The generalised rainbow Turán problem for cycles

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Abstract

Given an edge-coloured graph, we say that a subgraph is rainbow if all of its edges have different colours. Let ex(n, H, rainbow-F) denote the maximal number of copies of H that a properly edge-coloured graph on n vertices can contain if it has no rainbow subgraph isomorphic to F. We determine the order of magnitude of $ex(n, C_s, rainbow-C_t)$ for all s, t with $s \neq 3$. In particular, we answer a question of Gerbner, Mészáros, Methuku and Palmer by showing that $ex(n, C_{2k}, rainbow-C_{2k})$ is $\Theta(n^{k-1})$ if $k \geq 3$ and $\Theta(n^2)$ if k = 2. We also determine the order of magnitude of $ex(n, P_\ell, rainbow-C_{2k})$ for all $k, \ell \geq 2$, where P_ℓ denotes the path with ℓ edges.

1 Introduction

The problem of estimating the maximal possible size ex(n, F) of an F-free graph on n vertices is one of the most fundamental problems in extremal graph theory. It is a well known fact that $ex(n, F)/\binom{n}{2} \rightarrow 1 - 1/(r-1)$ as $n \rightarrow \infty$ if F has chromatic number r, determining the asymptotic behaviour of this function when F is not bipartite. However, much less is known in the bipartite case. See [6] for a survey on the topic.

Alon and Shikhelman introduced [1] the following generalisation of the problem above. Given two graphs H and F, let ex(n, H, F) denote the maximal number of copies of H that an F-free graph on n vertices can contain. Note that the usual Turán number ex(n, F) is the special case $ex(n, K_2, F)$. This problem has been studied for several different choices of H and F, see e.g. [1, 7, 9].

Another generalisation of the Turán problem was introduced by Keevash, Mubayi, Sudakov and Verstraëte [12]. Given an edge-coloured graph, we say that a subgraph is rainbow if all of its edges have different colours. Let $ex^*(n, F)$ denote the maximal numer of edges that a properly edge-coloured graph on n vertices can have if it contains no rainbow copy of F. Note that clearly $ex(n, F) \leq ex^*(n, F)$, and in fact $ex^*(n, F) = ex(n, F) + o(n^2)$, giving the asymptotic behaviour when F is not bipartite [12]. This rainbow Turán problem has been studied for graphs F including paths [11, 5], cycles [12, 3] and complete bipartite graphs [12], and for several graphs exact results are also known [12].

A common generalisation was studied by Gerbner, Mészáros, Methuku and Palmer [8]. Let ex(n, H, rainbow-F) denote the maximal number of copies of H that a properly edge-coloured graph on n vertices can contain if it has no rainbow subgraph isomorphic to F. The authors of [8] focused mainly on the case H = F, and obtained several results, for example when F is a path, cycle or a tree. Concerning cycles, they proved the following theorem.

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Theorem 1.1 (Gerbner, Mészáros, Methuku, Palmer [8]). If $k \ge 2$ is an integer, then

 $ex(n, C_{2k+1}, rainbow - C_{2k+1}) = \Theta(n^{2k-1})$

and

$$\Omega(n^{k-1}) \le \exp(n, C_{2k}, \operatorname{rainbow-} C_{2k}) \le O(n^k).$$

Moreover, if $\ell \geq 2$ is an integer with $\ell \neq k$, then

$$ex(n, C_{2\ell}, rainbow - C_{2k}) = \Theta(n^{\ell}).$$

(Throughout this paper, whenever we use the Ω , Θ or O notation, the implied constants may depend, as usual, on the other parameters present, such as k and ℓ above.) The authors of [8] asked what the correct order of magnitude is for $\exp(n, C_{2k}, \operatorname{rainbow-} C_{2k})$. (They were able to improve the lower bound to $\Omega(n^{3/2})$ when k = 2 and the upper bound to $O(n^{8/3})$ when k = 3.) The main aim of this paper is to obtain the following extension of Theorem 1.1.

Theorem 1.2. If $s \ge 4$ and $t \ge 3$ are positive integers, then

$$\operatorname{ex}(n, C_s, \operatorname{rainbow-}C_t) = \begin{cases} \Theta(n^{s/2}) & \text{if } t = 4\\ \Theta(n^{s/2}) & \text{if } s, t \text{ are even with } s \neq t\\ \Theta(n^{s/2-1}) & \text{if } s = t \ge 6 \text{ and } t \text{ is even}\\ \Theta(n^{(s-1)/2}) & \text{if } t \ge 6 \text{ is even and } s \text{ is odd}\\ \Theta(n^{s-2}) & \text{if } s, t \text{ are odd with } s \le t\\ \Theta(n^s) & \text{if } t \text{ is odd, and } s > t \text{ or } s \text{ is even.} \end{cases}$$

For comparison, we mention the order of magnitude of this function in the non-rainbow setting. We note that in many cases more precise bounds are known than the ones given below.

Theorem 1.3 (Gishboliner, Shapira [9], Gerbner, Győri, Methuku, Vizer [7]). If $s \ge 4$ and $t \ge 3$ are distinct positive integers, then

$$ex(n, C_s, C_t) = \begin{cases} \Theta(n^{s/2}) & \text{if } t = 4\\ \Theta(n^{s/2}) & \text{if } s, t \text{ are even} \\ \Theta(n^{(s-1)/2}) & \text{if } t \ge 6 \text{ is even and } s \text{ is odd} \\ \Theta(n^{(s-1)/2}) & \text{if } s, t \text{ are odd with } s < t \\ \Theta(n^s) & \text{if } t \text{ is odd, and } s > t \text{ or } s \text{ is even.} \end{cases}$$

As part of our proof, we will also determine the order of magnitude of the maximal number of paths of length ℓ if there is no rainbow copy of C_{2k} whenever $k, \ell \geq 2$. (By the path P_{ℓ} of length ℓ we mean the path with ℓ edges and $\ell+1$ vertices.) This result is given in the following theorem. Note that the answer is of the same order of magnitude as in the case of the corresponding (non-rainbow) generalised Turán problem [9], although our proof is rather different. Also, we trivially have $\exp(n, P_{\ell}, \operatorname{rainbow-}C_t) = \Theta(n^{\ell+1})$ if t is odd.

