UNIQUENESS, LIPSCHITZ STABILITY AND RECONSTRUCTION FOR THE INVERSE OPTICAL TOMOGRAPHY PROBLEM *

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Abstract. In this paper, we consider the inverse problem of recovering a diffusion σ and absorption coefficients q in steady-state optical tomography problem from the Neumann-to-Dirichlet map. We first prove a Global uniqueness and Lipschitz stability estimate for the absorption parameter provided that the diffusion σ is known and show how to quantify the Lipschitz stability constant for a given setting. Then, we prove a Lipschitz stability result for simultaneous recovery of σ and q. In both cases the parameters belong to a known finite subspace with a priori known bounds. The proofs rely on a monotonicity result combined with the techniques of localized potentials. To numerically solve the inverse problem, we propose a Kohn-Vogelius-type cost functional over a class of admissible parameters subject to two boundary value problems. The reformulation of the minimization problem via the Neumann-to-Dirichlet operator allows us to obtain the optimality conditions by using the Fréchet differentiability of this operator and its inverse. The reconstruction is then performed by means of an iterative algorithm based on a quasi-Newton method. Finally, we illustrate some numerical results.

Key words. Optical tomography, Inverse problem, Uniqueness, Lipschitz stability, Monotonicity, Localized potentials.

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1. Introduction. In this paper, we consider the inverse problem of recovering the parameters $\sigma(x)$ and $q(x)$ in the elliptic partial differential equation

$$-\nabla \cdot (\sigma \nabla u) + qu = 0 \text{ in } \Omega, \tag{1.1}$$

from the knowledge of all possible Cauchy data on the boundary $\partial\Omega$, $\sigma\partial_\nu u|_{\partial\Omega}, u|_{\partial\Omega}$. Problem (1.1) can be viewed as steady-state diffusion optical tomography, where light propagation is modeled by a diffusion approximation and the excitation frequency is set to zero. Here u represents the density of photons, σ the diffuse coefficient and q the optical absorption. This problem arises in medical imaging and in geophysics, for example, in reflection seismology assuming a description in terms of time-harmonic scalar waves. For a full description of optical tomography, we refer the reader to the topical reviews of Arridge [1] and Gibson, Hebden and Arridge [2].

Although it is common practice in optical tomography to use the Robin-to-Robin map to describe the boundary measurements (see [1, 3]), the Neumann-to-Dirichlet map will be employed here instead. This is justified by the fact that in optical tomography, prescribing the Neumnann to-Dirichlet map, is equivalent to prescribing the Robin-to-Robin boundary map as long as there are no additional unknown coefficients in the Robin conditions (see for instance [4]).
The paper is split into three parts. Part one is on proving uniqueness and Lipschitz stability of the the absorption coefficient q provided that the diffusion coefficient σ is known. Part two is on proving Lipschitz stability of σ and q simultaneously. Part three deals with the reconstruction of σ and q based on minimizing a Kohn-Vogelius type functional.

The inverse problem of recovering q from the knowledge of the Dirichlet-to-Neumann map was first introduced (in a slightly different setting) by Calderón in [5]. The uniqueness issue was treated by Sylvester and Uhlmann in [6]. For more recent

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result on uniqueness, we refer the reader to [7]. By virtue of the work of Alessandrini [8] it is known that both problems of recovering σ or q (in suitable regularity scales) enjoy logarithmic stability estimates under mild a priori assumptions on the data. As shown by Mandache [9], this log-type estimate is optimal. Thus for arbitrary potentials q, Lipschitz stability cannot hold. As discovered in [10], considering potentials or conductivities in certain finite-dimensional spaces provides improvements in terms of stability. Under certain assumptions, the authors prove Lipschitz stability estimates. Their argument relies on a combination of singular solutions and unique continuation estimates. This idea has been extended to more complex equations and systems (see for instance [11, 12, 13, 14, 15, 16]).

As a key novelty in this article, we present a different approach based on the monotonicity and the techniques of localized potentials instead of combining singular solutions with unique continuation results as previously done in the literature. Following analogous results in electrical impedance tomography and elasticity [17, 18, 19], here we will study the question whether the coefficient q can be uniquely and stably reconstructed. More precisely, we show that q is uniquely determined and depends upon the Neumaun-to-Dircihlet map of (1.1) in a Lipschitz way as long as $\operatorname{supp}(q) \Subset \Omega$ and σ is known. Moreover, we quantify the Lipschitz constant for a given setting by solving a finite number of well-posed PDEs which may be important to quantify the noise robustness in practical applications. To our best knowledge, this result of quantitative Lipschitz stability is new for the problem under consideration.

As mentioned in [20], the inverse problem of simultaneous reconstruction of σ and q is in general not uniquely solvable, i.e., it is not possible to uniquely determine both σ and q from boundary data of u provided that σ and q are smooth. The reason is that a diffusion coefficient can be transformed into an absorption coefficient by setting

$$v := \sqrt{\sigma} u,$$

which transforms equation (1.1) into

$$-\Delta v + cv = 0, \quad c = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{q}{\sigma}.$$

If $\sigma = 1$ in a neighborhood of $\partial\Omega$, then the boundary values remain unchanged. Hence, boundary measurements can only contain information about c, from which one cannot extract σ and q. Despite this negative theoretical result, a prominent result by Harrach [4] demonstrates that uniqueness holds for piecewise constant diffusion and piecewise analytic absorption coefficients. The author proves that under this condition both parameters are simultaneously uniquely determined by knowledge of all possible pairs of Neumann and Dirichlet boundary values $\sigma\partial_\nu|_S, u|_S$, of solutions u of (1.1), and S is a non-empty subset of $\partial\Omega$

In this paper, we go a step further and we prove a Lipschitz stability for the inverse problem of recovering q and σ simultaneously. The proof relies on a monotonicity estimates combined with the techniques of localized potentials. To the author's knowledge the Lipschitz stability presented in this work is the first result on simultaneous recovery for a class of real-valued diffusion and absorption coefficients.

The idea of using monotonicity and localized potentials method has lead to a several results for inverse coefficient problems; see for instance [21, 22, 23, 24, 25, 26, 27, 28]. Together with the recent results [29, 18, 17, 19], this work shows that this idea can also be used to prove Uniqueness and Lipschitz stability results for the inverse optical tomography problem.

Lipschitz stability estimates for inverse and ill-posed problems are usually based on constructive approaches involving Carleman estimates or quantitative estimates of unique continuation [30, 14, 31, 32, 33, 34, 35]. For some applications these constructive approaches also allowed to quantify the asymptotic behavior of the Lipschitz constant; see for instance [36].

Our approach on proving Lipschitz stability is relatively simple compared to previous works. The main tools are: standard (non quantitative) unique continuation, the monotonicity result and the method of localized potentials.

For the numerical solution, we reformulate the inverse problem into a minimization problem using a Kohn-Vogelius functional, and use a quasi-Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by BFGS scheme [37]. Let us stress that this numerical part approaches the problem from a heuristic numerical side to demonstrate that useful numerical reconstructions are indeed possible. It remains a challenging open task how to unite the theoretical and numerical approaches in order to find rigorously justified reconstruction methods that work well in practically relevant settings.

Let us recall that in [38, 39, 40, 41], the authors propose new algorithms for recovering optical material properties. These algorithmes are experimentally tested for two and three- dimensional cases. While these works, which address real-life three-dimensional problems are an important step towards practical applications, they still suffer from considerable cross-talk between absorption and scattering reconstructions. What we mean by cross-talk is that purely scattering (or purely absorbing) inclusions are often reconstructed with unphysical absorption (or scattering) properties. This behavior is well-understood from the theoretical viewpoint: Different optical distributions inside the medium can lead to the same measurements collected at the surface of the medium [20, 42]. To avoid such cross-talks for our numerical results, we have used a suitable regularization techniques for the proposed algorithm in order to better separate and estimate simultaneously the optical properties σ and q.

The paper is organized as follows. In section 2, we introduce the forward, the Neumann-to-Dirichlet operator and the inverse problem. Section 3 and 4 contain the main theoretical tools for this work. Section 3 is devoted to the reconstruction of the absorption coefficient assuming that the diffusion coefficient is known. We show a monotonicity relation and we prove a Runge approximation result. Then we deduce the existence of localized potentials and prove the global uniqueness and Lipschitz stability estimate and show how to calculate the Lipschitz stability constant for a given setting. Section 4 is concerned with the reconstruction of the diffusion and the absorption coefficients simultaneously. We first show a monotonicity result between the diffusion and absorption coefficients and the Neumann-to-Dirichlet operator and prove the existence of localized potentials. Then, we prove the Lipschitz stability estimate. In section 5, we introduce the minimization problem, and we compute the first order optimality condition. In section 6, satisfactory numerical results for two-dimensional problem are presented. The last section contains some concluding remarks.

2. Problem formulation. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$, be a bounded domain with smooth boundary $\partial\Omega$. For $\sigma, q \in L^\infty_+(\Omega)$, where L^∞_+ denotes the subset of L^∞-functions with positive essential infima, we consider the following problem with Neumann boundary data $g \in L^2(\partial\Omega)$:

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) + qu = 0 & \text{in } \Omega, \\ \sigma \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ν is the unit normal vector to $\partial\Omega$. The weak formulation of problem (2.1) reads

$$\int_\Omega \sigma \nabla u \cdot \nabla w \, dx + \int_\Omega quw \, dx = \int_{\partial\Omega} gw \, ds \text{ for all } w \in H^1(\Omega). \tag{2.2}$$

Using the Riesz representation theorem (or the Lax-Milgram-Theorem), it is easily seen that (2.2) is uniquely solvable and that the solution depends continuously on $g \in L^2(\partial\Omega)$ and $\sigma, q \in L^\infty_+(\Omega)$. Then, we can define the Neumann-to-Dirichlet operator (NtD):

$$\Lambda(\sigma, q) : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$$
$$g \longmapsto u_{|\partial\Omega},$$

The inverse problem we consider here, is the following:

(2.3) *Find the parameters σ, q from the knowledge of the map $\Lambda(\sigma, q)$.*

We will consider diffusion and absorption parameters that are a priori known to belong to a finite dimensional set of piecewise-analytic functions and that are bounded from above and below by a priori known constants. To that end, we first define piecewise-analyticity as in [17, Definition 2.1]

Definition 2.1. *(a) A Subset $\Gamma \subseteq \partial\Omega$ of the boundary of an open set $\Omega \subset \mathbb{R}^n$ is called a smooth boundary piece if it is a C^∞-surface and Ω lies on one side of it, i.e. if for each $z \in \Gamma$ there exists a ball $B_\epsilon(z)$ and function $\gamma \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R})$ such that*

$$\Gamma = \partial\Omega \cap B_\epsilon(z) = \{x \in B_\epsilon(z) : x_n = \gamma(x_1, \dots, x_{n-1})\},$$

$$\Omega \cap B_\epsilon(z) = \{x \in B_\epsilon(z) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

(b) Ω is said to have smooth boundary if $\partial\Omega$ is a union of smooth boundary pieces. Ω is said to have piecewise smooth boundary if $\partial\Omega$ is a countable union of the closures of smooth boundary pieces.

