RIGOROUS JUSTIFICATION OF THE FOKKER-PLANCK EQUATIONS OF NEURAL NETWORKS BASED ON AN ITERATION PERSPECTIVE

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ABSTRACT. In this work, the primary goal is to establish a rigorous connection between the Fokker-Planck equation of neural networks with its microscopic model: the diffusion-jump stochastic process that captures the mean field behavior of collections of neurons in the integrate-and-fire model. The proof is based on a novel iteration scheme: with an auxiliary random variable counting the firing events, both the density function of the stochastic process and the solution of the PDE problem admit series representations, and thus the difficulty in verifying the link between the density function and the PDE solution in each sub problem has been greatly mitigated. The iteration approach provides a generic frame in integrating the probability approach with PDE techniques, with which we prove that the density function of the diffusion-jump stochastic process is indeed the classical solution of the Fokker-Planck equation with a unique flux-shift structure.

1. INTRODUCTION

While various models emerge in neuroscience [23, 30, 36, 45], one of the most active disciplines at the present time, the level of mathematical rigor in understanding the rational connections between these models is usually formal or empirical. When it comes to modeling the dynamics of a large collection of interacting neurons, the integrate-and-fire model for the potential through the neuron cell membrane, which dates back to [30], has received great attention. In this model, the collective behavior of neuron networks can be predicted by the stochastic process of a single neuron [3, 4, 14, 15, 25, 29, 31, 34, 35, 41, 42, 44] where the influence from the network is given by an average synaptic input by the mean-field approximation [15, 29, 41, 44]. The time evolution of the probability density function (abbreviated by p.d.f.) of the potential voltage is governed by a Fokker-Planck equation on the half space with an unusual structure: constantly shifting the boundary flux to an interior point. This equation has been utilized by neuroscientists to explore the macroscopic behavior of neural networks, and has also attracted many mathematicians to investigate the unique solutions structures in the past decade [5, 6, 7, 8, 9, 11, 28, 37], which in turn have enriched the scientific interpretation of the integrate-and-fire model.

In this paper, we focus on the single neuron approximation of the celebrated noisy Leaky Integrate-and-Fire (LIF) model for neuron networks, where the state variable X_t denotes the membrane potential of a typical neuron within the network. In the LIF model, when the synaptic input of the network (denoted by $I(t)$) vanishes, the membrane potential relaxes to their resting potential V_L, and in the single neuron approximation, the synaptic input $I(t)$, which itself is another stochastic process, is replaced by a continuous-in-time counterpart $I_c(t)$ (see e.g. [3, 4, 31, 37, 41, 42]), which takes the drift-diffusion form

$$I\,dt \approx I_c\,dt = \mu_c\,dt + \sigma_c\,dB_t. \tag{1}$$

Here, B_t is the standard Brownian motion, and in principle the two processes $I_c(t)$ and $I(t)$ have the same mean and variance. Thus between the firing events, the evolution of the membrane potential is given by the following stochastic differential equation

$$dX_t = (-X_t + V_L + \mu_c)\,dt + \sigma_c\,dB_t. \tag{2}$$

The next key component of the model is the firing-and-reseting mechanism: whenever the membrane voltage X_t reaches a threshold value called the threshold or firing voltage V_F, it is immediately relax to a reset value V_R, where $V_R < V_F$. The readers may refer to [41] for a thorough introduction of this subject. It is worth mentioning that, numerous mathematical aspects of the LIF model and its variants have been studied (see e.g. [15, 18, 29, 38, 41, 44]) besides its enormous significance in neuroscience.

There has been a growing interest in studying the partial differential equation problem for the dynamics of the probability density function that the stochastic process X_t is associated with [12, 13, 15, 18, 29]. We denote the density of the distribution of neuron potential voltage at time $t \geq 0$ by $f(x,t)$, $x \in (-\infty, V_F]$. At least from a heuristic viewpoint, it is widely accepted that the p.d.f. $f(x,t)$ satisfies the following Fokker-Planck equation on the half line with a singular source term

$$\frac{\partial f}{\partial t}(x,t) + \frac{\partial}{\partial x}[hf(x,t)] - a\frac{\partial^2 f}{\partial x^2}(x,t) = N(t)\delta(x - V_R), \quad x \in (-\infty, V_F), \quad t > 0, \tag{3}$$

where $N(t)$ denotes the mean firing rate. By formal calculations via Ito's calculus, we obtain the drift velocity $h = -x + V_L + \mu_c$ and diffusion coefficient $a = \sigma_c^2/2$.

The firing-and-reset mechanism in the stochastic process has led to multiple consequences in the PDE model. First, since the neurons at the threshold voltage has instantaneous discharges where the density is supposed to vanish and due to the noisy leaky terms, we consider the following Dirichlet boundary conditions:

$$f(V_F, t) = 0, \quad f(-\infty, t) = 0, \quad \forall t \geq 0. \tag{4}$$

Second, due to the Dirichlet boundary condition at $x = V_F$, there is a time-dependent boundary flux escaping the domain, and a Dirac delta source term is added to the reset location $x = V_R$ to compensate the loss. Noting that (3) is the evolution of a p.d.f, therefore for all $t \geq 0$

$$\int_{-\infty}^{V_F} f(x,t)\,dx = \int_{-\infty}^{V_F} f_{\text{in}}(x)\,dx = 1.$$

The conservation of mass and the boundary condition characterize the magnitude of mean firing rate

$$N(t) := -a\frac{\partial f}{\partial x}(V_F, t) \geq 0. \tag{5}$$

The PDE problem is completed by an appropriate initial condition $f(x,0) = f_{\text{in}}(x)$.

Third, the firing events generate currents that propagate within the neuron networks, which is incorporated into this PDE model by expressing the drift velocity h and the diffusion coefficient a as functions of the mean-firing rate $N(t)$. For example, it is assumed in quite a few works (see e.g. [5, 6, 11, 28]) that

$$h(x,N) = -x + bN, \quad a(N) = a_0 + a_1 N,$$

where b, $a_0 > 0$ and $a_1 \geq 0$ are some modeling parameters. When $b > 0$, the neuron network is excitatory on average, and when $b < 0$ the network is inhibitory. In particular, when $b = 0$ and $a_1 = 0$, the PDE problem becomes linear, but the flux shift structure persists.

We remark that this delta source term on the right hand side of (3) is equivalent to setting the equation on $(-\infty, V_R) \cup (V_R, V_F)$ instead and imposing the following conditions

$$f(V_R^-, t) = f(V_R^+, t), \quad a\frac{\partial}{\partial x} f(V_R^-, t) - a\frac{\partial}{\partial x} f(V_R^+, t) = N(t), \quad \forall t \geq 0.$$

The equivalence can be checked by directly integration by parts and we choose to use this form for the rest of the paper.

Due to the unique structure of the PDE problem, most conventional analysis methods do not directly apply, and many recent works are devoted to investigate the solution properties of such model and its various modifications, including the finite-time blow-up of weak solutions, the multiplicity of the steady solutions, the relative entropy estimate, the existence of the classical solutions, the structure-preserving numerical approximation, etc. (see e.g. [5, 6, 7, 9, 11, 28] and the references therein). For the stochastic process (2), as the jumping time for X_t is determined by its hitting time, the classical Itô calculus is not directly applicable.

The primary goal of this paper is to show the rigorous derivation of the Fokker-Planck equation from the stochastic process. More specifically, we investigate whether and in which sense the probability density function $f(x, t)$ of the stochastic process X_t satisfies the PDE model. In this paper, we choose the model parameters as follows

$$V_L = V_R = 0, \quad \mu_c = 0, \quad \sigma_c = \sqrt{2}, \quad \text{and} \quad V_F = 1. \tag{6}$$

Let the distribution of X_0 be denoted by ν, which is a probability measure compactly support on $(-\infty, 1)$ and let $f_{\text{in}}(x)$ to denote the density function of ν. Then $X_t \in (-\infty, 1)$ is a stochastic process whose trajectory is *càdlàg* in time, and it evolves as an Ornstein–Uhlenbeck process

$$dX_t = -X_{t-}\, dt + \sqrt{2}\, dB_t, \tag{7}$$

until it hits 1. Whenever at time t, X_t hits 1, it immediately jumps to 0, i.e.

$$\text{if} \quad X_{t-} = 1, \quad X_t = 0. \tag{8}$$

Then we restart the O-U-like evolution independent of the past. We remark that (7) and (8) serve as a formal definition of the diffusion-jump process only for heuristic purposes and the rigorous definition shall be presented in Section 2.2. In this paper, we aim to show for any fixed $T > 0$ that the associated density function $f(x, t)$ is indeed a classical solution to the PDE problem

$$\begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\partial}{\partial x}(xf) - \dfrac{\partial^2 f}{\partial x^2} = 0, \quad x \in (-\infty, 0) \cup (0, 1), t \in (0, T], \\ f(0^-, t) = f(0^+, t), \quad \dfrac{\partial}{\partial x} f(0^-, t) - \dfrac{\partial}{\partial x} f(0^+, t) = -\dfrac{\partial}{\partial x} f(1^-, t), \quad t \in (0, T], \\ f(-\infty, t) = 0, \quad f(1, t) = 0, \quad t \in [0, T], \\ f(x, 0) = f_{\text{in}}(x), \quad x \in (-\infty, 1). \end{cases} \tag{9}$$

The processes of such type (7) and (8) were first introduced by Feller [19, 20] (in terms of transition semigroups). In particular, [20] presents the Fokker-Planck equation of such processes (dubbed "elementary return process" there) in a weak form, of which the proof is based on a Markov semigroup argument in [19]. See Theorem 9 of [20] for details. Such processes have also been studied in later works such as [2, 24, 38, 39, 40, 43]. More specifically, in [1, 2, 38, 39], the authors are concerned with the spectral properties of the generator of the stochastic process or related models, and have shown the exponential convergences in time towards the stationary distribution. In particular, [38] applied their results to a neuronal firing model driven by a Wiener process and computed the distribution of the first passage time. In the works [40, 43], the authors made more relaxed or modified assumptions on the stochastic process than those in [24], and proved the existence of pathwise solution of such process in a generalized sense.

Following the spirit of the pioneering work of Feller [20], the focus of this paper is to rigorously establish the bridge between the density functions of such processes and the classical solutions of the Fokker-Planck equations to be specified as in (9). From the technical perspective, there is no available mathematical tools to link the boundary condition at the firing voltage and the jump condition at the reset voltage (or equivalently, the singular delta source term) of the PDE model to stochastic model for a single neuron model. In [5, 11], some heuristic arguments are provided to connect $N(t)$ to the rate of change of the expectation of the number of firing events, which is related to the synchronization behavior of the neuron networks. Whereas, such an interpretation is not applicable for a single neuron model. In this paper, we rigorously prove that for a single neuron the mean firing rate $N(t) = \sum_{n=1}^{\infty} f_{T_n}(t)$ where f_{T_n} stands for the p.d.f. of the n-th jumping time of X_t.

The key strategy of our proofs in this paper is based on an iterated scheme: with the introduction of an auxiliary random variable counting the number of firing events, the probability density function of potential voltage $f(x,t)$ allows a decomposition as a summation of sub-density functions $\{f_n(x,t)\}_{n=0}^{\infty}$. Each sub-density naturally links to a less singular sub-PDE problem, and all the sub-PDE problems are connected successively by iteration: the escaping boundary flux of $f_n(x,t)$ serves as the singular source for $f_{n+1}(x,t)$. Among all the iterations, the first step from f_0 to f_1 exhibits strongest singularity at the source of the flux, and thus turns out to be the major technical difficulty in our proof. In order to tackle this obstacle, elaborated estimates on the regularities of f_0 have to be established. The first sub-PDE problem corresponds to the stochastic process killed at the first hitting time and there is a vast literature [16, 26, 27, 32, 33] concerned with the stochastic processes with no reset for the killed particles. In [17, 18], the authors consider the process with firing-and-reseting as in this paper and have established the connections between the sub-density function and the PDE solution. They have proved that $f_0(t,x)$ is continuous in (t,x) and continuously differentiable in x on $(0,T] \times (-\infty, 1]$ and admits Sobolev derivatives of order 1 in t and of order 2 in x on any compact subset of $(0,T] \times (-\infty, 1)$. However, these results are not strong enough to guarantee the existence of the classical solution to the whole problem (9). In fact, by analysing the Green's function for the parabolic equation on the half space, we get estimates for classical derivatives and high order regularity for t in Proposition 3.1 and 3.2, which is essential for the iteration from f_0 to f_1. Besides, all the desired smoothness properties are maintained by the iteration scheme, and thanks to the decomposition, rigorous justification of the jump condition for each sub-PDE problem becomes tractable. Finally, with the exponential convergence of decomposition, we can pass to the limit, and conclude the preserved properties on the original problem. This iteration scheme is inspired by the renewal nature of the stochastic process, which shares the spirit of Feller's original work in [20], and provides a platform to combine the techniques from both the probability theory and the differential equations.

It is worth noting that, as the first attempt to study the rigorous justification of the Fokker-Planck equations of neural networks from the stochastic model, we have only obtained the results for the linear cases. In particular, we could not incorporate the dependence on the mean firing rate in the drift velocity and in the diffusion coefficient yet, but we shall investigate those directions in the future.

The rest of the paper is outlined as follows. In Section 2, we summarize the main results of this work as well as give precise definition of the stochastic process and lay out the iterated scheme. In Section 3, we show that the density function of the stochastic process is indeed the mild solution of the PDE problem with certain smoothing properties, and we give a few remarks on the implications in the weak solution. For the rest of this work, we use C, C_0, C_k and C_T to denote generic constants.

2. Preliminaries and Main Results

In this section, we present the main results of this paper in details, and also provide some technical preparations for the proofs, including the construction of the stochastic process, which serves as the precise definition, and the elaboration of the iterated strategy, accompanied by some elementary estimates.

2.1. **Main Results.**

The stochastic process X_t has been formally defined in (7) and (8), but note that the rigorous construction of such a process can be found in (18) below of Section 2.2.

We first suppose that the process X_t starts from 0, i.e. the distribution of X_0 is $f_{\text{in}}(x) = \delta(x)$. We state the first main result in the following

Theorem 1. *The process X_t as in (18) that starts from 0 has a continuously evolved probability density function denoted by $f(x,t)$. $f(x,t)$ is a solution of (9) in the time interval $(0,T]$ for any given $0 < T < +\infty$ and with initial condition $\delta(x)$ in the following sense:*

(i) *$N(t) := -\frac{\partial}{\partial x} f(1^-,t)$ is a continuous function for $t \in [0,T]$,*
(ii) *f is continuous in the region $\{(x,t) : -\infty < x \leq 1,\ t \in (0,T]\}$,*
(iii) *f_{xx} and f_t are continuous in the region $\{(x,t) : x \in (-\infty,0) \cup (0,1),\ t \in (0,T]\}$,*
(iv) *$f_x(0^-,t)$, $f_x(0^+,t)$ are well defined for $t \in (0,T]$*
(v) *For $t \in (0,T]$, $f_x(x,t) \to 0$ when $x \to -\infty$,*
(vi) *Equations (9) are satisfied with $f(x,0) = \delta(x)$ in the following sense: for any $\varphi \in C_b(-\infty,1)$,*

$$\lim_{t \to 0^+} \int_{-\infty}^{1} \varphi(x) f(x,t) dx = \varphi(0). \tag{10}$$

The proof of Theorem 1 is shown in Section 3, which relies on an iteration approach. In fact, we decompose both the probability density of the stochastic process and the solution to equation (9) into series and show that there is a one-to-one correspondence between the two series representations.

Next we let the process start from any fixed $y < 1$, this time we use $f^y(x,t)$ to denote the p.d.f the process X_t in (18) start from y and now the distribution of X_0 is $f_{\text{in}}(x) = \delta(x-y)$. With the same method, we get the following corollary immediately.

Corollary 2.1. *For any fixed $y \in (-\infty,1)$, the process X_t as in (18) that starts from y has a continuously evolved probability density function denoted by $f^y(x,t)$. $f^y(x,t)$ is a solution of (9) in the time interval $(0,T]$ for any given $0 < T < +\infty$ and with initial condition $\delta(x-y)$ in the following sense:*

(i) *$N^y(t)$ is a continuous function for $t \in [0,T]$,*
(ii) *f^y is continuous in the region $\{(x,t) : -\infty < x \leq 1,\ t \in (0,T]\}$,*
(iii) *$\partial_{xx} f^y$ and $\partial_t f^y$ are continuous in the region $\{(x,t) : x \in (-\infty,0) \cup (0,1),\ t \in (0,T]\}$,*
(iv) *$\partial_x f^y(0^-,t)$, $\partial_x f^y(0^+,t)$ are well defined for $t \in (0,T]$*
(v) *For $t \in (0,T]$, $\partial_x f^y(x,t) \to 0$ when $x \to -\infty$,*
(vi) *Equations (9) are satisfied with $f(x,0) = \delta(x-y)$ in the following sense: for any $\varphi \in C_b(-\infty,1)$,*

$$\lim_{t \to 0^+} \int_{-\infty}^{1} \varphi(x) f^y(x,t) dx = \varphi(y). \tag{11}$$

Moreover, for any fixed $\varepsilon_0 > 0$, the continuity in (i), (ii), (iii) and the convergence in (v) and (vi) are uniform for $y \le 1 - \varepsilon_0$.

The proof of Corollary 2.1 is exactly same as in Theorem 1, and is thus skipped.