Theorem 1.4. If $k, \ell \geq 2$ are integers, then

$$\operatorname{ex}(n, P_{\ell}, \operatorname{rainbow-}C_{2k}) = \begin{cases} \Theta(n^{\lceil (\ell+1)/2 \rceil}) & \text{if } k \ge 3\\ \Theta(n^{\ell/2+1}) & \text{if } k = 2. \end{cases}$$

Note that a path of length $\ell = 1$ is just an edge, so the corresponding generalised rainbow Turán number $ex(n, P_1, rainbow-C_{2k})$ is $ex^*(n, C_{2k})$. The correct order of magnitude of this is unknown for $k \ge 4$, but is conjectured to be $\Theta(n^{1+1/k})$ for all k (and a corresponding lower bound is known) [12, 3]. We mention that we believe that the most difficult (new) results in this paper are Theorem 1.4 and the closely related s = t = 2k case of Theorem 1.2.

Theorem 1.2 deals with all cases except when s = 3. In that case the correct order of magnitude is unknown in general even in the non-rainbow setting, where the following bounds are known.

Theorem 1.5 (Győri, Li [10], Alon, Shikhelman [1], Gishboliner, Shapira [9]). For every $k \ge 2$, we have

$$\Omega(\exp(n, \{C_4, C_6, \dots, C_{2k}\})) \le \exp(n, C_3, C_{2k}) \le O(\exp(n, C_{2k}))$$

and

$$\Omega(\exp(n, \{C_4, C_6, \dots, C_{2k}\})) \le \exp(n, C_3, C_{2k+1}) \le O(\exp(n, C_{2k})).$$

Note that the lower and upper bounds are only known to be of the same order of magnitude when $k \in \{2, 3, 5\}$, in which case both bounds are $\Theta(n^{1+1/k})$. For the rainbow version, we have the following.

Theorem 1.6. If $k \ge 2$ is odd then $ex(n, C_3, rainbow-C_{2k}) = \Omega(n^{1+1/k})$, and if k is even then $ex(n, C_3, rainbow-C_{2k+1}) = \Omega(n^{1+1/k})$. Furthermore, for every $k \ge 2$ integer, we have

 $ex(n, C_3, rainbow-C_{2k}) = O(ex^*(n, C_{2k}))$

and

$$\exp(n, C_3, \operatorname{rainbow-}C_{2k}) \ge \exp(n, C_3, C_{2k}) = \Omega(\exp(n, \{C_4, C_6, \dots, C_{2k}\})),$$

$$\exp(n, C_3, \operatorname{rainbow-}C_{2k+1}) \ge \exp(n, C_3, C_{2k+1}) = \Omega(\exp(n, \{C_4, C_6, \dots, C_{2k}\})).$$

Note that, as mentioned before, $ex^*(n, C_{2k})$ is conjectured [12, 3] to be $O(n^{1+1/k})$, and $ex(n, \{C_4, C_6, \ldots, C_{2k}\})$ is only known to be $\Omega(n^{1+1/k})$ when k = 2, 3, 5.

2 Forbidden rainbow C_{2k}

In this section we consider graphs having no rainbow C_{2k} subgraph, and prove the corresponding cases of Theorem 1.2, as well as Theorem 1.4 concerning the number of paths. We will use the following lemma of Gerbner, Mészáros, Methuku and Palmer [8]. We also include its proof below for completeness.

Lemma 2.1 (Gerbner, Mészáros, Methuku, Palmer [8]). Let G be a properly edge-coloured graph on n vertices containing no rainbow C_{2k} . Then for every $a \in V(G)$, the number of paths axy of length 2 starting at a is O(n).

Proof. We may assume that G is bipartite, since a random bipartition is expected to preserve a quarter of all paths of length 2 starting at a. Let X = N(a) and $Y = N(N(a)) \setminus \{a\}$. Observe that the number of paths axy is e(X, Y), that is, the number of edges between X and Y. So it

suffices to show that the induced subgraph $G[X \cup Y]$ does not contain a (100k)-ary tree of depth 2k.

Assume that it does contain such a tree. Then it also contains a (100k)-ary tree of depth 2k-1 rooted at some $x_1 \in X$. Then we can recursively find distinct vertices $y_1, x_2, y_2, \ldots, y_{k-1}, x_k$ (with $x_i \in X, y_j \in Y$) such that for all $i, x_i y_i, y_i x_{i+1} \in E(G)$, and the colours $c(x_i y_i), c(y_i x_{i+1}), c(ax_i)$ are all distinct. (Here c denotes the edge-colouring.) But then $ax_1y_1x_2y_2\ldots y_{k-1}x_ka$ is a rainbow cycle of length 2k, giving a contradiction.

We now state explicitly the cases of Theorem 1.2 we deal with in the next two subsections.

Theorem 2.2. Let $k \geq 2$ be an integer. Then

$$\exp(n, C_{2k}, \operatorname{rainbow-} C_{2k}) = \begin{cases} \Theta(n^{k-1}) & \text{if } k \ge 3\\ \Theta(n^2) & \text{if } k = 2. \end{cases}$$

Theorem 2.3. If $k, \ell \geq 2$ are integers, then

$$ex(n, C_{2\ell+1}, \text{rainbow-}C_{2k}) = \begin{cases} \Theta(n^{\ell}) & \text{if } k \ge 3\\ \Theta(n^{\ell+1/2}) & \text{if } k = 2 \end{cases}$$

For the remainder of this section, unless otherwise stated, we will assume that $k \geq 2$ is an integer, G is a properly edge-coloured graph on n vertices with no rainbow copy of C_{2k} , and $c: E(G) \to \mathbb{Z}$ denotes the edge-colouring.

2.1 Paths and even cycles

In this subsection, we will prove Theorems 1.4 and 2.2. Note that for the upper bounds in Theorems 1.4 and 2.2 it suffices to consider bipartite graphs G, since a random bipartition is expected to preserve a fixed positive proportion of subgraphs isomorphic to a given bipartite graph, so from now on we assume that G is bipartite.

