{i}
$$

The standard basis vectors $e_{1}, \ldots, e_{m}$ form in this way an orthogonal basis which does not need to be orthonormal. A regular lattice $L$ with inner product $(\cdot, \cdot)_{g}$ is called a zonotopal lattice. As we explain below, this terminology refers to the fact that the Voronoi cell of a zonotopal lattice is a zonotope. The Voronoi cell of $L$ is

$$
\mathcal{V}(L)=\left\{x \in \mathcal{L}:(x, x)_{g} \leq(v-x, v-x)_{g} \text { for all } v \in L\right\}
$$

which is a centrally symmetric polytope. Lattice vectors $v \in L$ which determine a facet defining hyperplane

$$
H_{v}=\left\{x \in \mathcal{L}:(x, x)_{g}=(v-x, v-x)_{g}\right\}=\left\{x \in \mathcal{L}:(x, v)_{g}=\frac{1}{2}(v, v)_{g}\right\}
$$

of $\mathcal{V}(L)$ are called strict Voronoi vectors (sometimes also called relevant vectors). We denote the set of all strict Voronoi vectors by $\operatorname{Vor}(L)$.

Voronoi showed (see for example [7, Chapter 21, Theorem 10] or [9]), for arbitrary lattices $L$, that a nonzero vector $v \in L$ is a strict Voronoi vector if and only if $\pm v$ are the only shortest vectors in $v+2 L$.

In the following let $\mathcal{L} \subseteq \mathbb{R}^{m}$ be a regular subspace and let $L=\mathcal{L} \cap \mathbb{Z}^{m}$ be the corresponding regular lattice with positive vector $g \in \mathbb{R}_{>0}^{m}$. Essentially, the arguments given below can also be found in [13] in the special case of cographic lattices with constant $g$.

Applying Voronoi's characterization to $L$ yields:
Proposition 2.1. A lattice vector of $L$ is a primitive chain if and only if it is a strict Voronoi vector of $L$.

Proof. Let $v \in L$ be a primitive chain and let $u \in v+2 L$ be a lattice vector with $u \neq$ $\pm v$. We have $v-u \in 2 L \subseteq 2 \mathbb{Z}^{n}$ and $v_{i} \in\{-1,0,+1\}$, for all $i \in[m]$, which shows $\operatorname{supp} v \subseteq \operatorname{supp} u$. If $\operatorname{supp} v \neq \operatorname{supp} u$, then $(v, v)_{g}<(u, u)_{g}$. If $\operatorname{supp} v=\operatorname{supp} u$, then there exists a factor $\alpha \in \mathbb{Z} \backslash\{-1,+1\}$ so that $u=\alpha v$, hence $(v, v)_{g}<(u, u)_{g}$. In both cases $\pm v$ are the only shortest vectors in $v+2 L$. Hence, $v$ is a strict Voronoi vector.

Conversely, let $v \in L$ be a strict Voronoi vector. Write $v$ as a conformal sum of primitive chains as in (1). Set $u=v-2 w_{1}$. Then

$$
(u, u)_{g}=(v, v)_{g}-4\left(v-w_{1}, w_{1}\right)_{g} \leq(v, v)_{g}
$$

since $\left(v-w_{1}, w_{1}\right)_{g} \geq 0$ by (1). Hence, $\pm v$ is the unique shortest vector in the coset $v+2 L$ if and only if $s=1$.

The following special case of Farkas lemma is proved e.g. in [30, Theorem 22.6].
Lemma 2.2. Let $x \in \mathbb{R}^{m}$ be a vector, and let $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R} \cup\{ \pm \infty\}$. Exactly one of the following two alternatives holds:
(1) There exists a vector $y^{\prime}$ with $\left(y^{\prime}, z\right)_{g}=0$ for all $z \in \mathcal{L}$ so that

$$
y^{\prime} \in x+\prod_{i=1}^{m}\left[-\alpha_{i}, \alpha_{i}\right]
$$

(2) There exists a vector $y \in \mathcal{L}$ such that

$$
\text { for all } z \in x+\prod_{i=1}^{m}\left[-\alpha_{i}, \alpha_{i}\right] \text { we have }(y, z)_{g}>0
$$

If the second condition holds, then one can choose $y$ to be a primitive chain of $L$.
Theorem 2.3. Let $\pi_{g}: \mathbb{R}^{m} \rightarrow \mathcal{L}$ be the orthogonal projection of $\mathbb{R}^{m}$ onto $\mathcal{L}$. Then, $\mathcal{V}(L)=\pi_{g}\left([-1 / 2,1 / 2]^{m}\right)$.

