Ample completions of oriented matroids and complexes of uniform oriented matroids

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Abstract This paper considers completions of tope graphs of COMs (complexes of oriented matroids) to ample partial cubes of the same VC-dimension. We show that these exist for OMs (oriented matroids) and CUOMs (complexes of uniform oriented matroids). This implies that tope graphs of OMs and CUOMs satisfy the sample compression conjecture – one of the central open questions of learning theory. We conjecture that the tope graph of every COM can be completed to an ample partial cube without increasing the VC-dimension.

1. Introduction

Oriented matroids (OMs), co-invented by Bland and Las Vergnas [7] and Folkman and Lawrence [20], represent a unified combinatorial theory of orientations of ordinary matroids. They capture the basic properties of sign vectors representing the circuits in a directed graph and the regions in a central hyperplane arrangement in $\mathbb{R}^m$. Oriented matroids are systems of sign vectors satisfying three simple axioms (composition, strong elimination, and symmetry) and may be defined in a multitude of ways, see the book by Björner et al. [6]. The tope graphs of OMs can be viewed as subgraphs of the hypercube Q_m satisfying two strong properties: they are centrally-symmetric and are isometric subgraphs of Q_m, i.e., are antipodal partial cubes [6].

Ample set systems (AMPs) have been introduced by Lawrence [24] as asymmetric counterparts of oriented matroids and have been re-discovered independently by several works in different contexts [3, 8, 36]. Consequently, they received different names: lopsided [24], simple [36], extremal [8], and ample [3, 17]. Lawrence [24] defined ample set systems for the investigation of the possible sign patterns realized by a convex set in $\mathbb{R}^m$. Ample sets admit a multitude of combinatorial and geometric characterizations [3, 8, 24] and comprise many natural examples arising from discrete geometry, combinatorics, and geometry of groups [3, 24]. Analogously to tope graphs of OMs, AMPs induce isometric subgraphs of Q_m. In fact, they satisfy a stronger property: any two parallel cubes are connected in the set by a shortest path of parallel cubes.

Complexes of oriented matroids (COMs) have been introduced and investigated in [5] as a far-reaching natural common generalization of oriented matroids and ample set systems. COMs are defined in a similar way as OMs, simply replacing the global axiom of symmetry by a local axiom of face symmetry. This simple alteration leads to a rich combinatorial and geometric structure that is build from OM faces but is quite different from OMs. Replacing each face by a PL-ball, each COM leads to a contractible cell complex (topologically, OMs are spheres and AMPs are contractible cubical complexes). The tope graphs of COMs are still isometric subgraphs of hypercubes; as such, they have been characterized in [22].

Set families are fundamental objects in combinatorics, algorithmics, machine learning, discrete geometry, and combinatorial optimization. The Vapnik-Chervonenkis dimension (*VC-dimension* for short) of a set family was introduced by Vapnik and Chervonenkis [35] and plays a central role in the theory of PAC-learning. The VC-dimension was adopted in the above areas as a complexity measure and as a combinatorial dimension of the set family. The topes of

OMs, COMs, and AMPs (viewed as isometric subgraphs of hypercubes) give rise to set families for which the VC-dimension has a particular significance: the VC-dimension of an AMP is the largest dimension of a cube of its cube complex, the VC-dimension of an OM is its rank, and the VC-dimension of a COM is the largest VC-dimension of its faces.

Littlestone and Warmuth [25] introduced the sample compression technique for deriving generalization bounds in machine learning. Floyd and Warmuth [19] asked whether any set family of VC-dimension d has a sample compression scheme of size $O(d)$. This question, known as the *sample compression conjecture*, remains one of the oldest open problems in machine learning. It was shown in [28] that labeled compression schemes of size $O(2^d)$ exist. Moran and Warmuth [27] designed labeled compression schemes of size d for ample set systems. Chalopin et al. [9] designed (stronger) unlabeled compression schemes of size d for maximum families and characterized such schemes for ample set systems. In view of the above, it was noticed in [30] and [27] that the sample compression conjecture would be solved if *any set family of VC-dimension d can be completed to an ample (or maximum) set system of VC-dimension $O(d)$* or *covered by $O(2^d)$ ample set systems of VC-dimension $O(d)$.*

This opens a perspective that apart from its application to sample compression, is interesting in its own right: ample completions of structured set families. This is extending a given set system to an ample system by adding sets. A natural problem is ample completions of set families defined by *partial cubes* (i.e., isometric subgraphs of hypercubes). In [14], we prove that any partial cube of VC-dimension 2 admits an ample completion of VC-dimension 2. Moreover, we give a set family of VC-dimension 2 which has no ample completion of the same VC-dimension. In the present paper, we give an example of a partial cube of VC-dimension 3 which cannot be completed to an ample set system of VC-dimension 3. Hence, in higher dimension we cannot complete all partial cubes without increasing the VC-dimension. In the light of the above perspective, one may ask if *there exists a constant c such that every partial cube of VC-dimension d admits an ample completion of VC-dimension $\leq cd$?* Even stronger, we wonder if *partial cubes of VC-dimension d admit an ample completion of VC-dimension $d + c$.* Note that no such additive constant c exists for general set families [30]. In [14], we perform the ample completion of a partial cube of VC-dimension 2 in two steps. First, we show that they can be completed to tope graphs of COMs and next we complete the resulting graphs into ample ones, in both cases, without increasing the VC-dimension. In the present article, we are interested in the completion of this intermediate class of partial cubes. For COMs, we are inclined to believe that the following stronger result holds:

Conjecture 1. The tope graph of any COM of VC-dimension d has an ample completion of VC-dimension d.

COMs of rank 2 have the nice property that their faces are uniform OMs. This is not longer true in higher dimensions: COMs whose faces are uniform OMs constitute a proper subclass of COMs (that we call CUOMs). In this paper, we prove that Conjecture 1 holds for all tope graphs of OMs and CUOMs. This proves that set families arising from topes of OMs and CUOMs satisfy the sample compression conjecture. In Fig. 1, we present an example of the tope graph of a CUOM of VC-dimension 3, which we further use as a running example.

2. Preliminaries

2.1. **VC-dimension.** Let $\mathcal{S}$ be a family of subsets of an m-element set U. A subset X of U is *shattered* by $\mathcal{S}$ if for all $Y \subseteq X$ there exists $S \in \mathcal{S}$ such that $S \cap X = Y$. The *Vapnik-Chervonenkis dimension* [35] (the *VC-dimension* for short) VC-dim($\mathcal{S}$) of $\mathcal{S}$ is the cardinality of the largest subset of U shattered by $\mathcal{S}$. Any set family $\mathcal{S} \subseteq 2^U$ can be viewed as a subset of vertices of the m-dimensional hypercube $\mathrm{Q}_m = \mathrm{Q}(U)$. Denote by $G(\mathcal{S})$ the subgraph of Q_m

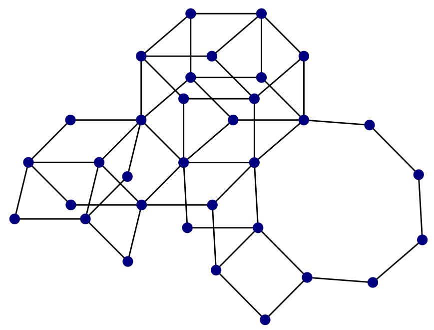


FIGURE 1. The tope graph M of a CUOM

induced by the vertices of Q_m corresponding to the sets of $\mathcal{S}$; $G(\mathcal{S})$ is called the *1-inclusion graph* of $\mathcal{S}$; each subgraph of Q_m is the 1-inclusion graph of a family of subsets of U. A subgraph G of Q_m has VC-dimension d if G is the 1-inclusion graph of a set family of VC-dimension d. For a subgraph G of Q_m we denote by $C(G)$ the smallest cube of Q_m containing G.

An X-*cube* of Q_m is the 1-inclusion graph of the set family $\{Y \cup X' : X' \subseteq X\}$, where Y is a subset of $U \setminus X$. If $|X| = m'$, then any X-cube is an m'-dimensional subcube of Q_m and Q_m contains $2^{m-m'}$ X-cubes. We call any two X-cubes *parallel cubes*. A subset X of U is *strongly shattered* by $\mathcal{S}$ if the 1-inclusion graph $G(\mathcal{S})$ of $\mathcal{S}$ contains an X-cube. Denote by $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ the families consisting of all shattered and of all strongly shattered sets of $\mathcal{S}$, respectively. Clearly, $\underline{\mathcal{X}}(\mathcal{S}) \subseteq \overline{\mathcal{X}}(\mathcal{S})$ and both $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ are closed under taking subsets, i.e., $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ are *abstract simplicial complexes.* The VC-dimension VC-dim$(\mathcal{S})$ of $\mathcal{S}$ is thus the size of a largest set shattered by $\mathcal{S}$, i.e., the dimension of the simplicial complex $\overline{\mathcal{X}}(\mathcal{S})$.

Two important inequalities relate a set family $\mathcal{S} \subseteq 2^U$ with its VC-dimension. The first one, the *Sauer-Shelah lemma* [31, 32] establishes that if $|U| = m$, then the number of sets in a set family $\mathcal{S} \subseteq 2^U$ with VC-dimension d is upper bounded by $\Phi_d(m) := \sum_{i=0}^{d} \binom{m}{i}$. The second stronger inequality, called the *sandwich lemma* [2,8,17,29], proves that $|\mathcal{S}|$ is sandwiched between the number of strongly shattered sets and the number of shattered sets, i.e., $|\underline{\mathcal{X}}(\mathcal{S})| \leq |\mathcal{S}| \leq |\overline{\mathcal{X}}(\mathcal{S})|$. If $d =$ VC-dim$(\mathcal{S})$ and $m = |U|$, then $\overline{\mathcal{X}}(\mathcal{S})$ cannot contain more than $\Phi_d(m)$ simplices, thus the sandwich lemma yields the Sauer-Shelah lemma. The set families for which the Sauer-Shelah bounds are tight are called *maximum sets* [21, 19] and the set families for which the upper bounds in the sandwich lemma are tight are called *ample, lopsided, and extremal sets* [3, 8, 24]. Every maximum set system is ample, but not vice versa.

2.2. **Partial cubes.** All graphs $G = (V, E)$ in this paper are finite, connected, and simple. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths: $I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. An induced subgraph H of G (or its vertex set S) is called *convex* if it includes the interval of G between any two vertices of H. A subset $S \subseteq V$ or the subgraph H of G induced by S is called *gated* (in G) [18] if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'. It is easy to see that if x has a gate in H, then it is unique and that gated sets are convex.

A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$, i.e., $\varphi(G)$ is an isometric subgraph of H. A graph G is called a *partial cube* if it admits an isometric embedding into some hypercube Q_m. For an edge uv of G, let $W(u,v) = \{x \in V : d(x,u) < d(x,v)\}$. For an edge uv, the sets $W(u,v)$ and $W(v,u)$ are called *complementary halfspaces* of G.

Theorem 1. [16] *A graph G is a partial cube if and only if G is bipartite and for any edge $uv$the sets $W(u,v)$ and $W(v,u)$ are convex.*

Djoković [16] introduced the following binary relation Θ on the edges of G: for two edges $f = uv$ and $f' = u'v'$, we set $f\Theta f'$ if and only if $u' \in W(u,v)$ and $v' \in W(v,u)$. Under the conditions of the theorem, $f\Theta f'$ if and only if $W(u,v) = W(u',v')$ and $W(v,u) = W(v',u')$, i.e. Θ is an equivalence relation. Let $E_1,\ldots,E_m$ be the equivalence classes of Θ and let b be an arbitrary vertex taken as the basepoint of G. For a Θ-class E_i, let $\{G_i^-, G_i^+\}$ be the pair of complementary halfspaces of G defined by setting $G_i^- := G(W(u,v))$ and $G_i^+ := G(W(v,u))$ for an arbitrary edge $uv \in E_i$ such that $b \in G_i^-$. The isometric embedding φ of G into the m-dimensional hypercube Q_m is obtained by setting $\varphi(v) := \{i : v \in G_i^+\}$ for any vertex $v \in V$.

Running example. The complementary halfspaces M_i^- and M_i^+ of the running example M defined by the Θ-class E_i are illustrated in Fig. 2(a).

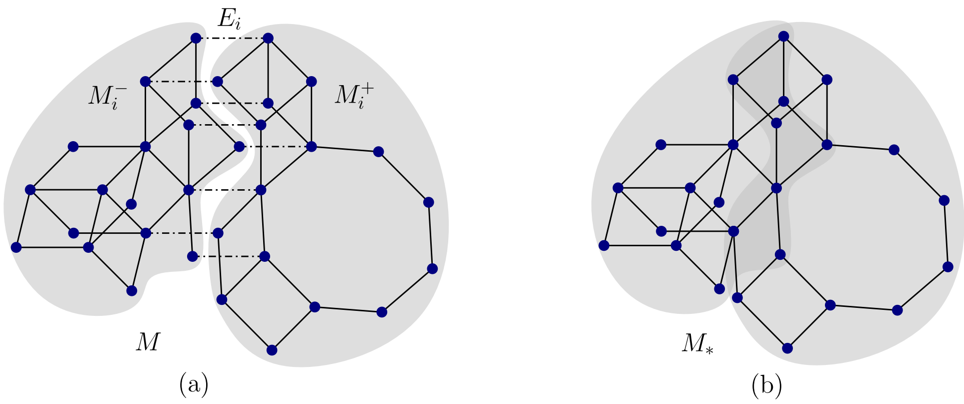


FIGURE 2. (a) Two complementary halfspaces of M. (b) The contraction M_* of a Θ-class of M.

2.3. **OMs, COMs, and AMPs.** We recall the basic notions and results from the theory of oriented matroids (OMs), complexes of oriented matroids (COMs), and ample set systems (AMPs). We follow [6] for OMs, [5] for COMs, and [3, 24] for AMPs.