In light of Lemma 2.1, to prove the upper bound in Theorem 1.4 for $k \ge 3$, it is sufficient to show that the number of paths of length 3 is $O(n^2)$. Let us say that a pair x, y of vertices of Gis bad if x and y have at least 100k common neighbours, and it is good otherwise. Then there are three types of paths axyz of length 3: either ay and xz are both good, or both bad, or one of them is good and the other one is bad. We will treat these cases in separate lemmas. It will be important later that for two of these cases we prove not only that the number of P_{3s} of that type is $O(n^2)$, but also that any vertex is a certain endpoint of O(n) such P_{3s} . However, it is not true that for any vertex a the number of paths axyz of length 3 starting at a has to be O(n). To see this, take a C_{2k} -free bipartite graph G_0 on vertex classes X, Y with |X| = |Y| = n/4 and $|E(G_0)| = \omega(n)$. For each $x \in X$ add a new vertex x', and join each pair xx' by an edge of the same colour. Finally, add a vertex a and join it to all vertices x'. Then the (bipartite) graph we get contains no rainbow C_{2k} , and the number of paths of length 3 starting at a is $|E(G_0)|$.

Lemma 2.4. Let $k \ge 3$. For every $a \in V(G)$, the number of paths axyz such that ay and xz are both bad is O(n).

Proof. Let $Y = \{y \in N(N(a)) \setminus \{a\} : ay \text{ is bad}\}$, and let Z = N(Y). Observe that $G[Y \cup Z]$ cannot contain a rainbow path of length 2k - 3. Indeed, if there is such a rainbow path,

then there is a rainbow path $y_1z_1 \ldots y_{k-2}z_{k-2}y_{k-1}$ of length 2k - 4 with $y_i \in Y, z_j \in Z$. Since ay_1 and ay_{k-1} are bad, we can choose $b \in N(a) \cap N(y_1)$ and $b' \in N(a) \cap N(y_{k-1})$ such that $aby_1z_1 \ldots y_{k-2}z_{k-2}y_{k-1}b'a$ is a rainbow 2k-cycle, giving a contradiction. It follows that e(Y,Z) = O(n), i.e., $\sum_{y \in Y} \deg_G(y) = O(n)$. (We are using the fact that for any ℓ we have $ex^*(n, P_\ell) = O(n)$. See [5] for the best known upper bound.)

For each $y \in Y$, define an auxiliary graph H_y on vertex set N(y) by letting zz' be an edge if and only if zz' is bad. Note that H_y cannot contain a path of length k-1. Indeed, if $z_1 \ldots z_k$ is such a path, then we can choose $b_i \in N_G(z_i) \cap N_G(z_{i+1})$ in such a way that $az_1b_1z_2 \ldots b_{k-1}z_ka$ is a rainbow 2k-cycle in G, giving a contradiction. It follows that $|E(H_y)| \leq k|H_y| = k \deg_G(y)$. But the number of triples (x, y, z) such that xyz is a path, xz is bad and $y \in Y$ is $2\sum_{y \in Y} |E(H_y)| \leq 2k \sum_{y \in Y} \deg_G(y) = O(n)$. The statement of the lemma follows.

Lemma 2.5. Let $k \ge 3$. For every $a \in V(G)$, the number of paths axyz such that ay is good and xz is bad is O(n).

Proof. Let $Y = \{y \in N(N(A)) \setminus \{a\} : ay \text{ is good}\}$, and let

 $Z = \{z \in V(G) : \text{for any set } S \subseteq V(G) \text{ with } |S| \le 100k \text{ there is a path } axyz \text{ of length } 3 \text{ such that } x \notin S, ay \text{ is good and } xz \text{ is bad} \}.$

Consider first the number of paths axyz with $z \in Z$ such that ay is good (and xz is bad). The number of these is at most $100k \cdot e(Y, Z)$, as after picking yz there are at most 100k possible choices for x.

Claim. $G[Y \cup Z]$ cannot contain a rainbow path of length 2k - 5.

Proof of Claim. Suppose it contains such a rainbow path. Then it also contains a rainbow path $P: z_1y_1 \dots z_{k-3}y_{k-3}z_{k-2}$ of length 2k - 6 such that $z_i \in Z, y_j \in Y$. Let

$$S_1 = V(P) \cup \{x \in N(a) : c(ax) = c(z_i y_i) \text{ or } c(ax) = c(y_i z_{i+1}) \text{ for some } i\}.$$

Then $|S_1| < 100k$, so we can pick a $P_3 axyz_1$ from a to z_1 such that $x \notin S_1$, ay is good and xz_1 is bad. Let $S_2 = S_1 \cup \{x\}$ and pick a path $ax'y'z_{k-2}$ such that $x' \notin S_2$, ay' is good and $x'z_{k-2}$ is bad. Then we can pick $y'' \in N(x) \cap N(z_1)$ such that c(xy'') and $c(y''z_1)$ are distinct from all $c(z_iy_i), c(y_iz_{i+1}), c(ax), c(ax')$, and y'' is distinct from a and each y_i . Similarly, we can pick y''' such that c(xy''') and $c(y''z_1)$ are distinct from all $c(z_iy_i), c(y_iz_{i+1}), c(ax), c(ax'), c(y''z_1)$, and y''' is distinct from all $c(z_iy_i), c(y_iz_{i+1}), c(ax), c(xy''), c(y''z_1)$, and y''' is distinct from a, y'' and each y_i . Then $axy''z_1y_1z_2 \dots y_{k-3}z_{k-2}y'''x'a$ is a rainbow C_{2k} , giving a contradiction. The claim follows.

So $G[Y \cup Z]$ contains no rainbow P_{2k-5} , so e(Y,Z) = O(n). So there are O(n) P_{3s} axyz with $z \in Z$ such that ay is good (and xz is bad).

Now consider the number of P_{3s} axyz with $z \notin Z$ such that ay is good and xz is bad. Given $z \notin Z$, there is a set S with $|S| \leq 100k$ such that any P_3 axyz such that ay is good and xz is bad must have $x \in S$. So for each $z \in Z$ we can pick $x_z \in N(a)$ such that at least a proportion of 1/(100k) of all such P_{3s} from a to z go through x_z . For each $x \in N(a)$ let $Z_x = \{z \notin Z : x_z = x\}$.

