SEMICLASSICAL LIMIT OF GROSS-PITAEVSKII EQUATION WITH DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper, we justify the semiclassical limit of Gross-Pitaevskii equation with Dirichlet boundary condition on the 3-D upper space under the assumption that the leading order terms to both initial amplitude and initial phase function are sufficiently small in some high enough Sobolev norms. We remark that the main difficulty of the proof lies in the fact that the boundary layer appears in the leading order terms of the amplitude functions and the gradient of the phase functions to the WKB expansions of the solutions. In particular, we partially solved the open question proposed in [6, 18] concerning the semiclassical limit of Gross-Pitaevskii equation with Dirichlet boundary condition.

Keywords: Semiclassical limit, Schrödinger equation, Boundary layer, Successive complementary expansion method

AMS Subject Classification (2000): 35Q40, 35Q55.

1. Introduction

We consider here the semiclassical limit of Gross-Pitaevskii equation (GP equation for short) with the Dirichlet boundary condition in the three-dimensional upper space $\mathbb{R}^3_+$:

$$\begin{cases} i\,\varepsilon\,\partial_t\Psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\Psi^\varepsilon - (|\Psi^\varepsilon|^2-1)\Psi^\varepsilon = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\ \Psi^\varepsilon|_{z=0}=1, \quad \Psi^\varepsilon|_{t=0} = a_0^\varepsilon\exp\Big(i\frac{\varphi_0^\varepsilon}{\varepsilon}\Big), \end{cases} \tag{1.1}$$

where $x=(y,z)\in\mathbb{R}^2_{\rm h}\times\mathbb{R}_+$, $a_0^\varepsilon\geq 0$ and φ_0^ε are real-valued functions. We assume that

$$\begin{aligned} a_0^\varepsilon =& a_{0,0}^{\rm in} + \sum_{j=0}^{m+2}\varepsilon^j a_{j,0}^{\rm in} + \varepsilon^{m+2}R^\varepsilon_{a,0} \quad \text{with} \quad \lim_{\varepsilon\to0}\|R^\varepsilon_{a,0}\|_{H^{s_0-2m-5}} = 0,\\ \varphi_0^\varepsilon =& \varphi_{0,0}^{\rm in} + \sum_{j=0}^{m+2}\varepsilon^j \varphi_{j,0}^{\rm in} + \varepsilon^{m+2}R^\varepsilon_{\varphi,0} \quad \text{with} \quad \lim_{\varepsilon\to0}\|\nabla R^\varepsilon_{\varphi,0}\|_{H^{s_0-2m-5}} = 0, \end{aligned} \tag{1.2}$$

for some s_0 large enough. We also impose the following condition at infinity:

$$\Psi^\varepsilon(t,x)\to e^{\frac{i}{\varepsilon}\big(u^\infty\cdot x-\frac{t}{2}|u^\infty|^2\big)} \quad \text{as} \quad |x|\to+\infty. \tag{1.3}$$

In what follows, we assume that the constant vector $u^\infty=0$ for simplicity.

The motivation for us to study the problem (1.1) comes from many interesting issues concerning a superfluid passing an obstacle (see for example [9, 12, 18]). Classical Madelung transform introduces two real variables: $a^\varepsilon\geq0$ and φ^ε so that

$$\Psi^\varepsilon = a^\varepsilon\,\exp\Big(i\frac{\varphi^\varepsilon}{\varepsilon}\Big). \tag{1.4}$$

By substituting (1.4) into (1.1) and separating the real and imaginary parts, we find

$$\begin{cases} \partial_t a^\varepsilon + \nabla\varphi^\varepsilon\cdot\nabla a^\varepsilon + \frac12 a^\varepsilon\Delta\varphi^\varepsilon = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\ a^\varepsilon\big(\partial_t\varphi^\varepsilon + \frac12|\nabla\varphi^\varepsilon|^2 + (a^\varepsilon)^2 - 1\big) = \frac{\varepsilon^2}{2}\Delta a^\varepsilon, \end{cases} \tag{1.5}$$

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with the initial-boundary conditions

$$a^\varepsilon|_{z=0} = 1, \quad \varphi^\varepsilon|_{z=0} = 0, \quad a^\varepsilon \to 1 \quad \text{as} \quad |x| \to \infty \quad \text{and} \quad a^\varepsilon|_{t=0} = a_0^\varepsilon, \quad \varphi^\varepsilon|_{t=0} = \varphi_0^\varepsilon. \tag{1.6}$$

We denote $\rho^\varepsilon \stackrel{\text{def}}{=} (a^\varepsilon)^2$ and $\rho^\varepsilon u^\varepsilon \stackrel{\text{def}}{=} \rho^\varepsilon \nabla \varphi^\varepsilon$, which corresponds to the quantum density and momentum respectively in quantum mechanics (see [14] for instance). This allows to rewrite (1.5-1.6) as the following hydrodynamical system:

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0, \\ \rho^\varepsilon \big(\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon\big) + \nabla\, p(\rho^\varepsilon) = \frac{\varepsilon^2}{2} \rho^\varepsilon \nabla \left(\frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right), \end{cases} \tag{1.7}$$

with the initial-boundary conditions

$$\begin{cases} \rho^\varepsilon|_{z=0} = 1, \quad \varphi^\varepsilon|_{z=0} = 0 \quad \text{and} \quad \rho^\varepsilon \to 1 \quad \text{as} \quad |x| \to \infty \\ \rho^\varepsilon|_{t=0} = (a_0^\varepsilon)^2, \quad u^\varepsilon|_{t=0} = \nabla \varphi_0^\varepsilon, \end{cases} \tag{1.8}$$

and the pressure law $p(\rho^\varepsilon) = \frac{1}{2}(\rho^\varepsilon)^2$.

The system (1.7) is called quantum compressible Euler system, and the right hand-side of the u^ε equation in (1.7) is called quantum pressure. As ε approaches to 0, the quantum pressure is formally negligible and the system (1.7-1.8) approaches to the classical compressible Euler equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, u) = 0, \\ \rho \left(\partial_t u + u \cdot \nabla u\right) + \nabla\, p(\rho) = 0, \end{cases} \tag{1.9}$$

with the initial-boundary conditions

$$\int_0^{+\infty} u \cdot \mathbf{n}\, dz = 0, \quad \rho \to 1 \quad \text{as} \quad |x| \to \infty \quad \text{and} \quad \rho|_{t=0} = (a_{0,0}^{\text{in}})^2, \quad u|_{t=0} = \nabla \varphi_{0,0}^{\text{in}}. \tag{1.10}$$

The justification of the above formal limit has attracted the interests by many authors. In the whole space case, Gérard [10] proved the limit with analytical initial data. Grenier [11] solved the limit problem before the formation of singularity in the limit system with initial data in Sobolev spaces. The main idea in [11] is to use the symmetrizer of the limit system (1.9) to get H^s energy estimates which are uniform in ε for a singularly perturbed system. Nevertheless this method does not work for the semiclassical limit of Schrödinger-Poisson equations, as the resulting limit system is not a symmetric hyperbolic one. Motivated by the work of Brenier [4], where the author proved the local-in time convergence of the scaled Vlasov-Poisson equations to the incompressible Euler equations, the second author [20] used the Wigner measure approach (see [16, 22]) to study the semiclassical limit of Schrödinger-Poisson equation (see [21] for more general nonlinearity).

In order to solve the semiclassical limit of GP equation in the exterior domain with Neumann boundary condition (which corresponds to the non-slip boundary condition $u \cdot \mathbf{n} = 0$ for the limit system (1.9)), where we can not use Wigner transform, the authors [15] simplified the modulated energy functional in [20, 21] and proved that

$$|\Psi^\varepsilon|^2 - \rho \to 0 \quad \text{in} \quad L^\infty(]0,T[;L^2) \quad \text{and} \quad \varepsilon \mathrm{Im}\left(\bar{\Psi}^\varepsilon \nabla \Psi^\varepsilon\right) - \rho u \to 0 \quad \text{in} \quad L^\infty(]0,T[;L^1_{\text{loc}}),$$

before the formation of singularity in the limit system. This idea has been used and extended by the authors in [1, 19]. Interested readers may check [2, 6] for the so-called supercritical geometric optics where they allow $p'(0) = 0$ for the pressure function in (1.9). One may also check the books [5, 22] and references therein for more information in this context.

For the problem (1.1), by comparing (1.8) with (1.10), we find that the boundary condition $\rho^\varepsilon|_{z=0} = 1$ in (1.8) does not match the boundary condition for ρ in (1.10) at the boundary $\{z = 0\}$, where we do not have any restriction on ρ. This results in a strong boundary layer near the boundary $\{z=0\}$. In fact, if we formally seek for WKB expansions $\Psi^\varepsilon = a^\varepsilon \exp\left(i\frac{\varphi^\varepsilon}{\varepsilon}\right)$ of the form

$$\left\{\begin{array}{l} a^\varepsilon(t,x) = \sum_{k=0}^{\infty} \varepsilon^k \left(a_k(t,y,z) + A_k(t,y,Z)\right), \\ \varphi^\varepsilon(t,x) = \sum_{k=0}^{\infty} \varepsilon^k \left(\varphi_k(t,y,z) + \Phi_k(t,y,Z)\right), \end{array}\right. \tag{1.11}$$

we shall find below that $\Phi_0 \equiv 0$, and a_0, a_k, A_k, φ_k, Φ_k with $k = 1,2,3,...$ are non-trivial. In this case, we have $\nabla u^\varepsilon = \nabla^2\varphi^\varepsilon \sim O(\frac{1}{\varepsilon})$ and $\nabla a^\varepsilon \sim O(\frac{1}{\varepsilon})$. In some sense, this phenomena has some similarity with the strong boundary layer caused by vanishing viscosity of incompressible Navier-Stokes system to Euler system (see [17]). Lately there are a lot of progresses on this topic (see for instance [8, 13] and the references therein).

On the other hand, for the case of the semiclassical limit of GP equation with the Neumann boundary condition in $\mathbb{R}^3_+$, that is, $\partial_z \Psi^\varepsilon|_{\partial\,\mathbb{R}^3_+} = 0$, Chiron and Rousset [6] justified the validity of the WKB expansions on some finite time interval $[0,T]$. We remark that in this case, the boundary layer profiles $A_0 = \Phi_0 = \Phi_1 \equiv 0$ in (1.11), which implies $\nabla u^\varepsilon \sim O(1)$ and $\nabla a^\varepsilon \sim O(1)$. This weak boundary layer plays a key role in the study of the nonlinear stability to the WKB expansions. Nevertheless, the semiclassical limit of nonlinear Schrödinger equation with Dirichlet boundary condition was left open in [6, 18].

In this paper, we are going to answer this question proposed in [6, 18] under the condition that both $a^{\rm in}_{0,0} - 1$ and $\nabla\varphi_0$ are sufficiently small in some regular enough Sobolev space.

Let us end this introduction by some notations that will be used in all that follows.

For operators A, B, we denote $[A;B] = AB - BA$ to be the commutator of A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We denote by $\int_{\mathbb{R}^3} f|g\,dx$ the $L^2(\mathbb{R}^3_+)$ inner product of f and g, and $L^p(\mathbb{R}^3_+)$ by L^p_+. Finally we shall always denote ∇_y by $\nabla_{\rm h}$ and $\mathcal{T} \stackrel{\rm def}{=} (\partial_t,\, \nabla_{\rm h})$.

2. Formal asymptotic analysis

2.1. Outer expansion.

Since the boundary layer is concentrated in the $\varepsilon-$ neighborhood of $\{z=0\}$, we call the domain $\mathfrak{O} \stackrel{\rm def}{=} \left\{\, x = (y,z) \,:\, y \in \mathbb{R}^2,\, z > \varepsilon \,\right\}$ the outer region, and the associated vertical variable z is called outer variable. We also denote the inner region by $\mathfrak{I} \stackrel{\rm def}{=} \left\{\, x = (y,z) : \, y \in \mathbb{R}^2,\, 0 \leq z < \varepsilon \,\right\}$, and call $Z = \frac{z}{\varepsilon}$ the inner variable, which makes us to specify the so-called "inner limit process".

In the outer region $\mathfrak{O}$, we formally seek the solution $(a^\varepsilon, \varphi^\varepsilon)$ of (1.5) with the form:

$$a^\varepsilon(t,x) = \sum_{k=0}^{\infty} \varepsilon^k a_k(t,y,z) \quad \text{and} \quad \varphi^\varepsilon(t,x) = \sum_{k=0}^{\infty} \varepsilon^k \varphi_k(t,y,z). \tag{2.1}$$

By substituting (2.1) into (1.5), we get

$$\sum_{k=0}^{\infty} \varepsilon^k \partial_t a_k + \sum_{k_1,k_2=0}^{\infty} \varepsilon^{k_1+k_2}\big(\nabla\varphi_{k_1}\cdot\nabla a_{k_2} + \frac{1}{2} a_{k_1}\Delta\varphi_{k_2}\big) = 0 \tag{2.2}$$

and

$$\begin{aligned} \sum_{k_1,k_2=0}^{\infty} \varepsilon^{k_1+k_2} a_{k_1}\partial_t\varphi_{k_2} + \sum_{k_1,k_2,k_3=0}^{\infty} \varepsilon^{k_1+k_2+k_3}\frac{a_{k_1}}{2}\big(\nabla\varphi_{k_2}\cdot\nabla\varphi_{k_3} + 2a_{k_2}a_{k_3}\big) \\ - \sum_{k=0}^{\infty}\varepsilon^k a_k = \frac{1}{2}\sum_{k=0}^{\infty}\varepsilon^{k+2}\Delta a_k. \end{aligned} \tag{2.3}$$

Vanishing the coefficients to the zeroth order of ε in (2.2) and (2.3) gives

$$\left\{\begin{array}{l}\partial_t a_0 + \nabla\varphi_0\cdot\nabla a_0 + \frac{1}{2}a_0\Delta\varphi_0 = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\ \partial_t\varphi_0 + \frac{1}{2}|\nabla\varphi_0|^2 + (a_0^2-1) = 0,\end{array}\right. \tag{2.4}$$

where we used the fact that a_0 has a positive lower bound which will be justified in Section 4.