2.3.1. *OMs: oriented matroids.* Oriented matroids (OMs) are abstractions of systems of sign vectors of all cells in the partition of $\mathbb{R}^m$ by a central arrangement of hyperplanes. Let U be a set with m elements and let $\mathcal{L}$ be a *system of sign vectors*, i.e., maps from U to $\{-1,0,+1\}$. The elements of $\mathcal{L}$ are referred to as *covectors* and denoted by capital letters X,Y,Z, etc. For $X \in \mathcal{L}$, the subset $\underline{X} = \{e \in U : X_e \neq 0\}$ is called the *support* of X and its complement $X^0 = E \setminus \underline{X} = \{e \in E : X_e = 0\}$ the *zero set* of X. For a sign vector X and a subset $A \subseteq U$, let X_A be the restriction of X to A. Let $\leq$ be the product ordering on $\{-1,0,+1\}^U$ relative to the standard ordering of signs with $0 \leq -1$ and $0 \leq +1$. For $X,Y \in \mathcal{L}$, we call $S(X,Y) = \{e \in U : X_eY_e = -1\}$ the *separator* of X and Y. The *composition* of X and Y is the sign vector $X \circ Y$, where for all $e \in U$, $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$.

Definition 1. An *oriented matroid* (OM) is a system of sign vectors $(U,\mathcal{L})$ satisfying

- **(C)** (Composition) $X \circ Y \in \mathcal{L}$ for all $X,Y \in \mathcal{L}$.
- **(SE)** (Strong elimination) for each pair $X,Y \in \mathcal{L}$ and for each $e \in S(X,Y)$, there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in U \setminus S(X,Y)$.

(Sym) (Symmetry) $-\mathcal{L} = \{-X : X \in \mathcal{L}\} = \mathcal{L}$, that is, $\mathcal{L}$ is closed under sign reversal.

A system of sign-vectors $(U, \mathcal{L})$ is *simple* if it has no "redundant" elements, i.e., for each $e \in U$, $\{X_e : X \in \mathcal{L}\} = \{-1, 0, +1\}$ and for each pair $e \neq f$ there exist $X, Y \in \mathcal{L}$ with $\{X_e X_f, Y_e Y_f\} = \{+, -\}$. We will only consider simple OMs, without explicitly stating it every time. The poset $(\mathcal{L}, \leq)$ of an OM $\mathcal{L}$ together with an artificial global maximum $\hat{1}$ forms a graded lattice, called the *big face lattice* $\mathcal{F}_{\text{big}}(\mathcal{L})$. The length of the maximal chains of $\mathcal{F}_{\text{big}}$ minus one is called the *rank* of $\mathcal{L}$ and denoted $\text{rank}(\mathcal{L})$. Note that $\text{rank}(\mathcal{L})$ equals the rank of the underlying unoriented matroid [6, Theorem 4.1.14]. The *topes* of $\mathcal{L}$ are the co-atoms of $\mathcal{F}_{\text{big}}(\mathcal{L})$.

From (C), (Sym), and (SE) it follows that the set $\mathcal{T}$ of topes of any simple OM $\mathcal{L}$ are $\{-1, +1\}$-vectors. Thus, $\mathcal{T}$ can be viewed as a family of subsets of U, where for each $T \in \mathcal{T}$ an element $e \in U$ belongs to the corresponding set if $T_e = +$ and does not belong to the set otherwise. The *tope graph* $G(\mathcal{L})$ of an OM $\mathcal{L}$ is the 1-inclusion graph of the set $\mathcal{T}$ of topes of $\mathcal{L}$. In *realizable OMs* (i.e., of OMs arising from central hyperplane arrangements of $\mathbb{R}^m$), $X \leq Y$ for two covectors X, Y if and only if the cell corresponding to X is contained in the cell corresponding to Y. Consequently, the topes of realizable OMs are the covectors of the inclusion maximal cells (which all have dimension m), called *regions*. Therefore, the tope graph of a realizable OM can be viewed as the adjacency graph of regions: the vertices of this graph are the regions of a hyperplane arrangement and two regions are adjacent in this graph if they are separated by a unique hyperplane of the arrangement. The *Topological Representation Theorem of Oriented Matroids* of [20], generalizes this correspondence to all OMs: tope graphs of OMs can be characterized as the adjacency graphs of regions (inclusion maximal cells) of pseudo-sphere arrangements in a sphere S^{m-1} [6], where m is the rank of the OM. More precisely, two topes are adjacent if and only if the corresponding regions are separated by a unique pseudo-sphere, see Fig. 3. It is also well-known (see for example [6]) that tope graphs of OMs are partial cubes and that $\mathcal{L}$ can be recovered from its tope graph $G(\mathcal{L})$ (up to isomorphism). Therefore, *we can define all terms in the language of tope graphs.* One instance of this is the correspondence between the rank of an OM and the VC-dimension of its tope graph, see Lemma 13.

Another important axiomatization of OMs is in terms of cocircuits. The *cocircuits* of $\mathcal{L}$ are the minimal non-zero elements of $\mathcal{F}_{\text{big}}(\mathcal{L})$, i.e., its atoms. The collection of cocircuits is denoted by $\mathcal{C}^*$ and can be axiomatized as follows: a system of sign vectors $(U, \mathcal{C}^*)$ is called an *oriented matroid* (OM) if $\mathcal{C}^*$ satisfies **(Sym)** and the following two axioms:

(Inc) (Incomparability) $\underline{X} \subseteq \underline{Y}$ implies $X = \pm Y$ for all $X, Y \in \mathcal{C}^*$.

(E) (Elimination) for each pair $X, Y \in \mathcal{C}^*$ with $X \neq -Y$ and for each $e \in S(X, Y)$, there exists $Z \in \mathcal{C}^*$ such that $Z_e = 0$ and $Z_f \in \{0, X_f, Y_f\}$ for all $f \in U$.

The set $\mathcal{L}$ of covectors can be derived from $\mathcal{C}^*$ by taking the closure of $\mathcal{C}^*$ under composition. The axiomatization of OMs via cocircuits is used to define uniform oriented matroids.

Definition 2. [6] A *uniform oriented matroid* (UOM) of rank r on a set U of size m is an OM $(U, \mathcal{C}^*)$ such that $\mathcal{C}^*$ consists of two opposite signings of each subset of U of size $m - r + 1$.

2.3.2. *COMs: complexes of oriented matroids.* Complexes of oriented matroids (COMs) are abstractions of sign vectors of all cells in the partition of an open convex set C of $\mathbb{R}^m$ by an arrangement of affine hyperplanes. COMs are defined in a similar way as OMs, simply replacing the global axiom (Sym) by a weaker local axiom (FS) of face symmetry:

Definition 3. A *complex of oriented matroids* (COMs) is a system of sign vectors $(U, \mathcal{L})$ satisfying (SE) and the following axiom:

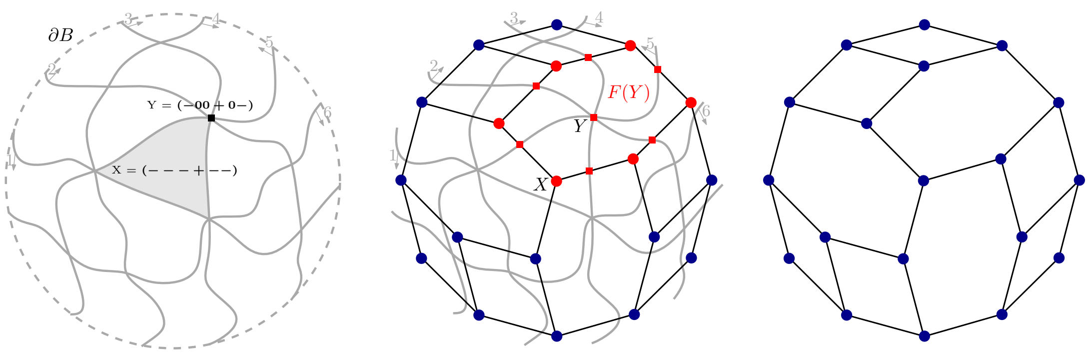
(FS) (Face symmetry) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

As for OMs, we restrict ourselves to *simple* COMs, i.e., COMs defining simple systems of sign vectors. One can see that (FS) implies (C), thus OMs are exactly the COMs containing the zero vector $\mathbf{0}$, see [5]. A COM $\mathcal{L}$ is *realizable* if $\mathcal{L}$ is the system of sign vectors of all cells in an arrangement U of (oriented) affine hyperplanes restricted to an open convex set of $\mathbb{R}^m$. For other examples of (tope graphs of) COMs, see [5, 13, 22].

The simple twist between (Sym) and (FS) leads to a rich combinatorial and geometric structure that is build from OMs but is quite different from OMs. Let $(U, \mathcal{L})$ be a COM and X be a covector of $\mathcal{L}$. The *face* of X is $\mathrm{F}(X) := \{X \circ Y : Y \in \mathcal{L}\}$; a *facet* is a maximal proper face. From the definition, any face $\mathrm{F}(X)$ consists of the sign vectors of all faces of the subcube of $[-1, +1]^U$ with barycenter X. By [5, Lemma 4], each face $\mathrm{F}(X)$ of $\mathcal{L}$ is an OM. Since OMs are COMs, each face of an OM is an OM and the facets correspond to cocircuits. Furthermore, by [5, Section 11] replacing each combinatorial face $\mathrm{F}(X)$ of $\mathcal{L}$ by a PL-ball, we obtain a contractible cell complex associated to each COM. The *topes* and the *tope graphs* of COMs are defined in the same way as for OMs. Again, the topes are $\{-1, +1\}$-vectors, the tope graph $G(\mathcal{L})$ is a partial cube, and the COM $\mathcal{L}$ can be recovered from its tope graph, see [5, 22].

Example 1. In Fig. 3(a) we consider an oriented pseudoline arrangement U. The set of sign vectors of the cells of this arrangement restricted to the disk B is denoted by $\mathcal{L}$. Notice that $\mathcal{L}$ is a COM. This is due to the fact that U comes from an arrangement U' of pseudo-spheres of the sphere S^2, which together with the boundary ∂B of B define an OM $\mathcal{L}'$. Then $\mathcal{L}$ is one of the halfspaces of $\mathcal{L}'$ defined by ∂B, thus $\mathcal{L}$ is a COM by [5, Proposition 6].

In Fig. 3(a) we present the sign vectors X and Y of two cells, one included in another one. Note that X is a tope of $\mathcal{L}$ and $Y \leq X$. In Fig. 3(b), we present the adjacency graph of the regions of the arrangement U and draw in red the covectors belonging to the face $F(Y)$ of Y. In Fig. 3(c) the tope graph of the COM $\mathcal{L}$ is given. It is the adjacency graph of regions.



(a) (b) (c)

Figure 3. (a) A pseudoline arrangement U. (b) The adjacency graph of regions of U and the face $F(Y)$. (c) The tope graph of COM $\mathcal{L}$.

We continue with the definitions of AMPs and CUOMs. For $\mathcal{L} \subseteq \{-1, 0, +1\}^U$, let $\uparrow\mathcal{L} := \{Y \in \{-1, 0, +1\}^U : X \leq Y \text{ for some } X \in \mathcal{L}\}$.

Definition 4. [5] An *ample set system* (AMP) is a COM $(U, \mathcal{L})$ satisfying the following axiom:

(IC) (Ideal composition) $\uparrow\mathcal{L} = \mathcal{L}$.

Definition 5. A *complex of uniform oriented matroids* (CUOM) is a COM $(U, \mathcal{L})$ in which each facet is a UOM.

2.3.3. *AMPs: ample set systems.* Ample set systems (AMPs) are abstractions of sign vectors of all cells in a partition of an open convex set C of $\mathbb{R}^m$ by an arrangement of coordinate hyperplanes. We already defined ample set systems (1) as COMs $\mathcal{L}$ such that $\uparrow\mathcal{L} = \mathcal{L}$ and (2) as families of sets $\mathcal{S}$ for which the upper bounds in the sandwich lemma are tight: $|\mathcal{S}| = |\overline{\mathcal{X}}(\mathcal{S})|$. The set system arising in the second definition is the set of topes of the system of sign vectors from the first definition. As in the case of OMs and COMs, *we can consider AMPs as set systems or as partial cubes.* By [3, 8], $\mathcal{S}$ is ample if and only if $|\underline{\mathcal{X}}(\mathcal{S})| = |\mathcal{S}|$ and if and only if $\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$. This can be rephrased in the following combinatorial way: $\mathcal{S} \subseteq 2^U$ is *ample* if and only if each set shattered by $\mathcal{S}$ is strongly shattered. Consequently, the VC-dimension of an ample set system is the dimension of the largest cube in its 1-inclusion graph. A nice characterization of ample set systems was provided in [24]: $\mathcal{S}$ is ample if and only if for any cube Q of Q_m if $\mathrm{Q} \cap \mathcal{S}$ is closed under taking antipodes, then either $\mathrm{Q} \cap \mathcal{S} = \varnothing$ or Q is included in $G(\mathcal{S})$. The paper [3] provides metric and recursive characterizations of ample set systems, such as the following one.

Recall that any two X-cubes Q', Q'' of $\mathcal{S}$ are called *parallel cubes.* The distance $d(Q', Q'')$ is the distance between mutually closest vertices of Q' and Q''. A *gallery of length* k between Q' and Q'' is a sequence of X-cubes $(Q' = R_0, R_1 \dots, R_{k-1}, R_k = Q'')$ of $\mathcal{S}$ such that $R_{i-1} \cup R_i$ is a cube for every $i = 1, \dots, k$. A *geodesic gallery* is a gallery of length $d(Q', Q'')$.