The initial condition of the Fokker-Planck equation (9) corresponds to the initial distribution of the stochastic process X_0. We remark that, in the above cases the most arguments below are based on the initial condition of the process $X_0 = y$ for any $y < 1$, and the corresponding initial condition of the PDE problem becomes $f(x, 0) = \delta(x-y)$. Although the initial condition is a singular function, we have shown that PDE has an instantaneous smoothing effect, while the solution coincide with the density function of the stochastic process. Since the problem is linear, the natural extension to general and proper initial conditions can be obtained by integration against the initial distribution (see e.g. [17] for a careful discussion).

Theorem 2. *Let ν be a c.d.f. whose p.d.f. $f_{in}(x) \in C_c(-\infty, 1)$. We assume that $f_{\text{in}}(x)$ is continuous and supported in $(-\infty, 1-\varepsilon_0)$ for some $\varepsilon_0 > 0$. Then the process X_t as in (18) that starts from p.d.f. $f_{in}(x)$ has a continuously evolved probability density function denoted by $f^\nu(x,t)$ with*

$$f^\nu(x,t) = \int_{-\infty}^{1-\varepsilon_0} f^y(x,t)\nu(dy), \quad x \in (-\infty, 1], \quad t > 0, \tag{12}$$

and $f^\nu(x,t)$ is a classical solution of (9) in the time interval $(0, T]$ for any given $0 < T < +\infty$ with initial condition $f_{\text{in}}(x)$ in the following sense:

- (i) *$N^\nu(t) := -\frac{\partial}{\partial x} f^\nu(1^-, t)$is a continuous function for $t \in [0, T]$,*
- (ii) *f^ν is continuous in the region $\{(x,t) : -\infty < x \le 1,\ t \in [0,T]\}$,*
- (iii) *$\partial_{xx} f^\nu$ and $\partial_x f^\nu$ are continuous in the region $\{(x,t) : x \in (-\infty, 0) \cup (0, 1),\ t \in [0, T]\}$,*
- (iv) *$\partial_x f^\nu(0^-, t)$, $\partial_x f^\nu(0^+, t)$ are well defined for $t \in [0, T]$,*
- (v) *For $t \in (0, T]$, $\partial_x f^\nu(x, t) \to 0$ when $x \to -\infty$.*
- (vi) *Equations (9) are satisfied with the L^2 convergence to the initial condition as $t \to 0^+$, i.e.*

$$\lim_{t \to 0^+} \int_{-\infty}^{1} |f^\nu(x,t) - f_{in}(x)|^2 dx = 0. \tag{13}$$

.

A proof can be found at the end of Section 3.1.

Remark 1. It is not clear yet how to get the uniform estimates near the boundary of the domain and thus we suppose that the initial distribution compactly support on $(-\infty, 1)$ in this paper. Actually, some recent works [27] concerning related models progressed towards more general assumptions, from compactly supported to $o(1-x)$ decay near 1 and more recently $\mathcal{O}\left((1-x)^\beta\right)$ with $\beta \in (0,1)$. Usually, the literature assumes $\mathcal{O}(1-x)$ decay near 1 (e.g. [10]) and in Theorem 1.1 of [16], this boundary decay is linked with short-term regularity of the solutions. Thus the hypothesis of a compactly supported initial condition has deep consequences upon the smoothness of the solution in short time.

2.2. **Construction of the Process.**

For the rest of this section, we shall present some preliminaries of the stochastic process. Firstly, we should give the process X_t a precise definition in probability by following the construction by Gihman and Skorohod [24]. We emphasize that, an addition process n_t is introduced to count the number of jumping events of a trajectory that has taken place before time t.

On a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we consider a sequence of independent O-U processes

$$\left\{Y_t^{(n)}\right\}_{n=1}^{\infty}$$

with $Y_0^{(n)} = 0$ for all $n \geq 1$. Note that an O-U process Y_t starting from initial value y_0 is an SDE with a.s. pathwisely continuous strong solution. That is

$$Y_t = e^{-t}y_0 + \sqrt{2}\int_0^t e^{-(t-s)}dB_s \tag{14}$$

with a normal p.d.f.:

$$N(e^{-t}y_0, 1-e^{-2t}). \tag{15}$$

For each $n \in \mathbb{N}$, $t \in [0, \infty]$, define the natural filtration

$$\mathcal{F}_t^{(n)} = \sigma\left(Y_s^{(n)} : s \in [0,t)\right).$$

I.e., $\mathcal{F}_t^{(n)}$ represents the information carried by the path of the nth copy of O-U process by time t. For all n, $\mathcal{F}_\infty^{(n)}$ are abbreviated to $\mathcal{F}^{(n)}$, which are easy to see to be jointly independent. Now define their filtration

$$\mathcal{G}_n = \sigma(\mathcal{F}^{(k)},\ k \leq n),\ \mathcal{G}^n = \sigma(\mathcal{F}^{(k)},\ k \geq n)$$

with the convention $\mathcal{G}_\infty = \mathcal{G}$.

For each n, let

$$\tau_n = \inf\left\{t \geq 0 : Y_t^{(n)} = 1\right\} = \inf\left\{t \geq 0 : \lim_{h \to t^-} Y_h^{(n)} = 1\right\} \tag{16}$$

be the first time $Y_t^{(n)}$ hits 1, with the convention $\tau_0 = 0$. Moreover, for all $n \geq 0$ and $k \leq n$, define

$$T_n = \sum_{i=0}^{n} \tau_i, \quad T_{n,k} = \sum_{i=k+1}^{n} \tau_i. \tag{17}$$

By definition, τ_n is a stopping time with respect to the natural filtration $\{\mathcal{F}_t^{(n)}\}_{t\geq 0}$. And we have that $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of i.i.d. r.v.'s with strictly positive expectation. Thus by the law of large numbers, $(\sum_{i=1}^n \tau_i)/n \to E[\tau_1] > 0$ a.s., which implies that

$$\mathbf{P}\left(\sum_{i=k}^{\infty} \tau_i = \infty,\ \forall k \geq 1\right) = 1.$$

Particularly, we have $T_n \to \infty$ a.s. as $n \to \infty$. Then within the almost sure event $A_0 = \{\sum_{i=k}^{\infty} \tau_i = \infty,\ \forall k \geq 1\}$, we define (X_t, n_t) as follows: for any $k \geq 1$

$$(X_t, n_t) = \left(Y_{t-T_{k-1}}^{(k)}, k-1\right) \tag{18}$$

on $[T_{k-1}, T_k)$. And thus T_k is interpreted as the k-th jumping time associated with X_t.

By definition, we have constructed a piecewise continuous path on $[0,\infty)$ for each $\omega \in A_0$, and thus a mapping from A_0 to $(D[0,\infty) \times \mathbb{N}, \mathcal{D} \times \mathcal{N})$ is clearly measurable with respect to $\mathcal{G}$, where $D[0,\infty)$ is the space of *càdlàg* paths. Here $\mathcal{D}$ is the smallest sigma field generated by all coordinate mappings and $\mathcal{N}$ is the trivial sigma field on $\mathbb{N}$. In the rest of this paper, we will use the construction above as the formal definition of (X_t, n_t), which is the stochastic process of interest.

Similarly, we can define the process X_t starts from $y < 1$ or starts from a distribution ν. We denote the probability measure of (X_t, n_t) by $\mathbf{P}^y(\cdot)$ and the expectation by $\mathbf{E}^y[\cdot]$. The meaning of $\mathbf{P}^\nu(\cdot)$ and $\mathbf{E}^\nu[\cdot]$ are analogous. Using F_{τ_k}/F_{T_k} to denote the cumulative distribution function of τ_k/T_k, it is immediate to see that for any k and t, $\mathbf{P}(\tau_k = t) \le \mathbf{P}(Y_t^{(k)} = 1) = 0$. So F_{τ_k} and F_{T_k} are always continuous.

2.3. **Properties of the Process and the Iteration Approach.**

We derive some preliminary estimates for the process (X_t, n_t), which manifest the solution properties and also motivate us to propose the iterated scheme.

It has been shown in [24] the process X_t constructed above is always Markovian. Now we are ready to show the following "Strong Markovian" type result that allows us to later calculate the probability distribution of (X_t, n_t) in an iterative fashion: for each integer $k \ge 0$, define

$$F_k(x,t) = \mathbf{P}^0(X_t \le x, n_t = k), \tag{19}$$

then we have

Proposition 2.1. *For any $x < 1$, $k \ge 1$, and $t > 0$,*

$$F_k(x,t) = \mathbf{E}_0\left[\mathbf{P}\left(Y_{t-T_k}^{(k+1)} \le x, \tau_{k+1} > t - T_k\right)\mathbb{1}_{T_k<t}\right]. \tag{20}$$

And thus

$$F_k(x,t) = \int_0^t F_0(x, t-s)dF_{T_k}(s). \tag{21}$$

Proof. We only prove (20) and then (21) is obvious. First, note that $T_{k+1} = T_k + \tau_{k+1}$ and that the event

$$\{n_t = k\} = \{T_k \le t, T_{k+1} > t\}.$$

By Fubini's formula,

$$\mathbf{P}^0(n_t = k) = \mathbf{E}_0\left[\mathbf{P}\left(\tau_{k+1} > t - T_k\right)\mathbb{1}_{T_k<t}\right].$$

Thus it suffices to prove

$$\mathbf{P}^0(X_t > x, n_t = k) = \mathbf{E}_0\left[\mathbf{P}\left(Y_{t-T_k}^{(k+1)} > x, \tau_{k+1} > t - T_k\right)\mathbb{1}_{T_k<t}\right].$$

Let $A = \{X_t > x, n_t = k\}$ be our event of interest. For any $n \ge 1$ and any $0 \le i \le 2^n - 1$, we define interval

$$I_n^{(i)}(t) = \left(2^{-n}it, 2^{-n}(i+1)t\right].$$

Moreover, for any $s \in (0,t]$ and any n, one may define $\mathrm{Id}(n,s)$ be the unique $i \le 2^n - 1$ such that $s \in I_n^{(i)}(t)$. Now we define event

$$A_n^{(i)} = \left\{\inf_{s\in t-I_n^{(i)}(t)} Y_s^{(k+1)} > x,\ \ \tau_{k+1} > (1-2^{-n}i)t\right\} \cap \{T_k \in I_n^{(i)}(t)\}$$

and $A_n = \cup_{i=0}^{2^n-1} A_n^{(i)}$. By definition, $A_n^{(i)} \subset A$ for every feasible n and i. Thus $\mathbf{P}(A_n) \le \mathbf{P}(A)$. On the other hand, for any $\omega \in \bar{A} = \{X_t > x, n_t = k, T_k < t\}$, The continuity of path in $Y^{(k+1)}$ guarantees that there has to be some $N < \infty$ such that for all $n \ge N$, $\omega \in A_n^{(\mathrm{Id}(n,T_k(\omega)))}$ and thus $\mathbf{P}^0(A_n) \to \mathbf{P}^0(\bar{A}) = \mathbf{P}^0(A)$ as $n \to \infty$. The last equality follows from the fact that F_{T_k} is continuous.

Meanwhile, note that T_k is independent to $Y^{(k+1)}$. We have

$$\begin{aligned}\mathbf{P}^0(A_n) &= \sum_{i=0}^{2^n-1}\mathbf{P}^0\left(T_k \in I_n^{(i)}(t)\right)\mathbf{P}\left(\inf_{s\in t-I_n^{(i)}(t)} Y_s^{(k+1)} > x,\ \ \tau_{k+1} > (1-2^{-n}i)t\right)\\ &= \mathbf{E}^0\left[\mathbf{P}\left(\inf_{s\in t-I_n^{(\mathrm{Id}(n,T_k))}(t)} Y_s^{(k+1)} > x,\ \ \tau_{k+1} > (1-2^{-n}\mathrm{Id}(n,T_k))t\right)\mathbb{1}_{T_k<t}\right].\end{aligned}$$

Now noting that for any $0 < h < t$, one may similarly have from the continuity of $Y^{(k+1)}$

$$\mathbf{P}\left(\inf_{s\in t-I_n^{(\mathrm{Id}(n,\tau_k))}(t)} Y_s^{(k+1)} > x,\ \tau_{k+1} > (1-2^{-n}\mathrm{Id}(n,h))t\right) \to \mathbf{P}\left(Y_{t-h}^{(k+1)} > x, \tau_{k+1} > t-h\right),$$

we have (20) follows from monotone convergence.

□

For any $t > 0$, we first consider the case where no jumps have been made by time t. Note that $F_0(x,t) = P(X_t \le x, T_1 > t) = \mathbf{P}(Y_t^{(1)} \le x, \tau_1 > t)$ for all $x \in (-\infty, 1)$. It is clear that $F_0(\cdot, t)$ induces a measure on $((-\infty,1), \mathcal{B})$, which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. The assertion above can be seen from the fact that for any measurable A, $\mathbf{P}(Y_t^{(1)} \in A, \tau_1 > t) \le \mathbf{P}(Y_t^{(1)} \in A)$ and that $Y_t^{(1)}$ is a continuous random variable. Here we also use $F_0(\cdot, t)$ to denote the corresponding measure on $((-\infty,1), \mathcal{B})$. And let $f_0(x,t)$ be its density and $p_{\mathrm{ou}}(x,t)$ denotes the p.d.f of $Y_t^{(1)}$. Thus we have

$$f_0(x,t) \le p_{\mathrm{ou}}(x,t) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\{\frac{-x^2}{2(1-e^{-2t})}\}, \tag{22}$$

which together with (15) derives

$$f_0(x,t) \le \frac{1}{\sqrt{2\pi(1-e^{-2t})}}. \tag{23}$$

Lemma 2.1. *$F_0(x,t)$ is a bi-variate continuous function on $(-\infty, 1] \times (0,\infty)$. Moreover, for any bounded continuous function $\varphi(x)$,*

$$\lim_{t\to 0^+} \mathbf{E}^0[\varphi(X_t)\mathbb{1}_{n_t=0}] = \lim_{t\to 0^+} \int_{-\infty}^1 \varphi(x) f_0(x,t)dx = \varphi(0). \tag{24}$$

Proof. In order to prove this lemma, one may first show that for any $(x,t) \in (-\infty,1)\times(0,\infty)$, $F_0(\cdot,\cdot)$ is continuous at (x,t) on both directions.

The continuity on the direction of x is obvious since that for all $x' > x$,

$$0 \le F_0(x',t) - F_0(x,t) \le \mathbf{P}\left(Y_t^{(1)} \in [x,x']\right) = \int_x^{x'} p_{\mathrm{ou}}(y,t)dy$$

and the last term goes to 0 as $x' \to x^+$.

Thus one may concentrate on proving continuity on the direction of t. Let Δ be the symmetric difference between events. One may first note that for any events $A = A_1 \cap A_2$, and $B = B_1 \cap B_2$,

$$\begin{aligned} A\Delta B &= (A_1 \cap A_2 \cap B_1^c) \cup (A_1 \cap A_2 \cap B_2^c) \cup (A_1^c \cap B_1 \cap B_2) \cup (A_2^c \cap B_1 \cap B_2) \\ &\subset (A_1 \cap B_1^c) \cup (A_2 \cap B_2^c) \cup (A_1^c \cap B_1) \cup (A_2^c \cap B_2) \\ &= (A_1 \Delta B_1) \cup (A_2 \Delta B_2). \end{aligned} \tag{25}$$

For any $t > 0$, fixed x_0 and any Δt sufficiently close to 0 (without loss of generality, one may assume $\Delta t > 0$)

$$F_0(x_0,t) = \mathbf{P}(Y_t^{(1)} \le x_0,\ \tau_1 > t)$$
$$F_0(x_0,t+\Delta t) = \mathbf{P}(Y_{t+\Delta t}^{(1)} \le x_0,\ \tau_1 > t+\Delta t).$$

Now let $A_1 = \{Y_t^{(1)} \le x_0\}$, $A_2 = \{\tau_1 > t\}$, and $B_1 = \{Y_{t+\Delta t}^{(1)} \le x_0\}$, $B_2 = \{\tau_1 > t + \Delta t\}$. By (25) we have

$$\begin{aligned}
&|F_0(x_0,t) - F_0(x_0,t+\Delta t)| \\
\le& \mathbf{P}(A\Delta B) \le \mathbf{P}(A_1\Delta B_1) + \mathbf{P}(A_2\Delta B_2) \\
=& \mathbf{P}\left(Y_t^{(1)} \le x_0, Y_{t+\Delta t}^{(1)} > x_0\right) + \mathbf{P}\left(Y_t^{(1)} > x_0, Y_{t+\Delta t}^{(1)} \le x_0\right) + \mathbf{P}(\tau_1 \in (t, t+\Delta t]) \\
\le& \mathbf{P}\left(\exists s \in [t, t+\Delta t],\ s.t.\ Y_s^{(1)} = x_0\right) + F_{\tau_1}(t+\Delta t) - F_{\tau_1}(t).
\end{aligned}$$

Recalling that $F_{\tau_1}(\cdot)$ is continuous,

$$\lim_{\Delta t \to 0} F_{\tau_1}(t+\Delta t) - F_{\tau_1}(t) = 0.$$

At the same time, for any positive integer n, define event

$$\Delta_n = \left\{\exists s \in \left[t, t+n^{-1}\right],\ s.t.\ Y_s^{(1)} = x_0\right\}.$$

Note that

$$\mathbf{P}(\Delta_n) \to \mathbf{P}(Y_t^{(1)} = x_0) = 0 \quad as \quad n \to \infty.$$

We can got the continuity of t.