In view of (1.2) and (1.6), we implement the system (2.4) with the initial and boundary conditions:

$$a_0|_{t=0} = a_{0,0}^{\mathrm{in}}, \quad \varphi_0|_{t=0} = \varphi_{0,0}^{\mathrm{in}} \quad \text{and} \quad \varphi_0|_{z=0} = 0, \quad a_0\to 1 \quad \text{as} \quad |x|\to\infty. \tag{2.5}$$

We shall prove the local well-posedness of the above problem with sufficiently smooth initial data in Section 4.

Vanishing the coefficients of ε^1 in (2.2) and (2.3) leads to the coupled system $(a_1,\,\varphi_1)$:

$$\left\{\begin{array}{l}\partial_t a_1 + \nabla a_0\cdot\nabla\varphi_1 + \nabla\varphi_0\cdot\nabla a_1 + \frac{1}{2}a_1\Delta\varphi_0 + \frac{1}{2}a_0\Delta\varphi_1 = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\ \partial_t\varphi_1 + \nabla\varphi_0\cdot\nabla\varphi_1 + 2a_0a_1 = 0,\\ a_1|_{t=0} = a_{1,0}^{\mathrm{in}}, \quad \varphi_1|_{t=0} = \varphi_{1,0}^{\mathrm{in}},\end{array}\right. \tag{2.6}$$

where $(a_{1,0}^{\mathrm{in}}, \varphi_{1,0}^{\mathrm{in}})$ is given by (1.2).

In general, by vanishing the coefficients of ε^{k+2} in (2.2) and (2.3) for $k=0,\cdots,m$, and using (1.2), we find

$$\left\{\begin{array}{l}\partial_t a_{k+2} + \nabla a_0\cdot\nabla\varphi_{k+2} + \nabla\varphi_0\cdot\nabla a_{k+2} + \frac{\Delta\varphi_0}{2}a_{k+2} + \frac{a_0}{2}\Delta\varphi_{k+2} = f^a_{k+1}\\ \partial_t\varphi_{k+2} + \nabla\varphi_0\cdot\nabla\varphi_{k+2} + 2a_0a_{k+2} = \frac{1}{2a_0}\left(\Delta a_k + g^\varphi_{k+1}\right),\\ a_{k+2}|_{t=0} = a_{k+2,0}^{\mathrm{in}}, \quad \varphi_{k+2}|_{t=0} = \varphi_{k+2,0}^{\mathrm{in}},\end{array}\right. \tag{2.7}$$

where the source terms $(f^a_{k+1},\,g^\varphi_{k+1})$ are determined by

$$\begin{aligned} f^a_{k+1} &\stackrel{\text{def}}{=} -\sum_{k_1=1}^{k+1}\big(\nabla\varphi_{k_1}\cdot\nabla a_{k+2-k_1} + \frac{1}{2}a_{k_1}\Delta\varphi_{k+2-k_1}\big),\\ g^\varphi_{k+1} &\stackrel{\text{def}}{=} -\sum_{k_1=1}^{k+1} a_{k_1}\partial_t\varphi_{k+2-k_1} - \sum_{\substack{k_1+k_2+k_3=k+2\\ 0\le k_1,k_2,k_3\le k+1}}\frac{a_{k_1}}{2}\big(\nabla\varphi_{k_2}\cdot\nabla\varphi_{k_3} + 2a_{k_2}a_{k_3}\big).\end{aligned} \tag{2.8}$$

We shall implement the above systems with boundary conditions in Subsection 2.2.

2.2. **Uniformly valid approximation.** In all that follows, we shall always denote

$$[f]_\varepsilon(x) \stackrel{\text{def}}{=} f\big(y,\frac{z}{\varepsilon}\big). \tag{2.9}$$

With the outer solution $(a_j,\,\varphi_j)$ for $j=0,1,2,...$ in hand, we shall use the Successive Complementary Expansion Method (SCEM for short, see [7]) to seek a Uniformly Valid Approximate solutions (UVA for short) to (1.5). In order to do so, we take the following ansatz

$$\begin{aligned} a^\varepsilon(t,x) &= \sum_{k=0}^\infty \varepsilon^k\left(a_k(t,x) + [A_k]_\varepsilon(t,x)\right) \quad \text{and}\\ \varphi^\varepsilon(t,x) &= \sum_{k=0}^\infty \varepsilon^k\left(\varphi_k(t,x) + [\Phi_k]_\varepsilon(t,x)\right).\end{aligned} \tag{2.10}$$

We require that both $A_k(t,y,Z)$ and $\Phi_k(t,y,Z)$ together with all of their derivatives are rapidly vanishing as $Z\to+\infty$.

We denote $\overline{g}(t,y)$ the trace of $g(t,y,z)$ on the boundary $\{z=0\}$, that is,

$$\overline{g}(t,y) \stackrel{\text{def}}{=} g(t,y,z=0). \tag{2.11}$$

Formally let us write that

$$g(t,y,z)=\sum_{j=0}^{\infty}\frac{z^j}{j!}\overline{\partial_z^j g}(t,y). \tag{2.12}$$

By plugging the ansatz (2.10) into (1.5) and using (2.2), (2.3) and (2.12), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\varepsilon^k\partial_t A_k+\sum_{k_1,k_2=0}^{\infty}\varepsilon^{k_1+k_2}\Big(\big(\nabla_{\rm h}\Phi_{k_1}\cdot\nabla_{\rm h}A_{k_2}+\frac12 A_{k_1}\Delta_{\rm h}\Phi_{k_2}\big)\\
&\qquad\qquad\qquad\qquad\qquad+\varepsilon^{-2}\big(\partial_Z\Phi_{k_1}\partial_Z A_{k_2}+\frac12 A_{k_1}\partial_Z^2\Phi_{k_2}\big)\Big)\\
&+\sum_{k_1,k_2,j=0}^{\infty}\varepsilon^{k_1+k_2+j}\frac{Z^j}{j!}\Big(\overline{\nabla_y\partial_z^j\varphi_{k_1}}\cdot\nabla_{\rm h}A_{k_2}+\nabla_{\rm h}\Phi_{k_1}\cdot\overline{\nabla_{\rm h}\partial_z^j a_{k_2}}+\frac{\overline{\partial_z^j a_{k_1}}}{2}\Delta_{\rm h}\Phi_{k_2}\\
&\quad+\frac{A_{k_1}}{2}\overline{\Delta\partial_z^j\varphi_{k_2}}+\varepsilon^{-1}\big(\overline{\partial_z^{j+1}\varphi_{k_1}}\partial_Z A_{k_2}+\partial_Z\Phi_{k_1}\overline{\partial_z^{j+1}a_{k_2}}\big)+\varepsilon^{-2}\frac{\overline{\partial_z^j a_{k_1}}}{2}\partial_Z^2\Phi_{k_2}\Big)=0,
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&\sum_{k_1,k_2,j=0}^{\infty}\varepsilon^{k_1+k_2+j}\frac{Z^j}{j!}\big(\overline{\partial_z^j a_{k_1}}\partial_t\Phi_{k_2}+A_{k_1}\overline{\partial_t\partial_z^j\varphi_{k_2}}\big)+\sum_{k_1,k_2=0}^{\infty}\varepsilon^{k_1+k_2}(A_{k_1}\partial_t\Phi_{k_2})\\
&-\sum_{k=0}^{\infty}\varepsilon^k A_k+\sum_{k_1,k_2,k_3,j_1,j_2=0}^{\infty}\varepsilon^{k_1+k_2+k_3+j_1+j_2}\frac{Z^{j_1+j_2}}{j_1!j_2!}\Big(\overline{\partial_z^{j_1}a_{k_1}\nabla_{\rm h}\partial_z^{j_2}\varphi_{k_2}}\cdot\nabla_{\rm h}\Phi_{k_3}+\\
&\quad+\frac{A_{k_1}}{2}\overline{\nabla\partial_z^{j_1}\varphi_{k_2}}\cdot\overline{\nabla\partial_z^{j_2}\varphi_{k_3}}+3\overline{\partial_z^{j_1}a_{k_1}\partial_z^{j_2}a_{k_2}}A_{k_3}+\varepsilon^{-1}\overline{\partial_z^{j_1}a_{k_1}\partial_z^{j_2+1}\varphi_{k_2}}\partial_Z\Phi_{k_3}\Big)\\
&+\sum_{k_1,k_2,k_3,j_1=0}^{\infty}\varepsilon^{k_1+k_2+k_3+j_1}\frac{Z^{j_1}}{j_1!}\Big(A_{k_1}\overline{\nabla_{\rm h}\partial_z^{j_1}\varphi_{k_2}}\cdot\nabla_{\rm h}\Phi_{k_3}+\frac{\overline{\partial_z^{j_1}a_{k_1}}}{2}\nabla_{\rm h}\Phi_{k_2}\cdot\nabla_{\rm h}\Phi_{k_3}+\\
&\quad+3\overline{\partial_z^{j_1}a_{k_1}}A_{k_2}A_{k_3}+\varepsilon^{-1}A_{k_1}\overline{\partial_z^{j_1+1}\varphi_{k_2}}\partial_Z\Phi_{k_3}+\varepsilon^{-2}\frac{\overline{\partial_z^{j_1}a_{k_1}}}{2}\partial_Z\Phi_{k_2}\partial_Z\Phi_{k_3}\Big)\\
&+\sum_{k_1,k_2,k_3=0}^{\infty}\varepsilon^{k_1+k_2+k_3}\frac{A_{k_1}}{2}\Big(\nabla_{\rm h}\Phi_{k_2}\cdot\nabla_{\rm h}\Phi_{k_3}+2A_{k_2}A_{k_3}+\varepsilon^{-2}\partial_Z\Phi_{k_2}\partial_Z\Phi_{k_3}\Big)\\
&=\frac12\sum_{k=0}^{\infty}\varepsilon^k\partial_Z^2A_k+\frac12\sum_{k=0}^{\infty}\varepsilon^{k+2}\Delta_{\rm h}A_k\quad\text{with}\quad\Delta_{\rm h}=\partial_{y_1}^2+\partial_{y_2}^2.
\end{aligned} \tag{2.14}$$

The coefficients of ε^{-2} in (2.13) and (2.14) yields

$$\frac12(A_0+\overline{a_0})\partial_Z^2\Phi_0+\partial_Z\Phi_0\partial_Z A_0=0, \tag{2.15}$$

and

$$\frac12(A_0+\overline{a}_0)|\partial_Z\Phi_0|^2=0, \tag{2.16}$$

respectively.

We first assume that $A_0+\overline{a}_0$ has a positive lower bound, which we shall justify in Section 5. Then due to $\Phi_0|_{Z=\infty}=0$, we deduce from (2.16) that

$$\Phi_0(t,y,Z)\equiv 0. \tag{2.17}$$

Similarly by virtue of (2.4) and (2.17), we find that the coefficients of ε^{-1} in (2.13) and of ε^0 in (2.14) give respectively

$$\frac{1}{2}(A_0+\overline{a}_0)\partial_Z^2\Phi_1+(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})\partial_Z A_0=0, \tag{2.18}$$

and

$$\frac{1}{2}\partial_Z^2 A_0=(A_0+\overline{a}_0)\partial_Z\Phi_1\overline{\partial_z\varphi_0}+\frac{\overline{a}_0+A_0}{2}(\partial_Z\Phi_1)^2+A_0^3+3\overline{a}_0A_0^2+2\overline{a}_0^2A_0. \tag{2.19}$$

According to the boundary condition $a^\varepsilon|_{z=0}=1$ in (1.6), we have the matched condition on the boundary $\{z=0\}$ that $A_0(Z=0)+a_0(z=0)=1$. So that we impose the following boundary condition for A_0 :

$$A_0|_{Z=0}=1-a_0(t,y,0). \tag{2.20}$$

Furthermore, we require that both $A_0(t,y,Z)$ and $\Phi_1(t,y,Z)$ along with all of their derivatives are rapidly vanishing as $Z\to+\infty$. We shall present the unique solvability of the required solution to the system (2.18-2.19) in Section 5.

Notice the boundary condition $\varphi^\varepsilon|_{z=0}=0$ in (1.6) and $\varphi_0|_{z=0}=0$ in (2.5), we have the following matched condition of φ_1 on the boundary $\{z=0\}$

$$\varphi_1|_{z=0}=-\Phi_1(t,y,0). \tag{2.21}$$

We implement the system (2.6) with the Dirichlet boundary condition (2.21), and we shall prove its unique solvability in Section 7.