(c) A function $\varphi \in L^\infty(\Omega)$ is called piecewise constant if there exists finitely many pairwise disjoint subdomains $\Omega_1, \dots, \Omega_N \subset \Omega$ with piecewise smooth boundaries, such that $\overline{\Omega} = \overline{\Omega_1 \cup, \dots, \cup \Omega_N}$ and $\varphi|_{\Omega_i}$ is constant, $i = 1, \dots, N$.

(d) A function $\varphi \in L^\infty(\Omega)$ is called piecewise analytic if there exists finitely many pairwise disjoint subdomains $\Omega_1, \dots, \Omega_N \subset \Omega$ with piecewise smooth boundaries, such that $\overline{\Omega} = \overline{\Omega_1 \cup, \dots, \cup \Omega_N}$, and $\varphi|_{\Omega_i}$ has an extension which is (real-)analytic in a neighborhood of $\overline{\Omega_i}$, $i = 1, \dots, N$.

As mentioned in [17], it is not clear whether the sum of two piecewise-analytic functions is always piecewise-analytic, i.e. whether the set of piecewise-analytic functions is a vector space. However, this can be guaranteed with a slightly stronger definition of piecewise analyticity (see [43, lemma 1]). Therefore, we make the following definition.

DEFINITION 2.2. *A set $\mathcal{F} \subseteq L^\infty(\Omega)$ is called a finite-dimensional subset of piecewise-analytic functions if its linear span*

$$\text{span}\ \mathcal{F} = \left\{ \sum_{j=1}^{k} \lambda_j f_j : k \in \mathbb{N}, \lambda_j \in \mathbb{R}, f_j \in \mathcal{F} \right\} \subseteq L^\infty(\Omega,$$

contains only piecewise-analytic functions and dim(span $\mathcal{F}$) $< \infty$.

3. Recovery of the absorption coefficient. In this section, we assume that $\sigma = \sigma_0 \chi_{\Omega\setminus\omega} + \sigma_1 \chi_\omega$, and $q = q\chi_\omega$, where σ_0, σ_1 are positive constants and $\omega \Subset \Omega$. We aim to recover the absorption parameter $q \in L^\infty_+(\omega)$ from the NtD operator

$$\Lambda(q) : L^2(\partial\Omega) \to L^2(\partial\Omega) : g \mapsto u|_{\partial\Omega}.$$

provided that σ is known.

Given a finite-dimensional subset $\mathcal{F}$ of piecewise analytic functions and two constants $b > a > 0$, we denote the set

$$\mathcal{F}_{[a,b]} := \{q \in \mathcal{F} : \quad a \leq q(x) \leq b, \quad \text{for all } x \in \omega\}.$$

Throughout this paper, the domain ω, the finite-dimensional subset $\mathcal{F}$ and the bounds $b > a > 0$ are fixed, and the constants in the Lipschitz stability results will depend on them. Our first results show Uniqueness and Lipschitz stability for the inverse absorption problem in $\mathcal{F}_{[a,b]}$, when the complete infinite-dimensional NtD-operator is measured.

The outline of this section is the following

(i) In Subsection 3.1, we prove a runge approximation result and we deduce a global uniqueness for determining q from $\Lambda(q)$.

(ii) In Subsection 3.2, we show a monotonicity and localized potentials results and we deduce a Lipschitz stability estimate for determining q from $\Lambda(q)$.

(iii) In Subsection 3.3, we show how to quantify the Lipschitz constant.

3.1. Runge approximation and uniqueness.. We first note the following unique continuation property. For every open connected subset $\mathcal{O} \subset \Omega$, only the trivial solution of

$$-\text{div}(\sigma\nabla u) + qu = 0 \text{ in } \mathcal{O},$$

vanishes on an open subset of $\mathcal{O}$ or possesses zero Cauchy data on a smooth, open part of $\partial\mathcal{O}$. When σ is Lipschitz and q is bounded, this property is proven in Miranda [44, Thm. 19, II]. It can be extended to the case of piecewise analytic σ and q by sequentially solving Cauchy problems (see [45]).

We will deduce the uniqueness theorem 3.2 from the following Runge approximation result.

THEOREM 3.1 (Runge approximation). *Let $q \in L^\infty_+(\omega)$ be piecewise analytic. For all $f \in L^2(\omega)$ there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^2(\partial\Omega)$ such that the corresponding solutions $u^{(g_n)}$ of (2.1) with boundary data g_n, $n \in \mathbb{N}$, fulfill*

$$u^{(g_n)}|_\omega \to f \quad \textit{in } L^2(\omega).$$

Proof. We introduce the operator

$$A : L^2(\omega) \to L^2(\partial\Omega), \quad f \mapsto Af := v_{|\partial\Omega},$$

where $v \in H^1(\Omega)$ solves

$$\int_\Omega \sigma \nabla v \cdot \nabla w \, dx + \int_\omega q v w \, dx = \int_\omega f w \, dx \quad \text{for all } w \in H^1(\Omega). \tag{3.1}$$

Let $g \in L^2(\partial\Omega)$ and $u \in H^1(\Omega)$ be the corresponding solution of problem (2.1). Then the adjoint operator of A is characterized by

$$\begin{aligned} \int_\omega (A^* g)\, f \, dx = \int_{\partial\Omega} (Af)\, g \, ds = \int_{\partial\Omega} v g \, ds = \int_\Omega \sigma \nabla u \cdot \nabla v \, dx + \int_\omega q u v \, dx \\ = \int_\omega f u \, dx, \quad \text{for all } f \in L^2(\omega), \end{aligned} \tag{3.2}$$

which shows that $A^* : \; L^2(\partial\Omega) \to L^2(\omega)$ fulfills $A^* g = u|_\omega$. The assertion follows if we can show that A^* has dense range, which is equivalent to A being injective.

To prove this, let $v|_{\partial\Omega} = Af = 0$ with $v \in H^1(\Omega)$ solving (3.1). Since (3.1) also implies that $\sigma \partial_\nu v|_{\partial\Omega} = 0$, and $\Omega \setminus \omega$ is connected, it follows by unique continuation that $v|_{\Omega\setminus\omega} = 0$ and thus $v^+|_{\partial\omega} = 0$. Since $v \in H^1(\Omega)$ this also implies that $v^-|_{\partial\omega} = 0$, and together with (3.1) we obtain that $v|_\omega \in H^1(\omega)$ solves

$$-\nabla \cdot (\sigma \nabla v) + q v = 0 \quad \text{in } \omega,$$

with homogeneous Dirichlet boundary data $v|_{\partial\omega} = 0$. Hence, $v|_\omega = 0$, so that $v = 0$ almost everywhere in Ω. From (3.1) it then follows that $\int_\omega f w \, dx = 0$ for all $w \in H^1(\Omega)$ and thus $f = 0$. ◻

Theorem 3.2 (Global uniqueness). *For $q_1, q_2 \in L^\infty_+(\omega)$ that are piecewise analytic,*

$$\Lambda(q_1) = \Lambda(q_2) \quad \textit{if and only if} \quad q_1 = q_2.$$

Proof. For absorption parameters $q_1, q_2 \in L^\infty_+(\omega)$ and Neumann data $g, h \in L^2(\partial\Omega)$ we denote the corresponding solutions of (2.1) by u_1^g, u_1^h, u_2^g, and u_2^h respectively. The variational formulation (2.2) yields the orthogonality relation

$$\begin{aligned} &\int_{\partial\Omega} h \left(\Lambda(q_2) - \Lambda(q_1)\right) g \, ds \\ &= \int_{\partial\Omega} h \Lambda(q_2) g \, ds - \int_{\partial\Omega} g \Lambda(q_1) h \, ds = \int_{\partial\Omega} h u_2^g \, ds - \int_{\partial\Omega} g u_1^h \, ds \\ &= \int_\Omega \sigma \nabla u_1^h \cdot \nabla u_2^g \, dx + \int_\omega q_1 u_1^h u_2^g \, dx - \left(\int_\Omega \sigma \nabla u_2^g \cdot \nabla u_1^h \, dx + \int_\omega q_2 u_2^g u_1^h \, dx \right) \\ &= \int_\omega (q_1 - q_2) u_1^h u_2^g \, dx. \end{aligned}$$

This shows that $\Lambda(q_1) = \Lambda(q_2)$ implies that

$$\int_\omega (q_1 - q_2) u_1^h u_2^g \, dx = 0, \quad \text{for all } g, h \in L^2(\partial\Omega).$$

Using the Runge approximation result in theorem 3.1, this yields that $(q_1 - q_2) u_1^h = 0$ (a.e.) in ω for all $h \in L^2(\partial\Omega)$, and using theorem 3.1 again, this implies $q_1 = q_2$. ◻

3.2. Monotonicity, localized potentials and Lipschitz stability. To prove the Lipschitz stability result in Theorem 4.3, we first show a monotonicity estimate between the absorption coefficient and the Neumann-to-Dirichlet operator, and deduce the existence of localized potentials from the Runge approximation result.