Thus, one can show that $F_0(x,t)$ is binary continuous at (x,t) as follows: given $(x,t) \in (-\infty,1) \times (0,+\infty)$ and any $\epsilon > 0, \exists 0 < \delta < \frac{t}{2}$ such that for any $|t'-t| \le \delta$,

$$|F_0(x,t') - F_0(x,t)| < \frac{\epsilon}{2}.$$

And for any $s > \frac{t}{2}$ and any $|x'-x| \le \delta$ (Here without loss of generality, we ask $x < x'$)

$$|F_0(x',s) - F_0(x,s)| \le \mathbf{P}(Y_s^{(1)} \in [x,x']) < \frac{\epsilon}{2}.$$

(The last inequality is because when $s < \frac{t}{2}$, the density of $Y_s^{(1)}$ can be bounded by a big enough constant C.)

Then for all $(x',t') \in (-\infty,1) \times (0,\infty)$ such that $|t'-t| \le \delta, |x'-x| \le \delta$, we have

$$|F_0(x',t') - F_0(x,t)| \le |F_0(x',t') - F_0(x,t')| + |F_0(x,t') - F_0(x,t)| < \epsilon$$

Finally, we show that $F_0(x,t)$ is continuous at $x = 1$. It suffice to prove that for any $t_n \to t$ and $\varepsilon_n \to 0^+$, we have $\lim_{n\to\infty} F_0(1-\varepsilon_n, t_n) = F_0(1,t) = \mathbf{P}(\tau_1 > t)$, i.e. $\lim_{n\to\infty} \mathbf{P}(X_{t_n} \le 1-\varepsilon_n, \tau_1 > t_n) = \mathbf{P}(\tau_1 > t)$, which is equivalent to

$$\lim_{n\to\infty} \mathbf{P}(X_{t_n} > 1-\varepsilon_n, \tau_1 > t_n) = 0.$$

Set event $A_n = \{X_{t_n} > 1-\varepsilon_n, \tau_1 > t_n\}$ and we have

$$\mathbf{P}(\cup_{m\ge n}^{\infty} A_m) \le \mathbf{P}(\exists s \in [\max_{m\ge n} t_m, \max_{m\ge n} t_m],\ s.t.\ X_s > 1-\varepsilon_n, \tau_1 > \min_{m\ge n} t_m).$$

Note that $\limsup_{n\to\infty} \mathbf{P}(A_n) \le \mathbf{P}(\limsup A_n) \le \mathbf{P}(X_t \ge 1, \tau_1 \ge t) = 0$. Thus we get $\lim_{n\to\infty} \mathbf{P}(A_n) = 0$ and get the result we want.

Finally to prove (24), recall that φ is a bounded and continuous function. Thus $|\varphi(x)| \le M$ for all x, and for each $\varepsilon > 0$, there is a $0 < \delta < 1$ such that for all $x \in [-\delta,\delta]$, $|\varphi(x) - \varphi(0)| < \varepsilon$. So we have

$$|\mathbf{E}_0[\varphi(X_t)\mathbb{1}_{n_t=0}] - \varphi(0)| \le \varepsilon + 2M\mathbf{P}\left(\max_{s\le t} |Y_s^{(1)}| \ge \delta\right).$$

Now recalling (14)

$$|Y_t^{(1)}| \stackrel{d}{=} \left|\sqrt{2}\int_0^t e^{-(t-s)} dB_s\right| \stackrel{d}{\le} \left|\sqrt{2}\int_0^t e^{s} dB_s\right|, \tag{26}$$

where the d means the probability distribution. Note that the right hand side of (26) forms a martingale. One immediately have

$$\lim_{t\to 0^+} \mathbf{P}\left(\max_{s\le t} |Y_s^{(1)}| \ge \delta\right) = 0$$

by Doob's inequality. Thus we have shown (24) and then completed the proof.

$\square$

Remark 2. With Lemma 2.1, one may immediately have that $F(x_0, t)$ is a bounded and measurable function of $t \in [0, \infty)$.

Moreover, the following corollary follows directly from Proposition 2.1, Lemma 2.1, and a standard measure theory argument:

Corollary 2.2. *For any bounded measurable function f, any integer $k \ge 1$ and any $t > 0$, $\mathbf{E}\left[f(Y_t^{(1)})\mathbb{1}_{\tau_1 > t}\right]$ is measurable with respect to t, and*

$$\mathbf{E}^0\left[f(X_t)\mathbb{1}_{n_t=k}\right] = \mathbf{E}^0\left[\mathbf{E}\left[f(Y_{t-T_k}^{(k+1)})\mathbb{1}_{\tau_{k+1}>t-T_k}\right]\mathbb{1}_{T_k<t}\right]. \tag{27}$$

Note that

$$F_{\tau_1}(t) = 1 - P(\tau_1 > t) = 1 - F_0(1, t) = 1 - \int_{-\infty}^{1} f_0(x, t)dx \tag{28}$$

and

$$F_{T_n} = F_{\tau_1} * F_{\tau_2} * \cdots * F_{\tau_n}. \tag{29}$$

Moreover, for each n, $F_n(\cdot, t)$ is absolutely continuous and let $f_n(x, t)$ denotes its density.

In the rest of this section, we use Proposition 2.1 and the similar renewal argument as in [20] to calculate the distribution of X_t. First one has the following lemma

Lemma 2.2. *For all $n \ge 1$, $t > 0$, and $x < 1$,*

$$F_n(x, t) = \int_0^t F_{n-1}(x, t-s)dF_{\tau_1}(s). \tag{30}$$

Moreover, $F_n(x, t)$ is also bi-variate continuous on $(-\infty, 1] \times (0, \infty)$.

Proof. Suppose the lemma holds for $n - 1 \ge 0$, which has been shown true for $n = 1$. By Proposition 2.1, Lemma 2.1, and Fubini's formula on the independent random variables T_{n-1} and τ_n

$$\begin{aligned} F_n(x, t) &= \mathbf{P}(X_t \le x, n_t = n) = \mathbf{E}_0\left[\mathbf{P}\left(Y_{t-T_n}^{(n+1)} \le x, \tau_{n+1} > t - T_n\right)\mathbb{1}_{T_n<t}\right] \\ &= \mathbf{E}_0\left[F_0(x, t - T_n)\mathbb{1}_{T_n<t}\right] = \mathbf{E}_0\left[F_0(x, t - T_{n-1} - \tau_n)\mathbb{1}_{T_{n-1}+\tau_n<t}\right] \\ &= \int_0^t \int_0^{t-s} F_0(x, t-s-h)dF_{T_{n-1}}(h)dF_{\tau_1}(s) \\ &= \int_0^t F_{n-1}(x, t-s)dF_{\tau_1}(s) \end{aligned}$$

and thus we have got (30). With (30), for any $t_0 > 0$ and $x_0 < 1$, the continuity of $F_n(x, t)$ at (x_0, t_0) with respect to t can be shown as follows: For any $\varepsilon > 0$, by the continuity of $F_{\tau_1}(t)$, there is a $\delta_1 \in (0, t_0)$ such that

$$F_{\tau_1}(t_0 + \delta_1) - F_{\tau_1}(t_0 - \delta_1) < \varepsilon.$$

Now note that $F_{n-1}(x_0, t)$ is continuous on $(0, \infty)$ and thus uniformly continuous on $[\delta_1/2, t_0 + \delta_1]$. Thus there is a $\delta_2 > 0$ such that for all $t_1, t_2 \in [\delta_1/2, t_0 + \delta_1]$, $|t_1 - t_2| < \delta_2$,

$$|F_{n-1}(x_0, t_1) - F_{n-1}(x_0, t_2)| < \varepsilon.$$

Thus for any t such that $|t - t_0| < \min\{\delta_1/2, \delta_2\}$ (here we may without loss of generality assume that $t < t_0$), one has

$$\begin{aligned} |F_n(x_0, t_0) - F_n(x_0, t)| &\leq \int_0^{t_0-\delta_1} |F_{n-1}(x_0, t_0 - s) - F_{n-1}(x_0, t - s)| dF_{\tau_1}(s) \\ &+ \int_{t_0-\delta_1}^{t} F_{n-1}(x_0, t - s) dF_{\tau_1}(s) + \int_{t_0-\delta_1}^{t_0} F_{n-1}(x_0, t_0 - s) dF_{\tau_1}(s) \\ &\leq \varepsilon + 2[F_{\tau_1}(t_0 + \delta) - F_{\tau_1}(t_0 - \delta)] \leq 3\varepsilon. \end{aligned}$$

Similarly, the continuity of $F_n(x, t)$ at (x_0, t_0) with respect to x is guaranteed by that $F_{n-1}(x, t)$ is continuous and thus uniformly continuous on $[x, x'] \times [\varepsilon, t]$ for all $\varepsilon > 0$ and that $F_{\tau_1}(\cdot)$ put no mass on point t_0. And with the similar argument in the last lemma to show $F_n(\cdot, \cdot)$ is bi-variate continuous, we complete the proof.

□

With the same argument as before, we have

Corollary 2.3. *For any bounded measurable function f, any integer $k \geq 1$ and any $t > 0$,*

$$\mathbf{E}_0 \left[f(X_t) \mathbb{1}_{n_t = k} \right] = \int_{-\infty}^{1} f(x) dF_k(x, t)$$

is measurable with respect to t, and

$$\mathbf{E}_0 \left[f(X_t) \mathbb{1}_{n_t = k} \right] = \int_{-\infty}^{1} f(x) dF_k(x, t) = \int_0^t \int_{-\infty}^{1} f(x) dF_{k-1}(x, t - s) dF_{T_1}(s). \tag{31}$$

Our next lemma gives the exponential decay of $F_n(x, t)$ on a compact set of t, which is useful in our later calculations especially when we need to deal with the convergence of some series.

Lemma 2.3. *There is a $\theta > 0$ such that $T \in (0, \infty)$*

$$F_n(x, t) \leq \exp(-\theta n + T) \tag{32}$$

for all $n \in \mathbb{N}$, $t \leq T$ and $x \in (-\infty, 1]$.

Proof. For any $t \leq T$ and $x \in (-\infty, 1]$,

$$F_n(x, t) = \mathbf{P}(X_t \leq x, n_t = n) \leq \mathbf{P}(n_t \geq n) = \mathbf{P}(T_n \leq t) \leq \mathbf{P}(T_n \leq T).$$

Thus it suffices to show that

$$\mathbf{P}(T_n \leq T) \leq \exp(-\theta n + T).$$

Now recalling that $T_n = \sum_{i=1}^{n} \tau_i \in (0, \infty)$, define

$$Y_n = \exp(-T_n) \in (0, 1)$$

where by the independence of $\{\tau_i,\ i \geq 1\}$

$$\mathbf{E}[Y_n] = \left(\mathbf{E}[\exp(-\tau_1)]\right)^n.$$

Note that for a.s. $\omega, Y_t^{(1)}(\omega)$ is a continuous trajectory, which implies $\tau_1(\omega) > 0$ *a.s.*. Thus we have $\mathbf{P}(\tau_1 > 0) = 1$, which implies

$$\mathbf{E}[\exp(-\tau_1)] = \exp(-\theta) < 1$$

for some $\theta > 0$. Then the desired result follows from the Markov inequality for Y_n and the fact that $\{T_n \leq T\} = \{Y_n \geq \exp(-T)\}$. $\square$

Remark 3. The upper bound found in Lemma 2.3 is clearly not sharp, although it suffices the purpose in the later context.

In light of the properties of joint process (X_t, n_t) defined in (18) above, we have a new perspective to investigate the distribution of X_t. Let $F(x,t)$ denote the cumulative distribution function of X_t. Based on the number of jumping times, it admits the following decomposition

$$F(x,t) = \sum_{n=0}^{\infty} F_n(x,t). \tag{33}$$

There are two major types of results that we could obtain from the decomposition above.

On one hand, we immediately get the wellposedness and regularity properties of the distribution of X_t at a given time, which are not easily achievable due to the complication of jumps. We observe the right hand side of (33) converges by the bounded convergence theorem, and, moreover, it is clear that by the previous lemmas $F(x,t)$ is continuous on $(-\infty, 1] \times (0, \infty)$. Besides, due to the exponential decay of $F_n(x,t)$ with respect to n, we know that the measure induced by $F(\cdot, t)$ is absolutely continuous with respect to the Lebesgue measure, whose density function we shall denote by $f(x,t)$.

On the other hand, such a decomposition provides an auxiliary degree of freedom in the representation of the density function, which facilitates analyzing the time evolution of the density function. While the flux shift mechanism makes the evolution of $F(x,t)$ nonlocal, the decomposition unfold the distribution by adding one more dimension such that the evolution has a simpler structure: the evolution of F_0 is self-contained without any nonlocality, and for $n \geq 1$, the evolution of F_n is also local, although it has a tractable dependence on F_{n-1}. Recall that, we have used $f_n(x,t)$ to denote the density function of $F_n(x,t)$ respectively. In fact, we are able to show that $f_n(x,t)$ is a solution to a sub-PDE problem, and eventually, the exponential convergence in n can help conclude that

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t) \tag{34}$$

is a solution of the PDE problem of interest satisfying the properties in Theorem 1.

3. Iteration Approach

In this section we aim to prove the theorems in Section 2.1. First, we prove the density of the process X_t that starts from 0 is an instantaneous smooth mild solution of (9) with initial condition $f_{\text{in}}(x) = \delta(x)$. Then with similar treatment we can get Corollary 2.1 easily, which together with the integral representation (12) derive Theorem 2. Finally, we show that the mild solution is consistent with the definition of the weak solution of (9) defined in [5].

3.1. Solutions in Iteration.

Recalling the process (X_t, n_t) defined in (18) above, we first focus on the case $X_0 = 0$, i.e. the initial condition PDE (9) is $f(x,0) = \delta(x)$. In the previous section, we have decomposed the distribution $F(x,t)$ of the stochastic process X_t into a summation of series $\{F_n(x,t)\}_{n=0}^{+\infty}$ according to (19) and (33). We also decompose the original PDE problem (9) into a sequence of sub-PDE problems: for $n = 0$

$$\begin{cases} \dfrac{\partial f_0}{\partial t} - \dfrac{\partial}{\partial x}(x f_0) - \dfrac{\partial^2 f_0}{\partial x^2} = 0, \quad x \in (-\infty, 1), t \in (0, T], \\ f_0(-\infty, t) = 0, \quad f_0(1, t) = 0, \quad t \in [0, T], \\ f_0(x, 0) = \delta(x) \quad \text{in } \mathcal{D}'(-\infty, 1) \end{cases} \tag{35}$$

where $\mathcal{D}(-\infty, 1) = C_c^\infty(-\infty, 1)$ and for $n \geq 1$ define $N_{n-1}(t) = -\frac{\partial}{\partial x} f_{n-1}(1, t)$, we solve

$$\begin{cases} \dfrac{\partial f_n}{\partial t} - \dfrac{\partial}{\partial x}(x f_n) - \dfrac{\partial^2 f_n}{\partial x^2} = 0, \quad x \in (-\infty, 0) \cup (0, 1), t \in (0, T], \\ f_n(0^-, t) = f_n(0^+, t), \quad \dfrac{\partial}{\partial x} f_n(0^-, t) - \dfrac{\partial}{\partial x} f_n(0^+, t) = N_{n-1}(t), \quad t \in (0, T], \\ f_n(-\infty, t) = 0, \quad f_n(1, t) = 0, \quad t \in [0, T], \\ f_n(x, 0) = 0, \quad x \in (-\infty, 1). \end{cases} \tag{36}$$

In particular, we find the PDE problem for f_0 (35) is self-contained with a singular initial data, and thus only a mild solution can be expected, which, however, can be shown to be instantaneously smooth. For $n \geq 1$ the PDE problems for f_n (36) are defined when $x \in (-\infty, 0) \cup (0, 1)$, and the time-dependent interface boundary data N_{n-1} at $x = 1$ is determined by f_{n-1}, the solution to the previous PDE problem in the sequence, but the classical solution of such problems can be understood in the usual sense.

Here is a bit ambiguity in the notations, since we have used $f_n(x, t)$ to denote the sub density function of the stochastic process and also the solution to the PDE problem. In fact, we shall show those two functions coincide, of which the precise meaning shall be specified. In the following, we show that sub density function f_0 with delta initial data is an instantaneous smooth mild solution of (35), and then following the iteration scheme, we prove that for each $n \geq 1$, the sub density function f_n is the classical solution of (36). We conclude with the proof of Theorem 1 by the end of this section.