Inductively, assuming that we already obtain (a_0,φ_0) , $(a_{j+1},\ \varphi_{j+1})$ and (A_j,Φ_{j+1}) with $0\le j\le k\le m-1$, we get, by vanishing of the coefficients of ε^k in (2.13) and of ε^{k+1} in (2.14), that

$$\frac{\overline{a}_0+A_0}{2}\partial_Z^2\Phi_{k+2}+\partial_Z A_0\partial_Z\Phi_{k+2}+(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})\partial_Z A_{k+1}+\frac{A_{k+1}}{2}\partial_Z^2\Phi_1=F_k, \tag{2.22}$$

and

$$\begin{aligned}\frac{1}{2}\partial_Z^2 A_{k+1}=&(A_0+\overline{a}_0)(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})\partial_Z\Phi_{k+2}\\&+(3A_0^2+6\overline{a}_0A_0+2\overline{a}_0^2+\partial_Z\Phi_1\overline{\partial_z\varphi_0}+\frac{1}{2}|\partial_Z\Phi_1|^2)A_{k+1}+G_k,\end{aligned} \tag{2.23}$$

where the source terms $(F_k,\ G_k)$ depend only on $\left(\overline{\partial_z^\ell a_0},\overline{\partial_z^\ell\varphi_0}\right)$ for $\ell\le k{+}2$, and $\left(\overline{\partial_z^\ell a_{j+1}},\overline{\partial_z^k\varphi_{\ell+1}}\right)$ and $(A_j,\nabla\Phi_{j+1})$ for $0\le j\le k$ and $0\le\ell+j\le k+1$, the explicit form of which will be presented in the Appendix A.

Thanks to the matched boundary condition on $\{z=0\}$ for the outer and inner solutions, we impose the following condition for A_{k+1} :

$$A_{k+1}|_{Z=0}=-a_{k+1}(t,y,0). \tag{2.24}$$

We also require that both $A_{k+1}(t,y,Z)$ and $\Phi_{k+2}(t,y,Z)$ together with all of their derivatives are rapidly vanishing as $Z\to+\infty$.

Finally, with thus obtained Φ_{k+2}, according to the matched boundary condition on $\{z=0\}$ for the outer and inner solutions, we implement the system (2.7) with the Dirichlet boundary condition

$$\varphi_{k+2}|_{z=0}=-\Phi_{k+2}(t,y,0). \tag{2.25}$$

The unique solvability of the systems (2.22-2.23) and (2.7) with the boundary condition (2.25) will be outlined in Section 7.

3. The main result and its sketch of the proof

We first observe from (2.4) that

$$\begin{aligned}\partial_t\varphi_0|_{t=0} =& -\Big(\frac{1}{2}|\nabla\varphi_0|^2+(\rho_0-1)\Big)|_{t=0}\\ =&1-\big(a_{0,0}^{\rm in}\big)^2-\frac{1}{2}|\nabla\varphi_{0,0}^{\rm in}|^2 \stackrel{\rm def}{=} \varphi_{0,1}^{\rm in}.\end{aligned} \tag{3.1}$$

To guarantee the local existence of smooth solutions to (2.4-2.5), we need the following compatibility conditions for the initial data:
$(\mathcal{A}_0)$: Let $(\varphi_{0,0}^{\rm in}, \varphi_{0,1}^{\rm in}) \in H^{s_0}\times H^{s_0-1}$ for $4\leq s_0\in\mathbb{N}$. We assume that the data satisfies $\partial_t^j\varphi_0(0,y,0)=0$ for $y\in\mathbb{R}^2$ and $j=0,\cdots,s_0-1$.

Definition 3.1. *Let $s, T>0$, we define the functional space $W^s_T \stackrel{def}{=} \bigcap_{j=0}^{[s]} C^j([0,T];\, H^{s-j}(\mathbb{R}^3_+))$, where $[s]$ denotes the integer part of s and its norm is given by*

$$\|\varphi(t)\|_{W^s}^2 \stackrel{def}{=} \sum_{j=0}^{[s]}\|\partial_t^j\varphi(t)\|_{H^{s-j}}^2 \quad \textit{and} \quad \|\varphi\|_{W^s_T}\stackrel{def}{=}\sup_{t\in[0,T]}\|\varphi(t)\|_{W^s}. \tag{3.2}$$

We shall prove in Section 4 that

Proposition 3.1. *Let $4\leq s_0\in\mathbb{N}$ and $a_{0,0}^{\rm in}-1\in H^{s_0-1}$, $\varphi_{0,0}^{\rm in}\in H^{s_0}$ which satisfies the compatibility condition $(\mathcal{A}_0)$. We assume that*

$$\big\|\big(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in}\big)\big\|_{H^{s_0-1}}\leq c, \tag{3.3}$$

for some sufficiently small positive constant c, then there exists a positive constant $\mathcal{C}$ so that for $T_0\stackrel{def}{=}\mathcal{C}c^{-1}$, the system (2.4-2.5) has a unique solution (a_0,φ_0) on $[0,T_0]$, which satisfies

$$\|(a_0-1,\partial_t\varphi_0,\nabla\varphi_0)\|_{W^{s_0-1}_{T_0}}\leq C\big\|\big(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in}\big)\big\|_{H^{s_0-1}}. \tag{3.4}$$

We remark that the main difficulty in the proof of Proposition 3.1 lies in the boundary condition $\varphi_0|_{z=0}=0$ in (2.5) so that one can not apply the standard theory on symmetric hyperbolic system to prove its local well-posedness. The new idea here is to reformulate (2.4-2.5) to be an initial and boundary value problem of a nonlinear wave equation (4.4-4.6) (see Section 4) under the smallness initial condition (3.3).

In order to solve the boundary layer equation, we recall the boundary layer profile space from [6]:

Definition 3.2. *Let $s\in\mathbb{R}^+$ and $\gamma_0>0$ be a positive constant, we define $W^s_{\gamma_0}(\mathbb{R}^3_+)$ as the completion of $\big\{F(y,Z)\in H^s(\mathbb{R}^2_{\rm h};\,H^\infty(\mathbb{R}^+_Z))\big\}$ with $\|F\|_{W^s_{\gamma_0}(\mathbb{R}^3_+)}$ being finite, and*

$$\mathcal{W}^s_{\gamma_0,T}=\mathcal{W}^s_{\gamma_0}([0,T]\times\mathbb{R}^3_+)\stackrel{def}{=}\bigcap_{j=0}^{[s]}C^j([0,T];\,W^{s-j}_{\gamma_0}(\mathbb{R}^3_+)),$$

where the norms are given by

$$\begin{aligned}&\|F\|_{W^s_{\gamma_0}}\stackrel{def}{=}\max_{0\leq\ell\leq[s]}\sup_{Z\in\mathbb{R}^+}\big(e^{\gamma_0 Z}\|\partial_Z^\ell F(\cdot,Z)\|_{H^s(\mathbb{R}^2_{\rm h})}\big) \quad \textit{and}\\ &\|G\|_{\mathcal{W}^s_{\gamma_0,T}}\stackrel{def}{=}\max_{0\leq j\leq[s]}\sup_{t\in[0,T]}\|\partial_t^jG(t)\|_{W^{s-j}_{\gamma_0}}.\end{aligned} \tag{3.5}$$

We shall prove in Section 5 that

Proposition 3.2. *Under the assumptions of Proposition 3.1, the coupled equations (2.18-2.19) with the boundary condition* (2.20) *has a unique solution* $(A_0,\,\Phi_1)$ *in* $\mathcal{W}_{1,T_0}^{s_0-\frac{3}{2}}$, *where* T_0 *is determined by Proposition 3.1. Furthermore, there holds*

$$\big\|(A_0,\,\Phi_1)\big\|_{\mathcal{W}_{1,T_0}^{s_0-\frac{3}{2}}}\leq C\big\|\big(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in}\big)\big\|_{H^{s_0-1}}. \tag{3.6}$$

To solve the systems (2.6) and (2.7) for the inner expansions, we are going to solve first a linear wave equation in Section 6. More precisely, let φ_0 be determined by Proposition 3.1 and $f\in W_T^{s-1}(\mathbb{R}^3_+)$ for $s\leq s_0-1$ and $T\leq T_0$, we are going to solve the following linear wave equation

$$\begin{aligned}P(\varphi_0,D)\varphi\overset{\rm def}{=}&\partial_t^2\varphi-\operatorname{div}\big(a_0^2\nabla\varphi\big)+2\nabla\varphi_0\cdot\nabla\partial_t\varphi\\&+\operatorname{div}\big((\nabla\varphi_0\cdot\nabla\varphi)\nabla\varphi_0\big)+\nabla\partial_t\varphi_0\cdot\nabla\varphi+\Delta\varphi_0\partial_t\varphi=f,\end{aligned} \tag{3.7}$$

together with the following initial and boundary conditions:

$$\varphi(t,y,0)=g(t,y)\in W_T^{s+\frac{1}{2}}(\mathbb{R}^2)\quad\text{and}\quad\varphi|_{t=0}=\varphi_{,0}^{\rm in},\quad\partial_t\varphi|_{t=0}=\varphi_{,1}^{\rm in}. \tag{3.8}$$

The result about the unique solvability of the system (3.7-3.8) states as follows:

Theorem 3.1. *Let* $(\varphi_{,0}^{\rm in},\varphi_{,1}^{\rm in})$ *satisfy* $\nabla\varphi_{,0}^{\rm in},\varphi_{,1}^{\rm in}\in H^{s-1}$ *and the compatibility condition:* $\partial_t^\ell\left(\varphi-g\right)(0,y,0)=0$ *for* $y\in\mathbb{R}^2$ *and* $\ell=0,\cdots,s-1$. *Let* $f\in W_T^{s-1}$ *for some integer* $s\in[4,s_0]$. *Then under the assumptions of Proposition 3.1, the system (3.7-3.8) has a unique solution* φ *on* $[0,T]$, *which satisfies*

$$\|(\partial_t\varphi,\nabla\varphi)\|_{W_T^{s-1}}\leq C\big(\|g\|_{W_T^{s+\frac{1}{2}}(\mathbb{R}^2)}+\|\nabla\varphi_{,0}^{\rm in}\|_{H^{s-1}}+\|\varphi_{,1}^{\rm in}\|_{H^{s-1}}+\|f\|_{W_T^{s-1}}\big).$$

With (a_0,φ_0) being determined by Proposition 3.1 and $(A_0,\,\Phi_1)$ being determined by Proposition 3.2, we are going to solve the system (2.6) with the boundary condition (2.21).

We first observe from (2.4) and (2.6) that

$$\partial_t(a_1a_0)+\operatorname{div}\left(a_0a_1\nabla\varphi_0\right)+\frac{1}{2}\operatorname{div}\left(a_0^2\nabla\varphi_1\right)=0.$$

Then we get, by taking ∂_t to the φ_1 equation of (2.6) and inserting the above equation to the resulting one, that

$$\begin{aligned}\partial_t^2\varphi_1-\Delta\varphi_1+&\operatorname{div}\big((\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_1\big)+\partial_t(\nabla\varphi_0\cdot\nabla\varphi_1)\\&+\operatorname{div}(\partial_t\varphi_1\nabla\varphi_0)+\operatorname{div}\big((\nabla\varphi_0\cdot\nabla\varphi_1)\nabla\varphi_0\big)=0.\end{aligned} \tag{3.9}$$

Noticing from (2.6) that

$$\partial_t\varphi_1\big|_{t=0}=-\nabla\varphi_0\cdot\nabla\varphi_1\big|_{t=0}-2a_0a_1\big|_{t=0}=\nabla\varphi_{0,0}^{\rm in}\cdot\nabla\varphi_{1,0}^{\rm in}-2a_{0,0}^{\rm in}a_{1,0}^{\rm in}\overset{\rm def}{=}\varphi_{1,1}^{\rm in}, \tag{3.10}$$

we complement the equation (3.9) with the boundary condition (2.21) and the initial data

$$\varphi_1|_{t=0}=\varphi_{1,0}^{\rm in},\quad\partial_t\varphi_1|_{t=0}=\varphi_{1,1}^{\rm in}. \tag{3.11}$$

By applying Theorem 3.1, we shall prove in Section 7 that

Proposition 3.3. *Let* $s_0\geq 6$ *be an integer. Let* $a_{1,0}^{\rm in},\nabla\varphi_{1,0}^{\rm in}\in H^{s_0-3}$ *which satisfy the compatibility condition:* $\partial_t^\ell\left(\varphi_1+\Phi_1\right)(0,y,0)=0$ *for* $y\in\mathbb{R}^2$ *and* $\ell=0,\cdots,s_0-3$. *Then under the assumptions of Proposition 3.1, the system* (2.6) *with boundary condition* (2.21) *has a unique solution* (a_1,φ_1) *on* $[0,T_0]$ *such that*

$$\big\|\big(a_1,\partial_t\varphi_1,\nabla\varphi_1\big)\big\|_{W_{T_0}^{s_0-3}}\leq C\big(\big\|\big(a_{1,0}^{\rm in},\nabla\varphi_{1,0}^{\rm in}\big)\big\|_{H^{s_0-3}}+\big\|\big(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in}\big)\big\|_{H^{s_0-1}}\big). \tag{3.12}$$

for T_0 *being determined by Proposition 3.1.*

Let us turn to the solvability of the boundary layer problem (2.22-2.23) with the boundary condition (2.24) for $0 \leq k \leq m$. In fact, we shall prove in Section 7 that