Also let $Y_x = Y \cap N(x)$. Then the number of such P_{3s} starting in a and ending outside Z is at most

$$\sum_{z \notin Z} 100k \cdot |N(x_z) \cap N(z) \cap Y| = \sum_{z \notin Z} 100k \cdot e(Y_{x_z}, \{z\})$$
$$= \sum_{x \in N(a)} 100k \cdot e(Y_x, Z_x).$$

Note that $e(Y_x, Z_x)$ is the number of paths of length 2 starting at x in the graph $G[\{x\} \cup Y_x \cup Z_x]$. Since that graph contains no rainbow C_{2k} , Lemma 2.1 gives that $e(Y_x, Z_x) = O(|Y_x| + |Z_x| + 1)$. Note, however, that

$$\sum_{x \in N(a)} |Y_x| = \sum_{y \in Y} |N(y) \cap N(a)| \le 100k|Y| = O(n)$$

and

$$\sum_{x \in N(a)} |Z_x| = \sum_{z \notin Z} 1 = O(n).$$

Putting together, we get that the number of such P_{3s} starting at a and ending outside Z is O(n). The statement of the lemma follows.

Lemma 2.6. Let $k \ge 3$. The number of paths axyz such that ay and xz are both good is $O(n^2)$.

Some parts of the proof below will be similar to the proof of the fact $ex^*(n, C_6) = O(n^{4/3})$ in [12].

Proof. We start similarly as in the proof of Lemma 2.5. Let

 $W = \{(a, z) \in V(G) \times V(G) : \text{ for any set } S \text{ of at most } (100k)^2 \text{ colours there is a rainbow path } axyz \text{ such that } c(ax), c(xy), c(yz) \notin S \text{ and } ay, xz \text{ are good.} \}$

Given $a \in V(G)$, let $Z_a = \{z : (a, z) \in W\}$, and let $Y_a = \{y \in N(N(a)) \setminus \{a\} : ay \text{ is good}\}.$

Claim. $G[Y_a, Z_a]$ contains no rainbow path of length 2k - 5.

Proof of Claim. Suppose it does. Then it also contains a rainbow path $P: z_1y_1 \dots z_{k-3}y_{k-3}z_{k-2}$ of length 2k - 6 with $z_i \in Z_a, y_j \in Y_a$. Let

$$S_1 = \bigcup_i \{c(z_i y_i), c(y_i z_{i+1})\} \cup \bigcup_i \{c(ax) : x \in N(a) \cap N(y_i)\} \cup \bigcup_i \{c(xy_i) : x \in N(a) \cap N(y_i)\} \cup \{c(az_i) : z_i \in N(a)\}$$

Note that $|S_1| \leq 2k + k \cdot 100k + k \cdot 100k + k < (100k)^2$, so we can pick a rainbow path $axyz_1$ such that ay, xz_1 are good and $c(ax), c(xy), c(yz_1) \notin S_1$. Note that $y \neq y_i$ for all i and $x \neq z_j$ for all j. Let

$$S_2 = S_1 \cup \{c(ax), c(xy), c(yz_1)\} \cup \{c(aw) : w \in N(a) \cap N(y)\} \cup \{c(wy) : w \in N(a) \cap N(y)\}$$

We have $|S_2| < (100k)^2$, so we can pick a rainbow path $ax'y'z_{k-2}$ such that $ay', x'z_{k-2}$ are good and $c(ax'), c(x'y'), c(x'z_{k-2}) \notin S_2$. Note that $y' \neq y_i, y$ and $x' \neq z_j, x$. But then $axyz_1y_1 \dots z_{k-3}y_{k-3}z_{k-2}y'x'a$ is a rainbow C_{2k} , giving a contradiction. The claim follows. By the Claim, we have $e(Y_a, Z_a) = O(n)$ for all a. Hence the number of paths axyz such that ay and xz are good and $(a, z) \in W$ is $O(n^2)$ (since for any a, each edge yz extends to at most 100k such paths axyz).

Now consider $P_{3s} axyz$ with $(a, z) \notin W$. For any a and z, let f(a, z) denote the number of rainbow $P_{3s} axyz$ from a to z such that ay and xz are both good. If $(a, z) \notin W$, we can pick a colour c_{az} such that there are at least $\lceil f(a, z)/(100k)^2 \rceil P_{3s} axyz$ such that ay, xz are good and $c_{az} \in \{c(ax), c(xy), c(yz)\}$. Note that at most 100k of these P_{3s} have $c(ax) = c_{az}$, since the colouring is proper and xz is good. Similarly, at most 100k of these P_{3s} have $c(yz) = c_{az}$. We deduce that there are at least $N_{az} = \lceil f(a, z)/(100k)^2 \rceil - 200k P_{3s} axyz$ such that $c(xy) = c_{az}$ and ay, xz are good. Note that these paths must be internally vertex-disjoint. So we can list N_{az} such paths as ax_iy_iz for $i = 1, 2, \ldots, N_{az}$ such that if $i \neq j$ then $x_i \neq x_j$ and $y_i \neq y_j$.

Using the observations above, we now show that there are 'many' 6-cycles $ax_iy_izy_jx_ja$ such that $c(x_iy_i) = c(x_jy_j) = c_{az}$ and each pair (of distance 2) in the 6-cycle is good. (Note that if we did not require that xx' and yy' are good then we would immediately get at least $\binom{N_{az}}{2}$ such 6-cycles if $N_{az} > 0$). Write $N = N_{az}$. Define an auxiliary graph H on vertex set $\{x_1, \ldots, x_N\}$ such that x_ix_j is an edge if and only if x_ix_j is bad. Observe that H contains no path of length k-1. Indeed, if $x_{i_1}x_{i_2}\ldots x_{i_k}$ is such a path in H, then we can choose some vertices b_1, \ldots, b_{k-1} in G such that $ax_{i_1}b_1x_{i_2}b_2\ldots x_{i_{k-1}}b_{k-1}x_{i_k}a$ is a rainbow cycle of length 2k, giving a contradiction. It follows that $|E(H)| \leq kN$. So there are at most kN pairs $\{i, j\}$ such that x_ix_j is bad. Similarly, there are at most kN pairs $\{i, j\}$ such that $if N \geq 1$ then there are at least $\binom{N}{2} - 2kN$ 6-cycles $ax_iy_izy_jx_ja$ in which each pair of vertices of distance 2 is good.

Write $T = \{(a, z) \notin W : f(a, z) > (100k)^2 + 200k\}$. By the argument above, the number of 6-cycles axyzy'x'a in which c(xy) = c(x'y') and each pair of vertices of distance 2 is good is at least

$$\frac{1}{6} \sum_{(a,z)\in T} \left[\binom{N_{az}}{2} - 2kN_{az} \right],$$

which is at least

$$\sum_{(a,z)\in T} (\alpha f(a,z)^2 - \beta f(a,z))$$

for some positive constants α, β .