Before we start to prove our main theorem, we first discuss the Green function of the Fokker-Planck equation (35). According to Theorem 1.10 in Chapter VI of [22] by Garroni and Menaldi, we know that the generator of the O-U process (14), i.e.,

$$\mathcal{L}_y := (-y)\partial_y \cdot + \partial_{yy}^2 \cdot,$$

admits a Green's function $G : (-\infty, 1] \times [0, T] \times (-\infty, 1] \times [0, T] \ni (y, s, x, t) \mapsto G(y, s, x, t)$. For a given $(x, t) \in (-\infty, 1] \times [0, T]$, the function $(-\infty, 1] \times [0, t) \ni (y, s) \mapsto G(y, s, x, t)$ is a solution of the PDE

$$\begin{cases} \partial_s G(y, s, x, t) + \mathcal{L}_y G(y, s, x, t) = 0, \quad y \in (-\infty, 1), s \in [0, t), \\ G(1, s, x, t) = 0, \quad s \in [0, t], \\ G(y, t, x, t) = \delta(y - x) \text{ in } \mathcal{D}'(-\infty, 1) \end{cases} \tag{37}$$

Following Theorem 5 in Chap.9 of [21], for a given $(y, s) \in (-\infty, 1) \times [0, T)$, the function $(-\infty, 1] \times (s, T] \ni (x, t) \mapsto G(y, s, x, t)$ is also known to be Green's function of the adjoint operator

$$\mathcal{L}_x^* = \partial_x[x \cdot] + \partial_{xx}^2 \cdot,$$

i.e. the function $(-\infty,1]\times(s,T]\ni(x,t)\mapsto G(y,s,x,t)$ is a classical solution of the PDE

$$\begin{cases}\partial_t G(y,s,x,t)=\mathcal{L}_x^* G(y,s,x,t), & x\in(-\infty,1), t\in(s,T],\\ G(y,s,1,t)=0, \quad t\in[s,T], & \\ G(y,s,x,s)=\delta(x-y) \text{ in } \mathcal{D}'(-\infty,1), & \end{cases} \tag{38}$$

which is consistent with (35). Now we give an important lemma that connects the density function of the stochastic process before the first jumping time with the Green function of PDE problem (35), which is the starting point of our iteration strategy. And for Green function G, although we can not find a closed formula for it, there exists the following estimation.

Lemma 3.1. *There exists a unique Green function* $G:(-\infty,1]\times[0,T]\times(-\infty,1]\times[0,T]\ni(y,s,x,t)\mapsto G(y,s,x,t)$ *for equation* (35). *Let* $f_0(x,t)$ *denotes the density of the distribution* $F_0(x,t)$ *defined in* (19), *then* $f_0(x,t)=G(0,0,x,t)$, *i.e., it is a mild solution of* (35) *on* $(-\infty,1]\times[0,T]$. *Besides, we have the estimation:*

$$\left|\partial^\ell G(y,s,x,t)\right|\le C(t-s)^{-\frac{1+\ell}{2}}\exp\left(-C_0\frac{(x-y)^2}{t-s}\right),\quad 0\le s<t\le T. \tag{39}$$

where $\ell=0,1,2$, $\partial^\ell=\partial^\ell_{tx}=\partial^m_t\partial^n_x$, $\ell=2m+n$, *for* $m,n\in\mathbb{N}_0$.

Proof. Set

$$p(x,t):=G(0,0,x,t),\qquad x\in(-\infty,1],t\in(0,T].$$

now we prove that $p(x,t)$ coincides with $f_0(x,t)$, which immediately derives that $f_0(x,t)$ is a mild solution of equation (35). Given a smooth function $\phi:(-\infty,1]\times[0,T]\to\mathbb{R}$ with a compact support, noting that Green's function satisfies (38), we have that the PDE

$$\begin{cases}\partial_s u(y,s)-y\partial_y u(y,s)+\partial_{yy}u(y,s)+\phi(y,s)=0, & (y,s)\in(-\infty,1)\times(0,T]\\ u(1,s)=0, & s\in[0,T],\\ u(y,T)=0 & y\in(-\infty,1)\end{cases} \tag{40}$$

admits a (unique) classical solution

$$u(y,s)=\int_s^T\int_{-\infty}^1 G(y,s,x,t)\phi(x,t)dxdt,\quad s\in[0,T),\ y\le 1. \tag{41}$$

Moreover, u is bounded and continuous on $(-\infty,1]\times[0,T]$ and is once continuously differentiable in time and twice differentiable in space on $(-\infty,1)\times[0,T]$. Let (X_t,n_t) be the process defined in (18) and $\tau:=\inf\{t\ge0: X_{t\wedge T}\ge1\}$. By $It\hat{o}'s$ formula, we have

$$du(X_{t\wedge\tau},t\wedge\tau)=-\phi(X_{t\wedge\tau},t\wedge\tau)dt+\sqrt{2}u_x(X_{t\wedge\tau},t\wedge\tau)dB_t$$

Integrating above formula from 0 to T and take the expectation, with the boundary condition in (40), we then have the representation formula:

$$u(0,0)=\mathbf{E}\left[\int_0^{T\wedge\tau}\phi(X_t,t)dt\right] \tag{42}$$

And with the two presentations for $u(0,0)$ above, i.e. (41) and (42), we obtain

$$\mathbf{E}\left[\int_0^{T\wedge\tau}\phi(X_t,t)dt\right]=\int_0^T\int_{-\infty}^1 p(x,t)\phi(x,t)dxdt.$$

We further rewrite (42) as follows.

$$\mathbf{E}\left[\int_0^{T\wedge\tau}\phi(X_t,t)dt\right]=\int_0^T\mathbf{E}\left[\phi(X_t,t)\mathbb{1}_{\{t\le\tau\}}\right]dt=\int_0^T\int_{-\infty}^1\phi(x,t)\mathbf{P}(X_t\in dx,\tau>t)dt,$$

Clearly, for $t \in [0,T]$, $\{\tau > t\} = \{T_1 > t\} = \{n_t = 0\}$ and thus

$$\int_0^T \int_{-\infty}^1 \phi(x,t) f_0(x,t) dx dt = \int_0^T \int_{-\infty}^1 \phi(x,t) p(x,t) dx dt \tag{43}$$

By (22) and (39), $p(x,t)$ and $f_0(x,t)$ decay at $-\infty$ and thus (43) is also valid for any smooth function ϕ that is only bounded, which derives that the density function $f_0(x,t)$ coincides with $p(x,t)$. With (24), we conclude that $f_0(x,0) = \delta(x)$ and thus $f_0(x,t)$ is a mild solution of (35). The complete proof of estimation (39) can be found in Theorem 1.10 in Chapter VI of [22] by Garroni and Menaldi and the proof is complete.

□

Remark 4. The proof of Lemma 3.1 is essentially implied from the results in [17, 18, 22], in particular, Lemma 2.1 of [17] and Theorem 1.10 in Chapter VI of [22].

Next, we prove some regularities of the sub-density $f_0(x,t)$ that are useful in our later calculations.

Proposition 3.1. *Let X_t be the process defined in (18) and T_1 be the stopping time defined in (17). Let $F_0(x,t)$ be defined in (19) and its density is denoted as $f_0(x,t)$. Let $f_{T_1}(t)$ denotes the p.d.f. of T_1. For any fixed $T > 0$, we have*

(i)
$$\lim_{x \to -\infty} \partial_x f_0(x,t) = 0, \quad t \in (0,T]. \tag{44}$$

(ii) *For any $x_0 \in (0,1)$, $f_0(x,t) \in C^{2,1}((-\infty,-x_0] \cup [x_0,1] \times [0,T])$. Moreover for all $|x| \geq |x_0|$, $\lim_{t \to 0^+} f_0(x,t) = 0$.*

(iii) *For any $0 < \varepsilon_0 < T < \infty$, $f_0(x,t) \in C^{2,1}((-\infty,1] \times [\varepsilon_0,T])$. With the following uniform gradient estimations*

$$\begin{aligned} &\sup_{(-\infty,1]\times[\varepsilon_0,T]} |f_0| < \infty, \quad \sup_{(-\infty,1]\times[\varepsilon_0,T]} \left|\frac{\partial f_0}{\partial t}\right| < \infty, \quad \sup_{(-\infty,1]\times[\varepsilon_0,T]} \left|\frac{\partial f_0}{\partial x}\right| < \infty, \\ &\sup_{(-\infty,1]\times[\varepsilon_0,T]} \left|\frac{\partial (x f_0)}{\partial x}\right| < \infty, \quad \sup_{(-\infty,1]\times[\varepsilon_0,T]} \left|\frac{\partial^2 f_0}{\partial x^2}\right| < \infty. \end{aligned} \tag{45}$$

(iv) *We have the coupling relation between $f_{T_1}(t)$ and $f_0(x,t)$: $\forall t \in (0,T]$, it satisfies*

$$f_{T_1}(t) = -\int_{-\infty}^1 \frac{\partial f_0(x,t)}{\partial t} dx = -\frac{\partial}{\partial x} f_0(1,t) \tag{46}$$

and $f_{T_1}(t) \in C[0,T]$ with $f_{T_1}(0) = 0$.

Proof. (i) is the direct corollary of estimate (39). And from (39), we know that the Green function of (35) is continuous differentiable and decays exponentially fast as t tends to 0^+ when x stay away from 0. Thus we immediately obtain the properties in (ii). Also by the estimation (39) for the Green function, we can easily get the bound for f_0 in (iii) when t stay away from 0. Finally, to prove (iv), recall that $f_0(x,t)dx = \mathbb{P}(X_t \in dx, T_1 > t)$, thus the c.d.f of T_1 is given by

$$\mathbf{P}(T_1 \leq t) = 1 - \mathbf{P}(T_1 > t) = 1 - \int_{-\infty}^1 f_0(x,t) dx.$$

By (39), we can differentiate the above formula w.r.t t and exchange the derivative and the integral. Using (i) and the boundary condition of f_0, we have for any $t \in (0,T]$,

$$f_{T_1}(t) = \frac{d}{dt}\mathbf{P}(T_1 \leq t) = -\int_{-\infty}^1 \frac{\partial f_0(x,t)}{\partial t} dx = -\int_{-\infty}^1 \frac{\partial}{\partial x}(x f_0) + \frac{\partial^2 f_0}{\partial x^2} dx = -\frac{\partial}{\partial x} f_0(1,t).$$

And with Lemma 3.1:

$$|f_{T_1}(t)| = |\partial_x f_0(1,t)| \le \frac{C}{t}\exp(-\frac{C_0}{t}),$$

we conclude $f_{T_1}(t) \in C(0,T]$ and $\lim_{t\to 0^+} f_{T_1}(t) = 0$ and thus $f_{T_1}(t) \in C[0,T]$.

□

In order to make the iteration strategy successful, we need to further show that $f_{T_1}(t)$ is continuously differentiable, which is not a direct consequence of estimating Green's function. Thus next we shall prove that $f_{T_1}(t) \in C^1[0,T]$ and the following estimation is useful in the further calculations.

Corollary 3.1. *For any $T > 0$ and $\forall 0 < \varepsilon_0 < \min\{\frac{1}{T}, T\}$, $f_{T_1}(t) \in C^1(0,T]$ and for any $t \ge \varepsilon_0$, we have*

$$\left|f'_{T_1}(t)\right| \le C\varepsilon_0^{-3}. \tag{47}$$

Proof. By Proposition 3.1, we know that $f_0(x,t) \in C^{2,1}((-\infty,1]\times[\varepsilon_0,T])$ and $f_{T_1}(t) = -\frac{\partial}{\partial x} f_0(1,t) \in C[0,T]$. Then for any $x \in (-\infty,1]$, $t \in [\varepsilon_0,T]$, set $g_0(x,t) = \frac{\partial}{\partial t} f_0(x,t)$ and it satisfies

$$\begin{cases} \dfrac{\partial g_0}{\partial t} - \dfrac{\partial}{\partial x}(x g_0) - \dfrac{\partial^2 g_0}{\partial x^2} = 0, \quad x \in (-\infty,1), t \in (\varepsilon_0,T], \\ g_0(-\infty,t) = 0, \quad g_0(1,t) = 0, \quad t \in [\varepsilon_0,T], \\ g_0(x,\varepsilon_0) = \dfrac{\partial}{\partial t} f_0(x,\varepsilon_0) \quad x \in (-\infty,1). \end{cases} \tag{48}$$

Defining $\varphi(x) := \frac{\partial}{\partial t} f_0(x,\varepsilon_0)$, we immediately get that $\varphi(x) \in C^2(-\infty,1] \cap L^\infty(-\infty,1]$ and by (39)

$$|\varphi(x)| \le C\varepsilon_0^{-\frac{3}{2}}.$$

For any $t \ge 0$, $x \in (-\infty,1]$, define $h(x,t) := g_0(x,t+\varepsilon_0)$ and then $h(x,0) = \varphi(x)$. Recalling the Green function $G(s,y,x,t)$ in PDE (38), we have

$$h(x,t) = \int_{-\infty}^{1} G(y,0,t,x)\varphi(y)dy, \quad t \ge 0.$$

Then

$$g_0(x,t) = \int_{-\infty}^{1} G(y,0,t-\varepsilon_0,x)\varphi(y)dy, \quad t \ge \varepsilon_0. \tag{49}$$

By (39) and Lemma 3.1, we have

$$f'_{T_1}(t) = -\frac{\partial}{\partial t}\frac{\partial}{\partial x} f_0(1,t) = -\frac{\partial}{\partial x} g_0(1,t) = -\int_{-\infty}^{1} \frac{\partial}{\partial x} G(y,0,t-\varepsilon_0,1)\varphi(y)dy, \quad t > \varepsilon_0 \tag{50}$$

and thus $f_{T_1}(t) \in C^1(\varepsilon_0, T]$.
When $t \geq 2\varepsilon_0$,

$$\begin{aligned}
|f'_{T_1}(t)| &\leq \int_{-\infty}^{1} \left| \frac{\partial}{\partial x} G(y, 0, t-\varepsilon_0, 1) \right| C\varepsilon_0^{-\frac{3}{2}} dy \\
&\leq C\varepsilon_0^{-\frac{3}{2}} \int_{-\infty}^{1} \frac{C}{t-\varepsilon_0} \exp\left(-C_0 \frac{(1-y)^2}{t-\varepsilon_0}\right) dy \\
&= C\varepsilon_0^{-\frac{5}{2}} \int_0^{+\infty} \exp\left(-C_0 \frac{\xi^2}{t-\varepsilon_0}\right) d\xi \\
&\leq C\varepsilon_0^{-\frac{5}{2}} \frac{\sqrt{T-\varepsilon_0}}{\sqrt{C_0}} \frac{\sqrt{\pi}}{2} \\
&\leq C\varepsilon_0^{-3}.
\end{aligned} \tag{51}$$

where the second inequality is by the change of variable $\xi = 1 - y$ and the third inequality is from the fact $\varepsilon_0 \leq \frac{1}{T}$. And because ε_0 can be arbitrarily small, we complete the proof.

□

Now we focus on the behavior of $f'_{T_1}(t)$ when t is small. This proof is partially inspired by the reformulation and the representation proposed in [11].

Proposition 3.2. *The p.d.f. $f_{T_1}(t)$ of the first hitting time T_1 is $C^1[0,T]$ for any fixed $T > 0$.*

Proof. By Proposition 3.1 and Corollary 3.1, we know $f_{T_1}(t) \in C^1(0,T] \cap C[0,T]$ and thus we only need to prove that $\lim_{t\to 0^+} f'_{T_1}(t)$ exists. We prove it in the following steps. *Step 1:* We rewrite the problem (35) as a moving boundary problem and rewrite $f_{T_1}(t)$ as $M(s)$. With the heat kernel Γ, we derive an integral representation of $M(s)$. *Step 2:* We analyse the decay rate of $M(s)$ and $M'(s)$ at 0 by utilizing the decay property of heat kernel Γ. *Step 3:* Using the estimations of $M(s)$, $M'(s)$ and heat kernel Γ, we derive $\lim_{t\to 0^+} f'_{T_1}(t) = 0$.