Proposition 3.4. *Let $s_0 \geq 2k+5$ be an integer. Let $a_{j,0}^{\rm in}, \nabla\varphi_{j,0}^{\rm in} \in H^{s_0-2j-1}$ with $j = 1,...,k+1$, which satisfy the compatibility conditions:*
$(\mathcal{A}_{k+1})$: $\partial_t^\ell \left(\varphi_j + \Phi_j\right)(0,y,0) = 0$ for $y \in \mathbb{R}^2$, $\ell = 0,\cdots,s_0-2j-1$ and $j = 1,\cdots,k+1$.
Then under the assumptions of Proposition 3.1, the system (2.22-2.23) with the boundary condition (2.24) has a unique solution $(A_{k+1},\,\Phi_{k+2})$ in $\mathcal{W}_{1,T_0}^{s_0-2(k+2)+\frac12}$. Moreover, there holds

$$\big\|(A_{k+1},\,\Phi_{k+2})\big\|_{\mathcal{W}_{1,T_0}^{s_0-2(k+2)+\frac12}} \leq C\Big(\|(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in})\|_{H^{s_0-1}} + \sum_{j=1}^{k+1}\|(a_{j,0}^{\rm in},\nabla\varphi_{j,0}^{\rm in})\|_{H^{s_0-2j-1}}\Big). \tag{3.13}$$

Then along the same line to the proof of Proposition 3.3, we have

Proposition 3.5. *Let $s_0 \geq 2k+7$ be an integer. Let $a_{k+2,0}^{\rm in}, \nabla\varphi_{k+2,0}^{\rm in} \in H^{s_0-2k-5}$ which satisfy the compatibility condition $(\mathcal{A}_{k+2})$. Then under the assumptions of Propositions 3.1 and 3.4, the system (2.7) with boundary condition (2.25) has a unique solution (a_{k+2},φ_{k+2}) on $[0,T_0]$ such that*

$$\begin{aligned}\big\|\big(a_{k+2},\,\partial_t\varphi_{k+2},\,\nabla\varphi_{k+2}\big)\big\|_{W_{T_0}^{s_0-1-2(k+2)}} \leq C\Big(&\|(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in})\|_{H^{s_0-1}}\\ &+\sum_{j=1}^{k+2}\|(a_{j,0}^{\rm in},\nabla\varphi_{j,0}^{\rm in})\|_{H^{s_0-2j-1}}\Big)\end{aligned} \tag{3.14}$$

for T_0 being determined by Proposition 3.1.

Let (a_j,φ_j) for $j = 0,\cdots,m+2$, and (A_j,Φ_{j+1}) for $j = 0,\cdots,m+1$, be constructed in the previous propositions. We denote

$$\begin{aligned}&\Psi^{a,m} \stackrel{\rm def}{=} a^{\varepsilon,m}e^{\frac{i}{\varepsilon}\varphi^{\varepsilon,m}} \quad \text{with} \quad a^{\varepsilon,m} = a^{{\rm int},\varepsilon,m} + [a^{{\rm b},\varepsilon,m}]_\varepsilon, \quad \varphi^{\varepsilon,m} = \varphi^{{\rm int},\varepsilon,m} + [\varphi^{{\rm b},\varepsilon,m}]_\varepsilon,\\ &a^{{\rm int},\varepsilon,m} = \sum_{j=0}^{m+1}\varepsilon^j a_j, \quad a^{{\rm b},\varepsilon,m} = \sum_{j=0}^{m+1}\varepsilon^j A_j, \quad \varphi^{{\rm int},\varepsilon,m} = \sum_{j=0}^{m+2}\varepsilon^j\varphi_j, \quad \varphi^{{\rm b},\varepsilon,m} = \sum_{j=1}^{m+2}\varepsilon^j\Phi_j,\end{aligned} \tag{3.15}$$

and

$$\mathcal{E}_0 \stackrel{\rm def}{=} \|(a_{0,0}^{\rm in}-1,\nabla\varphi_{0,0}^{\rm in})\|_{H^{s_0-1}}^2 + \sum_{j=1}^{m+2}\|(a_{j,0}^{\rm in},\nabla\varphi_{j,0}^{\rm in})\|_{H^{s_0-2j-1}}^2. \tag{3.16}$$

Next let w and ϕ be real-valued functions, we are going to seek the true solution of (1.1) with the form:

$$\Psi^\varepsilon = \big(a^{\varepsilon,m} + w\big)e^{i\left(\frac{\varphi^{\varepsilon,m}}{\varepsilon}+\phi\right)}, \tag{3.17}$$

where (w,ϕ) satisfy the boundary conditions:

$$w|_{z=0} = 0, \quad \phi|_{z=0} = 0.$$

In view of (3.17), we write

$$\Psi^\varepsilon = \Psi^{a,m} + \mathfrak{w}e^{\frac{i}{\varepsilon}\varphi^{\varepsilon,m}} \quad \text{with} \quad \mathfrak{w} = w + (a^{\varepsilon,m}+w)(e^{i\phi}-1) \stackrel{\rm def}{=} w_{\rm R} + iw_{\rm I}. \tag{3.18}$$

It turns out that it is more convenient to handle the estimate of $\mathfrak{w}$ than that of (w,ϕ). As a matter of fact, we shall derive in Section 9 that $(w_{\mathrm{R}}, w_{\mathrm{I}})$ verifies

$$
\left\{
\begin{aligned}
&\varepsilon\big(\partial_t w_{\mathrm{R}} + \mathcal{S}_{u^{\varepsilon,m}}(w_{\mathrm{R}})\big) + \tfrac{\varepsilon^2}{2}\,\Delta w_{\mathrm{I}}\\
&\quad = \Big(\tfrac{\varepsilon^2}{2}\tfrac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2} r_{\varphi}^{m}\Big) w_{\mathrm{I}} - \varepsilon^{m+2} r_{\varphi}^{m} + \mathrm{Im}(Q^{\varepsilon}(\mathfrak{w})) \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3_+,\\
&\varepsilon\big(\partial_t w_{\mathrm{I}} + \mathcal{S}_{u^{\varepsilon,m}}(w_{\mathrm{I}})\big) - \tfrac{\varepsilon^2}{2}\Delta w_{\mathrm{R}} + \Big(2(a^{\varepsilon,m})^2 + \tfrac{\varepsilon^2}{2}\tfrac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2} r_{\varphi}^{m}\Big) w_{\mathrm{R}}\\
&\quad = \varepsilon^{m+1} a^{\varepsilon,m} r_a^m - \mathrm{Re}(Q^{\varepsilon}(\mathfrak{w})),\\
&w_{\mathrm{R}}|_{z=0} = 0, \quad w_{\mathrm{I}}|_{z=0} = 0,
\end{aligned}
\right. \tag{3.19}
$$

where $u^{\varepsilon,m} \stackrel{\text{def}}{=} \nabla\varphi^{\varepsilon,m}$, $\mathcal{S}_f(g) \stackrel{\text{def}}{=} f\cdot\nabla g + \frac{1}{2} g\nabla\cdot f$, and $Q^{\varepsilon}(\mathfrak{w}), r_a^m$ and r_{φ}^m are given respectively by (9.9) and (9.11).

By crucially using the symmetric property of the operator $\mathcal{S}_f(g)$ (see Lemma 8.2) and the special structure of the system (3.19) (especially that we can have the estimate of $\|w_{\mathrm{R}}\|_{L^2_+}$), we shall prove in Section 9 the following proposition:

Proposition 3.6. *Let m, N and s_0 be integers so that $m, N \geq 4$ and $s_0 \geq 2m+9+N$. Let $\Psi^{a,m} = a^{\varepsilon,m} e^{\frac{i}{\varepsilon}\varphi^{\varepsilon,m}}$ be the approximate solutions of* (1.1) *constructed in* (3.15). *Then under the assumptions of Proposition 3.5, there exists a small enough positive constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, the system* (1.1) *has a unique solution $\Psi^{\varepsilon} = (a^{\varepsilon,m} + \mathfrak{w}) e^{i\frac{\varphi^{\varepsilon,m}}{\varepsilon}}$ on $[0,T_0]$. Moreover, for all $t\in[0,T_0]$ and $\mathcal{T} \stackrel{\text{def}}{=} (\partial_t,\, \nabla_{\mathrm{h}})$, there holds*

$$
\sum_{j=0}^{N-1} \big(\|\mathcal{T}^j w_{\mathrm{R}}\|^2_{L^2_+} + \|\varepsilon\mathcal{T}^j \mathfrak{w}\|^2_{H^1}\big) \lesssim \mathcal{E}_0 \varepsilon^{2m+2}. \tag{3.20}
$$

The main result of this paper states as follows, the proof of which will be presented in Section 9.

Theorem 3.2. *Let $m \geq 4$ and $s_0 \geq 2m+13$ be integers. Let $a^{\varepsilon,m}, \varphi^{\varepsilon,m}$ and $\Psi^{a,m}$ be constructed in* (3.15). *Let $a^{\mathrm{in}}_{j,0}, \nabla\varphi^{\mathrm{in}}_{j,0} \in H^{s_0-2j-1}$ with $j = 1,\ldots,m+2$, which satisfy the compatibility conditions, $\mathcal{A}_{m+2}$. Then there exist sufficiently small positive constants c and ε_0 such that under the condition* (3.3), *for any $\varepsilon\in(0,\varepsilon_0)$,* (1.1) *has a unique solution Ψ^{ε}, which satisfies*

$$
\Big\| e^{-i\frac{\varphi^{\varepsilon,m}}{\varepsilon}} (\Psi^{\varepsilon} - \Psi^{a,m}) \Big\|_{L^{\infty}_{T_0}(W^{1,\infty})} \leq C\,\mathcal{E}_0^{\frac{1}{2}} \varepsilon^{m-1}, \tag{3.21}
$$

for the positive time T_0 being determined by Proposition 3.1.

4. The local well-posedness of the limit system (2.4-2.5)

In this section, we shall prove the local existence of smooth solutions to the initial-boundary value problem of the limit system (2.4-2.5). Let us denote $\rho_0 \stackrel{\text{def}}{=} a_0^2$. We rewrite (2.4-2.5) as

$$
\left\{
\begin{aligned}
&\partial_t\rho_0 + \nabla\cdot(\rho_0\,\nabla\varphi_0) = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,\\
&\partial_t\varphi_0 + \tfrac{1}{2}|\nabla\varphi_0|^2 + (\rho_0 - 1) = 0,\\
&\varphi_0|_{z=0} = 0 \quad \text{and} \quad \rho_0 \to 1 \quad \text{as} \quad |x|\to\infty,\\
&\rho_0|_{t=0} = \big(a^{\mathrm{in}}_{0,0}\big)^2, \quad \varphi|_{t=0} = \varphi^{\mathrm{in}}_{0,0}.
\end{aligned}
\right. \tag{4.1}
$$

By substituting the equivalent form of the second equation in (4.1)

$$
\rho_0 = -\big(\partial_t\varphi_0 + \frac{1}{2}|\nabla\varphi_0|^2 - 1\big) \tag{4.2}
$$

into the first equation of (4.1), we obtain

$$\partial_t^2\varphi_0+\nabla\varphi_0\cdot\nabla\partial_t\varphi_0-\nabla\cdot(\rho_0\,\nabla\,\varphi_0)=0, \tag{4.3}$$

which can also be equivalently written as

$$\partial_t^2\varphi_0-\Delta\varphi_0+\nabla\varphi_0\cdot\nabla\partial_t\varphi_0+\operatorname{div}\big((\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big)=0, \tag{4.4}$$

or

$$\partial_t^2\varphi_0+\nabla\varphi_0\cdot\nabla\partial_t\varphi_0+\partial_t\rho_0=0. \tag{4.5}$$

Let $\varphi_{0,1}^{\rm in}$ be given by (3.1). We implement the wave equation (4.4) with the initial-boundary conditions:

$$\varphi_0|_{t=0}=\varphi_{0,0}^{\rm in},\quad \partial_t\varphi_0|_{t=0}=\varphi_{0,1}^{\rm in}\quad\text{and}\quad \varphi_0|_{z=0}=0. \tag{4.6}$$

Before proceeding, let us first present the following product law in the space W_T^s, the proof of which will be postponed in the Appendix B.

Lemma 4.1. *Let $2\leq s$ and W_T^s be given by Definition 3.1. Then for any $f,g\in W_T^s$, one has*

$$\|fg\|_{W_T^s}\leq C_s\|f\|_{W_T^s}\|g\|_{W_T^s}. \tag{4.7}$$

The main result of this section states as follows:

Theorem 4.1. *Let $4\leq s_0\in\mathbb{N}$ and $(\varphi_{0,0}^{\rm in},\,\varphi_{0,1}^{\rm in})\in H^{s_0}(\mathbb{R}^3_+)\times H^{s_0-1}(\mathbb{R}^3_+)$ which satisfies the compatibility condition $(\mathcal{A}_0)$. We assume that*

$$\|\nabla\varphi_{0,0}^{\rm in}\|_{H^{s_0-1}}+\|\varphi_{0,1}^{\rm in}\|_{H^{s_0-1}}\leq c_0, \tag{4.8}$$

for some c_0 sufficiently small, then there exists a positive constant $\mathcal{C}$ so that for $T=\mathcal{C}c_0^{-1}$, (4.4-4.6) has a unique solution φ_0 on $[0,T]$, which satisfies

$$\|(\partial_t\varphi_0,\nabla\varphi_0)\|_{W_T^{s_0-1}}\leq C\,(\|\nabla\varphi_{0,0}^{\rm in}\|_{H^{s_0-1}}+\|\varphi_{0,1}^{\rm in}\|_{H^{s_0-1}}). \tag{4.9}$$

Proof. It is well-known that the existence of solutions to a nonlinear partial differential equation can be obtained by first constructing the appropriate approximate solutions, and then performing uniform estimates for such approximate solutions, and finally applying a compactness argument. For simplicity, here we just present the *a priori* estimates for sufficiently smooth solutions of (4.4-4.6) on $[0,T^*[$ with T^* being the maximal time of existence.