Proposition 1. [3] *$\mathcal{S}$ is ample if and only if any two parallel cubes of $\mathcal{S}$ can be connected in $\mathcal{S}$ by a geodesic gallery.*

Thus, the 1-inclusion graph $G(\mathcal{S})$ of an ample set $\mathcal{S}$ is a partial cube and we will speak about *ample partial cubes.* We conclude with the definition of ample completions.

Definition 6. An *ample completion* of a subgraph G of VC-dimension d of Q_m is an ample partial cube $\mathrm{amp}(G)$ containing G as a subgraph and such that VC-dim$(\mathrm{amp}(G)) = d$.

2.4. **Sample compression schemes.** The language of sign vectors is perfectly suited for defining sample compression schemes. This reformulation is due to [10], which we closely follow (for classical formulations, see [25, 27, 28]). A *concept class* is any family $\mathcal{C}$ of subsets of a finite set U; the elements of $\mathcal{C}$ are called *concepts.* As we noticed above, $\mathcal{C}$ can be viewed as a subset of $\{-1, +1\}^U$. For a concept $C \in \mathcal{C}$, any covector $X \in \{-1, 0, +1\}^U$ such that $X \leq C$ is called a *sample realizable by* C. Let $\downarrow \mathcal{C}$ denote the set of all samples realizable by concepts of $\mathcal{C}$, i.e., $\downarrow \mathcal{C} = \{X \in \{-1, 0, +1\}^U : \exists C \in \mathcal{C} \text{ such that } X \leq C\}$.

A *labeled sample compression scheme* for a concept class $\mathcal{C}$ is best viewed as a protocol between a *compressor* and a *reconstructor.* The compressor gets a realizable sample X from which it picks a small *subsample* $X' \leq X$. The compressor sends X' to the reconstructor. Based on X', the reconstructor outputs a set $C \in \{-1, +1\}^U$ (not necessarily belonging to $\mathcal{C}$) that needs to be *consistent* with the entire sample X, i.e., $X \leq C$. An *unlabeled sample compression scheme* is a sample compression scheme in which the compressed subsample X' is unlabeled. So, the compressor removes the labels before sending the subsample to the reconstructor. In both the labeled and unlabeled settings, the goal is to minimize the size of the support of the compressed subsample X' with X running over $\downarrow \mathcal{C}$.

Formally, a *labeled sample compression scheme* of size k for a concept class $\mathcal{C} \subseteq \{-1, +1\}^U$ is defined by a compressor function $\alpha : \{-1, 0, +1\}^U \to \{-1, 0, +1\}^U$ and a reconstructor function $\beta : \{-1, 0, +1\}^U \to \{-1, +1\}^U$ such that for any $X \in \downarrow \mathcal{C}$, it holds:

$$\alpha(X) \leq X \leq \beta(\alpha(X)) \text{ and } |\underline{\alpha}(X)| \leq k,$$

where $\leq$ is the order between sign vectors and $\underline{\alpha}(X)$ is the support of the sign vector $\alpha(X)$ as defined above. The condition $X \leq \beta(\alpha(X))$ means that the restriction of $\beta(\alpha(X))$ on the support of X coincides with the input sample X. In particular, if $X \in \mathcal{C}$, then $\beta(\alpha(X)) = X$. The *unlabeled sample compression schemes* are defined analogously, with the difference that in the unlabeled case $\alpha(X)$ is a subset of size at most k of the support of X.

From the definition it follows that if $\mathcal{C}$ is a completion of a concept class $\mathcal{C}'$ and (α, β) is a labeled, respectively unlabeled, sample compression scheme for $\mathcal{C}$, then (α, β) is a labeled, respectively unlabeled, sample compression scheme for $\mathcal{C}'$. (This conclusion is no longer true if the reconstructor map β takes values in $\mathcal{C}$ and not in $\{-1, +1\}^U$.)

The sample compression conjecture of [19] states that *any set family of VC-dimension d has a sample compression scheme of size $O(d)$.*

3. Auxiliary results

In this section we recall or prove some auxiliary results used in the proofs of the main results.

3.1. **Partial cubes and VC-dimension.**

In this subsection, we closely follow [13] and [14]. Let G be a partial cube, isometrically embedded in the hypercube Q_m.

3.1.1. *pc-Minors and VC-dimension.* For a Θ-class E_i of a partial cube G, an *elementary restriction* consists of taking one of the halfspaces G_i^- and G_i^+. More generally, a *restriction* is a convex subgraph of G induced by the intersection of a set of halfspaces of G. Since any convex subgraph of a partial cube G is the intersection of halfspaces [1, 11], the restrictions of G coincide with the convex subgraphs of G. For a Θ-class E_i, the graph $\pi_i(G)$ obtained from G by contracting the edges of E_i is called an *(i-)contraction* of G. For a vertex v of G, let $\pi_i(v)$ be the image of v under the i-contraction. We will apply π_i to subsets $S \subseteq V$, by setting $\pi_i(S) := \{\pi_i(v) : v \in S\}$. In particular, we denote the *i-contraction* of G by $\pi_i(G)$.

Running example. The i-contraction $M_* = \pi_i(M)$ of the running example M is given in Fig. 2(b); M_* is a CUOM and thus is a partial cube.

By [12, Theorem 3], $\pi_i(G)$ is an isometric subgraph of Q_{m-1}, thus the class of partial cubes is closed under contractions. Since edge contractions in graphs commute, if E_i, E_j are two distinct Θ-classes, then $\pi_j(\pi_i(G)) = \pi_i(\pi_j(G))$. Consequently, for a set A of Θ-classes, we can denote by $\pi_A(G)$ the isometric subgraph of $\mathrm{Q}_{m-|A|}$ obtained from G by contracting the equivalence classes of edges from A. Contractions and restrictions commute in partial cubes: any set of restrictions and contractions of a partial cube G provide the same result, independently on the order they are performed in. The resulting partial cube is called a *partial cube minor* (or *pc-minor*) of G. For a partial cube H, let $\mathcal{F}(H)$ denote the class of all partial cubes not having H as a pc-minor. For partial cubes, the VC-dimension can be formulated in terms of pc-minors:

Lemma 1. [14, Lemma 2] *A partial cube G has VC-dimension $\leq d$ if and only if $G \in \mathcal{F}(\mathrm{Q}_{d+1})$.*

Running example. The rhombododecahedron D from Fig. 4 is a tope graph of a UOM and is a facet of the running example M. In Fig. 4, D is isometrically embedded in the 4-cube Q_4. Since D is a proper subgraph of Q_4, VC-dim$(D) < 4$. On the other hand, contracting any Θ-class of D, we obtain the 3-cube Q_3. In Fig. 4, we represent the contraction of the Θ-class E_i constituted by oblique edges. The vertices that are merged by this contraction are darker. Thus Q_3 is a pc-minor of D, establishing that VC-dim$(D) = 3$. The 8-cycle C_8 is another face of M. Analogously to D one can show that VC-dim$(C_8) = 2$: the contraction of any two of its Θ-classes results in the square Q_2 and the contraction of any single Θ-class does not yield Q_3.

An *antipode* of a vertex v in a partial cube G is the (necessarily unique) vertex $-v$ such that $G = I(v, -v)$. A partial cube G is *antipodal* if all its vertices have antipodes. For a subgraph H of an antipodal partial cube G we denote by $-H$ the set of antipodes of H in G. We will use several times the following two results:

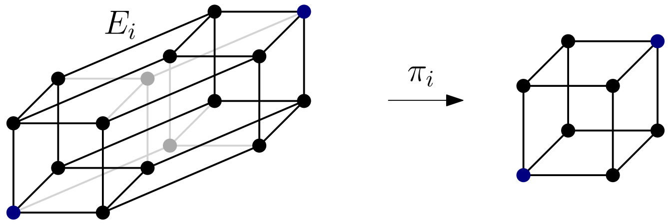


FIGURE 4. The rhombododecahedron D isometrically embedded in Q_4 and its resulting pc-minor $\pi_i(D)$ after contracting the Θ-class E_i represented by the oblique edges.

Lemma 2. [14, Lemma 16] *If G is a proper convex subgraph of an antipodal partial cube $H \in \mathcal{F}(\mathrm{Q}_{d+1})$, then $G \in \mathcal{F}(\mathrm{Q}_d)$.*

Lemma 3. [22] *Antipodal partial cubes are closed under contractions.*

The following lemma (a direct consequence of Theorem 1) is used in Proposition 5:

Lemma 4. *Intervals of partial cubes are convex.*

3.1.2. *Shattering via Cartesian products.* Recall that the *Cartesian product* of m graphs $G_1, \dots, G_m$ is the graph $G = G_1 \square \dots \square G_m$ whose vertex-set consists of all m-tuples $(v_1, \dots, v_m)$ with $v_i \in V(G_i)$ and two m-tuples $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ are adjacent in G if and only if there exists an index $1 \leq i \leq m$ such that u_i is adjacent to v_i in G_i and $u_j = v_j$ for all $j \neq i$. The m-cube $\mathrm{Q}_m = \mathrm{Q}(U)$ is the Cartesian product of m copies of K_2, i.e., $\mathrm{Q}_m = K_2 \square \cdots \square K_2$. For a subset $X \subseteq U$ of size d, denote by $\mathrm{Q}(X)$ the Cartesian product of the factors of $\mathrm{Q}(U)$ indexed by the elements of X; clearly, $\mathrm{Q}(X)$ is a d-cube Q_d. Analogously, let $\mathrm{Q}(U \setminus X)$ be the $(m-d)$-cube defined by $U \setminus X$. Then $\mathrm{Q}(U) = \mathrm{Q}(X) \square \mathrm{Q}(U \setminus X)$. For $Y \subseteq U$, let Q_Y be the subgraph of $\mathrm{Q}(U)$ induced by the sets $\{Y \cup Z : Z \subseteq U \setminus Y\}$; each Q_Y is isomorphic to the cube $\mathrm{Q}(U \setminus X) = \mathrm{Q}_{m-d}$. Let G be an isometric subgraph of $\mathrm{Q}(U)$. We also denote by G_Y the intersection of G with the cube $\mathrm{Q}(Y)$ and we call G_Y the *Y-fiber* of G. Since each Q_Y is a convex subgraph of $\mathrm{Q}(U)$ and G is an isometric subgraph of $\mathrm{Q}(U)$, each fiber F_Y is either empty or a non-empty convex subgraph of G. Then the definition of shattering can be rephrased in the following way:

Lemma 5. *A subset X of U is shattered by an isometric subgraph G of $\mathrm{Q}(U)$ if and only if each fiber G_Y, $Y \subseteq X$, is a nonempty convex subgraph of G.*

If $X \subseteq U$ is shattered by G, we call the map $\psi : V(G) \to 2^X = V(\mathrm{Q}(X))$ such that $\psi^{-1}(Y) = G_Y$ for all $Y \subseteq X$, a *shattering map.* The edges of G between two different fibers are called *X-edges.* Note that for any X-edge uv of G there exists $e \in X$ and $Y \subset X$ such that u corresponds to Y and v corresponds to $X \cup \{e\}$. Since the fibers define a partition of the vertex-set of G, any path connecting two vertices from different fibers of G contains X-edges.

The following simple lemmas are well-known and will be used in the proof of Theorem 3:

Lemma 6. *A (u, v)-path P of a partial cube G is a shortest path if and only if all edges of P belong to different Θ-classes of G.*

Proof. Suppose $|P \cap E_i| \geq 2$ and let xy and $y'x'$ be edges of $P \cap E_i$ that are consecutive with respect to P. Then x, x' belong to the same halfspace H_i^- or H_i^+ and y, y' belong to the complementary halfspace. Since $y, y' \in I(x, x')$, this contradicts Theorem 1. □

Lemma 7. *If G is a partial cube, H a gated subgraph of G, v a vertex of G, and v' the gate of v in H, then no shortest (v, v')-path of G contains edges of a Θ-class of H.*

Proof. Suppose a shortest (v, v')-path P contains an edge zz' of a Θ-class E_i of H. Let xy be an edge of H belonging to E_i. Since G is bipartite, let $d(v', x) < d(v', y)$. Since v' is the gate of v in H, the path R constituted by P, a shortest (v', x)-path of H, and the edge xy is a shortest (v, y)-path of G. Since R contains two edges of E_i, R cannot be a shortest path. $\square$

3.1.3. *Isometric expansions and VC-dimension.* A triplet (G^1, G^0, G^2) is called an *isometric cover* of a connected graph G, if the following conditions are satisfied:

- G^1 and G^2 are two isometric subgraphs of G;
- $V(G) = V(G^1) \cup V(G^2)$ and $E(G) = E(G^1) \cup E(G^2)$;
- $V(G^1) \cap V(G^2) \neq \varnothing$ and G^0 is the subgraph of G induced by $V(G^1) \cap V(G^2)$.

A graph G' is an *isometric expansion* of G with respect to an isometric cover (G^1, G^0, G^2) of G (notation $G' = \psi(G)$) if G' is obtained from G by replacing each vertex v of G^1 by a vertex v_1 and each vertex v of G^2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$ are adjacent in G' if and only if u and v are adjacent vertices of G^i and v_1v_2 is an edge of G' if and only if v is a vertex of G^0. If $G^1 = G^0$ (and thus $G^2 = G$), then the isometric expansion is called *peripheral* and we say that G' is obtained from G by a peripheral expansion with respect to G^0.

Running example. Recall that M_* is a pc-minor of the graph M and is obtained by contracting a single Θ-class of M. In Fig. 5 we present an isometric expansion of M_*, resulting in a partial cube M'_* different from M. The initial graph M can be retrieved from M_* by an isometric expansion with respect to the isometric cover of M_* given in Fig. 2(b).