Step 1: Inspired from [11], we introduce a change of variable to transform (35) to a moving boundary problem. Let

$$y = e^t x, \quad s = (e^{2t} - 1)/2, \quad u(y,s) = e^{-t} f(x,t). \tag{52}$$

Note that PDE (35) is for the O-U process killed at a stopping time and thus has the Dirichlet boundary condition. By the standard change of variable (52), we can transform (35) into a heat equation with the moving boundary $b(s) = \sqrt{2s+1}$. Actually, we have the new equation

$$\begin{cases} u_s = u_{yy}, & y \in (-\infty, b(s)), s > 0 \\ u(-\infty, s) = 0, \quad u(b(s), s) = 0, & s \geq 0, \\ u(y,0) = \delta(y) \text{ in } \mathcal{D}'(-\infty, b(s)). \end{cases} \tag{53}$$

Let Γ be the Green's function for the heat equation on the real line:

$$\Gamma(y,s,\xi,\tau) = \frac{1}{\sqrt{4\pi(s-\tau)}} \exp\{-\frac{(y-\xi)^2}{4(s-\tau)}\}, \quad s > \tau. \tag{54}$$

In the region $-\infty < \xi < b(\tau),\ 0 < \tau < h$, recall the Green's identity

$$\frac{\partial}{\partial \xi}(\Gamma u_\xi - u\Gamma_\xi) - \frac{\partial}{\partial \tau}(\Gamma u) = 0. \tag{55}$$

To derive an expression of u, we consider the integration of (55) over such a region and let

$$I = \int_0^s \int_{-\infty}^{b(\tau)} (\Gamma u_\xi)_\xi d\xi d\tau, \quad II = \int_0^s \int_{-\infty}^{b(\tau)} (u\Gamma_\xi)_\xi d\xi d\tau, \quad III = \int_0^s \int_{-\infty}^{b(\tau)} (\Gamma u)_\tau d\xi d\tau.$$

We have

$$I=\int_0^s \Gamma u_\xi|_{\xi=b(\tau)}d\tau.$$

Using the boundary condition of $u(y,s)$ in (53), we have

$$II=0$$

and

$$III=\int_{-\infty}^{b(s)}\Gamma u|_{\tau=s^-}d\xi-\int_{-\infty}^{b(0)}\Gamma u|_{\tau=0}d\xi=u(y,s)-\int_{-\infty}^{b(0)}\Gamma u|_{\tau=0}d\xi.$$

Plugging in (55),

$$\begin{aligned}u(y,s)&=\int_{-\infty}^{b(0)}\Gamma(y,s,\xi,0)\delta(\xi)d\xi+\int_0^s\Gamma(y,s,b(\tau),\tau)u_\xi(b(\tau),\tau)d\tau\\&=\Gamma(y,s,0,0)-\int_0^s\Gamma(y,s,b(\tau),\tau)M(\tau)d\tau.\end{aligned}\tag{56}$$

where $M(\tau)=-u_\xi(b(\tau),\tau)$. Note that the Green function Γ is infinitely continuously differentiable, thus the regularity of u depends on M. Using Lemma 1 on Page 217 of [21], we know that for any continuous function ρ the following limit holds:

$$\lim_{y\to b(s)^-}\frac{\partial}{\partial y}\int_0^s\rho(\tau)\Gamma(y,s,b(\tau),\tau)d\tau=\frac{1}{2}\rho(s)+\int_0^s\rho(\tau)\Gamma_y(b(s),s,b(\tau),\tau)d\tau.$$

So differentiating (56) at $y=b(s)^-$, we can get the following integral equation

$$-M(s)=\Gamma_y(b(s),s,0,0)-\frac{1}{2}M(s)-\int_0^s\Gamma_y(b(s),s,b(\tau),\tau)M(\tau)d\tau.$$

That is

$$\begin{aligned}M(s)&=-2\Gamma_y(b(s),s,0,0)+2\int_0^s\Gamma_y(b(s),s,b(\tau),\tau)M(\tau)d\tau\\&=:2J_1(s)+2J_2(s).\end{aligned}\tag{57}$$

Recalling the change of variable in (52) and taking derivatives directly, we know that

$$f_{T_1}(t)=e^{2t}M(s)\quad\text{and}\quad f_{T_1}'(t)=2e^{2t}M(s)+e^{4t}M'(s).\tag{58}$$

Step 2: We shall analyse the decay rate of $M(s)$ at 0. By heat kernel (54) $\Gamma(y,s,0,0)=\frac{1}{\sqrt{4\pi s}}\exp(\frac{-y^2}{4s})$ and $b(s)=\sqrt{2s+1}$, we have that for any $n\geq 0$, $\lim_{s\to 0^+}\frac{J_1(s)}{s^n}=0$ and thus there exists a constant C such that for $s\in[0,T]$, $n\geq 0$

$$|J_1(s)|\leq Cs^n.\tag{59}$$

Note that

$$\Gamma_y(b(s),s,b(\tau),\tau)=\frac{1}{\sqrt{4\pi(s-\tau)}}\exp\{-\frac{(b(s)-b(\tau))^2}{4(s-\tau)}\}\{\frac{b(s)-b(\tau)}{-2(s-\tau)}\},\tag{60}$$

thus we have

$$|\Gamma_y(b(s),s,b(\tau),\tau)|\leq\frac{C}{(s-\tau)^{\frac{1}{2}}}.$$

By (iv) of Proposition 3.1 and (58), there exists another big enough constant K s.t. $|M(s)|\leq K$, $\forall s\in[0,T]$. Thus

$$|J_2(s)|\leq C\int_0^s\frac{K}{(s-\tau)^{\frac{1}{2}}}=C\sqrt{s}.$$

Combining with (59), we also have $|M(s)| \le |J_1(s)| + |J_2(s)| \le C\sqrt{s}$, and thus

$$|J_2(s)| \le C\int_0^s \frac{\sqrt{\tau}}{(s-\tau)^{\frac{1}{2}}} = Cs.$$

Using (59) again, we have $|M(s)| \le Cs$ and thus

$$|J_2(s)| \le C\int_0^s \frac{\tau}{(s-\tau)^{\frac{1}{2}}} = Cs^{\frac{3}{2}}.$$

Using (59) for the third time, we can get $|M(s)| \le Cs^{\frac{3}{2}}$, which together with $M(0) = 0$ leads to the right derivative of M at 0 exists and

$$M'(0^+) = \lim_{s\to 0^+} \frac{M(s)}{s} = 0.$$

Repeating the above calculations step by step, we can get for any $n \ge 0$, there exists a constant that depends on n, such that

$$|M(s)| \le Cs^n. \tag{61}$$

By (47) and (58), we know that for any sufficiently small $\varepsilon_0 > 0$, there is a constant $C < +\infty$ such that

$$|M'(s)| \le C\varepsilon_0^{-3}, \quad \forall s \in [\varepsilon_0, 1]. \tag{62}$$

Step 3: In order to prove $f_{T_1}(t) \in C^1[0,T]$, which is equivalent to prove that $\lim_{s\to 0^+} M'(s)$ exists by (58), now we prove that $\lim_{s\to 0^+} M'(s) = 0$. Using (57) and the fact $\lim_{s\to 0^+} J_1'(s) = 0$, we only need to to prove that

$$\lim_{s\to 0^+} J_2'(s) = 0. \tag{63}$$

Using the estimations (61), (62) and heat kernel Γ, we compute the difference between $A := \int_0^s \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau$ and $B := \int_0^{s+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau$.

A can have the following decomposition

$$A := \left(\int_0^{\frac{s}{2}} + \int_{\frac{s}{2}}^s\right)\Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau.$$

and for B,

$$B := \left(\int_0^{\frac{s}{2}} + \int_{\frac{s}{2}}^{\frac{s}{2}+\Delta s} + \int_{\frac{s}{2}+\Delta s}^{s+\Delta s}\right)\Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau.$$

Define

$$\frac{J_2(s+\Delta s) - J_2(s)}{\Delta s} =: I_1 + I_2 + I_3$$

where

$$I_1 := \int_0^{\frac{s}{2}} \left[\frac{\Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau) - \Gamma_y(b(s), s, b(\tau), \tau)}{\Delta s}\right] M(\tau)d\tau,$$

$$I_2 := \frac{1}{\Delta s}\int_{\frac{s}{2}}^{\frac{s}{2}+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau$$

and

$$I_3 := \frac{1}{\Delta s}\left[\int_{\frac{s}{2}+\Delta s}^{s+\Delta s} \Gamma_y(b(s+\Delta s), s+\Delta s, b(\tau), \tau)M(\tau)d\tau - \int_{\frac{s}{2}}^{s} \Gamma_y(b(s), s, b(\tau), \tau)M(\tau)d\tau\right].$$

Thus to get (63), now it suffices to show that

$$\lim_{\Delta s\to 0}|I_1| \le \int_0^{\frac{s}{2}} |\partial_s\Gamma_y(b(s),s,b(\tau),\tau)M(\tau)|\,d\tau = o(1), \tag{64}$$

$$\lim_{\Delta s\to 0}|I_2| = o(1) \tag{65}$$

and

$$\lim_{\Delta s\to 0}|I_3| = o(1). \tag{66}$$

The above $= o(1)$ means that the left side goes to 0 as $s \to 0^+$.

Note that for $\tau \le \frac{3}{4}s$, then Γ_y and $\partial_s\Gamma_y$ terms in (64) and (65) can be bounded by a polynomial order with respect to s^{-1}, which together with (61) immediately derives (64) and (65). Thus we only need to focus on proving (66). With a simple change of variable, we have

$$\begin{aligned}
&\int_{\frac{s}{2}+\Delta s}^{s+\Delta s} \Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau),\tau)M(\tau)d\tau\\
&=\int_{\frac{s}{2}}^{s} \Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s)M(\tau+\Delta s)d\tau\\
&=\int_{\frac{s}{2}}^{s} \Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s)M(\tau)d\tau\\
&+\int_{\frac{s}{2}}^{s} \Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s)\left[M(\tau+\Delta s)-M(\tau)\right]d\tau.
\end{aligned}$$

We define

$$I_3 := I_{3,1} + I_{3,2}$$

where

$$I_{3,1} = \frac{1}{\Delta s}\int_{\frac{s}{2}}^{s} \left[\Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s) - \Gamma_y(b(s),s,b(\tau),\tau)\right]M(\tau)d\tau$$

and

$$I_{3,2} = \int_{\frac{s}{2}}^{s} \Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s)\frac{M(\tau+\Delta s)-M(\tau)}{\Delta s}d\tau.$$

Thus to show (66), it suffices to prove

$$\lim_{\Delta s\to 0}|I_{3,1}| = o(1) \tag{67}$$

and

$$\lim_{\Delta s\to 0}|I_{3,2}| = o(1). \tag{68}$$

For (67), by (60) we have

$$\begin{aligned}
&\Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s) - \Gamma_y(b(s),s,b(\tau),\tau)\\
&=\frac{1}{\sqrt{4\pi(s-\tau)}}\exp\{-\frac{1}{2}\frac{b(s+\Delta s)-b(\tau+\Delta s)}{b(s+\Delta s)+b(\tau+\Delta s)}\}\{\frac{-1}{b(s+\Delta s)+b(\tau+\Delta s)}\}\\
&-\frac{1}{\sqrt{4\pi(s-\tau)}}\exp\{-\frac{1}{2}\frac{b(s)-b(\tau)}{b(s)+b(\tau)}\}\{\frac{-1}{b(s)+b(\tau)}\},
\end{aligned}$$

and thus there exists a constant $C < +\infty$ independent of the choices of s, τ and Δs such that

$$|\Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s) - \Gamma_y(b(s),s,b(\tau),\tau)| \le C\cdot\Delta s\cdot\frac{1}{\sqrt{s-\tau}},$$

which together with (61) derive $\lim_{\Delta s\to 0}|I_{3,1}|=o(1)$.

Finally, for (68), note that $M(\tau)\in C^1[\frac{s}{2},s]$ and that $|\Gamma_y(b(s+\Delta s),s+\Delta s,b(\tau+\Delta s),\tau+\Delta s)|\le\frac{C}{\sqrt{s-\tau}}$. By the dominated convergence theorem, we have the limit in (68) exists and equals to

$$I_{3,3}:=\int_{\frac{s}{2}}^{s}\Gamma_y(b(s),s,b(\tau),\tau)M'(\tau)d\tau. \tag{69}$$

To prove $|I_{3,3}|=o(1)$, one may further decompose it as

$$\begin{aligned}I_{3,3}&=\int_{\frac{s}{2}}^{s-s^7}\Gamma_y(b(s),s,b(\tau),\tau)M'(\tau)d\tau+\int_{s-s^7}^{s}\Gamma_y(b(s),s,b(\tau),\tau)M'(\tau)d\tau\\&=:I_4+I_5.\end{aligned}$$

For I_4, note that $\Gamma_y(b(s),s,b(\tau),\tau)$ and $M(\tau)$ are both smooth on $[\frac{s}{2},s-s^7]$, we may use the integration by part and have

$$\begin{aligned}|I_4|\le&\left|\Gamma_y(b(s),s,b(s-s^7),s-s^7)\cdot M(s-s^7)\right|+\left|\Gamma_y(b(s),s,b(\frac{s}{2}),\frac{s}{2})\cdot M(\frac{s}{2})\right|\\&+\left|\int_{\frac{s}{2}}^{s-s^7}\partial_\tau\Gamma_y(b(s),s,b(\tau),\tau)M'(\tau)d\tau\right|.\end{aligned}$$

where all the terms are small since $|M(\tau)|$ is much less than any polynomial of τ and thus $I_4=o(1)$. For I_5, recall that $|\Gamma_y(b(s),s,b(\tau),\tau)|\le\frac{C}{\sqrt{s-\tau}}$ and $|M'(\tau)|\le Cs^{-3}$ on $[s-s^7,s]$, we have

$$|I_5|\le s^{-3}\int_{s-s^7}^{s}\frac{C}{\sqrt{s-\tau}}d\tau\le C\sqrt{s}=o(1).$$

which derives $\lim_{\Delta s\to 0}|I_{3,2}|=o(1)$ and thus $\lim_{\Delta s\to 0}|I_3|=o(1)$. Combining (64), (65) and (66), we got $\lim_{s\to 0^+}J_2'(s)=0$ and then $\lim_{s\to 0^+}M'(s)=0$, which together with (58) derive $\lim_{t\to 0^+}f_{T_1}'(t)=0$ and $f_{T_1}(t)\in C^1[0,T]$.

□

Next, we can do the first iteration.

Proposition 3.3. *Let $f_1(x,t)$ be the density function of the measure induced by $F_1(\cdot,t)$ defined in (19), it satisfies the following initial condition and the recursive relation*

$$\begin{aligned}&f_1(x,0)=0,\quad\forall x\in(-\infty,1),\\&f_1(x,t)=\int_0^t f_0(x,t-s)f_{T_1}(s)ds,\quad\forall x\in(-\infty,0)\cup(0,1),t>0.\end{aligned}\tag{70}$$

For any fixed $T>0$, we have

(i) *$f_1(x,t)$ is the classical solution of the following PDE on $(-\infty,1]\times[0,T]$:*

$$\begin{cases}\dfrac{\partial f_1}{\partial t}-\dfrac{\partial}{\partial x}(xf_1)-\dfrac{\partial^2}{\partial x^2}f_1=0,\quad x\in(-\infty,0)\cup(0,1),t\in(0,T], & (71)\\ f_1(0^-,t)=f_1(0^+,t),\quad\dfrac{\partial}{\partial x}f_1(0^-,t)-\dfrac{\partial}{\partial x}f_1(0^+,t)=f_{T_1}(t),\quad t\in(0,T], & (72)\\ f_1(-\infty,t)=0,\quad f_1(1,t)=0,\quad t\in[0,T], & (73)\\ f_1(x,0)=0,\quad x\in(-\infty,1) & (74)\end{cases}$$

with

$$\lim_{x\to-\infty}\partial_x f_1(x,t)=0,\quad t\in[0,T]. \tag{75}$$

(ii) *There is a big enough constant C_T depending only on T such that*

$$|f_1(x,t)| \leq C_T, \quad \forall x \in (-\infty,0)\cup(0,1), t\in[0,T], \tag{76}$$

$$\left|\frac{\partial}{\partial x} f_1(x,t)\right| \leq C_T, \quad \forall x \in (-\infty,0)\cup(0,1), t\in[0,T]. \tag{77}$$

And at the domain boundary:

$$\left|\frac{\partial}{\partial x} f_1(0^-,t)\right| \leq C_T, \quad \left|\frac{\partial}{\partial x} f_1(0^+,t)\right| \leq C_T, \quad \left|\frac{\partial}{\partial x} f_1(1^-,t)\right| \leq C_T, \quad t\in[0,T]. \tag{78}$$

(iii) *For $t>0$, recalling that the density of the second jumping time*

$$f_{T_2}(t) = \int_0^t f_{T_1}(t-s) f_{T_1}(s) ds, \tag{79}$$

we have

$$-\frac{\partial f_1}{\partial x}(1,t) = f_{T_2}(t). \tag{80}$$

Proof. By (30) and the Fubini formula, we immediately get (70). As we have already known $f_0(x,t)$ satisfies PDE (35), thus from iteration relationship (70) and the regularities for $f_0(x,t)$ in Proposition 3.1, we can check that $f_1(x,t)$ satisfies PDE (36) with $n=1$ and the estimations for $f_1(x,t)$ are valid.

To prove (i), by the regularities of f_0 in Proposition 3.1, we have $\forall x\in(-\infty,0)\cup(0,1)$,

$$\frac{\partial}{\partial x} f_1(x,t) = \int_0^t \frac{\partial}{\partial x} f_0(x,t-s) f_{T_1}(s)ds \quad \text{and} \quad \frac{\partial^2}{\partial x^2} f_1(x,t) = \int_0^t \frac{\partial^2}{\partial x^2} f_0(x,t-s) f_{T_1}(s)ds,$$

which together with the decay property (44) for f_0 derive (75).
Moreover,

$$\begin{aligned}
\frac{\partial}{\partial t} f_1(x,t) &= \frac{\partial}{\partial t}\int_0^t f_0(x,t-s) f_{T_1}(s)ds \\
&= \lim_{\Delta t\to 0}\int_0^t \frac{f_0(x,t+\Delta t-s)-f_0(x,t-s)}{\Delta t} f_{T_1}(s)ds \\
&+ \lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f_0(x,t+\Delta t-s) f_{T_1}(s)ds}{\Delta t} \\
&= \int_0^t \frac{\partial}{\partial t} f_0(x,t-s) f_{T_1}(s)ds.
\end{aligned}$$

Thus we have checked (71) and also got the continuity of $\frac{\partial^2}{\partial x^2} f_1(x,t)$ and $\frac{\partial}{\partial t} f_1(x,t)$. At the same time, (73) and (74) are obvious because of the boundary conditions of f_0 and the formula (70). So for the rest of the proof we concentrate on verifying (72), which is composed of

$$f_1(0^-,t) = f_1(0^+,t), \quad t\in(0,T] \tag{81}$$

and

$$\frac{\partial}{\partial x} f_1(0^-,t) - \frac{\partial}{\partial x} f_1\left(0^+,t\right) = f_{T_1}(t), \quad t\in(0,T]. \tag{82}$$

To show (81), note that $\int_0^1 \frac{1}{\sqrt{1-e^{-2s}}}ds < \infty$ and thus for any $\varepsilon>0$, $\exists 0<\delta<t$ s.t. $\int_0^\delta \frac{1}{\sqrt{1-e^{-2s}}}ds < \frac{\varepsilon}{c}$, where the constant c is the same as in (23). With (70), we have for any $x\neq 0$,

$$f_1(x,t) = \int_0^t f_0(x,t-s) f_{T_1}(s)ds = \int_0^{t-\delta} f_0(x,t-s) f_{T_1}(s)ds + \int_{t-\delta}^t f_0(x,t-s) f_{T_1}(s)ds. \tag{83}$$

For the 2nd term above, using (23):

$$\int_{t-\delta}^{t} f_0(x,t-s)f_{T_1}(s)ds \leq \left\|f_{T_1}\right\|_{L^\infty[0,t]} \int_0^\delta \frac{c}{\sqrt{1-e^{-2s}}}ds \leq \left\|f_{T_1}\right\|_{L^\infty[0,t]} \cdot \varepsilon.$$

While for the first term, one may use (45) and see that

$$\lim_{x_1\to 0^+, x_2\to 0^+} \int_0^{t-\delta} \left|f_0(x_1,t-s) - f_0(-x_2,t-s)\right| f_{T_1}(s)ds = 0.$$

Since ε is arbitrary, we get (81).