On the other hand, if L denotes the number of paths axyz in which ay, xz are both good, then the number of such 6-cycles is at most 100kL. Indeed, there are L ways to choose xyzy', then x' is uniquely determined by the condition c(xy) = c(x'y'), and then there are at most 100kpossible choices for a, since we need xx' to be good. Hence

$$\sum_{(a,z)\in T} (\alpha f(a,z)^2 - \beta f(a,z)) \le 100kL$$

But we have

$$L \le \sum_{(a,z)\in T} f(a,z) + O(n^2).$$
 (1)

Indeed, we know that the number of P_{3} s axyz (such that ay and xz are good) having $(a, z) \in W$ is $O(n^2)$, the number of such rainbow P_{3} s axyz with $(a, z) \in T$ is $\sum_{(a,z)\in T} f(a, z)$, the number of such rainbow P_{3} s axyz with $(a, z) \notin T$, $(a, z) \notin W$ is at most $((100k)^2 + 200k)n^2$, and finally, the number of such non-rainbow P_{3s} is at most the number of $P_{2s} xyz$ with xz good, which is $O(n^2)$. It follows that

$$\sum_{(a,z)\in T} (\alpha f(a,z)^2 - \beta f(a,z)) \le 100k \sum_{(a,z)\in T} f(a,z) + O(n^2),$$

and hence

$$\sum_{(a,z)\in T} f(a,z)^2 \le A \sum_{(a,z)\in T} f(a,z) + Bn^2$$

for some positive constants A, B > 0. But we have

$$\sum_{(a,z)\in T} f(a,z)^2 \ge \left[\sum_{(a,z)\in T} f(a,z)\right]^2 \cdot \frac{1}{|T|} \ge \left[\sum_{(a,z)\in T} f(a,z)\right]^2 \cdot \frac{1}{n^2}.$$

We get

$$\left[\sum_{(a,z)\in T} f(a,z)\right]^2 \le An^2 \sum_{(a,z)\in T} f(a,z) + Bn^4,$$

which gives $\sum_{(a,z)\in T} f(a,z) = O(n^2)$. The statement of the lemma then follows using (1).

Proof of Theorem 1.4. For $k \ge 3$, Lemma 2.1 shows that there are $O(n^2)$ copies of P_2 , and Lemmas 2.4, 2.5 and 2.6 show that there are $O(n^2)$ copies of P_3 . The required upper bound then follows by repeated application of Lemma 2.1. For the lower bound, take an $(\ell+1)$ -partite graph with vertex classes $X_1, \ldots, X_{\ell+1}$ such that $|X_i| = 1$ if *i* is even and $|X_i| = \Theta(n)$ if *i* is odd, and join vertices *x* and *y* if and only if $x \in X_i$ and $y \in X_j$ with $i - j = \pm 1$. (The edge-colouring is arbitrary.)

When k = 2, the number of paths of length 2 is $O(n^2)$ by Lemma 2.1, and the number of paths of length 1 is at most $ex^*(n, C_4) = \Theta(n^{3/2})$ (see [12]). The required upper bound then follows by repeated application of Lemma 2.1. For the lower bound, we can take a C_4 -free *d*-regular graph on $\Theta(n)$ vertices with $d = \Theta(n^{1/2})$.

We now prove Theorem 2.2. Although the upper bound is proved for k = 2 and the lower bound is proved for $k \ge 3$ in [8], we include proofs of these for completeness.

Proof of Theorem 2.2. Consider first the case k = 2. For the upper bound, observe that there can be no bad pair if there is no rainbow C_4 , thus any two vertices x and z are contained in O(1)4-cycles of the form xyzw. The upper bound $ex(n, C_4, rainbow-C_4) = O(n^2)$ follows. For the lower bound when k = 2, let A be a Sidon set in \mathbb{Z}_n of size $\Theta(\sqrt{n})$, i.e., a set such that whenever $a, b, a', b' \in A$ with a + b = a' + b' then (a, b) = (a', b') or (a, b) = (b', a'). (See e.g. [4] for the construction of such sets.) Partition A into two subsets A_1, A_2 of size $\Theta(\sqrt{n})$ each. Let G be a 4-partite graph with vertex classes $X_{00}, X_{01}, X_{10}, X_{11}$ each being copies of \mathbb{Z}_n , and edges given as follows. If $x_{00} \in X_{00}, x_{01} \in X_{01}, x_{10} \in X_{10}, x_{11} \in X_{11}$, then we join:

- x_{00} to x_{10} by an edge of colour a_1 if $x_{10} x_{00} = a_1 \in A_1$;
- x_{00} to x_{01} by an edge of colour a_2 if $x_{01} x_{00} = a_2 \in A_2$;
- x_{10} to x_{11} by an edge of colour a_2 if $x_{11} x_{10} = a_2 \in A_2$;

• x_{01} to x_{11} by an edge of colour a_1 if $x_{11} - x_{01} = a_1 \in A_1$.

It is easy to check that the graph we get is properly edge-coloured with no rainbow C_4 , has 4n vertices, and the number of 4-cycles is $n|A_1||A_2| = \Theta(n^2)$.

Now consider the lower bound for $k \geq 3$. Take a (2k)-partite graph with vertex classes X_1, \ldots, X_{2k} , where $|X_1| = |X_2| = |X_4| = |X_5| = 1$, $|X_6| = |X_8| = |X_{10}| = \cdots = |X_{2k}| = n$, $|X_3| = n$ and $|X_7| = |X_9| = \cdots = |X_{2k-1}| = 1$. Join two vertices x and y by an edge if and only if $x \in X_i, y \in X_j$ with $i - j \equiv \pm 1 \mod 2k$. Give the unique edge X_1 to X_2 and the unique edge X_4 to X_5 colour 1, and arbitrary distinct colours to the remaining edges. It is easy to see that any 2k-cycle must contain both of the edges of colour 1, there are $\Theta(n)$ vertices and $\Theta(n^{k-1})$ copies of C_{2k} .