LEMMA 3.3 (Monotonicity estimate). *Let $q_1, q_2 \in L^\infty_+(\omega)$ be two absorption parameters, let $g \in L^2(\partial\Omega)$ be an applied boundary current, and let $u_2 := u^g_{q_2} \in H^1(\Omega)$ solve (2.1) for the boundary current g and the absorption parameter q_2. Then*

$$\int_\omega (q_1 - q_2) u_2^2 \, dx \geq \int_{\partial\Omega} g \left(\Lambda(q_2) - \Lambda(q_1)\right) g \, ds \geq \int_\omega \left(q_2 - \frac{q_2^2}{q_1}\right) u_2^2 \, dx. \tag{3.3}$$

Proof. Let $u_1 := u^g_{q_1} \in H^1(\Omega)$. From the variational equation, we deduce

$$\int_\Omega \sigma \nabla u_1 \cdot \nabla u_2 \, dx + \int_\omega q_1 u_1 u_2 \, dx = \int_{\partial\Omega} g\Lambda(q_2) g \, ds = \int_\Omega \sigma |\nabla u_2|^2 \, dx + \int_\omega q_2 u_2^2 \, dx.$$

Thus

$$\begin{aligned} &\int_\Omega \sigma |\nabla (u_1 - u_2)|^2 \, dx + \int_\omega q_1 (u_1 - u_2)^2 \, dx \\ &\qquad = \int_\Omega \sigma |\nabla u_1|^2 \, dx + \int_\omega q_1 u_1^2 \, dx + \int_\Omega \sigma |\nabla u_2|^2 \, dx + \int_\omega q_1 u_2^2 \, dx \\ &\qquad - 2\int_\Omega \sigma |\nabla u_2|^2 \, dx - 2\int_\omega q_2 u_2^2 \, dx \\ &\qquad = \int_{\partial\Omega} g\Lambda(q_1) g \, ds - \int_{\partial\Omega} g\Lambda(q_2) g \, ds + \int_\omega (q_1 - q_2) u_2^2 \, dx. \end{aligned}$$

Since the left-hand side is nonnegative, the first asserted inequality follows. Interchanging q_1 and q_2, we get

$$\begin{aligned} &\int_{\partial\Omega} g\Lambda(q_2) g \, ds - \int_{\partial\Omega} g\Lambda(q_1) g \, ds \\ &= \int_\Omega \sigma |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega q_2 (u_2 - u_1)^2 \, dx - \int_\omega (q_2 - q_1) u_1^2 \, dx \\ &= \int_\Omega \sigma |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega \left(q_2 u_2^2 - 2 q_2 u_1 u_2 + q_1 u_1^2\right) \, dx \\ &= \int_\Omega \sigma |\nabla (u_2 - u_1)|^2 \, dx + \int_\omega q_1 \left(u_1 - \frac{q_2}{q_1} u_2\right)^2 \, ds + \int_\omega \left(q_2 - \frac{q_2^2}{q_1}\right) u_2^2 \, dx. \end{aligned}$$

Since the first two integrals on the right-hand side are non negative, the second asserted inequality follows. ∎

Note that we call Lemma 3.3 a monotonicity estimate because of the following corollary:

COROLLARY 3.4 (Monotonicity). *For two absorption parameters $q_1, q_2 \in L^\infty_+(\omega)$*

$$q_1 \leq q_2 \quad \text{implies} \quad \Lambda(q_1) \geq \Lambda(q_2) \quad \text{in the sense of quadratic forms.} \tag{3.4}$$

Let us stress, however, that Lemma 3.3 holds for any $q_1, q_2 \in L^\infty_+(\omega)$ and does not require $q_1 \leq q_2$ or $q_1 \geq q_2$.

The existence of localized potentials follows from the Runge approximation property as in [18, Lemma 4.3].

LEMMA 3.5 (Localized potentials). *Let $q \in L^\infty_+(\omega)$ be piecewise analytic, and let $\mathcal{O} \subseteq \omega$ be a subset with positive boundary measure. Then there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^2(\partial\Omega)$ such that the corresponding solutions $u^{(g_n)}$ of (2.1) fulfill*

$$\lim_{n\to\infty} \int_{\mathcal{O}} |u^{(g_n)}|^2 \, ds = \infty \quad \text{and} \quad \lim_{n\to\infty} \int_{\omega\setminus\mathcal{O}} |u^{(g_n)}|^2 \, ds = 0.$$

Proof. Using the Runge approximation property in Theorem 3.1, we find a sequence $\tilde{g}_n \in L^2(\partial\Omega)$ so that the corresponding solutions $u^{(\tilde{g}_n)}$ fulfill

$$u^{(\tilde{g}_n)}|_\omega \to \frac{\chi_{\mathcal{O}}}{\left(\int_{\mathcal{O}} dx\right)^{1/2}} \quad \text{in } L^2(\omega).$$

Hence

$$\lim_{n\to\infty} \int_{\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{\omega\setminus\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx = 0,$$

so that

$$g_n := \frac{\tilde{g}_n}{\left(\int_{\omega\setminus\mathcal{O}} \tilde{u}_n^2 \, dx\right)^{1/4}},$$

has the desired property

$$\lim_{n\to\infty} \int_{\mathcal{O}} |u^{(g_n)}|^2 \, dx = \lim_{n\to\infty} \frac{\int_{\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx}{\left(\int_{\omega\setminus\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx\right)^{1/2}} = \infty,$$

$$\lim_{n\to\infty} \int_{\omega\setminus\mathcal{O}} |u^{(g_n)}|^2 \, dx = \lim_{n\to\infty} \left(\int_{\omega\setminus\mathcal{O}} |u^{(\tilde{g}_n)}|^2 \, dx\right)^{1/2} = 0.$$

□

THEOREM 3.6 (Lipschitz stability). *There exists a constant $C > 0$ such that*

$$\|q_1 - q_2\|_{L^\infty(\omega)} \leq C\|\Lambda(q_1) - \Lambda(q_2)\|_{\mathcal{L}(L^2(\partial\Omega))}, \quad \textit{for all } q_1, q_2 \in \mathcal{F}_{[a,b]}.$$

Proof. Let $\mathcal{F} \subset L^\infty(\omega)$ be a finite dimensional subspace of piecewise analytic functions, $b > a > 0$, and

$$q_1, q_2 \in \mathcal{F}_{[a,b]} = \{q \in \mathcal{F} : \quad a \leq q(x) \leq b \text{ for all } x \in \omega\}.$$

For the ease of notation, we write in the following

$$\|q_1 - q_2\| := \|q_1 - q_2\|_{L^\infty(\Omega)} \quad \text{and} \quad \|g\| := \|g\|_{L^2(\partial\Omega)}.$$

Since $\Lambda(q_1)$ and $\Lambda(q_2)$ are self-adjoint, we have that

$$\begin{aligned}
&\|\Lambda(q_2) - \Lambda(q_1)\|_* \\
&= \sup_{\|g\|=1} \left| \int_{\partial\Omega} g\,(\Lambda(q_2) - \Lambda(q_1))\, g \, ds \right| \\
&= \sup_{\|g\|=1} \max\left\{ \int_{\partial\Omega} g\,(\Lambda(q_2) - \Lambda(q_1))\, g \, ds, \int_{\partial\Omega} g\,(\Lambda(q_1) - \Lambda(q_2))\, g \, ds \right\}.
\end{aligned}$$

Using the first inequality in the monotonicity relation (3.3) in Lemma 3.3 in its original form, and with q_1 and q_2 interchanged, we obtain for all $g \in L^2(\partial\Omega)$

$$\int_{\partial\Omega} g\,(\Lambda(q_2) - \Lambda(q_1))\,g\,ds \geq \int_\omega (q_1 - q_2)|u_{q_1}^{(g)}|^2\,dx,$$
$$\int_{\partial\Omega} g\,(\Lambda(q_1) - \Lambda(q_2))\,g\,ds \geq \int_\omega (q_2 - q_1)|u_{q_2}^{(g)}|^2\,dx,$$

where $u_{q_1}^{(g)}, u_{q_2}^{(g)} \in H^1(\Omega)$ denote the solutions of (2.1) with Neumann data g and absorption parameter q_1 and q_2, resp. Hence, for $q_1 \neq q_2$, we have

$$\frac{\|\Lambda(q_2) - \Lambda(q_1)\|_*}{\|q_1 - q_2\|} \geq \sup_{\|g\|=1} \phi\left(g, \frac{q_1 - q_2}{\|q_1 - q_2\|_{L^\infty(\omega)}}, q_1, q_2\right),$$

where (for $g \in L^2(\partial\Omega)$, $\zeta \in \mathcal{F}$, and $\kappa_1, \kappa_2 \in \mathcal{F}_{[a,b]}$)

$$\phi\,(g, \zeta, \kappa_1, \kappa_2) := \max\left\{\int_\omega \zeta|u_{\kappa_1}^{(g)}|^2\,dx, \int_\omega (-\zeta)|u_{\kappa_2}^{(g)}|^2\,dx\right\}. \tag{3.5}$$

Introduce the compact set

$$\mathcal{C} = \left\{\zeta \in \ \text{span}\,\mathcal{F} : \quad \|\zeta\|_{L^\infty(\omega)} = 1\right\}. \tag{3.6}$$

Then, we have

$$\begin{aligned}\frac{\|\Lambda(q_2) - \Lambda(q_1)\|_*}{\|q_1 - q_2\|} &\geq \sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2)\\ &\geq \inf_{\substack{\zeta\in\mathcal{C}\\ \kappa_1,\kappa_2\in\mathcal{F}_{[a,b]}}} \sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2).\end{aligned} \tag{3.7}$$

The assertion of Theorem 3.6 follows if we can show that the right hand side of (3.7) is positive. Since ϕ is continuous, the function

$$(\zeta, \kappa_1, \kappa_2) \mapsto \sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2)$$

is semi-lower continuous, so that it attains its minimum on the compact set $\mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}$. Hence, to prove Theorem 3.6, it suffices to show that

$$\sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2) > 0 \quad \text{ for all } (\zeta, \kappa_1, \kappa_2) \in \mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}.$$

To show this, let $(\zeta, \kappa_1, \kappa_2) \in \mathcal{C} \times \mathcal{F}_{[a,b]} \times \mathcal{F}_{[a,b]}$. Since $\|\zeta\|_{L^\infty(\omega)} = 1$, there exists a subset $\mathcal{O} \subseteq \omega$ with positive measure and $0 < \Theta < 1$ such that either

$$\text{(a) } \zeta(x) \geq \Theta \text{ for all } x \in \mathcal{O}, \quad \text{ or } \quad \text{(b) } -\zeta(x) \geq \Theta \text{ for all } x \in \mathcal{O}.$$

In case (a), we use the localized potentials sequence in Lemma 3.5, to obtain a boundary current $\hat{g} \in L^2(\partial\Omega)$ with

$$\int_{\mathcal{O}} \left|u_{\kappa_1}^{(\hat{g})}\right|^2\,dx \geq \frac{1}{\Theta} \quad \text{ and } \quad \int_{\omega\setminus\mathcal{O}} \left|u_{\kappa_1}^{(\hat{g})}\right|^2\,dx \leq \frac{1}{2},$$

so that (using again $\|\zeta\|_{L^\infty(\omega)} = 1$)

$$\phi(\hat{g}, \zeta, \kappa_1, \kappa_2) \geq \int_\omega \zeta \left| u_{\kappa_1}^{(\hat{g})} \right|^2 dx \geq \Theta \int_{\mathcal{O}} \left| u_{\kappa_1}^{(\hat{g})} \right|^2 dx - \int_{\omega \setminus \mathcal{O}} \left| u_{\kappa_1}^{(\hat{g})} \right|^2 dx \geq \frac{1}{2}.$$