Now to prove (82): noting that

$$\frac{\partial}{\partial x} f_1(x_1,t) = \int_0^t \frac{\partial}{\partial x} f_0(x_1,t-s)f_{T_1}(s)ds, \quad x_1 \in (0,1)$$

and for any $t-s \neq 0$,

$$\begin{aligned}
\frac{\partial}{\partial x} f_0(x_1,t-s) &= -\int_{x_1}^1 \frac{\partial^2}{\partial x^2} f_0(x,t-s)dx + \frac{\partial}{\partial x} f_0(1,t-s) \\
&= -\int_{x_1}^1 \left[\frac{\partial f_0}{\partial t}(x,t-s) - \frac{\partial}{\partial x}(xf_0)(x,t-s)\right] dx + \frac{\partial}{\partial x} f_0(1,t-s) \\
&= \int_{x_1}^1 \left[\frac{\partial}{\partial x}(xf_0)(x,t-s) - \frac{\partial f_0}{\partial t}(y,t-s)\right] dx + \frac{\partial}{\partial x} f_0(1,t-s) \\
&= f_0(1,t-s) - x_1 f_0(x_1,t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x,t-s)dx + \frac{\partial}{\partial x} f_0(1,t-s) \\
&= -x_1 f_0(x_1,t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x,t-s)dx - f_{T_1}(t-s).
\end{aligned}$$

Thus:

$$\begin{aligned}
&\frac{\partial}{\partial x} f_1(x_1,t) = \int_0^t \frac{\partial}{\partial x} f_0(x_1,t-s)f_{T_1}(s)ds \\
&= \int_0^t \left[-x_1 f_0(x_1,t-s) - \int_{x_1}^1 \frac{\partial f_0}{\partial t}(x,t-s)dx - f_{T_1}(t-s)\right] f_{T_1}(s)ds.
\end{aligned} \tag{84}$$

Similarly for any $x_2 > 0$, we have

$$\begin{aligned}
&\frac{\partial}{\partial x} f_0(-x_2,t-s) \\
&= \int_{-\infty}^{-x_2} \frac{\partial^2}{\partial x^2} f_0(x,t-s)dx + 0 \\
&= \int_{-\infty}^{-x_2} \left[\frac{\partial f_0}{\partial t}(x,t-s) - \frac{\partial}{\partial z}(xf_0(x,t-s))\right] dx \\
&= x_2 f_0(-x_2,t-s) + \int_{-\infty}^{-x_2} \frac{\partial f_0}{\partial t}(x,t-s)dx.
\end{aligned}$$

And thus

$$\frac{\partial}{\partial x} f_1(-x_2,t) = \int_0^t \left[x_2 f_0(-x_2,t-s) + \int_{-\infty}^{-x_2} \frac{\partial f_0}{\partial t}(x,t-s)dx\right] f_{T_1}(s)ds. \tag{85}$$

Combining (84) and (85), we have for all $x_1 \in (0,1)$ and $x_2 > 0$,

$$
\begin{aligned}
&\frac{\partial}{\partial x} f_1(-x_2,t) - \frac{\partial}{\partial x} f_1(x_1,t) \\
=&x_2 \int_0^t f_0(-x_2, t-s) f_{T_1}(s) ds + x_1 \int_0^t f_0(x_1, t-s) f_{T_1}(s) ds \\
&+ \int_0^t \left[\int_{\mathbb{R}\setminus[-x_2,x_1]} \frac{\partial f_0}{\partial t}(x, t-s) dx + f_{T_1}(t-s) \right] f_{T_1}(s) ds \\
=:& I_6 + I_7 + I_8.
\end{aligned}
\tag{86}
$$

For I_6, we have by (23):

$$
I_6 \leq \|f_{T_1}\|_{L^\infty[0,t]} \cdot x_2 \cdot \int_0^t \frac{c}{\sqrt{1-e^{-2s}}} ds \to 0 \quad as \quad x_2 \to 0^+.
$$

And $I_7 \to 0$ by the same argument, it now suffices to show

$$
I_8 \to f_{T_1}(t) \quad as \quad x_1, x_2 \to 0^+. \tag{87}
$$

In the rest of our calculations, integrand of I_8 will be called $H(s)$. As a result of Proposition 3.2, for any $\varepsilon > 0$, we let the chosen δ small enough such that

$$
\delta \, \|f_{T_1}\|^2_{L^\infty[0,t]} < \varepsilon \tag{88}
$$

$$
\int_{t_1}^{t_2} \left|f'_{T_1}(s)\right| ds < \varepsilon, \quad \forall t_1 < t_2 < t,\ t_2 - t_1 < \delta \tag{89}
$$

$$
P(T_1 < \delta) < \varepsilon. \tag{90}
$$

Then for the fixed $\delta > 0$ defined above,

$$
I_8 = \int_0^{t-\delta} H(s) ds + \int_{t-\delta}^t H(s) ds =: I_{8,1} + I_{8,2}. \tag{91}
$$

For $I_{8,1}$, we have by (45) and (46),

$$
\begin{aligned}
|I_{8,1}| =& \left| \int_0^{t-\delta} \left[\int_{R\setminus[-x_2,x_1]} \frac{\partial f_0}{\partial t}(y, t-s) dy + f_{T_1}(t-s) \right] f_{T_1}(s) ds \right| \\
=& \left| \int_0^{t-\delta} \int_{-x_2}^{x_1} \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) dy ds \right| \\
\leq& \|f_{T_1}\|_{L^\infty[0,t]} \int_0^{t-\delta} \int_{-x_2}^{x_1} \left\| \frac{\partial f_0}{\partial t} \right\|_{L^\infty(-\infty,1]\times[\delta,T]} dy ds \\
\leq& t \cdot \|f_{T_1}\|_{L^\infty[0,T]} \cdot \left\| \frac{\partial f_0}{\partial t} \right\|_{L^\infty(-\infty,1]\times[\delta,T]} \cdot (x_1 + x_2)
\end{aligned}
$$

which $\to 0$ as $x_1, x_2 \to 0$. As for $I_{8,2}$,

$$
I_{8,2} = \int_{t-\delta}^t \left[\int_{\mathbb{R}\setminus[-x_2,x_1]} \frac{\partial f_0}{\partial t}(y, t-s) dy + f_{T_1}(t-s) \right] f_{T_1}(s) ds.
$$

One may first see by (88), we have $\int_{t-\delta}^t f_{T_1}(t-s) f_{T_1}(s) ds \leq \varepsilon$. Moreover, for any $x_1, x_2 > 0$, note that function $\frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s)$ is bounded and continuous on the region $(R \setminus [-x_2, x_1]) \times$

$[t-\delta, t]$. One may apply Fubini's formula and have:

$$I_{8,2} = \int_{R\setminus[-x_2,x_1]} \int_{t-\delta}^{t} \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) ds dy. \tag{92}$$

At the same time, by (39) we have for any fixed $t > 0, y \notin [-x_2, x_1]$,

$$f_0(y, t-s) f_{T_1}(s) \in C^1[t-\delta, t)$$

and

$$\lim_{s \to t^-} f_0(y, t-s) f_{T_1}(s) = 0.$$

Thus, one may apply integration by parts and have

$$\begin{aligned}
&\int_{t-\delta}^{t} \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) ds \\
=&(-f_0(y, t-s) f_{T_1}(s)) \Big|_{t-\delta}^{t} + \int_{t-\delta}^{t} f_0(y, t-s) f'_{T_1}(s) ds \\
=&f_0(y, \delta) f_{T_1}(t-\delta) + \int_{t-\delta}^{t} f_0(y, t-s) f'_{T_1}(s) ds.
\end{aligned} \tag{93}$$

Plugging (93) back to (92) and applying the Fubini theorem once again, we have

$$\begin{aligned}
I_{8,2} =& \left[\int_{R\setminus[-x_2,x_1]} f_0(y, \delta) dy\right] f_{T_1}(t-\delta) + \int_{t-\delta}^{t} \int_{R\setminus[-x_2,x_1]} f_0(y, t-s) dy f'_{T_1}(s) ds \\
=:& I_9 + I_{10}.
\end{aligned} \tag{94}$$

First for I_{10}, noting that f_0 is a p.d.f., for any $s \in (t-\delta, t)$ we have

$$\int_{\mathbb{R}\setminus[-x_2,x_1]} f_0(y, t-s) dy \leq 1,$$

which together with (89) derive

$$|I_{10}| \leq \int_{t-\delta}^{t} \left|f'_{T_1}(s)\right| ds < \varepsilon. \tag{95}$$

Then for I_9, by (90) we have

$$\lim_{x_1 \to 0^+, x_2 \to 0^+} \int_{R\setminus[-x_2,x_1]} f_0(y, \delta) dy = \int_{-\infty}^{1} f_0(y, \delta) dy = P(T_1 > \delta) \in [1-\varepsilon, 1]$$

and

$$\left|f_{T_1}(t-\delta) - f_{T_1}(t)\right| < \varepsilon.$$

Thus we have for all sufficiently small $x_1 > 0, x_2 > 0$,

$$|I_9 - f_{T_1}(t)| = |I_9 - f_{T_1}(t-\delta)| + |f_{T_1}(t-\delta) - f_{T_1}(t)| < (\|f_{T_1}\|_{L^\infty[0,t]} + 1)\varepsilon. \tag{96}$$

Now combing from (91) to (96), we have concluded that $I_8 \to f_{T_1}(t)$ as $x_1, x_2 \to 0^+$, which together with $I_6 \to 0$ and $I_7 \to 0$ derive (82).

As for (ii), we first derive (76) and (77) that are essential in getting (80) and also set the basis for subsequent iterations. First we verify (76) and without loss of generality, one may assume $T > 1$. So when $t \in (0, T]$ and $x \in (-\infty, 0) \cup (0, 1)$,

$$\begin{aligned} f_1(x,t) &= \int_0^t f_0(x, t-s) f_{T_1}(s) ds \\ &\leqslant \int_0^{t-1} f_0(x, t-s) f_{T_1}(s) ds + \|f_{T_1}\|_{L^\infty[0,T]} \int_0^1 \frac{1}{\sqrt{1-e^{-2s}}} ds \\ &\leqslant \|f_0(x,t)\|_{L^\infty(-\infty,1]\times[1,\infty)} + C_T' = C_T. \end{aligned}$$

And for (77), without loss of generality, one may assume that $x > 0$ and by (84) we have:

$$\frac{\partial f_1}{\partial x}(x,t) = \int_0^t \left[-x f_0(x, t-s) - \int_x^1 \frac{\partial f_0}{\partial t}(y, t-s) dy - f_{T_1}(t-s) \right] f_{T_1}(s) ds =: I_{11} + I_{12} + I_{13}.$$

Using the estimate (22) for $f_0(x,t)$, one have

$$\begin{cases} |I_{11}| \leq C \cdot F_{T_1}(t) \leq C_T \\ |I_{13}| \leq \|f_{T_1}\|_{L^\infty[0,T]} F_{T_1}(t) \leq C_T. \end{cases}$$

For the remaining I_{12}, formula twice and integration by parts together with the fact that $f_0(\cdot, t)$ is a p.d.f. to have:

$$\begin{aligned} |I_{12}| &= \left| \int_x^1 \int_0^t \frac{\partial f_0}{\partial t}(y, t-s) f_{T_1}(s) ds dy \right| \\ &= \left| \int_x^1 \int_0^t f_0(y, t-s) f_{T_1}'(s) ds dy \right| \\ &= \left| \int_0^t \int_x^1 f_0(y, t-s) dy f_{T_1}'(s) ds \right| \\ &\leq \int_0^t \left| f_{T_1}'(s) \right| ds \\ &\leq C_T. \end{aligned}$$

Because of the proof of (72) in (i), property (ii) of Proposition 3.1 and representation (70), we know that $\frac{\partial}{\partial x} f_1(0^-, t)$, $\frac{\partial}{\partial x} f_1(0^+, t)$ and $\frac{\partial}{\partial x} f_1(1^-, t)$ are well-defined, and thus by taking the one side limit in (77), we immediately get (77) and thus we complete the proof of (ii).

Finally for (iii), using integral representation (29), we immediately get (79). Recalling that $f_1(1, t) = 0$, $\forall t > 0$, it suffices to prove

$$\lim_{x_1 \to 0^+} \frac{f_1(1 - x_1, t)}{x_1} = f_{T_2}(t), \quad \forall t > 0. \tag{97}$$

Now note that for all $0 < x_1 < \frac{1}{2}$,

$$f_1(1 - x_1, t) = \int_0^t f_0(1 - x_1, t - s) f_{T_1}(s) ds$$

While at the same time by mean value theorem on f_0, for all $s \in [0, t]$, $\exists\, \xi_{t-s}(x_1) \in [1 - x_1, 1] \subset [\frac{1}{2}, 1]$ $\quad s.t.$

$$\begin{aligned} \frac{f_0(1 - x_1, t - s)}{x_1} &= -\frac{f_0(1, t-s) - f_0(1 - x_1, t - s)}{x_1} \\ &= -\frac{\partial}{\partial x} f_0 \left(\xi_{t-s}(x_1), t - s \right). \end{aligned}$$

Note for all $0 < x_1 < \frac{1}{2}$, by (ii) of Proposition 3.1 for f_0:

$$\frac{\partial}{\partial x} f_0(\xi_{t-s}(x_1), t-s) \leqslant \left\| \frac{\partial f_0}{\partial x} \right\|_{L^\infty_{\left[\frac{1}{2},1\right]\times[0,T]}}$$

and we have

$$\lim_{x_1 \to 0^+} \frac{\partial}{\partial x} f_0\left(\xi_{t-s}(x_1), t-s\right) = \frac{\partial}{\partial x} f_0(1, t-s).$$

By the dominated convergence theorem,

$$\lim_{x_1 \to 0^+} \frac{f_1(1-x_1,t)}{x_1} = -\int_0^t \frac{\partial}{\partial x} f_0(1,t-s) f_{T_1}(s)ds = \int_0^t f_{T_1}(t-s) f_{T_1}(s)ds = f_{T_2}(t).$$

and thus

$$-\frac{\partial f_1}{\partial x}(1,t) = f_{T_2}(t).$$

The proof of Proposition 3.3 is complete.