In what follows, we shall separate the proof into the following steps:

Step 1. $H^1_{\rm tan}$ estimate

Due to $\partial_t\varphi|_{z=0}=0$, by taking the L^2 inner product of (4.3) with $\partial_t\varphi_0$ and using integration by parts, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3_+}(|\partial_t\varphi_0|^2+\rho_0|\nabla\varphi_0|^2)\,dx=\int_{\mathbb{R}^3_+}\big(\partial_t\rho_0|\nabla\varphi_0|^2-2(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)\partial_t\varphi_0\big)\,dx,$$

from which and (4.2), we infer

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3_+}(|\partial_t\varphi_0|^2+\rho_0|\nabla\varphi_0|^2)\,dx\lesssim&\big(\|\partial_t\varphi_0\|_{L^2_+}^2+\|\nabla\varphi_0\|_{L^2_+}^2\big)\\&\times\big(\|\partial_t^2\varphi_0\|_{L^\infty_+}+(1+\|\nabla\varphi_0\|_{L^\infty_+})\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\big).\end{aligned} \tag{4.10}$$

Step 2. $H^2_{\rm tan}$ estimate

Recall that $\mathcal{T}\overset{\rm def}{=}(\partial_t,\,\nabla_{\rm h})$. Applying $\mathcal{T}$ to (4.3) gives

$$\partial_t^2\mathcal{T}\varphi_0-\nabla\cdot(\rho_0\nabla\mathcal{T}\varphi_0)+\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0+\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}\varphi_0-\nabla\cdot(\mathcal{T}\rho_0\nabla\varphi_0)=0. \tag{4.11}$$

Due to $\partial_t \mathcal{T}\varphi|_{z=0}=0$, by taking the L^2 inner product of (4.11) with $\partial_t\mathcal{T}\varphi_0$ and using integration by parts, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}\varphi_0|^2+\rho_0|\nabla\mathcal{T}\varphi_0|^2\big)\,dx-\frac{1}{2}\int_{\mathbb{R}^3_+}\partial_t\rho_0|\nabla\mathcal{T}\varphi_0|^2\,dx\\
&\quad+\int_{\mathbb{R}^3_+}(\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0)\partial_t\mathcal{T}\varphi_0\,dx+\int_{\mathbb{R}^3_+}(\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}\varphi_0)\,\partial_t\mathcal{T}\varphi_0\,dx\\
&\quad+\int_{\mathbb{R}^3_+}\mathcal{T}\rho_0\,\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0\,dx=0.
\end{aligned}
\tag{4.12}
$$

According to (4.2), it is easy to observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+}(\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0)\partial_t\mathcal{T}\varphi_0\,dx+\int_{\mathbb{R}^3_+}\mathcal{T}\rho_0\;\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0\,dx\\
&=-\int_{\mathbb{R}^3_+}(\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)(\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}\varphi_0)\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}(\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)^2\,dx+\int_{\mathbb{R}^3_+}(\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)(\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}\varphi_0)\,dx.
\end{aligned}
$$

Plugging the above estimate into (4.12) and using the equation (4.5) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}\varphi_0|^2+\rho_0|\nabla\mathcal{T}\varphi_0|^2-|\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0|^2\big)\,dx\\
&=\int_{\mathbb{R}^3_+}\big(\partial_t\rho_0|\nabla\mathcal{T}\varphi_0|^2-2(\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}\varphi_0)(\partial_t\mathcal{T}\varphi_0+\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)\big)\,dx,
\end{aligned}
$$

which together with (4.2) ensures that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}\varphi_0|^2+\rho_0|\nabla\mathcal{T}\varphi_0|^2-|\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0|^2\big)\,dx\\
&\quad\lesssim\big(\|\partial_t^2\varphi_0\|_{L^\infty_+}+(1+\|\nabla\varphi_0\|_{L^\infty_+})\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\big)\big(\|\partial_t\mathcal{T}\varphi_0\|^2_{L^2_+}+\|\nabla\mathcal{T}\varphi_0\|^2_{L^2_+}\big).
\end{aligned}
\tag{4.13}
$$

Step 3. High-order tangential derivatives estimate

Let $\ell\in\mathbb{N}$, by applying the operator $\mathcal{T}^\ell$ (with $\mathcal{T}=(\partial_t,\nabla_{\mathrm{h}})$ and $\mathcal{T}^\ell=\partial_t^{\alpha_1}\nabla_{\mathrm{h}}^{\alpha_2}$ for $\alpha_1+|\alpha_2|=\ell\in\mathbb{N}$) to (4.11), we find

$$
\partial_t^2\mathcal{T}^{\ell+1}\varphi_0-\nabla\cdot(\rho_0\nabla\mathcal{T}^{\ell+1}\varphi_0)+\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}^{\ell+1}\varphi_0-\nabla\cdot(\mathcal{T}^{\ell+1}\rho_0\nabla\varphi_0)=g_\ell
\tag{4.14}
$$

with

$$
\begin{aligned}
g_\ell\overset{\mathrm{def}}{=}\nabla\cdot\big([\mathcal{T}^\ell;\rho_0]\nabla\mathcal{T}\varphi_0\big)-[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\partial_t\mathcal{T}\varphi_0&\\
-\mathcal{T}^\ell(\nabla\mathcal{T}\varphi_0\cdot\nabla\partial_t\varphi_0)+\nabla\cdot([\mathcal{T}^\ell;\nabla\varphi_0]\mathcal{T}\rho_0)&.
\end{aligned}
$$

Noticing that $\partial_t\mathcal{T}^{\ell+1}\varphi_0|_{z=0}=0$, by taking the L^2 inner product of (4.14) with $\partial_t\mathcal{T}^{\ell+1}\varphi_0$ and using integration by parts, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}^{\ell+1}\varphi_0|^2+\rho_0|\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\big)\,dx-\frac{1}{2}\int_{\mathbb{R}^3_+}\partial_t\rho_0|\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\,dx\\
&\quad-\int_{\mathbb{R}^3_+}\Delta\varphi_0|\partial_t\mathcal{T}^{\ell+1}\varphi_0|^2\,dx-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)|(\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}^{\ell+1}\varphi_0)\,dx\\
&=\int_{\mathbb{R}^3_+}g_\ell\,|\partial_t\mathcal{T}^{\ell+1}\varphi_0\,dx.
\end{aligned}
\tag{4.15}
$$

By using integration by parts, one has

$$
\begin{aligned}
&-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\nabla\varphi_0\cdot\nabla\mathcal{T}\varphi_0)|(\nabla\varphi_0\cdot\nabla\partial_t\mathcal{T}^{\ell+1}\varphi_0)\,dx\\
&=-\frac12\frac{d}{dt}\int_{\mathbb{R}^3_+}|\nabla\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\,dx+\int_{\mathbb{R}^3_+}\Big((\nabla\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0)(\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0)\\
&\quad+\Big(\nabla([\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0)\cdot\nabla\varphi_0+[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0\Delta\varphi_0\Big)\,\partial_t\mathcal{T}^{\ell+1}\varphi_0\Big)\,dx.
\end{aligned}
$$

Plugging the above equality into (4.15) yields

$$
\frac12\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}^{\ell+1}\varphi_0|^2+\rho_0|\nabla\mathcal{T}^{\ell+1}\varphi_0|^2-|\nabla\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\big)\,dx=\mathfrak{R}_\ell, \tag{4.16}
$$

with

$$
\begin{aligned}
\mathfrak{R}_\ell\overset{\text{def}}{=}&\frac12\int_{\mathbb{R}^3_+}\partial_t\rho_0|\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\,dx+\int_{\mathbb{R}^3_+}\Delta\varphi_0|\partial_t\mathcal{T}^{\ell+1}\varphi_0|^2\,dx\\
&-\int_{\mathbb{R}^3_+}\Big((\nabla\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0)(\nabla\partial_t\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0)\\
&+\Big(g_\ell-\nabla([\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0)\cdot\nabla\varphi_0-[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0\Delta\varphi_0\Big)\,\partial_t\mathcal{T}^{\ell+1}\varphi_0\Big)\,dx,
\end{aligned}
$$

from which, we infer

$$
\begin{aligned}
|\mathfrak{R}_\ell|\lesssim&\big(\|\partial_t\rho_0\|_{L^\infty_+}+\|\nabla\varphi_0\|_{L^\infty_+}\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\big)\|\nabla\mathcal{T}^{\ell+1}\varphi_0\|_{L^2_+}^2+\|\Delta\varphi_0\|_{L^\infty_+}\|\partial_t\mathcal{T}^{\ell+1}\varphi_0\|_{L^2_+}^2\\
&+\big\|\big(g_\ell-\nabla([\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0)\cdot\nabla\varphi_0-[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0\Delta\varphi_0\big)\big\|_{L^2_+}\|\partial_t\mathcal{T}^{\ell+1}\varphi_0\|_{L^2_+}.
\end{aligned}
$$

Recall that for $s\in\mathbb{N}$, $\|\varphi_0(t)\|_{W^s}^2=\sum_{j=0}^s\|\partial_t^j\varphi_0(t)\|_{H^{s-j}}$. Then by virtue of (4.2), and the Sobolev embedding theorem: $H^2(\mathbb{R}^3_+)\hookrightarrow L^\infty(\mathbb{R}^3_+)$, we deduce that

$$
\|\partial_t\rho_0\|_{L^\infty_+}+\|\nabla\varphi_0\|_{L^\infty_+}\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\lesssim\big(1+\|\nabla\varphi_0\|_{L^\infty_+}\big)\|\partial_t\varphi_0\|_{W^3}. \tag{4.17}
$$

Next for $4\le s\in\mathbb{N}$, we claim that

$$
\|fg\|_{H^1}\lesssim\|f\|_{H^1}\|g\|_{H^2}, \tag{4.18}
$$

and

$$
\sum_{\ell=1}^{s-2}\|\nabla([\mathcal{T}^\ell;f]g)\|_{L^2_+}\lesssim\|f\|_{W^{s-1}}\|g\|_{W^{s-2}}. \tag{4.19}
$$

Indeed, it follows from Sobolev embedding theorem that

$$
\begin{aligned}
\|fg\|_{H^1}=&\|fg\|_{L^2_+}+\|g\nabla f\|_{L^2_+}+\|f\nabla g\|_{L^2_+}\\
\le&\big(\|f\|_{L^2_+}+\|\nabla f\|_{L^2_+}\big)\|g\|_{L^\infty_+}+\|f\|_{L^6_+}\|\nabla g\|_{L^3_+}\lesssim\|f(t)\|_{H^1}\|g(t)\|_{H^2},
\end{aligned}
$$

which yields (4.18).

By applying (4.18), we find

$$
\begin{aligned}
\|\nabla\cdot([\mathcal{T}^\ell;f]g)\|_{L^2_+}\lesssim&\sum_{i=0}^{\ell-1}\|\mathcal{T}^{\ell-i}f\mathcal{T}^ig\|_{H^1}\\
\lesssim&\sum_{i=1}^{\ell-1}\|\mathcal{T}^{\ell-i}f\|_{H^2}\|\mathcal{T}^ig\|_{H^1}+\|\mathcal{T}^\ell f\|_{H^1}\|g\|_{H^2}\\
\lesssim&\sum_{i=1}^{\ell-1}\|f\|_{W^{\ell-i+2}}\|g\|_{W^{i+1}}+\|f\|_{W^{\ell+1}}\|g\|_{H^2},
\end{aligned}
$$

which leads to (4.19).