Proof. For a vector $x \in[-1 / 2,1 / 2]^{m}$ inequality $(x, v)_{g} \leq \frac{1}{2}(v, v)_{g}$ holds for all $v \in \mathbb{Z}^{m} \backslash\{0\}$. Decompose $x$ orthogonally $x=y+y^{\prime}$ with $y=\pi_{g}(x) \in \mathcal{L}$. For all $v \in L \backslash\{0\}$ we have

$$
(y, v)_{g}=(x, v)_{g}-\left(y^{\prime}, v\right)_{g}=(x, v)_{g} \leq \frac{1}{2}(v, v)_{g}
$$

Thus, $\pi_{g}(x) \in \mathcal{V}(L)$.
Let $x \in \mathcal{V}(L)$ be a vector of the Voronoi cell. If there exists $y^{\prime}$ with $\left(y^{\prime}, z\right)_{g}=0$ for all $z \in \mathcal{L}$ so that $y^{\prime} \in-x+[-1 / 2,1 / 2]^{m}$, then $x+y^{\prime} \in[-1 / 2,1 / 2]^{m}$ and $\pi_{g}\left(x+y^{\prime}\right)=\pi_{g}(x)=x$. Suppose that such a vector $y^{\prime}$ does not exist. Then by Lemma 2.2 there is a primitive chain $v \in L$ so that

$$
\left(v,-x+[-1 / 2,1 / 2]^{m}\right)_{g}>0
$$

This implies $\left(v,-x-\frac{1}{2} v\right)_{g}>0$. Hence, $-x \notin \mathcal{V}(L)$; a contradiction because $\mathcal{V}(L)$ is centrally symmetric.

This theorem proves that the Voronoi cell of a zonotopal lattice is indeed a zonotope. The operations deleting or contracting correspond to contracting the corresponding zones or projecting along the corresponding zones of $\mathcal{V}(L)$, as mentioned in [5, Proposition 2.2.6]. Also the combinatorial structure of $\mathcal{V}(L)$, which is independent of $g$, is completely encoded in the covectors of the oriented matroid defined by $\mathcal{L}$, see [5, Proposition 2.2.2].

Conversely, Erdahl [16], see also [38], [40] for an alternative proof, showed that every zonotope which tiles spaces by translates is the affine linear image of the Voronoi cell of a zonotopal lattice.
2.3. Example: Lattices of Voronoi's first kind. McKilliam, Grant, and Clarkson [25] gave a polynomial time algorithm for solving the closest vector problem for lattices of Voronoi's first kind (with known obtuse superbasis, see below). Now we show that these lattices correspond to cographic lattices.

Following Conway and Sloane [8] we say that a lattice $L$ is of Voronoi's first kind if $L$ has an obtuse superbasis: These are $n+1$ vectors $b_{0}, b_{1}, \ldots, b_{n}$ so that the following three conditions hold:
(i) $b_{1}, \ldots, b_{n}$ is a basis of $L$,
(ii) $b_{0}+b_{1}+\cdots+b_{n}=0$,
(iii) $b_{i}^{\top} b_{j} \leq 0$ for $i, j=0, \ldots, n$ and $i \neq j$.

A classical theorem of Voronoi states that every lattice in dimensions 2 and 3 has an obtuse superbasis, see Conway and Sloane [8, Section 7]. However, starting from dimension $n=4$ on, not every lattice is of Voronoi's first kind.