In case (b), we can analogously use a localized potentials sequence for κ_2, and find $\hat{g} \in L^2(\partial\Omega)$ with

$$\phi(\hat{g}, \zeta, \kappa_1, \kappa_2) \geq \int_\omega (-\zeta) \left| u_{\kappa_2}^{(\hat{g})} \right|^2 dx \geq \Theta \int_{\mathcal{O}} \left| u_{\kappa_2}^{(\hat{g})} \right|^2 dx - \int_{\omega \setminus \mathcal{O}} \left| u_{\kappa_2}^{(\hat{g})} \right|^2 dx \geq \frac{1}{2}.$$

Hence, in both cases,

$$\sup_{\|g\|=1} \phi(g, \zeta, \kappa_1, \kappa_2) \geq \phi\left(\frac{\hat{g}}{\|\hat{g}\|}, \zeta, \kappa_1, \kappa_2 \right) = \frac{1}{\|\hat{g}\|^2} \phi(\hat{g}, \zeta, \kappa_1, \kappa_2) > 0,$$

so that Theorem 3.6 is proven. ∎

3.3. Quantitative Lipschitz stability. In this subsection, we restrict ourself to the case where $\mathcal{F}$ is a set of piecewise constant functions on a given partition $\cup_{j=1}^N D_j = \omega$, i.e,

$$\mathcal{F} = \left\{ q(x) = \sum_{j=1}^N q_j \chi_{D_j}, \quad q_1, \ldots q_N \in \mathbb{R} \right\} \subset L^\infty(\omega),$$

and for $0 < a < b$, $\mathcal{F}_{[a,b]}$ is the set of $q \in \mathcal{F}$ such that $a \leq q_j \leq b$ for all $j = 1, \ldots, N$. The structure assumed for q fits well in several problems arising in practical applications.
For our quantitative Lipschitz stability estimate, we need a finite numbers of localized potentials and we show how to reconstruct them.

Lemma 3.7. *Let $b > a > 0$ be given constants. For $j = 1, \ldots, N$ anf $k = 1, \ldots, K$, with $K = \left(\lfloor 3 \left(\frac{b}{a} - 1 \right) \rfloor + 3 \right)$, we define the piecewise constant function $\eta^{(j,k)} \in L^\infty_+(\omega)$ by*

$$\eta^{(j,k)}(x) = \begin{cases} (k+4)\dfrac{a}{3} & \text{if } x \in D_j, \\ \dfrac{a}{3} & \text{if } x \in \omega \setminus D_j. \end{cases}$$

(i) There exist boundary data $g^{(j,k)} \in L^2(\partial\Omega)$, so that the corresponding solutions $u_{\eta^{(j,k)}}^{g^{(j,k)}} \in H^1(\Omega)$ of (2.1) with $g = g^{(j,k)}$ and $q = \eta^{(j,k)}$ fulfill

$$\beta^{(i,k)} := \frac{1}{2} \int_{D_j} |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 dx - \left(\frac{3b}{2a} - \frac{1}{2} \right) \int_{\omega \setminus D_j} |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 dx > 1. \tag{3.8}$$

(ii) For arbitrary $q \in \mathcal{F}_{[a,b]}$, the solutions $u_q^{g^{(j,k)}} \in H^1(\Omega)$ of (2.1) with $g = g^{(j,k)}$ fulfill

$$\int_{D_j} |u_q^{g^{(j,k)}}|^2 dx - \int_{\omega \setminus D_j} |u_q^{g^{(j,k)}}|^2 dx \geq \beta^{(j,k)} > 1.$$

(iii) $g^{(j,k)}$ can be computed by solving a finite number of well-posed PDEs.

Proof. (i) follows immediately from the localized potentials result in Lemma 3.5. To prove (b), we need the following monotonicity result which follows from Lemma 3.3 with $q_2 = q + \delta$ and $q_1 = q$ and from using the same inequality again with interchanged roles of q_1 and q_2. For $g \in L^2(\partial\Omega)$, $q \in L^\infty_+(\omega)$, and $\delta \in L^\infty_+(\omega)$ such that $(q+\delta) \in L^\infty_+(\omega)$, we have

$$\int_\omega \delta u_q^g \, dx \geq \int_\omega \delta u_{q+\delta}^g \, dx. \tag{3.9}$$

Let $j = 1, \ldots, N$ and $q \in \mathcal{F}_{[a,b]}$. Since K fullfils $b < (K+3)\frac{a}{3}$, there exists $k \in \{1, \ldots, K\}$ such that $q_j = q|_{D_j}$ fullfils

$$(k+2)\frac{a}{3} \leq q_j < (k+3)\frac{a}{3}.$$

Using the monotonicity-based inequality (3.9), with

$$\frac{a}{3} \leq (k+4)\frac{a}{3} - q_j < \frac{2a}{3} \quad \text{ and } \quad -b + \frac{a}{3} \leq \frac{a}{3} - q_j < -\frac{2a}{3},$$

we obtain

$$\begin{aligned}
&\int_{D_j} |u_q^{g^{(j,k)}}|^2 \, dx - \int_{\omega \setminus D_j} |u_q^{g^{(j,k)}}|^2 \, dx \\
&= \frac{3}{2a}\left(\int_{D_j} \frac{2a}{3} |u_q^{g^{(j,k)}}|^2 \, dx - \frac{2a}{3} \int_{\omega \setminus D_j} |u_q^{g^{(j,k)}}|^2 \, dx \right) \\
&\geq \frac{3}{2a}\left(\int_{D_j} \left((k+4)\frac{a}{3} - q_j \right) |u_q^{g^{(j,k)}}|^2 \, dx + \int_{\omega \setminus D_j} \left(\frac{a}{3} - q_j \right) |u_q^{g^{(j,k)}}|^2 \, dx \right) \\
&= \frac{3}{2a}\left(\int_{D_j} \left(\eta^{(j,k)} - q_j \right) |u_q^{g^{(j,k)}}|^2 \, dx + \int_{\omega \setminus D_j} \left(\eta^{(j,k)} - q_j \right) |u_q^{g^{(j,k)}}|^2 \, dx \right) \\
&\geq \frac{3}{2a}\left(\int_{D_j} \left(\eta^{(j,k)} - q_j \right) |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx + \int_{\omega \setminus D_j} \left(\eta^{(j,k)} - q_j \right) |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx \right) \\
&\geq \frac{3}{2a}\left(\int_{D_j} \frac{a}{3} |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx - \int_{\omega \setminus D_j} \left(b - \frac{a}{3} \right) |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx \right) \\
&= \frac{1}{2} \int_{D_j} |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx - \left(\frac{3b}{2a} - \frac{1}{2} \right) \int_{\omega \setminus D_j} |u_{\eta^{(j,k)}}^{g^{(j,k)}}|^2 \, dx = \beta^{(j,k)} > 1,
\end{aligned}$$

and (ii) is proved. To prove (iii) we use a similar approach as in the construction of localized potentials in [18]. For $j = 1, \ldots, N$ and $k = 1, \ldots, K$, we introduce the operator A as in Theorem 3.1

$$A : L^2(\omega) \to L^2(\partial\Omega), \quad f \mapsto Af := v_{|\partial\Omega},$$

where $v \in H^1(\Omega)$ solves

$$\int_\Omega \sigma \nabla v \cdot \nabla w \, dx + \int_\omega \eta^{(j,k)} v w \, dx = \int_\omega f w \, dx \quad \text{ for all } w \in H^1(\Omega).$$

We have shown that the adjoint operator A^* of A is given by

$$A^* : L^2(\partial\Omega) \to L^2(\omega) : g \mapsto u|_\omega,$$

where u is the solution of (2.1) with $q = \eta^{(j,k)}$, and that A^* has dense range. Consider the linear ill-posed equation

$$A^* g = 3\chi_{D_j}.$$

Sine $3\chi_{D_j} \in \overline{\mathcal{R}(A^*)}$, the conjugate gradient method [46, III.15], yields a sequence of iterates $(g_n)_{n\in\mathbb{N}} \subset L^2(\partial\Omega)$ for which

$$A^* g_n \to 3\chi_{D_j}.$$

Therefore, the solutions u_n of (2.1) with $q = \eta^{(j,k)}$ and $g = g_n$ fulfill

$$\frac{1}{2}\int_{D_j} |u_n|^2\,dx - \left(\frac{3b}{2a} - \frac{1}{2}\right)\int_{\omega\setminus D_j} |u_n|^2\,dx \to \frac{3}{2},$$

so that after finitely many iteration steps, (3.8) is fulfilled. ∎

Now, we state the main result of this subsection.

THEOREM 3.8 (Quantitative Lipschitz stability). *Let $g^{(j,k)} \in L^2(\Omega)$ defined as in Lemma 3.7. Set*

$$L = \left(\max\left\{\|g^{(j,k)}\|^2_{L^2(\partial\Omega)}, \quad j = 1,\dots,N, k = 1,\dots,K\right\}\right)^{-1}.$$

Then

$$\|q_1 - q_2\|_\infty \le L\|\Lambda(q_1) - \Lambda(q_2)\|_\infty \quad \textit{for all} \quad q_1, q_2 \in \mathcal{F}_{[a,b]}. \tag{3.10}$$

Proof. From Lemma 3.7, we have for all $q \in \mathcal{F}_{[a,b]}$, and for all $j \in \{1,\dots,N\}$

$$\begin{aligned}&\sup_{\|g\|=1}\left(\int_{D_j} |u_q^g|^2\,dx - \int_{\omega\setminus D_j} |u_q^g|^2\,dx\right)\\ &= \sup_{0\neq g\in L^2(\partial\Omega)} \frac{1}{\|g\|^2}\left(\int_{D_j} |u_q^g|^2\,dx - \int_{\omega\setminus D_j} |u_q^g|^2\,dx\right) \ge L.\end{aligned} \tag{3.11}$$

To prove Theorem 3.8, it suffices to show that

$$\sup_{\|g\|=1} \phi(g,\zeta,\kappa_1,\kappa_2) \ge L \quad \text{for all } (\zeta,\kappa_1,\kappa_2) \in \mathcal{C}\times\mathcal{F}_{[a,b]}\times\mathcal{F}_{[a,b]}, \tag{3.12}$$

where ϕ and $\mathcal{C}$ defined in (3.5) and (3.6). Since $\mathcal{F}$ contains only piecewise-constant functions, for every $\zeta \in \mathcal{C}$ there must exist a subset $D_j \subset \omega$ with either

$$\zeta|_{Dj} = 1, \quad \text{or} \quad \zeta|_{Dj} = -1,$$

Hence using (3.5) and (3.11), we obtain for the case $\zeta|_{Dj} = 1$,

$$\begin{aligned}\sup_{\|g\|=1} \phi(g,\zeta,\kappa_1,\kappa_2) &\ge \sup_{\|g\|=1}\int_\omega \zeta|u^g_{\kappa_1}|^2\,dx\\ &\ge \sup_{\|g\|=1}\left(\int_{D_j}|u^g_{\kappa_1}|^2\,dx - \int_{\omega\setminus D_j}|u^g_{\kappa_1}|^2\,dx\right) \ge L,\end{aligned}$$

and for the case $\zeta|_{Dj} = -1$,

$$\sup_{\|g\|=1} \phi(g,\zeta,\kappa_2,\kappa_2) \geq \sup_{\|g\|=1} \int_\omega (-\zeta)|u^g_{\kappa_2}|^2\,dx$$
$$\geq \sup_{\|g\|=1} \left(\int_{D_j} |u^g_{\kappa_2}|^2\,dx - \int_{\omega\setminus D_j} |u^g_{\kappa_2}|^2\,dx \right) \geq L.$$

so that (3.12) is proved and the proof is completed. □

4. Simultaneous recovery of diffusion and absorption. The inverse problem of recovering σ and q simultaneously is known to be an ill-posed problem and stability results can only be obtained under a-priori assumptions.