Notice that

$$[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\partial_t\mathcal{T}\varphi_0=\nabla\cdot\big([\mathcal{T}^\ell;\nabla\varphi_0]\partial_t\mathcal{T}\varphi_0\big)-[\mathcal{T}^\ell;\Delta\varphi_0]\partial_t\mathcal{T}\varphi_0,$$

we get, by applying (4.19) and Lemma 4.1, that

$$\begin{aligned}\sum_{\ell=1}^{s_0-2}\|g_\ell\|_{L^2_+}\lesssim\big(\|\partial_t\varphi_0\|_{W^{s_0-1}}&+\|\nabla\varphi_0\|_{W^{s_0-1}}+\|\nabla\varphi_0\|^2_{W^{s_0-1}}\big)\\&\times\big(\|\partial_t\mathcal{T}\varphi_0\|_{W^{s_0-2}}+\|\nabla\mathcal{T}\varphi_0\|_{W^{s_0-2}}\big).\end{aligned}$$

Along the same line, one has

$$\begin{aligned}&\sum_{\ell=1}^{s_0-2}\big\|\big(\nabla([\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0)\cdot\nabla\varphi_0+[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0\Delta\varphi_0\big)\big\|_{L^2_+}\\&\lesssim\|\nabla\varphi_0\|_{L^\infty_+}\|\nabla\varphi_0\|_{W^{s_0-1}}\|\nabla\mathcal{T}\varphi_0\|_{W^{s_0-2}}+\big\|[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\mathcal{T}\varphi_0\|_{L^6_+}\|\Delta\varphi_0\|_{L^3_+}\\&\lesssim\|\nabla\varphi_0\|_{H^2}\|\nabla\varphi_0\|_{W^{s_0-1}}\|\nabla\mathcal{T}\varphi_0\|_{W^{s_0-2}}.\end{aligned}$$

This together with (4.17) ensures that

$$\begin{aligned}\sum_{\ell=1}^{s_0-2}|\mathfrak{R}_\ell|\ \lesssim&\big(\|\partial_t\mathcal{T}\varphi_0\|^2_{W^{s_0-2}}+\|\nabla\mathcal{T}\varphi_0\|^2_{W^{s_0-2}}\big)\\&\times\big(\|\partial_t\varphi_0\|_{W^{s_0-1}}+\|\nabla\varphi_0\|_{W^{s_0-1}}+\|\nabla\varphi_0\|^2_{W^{s_0-1}}\big).\end{aligned}\tag{4.20}$$

Inserting the estimate (4.20) into (4.16) leads to

$$\begin{aligned}&\sum_{\ell=1}^{s_0-2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}^{\ell+1}\varphi_0|^2+\rho_0|\nabla\mathcal{T}^{\ell+1}\varphi_0|^2-|\nabla\varphi_0\cdot\nabla\mathcal{T}^{\ell+1}\varphi_0|^2\big)\,dx\\&\lesssim\big(\|\partial_t\varphi_0\|_{W^{s_0-1}}+\|\nabla\varphi_0\|_{W^{s_0-1}}+\|\nabla\varphi_0\|^2_{W^{s_0-1}}\big)\big(\|\partial_t\mathcal{T}\varphi_0\|^2_{W^{s_0-2}}+\|\nabla\mathcal{T}\varphi_0\|^2_{W^{s_0-2}}\big).\end{aligned}\tag{4.21}$$

Let us define three energy functionals of φ_0 as

$$\begin{aligned}E_s(t)&\stackrel{\text{def}}{=}\|\partial_t\varphi_0(t)\|^2_{W^{s-1}}+\|\nabla\varphi_0(t)\|^2_{W^{s-1}};\\E_{s,\tan}(t)&\stackrel{\text{def}}{=}\sum_{\ell=0}^{s-1}\big(\|\partial_t\mathcal{T}^\ell\varphi_0(t)\|^2_{L^2(\mathbb{R}^3_+)}+\|\nabla\mathcal{T}^\ell\varphi_0(t)\|^2_{L^2(\mathbb{R}^3_+)}\big);\\\widetilde{E}_{s,\tan}(t)&\stackrel{\text{def}}{=}\sum_{\ell=0}^{s-1}\int_{\mathbb{R}^3_+}\big(|\partial_t\mathcal{T}^\ell\varphi_0(t)|^2+\rho_0|\nabla\mathcal{T}^\ell\varphi_0(t)|^2-|\nabla\varphi_0\cdot\nabla\mathcal{T}^\ell\varphi_0|^2\big)\,dx.\end{aligned}\tag{4.22}$$

Then by summing up the estimates, (4.10), (4.13) and (4.21), we achieve

$$\frac{d}{dt}E_{s_0,\tan}(t)\le C\big(1+E_{s_0}(t)\big)E^{\frac{3}{2}}_{s_0}(t).\tag{4.23}$$

For $\delta>0$ being sufficiently small, which will be determined later on, we define

$$T_1^\star\stackrel{\text{def}}{=}\sup\Big\{\ t<T^*:\ E_{s_0}(t)\le\delta\ \ \Big\}.\tag{4.24}$$

Then for $t\le T_1^\star$, we observe from (4.2) that there exits a positive constant C_0 such that

$$C_0^{-1}E_{s_0,\tan}(t)\le\widetilde{E}_{s_0,\tan}(t)\le C_0E_{s_0,\tan}(t),\tag{4.25}$$

provided that δ is sufficiently small in (4.24).

Step 4. Full energy estimates

Let $E_s(t)$, $E_{s,\tan}(t)$ be given by (4.22). We claim that

$$E_\ell(t)\le C_\ell E_{\ell,\tan}(t)\quad\text{for}\quad t\le T_1^\star\quad\text{and}\quad\ell=2,\cdots,s_0,\tag{4.26}$$

provided that δ is sufficiently small in (4.24).

When $\ell = 2$, we have

$$E_2(t) \leq \|\partial_t\varphi_0\|_{H^1}^2 + \|\partial_t^2\varphi_0\|_{L^2_+}^2 + \|\nabla\nabla_{\mathrm{h}}\varphi\|_{L^2_+}^2 + \|\partial_3^2\varphi\|_{L^2_+}^2 + \|\nabla\partial_t\varphi_0\|_{L^2_+}^2. \tag{4.27}$$

Yet in view of (4.4), we have

$$\partial_3^2\varphi_0 = \partial_t^2\varphi_0 - \Delta_{\mathrm{h}}\varphi_0 + \nabla\varphi_0\cdot\nabla\partial_t\varphi_0 + \operatorname{div}\big((\partial_t\varphi_0 + \frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big), \tag{4.28}$$

which implies that

$$\begin{aligned}\|\partial_3^2\varphi_0\|_{L^2_+} \leq & \|\partial_t^2\varphi_0\|_{L^2_+} + \|\Delta_{\mathrm{h}}\varphi_0\|_{L^2_+} + C\big(\|\nabla\varphi_0\|_{L^\infty_+}^2\|\nabla^2\varphi_0\|_{L^2_+} \\ & + (\|\nabla\varphi_0\|_{L^\infty_+} + \|\Delta\varphi_0\|_{L^3_+})\|\nabla\partial_t\varphi_0\|_{L^2_+}\big) \\ \leq & C\big((1+\|\nabla\varphi_0\|_{H^2}^2)E_{2,\tan}^{\frac12}(t) + \|\nabla\varphi_0\|_{H^2}^2\|\partial_3^2\varphi_0\|_{L^2_+}\big).\end{aligned}$$

So that as long as δ is sufficiently small in (4.24), we obtain

$$\|\partial_3^2\varphi_0\|_{L^2_+} \leq CE_{2,\tan}^{\frac12}(t).$$

Inserting the above estimate into (4.27) gives rise to

$$E_2(t) \leq CE_{2,\tan}(t).$$

This proves (4.26) for $\ell = 2$.

Now we assume that (4.26) holds for $\ell = k$, we are going to prove that (4.26) holds for $\ell = k+1 \leq s_0$. We first notice by the definition that

$$E_{k+1}(t) \leq \sum_{j=0}^{k}\big(\|\partial_t^{j+1}\varphi_0(t)\|_{H^{k-j}}^2 + \|\nabla\partial_t^j\varphi_0(t)\|_{H^{k-j}}^2\big).$$

• When $j = k$.

We observe from (4.22) that

$$\|\partial_t^{k+1}\varphi_0(t)\|_{L^2_+}^2 + \|\nabla\partial_t^k\varphi_0(t)\|_{L^2_+}^2 \leq E_{k+1,\tan}(t).$$

• When $j = k-1$.

It follows from (4.22) that

$$\|\partial_t^k\varphi_0\|_{H^1}^2 = \|\partial_t^k\varphi_0\|_{L^2_+}^2 + \|\nabla\partial_t^k\varphi_0\|_{L^2_+}^2 \leq E_{k+1,\tan}(t).$$

Whereas notice that

$$\begin{aligned}\|\nabla\partial_t^{k-1}\varphi_0(t)\|_{H^1}^2 = & \|\nabla\partial_t^{k-1}\varphi_0(t)\|_{L^2_+}^2 + \|\nabla^2\partial_t^{k-1}\varphi_0(t)\|_{L^2_+}^2 \\ \leq & \|\partial_t\varphi_0(t)\|_{W^{k-1}}^2 + \|\nabla\nabla_{\mathrm{h}}\partial_t^{k-1}\varphi_0(t)\|_{L^2_+}^2 + \|\partial_3^2\partial_t^{k-1}\varphi_0(t)\|_{L^2_+}^2.\end{aligned}$$

We deduce from (4.28) that

$$\begin{aligned}\|\partial_3^2\partial_t^{k-1}\varphi_0\|_{L^2_+} \leq & \|\partial_t^{k+1}\varphi_0\|_{L^2_+} + \|\Delta_{\mathrm{h}}\partial_t^{k-1}\varphi_0\|_{L^2_+} + \|\partial_t^{k-1}(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)\|_{L^2_+} \\ & + \big\|\partial_t^{k-1}\operatorname{div}\big((\partial_t\varphi_0 + \frac12|\nabla\varphi_0|^2)\nabla\varphi_0\big)\big\|_{L^2_+}.\end{aligned}$$

Yet for $j \in [0, k-2]$, it follows from the law of product, Lemma 4.1, that

$$\begin{aligned}\|\partial_t^{j+1}(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)\|_{H^{k-j-2}} \leq & \|\nabla\varphi_0\cdot\nabla\partial_t\varphi_0\|_{W^{k-1}} \\ \leq & C\|\nabla\varphi_0\|_{W^{k-1}}\|\nabla\partial_t\varphi_0\|_{W^{k-1}},\end{aligned} \tag{4.29}$$

and

$$\|\partial_t^{j+1}\operatorname{div}\big((\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big)\|_{H^{k-j-2}} \le \big\|(\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big\|_{W^k} \le C\big(\|\partial_t\varphi_0\|_{W^k}+\|\nabla\varphi_0\|_{W^k}^2\big)\|\nabla\varphi_0\|_{W^k}. \tag{4.30}$$

Therefore, we obtain

$$\|\partial_3^2\partial_t^{k-1}\varphi_0\|_{L^2_+} \le C\left(E_{k+1,\tan}^{\frac{1}{2}}(t)+(1+\|\nabla\varphi_0\|_{W^k})\|\nabla\varphi_0\|_{W^k}E_{k+1}^{\frac{1}{2}}(t)\right).$$

$\|\nabla\partial_t^{k-1}\varphi_0(t)\|_{H^1}$ shares the same estimate.

As a result, it comes out

$$\|\partial_t^k\varphi_0\|_{H^1}^2+\|\nabla\partial_t^{k-1}\varphi_0(t)\|_{H^1}^2 \le C\Big(E_{k+1,\tan}(t)+(1+\|\nabla\varphi_0\|_{W^k}^2)\|\nabla\varphi_0\|_{W^k}^2E_{k+1}(t)\Big).$$

• <u>When $k-j\ge 2$.</u>

We have

$$\begin{aligned}\|\partial_t^{j+1}\varphi_0\|_{H^{k-j}}^2 =&\|\partial_t^{j+1}\varphi_0\|_{H^{k-j-1}}^2+\|\nabla^2\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}}^2\\ \le&\|\partial_t\varphi_0\|_{W^{k-1}}^2+\|\nabla\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}}^2+\|\partial_3^2\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}}^2.\end{aligned}$$

By virtue of (4.28), we find

$$\begin{aligned}\|\partial_3^2\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}} \le& \|\partial_t^{j+3}\varphi_0\|_{H^{k-j-2}}+\|\partial_t^{j+1}\Delta_{\mathrm h}\varphi_0\|_{H^{k-j-2}}\\ &+\|\partial_t^{j+1}(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)\|_{H^{k-j-2}}+\big\|\partial_t^{j+1}\operatorname{div}\big((\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big)\big\|_{H^{k-j-2}},\end{aligned}$$

which together with (4.29) and (4.30) ensures that

$$\begin{aligned}\|\partial_t^{j+1}\varphi_0\|_{H^{k-j}} \le&\|\partial_t\varphi_0\|_{W^{k-1}}+\|\nabla\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}}+\|\partial_t^{j+3}\varphi_0\|_{H^{k-j-2}}\\ &+\|\partial_t^{j+1}\Delta_{\mathrm h}\varphi_0\|_{H^{k-j-2}}+C\big(1+\|\nabla\varphi_0\|_{W^k}\big)\|\nabla\varphi_0\|_{W^k}E_{k+1}^{\frac{1}{2}}(t).\end{aligned} \tag{4.31}$$

In the case when $k-j\ge 3$, we have

$$\begin{aligned}\|\nabla\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-2}} =&\|\nabla\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}}+\|\nabla^2\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}}\\ \le&\|\partial_t\varphi_0\|_{W^{k-1}}+\|\nabla\nabla_{\mathrm h}^2\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}}+\|\partial_3^2\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}}.\end{aligned}$$

Yet it follows from (4.28) that

$$\begin{aligned}\|\partial_3^2\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}} \le& \|\nabla_{\mathrm h}\partial_t^{j+3}\varphi_0\|_{H^{k-j-3}}+\|\partial_t^{j+1}\nabla_{\mathrm h}^3\varphi_0\|_{H^{k-j-3}}\\ &+\|\nabla_{\mathrm h}\partial_t^{j+1}(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)\|_{H^{k-j-3}}+\big\|\nabla_{\mathrm h}\partial_t^{j+1}\operatorname{div}\big((\partial_t\varphi_0+\frac{1}{2}|\nabla\varphi_0|^2)\nabla\varphi_0\big)\big\|_{H^{k-j-3}},\end{aligned}$$

from which and (4.29), (4.30), we infer

$$\begin{aligned}\|\partial_3^2\nabla_{\mathrm h}\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}} \le&\|\nabla_{\mathrm h}\partial_t^{j+3}\varphi_0\|_{H^{k-j-3}}+\|\partial_t^{j+1}\nabla_{\mathrm h}^3\varphi_0\|_{H^{k-j-3}}\\ &+C\big(1+\|\nabla\varphi_0\|_{W^k}\big)\|\nabla\varphi_0\|_{W^k}E_{k+1}^{\frac{1}{2}}(t).\end{aligned}$$

Inserting the above estimate into (4.31) gives rise to

$$\begin{aligned}\|\partial_t^{j+1}\varphi_0\|_{H^{k-j}} \le&2\|\partial_t\varphi_0\|_{W^{k-1}}+\|\partial_t^{j+3}\varphi_0\|_{H^{k-j-2}}+\|\partial_t^{j+1}\Delta_{\mathrm h}\varphi_0\|_{H^{k-j-2}}\\ &+\|\nabla\nabla_{\mathrm h}^2\partial_t^{j+1}\varphi_0\|_{H^{k-j-3}}+\|\nabla_{\mathrm h}\partial_t^{j+3}\varphi_0\|_{H^{k-j-3}}\\ &+\|\partial_t^{j+1}\nabla_{\mathrm h}^3\varphi_0\|_{H^{k-j-3}}+C\big(1+\|\nabla\varphi_0\|_{W^k}\big)\|\nabla\varphi_0\|_{W^k}E_{k+1}^{\frac{1}{2}}(t).\end{aligned}$$

By finite steps of iteration and using the inductive assumption for $\ell=k$, we deduce that

$$\|\partial_t^{j+1}\varphi_0\|_{H^{k-j}}^2 \le C_kE_{k+1,\tan}(t)+C\big(1+\|\nabla\varphi_0\|_{W^k}^2\big)\|\nabla\varphi_0\|_{W^k}^2E_{k+1}(t). \tag{4.32}$$

The same estimate holds for $\|\nabla\partial_t^j\varphi_0(t)\|_{H^{k-j}}$.