In the setting of zonotopal lattices, lattices of Voronoi's first kind appear as cographic lattices: Let $L \subseteq \mathbb{R}^{n}$ be a lattice of Voronoi's first kind having an obtuse superbasis $b_{0}, \ldots, b_{n}$. Define the directed graph $D=(V, A)$ with vertex set $V=$ $\left\{b_{0}, \ldots, b_{n}\right\}$ where we draw an $\operatorname{arc} a_{i j}$ between vertices $b_{i}$ and $b_{j}$ whenever $b_{i}^{\top} b_{j}<0$ and $i<j$. We assign to the arc $a_{i j}$ the (positive) weight $g_{i j}=-b_{i}^{\top} b_{j}$.

The undirected graph which underlies $D$ is called Delone graph of $L$, see [8]. In fact, the choice of the directions of the arcs is arbitrary, as long as the graph does not contain a directed cycle.

Proposition 2.4. The cographic lattices are exactly the lattices of Voronoi's first kind.

Proof. The graph $D$ is weakly connected (i.e. the underlying undirected graph is connected) since $L$ has rank $n$ : For suppose not. Then one can partition the vertex set $V=V_{1} \cup V_{2}$ so that there is no arc between $V_{1}$ and $V_{2}$. Consider the spaces $U_{i}$ spanned by the vectors in $V_{i}$, with $i=1,2$. These spaces are orthogonal and we have $\operatorname{dim} U_{i}=\left|U_{i}\right|$ because of (i) and (ii). Hence, $\operatorname{dim}\left(U_{1}+U_{2}\right)=n+1$, contradicting that the rank of $L$ is $n$.

Consider the vertex-arc incidence matrix $M(D) \in\{-1,0,+1\}^{V \times A}$ of $D$ and let $v_{0}, \ldots, v_{n} \in \mathbb{R}^{A}$ be the row vectors of $M$. Their integral span coincides with the cographic lattice $L^{\prime}=\mathcal{L}(D)^{\perp} \cap \mathbb{Z}^{A}$. Furthermore, the vectors $v_{0}, \ldots, v_{n}$ form an obtuse superbasis of $L^{\prime}$ and $\left(v_{i}, v_{j}\right)_{g}=-g_{i j}=b_{i}^{\top} b_{j}$ holds when $v_{i}$ and $v_{j}$ are adjacent in $D$. Hence, the lattice $L$ which is of Voronoi's first kind is isometric to the cographic lattice $L^{\prime}$.

Clearly, this construction can be reversed. Starting from a vertex-arc incidence matrix of a weakly connected acyclic directed graph defining a cographic lattice one can get an obtuse superbasis of this lattice. If the graph defining the cographic lattice is not weakly connected, then one can make it weakly connected by identifying vertices of distinct connected components without changing the cographic lattice.

For instance, the root lattice

$$
\mathrm{A}_{n}=\left\{x \in \mathbb{Z}^{n+1}: \sum_{i=1}^{n+1} x_{i}=0\right\}
$$

is a lattice of Voronoi's first kind. Its Delone graph is the cycle graph $C_{n+1}$ of length $n+1$. The dual lattice $A_{n}^{*}$ is again a lattice of Voronoi's first kind. Its Delone graph is the complete graph $\mathrm{K}_{n+1}$ on $n+1$ vertices. The Voronoi cell of $\mathrm{A}_{n}^{*}$ is the $n$-dimensional permutahedron.
2.4. Example: Tensor product of root lattices of type A. Ducas and van Woerden [12] gave a polynomial time algorithm for solving the closest vector problem for tensor products of the form $\mathrm{A}_{m} \otimes \mathrm{~A}_{n}$. Now we show that these lattices correspond to the graphic lattices for the complete bipartite graph $K_{m+1, n+1}$.

Let $L_{1} \subseteq \mathbb{R}^{n_{1}}$ be a lattice of rank $r_{1}$ with basis $a_{1}, \ldots, a_{r_{1}}$ and let $L_{2} \subseteq \mathbb{R}^{n_{2}}$ be a lattice of rank $r_{2}$ with basis $b_{1}, \ldots, b_{r_{2}}$. Then their tensor product is the lattice $L_{1} \otimes L_{2} \subseteq \mathbb{R}^{n_{1} n_{2}}$ having basis $a_{i} \otimes b_{j}$ with $i=1, \ldots, r_{1}$ and $j=1, \ldots, r_{2}$.