It remains to prove the upper bound for $k \geq 3$. Given a 2k-cycle $x_1 \dots x_{2k} x_1$, define its pattern to be the list of i such that $x_i x_{i+2}$ is good (indices understood mod 2k), together with the list of pairs (i, j) such that $c(x_i x_{i+1}) = c(x_j x_{j+1})$. Note that there are finitely many patterns, so it suffices to show that for each pattern the number of 2k-cycles of that pattern is $O(n^{k-1})$.

Consider first the case $k \ge 4$. Assume that we have a pattern and an i such that $x_{i-1}x_{i+1}$ is good but $x_{i-3}x_{i-1}$ is bad in the pattern. Then we can choose vertices $x_{i+1}x_{i+2} \ldots x_{i+2k-4}$ in $O(n^{k-2})$ ways, since we have to pick a path of length 2k - 5. (Note that $x_{i+2k-4} = x_{i-4}$.) Then, by Lemmas 2.4 and 2.5, there are at most O(n) ways of choosing the path $x_{i-4}x_{i-3}x_{i-2}x_{i-1}$ according to the pattern (since $x_{i-3}x_{i-1}$ has to be bad). Then there are at most 100k possible ways of choosing x_i , since $x_{i-1}x_{i+1}$ is good. So we get $O(n^{k-1})$ 2k-cycles for these patterns.

So (when $k \ge 4$) it remains to consider the case when there is no *i* such that $x_{i-1}x_{i+1}$ is good but $x_{i-3}x_{i-1}$ is bad. Observe that for any 2*k*-cycle $x_1 \ldots x_{2k}x_1$, at least one (in fact, at least two) of the pairs $x_2x_4, x_4x_6, \ldots, x_{2k}x_2$ has to be good (otherwise we can find a rainbow C_{2k}). So it remains to consider patterns such that each of these pairs is good. Similarly, we may assume that each of $x_1x_3, \ldots, x_{2k-1}x_1$ is a good pair.

Now consider the colours for the pattern. We must have a pair of different edges with the same colour. We may assume that we have $c(x_1x_2) = c(x_ix_{i+1})$ for some *i* with $3 \le i \le k+1$. Then we can choose $x_2x_3 \ldots x_{2k-1}$ in $O(n^{k-1})$ ways (since it is a path of length 2k-3). Then x_1 is uniquely determined by the condition $c(x_1x_2) = c(x_ix_{i+1})$, and then there are at most 100k possible choices for x_{2k} (according to the pattern), since x_1x_{2k-1} is good. This gives $O(n^{k-1})$ 2k-cycles of this pattern, as required.

It remains to consider the case k = 3. Observe that if k = 3, then for any edge *ab* there is at most one way to extend this edge to a path *abc* such that *ac* is bad. Indeed, if we have two different extensions *abc* and *abc'* then there is a rainbow 6-cycle of the form axcbc'x'a. Consider any pattern, we show that there are $O(n^2)$ 6-cycles of that pattern. We may assume that $c(x_1x_2) = c(x_ix_{i+1})$ for some $i \in \{3,4\}$. If x_5x_1 is good in the pattern, then we are done exactly as above: we can choose $x_2x_3x_4x_5$ in $O(n^2)$ ways, then x_1 is determined by the condition $c(x_1x_2) = c(x_ix_{i+1})$, and there are at most 100k choices for x_6 . So we may assume that x_5x_1 is bad.

Case 1: i = 4. Then the same argument shows that we are done if x_2x_4 is good. So we may assume that x_2x_4 and x_5x_1 are both bad. Then we can choose $x_6x_1x_2x_3$ in $O(n^2)$ ways, and we can extend x_2x_3 to a path $x_2x_3x_4$ such that x_2x_4 is bad in at most one way, and similarly we can extend x_1x_6 in at most one way to get $x_1x_6x_5$. Then all the vertices are determined, so we get $O(n^2)$ copies.

Case 2: i = 3. There are $O(n^2)$ ways of choosing $x_3x_2x_1x_6$, and then there is at most one way of extending x_1x_6 to a path $x_1x_6x_5$ such that x_1x_5 is bad, and there is at most one way of picking x_4 such that $c(x_3x_4) = c(x_1x_2)$. So we get $O(n^2)$ copies of C_6 , as required.

2.2 Odd cycles

We now turn to the case of odd cycles. Once we have established Theorem 1.4, the proof of Theorem 2.3 is essentially the same as the proof of Gishboliner and Shapira [9] for the non-rainbow version of the problem.

Proof of Theorem 2.3. The lower bounds follow from the fact $ex(n, F, rainbow-H) \ge ex(n, F, H)$ and the corresponding results for the non-rainbow problem, see [9]. (Note that the only difficult case is when k = 2.)

For the upper bound when k = 2, observe that there can be no bad pair of vertices if there is no rainbow C_4 , hence the number of $(2\ell + 1)$ -cycles is at most 100k = 200 times the number of paths of length $2\ell - 1$, which is $O(n^{\ell+1/2})$ by Theorem 1.4.

Now consider the case $k \geq 3$. Given a path $P : x_1 x_2 \dots x_{2\ell-1}$ of length $2\ell - 2$ in G, write $X_P = N(x_1) \setminus V(P)$ and $Y_P = N(x_{2\ell-1}) \setminus V(P)$. Then the number of ways of extending path P to a cycle $x_1 x_2 \dots x_{2\ell+1} x_1$ is $e(X_P, Y_P)$. But this is at most the number of paths of length 2 starting at x_1 in the graph $G[\{x_1\} \cup X_P \cup Y_P]$, which is $O(1 + |X_P| + |Y_P|)$ by Lemma 2.1. It follows that P extends to at most $O(1 + |X_P| + |Y_P|)$ cycles of length $2\ell + 1$. But $|X_P|$ is the number of ways of extending P to a path $x_0 x_1 x_2 \dots x_{2\ell-1}$, and similarly, $|Y_P|$ is the number of ways of extending P to a path $x_1 \dots x_{2k}$. It follows that if the number of paths of length s is p_s , then $\sum_P |X_P| = O(p_{2\ell-1})$, and similarly for Y_P . Hence the number of cycles of length $2\ell + 1$ is $O(p_{2\ell-2}) + O(p_{2\ell-1})$, which is $O(n^\ell)$ by Theorem 1.4.

3 Forbidden rainbow C_{2k+1}

In this section we prove the following result, which is the only non-trivial case of Theorem 1.2 with t odd.