For our problem, we will prove a stability result under the assumption that the coefficients belong to an a-priori known finite-dimensional subspace, that upper and lower bounds are a-priori known, and that a definiteness condition holds.

As in the last section the main tools to prove the stability are the monotonicity and the existence of localized potentials, which are the subject of the following subsection.

4.1. Monotonicity and localized potentials.

Lemma 4.1 (Monotonivity). *Let $\sigma_1, \sigma_2, q_1, q_2 \in L^\infty_+(\Omega)$. Then*

(4.1)
$$\int_\Omega \left[(\sigma_2-\sigma_1)|\nabla u_1|^2 + (q_2-q_1)u_1^2\right] dx \geq \langle g, (\Lambda(\sigma_1,q_1) - \Lambda(\sigma_2,q_2))\, g\rangle$$
$$\geq \int_\Omega \left[(\sigma_2-\sigma_1)|\nabla u_2|^2 + (q_2-q_1)u_2^2\right] dx,$$

$$\langle g, (\Lambda(\sigma_1,q_1) - \Lambda(\sigma_2,q_2))\, g\rangle \geq \int_\Omega \left[\left(\sigma_1 - \frac{\sigma_1^2}{\sigma_2}\right)|\nabla u_1|^2 + \left(q_1 - \frac{q_1^2}{q_2}\right)u_1^2\right] dx$$
$$= \int_\Omega \left[\frac{\sigma_1}{\sigma_2}(\sigma_2-\sigma_1)|\nabla u_1|^2 + \frac{q_1}{q_2}(q_2-q_1)u_1^2\right] dx, \tag{4.2}$$

for all $g \in L^2(\partial\Omega)$ where $u_1, u_2 \in H^1(\Omega)$ are the solutions of (2.1) with Neumann boundary data g on $\partial\Omega$, and coefficients (σ_1, q_1), resp., (σ_2, q_2).

Proof. The proof of (4.1) is given in [4, Lemma 4.1]. Following the proof of Lemma 3.3, we can easily deduce (4.2). □

Theorem 4.2 (Localized potentials). *Let $\sigma, q \in L^\infty_+(\Omega)$ that are piecewise analytic and $D \Subset \Omega$ be non empty open set, such that $\Omega\setminus\overline{D}$ is connected. Let B be a subdomain of D with smooth boundary ∂B. Then there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$, such that the corresponding solutions $(u^{(g_n)})_{n\in\mathbb{N}}$ of (2.1) fulfill*

$$\lim_{n\to\infty} \|u^{(g_n)}\|^2_{L^2(B)} = \infty, \tag{4.3}$$

$$\lim_{n\to\infty} \|u^{(g_n)}\|^2_{H^1(D\setminus\overline{B})} = 0, \tag{4.4}$$

$$\lim_{n\to\infty} \|u^{(g_n)}\|^2_{L^2(\partial B)} = 0, \tag{4.5}$$

$$\lim_{n\to\infty} \|\nabla u^{(g_n)}\|^2_{L^2(B)} = \infty. \tag{4.6}$$

Proof. This proof is based on the UCP for Cauchy data. First, we define the virtual measurement operators A_j $(j = 1, 2)$ by

$$A_1 : L^2(B) \to L^2(\partial\Omega), \quad F \mapsto v|_{\partial\Omega},$$

where $v \in H^1(\Omega)$ solves

$$\int_\Omega \sigma\nabla v \cdot \nabla w \, dx + \int_\Omega qvw \, dx = \int_B Fw \, dx \quad \text{ for all } w \in H^1(\Omega), \tag{4.7}$$

$$A_2 : H^1(D \setminus \overline{B})' \to L^2(\partial\Omega), \quad G \mapsto v|_{\partial\Omega},$$

where $v \in H^1(\Omega)$ solves

$$\int_\Omega \sigma\nabla v \cdot \nabla w \, dx + \int_\Omega qvw \, dx = \langle G, w\rangle_{D\setminus\overline{B}} \quad \text{ for all } w \in H^1(\Omega). \tag{4.8}$$

Here $\langle ., .\rangle_{D\setminus\overline{B}}$ denotes the dual pairing on $H^1(D \setminus \overline{B})' \times H^1(D \setminus \overline{B})$. First, we show that the dual operators A_1' and A_2' are given by

$$\begin{aligned} A_1' &: L^2(\partial\Omega) \to L^2(B) : g \mapsto A_1' g = u|_B, \\ A_2' &: L^2(\partial\Omega) \to H^1(D \setminus \overline{B}) : g \mapsto A_2' g = u|_{D\setminus\overline{B}}. \end{aligned}$$

Let $F \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, u, $v \in H^1(\Omega)$ solve (2.1) and (4.7), respectively. Then,

$$\int_\Omega F A_1' g \, dx = \int_{\partial\Omega} g A_1 F \, ds = \int_\Omega \sigma\nabla v \cdot \nabla u \, dx + \int_\Omega qvu \, dx = \int_B Fu \, dx.$$

Let $G \in H^1(\Omega)$, $g \in L^2(\partial\Omega)$, u, $v \in H^1(\Omega)$ solve (2.1) and (4.8), respectively. Then,

$$\int_\Omega G A_2' g \, dx = \int_{\partial\Omega} g A_2 G \, ds = \int_\Omega \sigma\nabla v \cdot \nabla u \, dx + \int_\Omega qvu \, dx = \langle G, u\rangle_{D\setminus\overline{B}}.$$

Next, we will prove that

$$\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\} \quad \text{and} \quad \mathcal{R}(A_1) \neq \{0\}.$$

Let $\varphi \in \mathcal{R}(A_1) \cap \mathcal{R}(A_2)$. Then there exist $v_1, v_2 \in H^1(\Omega)$ such that $v_1|_{\partial\Omega} = v_2|_{\partial\Omega} = \varphi$, and

$$\int_\Omega \sigma\nabla v_j \cdot \nabla w \, dx + \int_\Omega qv_j w \, dx = 0,$$

for all $w \in H^1(\Omega)$ with $\operatorname{supp}(w) \subset \overline{\Omega} \setminus \overline{D}$. Hence,

$$\operatorname{div}(\sigma \ \nabla v_j) + qv_j = 0 \quad \text{ in } \Omega \setminus \overline{D},$$

and $(\sigma\partial_\nu v_1)|_{\partial\Omega} = (\sigma\partial_n v_2)|_{\partial\Omega} = 0$. The unique continuation principle for Cauchy data yields that $v_1 = v_2$ in $\Omega \setminus \overline{D}$. Hence $v := v_1\chi_{D\setminus\overline{B}} + v_2\chi_{\Omega\setminus(D\setminus\overline{B})} \in H^1(\Omega)$ and satisfies

$$\begin{cases} \operatorname{div}(\sigma\nabla v) + qv = 0 & \text{in } \Omega, \\ \sigma\partial_\nu v \ = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that $v = 0$ and thus $\varphi = v|_{\partial\Omega} = 0$, and consequently $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$.

Next, we will prove that $\mathcal{R}(A_1) \neq \{0\}$. We first prove the injectivity of the dual operator A_1'. Let $g \in L^2(\partial\Omega)$ be such that $A_1' g = u|_D = 0$. By the unique continuation principal, we conclude that $u = 0$ in Ω. This means that $g = \sigma\partial_\nu u|_{\partial\Omega} = 0$, which proves that A_1' is injective. Hence A_1 has a dense range, i.e., $\overline{\mathcal{R}(A_1)} = L^2(\partial\Omega)$.

A fortiori $\mathcal{R}(A_1) \neq \{0\}$, which together with $\mathcal{R}(A_1) \cap \mathcal{R}(A_2) = \{0\}$, implies the range non inclusion $\mathcal{R}(A_1) \not\subseteq \mathcal{R}(A_2)$. Using [47, Corollary 2.6], it follows that there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^2(\partial\Omega)$ such that

$$\lim_{n\to\infty} \|A_1' g_n\|^2_{L^2(B)} = \lim_{n\to\infty} \|u^{(g_n)}\|^2_{L^2(B)} = \infty,$$

and

$$\lim_{n\to\infty} \|A_2' g_n\|^2_{H^1(D\setminus\overline{B})} = \lim_{n\to\infty} \|u^{(g_n)}\|^2_{H^1(D\setminus\overline{B})} = 0. \tag{4.9}$$

i.e. (4.3) and (4.4) hold. Also (4.5), holds from (4.9). Since

$$\|u^{(g_n)}\|_{L^2(B)} \leq C\left(\|u^{(g_n)}\|_{L^2(\partial B)} + \|\nabla u^{(g_n)}\|_{L^2(B)}\right),$$

where $C > 0$ is a constant, this also imply (4.6). ∎

Let $\mathcal{G}$ be a finite dimensional subset of piecewise analytic functions. We consider four constants $0 < c_1 \leq c_2$ and $0 < c_3 \leq c_4$ which are the lower and upper bounds of the parameters and define the set

$$\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]} = \{(\sigma, q) \in \mathcal{G} : \quad c_1 \leq \sigma(x) \leq c_2, \quad c_3 \leq q(x) \leq c_4 \quad \text{for all } x \in \Omega\}.$$

In the following main result of this paper, the domain Ω, the finite-dimensional subset $\mathcal{G}$ and the bounds $0 < c_1 \leq c_2$ and $0 < c_3 \leq c_4$ are fixed, and the constant in the Lipschitz stability result will depend on them.