□

Similarly by (30), for all $n \geq 1$, we have

$$\begin{aligned} & f_n(x,0) = 0, \quad \forall x \in (\infty, 1), \\ & f_n(x,t) = \int_0^t f_{n-1}(x,t-s) f_{T_1}(s)ds, \quad \forall x \in (-\infty,0) \cup (0,1), t > 0 \end{aligned} \tag{98}$$

and

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(t-s) f_{T_1}(s)ds.$$

Hence, the iterative construction is feasible, and we can show

Proposition 3.4. *For each $n \geq 1$, let $f_n(x,t)$ be the density function of the measure induced by $F_n(\cdot,t)$ defined in (19). For any fixed $T > 0$, we have*

(i) *f_n is the classic solution of the following PDE:*

$$\begin{cases} \dfrac{\partial f_n}{\partial t} - \dfrac{\partial}{\partial x}(x f_n) - \dfrac{\partial^2}{\partial x^2} f_n = 0, \quad x \in (-\infty,0) \cup (0,1), t \in [0,T], & (99) \\ f_n(0^-,t) = f_n\left(0^+,t\right), \quad \dfrac{\partial}{\partial x} f_n\left(0^-,t\right) - \dfrac{\partial}{\partial x} f_n\left(0^+,t\right) = f_{T_n}(t), \quad t \in (0,T]. & (100) \\ f_n(-\infty,t) = 0, \quad f_n(1,t) = 0, \quad t \in [0,T] & (101) \\ f_n(x,0) = 0, \quad x \in (-\infty,1) & (102) \end{cases}$$

with

$$\lim_{x \to -\infty} \partial_x f_n(x,t) = 0, \quad t \in [0,T]. \tag{103}$$

(ii) *There is a C_T that depends only on T such that*

$$|f_n(x,t)| \leq C_T, \quad \forall x \in (-\infty,0) \cup (0,1), t \in [0,T], \tag{104}$$

$$\left| \frac{\partial}{\partial x} f_n(x,t) \right| \leq C_T, \quad \forall x \in (-\infty,0) \cup (0,1), t \in [0,T], \tag{105}$$

and at the domain boundary

$$\left| \frac{\partial}{\partial x} f_n(0^-,t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_n(0^+,t) \right| \leq C_T, \quad \left| \frac{\partial}{\partial x} f_n(1^-,t) \right| \leq C_T. \tag{106}$$

(iii) *For $t > 0$, f_n is differentiable at $x = 1$ and*

$$-\frac{\partial f_n}{\partial x}(1,t) = f_{T_{n+1}}(t). \tag{107}$$

Proof. The proof of Proposition 3.4 follows from induction. By Proposition 3.3, we have presented the inductive basis at $n = 1$. Now assuming the inductive hypothesis holds up to $n > 1$, To prove (i), by

$$f_{n+1}(x,t) = \int_0^t f_n(x, t-s) f_{T_1}(s) ds,$$

one may immediately see (99),(101),(102) and (103) hold. For (100), note that $f_n(0^-, t) = f_n(0^+, t)$, $\forall t > 0$, and that $|f_n(x,t)| \leq C_T$, $\forall x \in (-\infty, 0) \cup (0,1), t \leq T$. By the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{x_1 \to 0^+, x_2 \to 0^+} |f_{n+1}(x_1, t) - f_{n+1}(-x_2, t)| \\
\leq &\lim_{x_1 \to 0^+, x_2 \to 0^+} \int_0^t |f_n(x_1, t-s) - f_n(-x_2, t-s)| f_{T_1}(s) ds \\
= &0.
\end{aligned}$$

So we have

$$f_{n+1}(0^-, t) = f_{n+1}(0^+, t).$$

Similarly,

$$\begin{aligned}
&\lim_{x_1 \to 0^+, x_2 \to 0^+} \frac{\partial f_{n+1}}{\partial x}(x_1, t) - \frac{\partial f_{n+1}}{\partial x}(-x_2, t) \\
= &\lim_{x_1 \to 0^+, x_2 \to 0^+} \int_0^t \frac{\partial f_n}{\partial x}(x_1, t-s) - \frac{\partial f_n}{\partial x}(-x_2, t-s) f_{T_1}(s) ds.
\end{aligned}$$

By the inductive hypothesis and the dominated convergence theorem, we have

$$\frac{\partial}{\partial x} f_{n+1}\left(0^-, t\right) - \frac{\partial}{\partial x} f_{n+1}\left(0^+, t\right) = f_{T_{n+1}}(t).$$

As for (ii), to check the additional regularity conditions, note that inductive hypothesis,

$$0 \leq f_{n+1}(x,t) = \int_0^t f_n(x, t-s) f_{T_1}(s) ds \leq C_T.$$

And for any $y \in (-\infty, 0) \cup (0,1)$ and $t \leq T$,

$$\left| \frac{\partial f_{n+1}}{\partial x}(y,t) \right| \leq \int_0^t \left| \frac{\partial f_n}{\partial x}(y, t-s) \right| f_{T_1}(s) ds \leq C_T.$$

Using similar arguments as in Proposition 3.3, we have $\frac{\partial}{\partial x} f_{n+1}\left(0^-, t\right)$, $\frac{\partial}{\partial x} f_{n+1}\left(0^+, t\right)$ and $\frac{\partial}{\partial x} f_{n+1}\left(1^-, t\right)$ are individually bounded by C_T. Finally for (iii), noting that $\left|\frac{\partial f_n}{\partial x}(y,t)\right| \leq C_T$ for all $t \leq T$, $0 < y < 1$, the proof of

$$-\frac{\partial f_n}{\partial x}(1,t) = f_{T_{n+1}}(t), \quad \forall t > 0.$$

follows from the same treatment as in Proposition 3.3.

□

Now we can finish the proof of Theorem 1.

Proof of Theorem 1: Based on the previous analysis in Proposition 3.1-3.4, we have shown that for $n \geq 0$, f_n is the density function of the measure induced by $F_n(\cdot, t)$ defined in (19) as well as the solution to the sub PDE problems (35) and (36). Next, we consider the density function of the stochastic process X_t as in (18) that admits the series representation $f(x,t) = \sum_{n=0}^{+\infty} f_n(x,t)$.

In order to prove that $f(x,t)$ satisfies the properties in Theorem 1, we first show that the relevant derivatives of $f(x,t)$ also have the series representations and the series converge uniformly so that we can pass the regularity from $f_n(x,t)$ to $f(x,t)$. Besides, noting that f_n is the solution to the sub PDE problems (35) and (36), and thus we can show in the following $f = \sum_{n=0}^{+\infty} f_n$ satisfies the (9), which is the summation of sub PDE problems PDE (35) and (36).

For any fixed $T > 0$, we first show the uniform convergence of the relevant derivatives of $\sum_{n=0}^{+\infty} f_n(x,t)$ on $((-\infty,0) \cup (0,1]) \times [0,T]$. By (98), $\forall x_0 \in (-\infty,0) \cup (0,1]$, we have for any $0 \leq t \leq T$ and $n \geq 1$

$$\left|\frac{\partial}{\partial x} f_n(x_0,t)\right| \leq \int_0^T f_{T_1}(s)ds \cdot \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_{n-1}(x_0,t)\right| \leq \rho_T \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_{n-1}(x_0,t)\right|, \tag{108}$$

where

$$\rho_T = \int_0^T f_{T_1}(s)ds = \mathbf{P}_0(T_1 \leq T) \in (0,\, 1) \tag{109}$$

is a constant that depends only on T. The proof of (109) is quite standard in probability and thus we put the whole proof of it in Appendix. With (108), we have

$$\sum_{n=0}^{+\infty} \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_n(x_0,t)\right| \leq \sum_{n=0}^{+\infty} \rho_T^n \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_0(x_0,t)\right| = \frac{1}{1-\rho_T} \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_0(x_0,t)\right|, \tag{110}$$

which implies to show the uniform convergence of such series, it suffices to check the regularities of $f_0(x,t)$. In fact, with (ii) of Proposition 3.1, we know that for any $\varepsilon_0 \in (0,1)$, $f_0(x,t) \in C^{2,1}\left(((-\infty,-\varepsilon_0] \cup [\varepsilon_0,1]) \times [0,T]\right)$ and thus the last term in (110) has a uniform bound on any compact subset of $(-\infty,0) \cup (0,1]$, i.e., for any compact subset I of $(-\infty,0) \cup (0,1]$,

$$\sum_{n=0}^{+\infty} \max_{t\in[0,T]} \max_{x\in I} \left|\frac{\partial}{\partial x} f_n(x_0,t)\right| \leq \frac{1}{1-\rho_T} \max_{t\in[0,T]} \max_{x\in I} \left|\frac{\partial}{\partial x} f_0(x_0,t)\right| < +\infty.$$

With the same treatment, we know that

$$\sum_{n=0}^{+\infty} f_n(x,t),\ \sum_{n=0}^{+\infty} \frac{\partial}{\partial t} f_n(x,t),\ \sum_{n=0}^{+\infty} \frac{\partial}{\partial x}(x f_n(x,t)) \text{ and } \sum_{n=0}^{+\infty} \frac{\partial^2}{\partial x^2} f_n(x,t) \tag{111}$$

are inner closed uniformly convergent on $((-\infty,0) \cup (0,1]) \times [0,T]$, and thus we can exchange the derivative and the summation in (111). And by (110), we have

$$\max_{t\in[0,T]} |\partial_x f(x_0,t)| \leq \sum_{n=0}^{+\infty} \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_n(x_0,t)\right| \leq \frac{1}{1-\rho_T} \max_{t\in[0,T]} \left|\frac{\partial}{\partial x} f_0(x_0,t)\right|$$

With the same treatment, we can get the same bounds for the series in (111), from which we can analyse the regularities of $f(x,t)$ by estimating $f_0(x,t)$.

To check (i), we show that $N(t) = -\frac{\partial}{\partial x} f(1^-,t)$ is well-defined and $N(t)$ has a series representation in terms of the densities of jumping times. In fact, by uniform convergence, it is clear that $\sum_{n=0}^{+\infty} \frac{\partial}{\partial x} f_n(1^-,t)$ uniformly converges on $[0,T]$. In particular,

$$\frac{\partial f}{\partial x}(1^-,t) = \sum_{n=0}^{\infty} \frac{\partial f_n}{\partial x}(1^-,t).$$

Then by (46) and (107), we also have

$$N(t) = -\frac{\partial f}{\partial x}(1^-,t) = \sum_{n=0}^{\infty} f_{T_n}(t). \tag{112}$$

Note that $f_{T_n}(t) \in C[0,T]$, and thus $N(t) \in C[0,T]$. Hence, (i) is completely proved.

With the uniform convergence of the series representations and the regularities of $f_0(x,t)$ in Proposition 3.1, we can easily show (ii), (iii), (iv), (v) of Theorem 1. By (44) and (103), we have

$$\lim_{x\to-\infty} \partial_x f(x,t) = \sum_{n=0}^{+\infty} \lim_{x\to-\infty} \partial_x f_n(x,t) = 0, \quad t \in (0,T],$$

and thus (v) is valid. Similarly, the uniform convergence together with the continuity of f_n, $\partial_{xx} f_n$ and $\partial_t f_n$ on $((-\infty,0)\cup(0,1))\times(0,T]$ implies (ii) and (iii). To check (iv), we aim to show that $f_x(0^-,t)$ and $f_x(0^+,t)$ is well-defined for $t\in(0,T]$. With the similar analysis, we can prove that for fixed $0<t\le T$, $\sum_{n=0}^{\infty} \frac{\partial f_n}{\partial x}(x,t)$ uniformly converge on $[-1,0)$ and $(0,1]$, which together with Lemma 3.1 and the existence of one-side limits given in (45) and (106) derives (iv) of Theorem 1.

Finally, to prove (vi), that is, the density f satisfies the PDE problem (9), we need to show that the equation is satisfied as well as all the conditions are met. With uniform convergence, we can sum the equation (74) from $n=0$ to $+\infty$, and thus for any $(x,t)\in((-\infty,0)\cup(0,1))\times(0,T]$,

$$\begin{aligned}
&\frac{\partial f}{\partial t} - \frac{\partial}{\partial x}(xf) - \frac{\partial^2 f}{\partial x^2} \\
=&\frac{\partial}{\partial t}(\sum_{n=0}^{+\infty} f_n(x,t)) - \frac{\partial}{\partial x}\left(\sum_{n=0}^{+\infty} x f_n(x,t)\right) - \frac{\partial^2}{\partial x^2}(\sum_{n=0}^{+\infty} f_n(x,t)) \\
=&\sum_{n=0}^{+\infty}\left(\frac{\partial f_n}{\partial t} - \frac{\partial}{\partial x}(xf_n) - \frac{\partial^2}{\partial x^2} f_n\right) \\
=&0.
\end{aligned} \tag{113}$$

With the regularities of f proved above, all the initial and boundary conditions in (9) are trivially satisfied except that we need to prove the jump condition on f_x at $x=0$. Given any fixed $t>0$, for any $\epsilon>0$, due to the uniform convergence, there is a constant $N<\infty$ such that

$$\left|\sum_{n=N+1}^{\infty} \frac{\partial f_n}{\partial x}(x,t)\right| < \epsilon, \quad \forall x \in (-\infty,0]\cup[0,1], \tag{114}$$

where at $0,1$ the derivatives are understood in the one-sided sense. Moreover, for the now fixed N, by (100), there exists $\delta>0$, such that for all $y<0<x$, $|x|,|y|\le\delta$,

$$\left|\frac{\partial f_0}{\partial x}(x,t) - \frac{\partial f_0}{\partial x}(y,t)\right| \le \epsilon \tag{115}$$

and

$$\sum_{n=1}^{N}\left|\frac{\partial f_n}{\partial x}(x,t)-\frac{\partial f_n}{\partial x}(y,t)+f_{T_n}(t)\right|<\epsilon. \tag{116}$$

Combining (114)-(116), we have

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x}(x,t)-\frac{\partial f}{\partial x}(y,t)+\sum_{n=1}^{\infty}f_{T_n}(t)\right| \\
=&\left|\sum_{n=0}^{\infty}\left(\frac{\partial f_n}{\partial x}(x,t)-\frac{\partial f_n}{\partial x}(y,t)\right)+\sum_{n=1}^{\infty}f_{T_n}(t)\right| \\
\leq&\left|\frac{\partial f_0}{\partial x}(x,t)-\frac{\partial f_0}{\partial x}(y,t)\right|+\left|\sum_{n=1}^{N}\left(\frac{\partial f_n}{\partial x}(x,t)-\frac{\partial f_n}{\partial x}(y,t)+f_{T_n}(t)\right)\right| \\
+&\left|\sum_{n=N+1}^{\infty}\frac{\partial f_n}{\partial x}(x,t)\right|+\left|\sum_{n=N+1}^{\infty}\frac{\partial f_n}{\partial x}(y,t)\right|+\left|\sum_{n=N+1}^{\infty}f_{T_n}(t)\right|\leq 5\epsilon,
\end{aligned}$$

and thus we conclude that for $t>0$

$$\frac{\partial}{\partial x}f(0^-,t)-\frac{\partial}{\partial x}f(0^+,t)=-\frac{\partial}{\partial x}f(1^-,t).$$

Similarly, we can get for $t>0$

$$f(0^-,t)=f(0^+,t).$$

Now that we have thoroughly checked (vi) and hence, the proof of Theorem 1 is completed. $\square$

With the same steps as in proving Theorem 1, we can show Corollary 2.1. Next, we only focus on proving Theorem 2. Due to the results for the process X_t as in (18) that starting from $y<1$ are largely parallel to the one starts from 0 we have studied in details, only a sketch of proof will be given for those parts.

Noting that now ν is a c.d.f. whose p.d.f. $f_{\text{in}}(x)\in C_c(-\infty,1)$ and that $f_{\text{in}}(x)$ is continuous and compacted supported in $(-\infty,1-\varepsilon_0]$ for some $\varepsilon_0>0$. Without loss of generality, we assume $f_{\text{in}}(x)$ is supported in $[-C_0,1-\varepsilon_0]$ for some $C_0>0$. Thus for the fixed $T>0$ we have

(1) By conditional distribution, we have for any $x\in(-\infty,1]$, $t\in(0,T]$,

$$f^{\nu}(x,t)=\int_{-\infty}^{1-\varepsilon_0}f^{y}(x,t)f_{\text{in}}(y)dy.$$

(2) For all $t\in(0,T]$, $x\neq 0$ or 1, $f^y(x,t)$ is continuous with respect to y.

(3) All the regularities and convergences in Corollary 2.1 are uniform with respect to $y\in(-\infty,1-\varepsilon_0]$. Actually, for all $\varepsilon_1>0$, $t_0>0$, and any $x\in(-\infty,-\varepsilon_1]\cup[-\varepsilon_1,1)$, $t\in[t_0,T]$, $y\in(-\infty,1-\varepsilon_0]$, we have

$$\begin{aligned}
|f^y(x,t)|&\leq C^{(0)}_{\varepsilon_0,\varepsilon_1,t_0,T}, & |\partial_x f^y(x,t)|&\leq C^{(1)}_{\varepsilon_0,\varepsilon_1,t_0,T}, \\
|\partial_t f^y(x,t)|&\leq C^{(2)}_{\varepsilon_0,\varepsilon_1,t_0,T}, & |\partial_{xx} f^y(x,t)|&\leq C^{(3)}_{\varepsilon_0,\varepsilon_1,t_0,T}.
\end{aligned}$$

Moreover, for all $t\in[t_0,T]$, $y\in(-\infty,1-\varepsilon_0]$ and $x\in[-1,0)\cup(0,1)$

$$|\partial_x f^y(x,t)|\leq C_{\varepsilon_0,t_0,T}.$$

(4) Then we can take the derivative into the integral in (12), i.e. for $\ell = 0, 1, 2$, $\partial^\ell = \partial^\ell_{tx} = \partial^m_t \partial^n_x$, $\ell = 2m + n$,

$$\partial^\ell f^\nu(x,t) = \int_{-\infty}^{1-\varepsilon_0} \partial^\ell f^y(x,t)\nu(dy), \quad x \in (-\infty, 1], \quad t > 0,$$

and thus

$$N^\nu(t) := -\partial_x f^\nu(1^-, t) = -\int_{-\infty}^{1} \partial_x f^y(1^-, t)\nu(dy) = \int_{-\infty}^{1} N^y(t)\nu(dy).$$

And by the regularities and convergences for $f^y(x,t)$ in Corollary 2.1, we get the properties (i), (ii), (iii), (iv) and (v) for $f^\nu(x,t)$.