Theorem 3.1. If $k \ge \ell \ge 2$ are positive integers, then $ex(n, C_{2\ell+1}, rainbow C_{2k+1}) = \Theta(n^{2\ell-1})$.

From now on, unless otherwise stated, we will assume that $k \ge \ell \ge 2$ are integers, G is a properly edge-coloured graph of order n with no rainbow C_{2k+1} , and c denotes the edge-colouring. Also, we will say (as before) that a pair x, y of vertices is bad if $|N(x) \cap N(y)| \ge 100k$, and good otherwise.

We will deduce Theorem 3.1 from the following two lemmas.

Lemma 3.2. Let G be any properly edge-coloured graph, and let $\ell \geq 2$ be an integer. Then the number of non-rainbow copies of $C_{2\ell+1}$ in G is $O(n^{2\ell-1}) + O($ number of rainbow $C_{2\ell+1}$ s in G).

Lemma 3.3. Let $k \ge \ell \ge 2$ be integers and let G be a properly edge-coloured graph with no rainbow C_{2k+1} . Assume that every edge of G is contained in a rainbow $C_{2\ell+1}$. Then for every $a \in V(G)$ the number of paths axy of length 2 starting at a in G is O(n).

Deducing Theorem 3.1. For the lower bound, take a $(2\ell+1)$ -partite graph with vertex classes $X_1, \ldots, X_{2\ell+1}$ all being copies of $\{1, \ldots, n\}$. Join any $x \in X_1$ to $x \in X_2$ by an edge of colour

1, and also $x \in X_3$ to $x \in X_4$ by an edge of colour 1. For all $i \neq 1, 3$, join each pair of vertices x, y with $x \in X_i, y \in X_{i+1}$ by an edge of arbitrary unused colour (with indices understood mod $2\ell + 1$). It is clear that the graph we get is properly edge-coloured, there are $\Theta(n)$ vertices and $\Theta(n^{2\ell-1})$ copies of $C_{2\ell+1}$. Furthermore, no copy of C_{2k+1} is rainbow, since any C_{2k+1} must contain an edge between each pair of X_i, X_{i+1} (otherwise it would be a subgraph of a bipartite graph). The lower bound follows.

Now consider the upper bound. By Lemma 3.2, it suffices to show that if G contains no rainbow C_{2k+1} then the number of rainbow $C_{2\ell+1}$ s is $O(n^{2\ell-1})$. For this, we may assume that any edge is contained in a rainbow copy of $C_{2\ell+1}$. But then, by Lemma 3.3, for any vertex $a \in V(G)$ there are O(n) paths of length 2 starting at a. By repeated application of this fact, it follows that for any a there are $O(n^{\ell})$ paths of length 2ℓ starting at a, and hence there are $O(n^{\ell+1}) \leq O(n^{2\ell-1})$ copies of $C_{2\ell+1}$.

Proof of Lemma 3.2. We will consider patterns of $(2\ell + 1)$ -cycles. Recall that the pattern \mathcal{P} of a $(2\ell + 1)$ -cycle $x_1 \dots x_{2\ell+1}x_1$ is the list of *i* such that x_ix_{i+2} is good, together with the list of pairs (i, j) such that $c(x_ix_{i+1}) = c(x_jx_{j+1})$ (with the indices understood mod $2\ell + 1$). Since there are finitely many patterns, it suffices to show that for any non-rainbow pattern the required bound holds for cycles of that pattern.

Consider first the case when there are three edges with the same colour in a pattern \mathcal{P} , say $x_p x_{p+1}, x_q x_{q+1}, x_r x_{r+1}$. Then we can pick $(x_i)_{i \neq p,q}$ in $O(n^{2\ell-1})$ ways, and there is at most one way of extending those points to a $(2\ell + 1)$ -cycle of the appropriate pattern. This shows that there are $O(n^{2\ell-1})$ cycles with this pattern.

Now consider the case when there are two different colours such that each of them appears at least twice as the colour of an edge. For both of these colours, pick two edges of the appropriate colour. So we have c(e) = c(e') and c(f) = c(f') in our pattern for four different edges e, e', f, f'. Note that we must have $e \cup e' \neq f \cup f'$. So we can pick i, j such that $x_i \in (e \cup e') \setminus (f \cup f')$ and $x_j \in (f \cup f') \setminus (e \cup e')$. Then picking the vertices $(x_a)_{a \neq i, j}$ determines the $(2\ell + 1)$ -cycle uniquely by the colour conditions. It follows that there are $O(n^{2\ell-1})$ cycles of this pattern.

It remains to consider patterns \mathcal{P} in which there is only one pair of edges of the same colour, say $c(x_ix_{i+1}) = c(x_jx_{j+1})$, with $i \neq j-1, j, j+1$. Given a choice $X = \{x_a : a \neq i, j\}$ of all vertices except x_i, x_j , consider the number of ways of extending X to a $(2\ell + 1)$ -cycle. Write $d_1 = |N(x_{i-1}) \cap N(x_{i+1}) \setminus X|$ and $d_2 = |N(x_{j-1}) \cap N(x_{j+1}) \setminus X|$. Then the number of ways of extending X to a $(2\ell + 1)$ -cycle of pattern \mathcal{P} is at most $\min\{d_1, d_2\}$, whereas the number of ways of extending X to a rainbow $C_{2\ell+1}$ is at least $(d_1 - 5\ell)(d_2 - 5\ell)$. But we have $\min\{d_1, d_2\} \leq$ $10\ell + \max\{0, (d_1 - 5\ell)(d_2 - 5\ell)\}$, so the number of extensions of pattern \mathcal{P} is at most O(1) plus the number of rainbow extensions. Summing over all possible choices of X, we get the required bound.

Lemma 3.3 is proved similarly to Lemma 2.1.

Proof of Lemma 3.3. Given a bipartition $V(G) = X \cup Y$ of the vertex set of G, let $G_{X,Y}$ be the corresponding bipartite graph obtained from G (i.e., $G_{X,Y}$ is obtained by deleting all edges inside X and inside Y). Since a random bipartition is expected to preserve a quarter of all paths of length 2 starting at a, it suffices to show that for every bipartition $V(G) = X \cup Y$ with $a \in Y$, the number of paths of length 2 starting at a in $G_{X,Y}$ is O(n), where the implied constant is independent of the bipartition. So let $V(G) = X \cup Y$ be any bipartition. Write $X_1 = N_G(a) \cap X$

and $Y_1 = N_G(X_1) \cap Y \setminus \{a\}$, so that we would like to show $e_{G_{X,Y}}(X_1, Y_1) = O(n)$. It suffices to show that $G_{X,Y}[X_1 \cup Y_1]$ does not contain a (100k)-ary tree of depth 2k.