Theorem 4.3 (Lipschitz stability). *There exists a positive constant $C > 0$ such that for all $(\sigma_1, q_1), (\sigma_2, q_2) \in \mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}$ with either*

$$(a) \quad \sigma_1 \leq \sigma_2 \text{ and } q_1 \leq q_2 \quad \text{or}$$
$$(b) \quad \sigma_1 \geq \sigma_2 \text{ and } q_1 \geq q_2,$$

we have

$$\begin{aligned} d_\Omega((\sigma_1, q_1), (\sigma_2, q_2)) :=& \max\left(\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)}, \|q_1 - q_2\|_{L^\infty(\Omega)}\right) \\ &\leq C\|\Lambda(\sigma, q_1) - \Lambda(\sigma_2, q_2)\|_*. \end{aligned} \tag{4.10}$$

Here $\|.\|_*$ is the natural norm of $\|.\|_{\mathcal{L}(L^2(\partial\Omega))}$.

Proof. For the sake of brevity, we write $\|.\|$ for $\|.\|_{L^2(\partial\Omega)}$. We start with the reformulation of the right-hand side of estimate (4.10). Since $\Lambda(\sigma_1, q_1)$ and $\Lambda(\sigma_2, q_2)$ are self-adjoint, we have that

$$\begin{aligned} &\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_* \\ &= \sup_{\|g\|=1} |\langle g, (\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1))\, g\rangle| \\ &= \sup_{\|g\|=1} \max\{\langle g, (\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1))\, g\rangle, \langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2))\, g\rangle\}. \end{aligned}$$

Next, we apply both inequalities in the monotonicity relation (4.1) in Lemma 4.1 in order to obtain lower bounds for the corresponding integrals. We thus obtain for all $g \in L^2(\partial\Omega)$

(4.11)

$$\langle g, (\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1))\, g\rangle \geq \int_\Omega (\sigma_1 - \sigma_2)|\nabla u^g_{(\sigma_1,q_1)}|^2\, dx + \int_\Omega (q_1 - q_2)|u^g_{(\sigma_1,q_1)}|^2\, dx$$

and

(4.12)

$$\langle g, (\Lambda(\sigma_1, q_1) - \Lambda(\sigma_2, q_2))\, g\rangle \geq \int_\Omega (\sigma_2 - \sigma_2)|\nabla u^g_{(\sigma_2,q_2)}|^2\, dx + \int_\Omega (q_2 - q_2)|u^g_{(\sigma_2,q_2)}|^2\, dx$$

where $u^g_{\sigma_1,q_1}, u^g_{\sigma_2,q_2} \in H^1(\Omega)$ denote the solutions of (2.1) with Neumann data g and parameters (σ_1, q_1) and (σ_2, q_2), respectively. Based on the estimates (4.11) and (4.12), we obtain for $(\sigma_1, q_1) \neq (\sigma_2, q_2)$

(4.13)

$$\begin{aligned}&\frac{\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_*}{d_\Omega((\sigma_1, q_1), (\sigma_2, q_2))}\\ &\geq \sup_{\|g\|=1} \Phi\left(g, \frac{\sigma_1 - \sigma_2}{d_\Omega((\sigma_1, q_1), (\sigma_2, q_2))}, \frac{q_1 - q_2}{(d_\Omega((\sigma_1, q_1), (\sigma_2, q_2))}, (\sigma_1, q_1), (\sigma_2, q_2)\right),\end{aligned}$$

and define for $g \in L^2(\partial\Omega)$, $(\zeta_1, \zeta_2) \in \mathcal{G}$, and $(\kappa_1, \tau_1), (\kappa_2, \tau_2) \in \mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}$ the function $\Phi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right)$ by

$$\begin{aligned}&\Phi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right)\\ &:= \max\left(\Psi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1)\right), \Psi\left(g, (-\zeta_1, -\zeta_2), (\kappa_2, \tau_2)\right)\right),\end{aligned}$$

with

$$\Psi\left(g, (\beta, \gamma), (\kappa, \tau)\right) := \int_\Omega \beta|\nabla u^g_{(\kappa,\tau)}|^2\, dx + \int_\Omega \gamma|u^g_{(\kappa,\tau)}|^2\, dx.$$

We introduce the compact sets

$$\begin{aligned}\mathcal{K}_+ &= \left\{(\zeta_1, \zeta_2) \in \operatorname{span}\mathcal{G} : \quad \zeta_1, \zeta_2 \geq 0 \quad \text{and} \quad \max\left(\|\zeta_1\|_{\mathrm{L}^\infty(\Omega)}, \|\zeta_2\|_{\mathrm{L}^\infty(\Omega)}\right) = 1\right\},\\ \mathcal{K}_- &= \left\{(\zeta_1, \zeta_2) \in \operatorname{span}\mathcal{G} : \quad \zeta_1, \zeta_2 \leq 0 \quad \text{and} \quad \max\left(\|\zeta_1\|_{\mathrm{L}^\infty(\Omega)}, \|\zeta_2\|_{\mathrm{L}^\infty(\Omega)}\right) = 1\right\},\end{aligned}$$

and denote $\mathcal{K} := \mathcal{K}_+ \cup \mathcal{K}_-$. Then using that either assumption (a) or assumption (b) is fulfilled, we can rewrite (4.13) as

$$\begin{aligned}&\frac{\|\Lambda(\sigma_2, q_2) - \Lambda(\sigma_1, q_1)\|_*}{d_\Omega((\sigma_1, q_1), (\sigma_2, q_2))}\\ &\geq \inf_{\substack{(\zeta_1,\zeta_2)\in\mathcal{K}\\ (\kappa_1,\tau_1),(\kappa_2,\tau_2)\in\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}}} \sup_{\|g\|=1} \Phi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right).\end{aligned} \tag{4.14}$$

The assertion of Theorem 3.6 follows if we can show that the right-hand side of (4.14) is positive. Since Φ is continuous, we can conclude that the function

$$((\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)) \mapsto \sup_{\|g\|=1} \Phi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right),$$

is semi-lower continuous, so that it attains its minimum on the compact set $\mathcal{K} \times \mathcal{G}_{[c_1,c_2]\times[c_3,c_4]} \times \mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}$. Hence, to prove Theorem 3.6, it suffices to show that

$$\sup_{\|g\|=1} \Phi\left(g, (\zeta_1, \zeta_2), (\kappa_1, \tau_1), (\kappa_2, \tau_2)\right) > 0, \tag{4.15}$$

for all $((\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)) \in \mathcal{K}\times\mathcal{G}_{[c_1,c_2]\times[c_3,dc_4]}\times\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}$.

In order to prove that (4.15) holds true, let $((\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)) \in \mathcal{K}\times\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}\times\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}$.
We first treat the case that $(\zeta_1,\zeta_2)\in\mathcal{K}_+$. Then there exist an open subset $\emptyset\neq B\subset\Omega$ and a constant $0<\delta<1$, such that either

$$\begin{aligned}&\text{(i) } \zeta_1|_B \geq \delta, \text{ and } \zeta_2 \geq 0, \text{ or}\\ &\text{(ii) } \zeta_2|_B \geq \delta, \text{ and } \zeta_1 \geq 0.\end{aligned}$$

We use the localized potentials sequence in Theorem 4.2 to obtain a boundary load $\tilde g \in L^2(\partial\Omega)$ with

$$\int_B |u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq \frac{1}{\delta} \quad\text{and}\quad \int_B |\nabla u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq \frac{1}{\delta}\quad . \tag{4.16}$$

In case (i), this leads to

$$\begin{aligned}\Phi\left(\tilde g,(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)\right) &\geq \int_\Omega \zeta_1|\nabla u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx + \int_\Omega \zeta_2 |u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx\\ &\geq \int_B \zeta_1|\nabla u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq \delta\int_B |\nabla u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq 1,\end{aligned}$$

and in case (ii), we have

$$\begin{aligned}\Phi\left(\tilde g,(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)\right) &\geq \int_\Omega \zeta_1|\nabla u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx + \int_\Omega \zeta_2 |u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx\\ &\geq \int_B \zeta_2|u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq \delta\int_B |u^{\tilde g}_{(\kappa_1,\tau_1)}|^2\,dx \geq 1.\end{aligned}$$

Hence, in both cases,

$$\begin{aligned}\sup_{\|g\|=1}\Phi(g,(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)) &\geq \Phi\left(\frac{\tilde g}{\|\tilde g\|},(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)\right)\\ &= \frac{1}{\|\tilde g\|^2}\Phi(\tilde g,(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)) > 0.\end{aligned}$$

For $(\zeta_1,\zeta_2)\in\mathcal{K}_-$, we can analogously use a localized potentials sequence for (κ_2,τ_2), and prove that

$$\sup_{\|g\|=1}\Phi(g,(\zeta_1,\zeta_2),(\kappa_1,\tau_1),(\kappa_2,\tau_2)) > 0,$$

and the proof of Theorem 3.6 is completed. ∎

Remark 4.4. All the results of section 3 and section 4 stay valid when the Neumann-to-Dirichlet operator $\Lambda(\sigma,q)$ is extended to $H^{-\frac12}(\partial\Omega)\to H^{\frac12}(\partial\Omega)$. On these spaces, it is easily shown that $\Lambda(\sigma,q)$ is bijective, and its inverse is the Dirichlet-to-Neumann operator $\Lambda_D(\sigma,q): f\to u^{(f)}_{\sigma,q}|_{\partial\Omega}$, where $u^{(f)}_{\sigma,q}$ solves

$$\begin{cases} -\nabla\cdot(\sigma\nabla u^{(f)}_{\sigma,q}) + q u^{(f)}_{\sigma,q} = 0 & \text{in } \Omega,\\ u^{(f)}_{\sigma,q} = f & \text{on } \partial\Omega.\end{cases}$$

5. Numerical approach to solve the inverse problem. In this section, we are interested in the following inverse problem

$$\text{(5.1)} \qquad \textit{Find } \sigma, q \textit{ knowing measurements } f_k = \Lambda(\sigma,q)g_k,\ k=1,\dots K,$$

where $f_k \in L^2(\partial\Omega)$ is a measurement of the density of photons corresponding to the input flux g_k, and $K \in \mathbb{N}$ is the number of measurements.