(5) Finally we check the L^2 convergence (13). We first turn the problem into proving L^1 convergence by showing the uniform boundedness of $f^\nu(x,t)$ when t is sufficiently small. In fact, similar to the decomposition in (34), we have

$$f^y(x,t) = \sum_{n=0}^{+\infty} f^y_n(x,t) \tag{117}$$

where $f^y_n(x,t)dx = \mathbf{P}(X^y_t \in dx, n_t = n)$ as in (19). With (22), we have

$$f^y_0(x,t) \le f^y_{\mathrm{ou}}(x,t) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\{\frac{-(x-e^{-t}y)^2}{2(1-e^{-2t})}\}. \tag{118}$$

By the same method in Lemma 2.2, we get the iteration relationship for any $n \ge 1$

$$f^y_n(x,t) = \int_0^t f_{T^y_1}(t-s) f_{n-1}(x,s)ds. \tag{119}$$

Using (23), we know that for any $t > 0$, $f_0(x,t) \le f_{\mathrm{ou}}(x,t) \le \frac{C}{\sqrt{t}}$ and with the similar estimation in Proposition 3.2, we have for any $k \in \mathbb{N}$, all sufficiently small t and $s \le t$,

$$f_{T^y_1}(t-s) \le C_k t^k,$$

where the constant C_k is independent of all $y \le 1 - \varepsilon_0$. Thus

$$f^y_1(x,t) \le C_k t^k \int_0^t \frac{1}{\sqrt{s}} ds \le C_k t^{k+\frac{1}{2}}. \tag{120}$$

Repeat calculations in (120) and with the iteration (119), one has for all sufficiently small t,

$$f^y_n(x,t) \le (Ct)^n,$$

and thus for all sufficiently small t,

$$\sum_{n=1}^{+\infty} f^y_n(x,t) \le \frac{Ct}{1-Ct} \le C. \tag{121}$$

Combining (117), (118) and (121), we have

$$\begin{aligned} f^\nu(x,t) \le & \int_{-\infty}^{1-\varepsilon_0} \left[f^y_{\mathrm{ou}}(x,t) + C\right] f_{\mathrm{in}}(y)dy \\ \le & C + \|f_{\mathrm{in}}(y)\|_{L^\infty(-\infty, 1-\varepsilon_0]} \int_{-\infty}^{1-\varepsilon_0} f^y_{\mathrm{ou}}(x,t)dy. \end{aligned}$$

Noting that by (118) $\int_{-\infty}^{1-\varepsilon_0} f^y_{\text{ou}}(x,t)dy$ is uniformly bounded for any x and sufficiently small t, and so does f^ν. Noting that both $f_{\text{in}}(x)$ and $f^\nu(x,t)$ are uniformly bounded for all sufficiently small t, thus to prove (13), it suffices to prove

$$\lim_{t\to 0^+}\int_{-\infty}^{+\infty} |f^\nu(x,t)-f_{\text{in}}(x)|dx = 0. \tag{122}$$

To get (122), for a suitable constant M_0 whose value will be specified in the following, we have

$$\int_{-\infty}^{+\infty} |f^\nu(x,t)-f_{\text{in}}(x)|dx = \left(\int_{-\infty}^{-M_0} + \int_{-M_0}^{1}\right)|f^\nu(x,t)-f_{\text{in}}(x)|dx =: P_1+P_2. \tag{123}$$

First to bound P_1, we have

Lemma 3.2. *Now consider the process X_t as in (18) that starts from y. For any $\varepsilon>0$, there exists $t_0>0$ and $M_0<\infty$ such that for any $t\in[0,t_0]$ and any $y\in \operatorname{supp}(f_{\text{in}}) = [-C_0, 1-\varepsilon_0]$,*

$$\mathbf{P}^y(X_t\le -M_0)\le\varepsilon. \tag{124}$$

Proof. Note that according to the construction of the process X_t as in (18) that starts from y, we have

$$\{X_t>-M_0\}\supset\{Y_t^{(1)}>-M_0\}\cap\{T_1>t\}$$

which immediately implies

$$\mathbf{P}^y(X_t\le -M_0)\le \mathbf{P}^y(Y_t^{(1)}\le -M_0)+\mathbf{P}^y(T_1>t) := Q_1+Q_2. \tag{125}$$

For Q_2 when $t\le t_0$,

$$\mathbf{P}^y(T_1\le t)=\int_0^t f_{T_1^y}(s)ds\le C_k\int_0^t s^k ds.$$

So let $k=1$ and $t_0=\sqrt{\frac{\varepsilon}{C_1}}$, we have for all $t\le t_0$,

$$\mathbf{P}^y(T_1\le t)\le C_1\int_0^t sds\le\frac{1}{2}\varepsilon. \tag{126}$$

And for Q_1, noting that $Y_t^{(1)}$ is Gaussian, we can choose M_0 large enough to control Q_1 and then complete the proof.

□

Remark 5. Without loss of generality, we choose the constant M_0 in Lemma 3.2 larger than C_0.

Lemma 3.2 immediately implies that

$$F^\nu(-M_0,t)=\mathbf{P}^\nu(X_t\le -M_0)=\int_{-C_0}^{1-\varepsilon_0}\mathbf{P}^y(X_t\le -M_0)f_{\text{in}}(y)dy\le\varepsilon. \tag{127}$$

And for any $\varepsilon>0$, $\exists t_0>0$ and $M_0<\infty$ such that for all $t<t_0$,

$$P_1=\int_{-\infty}^{-M_0} f^\nu(x,t)dx=\mathbf{P}^\nu(X_t\le -M_0)<\varepsilon. \tag{128}$$

To estimate P_2 in (123), we show in the following

Lemma 3.3. *For any $\varepsilon > 0$, there is a $t_1 > 0$ such that for any $t \in (0, t_1]$ and $x \in \mathbb{R}$,*

$$f^\nu(x,t) \le f_{\rm in}(x) + \varepsilon.$$

Proof. Noing that when $x > 1$, $f^\nu(x,t) = f_{\rm in}(x) = 0$, thus we only need to focus on $x \in (-\infty, 1]$. By (117), (118) and (121), we already have

$$f^\nu(x,t) \le \int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy + \frac{Ct}{1-Ct} \tag{129}$$

and $\frac{Ct}{1-Ct} \to 0$ as $t \to 0^+$. Thus we only need to bound $\int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy$. To do this we will separate the case when $x \in [-C_0 - 1, 1]$ and $x \in (-\infty, -C_0 - 1)$.

(i) When x belongs to the compact set $[-C_0 - 1, 1]$, by (118) we have

$$f_{\rm ou}^y(x,t) = e^t \frac{1}{\sqrt{2\pi(1-e^{-2t})e^{2t}}} \exp\{-\frac{(y-xe^t)^2}{2(1-e^{-2t})e^{2t}}\} \tag{130}$$

which equals to the multiply of e^t and the p.d.f. of the normal distribution $N(xe^t, (1-e^{-2t})e^{2t})$. Noting that $f_{\rm in}(y)$ is uniformly continuous, thus for any $\varepsilon > 0$, there $\exists \delta > 0$, s.t. for all $|x_1 - x_2| \le \delta$, we have $|f_{\rm in}(x_1) - f_{\rm in}(x_2)| \le \varepsilon$. And there $\exists t_2 > 0$ s.t. for all $t < t_2$ and $x \in [-C_0 - 1, 1]$, $|x - e^t x| < \frac{\delta}{2}$. Moreover, for the fixed δ above, there $\exists t_3 > 0$ such that for all $t \in (0, t_3)$

$$\mathbf{P}(|N(0,1)| \ge \frac{\delta}{2\sqrt{(1-e^{-2t})e^{2t}}}) \le \frac{\varepsilon}{\|f_{\rm in}\|_{L^\infty}}, \tag{131}$$

where $N(0,1)$ stands for the standard normal distribution. Thus for $t_1 = t_2 \wedge t_3$,

$$\int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy = \left(\int_{xe^t-\frac{\delta}{2}}^{xe^t+\frac{\delta}{2}} + \int_{\mathbb{R}\setminus[xe^t-\frac{\delta}{2}, xe^t+\frac{\delta}{2}]}\right) f_{\rm ou}^y(x,t) f_{\rm in}(y)dy =: K_1 + K_2 \tag{132}$$

For K_1, we have by (130)

$$K_1 \le \max_{y\in[x-\delta,x+\delta]} f_{\rm in}(y) \cdot e^t \le \|f_{\rm in}\|_{L^\infty}(e^t - 1) + \max_{y\in[x-\delta,x+\delta]} f_{\rm in}(y) \le f_{\rm in}(x) + \varepsilon. \tag{133}$$

And for K_2, we have by (131)

$$K_2 \le \|f_{\rm in}\|_{L^\infty} \int_{\mathbb{R}\setminus[xe^t-\frac{\delta}{2}, xe^t+\frac{\delta}{2}]} f_{\rm ou}^y(x,t)dy \le e^t \frac{\varepsilon}{\|f_{\rm in}\|_{L^\infty}} = e^t \cdot \varepsilon. \tag{134}$$

Combing (133) and (134), the proof of case (i) is complete.

(ii) Note that $f_{\rm in}(x) = 0$ on $x \in (-\infty, -C_0 - 1)$ and $f^\nu(x,t) = 0$ on $x \ge 1$. We only need to prove that $\forall \varepsilon > 0$, $\exists t_1 > 0$ s.t. $\forall t \in (0, t_1]$ and any $x < -C_0 - 1$,

$$\int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy < \varepsilon. \tag{135}$$

By (118) and noting that for any $x < -C_0 - 1$ and $y \in [-C_0, 1]$, we have $|x - e^{-t}y| \ge 1$ and thus

$$\begin{aligned}\int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy \le& (C_0+1)\|f_{\rm in}\|_{L^\infty} \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\left(-\frac{1}{2(1-e^{-2t})}\right) \\ \le& (C_0+1)\|f_{\rm in}\|_{L^\infty} \frac{u}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right),\end{aligned}$$

where $u := (1-e^{-2t})^{-\frac{1}{2}}$. Thus we know $\int_{-\infty}^{+\infty} f_{\rm ou}^y(x,t) f_{\rm in}(y)dy \to 0$ as $t \to 0^+$ and the proof of Lemma 3.3 is complete.

$\square$

With Lemma 3.3, now we conclude the proof of the (13). Now for the fixed M_0 in Lemma 3.2, there exists $t_2 \geq 0$ s.t. for all $t \in (0, t_2]$ and $x \in \mathbb{R}$

$$f^\nu(x,t) \leq f_{\text{in}}(x) + \frac{\varepsilon}{M_0+1}$$

Noting that $|a-b| \leq b-a+2\max\{a-b,0\}$, we have

$$\begin{aligned}
P_2 \leq& \int_{-M_0}^{1} \left[f_{\text{in}}(x) - f^\nu(x,t) + \frac{2\varepsilon}{M_0+1} \right] dx \\
=& \int_{-M_0}^{1} f_{\text{in}}(x)dx - \int_{-M_0}^{1} f^\nu(x,t)dx + 2\varepsilon \\
\leq& 3\varepsilon.
\end{aligned} \tag{136}$$

Combining (128) and (136), we get (13). Then the proof of Theorem 2 is complete.

3.2. **Weak Solution.**

In this section, we show that the density of X_t, which we denote by $f(x,t)$ and $N(t) = \sum_{n=1}^{+\infty} F'_{T_n}(t)$ are the weak solution of the PDE problem (9). We adopt the definition of the weak solution of (9) as in [5] and the main theorem in this section is as follows:

Theorem 3. *Let $f^\nu(x,t)$ be the p.d.f of the process X_t as in (18) that starts from p.d.f. $f_{\text{in}}(x) \in C_c(-\infty,1)$ and $N^\nu(t) := \sum_{n=1}^{+\infty} F'_{T_n}(t)$. The pair (f,N) is a weak solution of (9) in the following sense: for any test function $\phi(x,t) \in C^\infty((-\infty,1] \times [0,T])$ such that $\frac{\partial^2 \phi}{\partial x^2}, x\frac{\partial \phi}{\partial x} \in L^\infty((-\infty,1] \times [0,T])$, we have*

$$\begin{aligned}
&\int_0^T \int_{-\infty}^1 \left(\frac{\partial}{\partial t}\phi - x\frac{\partial}{\partial x}\phi + \frac{\partial^2}{\partial x^2}\phi \right) f^\nu(x,t)dxdt \\
=& \int_0^T (\phi(1,t) - \phi(0,t)) N^\nu(t)dt - \int_{-\infty}^1 \phi(x,0) f_{\text{in}}(x)dx + \int_{-\infty}^1 \phi(x,T) f^\nu(x,T)dx
\end{aligned} \tag{137}$$

The convergence of the series $\sum_{n=1}^{+\infty} F'_{T_n}(t)$ relies on the proof of Theorem 2 with which we have already known that $f^\nu(x,t)$ is a solution to the PDE problem (9). To prove (f^ν, N^ν) is also a weak solution of (9), one simply multiplies the equation by the test function ϕ and carries out the integration by parts in space and in time respectively. Since the calculations is rather straightforward, we choose to omit the details in this work but we remark that the weak-strong uniqueness is still an open problem for such a Fokker-Planck equation with a flux-shift structure and we will continue research along this line in the future.

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Now we shall go back to show (109). In the following, we let X_t as in (14) denote an O-U process starting from 0 and define the stopping time T_1 be the first time that X_t hits 1, i.e., $T_1 = \inf\{t \geq 0, X_t = 1\}$. Now it suffices to prove that for all fixed $T \in (0, +\infty)$,

$$\mathbf{P}(T_1 > T) > 0. \tag{138}$$

In order to show (138), we show the probability of an event included in $\{T_1 > T\}$ is positive. Actually, we construct a sequence of stopping time and use the strong Markov property to decompose the process X_t such that each time $|X_t| > 1$, it escape from -1. By showing the product of the probability of an event sequence is positive, we complete the proof. Now we show a useful lemma.

Lemma 3.4. *For the O-U process X_t defined above, define a stopping time $\tau_1 = \inf\{t \geq 0, |X_t| = 1\}$, then*

$$\begin{cases} \mathbf{P}(\tau_1 < +\infty) = 1, & (139) \\ \mathbf{P}(\tau_1 > \frac{1}{16}, X_{\tau_1} = -1) = \mathbf{P}(\tau_1 > \frac{1}{16}, X_{\tau_1} = 1) > 0. & (140) \end{cases}$$

Proof. (139) follows from the fact that $\tau_1 < \inf\{n \in \mathbb{N}, |X_n| > 1\}$, the Markov property and the Gaussian transition distribution of X_t. As for (140), by symmetry, we only need to prove

$$\mathbf{P}(\tau_1 > \frac{1}{16}) > 0. \tag{141}$$

By (14), $X_t = \sqrt{2}\int_0^t e^{-(t-s)} dB_s$ and thus

$$\{\tau_1 \leq \frac{1}{16}\} = \{\max_{t \leq \frac{1}{16}} |X_t| \geq 1\} \subset \{\max_{t \leq \frac{1}{16}} \left|\sqrt{2}\int_0^t e^{-(t-s)} dB_s\right| \geq 1\}.$$

Now note that $\int_0^t e^s dB_s$ is a martingale and then

$$\begin{aligned} &\mathbf{P}(\tau_1 \leq \frac{1}{16}) \leq \mathbf{P}(\max_{t \leq \frac{1}{16}} \left|\sqrt{2}\int_0^t e^{-(t-s)} dB_s\right| \geq 1) \\ \leq &2\mathbf{E}\left(\max_{t \leq \frac{1}{16}} \left|\int_0^t e^s dB_s\right|\right)^2 \leq 8\mathbf{E}\left(\int_0^{\frac{1}{16}} e^s dB_s\right)^2, \end{aligned} \tag{142}$$

where the last two inequalities follows from the Markov inequality and Doob's inequality respectively. Noting that

$$\mathbf{E}\left(\int_0^{\frac{1}{16}} e^s dB_s\right)^2 = \int_0^{\frac{1}{16}} e^{2s} ds = \frac{1}{2}(e^{\frac{1}{8}} - 1) < \frac{1}{8}$$

and thus (141) is valid.

□

With the above lemma, now we prove (138) that is equivalent with (109).

Proof of (109)*:* We let Y_t be an O-U process starting at -1 and derive stopping time $\tau_1' = \inf\{t \geq 0, Y_t = 0\}$. Then by the recurrence of O-U process,

$$\mathbf{P}(\tau_1' < +\infty) = 1. \tag{143}$$

Next we define an increasing sequence of stopping times as follows:

$$\begin{aligned}
S_0' &= 0\\
S_1 &= \inf\{t \geq 0, |X_t| = 1\},\\
S_1' &= \inf\{t \geq S_1, X_t = 0\},\\
S_2 &= \inf\{t \geq S_1', |X_t| = 1\},\\
S_2' &= \inf\{t \geq S_2, X_t = 0\},\\
&\vdots
\end{aligned}$$

Combining (139), (143) and the Strong Markov Property of the O-U process $S_n, S_n' < +\infty$ for all n. At the same time,

$$S_1 - S_0',\ S_1' - S_1,\ S_2 - S_1',\ \cdots$$

are independent to each other while

$$\begin{aligned}
S_n - S_{n-1}' &\stackrel{d}{=} \tau_1,\\
S_n' - S_n &\stackrel{d}{=} \tau_1'.
\end{aligned}$$

Thus for the fixed $T \in (0, +\infty)$ above, let $N_0 = \lfloor T \rfloor + 1$ and then

$$\{T_1 > T\} \supset \cap_{i=1}^{16N_0}\{S_i - S_{i-1}' > \frac{1}{16}, X_{S_i} = -1, S_i' - S_i < +\infty\}$$

Using the strong Markov property, we have

$$\begin{aligned}
\mathbf{P}(T_1 > T) \geq& \mathbf{P}\left(\cap_{i=1}^{16N_0}\{S_i - S_{i-1}' > \frac{1}{16}, X_{S_i} = -1, S_i' - S_i < +\infty\}\right)\\
=& \prod_{n=1}^{16N_0} \mathbf{P}(\tau_1 > \frac{1}{16}, X_{\tau_1} = -1) > 0,
\end{aligned}$$

which completes the proof of (109).

$\square$

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