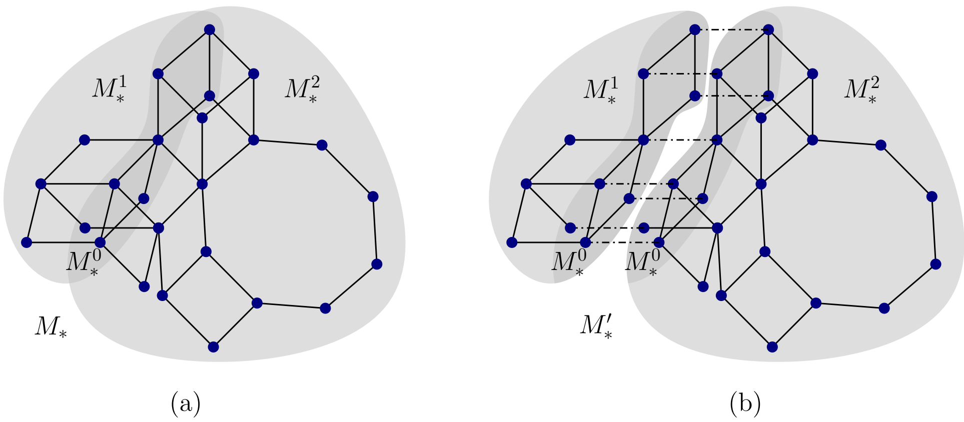


FIGURE 5. (a) The pc-minor M_* of M. (b) An isometric expansion M'_* of M_*.

If (G^1, G^0, G^2) is not peripheral, then G^0 is a separator of G. By [11, 12], G is a partial cube if and only if G can be obtained by a sequence of isometric expansions from a single vertex.

There is an intimate relation between contractions and isometric expansions. If G is a partial cube and E_i is a Θ-class of G, then contracting E_i we obtain the pc-minor $\pi_i(G)$ of G. Then G can be obtained from $\pi_i(G)$ by an isometric expansion with respect to $(\pi_i(G_i^+), G_0, \pi_i(G_i^-))$, where $\pi_i(G_i^+)$ and $\pi_i(G_i^-)$ are the images by the contraction of the halfspaces G_i^+ and G_i^- of G and G_0 is the contraction of the vertices of G incident to edges from E_i. The following result is used in the proof of Theorem 2:

Proposition 2. [14, Proposition 15] *Let G' be obtained from $G \in \mathcal{F}(\mathrm{Q}_{d+1})$ by an isometric expansion with respect to (G^1, G^0, G^2). Then $G' \in \mathcal{F}(\mathrm{Q}_{d+1})$ if and only if* $\text{VC-dim}(G^0) \leq d - 1$.

3.2. **OMs, COMs, and AMPs.** Here we recall some results about OMs, COMs, and AMPs.

3.2.1. *Faces.* First, since OMs satisfy the axiom (Sym), we obtain:

Lemma 8. *The tope graph of any OM is an antipodal partial cube.*

Let $(U, \mathcal{L})$ be a COM. For a covector $X \in \mathcal{L}$, recall that $\mathrm{F}(X)$ denotes the *face* of X. Let also $\mathrm{C}(X) := \mathrm{C}(F(X))$ denote the smallest cube of $\mathrm{Q}(U) := \{-1, +1\}^U$ containing $\mathrm{F}(X)$. Note that $\mathrm{F}(X)$ and $\mathrm{C}(X)$ are defined by the same set of Θ-classes. We continue with a fundamental property of faces of COMs:

Lemma 9. [5] *For each covector X of a COM $\mathcal{L}$, the face* $\mathrm{F}(X)$ *is an OM.*

The following lemma is implicit in [5], explicit in [22], and used in the proof of Theorem 3:

Lemma 10. *For each covector X of a COM $\mathcal{L}$, the face* $\mathrm{F}(X)$ *defines a gated subgraph of the tope graph G of $\mathcal{L}$. Moreover, for any tope Y of $\mathcal{L}$, $X \circ Y$ is the gate of Y in the cube* $\mathrm{C}(X)$.

Proof. Pick any $Y \in \{-1, +1\}^U \cap \mathcal{L}$. By the definition of $X \circ Y$, $X \circ Y \in \{-1, +1\}^U$, thus $X \circ Y$ is a tope of $\mathcal{L}$. By definition of $\mathrm{F}(X)$, $X \circ Y$ belongs to $\mathrm{F}(X)$ (and thus to $\mathrm{C}(X)$). Since $(X \circ Y)_e = Y_e$ for all $e \in U \setminus \underline{X}$, necessarily $X \circ Y$ is the gate of Y in $\mathrm{C}(X)$. □

3.2.2. *Minors and pc-minors.* In the present subsection we give two lemmas that are essential for inductive proofs. We start with the following result about pc-minors of tope graphs of COMs and AMPs, which follows from the results of [3] and [5]:

Lemma 11. *The classes of tope graphs of COMs and AMPs are closed under taking pc-minors. The class of tope graphs of OMs is closed under contractions.*

We continue with the notions of restriction, contraction, and minors for COMs (which can be compared with the similar notions for partial cubes). Let $(U, \mathcal{L})$ be a COM and $A \subseteq U$. Given a sign vector $X \in \{-1, 0, +1\}^U$ by $X \setminus A$ we refer to the *restriction* of X to $U \setminus A$, that is $X \setminus A \in \{-1, 0, +1\}^{U \setminus A}$ with $(X \setminus A)_e = X_e$ for all $e \in U \setminus A$. The *deletion* of A is defined as $(U \setminus A, \mathcal{L} \setminus A)$, where $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$. The *contraction* of A is defined as $(U \setminus A, \mathcal{L}/A)$, where $\mathcal{L}/A := \{X \setminus A : X \in \mathcal{L} \text{ and } \underline{X} \cap A = \varnothing\}$. If $\mathcal{L}'$ arises by deletions and contractions from $\mathcal{L}$, $\mathcal{L}'$ is said to be *minor* of $\mathcal{L}$. Deletion in a COM translates to pc-contraction in its tope graph, while contraction corresponds to what is called the zone graph, see [22].

Lemma 12. [5, Lemma 1] *The classes of COMs and AMPs are closed under taking minors.*

3.2.3. *Hyperplanes, carriers, and half-carriers.* For a COM $(U, \mathcal{L})$, a *hyperplane* of $\mathcal{L}$ is the set $\mathcal{L}_e^0 := \{X \in \mathcal{L} : X_e = 0\}$ for some $e \in U$. The *carrier* $N(\mathcal{L}_e^0)$ of the hyperplane $\mathcal{L}_e^0$ is the union of all faces $\mathrm{F}(X')$ of $\mathcal{L}$ with $X' \in \mathcal{L}_e^0$. The *positive and negative (open) halfspaces* supported by the hyperplane $\mathcal{L}_e^0$ are $\mathcal{L}_e^+ := \{X \in \mathcal{L} : X_e = +1\}$ and $\mathcal{L}_e^- := \{X \in \mathcal{L} : X_e = -1\}$. The carrier $N(\mathcal{L}_e^0)$ minus $\mathcal{L}_e^0$ splits into its positive and negative parts: $N^+(\mathcal{L}_e^0) := \mathcal{L}_e^+ \cap N(\mathcal{L}_e^0)$ and $N^-(\mathcal{L}_e^0) := \mathcal{L}_e^- \cap N(\mathcal{L}_e^0)$, which we call *half-carriers.*

Proposition 3. [5, Proposition 6] *In COMs and AMPs, all halfspaces, hyperplanes, carriers, and half-carriers are COMs and AMPs. In OMs, all hyperplanes and carriers are OMs.*

The result about AMPs was not stated in [5, Proposition 6], however it easily follows from the definition of AMPs as COMs satisfying the axiom (IC). The result also follows from [3]. Proposition 3 is used in the proof of the characterization of CUOMs from Proposition 6.

3.2.4. *Amalgams.* One important property of COMs is that they can be obtained by amalgams from their maximal faces, i.e., they are amalgams of OMs. Now we make this definition precise. Following [5], we say that a system $(U, \mathcal{L})$ of sign vectors is a *COM-amalgam* of two COMs $(U, \mathcal{L}')$ and $(U, \mathcal{L}'')$ if the following conditions are satisfied:

(1) $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ with $\mathcal{L}' \setminus \mathcal{L}'', \mathcal{L}'' \setminus \mathcal{L}', \mathcal{L}' \cap \mathcal{L}'' \neq \varnothing$;
(2) $(U, \mathcal{L}' \cap \mathcal{L}'')$ is a COM;
(3) $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ and $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$;
(4) for $X \in \mathcal{L}' \setminus \mathcal{L}''$ and $Y \in \mathcal{L}'' \setminus \mathcal{L}'$ with $X^0 = Y^0$ there exists a shortest path in the tope graph $G(\mathcal{L} \setminus X^0)$ of the deletion $\mathcal{L} \setminus X^0$ of X^0.

We continue with two propositions: Proposition 4 is used in the proof of Theorem 3 and Proposition 5 is used in the proofs of both main results.

Proposition 4. [5, Proposition 7 & Corollary 2] *The COM-amalgam of two COMs $\mathcal{L}', \mathcal{L}''$ is a COM $\mathcal{L}$ in which every facet is a facet of $\mathcal{L}'$ or $\mathcal{L}''$. Any COM which is not an OM is obtained via successive COM-amalgams from its facets.*

We will not make use of the following and state it here without proof just for completeness:

Corollary 1. *The COM-amalgam of two AMPs $\mathcal{L}', \mathcal{L}''$ such that $\mathcal{L}' \cap \mathcal{L}''$ is ample is an AMP. Any AMP which is not a cube is obtained via successive AMP-amalgams from its maximal faces.*

Now, we present a notion of AMP-amalgam formulated in terms of graphs. We say that a graph G is an *AMP-amalgam* of G_1 and G_2 if $(G_1, G_1 \cap G_2, G_2)$ is an isometric cover of G and G_1, G_2, and $G_0 = G_1 \cap G_2 \neq G_1, G_2$ are ample partial cubes. The main difference between this and COM-amalgams is that condition (4) in the definition of a COM-amalgam is replaced by the weaker condition that G is a partial cube. The next result from [4] was never published:

Proposition 5. [4] *Let G be a subgraph of the hypercube Q_m which is an AMP-amalgam of two ample isometric subgraphs G_1 and G_2 of Q_m. If G is an isometric subgraph of Q_m, then G is ample. Any ample partial cube can be obtained by AMP-amalgams from its facets.*

Proof. First we assert that any X-cube Q of G is contained either in G_1 or in G_2. We proceed by induction on $k := |X|$. Since $G_0 = G_1 \cap G_2$ is a separator, the assertion holds when $k = 1$. Suppose the assertion is true for all $X' \subset U$ with $|X'| < k$ and suppose that the X-cube Q of G contains two vertices $s \in V(G_1) \setminus V(G_2)$ and $t \in V(G_2) \setminus V(G_1)$. By induction hypothesis, any facet of Q containing s must be included in G_1 and any facet of Q containing t must be included in G_2. From this we conclude that all vertices of Q except s and t (which must be opposite in Q) belong to G_0. This is impossible since G_0 is ample.

By Proposition 1, we must show that any two X-cubes Q_1, Q_2 of G can be connected by a geodesic gallery. Since G_1 and G_2 are ample, this is true when Q_1 and Q_2 both belong to G_1 or to G_2. By previous assertion we can suppose that $Q_1 \subseteq G_1$ and $Q_2 \subseteq G_2$. By induction on $k = |X|$ we prove that Q_1 and Q_2 can be connected by a geodesic gallery containing an X-cube of G_0. If $k = 0$, then Q_1, Q_2 are vertices of G separated by G_0 and we are done. So, let $k > 0$. Pick any $e \in X$, set $X' := X \setminus \{e\}$, and let G^+, G^- be the halfspaces of G defined by e. Let Q_1^+, Q_1^- and Q_2^+, Q_2^- be the intersections of Q_1 and Q_2 with the halfspaces. By induction hypothesis, Q_1^+, Q_2^+ can be connected by a geodesic gallery $P(Q_1^+, Q_2^+)$ containing an X'-cube R^+ in G_0 and Q_1^-, Q_2^- can be connected by a geodesic gallery $P(Q_1^-, Q_2^-)$ containing an X'-cube R^- in G_0. Hence $d(Q_1^+, Q_2^+) = d(Q_1^+, R^+) + d(R^+, Q_2^+)$ and $d(Q_1^-, Q_2^-) = d(Q_1^-, R^-) + d(R^-, Q_2^-)$. Since G^+ and G^- are convex subgraphs of G, $P(Q_1^+, Q_2^+) \subseteq G^+$ and $P(Q_1^-, Q_2^-) \subseteq G^-$. Since G_0 is ample, the X'-cubes R^+ and R^- can be connected in G_0 by a geodesic gallery. Since $R^+ \subseteq G^+$ and $R^- \subseteq G^-$, on this gallery we can find two consecutive X'-cubes $Q^+ \subseteq G^+$ and $Q^- \subseteq G^-$ so that $Q = Q^+ \cup Q^-$ is an X-cube of G_0.

Since Q_1 and Q are two X-cubes of AMP G_1, they can be connected in G_1 by a geodesic gallery $P(Q_1, Q)$. Analogously, Q and Q_2 can be connected in G_2 by a geodesic gallery $P(Q, Q_2)$. We assert that the concatenation of the two galleries is a geodesic gallery $P(Q_1, Q_2)$ between Q_1 and

Q_2, i.e., $d(Q_1,Q_2)=d(Q_1,Q)+d(Q,Q_2)$. Since $d(Q_1^+,Q_2^+)=d(Q_1^-,Q_2^-)=d(Q_1,Q_2)$, it suffices to show that $d(Q_1^+,Q_2^+)=d(Q_1^+,Q^+)+d(Q^+,Q_2^+)$ and $d(Q_1^-,Q_2^-)=d(Q_1^-,Q^-)+d(Q^-,Q_2^-)$.