To solve the inverse problem (5.1) numerically, we consider a minimization problem of a Kohn-Vogelius type functional:
(5.2)

$$\min_{(\sigma,q)\in\mathcal{G}_{[c_1,c_2]\times[c_3,c_4]}} J(\sigma,q) = \sum_{k=1}^{K} \int_\Omega \left(\sigma|\nabla(u^{(g_k)} - u^{(f_k)})|^2 + q|u^{(g_k)} - u^{(f_k)}|^2\right) dx + \frac{\rho}{2}\int_\Omega (\sigma^2 + q^2)\, dx.$$

Here $u^{(g_k)}$ and $u^{(f_k)}$ solve the following problems:

$$\text{(5.3)} \qquad \begin{cases} -\nabla\cdot(\sigma\nabla u^{(g_k)}) + qu^{(g_k)} = 0 & \text{in } \Omega,\\ \sigma\partial_\nu u^{(g_k)} = g_k & \text{on } \partial\Omega, \end{cases}$$

$$\text{(5.4)} \qquad \begin{cases} -\nabla\cdot(\sigma\nabla u^{(f_k)}) + qu^{(f_k)} = 0 & \text{in } \Omega,\\ u^{(f_k)} = f_k & \text{on } \partial\Omega. \end{cases}$$

When dealing with reconstruction of the absorption parameter q where σ is assumed to be known, the minimization problem (5.2) is reduced to
(5.5)

$$\min_{q\in\mathcal{F}_{[a,b]}} \mathcal{J}(q) = \sum_{k=1}^{K} \int_\Omega \left(\sigma|\nabla(u^{(g_k)} - u^{(f_k)})|^2 + q|u^{(g_k)} - u^{(f_k)}|^2\right) dx + \frac{\rho}{2}\int_\Omega q^2\, dx.$$

THEOREM 5.1. *The functional $J : L^\infty_+(\Omega)^2 \to \mathbb{R}$, defined in (5.2) is Fréchet differentiable, and its Fréchet derivative at $(\sigma,q) \in L^\infty_+(\Omega)^2$ in the direction $(\tilde\sigma,\tilde q) \in L^\infty_+(\Omega)^2$ is given by*
(5.6)

$$J'(\sigma,q)(\tilde\sigma,\tilde q) = \sum_{k=1}^{K} \int_\Omega \left(\tilde\sigma\left(|\nabla u^{(f_k)}|^2 - |\nabla u^{(g_k)}|^2\right) + \tilde q\left((u^{(f_k)})^2 - (u^{(g_k)})^2\right)\right) dx + \rho\int_\Omega (\sigma\tilde\sigma + q\tilde q)\ dx.$$

We need the following lemma to prove Theorem 5.1.

LEMMA 5.2. *The non-linear operator*

$$\Lambda(\sigma,q) : L^\infty_+(\Omega)^2 \to \mathcal{L}(L^2(\partial\Omega)), \quad (\sigma,q) \to \Lambda(\sigma,q)$$

is Fréchet differentiable and its derivative

$$\Lambda' : L^\infty_+(\Omega)^2 \to \mathcal{L}(L^\infty(\Omega)^2, \mathcal{L}(L^2(\partial\Omega))$$

is given by the bilinear form

$$\int_{\partial\Omega} g(\Lambda'(\sigma,q)(\delta_1,\delta_2))h\,ds = -\int_\Omega \delta_1 \nabla u_{\sigma,q}^{(g)}\cdot\nabla u_{\sigma,q}^{(h)}\,dx - \int_\Omega \delta_2 u_{\sigma,q}^{(g)} u_{\sigma,q}^{(h)}\,dx, \tag{5.7}$$

for all $\sigma,q\in L^\infty_+(\Omega)$, $\delta_1,\delta_2\in L^\infty(\Omega)$, $g,h\in L^2(\partial\Omega)$ where $u_{\sigma,q}^{(g)}\in H^1(\Omega)$ is solution of the problem (2.1).

Proof. It follows from the monotonicity relation (3.3) that for all sufficiently small $\delta_1,\delta_2\in L^\infty(\Omega)$ such that $\sigma+\delta_1, q+\delta_2\in L^\infty_+(\Omega)$

$$\begin{aligned}&\int_\Omega (\delta_1|\nabla u_{\sigma,q}^{(g)}|^2+\delta_2(u_{\sigma,q}^{(g)})^2)\,dx \geq \int_{\partial\Omega} g\left(\Lambda(\sigma,q)-\Lambda(\sigma+\delta_1,q+\delta_2)\right)g\,ds\\ &\geq \int_\Omega\left(\sigma-\frac{\sigma^2}{\sigma+\delta_1}\right)|\nabla u_{\sigma,q}^{(g)}|^2\,dx+\int_\Omega\left(q-\frac{q^2}{q+\delta_2}\right)(u_{\sigma,q}^{(g)})^2\,dx.\end{aligned}$$

Thus

$$\begin{aligned}&\|\Lambda(\sigma,q)-\Lambda(\sigma+\delta_1,q+\delta_2)-\Lambda'(\sigma,q)(\delta_1,\delta_2)\|_{\mathcal{L}(L^2(\partial\Omega))}\\ &=\sup_{g\in L^2(\partial\Omega)}\left|\int_{\partial\Omega} g\left(\Lambda(\sigma,q)-\Lambda(\sigma+\delta_1,q+\delta_2)-\Lambda'(\sigma,q)(\delta_1,\delta_2)\right)\,ds\right|\\ &\leq \int_\Omega\left(\left(\frac{\delta_1^2}{\sigma+\delta_1}\right)|\nabla u_{\sigma,q}^{(g)}|^2+\left(\frac{\delta_2^2}{q+\delta_2}\right)(u_{\sigma,q}^{(g)})^2\right)\,dx = O\left(\|(\delta_1,\delta_2)\|_\infty^2\right).\end{aligned} \tag{5.8}$$

This shows that Λ is Fréchet differentiable, and its derivative is given by (5.7). ∎

Proof of Theorem 5.1. From the definition of the functional J, and applying Green's formula once, we have

$$\begin{aligned}J(\sigma,q) &= \sum_{k=1}^K\int_\Omega \sigma|\nabla u^{(g_k)}|^2\,dx+\sum_{k=1}^K\int_\Omega q|u^{(g_k)}|^2\,dx+\sum_{k=1}^K\int_\Omega\sigma|\nabla u^{(f_k)}|^2\,dx\\ &\quad+\sum_{k=1}^K\int_\Omega q|u^{(f_k)}|^2\,dx-2\sum_{k=1}^K\int_{\partial\Omega} g_kf_k\,ds+\frac{\rho}{2}\int_\Omega(\sigma^2+q^2)\,dx\\ &=\sum_{k=1}^K\langle g_k,\Lambda(\sigma,q)g_k\rangle+\sum_{k=1}^K\langle\Lambda_D(\sigma,q)f_k,f_k\rangle\\ &\quad-2\sum_{k=1}^K\int_{\partial\Omega} g_kf_k\,ds+\frac{\rho}{2}\int_\omega(\sigma^2+q^2)\,dx.\end{aligned} \tag{5.9}$$

From Lemma 5.2, $\Lambda(\sigma,q)$ is Fréchet differentiable with

$$\langle g_k,\Lambda'(\sigma,q)(\tilde\sigma,\tilde q)g_k\rangle = -\int_\Omega\left(\tilde\sigma|\nabla u^{(g_k)}|^2+\tilde q(u^{(g_k)})^2\right)\,dx,$$

and

$$\begin{aligned}\langle(\Lambda_D(\sigma,q))'(\tilde\sigma,\tilde q)f_k,f_k\rangle &= \langle(\Lambda(\sigma,q)^{-1})'(\tilde\sigma,\tilde q)f_k,f_k\rangle\\ &=\int_\Omega\left(\tilde\sigma|\nabla u^{(f_k)}|^2+\tilde q(u^{(f_k)})^2\right)\,dx.\end{aligned}$$

Since $\int_{\partial\Omega} g_kf_k\,ds$ is constant and $(\sigma,q)\to\int_\Omega(\sigma^2+q^2)\,dx$ is Fréchet differentiable, we conclude that J is Fréchet differentiable and its derivative is given by (5.6). ∎

Remark 5.3. Using the same techniques, we can prove that the functional $\mathcal{J}$ is Fréchet differentiable and its derivative is given by:

$$\mathcal{J}'(q)\tilde{q} = \sum_{k=1}^{K} \int_\Omega \tilde{q} \left((u^{(f_k)})^2 - (u^{(g_k)})^2 \right) \, dx + \rho \int_\Omega \tilde{q} q \, dx.$$

6. Implementation details and numerical examples. We provide in this section two numerical examples that illustrate the performance of our numerical method. In the first example, we reconstruct the spatial distribution of the absorption coefficient while keeping the diffusion coefficient fixed. In the second example, we show that both optical properties are reconstructed simultaneously.

When dealing with reconstruction with noise data, the choice of the regularization parameter ρ in (5.1) is crucial. Usually, it is determined using a knowledge of the noise level by, e.g., the discrepancy principle. However, in practice, the noise level may be unknown, rendering such rules inapplicable. To overcome this issue, we propose a heuristic choice rule based on the following balancing principle [48]: Choose ρ such that

(6.1)

$$(\beta - 1) \sum_{k=1}^{K} \int_\Omega \left(\sigma |\nabla (u^{(g_k)} - u^{(f_k)})|^2 + q |u^{(g_k)} - u^{(g_k)}|^2 \right) \, dx - \frac{\rho}{2} \int_\Omega (\sigma^2 + q^2) \, dx = 0.$$

The idea behind this principle is to balance the data fitting term with the penalty term and the weight $\beta > 1$ controls the trade-off between them. The choice rule does not require the knowledge of the noise level, and has been successfully applied to linear and non linear inverse problems [49, 48, 50, 51, 52].

When dealing only with the reconstruction of q, the balancing equation (6.1) is reduced to

$$(\beta - 1) \sum_{k=1}^{K} \int_\Omega \left(\sigma |\nabla (u^{(g_k)} - u^{(f_k)})|^2 + q |u^{(g_k)} - u^{(f_k)}|^2 \right) \, dx - \frac{\rho}{2} \int_\Omega q^2 \, dx = 0. \tag{6.2}$$

For our problem, we compute a solution ρ^* to the balancing equation (6.1) or (6.2) by the fixed point algorithm proposed in [50, 51].

We consider the following setup for our numerical examples: The domain Ω under consideration is the two dimensional unit disk centered at the origin. We use a Delaunay triangular mesh and a standard finite element method with piecewise finite elements to numerically compute the states for our problem. The measurements f_k are computed synthetically by solving the direct problem (2.1). To simulate noisy data, the measurements f_k are corrupted by adding a normal Gaussian noise with mean zero and standard deviation $\epsilon \|f_k\|_\infty$ where ϵ is a parameter. To avoid the so called "inverse crime", the inverse problem is solved using 1016 elements, while the data f_k is computed with 4064 elements. For all the computations we have used Matlab R2018a.