Suppose it contains such a tree, then it also contains a (100k)-ary tree T of depth 2k - 1 rooted at some $x \in X_1$. Since $ax \in E(G)$, the edge ax of G is contained in a rainbow cycle of length $2\ell + 1$ in G. Hence we can find a rainbow path $P : az_1z_2 \dots z_{2\ell-1}x$ of length 2ℓ from a to x in G. Then we can recursively find distinct vertices $x = x_1, x_2, \dots, x_{2(k-\ell)+1}$ on our tree T such that

- for all *i* we have $x_i x_{i+1} \in E(G_{X,Y})$
- for all *i* even we have $x_i \in Y_1 \setminus V(P)$;
- for all $i \ge 3$ odd we have $x_i \in X_1 \setminus V(P)$;
- for all $i, c(x_i x_{i+1})$ does not appear on the path $az_1 z_2 \dots z_{2\ell-1} x_1 \dots x_i$;
- the colour $c(ax_{2(k-\ell)+1})$ does not appear on the path $az_1z_2\ldots z_{2\ell-1}x_1\ldots x_{2(k-\ell)}$.

But then $az_1z_2...z_{2\ell-1}x_1x_2...x_{2(k-\ell)+1}a$ is a rainbow cycle of length 2k+1 in G, giving a contradiction.

4 Deducing Theorem 1.2 and Theorem 1.6

We now summarise how we deduce each case in Theorem 1.2.

Proof of Theorem 1.2. We have the following cases.

- If s = t = 4, then the result follows from Theorem 2.2. If $t = 4, s \neq 4$ and s is even, then it follows from Theorem 1.1. If t = 4 and s is odd, it follows from Theorem 2.3.
- If s, t are even with $s \neq t$, then the result follows from Theorem 1.1.
- If $s = t \ge 6$ is even, then the result follows from Theorem 2.2.
- If $t \ge 6$ is even and s is odd, then the result follows from Theorem 2.3.
- If s, t are odd with $s \leq t$, then the result follows from Theorem 3.1.
- If t is odd, and s is even or s > t, then the upper bound is trivial, and for the lower bound we can take a blowup of C_s , (i.e., we replace each vertex of C_s by n vertices and each edge by a complete bipartite graph. The edge-colouring is arbitrary.)

Finally, we prove Theorem 1.6 concerning triangles.

Proof of Theorem 1.6. For the upper bound $ex(n, C_3, rainbow-C_{2k}) = O(ex^*(n, C_{2k}))$, observe that the number of triangles containing a good pair is at most 100k|E(G)|, since we can pick the good pair in at most |E(G)| ways. So it suffices to show that the number of paths xyz with xz bad is O(|E(G)|). But for any $y \in V(G)$, if we define an auxiliary graph H_y with vertex set N(y) and edges being the bad pairs, then there can be no path $x_1 \dots x_k$ of length k-1 in H_y (otherwise we can find a rainbow cycle $yx_1b_1x_2b_2...x_ky$). It follows that H_y has at most $k|V(H_y)| = k \deg_G(y)$ edges, so each y is contained in at most $k \deg(y)$ paths xyz with xz bad. But $\sum_y \deg(y) = 2|E(G)|$, giving the required bound.

For the lower bound, the statements $ex(n, C_3, rainbow-C_{2k}) \ge ex(n, C_3, C_{2k})$ and $ex(n, C_3, rainbow-C_{2k+1}) \ge ex(n, C_3, C_{2k+1})$ are clear, and the lower bounds $ex(n, C_3, C_{2k}) = \Omega(ex(n, \{C_4, C_6, \dots, C_{2k}\})), ex(n, C_3, C_{2k+1}) = \Omega(ex(n, \{C_4, C_6, \dots, C_{2k+1}\}))$ follow from Theorem 1.5.

Finally, we prove that $ex(n, C_3, rainbow - C_{2k}) = \Omega(n^{1+1/k})$ when k is odd and $ex(n, C_3, rainbow - C_{2k+1}) = \Omega(n^{1+1/k})$ $\Omega(n^{1+1/k})$ when k is even. Take a B_k -set A of size $\Theta(n^{1/k})$ in \mathbb{Z}_n , that is, a set such that any $m \in \mathbb{Z}_n$ can be written as $a_1 + \cdots + a_k$ with $a_i \in A$ in at most one way (ignoring permutations of the summands). (See [2] for the construction of such 'dense' B_k -sets.) Then we take a tripartite graph G with vertex classes X_1, X_2, Y all being copies of \mathbb{Z}_n and edges given as follows. We join $x \in X_1$ to $x \in X_2$ by an edge of colour 0, and we join $x \in X_i$ to $x + a \in Y$ by an edge of colour (a,i) for i = 1,2. Clearly, G has $\Theta(n^{1+1/k})$ triangles. We claim that this graph contains no rainbow C_{2k} if k is odd and no rainbow C_{2k+1} if k is even. Indeed, assume that k is odd an there is a rainbow C_{2k} . Then it must be of the form $x_1y_1x_2y_2\ldots x_ky_kx_1$ with $y_i \in Y$ and $x_i \in X_1 \cup X_2$. Then we get a representation $0 = a_1 - b_1 + a_2 - b_2 + \cdots + a_k - b_k$ with $a_i, b_j \in A$ by letting $a_i = y_i - x_i$, $b_i = y_i - x_{i+1}$ (where $x_{k+1} = x_1$). So the a_i must be a permutation of the b_i . But k is odd, so we have $|\{x_1,\ldots,x_k\} \cap X_1| \neq |\{x_1,\ldots,x_k\} \cap X_2|$, and hence there exist i and j such that $a_i = b_j$ and x_i, x_{j+1} are in the same vertex class X_{ℓ} . But then $c(x_i y_i) = c(y_j x_{j+1})$, so the cycle is not rainbow, giving a contradiction. The case when k is even and G contains a rainbow (2k+1)-cycle is similar.

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