Proposition 2.5. The tensor product lattice $\mathrm{A}_{m} \otimes \mathrm{~A}_{n}$ coincides with the graphic lattice of the complete bipartite graph $\mathrm{K}_{m+1, n+1}$.
Proof. Recall that the Delone graph of $\mathrm{A}_{m}$ is the cycle graph $C_{m+1}$. A basis of $\mathrm{A}_{m}$ is

$$
b_{1}=e_{1}-e_{2}, b_{2}=e_{2}-e_{3}, \ldots, b_{m}=e_{m}-e_{m+1}
$$

where $e_{1}, \ldots, e_{m+1}$ are the standard basis vectors of $\mathbb{R}^{m+1}$. A basis of $\mathrm{A}_{n}$ is $c_{j}=$ $f_{j}-f_{j+1}$, with $j \in[n]$ where $f_{1}, \ldots, f_{n+1}$ are the standard basis vectors of $\mathbb{R}^{n+1}$. This defines the following basis of $\mathrm{A}_{m} \otimes \mathrm{~A}_{n}$

$$
\begin{equation*}
b_{i} \otimes c_{j}=e_{i} \otimes f_{j}-e_{i+1} \otimes f_{j}+e_{i+1} \otimes f_{j+1}-e_{i} \otimes f_{j+1} \quad \text { for } i \in[m], j \in[n] \tag{2}
\end{equation*}
$$

If one orients all the $\operatorname{arcs} A$ of $\mathrm{K}_{m+1, n+1}$ from the left $m+1$ vertices to the right $n+1$ vertices then the basis (2) lies in the graphic lattice $\mathcal{L}\left(\mathrm{K}_{m+1, n+1}\right) \cap \mathbb{Z}^{A}$ as it corresponds to the cycle with edges $\left(e_{i}, f_{j}\right),\left(f_{j}, e_{i+1}\right),\left(e_{i+1}, f_{j+1}\right),\left(f_{j+1}, e_{i}\right)$. Since the dimension of the graphic space of the complete bipartite graph is

$$
\operatorname{dim} \mathcal{L}\left(\mathrm{K}_{m+1, n+1}\right)=(m+1)(n+1)-(m+1+n+1)+1=m n
$$

we see that the basis (2) also forms a basis of the graphic lattice.
Example 2.6. We give a basis of the graphic lattice $\mathrm{A}_{2} \otimes \mathrm{~A}_{1}$ corresponding to the complete bipartite graph $\mathrm{K}_{3,2}$.

$$
\begin{aligned}
b_{1} \otimes c_{1} & =e_{1} \otimes f_{1}-e_{2} \otimes f_{1}+e_{2} \otimes f_{2}-e_{1} \otimes f_{2} \\
b_{2} \otimes c_{1} & =e_{2} \otimes f_{1}-e_{3} \otimes f_{1}+e_{3} \otimes f_{2}-e_{2} \otimes f_{2}
\end{aligned}
$$



## 3. Minimum mean cycle canceling algorithm for CVP

In this section we show how to derive the following theorem from the results of Karzanov and McCormick [23]; see also [22] for the conference version.
Theorem 3.1. Let $M \in\{-1,0,+1\}^{n \times m}$ be a given totally unimodular matrix and let $g \in \mathbb{Q}_{>0}^{m}$ be a given positive, rational vector. By $\mathcal{L}$ we denote the kernel of $M$ which is a regular subspace. This defines the zonotopal lattice $L=\mathcal{L} \cap \mathbb{Z}^{m}$ with inner product $(\cdot, \cdot)_{g}$. Let $t \in \mathcal{L} \cap \mathbb{Q}^{m}$ be a given rational (target) vector. Then one can compute a lattice vector $u \in L$ with $\|u-t\|=\min _{v \in L}\|v-t\|$ in polynomial time.