Therefore we conclude that

$$E_{k+1}(t) \leq C_k\Big(E_{k+1,\tan}(t) + \big(1+\|\nabla\varphi_0\|_{W^k}^2\big)\|\nabla\varphi_0\|_{W^k}^2 E_{k+1}(t)\Big).$$

Then in view of (4.24), as long as δ is small enough, we deduce (4.26) for $\ell = k+1$.

Now we are in a position to complete the proof of Theorem 4.1.

Thanks to (4.23) and (4.26), we obtain for $t \leq T_1^\star$ that

$$\frac{d}{dt}\widetilde{E}_{s_0,\tan}(t) \leq C_{s_0}\sqrt{\delta}E_{s_0\tan}(t) \leq C_{s_0}C_0\sqrt{\delta}\widetilde{E}_{s_0,\tan}(t),$$

where we used (4.25) in the last step. Applying Gronwall's inequality gives rise to

$$\begin{aligned}\widetilde{E}_{s_0,\tan}(t) \leq & \widetilde{E}_{s_0,\tan}(0)\exp\Big(C_{s_0}C_0\sqrt{\delta}t\Big)\\ \leq & C_0 E_{s_0,\tan}(0)\exp\Big(C_{s_0}C_0\sqrt{\delta}t\Big).\end{aligned} \tag{4.33}$$

On the other hand, we deduce from (4.4) and (4.8) that

$$E_{s_0,\tan}(0) \leq C_{s_0}(\|\nabla\varphi_{0,0}^{\mathrm{in}}\|_{H^{s_0-1}} + \|\varphi_{0,1}^{\mathrm{in}}\|_{H^{s_0-1}})^2 \leq C_{s_0}c_0^2,$$

which together with (4.33) ensures that

$$\widetilde{E}_{s_0,\tan}(t) \leq C_0C_{s_0}c_0^2\exp\Big(C_{s_0}C_0\sqrt{\delta}t\Big) \quad \text{for} \quad t \leq T_1^\star. \tag{4.34}$$

Let us denote $\bar{T} \stackrel{\text{def}}{=} \min\big(T_1^\star, (C_0C_{s_0}\sqrt{\delta})^{-1}\big)$. If we assume by a contradict argument that $T_1^\star < (C_0C_{s_0}\sqrt{\delta})^{-1}$, then for $t \leq \bar{T} = T_1^\star$, we deduce from (4.34) that

$$\widetilde{E}_{s_0,\tan}(t) \leq C_0C_{s_0}c_0^2 e,$$

from which, and (4.26), we infer

$$E_{s_0}(t) \leq C_{s_0}E_{s_0,\tan}(t) \leq C_{s_0}C_0\widetilde{E}_{s_0,\tan}(t) \leq C_0^2C_{s_0}^2\, e\, c_0^2. \tag{4.35}$$

Then as long as we take the positive c_0 to be so small that $C_0^2C_{s_0}^2\, e\, c_0^2 = \frac{\delta}{2}$, we find

$$E_{s_0}(t) \leq \frac{\delta}{2} \quad \text{for } t \leq \bar{T} = T_1^\star.$$

This contradicts with the definition of $T_1^\star$ given by (4.24). This in turn shows that $T_1^\star \geq (C_0C_{s_0}\sqrt{\delta})^{-1} = \big(\sqrt{2e}\, C_0^2C_{s_0}^2\, c_0\big)^{-1}$. This together with (4.35) completes the proof of Theorem 4.1. $\square$

Remark 4.1. *We remark that it is crucial to apply $\mathcal{T}^\ell$ to (4.11), and then perform the energy estimate for the equation (4.14). Otherwise, let $\ell \in \mathbb{N}$, by applying the operator $\mathcal{T}^\ell$ to (4.3), we find*

$$\partial_t^2\mathcal{T}^\ell\varphi_0 - \nabla\cdot\big(\rho_0\nabla\mathcal{T}^\ell\varphi_0\big) + \mathcal{T}^\ell(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0) = \nabla\cdot\big([\mathcal{T}^\ell;\rho_0]\nabla\varphi_0\big). \tag{4.36}$$

Due to $\partial_t\mathcal{T}^\ell\varphi_0|_{z=0} = 0$, by taking the L^2 inner product of (4.36) with $\partial_t\mathcal{T}^\ell\varphi_0$ and using integration by parts, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big((\partial_t\mathcal{T}^\ell\varphi_0)^2 + \rho_0|\nabla\mathcal{T}^\ell\varphi_0|^2\big)\,dx = \frac{1}{2}\int_{\mathbb{R}^3_+}\partial_t\rho_0|\nabla\mathcal{T}^\ell\varphi_0|^2\,dx\\ -\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)|\partial_t\mathcal{T}^\ell\varphi_0\,dx + \int_{\mathbb{R}^3_+}\nabla\cdot\big([\mathcal{T}^\ell;\rho_0]\nabla\varphi_0\big)|\partial_t\mathcal{T}^\ell\varphi_0\,dx.\end{aligned} \tag{4.37}$$

It is easy to observe that

$$
\begin{aligned}
-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\nabla\varphi_0\cdot\nabla\partial_t\varphi_0)|\partial_t\mathcal{T}^\ell\varphi_0\,dx =&\frac12\int_{\mathbb{R}^3_+}\Delta\varphi_0(\partial_t\mathcal{T}^\ell\varphi_0)^2\,dx\\
&-\int_{\mathbb{R}^3_+}[\mathcal{T}^\ell;\nabla\varphi_0]\nabla\partial_t\varphi_0|\partial_t\mathcal{T}^\ell\varphi_0\,dx.
\end{aligned}
$$

Plugging the above equality into (4.37) *and summing up the resulting inequalities for ℓ varying from* 1 *to* s_0-1 *yields*

$$
\begin{aligned}
&\sum_{\ell=1}^{s_0-1}\frac{d}{dt}\int_{\mathbb{R}^3_+}\big(|\mathcal{T}^\ell\partial_t\varphi_0|^2\;+\rho_0|\nabla\mathcal{T}^\ell\varphi_0|^2\big)\,dx\\
&\lesssim\sum_{\ell=1}^{s_0-1}\Big(\|\partial_t\rho_0\|_{L^\infty_+}\|\nabla\mathcal{T}^\ell\varphi_0\|_{L^2_+}^2+\|\Delta\varphi_0\|_{L^\infty_+}\|\partial_t\mathcal{T}^\ell\varphi_0\|_{L^2_+}^2\\
&\qquad+\big(\|[\mathcal{T}^\ell;\nabla\varphi_0]\nabla\partial_t\varphi_0)\|_{L^2_+}+\|\nabla\cdot([\mathcal{T}^\ell;\rho_0]\nabla\varphi_0)\|_{L^2_+}\big)\|\partial_t\mathcal{T}^\ell\varphi_0\|_{L^2_+}\Big).
\end{aligned}\tag{4.38}
$$

Applying (4.19) *gives*

$$
\begin{aligned}
\sum_{\ell=1}^{s_0-1}\|\nabla\cdot([\mathcal{T}^\ell;\rho_0]\nabla\varphi_0)\|_{L^2_+}\lesssim&\|\rho_0-1\|_{W^{s_0}}\|\nabla\varphi_0\|_{W^{s_0-1}}\\
\lesssim&\big(\|\partial_t\varphi_0\|_{W^{s_0}}+\|\nabla\varphi_0\|_{W^{s_0}}^2\big)\|\nabla\varphi_0\|_{W^{s_0-1}},
\end{aligned}
$$

which make us impossible to close the estimate in (4.38).

Now let us present the proof of Proposition 3.1.

Proof of Proposition 3.1. We first deduce from (3.1) and (3.3) that (4.8) holds as long as c is small enough in (3.3). Then it follows from Theorem 4.1 that (4.4-4.6) has a unique solution φ_0 on $[0,T_0]$ with $T_0=\mathcal{C}c_0^{-1}$ which satisfies (4.9) when we take $c_0=c$ in (4.8). Moreover, we deduce from (4.2) and Theorem 4.1 that

$$
1-\rho_0=\partial_t\varphi_0+\frac12|\nabla\varphi_0|^2\in W^{s_0-1}_{T_0},
$$

and

$$
\begin{aligned}
&\rho_0(t,x)-1\to0\quad\text{as}\quad|x|\to\infty\quad\text{and}\\
&\|(1-\rho_0)\|_{W^{s_0-1}_{T_0}}\le C_{s_0}\big(1+\|\nabla\varphi_0\|_{W^{s_0-1}_{T_0}}\big)E_{s_0}^{\frac12}(t)\le C_{s_0}\|(a^{\rm in}_{0,0}-1,\,\nabla\varphi^{\rm in}_{0,0})\|_{H^{s_0-1}}.
\end{aligned}\tag{4.39}
$$

Let us define $a_0\overset{\rm def}{=}\sqrt{\rho_0}$. Then we deduce from (4.39) that

$$
\begin{aligned}
&a_0(t,x)-1=\frac{\rho_0-1}{\sqrt{\rho_0}+1}\to0\quad\text{as}\quad|x|\to\infty\quad\text{and}\\
&\|(1-a_0)\|_{W^{s_0-1}_{T_0}}\le C_{s_0}\|(a^{\rm in}_{0,0}-1,\,\nabla\varphi^{\rm in}_{0,0})\|_{H^{s_0-1}}.
\end{aligned}
$$

It is easy to observe that thus obtained (a_0,φ_0) is indeed the unique solution of (2.4-2.5). Moreover there holds (3.4). This completes the proof of Proposition 3.1. $\square$

5. Solvability of the boundary layer equations (2.18-2.19)

The goal of this section is to prove the existence of smooth solutions to the boundary layer equations (2.18-2.19), namely the proof of Proposition 3.2.

Proof of Proposition 3.2. Once again, we shall only present the *a priori* estimates. In view of (2.18), we write

$$\frac{1}{2}(A_0+\overline{a}_0)\partial_Z(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})+\partial_Z(A_0+\overline{a}_0)(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})=0.$$

Multiplying the above equation by $A_0+\overline{a}_0$ yields

$$\partial_Z\left((A_0+\overline{a}_0)^2(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})\right)=0,$$

which together with the boundary conditions $A_0|_{Z=+\infty}=0=\partial_Z\Phi_1|_{Z=+\infty}$ ensures that

$$(A_0+\overline{a}_0)^2(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})=(\overline{a}_0)^2\overline{\partial_z\varphi_0}. \tag{5.1}$$

On the other hand, we deduce from (2.19) that

$$\begin{aligned}\frac{1}{2}\partial_Z^2A_0=&\frac{1}{2}(A_0+\overline{a}_0)(\partial_Z\Phi_1+\overline{\partial_z\varphi_0})^2\\&-\frac{1}{2}(A_0+\overline{a}_0)(\overline{\partial_z\varphi_0})^2+A_0(A_0+\overline{a}_0)(A_0+2\overline{a}_0).\end{aligned}$$

Inserting (5.1) into the above equation leads to

$$\frac{1}{2}\partial_Z^2A_0=\frac{1}{2}(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2(A_0+\overline{a}_0)^{-3}-\frac{1}{2}(\overline{\partial_z\varphi_0})^2(A_0+\overline{a}_0)+A_0(A_0+\overline{a}_0)(A_0+2\overline{a}_0). \tag{5.2}$$

Let us denote

$$\widetilde{A_0}\overset{\text{def}}{=}A_0+\overline{a}_0\quad\text{and}\quad q_0\overset{\text{def}}{=}\frac{d\widetilde{A_0}}{dZ}. \tag{5.3}$$

Then under the assumption that $A_0+\overline{a}_0>0$, (which we shall justify below), one has

$$\partial_Z^2A_0=q_0\partial_{\widetilde{A_0}}q_0,$$

and it follows from (5.2) that

$$\frac{1}{4}\frac{d\,q_0^2}{d\,\widetilde{A_0}}=\frac{1}{2}(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2\widetilde{A_0}^{-3}-\frac{1}{2}(\overline{\partial_z\varphi_0})^2\widetilde{A_0}+\widetilde{A_0}\big(\widetilde{A_0}^2-(\overline{a}_0)^2\big),$$

from which, we infer

$$q_0^2=-(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2\widetilde{A_0}^{-2}-(\overline{\partial_z\varphi_0})^2\widetilde{A_0}^2+\widetilde{A_0}^4-2(\overline{a}_0)^2\widetilde{A_0}^2+C_1(t,y). \tag{5.4}$$

Thanks to the conditions $\frac{dA_0}{dZ}|_{Z=+\infty}=A_0|_{Z=+\infty}=0$, one gets

$$0=-(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2(\overline{a}_0)^{-2}-\big(2(\overline{a}_0)^2+(\overline{\partial_z\varphi_0})^2\big)(\overline{a}_0)^2+(\overline{a}_0)^4+C_1(t,y),$$

which gives

$$C_1(t,y)=(\overline{a}_0)^2\left((\overline{a}_0)^2+2(\overline{\partial_z\varphi_0})^2\right).$$

By inserting the above equality into (5.4) and multiplying the resulting equality by $\widetilde{A_0}^2$, we find

$$\begin{aligned}\left(\widetilde{A_0}\frac{d\widetilde{A_0}}{dZ}\right)^2=&\widetilde{A_0}^6-\big(2(\overline{a}_0)^2+(\overline{\partial_z\varphi_0})^2\big)\widetilde{A_0}^4\\&+(\overline{a}_0)^2\big((\overline{a}_0)^2+2(\overline{\partial_z\varphi_0})^2\big)\widetilde{A_0}^2-(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2,\end{aligned}$$

that is

$$\frac{1}{4}\left(\frac{d\widetilde{A_0}^2}{dZ}\right)^2=\big(\widetilde{A_0}^2-(\overline{a}_0)^2\big)^2\big(\widetilde{A_0}^2-(\overline{\partial_z\varphi_0})^2\big). \tag{5.5}$$

Notice that according to Proposition 3.1, we have

$$(\overline{a}_0)^2-(\overline{\partial_z\varphi_0})^2\geq\frac{1}{4}$$

as long as c is small enough in (3.3).