In each $Q_1^+,Q_1^-,Q_2^+,Q_2^-,R^+,R^-$ pick a vertex, say $q_1^+\in Q_1^+,q_1^-\in Q_1^-,q_2^+\in Q_2^+,q_2^-\in Q_2^-,r^+\in R^+,r^-\in R^-$, such that each pair of vertices realizes the distance between the corresponding cubes. Then $d(q_1^+,q_1^-)=d(q_2^+,q_2^-)=1$ and $d(q_1^+,q_2^+)=d(q_1^+,r^+)+d(r^+,q_2^+)$ and $d(q_1^-,q_2^-)=d(q_1^-,r^-)+d(r^-,q_2^-)$. Let q^+ and q^- be two vertices of Q^+ and Q^-, respectively, belonging to a shortest (r^+,r^-)-path. Again, $d(q^+,q^-)=1$. Consequently, in G we have $r^+,r^-\in I(q_1^+,q_2^-)$ and $q^+,q^-\in I(r^+,r^-)$. Since G is a partial cube, the interval $I(q_1^+,q_2^-)$ is convex (Lemma 4), thus q^+ and q^- belong to a common shortest path between q_1^+ and q_2^-. Applying the same argument, we deduce that q^- and q^+ belong to a common shortest path between q_1^- and q_2^+. Hence $d(q_1^+,q_2^+)=d(q_1^+,q^+)+d(q^+,q_2^+)$ and $d(q_1^-,q_2^-)=d(q_1^-,q^-)+d(q^-,q_2^-)$, establishing that $d(Q_1^+,Q_2^+)=d(Q_1^+,Q^+)+d(Q^+,Q_2^+)$ and $d(Q_1^-,Q_2^-)=d(Q_1^-,Q^-)+d(Q^-,Q_2^-)$. Consequently, $d(Q_1,Q_2)=d(Q_1,Q)+d(Q,Q_2)$, i.e., $P(Q_1,Q_2)$ is a geodesic gallery. □

3.2.5. *VC-dimension.* The VC-dimension of OMs, COMs, and AMPs (all viewed as partial cubes) can be expressed in the following way. Subsequently, we will use this fundamental lemma without explicitly mentioning it.

Lemma 13. *If G is the tope graph of an OM $\mathcal{L}$, then* $\text{VC-dim}(G)=\text{rank}(\mathcal{L})$. *If G is the tope graph of a COM $\mathcal{L}$, then* $\text{VC-dim}(G)$ *is the largest VC-dimension among tope graphs of faces of $\mathcal{L}$. If G is the tope graph of an AMP $\mathcal{L}$, then* $\text{VC-dim}(G)$ *is the largest dimension among cubes of G.*

Proof. Let G be the tope graph of an OM $\mathcal{L}$. We have to prove that $\text{VC-dim}(G)=\text{rank}(\mathcal{L})$. If $G=\mathrm{Q}_m$, then $\mathcal{L}=\{-1,0,+1\}^m$ has rank m and the equality holds. Thus, let G be not a cube. First we show $\text{VC-dim}(G)\le\text{rank}(\mathcal{L})$. Since G is not a cube, it contains a Θ-class E_i whose contraction does not decrease the VC-dimension. If we set $G':=\pi_i(G)$ and $\mathcal{L}':=\mathcal{L}\backslash\{i\}$, then $\text{VC-dim}(G')=\text{VC-dim}(G)$ and G' is the tope graph of $\mathcal{L}'$ (Lemma 12). Since $\text{rank}(\mathcal{L}')\le\text{rank}(\mathcal{L})$, by induction hypothesis, $\text{VC-dim}(G)=\text{VC-dim}(G')\le\text{rank}(\mathcal{L}')\le\text{rank}(\mathcal{L})$.

To prove $\text{rank}(\mathcal{L})\le\text{VC-dim}(G)$, we contract any Θ-class E_i and set $G'=\pi_i(G)$ and $\mathcal{L}'=\mathcal{L}\backslash\{i\}$. By induction hypothesis, $\text{rank}(\mathcal{L}')\le\text{VC-dim}(G')\le\text{VC-dim}(G)$. Thus, if $\text{rank}(\mathcal{L}')=\text{rank}(\mathcal{L})$ or $\text{VC-dim}(G')=\text{VC-dim}(G)-1$, then we are obviously done. Thus suppose, that for any Θ-class E_i and $G'=\pi_i(G)$, we have $\text{rank}(\mathcal{L}')=\text{rank}(\mathcal{L})-1$ and $\text{VC-dim}(G')=\text{VC-dim}(G)$.

If a Θ-class E_i of G crosses the faces $\mathrm{F}(X)$ of all cocircuits $X\in\mathcal{C}^*$, then $\mathcal{L}$ is not simple. Therefore, for any cocircuit $X\in\mathcal{C}^*$ there is a Θ-class E_i not crossing $\mathrm{F}(X)$. However, since when we contract E_i the rank decreases by 1, we conclude that the resulting OM coincides with $F(X)$. Indeed, after contraction the rank of $\mathrm{F}(X)$ remains the same. Hence, if X would remain a cocircuit the global rank would not decrease. Consequently, G' is the tope graph of $\mathrm{F}(X)$. Thus, G and $G_i^+=F(X)$ are antipodal partial cubes and G_i^+ is gated (the latter because it is a face of G). Since G is antipodal, $G_i^-\cong G_i^+$ is antipodal as well. Since we are in a COM, G_i^- is also a gated subgraph of G by [22]. Since all Θ-classes of G_i^+,G_i^- coincide, the path from any vertex in G_i^+ to its gate in G_i^- consists of an edge from E_i, and vice versa. Thus, $G\cong G_i^+\Box K_2$. From the next claim we obtain that G must be a cube, contrary to our assumption.

Claim 1. If G is a partial cube and $G\cong G_i^+\Box K_2$ for any Θ-class E_i, then G is a hypercube.

Proof. First, note that any two Θ-classes E_i,E_j of G must *cross*, i.e., $G_i^+\cap G_j^+,G_i^-\cap G_j^+,G_i^+\cap G_j^-,G_i^-\cap G_j^-\neq\emptyset$. Indeed, since $G\cong G_i^+\Box K_2$, after contracting E_i we get a graph isomorphic to G_i^+ and G_i^-, which has the same Θ-classes as G except E_i. This implies that any other class E_j crosses both G_i^+ and G_i^-. We assert that G_i^+ satisfies the hypothesis of the claim. For each vertex in the halfspace $G_i^+\cap G_j^+$ of G_i^+ defined by E_j its unique neighbor with respect to the factorization $G \cong G_j^+ \square K_2$ is in $G_i^+ \cap G_j^-$ and vice versa. Therefore, $G_i^+ \cong (G_i^+ \cap G_j^+) \square K_2$. The same holds for G_i^-. By induction assumption, G_i^+, G_j^+ are hypercubes. Consequently, G is the Cartesian product of a hypercube with an edge, whence a hypercube itself. $\square$

That the VC-dimension of the tope graph of a COM is attained by a face is proved in [14, Lemma 42]. This also implies the result for AMPs. For AMPs, this also follows from the equality $\underline{\mathcal{X}}(G) = \overline{\mathcal{X}}(G)$. The equality for OMs is stated in [22] with a reference to [15]. $\square$

Running example. The running example M is the COM-amalgam of 12 maximal faces: ten C_4, one C_8, and one rhombododecahedron (see Fig. 6(b)). By Lemma 13, VC-dim$(M) = \max\{\text{VC-dim}(C_4), \text{VC-dim}(C_8), \text{VC-dim}\, D\} = \max\{2, 3\} = 3$.

4. Ample completions of OMs

The goal of this section is to prove the following result :

Theorem 2. *Let $\mathcal{L}$ be an oriented matroid of rank d and G its tope graph, which henceforth is of VC-dimension d. Then G can be completed to an ample partial cube* amp(G) *of VC-dimension d.*

This completion is done in two steps. First, we use the known result that any OM can be completed to a UOM of the same rank. Consequently, the tope graph of any OM can be completed to a tope graph of a UOM of the same VC-dimension. Second, we recursively complete the tope graph of any UOM to an ample partial cube of the same VC-dimension.

Example 2. The prism $\Pi = C_6 \square P_2$ is the tope graph of an OM and is a proper isometric subgraph of Q_4. Contracting any Θ-class of Π, except the vertical one, results into Q_3, thus VC-dim$(\Pi) = 3$. The rhombododecahedron D is obtained as a UOM-completion of Π. In Fig. 6(c) we present an ample completion of D (and thus of Π) obtained as in the proof of Lemma 19: first, the Θ-class of vertical edges of D are contracted to obtain the 3-cube Q_3. At the second stage, an ample completion of D is obtained by performing an ample expansion of Q_3 along a Q_3^- (Q_3 minus a vertex).

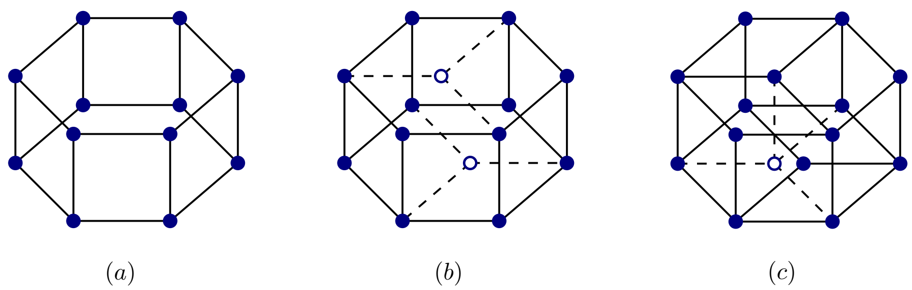


FIGURE 6. (a) The prism Π. (b) D as a UOM-completion of Π. (c) The ample completion of D and Π.

4.1. **UOMs: uniform oriented matroids.** In [6, Proposition 2.2.4] it is stated that the combinatorial types of cubical zonotopes, i.e., zonotopes in which all proper faces are cubes [33], are in one-to-one correspondence with realizable uniform matroids (up to reorientation). A way of generalizing this to general UOMs is basically due to Lawrence [24], also see [6, Exercise 3.28]. It has been restated in terms of tope graphs in [23]: the tope graphs of UOMs correspond to antipodal partial cubes in which all proper antipodal subgraphs are cubes. Let us give a proof.

Lemma 14. *For the tope graph G of an OM $\mathcal{L}$, the following conditions are equivalent:*

(i) *G is the tope graph of a UOM;*

(ii) *all proper faces, (i.e., all proper antipodal subgraphs,) of* G *are hypercubes;*
(iii) *all halfspaces (equivalently, all half-carriers) of* G *are ample partial cubes.*

Proof. (i)⇒(ii) By definition of a UOM of rank r, the cocircuits $\mathcal{C}^*$ are exactly orientations of sets of support of size $m-r+1$. In an OM the $\mathcal{L}$ can be obtained from $\mathcal{C}^*$ by taking all possible compositions. Thus, $\mathcal{L}$ consists of all possible sign vectors $Y \in \{+,-,0\}^U$ with $X \leq Y$ for some $X \in \mathcal{C}^*$. In other words, in a UOM we have $\mathcal{L} = \uparrow\mathcal{C}^* = \uparrow(\mathcal{L} \setminus \{\mathbf{0}\})$. In particular, for every face $\mathrm{F}(Y), Y \neq \{\mathbf{0}\}$ of $\mathcal{L}$, we have that $\uparrow\mathrm{F}(Y)$ is in $\mathcal{L}$, thus $\mathrm{F}(Y)$ is a hypercube.

(ii)⇒(iii): Since any OM is a COM, by [5], any halfspace and any half-carrier of G is the tope graph of a COM. Since all faces of this halfspace (respectively, half-carrier) are cubes, this COM satisfies the ideal composition axiom (IC) and thus is ample.

(iii)⇒(ii): Suppose that some proper face $\mathrm{F}(X)$ of G is not a cube. Then there exists a Θ-class E_i such that $\mathrm{F}(X)$ is contained in one of the halfspaces G_i^- or G_i^+, say $\mathrm{F}(X) \subseteq G_i^+$. Then $\mathrm{F}(X)$ is a face of G_i^+, thus G_i^+ does not satisfies (IC), thus is not ample.

(ii)&(iii)⇒(i): Let G be the tope graph of an OM $\mathcal{L}$ of rank r such that every proper face is a hypercube and all halfspaces are AMPs. Since all maximal antipodal subgraphs of the halfspace of the tope graph of an OM have the same VC-dimension and the VC-dimension of a hypercube is its dimension, all cocircuits X of $\mathcal{L}$ have the same support size. Since G is antipodal by Lemma 2, VC-dim(G) is one more than the VC-dimension of $F(X)$. Thus, all X have support of size $m-r+1$. This implies, that every set of size $m-r+1$ is the support of a cocircuit, since otherwise it has to be in containment relation with some cocircuit X, which then contradicts the support size property. □

Corollary 2. *If* G *is the tope graph of a UOM* $\mathcal{L}$*,* VC-dim$(G) = d$*, and* G' *is a proper convex subgraph of* G*, then* G' *is ample and* VC-dim$(G') \leq d-1$.

Proof. By Lemma 14, any halfspace H of G is ample. By Lemmas 2 and 8, VC-dim$(H) \leq d-1$. We are done, since any proper convex subgraph G' of G is an intersection of halfspaces. □

The following lemma is a well-known result in OM theory. We present a proof illustrating our tools.