For the proof define the separable convex function $w: \mathbb{R}^{m} \rightarrow \mathbb{R}$ by

$$
w_{i}\left(v_{i}\right)=g_{i}\left(v_{i}-t_{i}\right)^{2} \quad \text { so that } \quad w(v)=\sum_{i=1}^{m} g_{i}\left(v_{i}-t_{i}\right)^{2}=(v-t, v-t)_{g}
$$

Then solving the closest vector problem for $L$ given the target vector $t$ amounts to finding a minimizer for $w(v)$ among all lattice vectors $v \in L$. So we can apply the results of Karzanov and McCormick to solve the closest vector problem for zonotopal lattices.

The minimum mean cycle canceling method gives a polynomial time algorithm for solving the closest vector problem here. To see this we have to verify some technical conditions for $w$ which we will do now.

We describe how the minimum mean cycle canceling method works in our setting and discuss which arguments of the paper of Karzanov and McCormick have to be applied to prove that the algorithm runs in polynomial time.

We start by setting up notation. The (discrete) right derivative of $w_{i}$ is

$$
c_{i}^{+}\left(v_{i}\right)=w_{i}\left(v_{i}+1\right)-w_{i}\left(v_{i}\right)=g_{i}\left(v_{i}+1-t_{i}\right)^{2}-g_{i}\left(v_{i}-t_{i}\right)^{2}=g_{i}\left(2\left(v_{i}-t_{i}\right)+1\right) .
$$

Similarly the (discrete) left derivative of $w_{i}$ is

$$
c_{i}^{-}\left(v_{i}\right)=w_{i}\left(v_{i}\right)-w_{i}\left(v_{i}-1\right)=g_{i}\left(2\left(v_{i}-t_{i}\right)-1\right) .
$$

Technically we replace the quadratic objective function $w$ by its piecewise linear approximation at lattice points.

The cost of the strict Voronoi vector $u \in L$ at a lattice vector $v$ is

$$
c(v, u)=\sum_{i \in u^{+}} c_{i}^{+}\left(v_{i}\right)-\sum_{i \in u^{-}} c_{i}^{-}\left(v_{i}\right),
$$

where

$$
u^{+}=\left\{i \in \operatorname{supp} u: u_{i}=+1\right\} \quad \text { and } \quad u^{-}=\left\{i \in \operatorname{supp} u: u_{i}=-1\right\} .
$$

If the cost $c(v, u)$ is negative, then $v+u$ is closer to $t$ than $v$ because we have

$$
(v+u-t, v+u-t)_{g}=(v-t, v-t)_{g}+c(v, u)
$$

which is easily verified.
The mean cost of $u$ at $v$ is

$$
\bar{c}(v, u)=\frac{c(v, u)}{|\operatorname{supp} u|}
$$

A strict Voronoi vector $u$ is called a minimum mean strict Voronoi vector for $v$ if its mean cost $\bar{c}(v, u)$ is a small as possible. The following quantity is used to measure the progress of the algorithm:

$$
\lambda(v)=\max \left\{0,-\min _{u \in \operatorname{Vor}(L)} \bar{c}(v, u)\right\}
$$

where $\operatorname{Vor}(L)$ denotes the set of strict Voronoi vectors where we used Proposition 2.1. Now [23, Lemma 3.1] says that $v$ is a solution of the closest vector problem if and only if $\lambda(v)=0$. [23, Proof of Lemma 3.2] shows that the following linear program computes $-\lambda(v)$ :

$$
\begin{aligned}
& \min \sum_{i=1}^{m}\left(c_{i}^{+}\left(v_{i}\right) x_{i}^{+}-c_{i}^{-}\left(v_{i}\right) x_{i}^{-}\right) \\
& M\left(x^{+}-x^{-}\right)=0 \\
& e^{\top}\left(x^{+}+x^{-}\right)=1 \\
& x^{+}, x^{-} \in \mathbb{R}_{\geq 0}^{m}
\end{aligned}
$$

where $e=(1, \ldots, 1) \in \mathbb{R}^{m}$ is the all-ones vector. One can furthermore find a minimum mean strict Voronoi vector at $v$ by first determining $\lambda(v)$ and then finding a vector $u=x^{+}-x^{-}$with minimal support with $\lambda(v)=-\bar{c}(v, u)$. This can be done by solving at most $m$ auxiliary linear programs where one greedily probes to set coordinates to 0 , which is possible because every lattice vector is a conformal sum of primitive chains (1).