6.1. Example 1: Reconstructing q. In the following numerical results, the diffusion coefficient σ is assumed to be known, and is given by $\sigma = 1\chi_{\Omega \setminus \overline{\omega}} + 2\chi_\omega$, where ω is the disk of radius $1/2$ centered at the origin. The exact absorption coefficient to be recovered is given by

$$q^\dagger(x_1, x_2) = 1 + \cos(\pi x_1) \cos(\pi x_2) \chi_{(\|(x_1,x_2)\|_\infty < 0.5)}.$$

We obtain measurements f_k corresponding to the fluxes

$$g_k = 10 + \sin(k\theta), \quad \theta \in [0, 2\pi], \quad k = 1, \dots 5.$$

and we reconstruct q by minimizing the functional

$$\mathcal{J}(q) = \sum_{k=1}^{5} \int_\Omega \left(\sigma |\nabla (u^{(g_k)} - u^{(f_k)})|^2 + q |u^{(g_k)} - u^{(f_k)}|^2\right) dx + \frac{\rho}{2} \int_\Omega q^2 \, dx,$$

in the space of piecewise constant functions on the FEM mesh.
Figure 1 shows the true and the reconstructed absorption images with noise free synthetic data and without regularization. Figure 2 shows the reconstructed absorption images with respect to different initialization and noise levels. The quality of the reconstruction is satisfactory and depend on the initialization of the algorithm.

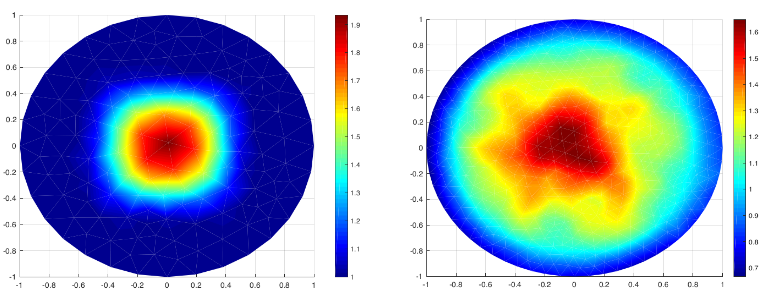

FIG. 1. *On the left the true absorption image and the right the reconstructed absorption image with $\epsilon = 0$, $\rho = 0$ and initialization $q(x_1, x_2) = 1$.*

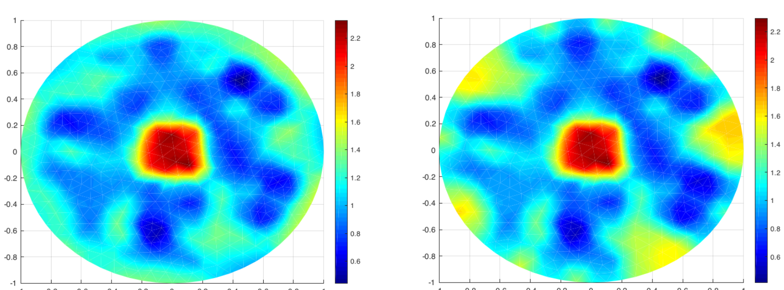

FIG. 2. *On the left the reconstructed absorption image with $\varepsilon = 0$ and $\rho = 0$. On the right reconstructed absorption image with $\varepsilon = 0.05$ and $\rho = 0.0000001672$. In both cases, the initialization is taken as $q(x_1, x_2) = (|x_1| < 0.2)(|x_2| < 0.2)$.*

6.2. Examples 2: Reconstructing σ and q simultaneously. In this example the exact parameters to be recovered are given by

$$\sigma^\dagger(x) = 2\chi_{D_1} + 3\chi_{D_2} + 1\chi_{\Omega \setminus \overline{D_1 \cup D_2}}, \quad q^\dagger(x) = 3\chi_{D_3} + 4\chi_{D_4} + 1\chi_{\Omega \setminus \overline{D_3 \cup D_4}},$$

where D_1, D_2, D_3 and D_4 are given by:

$$D_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 0.5)^2 + x_2^2 < 0.2^2\right\},$$

$$D_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : (x_1 + 0.5)^2 + x_2^2 < 0.2^2\right\},$$
$$D_3 = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - 0.5)^2 < 0.2^2\right\},$$
$$D_4 = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + 0.5)^2 < 0.2^2\right\}.$$

We use measurements f_k correspond to the fluxes $g_k(\theta) = \sin(k\theta), \theta \in [0, 2\pi], \quad k = 1, \dots 5$ and we reconstruct σ, q by minimizing the function

$$J(\sigma, q) = \sum_{k=1}^{5} \int_\Omega \left(\sigma|\nabla(u^{(g_k)} - u^{(f_k)}|^2 + q|u^{(g_k)} - u^{(f_k)}|^2\right) dx + \frac{\rho}{2} \int_\Omega (\sigma^2 + q^2)\, dx,$$

in the space of piecewise constant functions on the FEM mesh. The initialization is given by

$$(\sigma(x), q(x)) = (1.1\chi_{D_1} + 1.2\chi_{D_2} + 1\chi_{\Omega\setminus\overline{D_1 \cup D_2}}, 1.1\chi_{D_3} + 1.2\chi_{D_4} + 1\chi_{\Omega\setminus\overline{D_3 \cup D_4}}).$$

Figure 3. shows the true diffusion image and the reconstructed diffusion image with noise free synthetic data and without regularization. Figure 4. depicts the reconstructed diffusion images with different noise synthetic data and regularization. Figure 5. depicts the reconstructed absorption image with noise free synthetic data and without regularization. Figure 6. shows the reconstructed absorption image with different noise synthetic data and regularization.

In this example, the quality of reconstructions is satisfactory and the regularization technique that we have imposed here allows us to estimate the optical properties in the presence of moderate noise with accuracy. Let us mention that in [53] the authors introduced a gradient-based optimisation scheme to reconstruct the optical properties without regularization of the minimization problem. A crosstalk problem appeared in the reconstruction of the profiles. This is maybe due to the non uniqueness of the inverse problem which is know to be severally ill-posed.

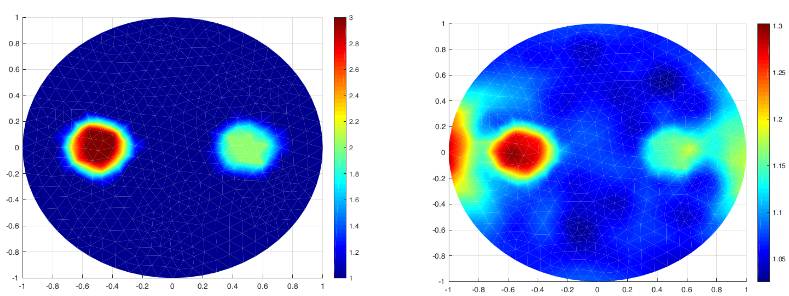

Fig. 3. *On the left the true diffusion image and on the right the reconstructed diffusion image with* $\varepsilon = 0$, $\rho = 0$.

7. Conclusion. In this paper, we have shown a global uniqueness and Lipschitz stability results when a-priori smoothness assumptions are imposed on the parameters (σ piecewise constant and q piecewise-analytic). We have also shown for a given setting that the Lipschitz constant can be computed by solving a finite numbers of well posed PDEs. The proofs rely on the monotonicity of the NtD operator combined with the techniques of localized potentials. These techniques seem simple compared to the

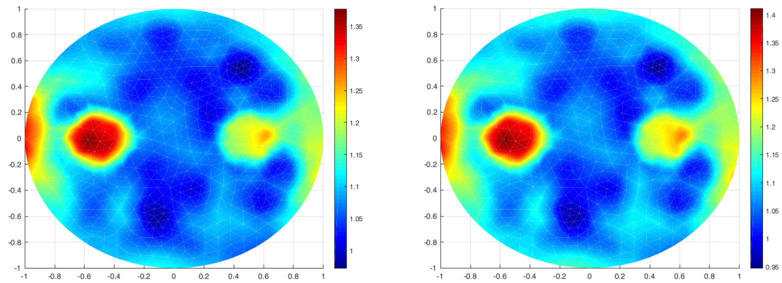

FIG. 4. *On the left the reconstructed diffusion image with* $\varepsilon = 0.03$, $\rho = 1.674 \times 10^{-6}$ *and on the right the reconstructed diffusion image with* $\varepsilon = 0.05$ *and* $\rho = 3.192 \times 10^{-7}$.

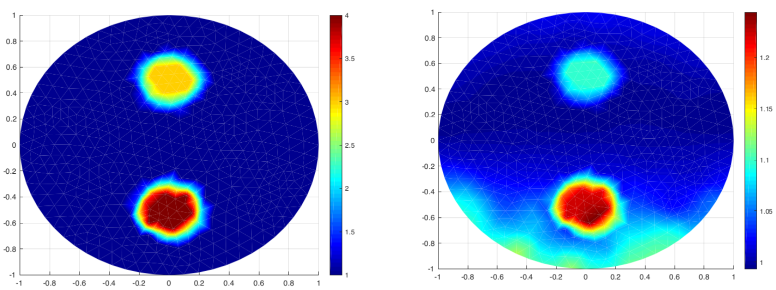

FIG. 5. *On the left the exact absorption and on the right the reconstructed absorption with* $\varepsilon = 0$, *and* $\rho = 0$.

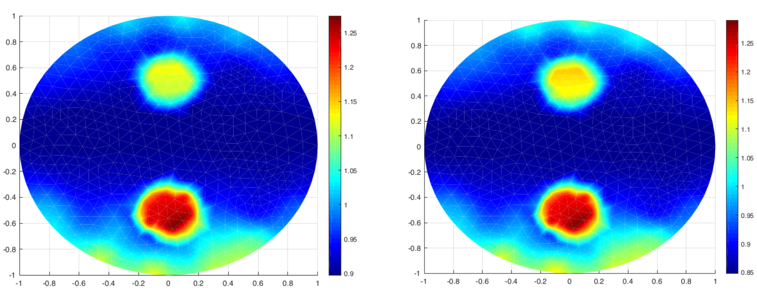

FIG. 6. *On the left the reconstructed absorption with* $\varepsilon = 0.03$ *and* $\rho = 5.82611 \times 10^{-6}$, *and on the right the reconstructed absorption with* $\varepsilon = 0.05$ *and* $\rho = 1.35438 \times 10^{-7}$.

techniques of Carleman estimates and complex geometrical optics(CGO) used in the litterature.

We have formulated the inverse problem as a regularized problem using a Khon-Vogelius functional. In the inversion procedure, the forward model is discretized using a finite element method. We solve the regularized problem by using a Quasi-Newton method with BFGS type updating rule for the Hessian matrix. Numerical reconstructions based on synthetic data provide results that are in agreement with the expected reconstructions and no crosstalk between the parameters is observed.

Let us mention that our numerical method depend strongly on the initialization, the measurements and the mesh size. When considering the reconstruction of σ and q simultaneously, our algorithm can't reconstruct the jump sets of the parameters. A shape optimization procedure may be used to reconstruct the parameters and their jump sets simultaneously.

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