Lemma 15. *The class of tope graphs of UOMs is closed under contractions.*

Proof. Let G be the tope graph of a UOM and E_i be a Θ-class of G. To show that $G' = \pi_i(G)$ is the tope graph of a UOM, by Lemma 14 we have to prove that all halfspaces of G' are ample partial cubes. Consider a Θ-class $E_j \neq E_i$ of G. Since $E_j \neq E_i$ there is a corresponding Θ-class in G'. Since G is the tope graph of a UOM, by Lemma 14 the halfspaces of G are ample partial cubes, in particular G_j^+ is ample. Moreover, as ample partial cubes are closed under contractions, $\pi_i(G_j^+)$ is ample. Since halfspaces can be viewed as restrictions and knowing that contractions and restrictions commute in partial cubes (see, for example [13]), we get that $\pi_i(G_j^+) = (\pi_i(G))_j^+ = (G')_j^+$ is ample. Consequently, the halfspaces of G' are ample. □

Lemma 16. *Let* G' *be a partial cube obtained from the tope graph* G *of a UOM* $\mathcal{L}$ *by an isometric expansion with respect to* (G^1, G^0, G^2) *such that* $G^1 = -G^2$*,* G^0 *is an isometric subgraph of* G*, and* G^0 *is the tope graph of a UOM. Then* G' *is the tope graph of a UOM. If* VC-dim$(G) = d$ *and* VC-dim$(G^0) \leq d-1$*, then* VC-dim$(G') \leq d$.

Proof. Since $G^1 = -G^2$, the graph G' is antipodal, see e.g. [22, Lemma 2.14]. By Lemma 14, to prove that G' is the tope graph of a UOM, we show that all antipodal subgraphs of G' are cubes. Let A' be an antipodal subgraph of G' and let E_i be the unique Θ-class of G' which does not exist in G, i.e., $\pi_i(G') = G$. If A' does not use the Θ-class E_i, then $\pi_i(A') = A'$ is a subgraph of $\pi_i(G') = G$, thus A' is an antipodal subgraph of G. As G is a tope graph of a UOM, by Lemma

14, A' is a cube. Otherwise, suppose that A' uses the Θ-class E_i. By Lemma 3, $A = \pi_i(A')$ is an antipodal subgraph of G. Since G is a tope graph of a UOM, using Lemma 14, A is a cube Q_k in G. Moreover, A' can be viewed as an isometric expansion (A^1, A^0, A^2) of $A = \mathrm{Q}_k$ with $A^1 = -A^2$. Moreover, since G^0 is an isometric subgraph of G, A^0 is a convex subgraph of G^0 that is closed under antipodes. Thus, A^0 is an antipodal subgraph of G^0. Finally, since A' is a proper subgraph of G, A^0 is a proper subgraph of G^0. Thus, A^0 is a cube since G^0 is a tope graph of a UOM. Thus, by the properties of isometric expansions, $G^0 \cap \mathrm{Q}_k = \mathrm{Q}_k$ and $A' = \mathrm{Q}_{k+1}$ is a cube. The statement about the VC-dimension follows straightforward from Lemma 2. $\square$

4.2. **Completions of tope graphs of OMs to tope graphs of UOMs.** Now, we use standard OM theory to obtain:

Lemma 17. *The tope graph G of any OM $\mathcal{L}$ can be completed to the tope graph of a UOM of the same VC-dimension.*

Proof. By [6, Definition 7.7.6], [6, Proposition 7.7.5], and some easy translation from topes to tope graphs there is a *weak map* from the tope graph G_1 of an OM to a tope graph G_2 of an OM, if G_2 is a subgraph of G_1, both are isometric subgraphs of the same hypercube, and both have the same isometric dimension. This implies, that G_2 is an isometric subgraph of G_1. Now [6, Corollary 7.7.9] says that every tope graph G_2 of an OM is the weak map image of the tope graph G_1 of a UOM of the same rank, i.e., the same VC-dimension. $\square$

4.3. **Ample completions of tope graphs of UOMs.**

Lemma 18. *A peripheral expansion G' of an ample partial cube G with respect to an ample subgraph H is ample.*

Proof. Clearly, $H' := H \square K_2$ is ample. Then G' is an AMP-amalgam of G and H' along H. Since G' is a partial cube (as an isometric expansion of G), by Proposition 5 G' is ample. $\square$

Lemma 19. *If G is the tope graph of a UOM of rank d, then G can be completed in* $\mathrm{C}(G)$ *to an ample partial cube* $\mathrm{amp}(G)$ *of VC-dimension d.*

Proof. Let E_i be any Θ-class of G and let G_i^+ and G_i^- be the halfspaces defined by E_i. By Lemma 14, G_i^+ and G_i^- are ample partial cubes. Let $G' = \pi_i(G)$ be the partial cube obtained by contracting the edges of E_i. By Lemma 15, G' is a tope graph of a UOM. Since $\pi_i(G_i^+)$ and $\pi_i(G_i^-)$ are isomorphic to G_i^+ and G_i^-, respectively, those subgraphs of G' are ample partial cubes. By Corollary 2, G_i^+, G_i^- and $\pi_i(G_i^+), \pi_i(G_i^-)$ have VC-dimension at most $d-1$.

By induction hypothesis, G' admits an ample completion $\mathrm{amp}(G')$ included in $\mathrm{C}(G')$ (where $\mathrm{C}(G')$ is considered in the hypercube of one less dimension) and having VC-dimension d. Define $\mathrm{amp}(G)$ as the peripheral expansion of $\mathrm{amp}(G')$ with respect to the ample partial cube $\pi_i(G_i^+)$. By Lemma 18, $\mathrm{amp}(G)$ is indeed ample. Notice also that $\mathrm{amp}(G)$ is contained in $\mathrm{C}(G)$ (considered in the original hypercube). It remains to show that $\mathrm{amp}(G)$ has VC-dimension d. The partial cube $\mathrm{amp}(G)$ is obtained from $\mathrm{amp}(G')$ by an isometric peripheral expansion with respect to $\pi_i(G_i^+)$ of VC-dimension $\leq d-1$. By Proposition 2, $\mathrm{amp}(G)$ has VC-dimension d. $\square$

This concludes the proof of Theorem 2.

5. Ample completions of CUOMs

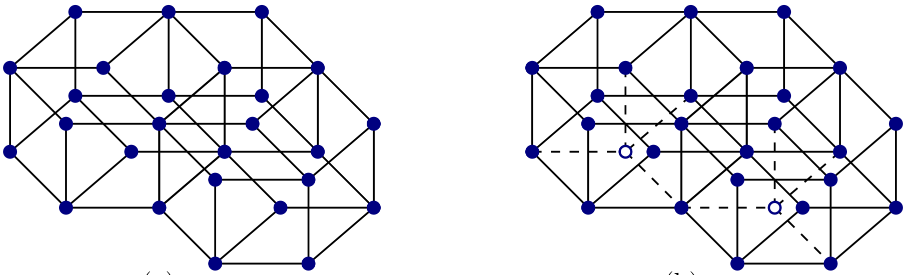
Recall that a COM $\mathcal{L}$ is called a *complex of uniform oriented matroids* (CUOM) if each facet of $\mathcal{L}$ is a UOM. The goal of this section is to prove the following result:

Theorem 3. *Let $\mathcal{L}$ be a complex of uniform oriented matroids and G its tope graph of VC-dimension d. Then G can be completed to an ample partial cube* $\mathrm{amp}(G)$ *of VC-dimension d.*

Remark 1. Note that in a COM of VC-dimension 2 the faces correspond to vertices, edges, and even cycles in its tope graph. Hence, 2-dimensional COMs are CUOMs and Theorem 3 generalizes the ample completion of 2-dimensional COMs presented in [14, Subsection 6.2].

The idea of the proof is to independently complete the facets of G to AMPs (using the recursive completion of tope graphs of UOMs) and show that their union is ample and has VC-dimension d.

Example 3. Fig. 7 presents the ample completion of the tope graph of a CUOM with two D-facets and one Q_3-facet. It is obtained by completing the two rhombododecahedra as UOMs (see Fig. 6 (b)& (c)).



(a) (b)

FIGURE 7. (a) The tope graph of a CUOM. (b) Its ample completion.

5.1. **A characterization of CUOMs.** We start with a characterization of CUOMs:

Proposition 6. *For the tope graph G of a COM $\mathcal{L}$ the following conditions are equivalent:*

(i) *G is the tope graph of a CUOM;*
(ii) *all non inclusion maximal faces of G are hypercubes;*
(iii) *all half-carriers of G are ample partial cubes.*

Proof. (i)$\Rightarrow$(ii): This trivially follows from the definitions of UOMs and CUOMs.

(ii)$\Rightarrow$(iii): From Proposition 3 it follows that the half-carriers $N_i^+(G)$ and $N_i^-(G)$ of the tope graph of a COM G are tope graphs of COMs. By the definition of half-carriers, each face $\mathrm{F}(Y)$ of a half-carrier, say of $N_i^+(G)$, is properly contained in a facet $\mathrm{F}(X)$ of the carrier $N_i(G)$. Then $\mathrm{F}(X)$ is a facet of G. Thus $\mathrm{F}(X)$ is a tope graph of a UOM and $\mathrm{F}(Y)$ is a cube. The tope graph of a COM in which all faces are cubes is ample because it satisfies (IC). This proves that all half-carriers of G are ample.

(iii)$\Rightarrow$(i): Suppose $\mathcal{L}$ is not a CUOM, i.e., its tope graph G contains a facet $\mathrm{F}(X)$ which is not the tope graph of a UOM. By Lemma 14(iii), $\mathrm{F}(X)$ contains a non-ample half-carrier, say $N_i^+(\mathrm{F}(X))$ defined by the Θ-class E_i' of $\mathrm{F}(X)$. This Θ-class E_i' can be extended to a Θ-class E_i of G and $N_i^+(\mathrm{F}(X))$ is included in the half-carrier $N_i^+(G)$ of G. Since $N_i^+(\mathrm{F}(X))$ is not ample, $N_i^+(G)$ is also not ample. $\square$

5.2. **Single gated extensions of partial cubes.** We mentioned already that all faces of a tope graph G of a COM are gated subgraphs of G and the completion method of G should first take care of completing its faces. In this subsection, we prove a general result about a partial completion of a partial cube G with respect to a gated subgraph. We suppose that G is isometrically embedded in the hypercube $\mathrm{Q}_m = \mathrm{Q}(U)$. Recall that $\mathrm{C}(G)$ is the smallest cube of Q_m containing G.

Proposition 7. *Let G be a partial cube and H be a gated subgraph of G. Let H' be an isometric subgraph of Q_m such that $H \subseteq H' \subseteq \mathrm{C}(H)$ and let G' be the subgraph of Q_m induced by $V(G) \cup V(H')$. Then the following holds:*

(i) *G' is an isometric subgraph of Q_m;*
(ii) *H' is a gated subgraph of G' and for each vertex v its gates in H and H' coincide;*
(iii) *$d := \text{VC-dim}(G') = \max\{\text{VC-dim}(G), \text{VC-dim}(H')\}$.*

In particular, if $\text{VC-dim}(H') \le \text{VC-dim}(G)$, then $\text{VC-dim}(G') = d$.

Proof. Since H is a gated and thus a convex subgraph of G, we have $\mathrm{C}(H) \cap V(G) = V(H)$. First we prove that G' is an isometric subgraph of Q_m. Since G and H' are isometric subgraphs of Q_m, it suffices to show that any vertex $v \in V(G) \setminus V(H)$ and any vertex $u \in V(H') \setminus V(H)$ can be connected in the graph G' by a shortest path of Q_m. Since H is a gated subgraph of G, let v' be the gate of v in H. Let P be any shortest (v, v')-path of G. Since v' is the gate of v in H, by Lemma 7, P does not use any Θ-class that appear in H. From the definition of $\mathrm{C}(H)$, the Θ-classes of H and $\mathrm{C}(H)$ coincide. Since H' is an isometric subgraph of $\mathrm{C}(H)$ and $v' \in V(H), u \in V(H')$, any shortest (v', u)-path S of G' can use only the Θ-classes of $\mathrm{C}(H)$, and thus of H. This implies that the concatenation of P and S is a (v, u)-path R of G' whose all Θ-classes are pairwise distinct. By Lemma 6, R is a shortest path of Q_m, establishing that G' is an isometric subgraph of Q_m. Moreover, the gate of v in H' is also v', because from v' (the gate of v in H) we can reach any vertex of H' using only Θ-classes belonging to H. We conclude that the gates of H' coincide with those of H. This proves the assertions (i) and (ii).

Before proving assertion (iii), we establish the following claim:

Claim 2. All shortest paths of G' from a vertex of $v \in V(G) \setminus V(H)$ to a vertex of $z \in V(H') \setminus V(H)$ traverse $V(H)$.

Proof. Suppose by way of contradiction that there exists a shortest (v, z)-path T of G' not intersecting $V(H)$. Since $v \in V(G) \setminus V(H)$, $z \in V(H') \setminus V(H)$, and $T \subset (V(G) \cup V(H')) \setminus V(H)$, the path T contains an edge xy with $x \in V(G) \setminus V(H)$ and $y \in V(H') \setminus V(H)$. We proved above that for any vertex of G its gates in H and in H' are the same. Since the vertices $x \in V(G) \setminus V(H)$ and $y \in V(H')$ are adjacent, y must be the gate of x in H'. Thus y is the gate of x in H, contrary to the assumption that $y \notin V(H)$. □

To prove (iii), suppose by way of contradiction that $d > \max\{\text{VC-dim}(G), \text{VC-dim}(H')\}$. This implies that G' shatters the d-cube $Q_d := \mathrm{Q}(X)$ for some $X \subseteq U$, $|X| = d$. By Lemma 5, each fiber $G'_{X'}, X' \subseteq X$ of G' is nonempty. Let $\psi : V(G') \to V(\mathrm{Q}_d)$ be the shattering map, mapping each X-fiber $G'_{X'}$ of G' to the subset X' of X. Since $d > \max\{\text{VC-dim}(G), \text{VC-dim}(H')\}$, neither G nor H' shatter Q_d, therefore the map ψ restricted to $V(G)$ and to $V(H')$ is no longer shattering. By Lemma 5, there exist two subsets Y, Z of X such that the fibers G_Y and H'_Z (of G and H', respectively) are empty. On the other hand, the fibers G'_Y and G'_Z are nonempty.