Now the minimum mean cycle canceling algorithm works as follows: We start at the origin $v=0$. As long as $\lambda(v)$ is positive, we improve $v$, moving it closer to the target vector $t$ by finding a minimum mean strict Voronoi vector $u$ at $v$ and
updating $v$ to $v+\varepsilon u$. The step size $\varepsilon$ is determined by [23, (16)] which is the minimum integer $\Delta$ so that

$$
\begin{equation*}
\Delta \in\left[\frac{\lambda(v)}{g_{i}}, \frac{\lambda(v)}{g_{i}}+1\right] \quad \text { for } i \in[m] \tag{3}
\end{equation*}
$$

Indeed, for instance if $u_{i}=+1$, then the bounds for $\Delta$ in [23, (16)] are

$$
c_{i}^{-}\left(v_{i}+\Delta\right) \leq c_{i}^{+}\left(v_{i}\right)+\lambda(v) \leq c_{i}^{+}\left(v_{i}+\Delta\right)
$$

yielding in our setting the interval in (3) which clearly always contains an integer.
Choosing the step size like this makes sure that $\lambda(v+\varepsilon u) \leq \lambda(v)$, see [23, Lemma 3.3]. By [23, Lemma 3.4] we see that after $m-n$ iterations the value of $\lambda$ decreases by a factor of at most $(1-1 /(2 m))$, so that we have a geometric decrease.

We start with $v=0$ and we assume that $\lambda(0)>0$. Let $u$ be a minimum mean strict Voronoi vector at 0 . If $g_{i}$ and $t_{i}$ are rational, it is immediate to see that the binary encoding length of $\lambda(0)$ is polynomial in the input size. If $g_{i}$ and $t_{i}$ are rational, then we can also derive a stopping criterion for the algorithm. Because of rationality, there exists an integer $K$ so that $K c(v, u)$ is an integer for all $v \in L$ and all strict Voronoi vectors $u$. The binary encoding length of $K$ is polynomial in the input size. If $\lambda(v)<\delta$ for $\delta<\frac{1}{K m}$ then $v$ is a closest vector to $t$ because from [23, Proof of Lemma 6.1] it follows that

$$
c(v, u)>-m \delta \geq-\frac{1}{K}
$$

and so $c(v, u)$ is nonnegative. The bound on $\lambda(0)$, the stopping criterion together with the geometric decrease of $\lambda$ show that only a polynomial number of iterations are needed to find a closest vector.

## 4. Further remarks and questions

After the first version of this paper was submitted to the arXiv repository, Rico Zenklusen pointed us to the paper [19] by Hochbaum and Shanthikumar. With their method it is possible, together with results of Artmann, Weismantel, and Zenklusen [3], to efficiently solve the closest vector problem for a larger class of lattices, for example for lattices of the form

$$
L=\left\{x \in \mathbb{Z}^{n}: M x=0\right\}
$$

where the matrix $M$ is totally bimodular, which means that every subdeterminant is bounded by 2 in absolute value. The combinatorics and the geometry of the Voronoi cells of these kind of lattices is currently unexplored.

Another very interesting open question is if detecting that a given lattice is isometric to a zonotopal lattice is computationally easy. To the best knowledge of the authors even for the class of cographical lattices this is not known.

## Acknowledgements

We thank Rico Zenklusen for his valuable remarks. We like to thank the reviewers for their constructive suggestions which helped to improve the presentation of the paper.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie agreement No 764759. The fourth named author is partially supported by the SFB/TRR 191
"Symplectic Structures in Geometry, Algebra and Dynamics" and by the project "Spectral bounds in extremal discrete geometry" (project number 414898050), both funded by the DFG.

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[^0]:    Date: June 18, 2021.
    2020 Mathematics Subject Classification. 68Q25, 52C07.
    Key words and phrases. closest vector problem, lattice, Voronoi cell, zonotope, totally unimodular matrix, minimum mean cycle canceling.