By Claim 2 all shortest paths from $V(G) \setminus V(H)$ to $V(H') \setminus V(H)$ pass through $V(H)$ and since $V(H) \subseteq V(H') \cap V(G)$, whence all vertices of the fiber G'_Y are included in $V(H') \setminus V(G)$. This implies that every X-edge of G' with one end in G'_Y must have the other end in H'. Since H' has the same Θ-classes as H, each such X-edge is defined by a Θ-class of H. Since in Q_d any vertex is incident to an edge from every Θ-class, G'_Y must be incident to all types of X-edges.

Now, applying again Claim 2, we conclude that the fiber G'_Z is included in $V(G') \setminus V(H')$. Pick any vertex $v \in V(G'_Z)$ and let v' be its gate in H (and in H'). Since $v \in V(G') \setminus V(H')$, necessarily $v' \neq v$. Let P be a shortest (v, v')-path of G'. Since v and v' belong to different fibers of G', necessarily P contains an X-edge xy. Since any X-edge is defined by a Θ-class of H (and H'), Lemma 7 yields a contradiction with the assertion (ii) that v' is the gate of v in H and H'. This contradiction shows that $d = \max\{\text{VC-dim}(G), \text{VC-dim}(H')\}$. □

Remark 2. If G' is obtained from a partial cube G via a single extension with respect to a gated subgraph H (as in Lemma 7), some gated subgraphs of G may no longer be gated in G'. Next we show that this phenomenon does not arise in tope graphs of CUOMs.

5.3. **Mutual projections between faces of COMs.** In the proof of Theorem 3 we use the following result of Dress and Scharlau [18] on mutual metric projections between gated sets. Recall that the *distance* $d(A, B)$ between two sets of vertices A, B of a graph G is $\min\{d(a, b) : a \in A, b \in B\}$. The *metric projection* $\mathrm{pr}_B(A)$ of B on A consists of all vertices a of A realizing the distance $d(A, B)$ between A and B, i.e., $\mathrm{pr}_B(A) = \{a \in A : d(a, B) = d(A, B)\}$.

Theorem 4. [18, Theorem] *Let A and B be two gated subgraphs of a graph G. Then $\mathrm{pr}_A(B)$ and $\mathrm{pr}_B(A)$ induce two isomorphic gated subgraphs of G such that for any vertex $a' \in \mathrm{pr}_B(A)$ if $b' = \mathrm{pr}_{a'}(B)$, then $d(a', b') = d(\mathrm{pr}_A(B), \mathrm{pr}_B(A)) = d(A, B)$, $\mathrm{pr}_{b'}(A) = a'$, and the map $a' \mapsto b'$ defines an isomorphism between $\mathrm{pr}_A(B)$ and $\mathrm{pr}_B(A)$.*

For $X, Y \in \mathcal{L}$, we denote by $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$ the metric projection of $\mathrm{F}(X)$ on $\mathrm{F}(Y)$ in the tope graph G of $\mathcal{L}$ and by $\mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ the metric projection of the cube $\mathrm{C}(X)$ on the cube $\mathrm{C}(Y)$ in the hypercube $\mathrm{Q}(U)$. Since by Lemma 10 the faces $\mathrm{F}(X)$ of $X \in \mathcal{L}$ are gated in G and all cubes $\mathrm{C}(X)$ are gated in $\mathrm{Q}(U)$, applying Theorem 4 to them we conclude that $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$ and $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ are isomorphic as well as $\mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ and $\mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$ and those isomorphisms map the pairs of vertices realizing the distances between $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$ and $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ and between $\mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ and $\mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$. We say that two faces $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ of $\mathcal{L}$ are *parallel* if $\mathrm{pr}_{\mathrm{F}(X)}\mathrm{F}(Y) = \mathrm{F}(Y)$ and $\mathrm{pr}_{\mathrm{F}(Y)}\mathrm{F}(X) = \mathrm{F}(X)$. A *gallery* between two parallel faces $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ of $\mathcal{L}$ is a sequence of faces $(\mathrm{F}(X) = \mathrm{F}(X_0), \mathrm{F}(X_1), \ldots, \mathrm{F}(X_{k-1}), \mathrm{F}(X_k) = \mathrm{F}(Y))$ such that any two faces of this sequence are parallel and any two consecutive faces $\mathrm{F}(X_{i-1}), \mathrm{F}(X_i)$ are facets of a common face of $\mathcal{L}$. A *geodesic gallery* between $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ is a gallery of length $d(\mathrm{F}(X), \mathrm{F}(Y)) = |S(X, Y)|$. Two parallel faces $\mathrm{F}(X), \mathrm{F}(Y)$ are called *adjacent* if $|S(X, Y)| = 1$, i.e., $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ are opposite facets of a face of $\mathcal{L}$. See Fig. 8 for an illustration.

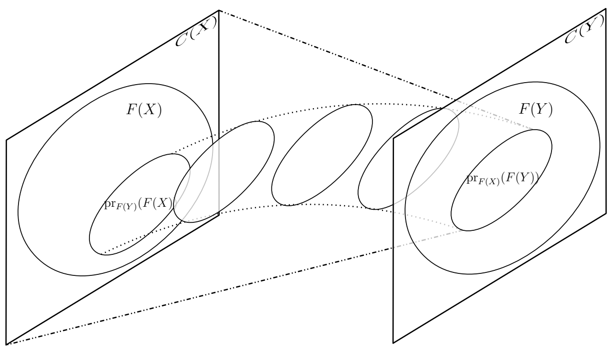


FIGURE 8. Two faces $F(X)$ and $F(Y)$, their mutual projections $\mathrm{pr}_{F(Y)}(F(X))$ and $\mathrm{pr}_{F(X)}(F(Y))$, and a geodesic gallery connecting them.

The most part of next result holds for all COMs. Therefore, we specify in the assertions where we require CUOMs. We use simultaneously the covector and the tope graph notations.

Proposition 8. *For any two covectors X, Y of a COM $\mathcal{L}$, the following properties hold:*

(i) $d(\mathrm{F}(X), \mathrm{F}(Y)) = d(\mathrm{C}(X), \mathrm{C}(Y)) = |S(X, Y)|$;
(ii) $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y)) \subseteq \mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ *and* $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) \subseteq \mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$;
(iii) $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) = \mathrm{F}(X \circ Y)$ *and* $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y)) = \mathrm{F}(Y \circ X)$;
(iv) $\mathrm{F}(X)$ *and* $\mathrm{F}(Y)$ *are parallel if and only if* $\underline{X} = \underline{Y}$ *(or, equivalently, if* $X^0 = Y^0$*)*;
(v) $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ *and* $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$ *are parallel faces of* $\mathcal{L}$;

(vi) *any two parallel faces* $\mathrm{F}(X)$ *and* $\mathrm{F}(Y)$ *can be connected in* $\mathcal{L}$ *by a geodesic gallery;*
(vii) *if* $\mathrm{F}(X)$ *is a facet of* $\mathcal{L}$*, then* $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ *is a proper face of* $\mathrm{F}(X)$*;*
(viii) *if* $\mathcal{L}$ *is a CUOM and* $\mathrm{F}(X), \mathrm{F}(Y)$ *are facets, then* $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$, $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$ *are cubes;*
(ix) *if* $\mathcal{L}$ *is a CUOM and* $\mathrm{F}(X), \mathrm{F}(Y)$ *are facets, then* $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y)) = \mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ *and* $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) = \mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$.

Proof. (i): From the definition of $\mathrm{C}(X)$ and $\mathrm{C}(Y)$ it follows that $\mathrm{F}(X)$ and $\mathrm{C}(X)$ have the same Θ-classes and $\mathrm{F}(Y)$ and $\mathrm{C}(Y)$ have the same Θ-classes. Therefore the set of Θ-classes separating the faces $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ is the same as the set of Θ-classes separating the cubes $\mathrm{C}(X)$ and $\mathrm{C}(Y)$ and coincides with $S(X,Y)$. Therefore, $d(\mathrm{F}(X),\mathrm{F}(Y)) = d(\mathrm{C}(X),\mathrm{C}(Y)) = |S(X,Y)|$.

(ii): $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y)) \subseteq \mathrm{pr}_{\mathrm{C}(X)}(\mathrm{C}(Y))$ and $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) \subseteq \mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$ follow from (i).

(iii): Note that for any two covectors X and Y, $\underline{X \circ Y} = \underline{Y \circ X}$ and $S(X,Y) = S(X \circ Y, Y \circ X)$ hold. Since $d(\mathrm{F}(X \circ Y), \mathrm{F}(Y \circ X)) = |S(X \circ Y, Y \circ X)|$ and $S(X \circ Y, Y \circ X) = S(X,Y)$ from property (i) we obtain that $d(\mathrm{F}(X),\mathrm{F}(Y)) = d(\mathrm{F}(X \circ Y), \mathrm{F}(Y \circ X))$, thus $\mathrm{F}(X \circ Y) \subseteq \mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ and $\mathrm{F}(Y \circ X) \subseteq \mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$. To prove the converse inclusions, suppose by way of contradiction that there exists a tope $Z \in \mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) \setminus \mathrm{F}(X \circ Y)$. Since $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ is gated, we can suppose that Z is adjacent to a tope Z' of $\mathrm{F}(X \circ Y)$. Let e be the element (a Θ-class) on which Z and Z' differ, say $Z_e = +1$ and $Z'_e = -1$. Since $X \leq Z$ and $X \leq Z'$, this implies that $X_e = 0$. If $Y_e = 0$, this would imply that $(X \circ Y)_e = 0$, thus Z would belong to $\mathrm{F}(X \circ Y)$, contrary to our choice of Z. Thus $Y_e = -1$. This implies that $d(Z, Y') \geq |S(X,Y)| + 1$ for any tope $Y' \in \mathrm{F}(Y)$. Indeed, $Y'_e = -1$ and $Y'_f = -Z_f$ for any $f \in S(X,Y)$. This contradiction shows that $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) = \mathrm{F}(X \circ Y)$ and $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y)) = \mathrm{F}(Y \circ X)$.

(iv): In view of (iii), we can rephrase the definition of parallel faces as follows: $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ are parallel if and only if $\mathrm{F}(X) = \mathrm{F}(X \circ Y)$ and $\mathrm{F}(Y) = \mathrm{F}(Y \circ X)$, i.e., if and only if $X = X \circ Y$ and $Y = Y \circ X$. Then one can easily see that $X = X \circ Y$ and $Y = Y \circ X$ hold if and only if $\underline{X} = \underline{Y}$ holds.

(v): This property follows from properties (iii) and (iv).

(vi): Let $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ be two parallel faces. By (iv), $\underline{X} = \underline{Y}$. We proceed by induction on $k := |S(X,Y)|$. Let $B := \underline{X} = \underline{Y}$. Set $A := U \setminus B$ and consider the COM $(B, \mathcal{L} \setminus A)$. Then $X' := X \setminus A$ and $Y' := Y \setminus A$ are topes of $\mathcal{L} \setminus A$. Note also that the distances between X' and Y' and between X and Y are equal to k. Since the tope graph $G(\mathcal{L} \setminus A)$ of the COM $\mathcal{L} \setminus A$ is an isometric subgraph of the cube $\{-1,+1\}^B$, X' and Y' can be connected in $G(\mathcal{L} \setminus A)$ by a shortest path of $\{-1,+1\}^B$, i.e., by a path of length k. Let Z' be the neighbor of X' in this path. Then there exists $e \in S(X,Y) = S(X',Y')$ such that $S(X',Z') = \{e\}$ and $S(Z',Y') = S(X,Y) \setminus \{e\}$. By the definition of $\mathcal{L} \setminus A$, there exists a covector $Z \in \mathcal{L}$ such that $(Z \setminus A)_f = Z'_f$ for each $f \in B$. Hence Z contains B in its support. Moreover, since $\underline{X} = \underline{Y} = B$, $S(X,Z) = \{e\}$ and $S(Z,Y) = S(X,Y) \setminus \{e\}$. In particular, $Z_f = X_f \neq 0$ for any $f \in B \setminus \{e\}$. Applying the axiom (SE) to X, Z and $e \in S(X,Z)$, we will find $X' \in \mathcal{L}$ such that $X'_e = 0$ and $X'_f = (X \circ Z)_f$ for all $f \in U \setminus S(X,Z)$. Since $\underline{X} = \underline{Y}$ and $S(X,Z) = \{e\}$, we conclude that $X'_f = X_f$ for any $f \in U \setminus \{e\}$. Consequently, $X' \leq X$, i.e., $\mathrm{F}(X)$ is a face of $\mathrm{F}(X')$. Since $S(X,Z) = \{e\}$, $\mathrm{F}(X)$ is a facet of $\mathrm{F}(X')$. By face symmetry (FS), $X'' := X' \circ (-X) \in \mathcal{L}$. Notice that $\mathrm{F}(X'')$ is a facet of $\mathrm{F}(X')$ symmetric to $\mathrm{F}(X)$, i.e., $\mathrm{F}(X)$ and $\mathrm{F}(X'')$ are adjacent parallel faces. Notice also that $\underline{X''} = \underline{X} = \underline{Y}$ and, since $X''_e = -X_e$, that $S(X'',Y) = S(X,Y) \setminus \{e\}$. By induction hypothesis, the parallel faces $\mathrm{F}(X'')$ and $\mathrm{F}(Y)$ can be connected in $\mathcal{L}$ by a geodesic gallery. Adding to this gallery the face $\mathrm{F}(X')$, we obtain a geodesic gallery connecting $\mathrm{F}(X)$ and $\mathrm{F}(Y)$.

(vii): This property follows from property (vi).

(viii): By (vii) and Proposition 6, $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ is a cube as a proper face of $\mathrm{F}(X)$.

(ix): By (viii), $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ is a cube and by (iii), $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) = \mathrm{F}(X \circ Y)$. By (ii) this cube $\mathrm{F}(X \circ Y)$ is included in the cube $\mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$. Suppose that this inclusion is proper.

Let e be an element (a Θ-class) of the support of $\mathrm{pr}_{\mathrm{C}(Y)}(\mathrm{C}(X))$ which does not belong to the support of $\mathrm{F}(X \circ Y)$. Suppose without loss of generality that $Z_e = +1$ for all $Z \in \mathrm{F}(X \circ Y)$, i.e., all topes Z of $\mathrm{F}(X \circ Y)$ belong to the halfspace G_e^+ of G. From the definition of the cubes $\mathrm{C}(X)$ and $\mathrm{C}(Y)$, we conclude that the halfspace G_e^- of G must contains a tope X' of $\mathrm{F}(X)$ and a tope Y' of $\mathrm{F}(Y)$. From the definition of the mutual gates, we must have a shortest path in G from X' to Y' traversing via a tope of $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X))$ and a tope of $\mathrm{pr}_{\mathrm{F}(X)}(\mathrm{F}(Y))$. But this is impossible because X', Y' belong to G_e^- while all the topes of $\mathrm{pr}_{\mathrm{F}(Y)}(\mathrm{F}(X)) = \mathrm{F}(X \circ Y)$ are included in G_e^+ and G_e^- and G_e^+ are convex because $G(\mathcal{L})$ is a partial cube. $\square$

5.4. **Proof of Theorem 3.** Let $\mathcal{L}$ be a CUOM and G be its tope graph. Let $\mathrm{F}_1 = \mathrm{F}(X_1), \ldots, \mathrm{F}_n = \mathrm{F}(X_n)$ be the facets of $\mathcal{L}$. Each F_i is a UOM and let $\mathrm{amp}(\mathrm{F}_i)$ be the ample completion of $G(\mathrm{F}_i)$ obtained by Lemma 19 ($\mathrm{amp}(\mathrm{F}_i)$ is contained in $\mathrm{C}(\mathrm{F}_i)$). Let $G_i^* = \mathrm{amp}(\mathrm{F}_1) \cup \cdots \cup \mathrm{amp}(\mathrm{F}_i) \cup \mathrm{F}_{i+1} \cup \cdots \cup \mathrm{F}_n$; in words, G_i^* is obtained from G by replacing the first i faces $\mathrm{F}_1, \ldots, \mathrm{F}_i$ by their ample completions $\mathrm{amp}(\mathrm{F}_1), \ldots, \mathrm{amp}(\mathrm{F}_i)$. Finally, set $G^* := G_n^*$. We assert that G^* is ample. For this we will use the amalgamation results for COMs and ample partial cubes and Proposition 7 about single gated set extensions of partial cubes. Proposition 7 will ensure that each partial completion G_i^* is a partial cube and its VC-dimension does not increase. As a result, the final graph G^* is a partial cube and has VC-dimension d.

To apply Proposition 7 to each G_i^*, we need that each not yet completed cell $\mathrm{F}_{i+1}, \ldots, \mathrm{F}_n$ of G remains gated in $G_1^*, \ldots, G_i^*$. By Proposition 8(ix), independently in which order the faces $\mathrm{F}_i = \mathrm{F}(X_i)$ and $\mathrm{F}_j = \mathrm{F}(X_j)$ are completed (F_i before F_j or F_j before F_i), the mutual projections of F_i and F_j initially coincide with those of the cubes $\mathrm{C}(X_i)$ and $\mathrm{C}(X_j)$, the gate of any vertex $Z \in \mathrm{amp}(\mathrm{F}_i)$ in F_j (or of any vertex $Z \in \mathrm{amp}(\mathrm{F}_j)$ in F_i) in each occurring partial completion will coincide with the gate of Z in the cube $\mathrm{C}(X_j)$ (respectively, with the gate of Z in the cube $\mathrm{C}(X_i)$). Hence, each partially completed graph G_i^* is a partial cube and all remaining faces $\mathrm{F}_{i+1}, \ldots, \mathrm{F}_n$ are gated in G_i^*. Thus we can apply Proposition 7 to the partial cube G_i^* and the remaining faces $\mathrm{F}_{i+1}, \ldots, F_n$.

Now we show that any edge uv of G^* is included in some completion $\mathrm{amp}(\mathrm{F}_i)$ of a facet F_i of G. Suppose $u \in \mathrm{amp}(\mathrm{F}_i)$ and $v \in \mathrm{amp}(\mathrm{F}_j)$. By construction, $\mathrm{amp}(\mathrm{F}_i) \subseteq \mathrm{C}(\mathrm{F}_i)$ and $\mathrm{amp}(\mathrm{F}_j) \subseteq \mathrm{C}(\mathrm{F}_j)$. Therefore, if u and v are adjacent, necessarily one of the vertices u, v, say v, belongs to $\mathrm{C}(\mathrm{F}_i) \cap \mathrm{C}(\mathrm{F}_j)$. Since G is a tope graph of a CUOM, $\mathrm{C}(\mathrm{F}_i) \cap \mathrm{C}(\mathrm{F}_j)$ is a proper (cube)face of F_i and of F_j. Consequently, $u \in \mathrm{amp}(F_i)$ and $v \in F_i$, and we are done.

To show that G^* is ample, we use induction on the number of faces of G and the amalgamation procedures for COMs and ample partial cubes, see Propositions 4 and 5. If G consists of a single maximal face, then we are done by Lemma 19. Otherwise, by Proposition 4 $\mathcal{L}$ is a COM-amalgam of two COMs $\mathcal{L}'$ and $\mathcal{L}''$ with tope graphs G' and G'' such that (1) every facet of G is a facet of G' or of G'' and (2) their intersection $G_0 = G' \cap G''$ is a the tope graph of the COM $\mathcal{L}' \cap \mathcal{L}''$. This implies that G', G'', and G_0 are tope graphs of CUOMs: each facet of each of them is either (a) a facet of G, and thus is the tope graph of a CUOM, or (b) is a proper face of G, and thus is a cube. We call the facets of type (a) *original facets* and the facets of type (b) *cube facets*.

Let $(G')^*$ be the union of all cube facets of G' and of the ample completions $\mathrm{amp}(\mathrm{F}_i)$ of all original facets F_i of the tope graph of a CUOM G'. Clearly, $(G')^*$ is obtained by the completion method described above and applied to the facets of G'. Analogously, we define the completions $(G'')^*$ and $(G_0)^*$ of G'' and G_0, respectively. Since G', G'', and G_0 are tope graphs of CUOMs with less vertices than G, by induction hypothesis, $(G')^*, (G'')^*$ and $(G_0)^*$ are ample completions of G', G'', and G_0, respectively. Moreover, since each facet of G is a facet of at least one of G' and G'', by the construction and by what has been proved above, the vertex-set and the edge-set of the partial cube G^* is the union of the vertex-sets and the edge-sets of ample partial cubes $(G')^*$ and $(G'')^*$. Consequently, $((G')^*, (G_0)^*, (G'')^*)$ is an isometric cover of G^*, i.e., G^* is an AMP-amalgam of $(G')^*$ and $(G'')^*$. By Proposition 5, G^* is ample. This concludes the proof of Theorem 3. □

Running example. In Fig. 9 we present an ample completion of the running example M (recall that M is the tope graph of a CUOM), obtained as in the proof of Theorem 3.

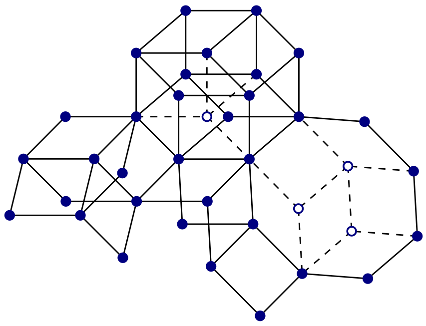


FIGURE 9. An ample completion of the running example M.

6. Discussion

We proved that the tope graph of every OM or CUOM of VC-dimension d has an ample completion of the same VC-dimension. For OMs, this result is proved in two stages: first, we complete the OM to a UOM (using the theory of oriented matroids) and then, recursively we complete the tope graph of the resulting UOM to an AMP. For CUOMs, the completion is obtained by completing each facet independently and by taking the union of those facet completions. Since ample set systems of VC-dimension d admit labeled compression schemes of size d [27] and this property is closed under taking subsystems, see Subsection 2.4, from Theorems 2 and 3 we obtain :

Corollary 3. *Concept classes defined by the topes of an OM or a CUOM of VC-dimension d admit labeled compression schemes of size d.*

For general COMs, one can envisage the same strategy: complete each facet of the tope graph and take their union. However, if the completion of faces is done as for OMs, then, as shown in the following example, a few difficulties arise.

Example 4. In Fig. 10(a) we present the tope graph G of a COM of VC-dimension 3, which is the Cartesian product $\mathrm{C}_8 \square P_3$. It consists of two facets (which are both prisms $\mathrm{C}_8 \square P_2$) glued together along a common face C_8. In Fig. 10(b) we complete each facet to the tope graph of a UOM. However, the resulting graph is not even a partial cube. This problem arises for any completion of the two facets to tope graphs of UOMs. Nevertheless, the graph has an ample completion of the same VC-dimension, see Fig. 10(c).

Let us discuss what we learn from the above example. The intersections of ample set systems with cubes is ample and all faces of the tope graph G of a COM are gated (and thus convex), thus if amp(G) is an ample completion of G and $\mathrm{F}(X)$ is a face of G, then the intersection of amp(G) with the smallest cube $\mathrm{C}(X)$ containing $\mathrm{F}(X)$ is ample and thus is an ample completion of $\mathrm{F}(X)$. This explains why we should take care of the completions of faces.

The completion of $C_8 \square P_3$ from Fig. 10(c) satisfies the following *parallel faces completion property*: any two parallel faces $\mathrm{F}(X)$ and $\mathrm{F}(Y)$ of G are completed in the same way, i.e., the isomorphism between $\mathrm{F}(X)$ to $\mathrm{F}(Y)$ (given by metric projection) extends to an isomorphism

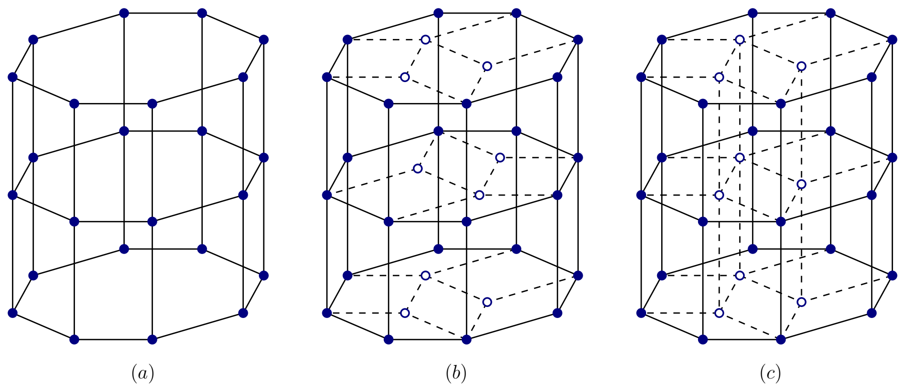


FIGURE 10. (a) The tope graph G of a COM corresponding to $C_8 \square P_3$. (b) The partial completion of G after completing the two facets. (c) The smallest ample completion of G.

between the completions $\mathrm{amp}(G) \cap \mathrm{C}(X)$ and $\mathrm{amp}(G) \cap \mathrm{C}(Y)$. Since in tope graphs of COMs parallel faces are not facets, in CUOMs they are cubes, and we conclude that the completion of CUOMs satisfies the parallel faces completion property. We believe, that *Conjecture 1 can be strengthened by furthermore imposing the parallel faces completion property.*

Example 5. In [14] we proved that any partial cube of VC-dimension 2 admits an ample completion of VC-dimension 2. The example from Fig. 11 shows that this is no longer true for partial cubes of VC-dimension 3. The graph is an isometric subgraph of Q_5, has VC-dimension 3, and all its ample completions have VC-dimension at least 4. There are six such subgraphs of Q_5 and the one in Fig. 11 is an isometric subgraph of all the others. On the other hand, all tope graphs of COMs in Q_5 satisfy Conjecture 1. The examples and their analysis have been obtained using SageMATH [34] and the database of partial cubes in Q_5 [26].

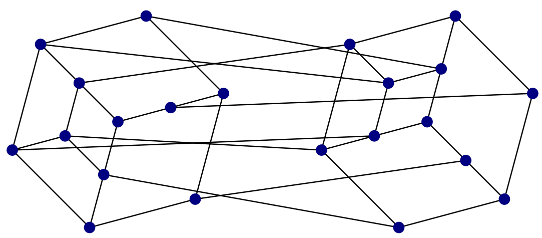


FIGURE 11. A partial cube of VC-dimension 3 that cannot be completed to an ample partial cube of the same VC-dimension.

The class of partial cubes, that can be completed to an ample partial cube of VC-dimension 3 is closed under pc-minors. What are the minimal excluded pc-minors of this class?

Acknowledgements. We are grateful to the anonymous referees for useful comments and improvements. This work was supported by ANR project DISTANCIA, ANR-17-CE40-0015. The second author was partially supported by the Spanish *Ministerio de Ciencia, Innovación y Universidades* through grants RYC-2017-22701 and PID2019-104844GB-I00.

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