Let us denote

$$h_0\stackrel{\text{def}}{=}\big((\overline{a}_0)^2-(\overline{\partial_z\varphi_0})^2\big)^{\frac12}\geq\frac12\quad\text{and}\quad B_0\stackrel{\text{def}}{=}\Big(\widetilde{A_0}^2-(\overline{\partial_z\varphi_0})^2\Big)^{\frac12}\tag{5.6}$$

in case of $\widetilde{A_0}^2-(\overline{\partial_z\varphi_0})^2>0$, which we shall justify later on.

By virtue of (5.5) and (5.6), we write

$$\frac14\left(\frac{dB_0^2}{dZ}\right)^2=\big(B_0^2-h_0^2\big)^2B_0^2,$$

that is,

$$\frac{dB_0}{dZ}=\pm\big(B_0^2-h_0^2\big),\tag{5.7}$$

from which, we infer

$$\frac{B_0-h_0}{B_0+h_0}=C_2(t,y)e^{\pm2h_0Z}.$$

Since B_0-h_0 is rapidly decaying to zero as $Z\to+\infty$, we have

$$\frac{B_0-h_0}{B_0+h_0}=C_2(t,y)e^{-2h_0Z},$$

which gives

$$B_0=h_0\frac{1+C_2(t,y)e^{-2h_0Z}}{1-C_2(t,y)e^{-2h_0Z}}.\tag{5.8}$$

While according to the boundary condition (2.20), one has

$$B_0^2|_{Z=0}=(A_0+\overline{a}_0)^2|_{Z=0}-(\overline{\partial_z\varphi_0})^2=1-(\overline{\partial_z\varphi_0})^2,$$

which together with (5.8) ensures that

$$\sqrt{1-(\overline{\partial_z\varphi_0})^2}=h_0\frac{1+C_2(t,y)}{1-C_2(t,y)}.$$

As a result, it comes out

$$\begin{aligned}C_2(t,y)=&\frac{\sqrt{1-(\overline{\partial_z\varphi_0})^2}-h_0}{\sqrt{1-(\overline{\partial_z\varphi_0})^2}+h_0}\\=&\frac{1-(\overline{a}_0)^2}{\big(\sqrt{1-(\overline{\partial_z\varphi_0})^2}+h_0\big)^2},\end{aligned}$$

from which, Proposition 3.1 and trace theorem, we deduce that

$$\|C_2(\cdot)\|_{W^{s_0-\frac32}_{T_0}(\mathbb{R}^2)}\leq C\|(a_{0,0}^{\rm in}-1,\,\nabla\varphi_{0,0}^{\rm in})\|_{H^{s_0-1}}\leq Cc.\tag{5.9}$$

Whereas it follows from (5.6) and (5.8) that

$$(A_0+\overline{a}_0)^2-(\overline{\partial_z\varphi_0})^2=h_0^2\Big(\frac{1+C_2e^{-2h_0Z}}{1-C_2e^{-2h_0Z}}\Big)^2,$$

which implies that

$$\begin{aligned}A_0 &= -\overline{a}_0 + \Big(\frac{(\overline{a}_0)^2(1+C_2^2e^{-4h_0Z}) + 2C_2(h_0^2 - (\overline{\partial_z\varphi_0})^2)e^{-2h_0Z}}{(1-C_2(t,y)e^{-2h_0Z})^2}\Big)^{\frac12}\\ &= 4C_2h_0^2e^{-2h_0Z}\Big(\overline{a}_0 + \Big(\frac{(\overline{a}_0)^2(1+C_2^2e^{-4h_0Z}) + 2C_2(h_0^2 - (\overline{\partial_z\varphi_0})^2)e^{-2h_0Z}}{(1-C_2e^{-2h_0Z})^2}\Big)^{\frac12}\Big)^{-1}.\end{aligned}$$

This together with (5.6) and (5.9) in particular shows that $A_0 \in \mathcal{W}^{s_0-\frac32}_{1,T_0}$, and there exists some positive constant c_1 such that

$$\|A_0\|_{\mathcal{W}^{s_0-\frac32}_{1,T_0}} \leq C\|(a^{\rm in}_{0,0}-1,\, \nabla\varphi^{\rm in}_{0,0})\|_{H^{s_0-1}} \quad \text{and} \quad A_0 + \overline{a}_0 \geq c_1 > 0. \tag{5.10}$$

Then we rigorously justify that $B_0 > 0$ for $Z \in \mathbb{R}^+$ as long as c is small enough in (3.3).

With such A_0, in view of (5.1), we write

$$\begin{cases} \partial_Z\Phi_1 = (A_0+\overline{a}_0)^{-2}(\overline{a}_0)^2\overline{\partial_z\varphi_0} - \overline{\partial_z\varphi_0},\\ \Phi_1|_{Z=+\infty} = 0.\end{cases}$$

It is easy to observe from the above equation that

$$\Phi_1(Z) = -\overline{\partial_z\varphi_0}\int_Z^\infty \frac{A_0(A_0+2\overline{a}_0)}{(A_0+\overline{a}_0)^2}\,dZ.$$

which together with (5.10) shows that

$$\|\Phi_1\|_{\mathcal{W}^{s_0-\frac32}_{1,T_0}} \leq C\|(a^{\rm in}_{0,0}-1,\, \nabla\varphi^{\rm in}_{0,0})\|^2_{H^{s_0-1}}. \tag{5.11}$$

This ends the proof of Proposition 3.2. □

6. The existence of solutions to a linear wave equation

The goal of this section is to present the proof of Theorem 3.1. Let $\chi(\tau) \in C^\infty_c(\mathbb{R})$ with $\chi(\tau) = 1$ in a neighborhood of 0. We denote

$$G \overset{\rm def}{=} \chi\Big(z(1+|D_{\rm h}|^2)^{\frac12}\Big)g(t,\cdot) \quad \text{and} \quad \varphi = \phi + G. \tag{6.1}$$

Then one has $G \in W^{s+1}_T(\mathbb{R}^3)$, and ϕ verifies

$$\begin{cases} P(\varphi_0, D)\phi = -P(\varphi_0,D)G + f \overset{\rm def}{=} F \in W^{s-1}_T(\mathbb{R}^3_+),\\ \varphi|_{z=0} = 0,\\ \phi|_{t=0} = \varphi^{\rm in}_{,0} - G|_{t=0} \overset{\rm def}{=} \phi^{\rm in}_0, \quad \text{and} \quad \partial_t\varphi|_{t=0} = \varphi^{\rm in}_{,1} - \partial_t G|_{t=0} \overset{\rm def}{=} \phi^{\rm in}_1.\end{cases} \tag{6.2}$$

And the proof of Theorem 3.1 is reduced to the following one:

Theorem 6.1. *Let $T \leq T_0$ and $4 \leq$ be an integer. Let $F \in W^{s-1}_T$ and $(\phi^{\rm in}_0, \phi^{\rm in}_1)$ satisfy $\nabla\phi^{\rm in}_0, \phi^{\rm in}_1 \in H^{s-1}$ and the compatibility condition that $\partial_t^\ell\phi(0,y,0) = 0$ for $\ell = 0, \cdots, s-1$. Then under the assumptions of Proposition 3.1,* (6.2) *has a unique solution ϕ on $[0,T]$, which satisfies*

$$\|(\partial_t\phi, \nabla\phi)\|_{W^{s-1}_T} \leq C\big(\|g\|_{W_T^{s+\frac12}(\mathbb{R}^2)} + \|(\nabla\phi^{\rm in}_0,\, \phi^{\rm in}_1)\|_{H^{s-1}} + \|F\|_{W^{s-1}_T}\big). \tag{6.3}$$

In what follows, we shall always denote $E_s(\phi)$, $E_{s,\tan}(\phi)$, and $\widetilde{E}_{s,\tan}(\phi)$ to be the energy functionals determined by (4.22).

Let us separate the proof of Theorem 6.1 into the following lemmas:

Lemma 6.1. *Let ϕ be a smooth enough solution of* (6.2) *on* $[0,T]$*. Then for $t \le T$, one has*

$$\frac{d}{dt}\int_{\mathbb{R}^3_+}\left((\partial_t\phi)^2+\rho_0|\nabla\phi|^2-(\nabla\varphi_0\cdot\nabla\phi)^2\right)\,dx \le C\left(1+E_4(\varphi_0)\right)E_1(\phi(t))+\|F\|^2_{L^2_+}. \tag{6.4}$$

Proof. By taking L^2 inner product of the ϕ equation of (6.2) with $\partial_t\phi$ and using integration by parts, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_+}\left((\partial_t\phi)^2+\rho_0|\nabla\phi|^2-(\nabla\varphi_0\cdot\nabla\phi)^2\right)\,dx\\
&=\int_{\mathbb{R}^3_+}\left(\frac{1}{2}\partial_t\rho_0|\nabla\phi|^2-(\nabla\varphi_0\cdot\nabla\phi)(\nabla\partial_t\varphi_0\cdot\nabla\phi)+(\nabla\partial_t\varphi_0\cdot\nabla\phi)\partial_t\phi+F\partial_t\phi\right)\,dx\\
&\le C\left(1+\|\partial_t^2\varphi_0\|_{L^\infty_+}+\big(1+\|\nabla\varphi_0\|_{L^\infty_+}\big)\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\right)\big(\|\partial_t\phi\|^2_{L^2_+}+\|\nabla\phi\|^2_{L^2_+}\big)+\|F\|^2_{L^2_+},
\end{aligned} \tag{6.5}$$

from which, and (4.22), we deduce (6.4). □

Lemma 6.2 (High-order tangential derivatives estimates)**.** *Let ϕ be a smooth enough solution of* (6.2) *on* $[0,T]$*. Then for $t\le T$, one has*

$$\frac{d}{dt}\widetilde{E}_{s,tan}(\phi(t))\lesssim\big(1+E_{s+1}(\varphi_0(t))\big)E_s(\phi(t))+\sum_{\ell=0}^{s-1}\|\mathcal{T}^\ell F\|^2_{L^2_+}. \tag{6.6}$$

Proof. Let $\ell\le s-1$ be an integer. By applying the operator $\mathcal{T}^\ell$ (with $\mathcal{T}=(\partial_t,\nabla_{\rm h})$ and $\mathcal{T}^\ell=\partial_t^{\alpha_1}\nabla_{\rm h}^{\alpha_2}$ for $\alpha_1+|\alpha_2|=\ell\in\mathbb{N}$) to the ϕ equation in (6.2), we find

$$P(\varphi_0,D)\mathcal{T}^\ell\phi=-[\mathcal{T}^\ell;P(\varphi_0,D)]\phi+\mathcal{T}^\ell F, \tag{6.7}$$

where

$$\begin{aligned}
[\mathcal{T}^\ell;P(\varphi_0,D)]\phi=&-\operatorname{div}\big([\mathcal{T}^\ell;\rho_0]\nabla\phi\big)+2[\mathcal{T}^\ell;\nabla\varphi_0]\cdot\nabla\partial_t\phi\\
&+\sum_{k=1}^{3}\operatorname{div}\big([\mathcal{T}^\ell;\partial_k\varphi_0\nabla\varphi_0]\partial_k\phi\big)+[\mathcal{T}^\ell;\nabla\partial_t\varphi_0]\nabla\phi+[\mathcal{T}^\ell,\Delta\varphi_0]\partial_t\phi.
\end{aligned}$$

Due to $\partial_t\mathcal{T}^\ell\phi|_{z=0}=0$, by taking the L^2 inner product of (6.7) with $\partial_t\mathcal{T}^\ell\phi$, we deduce from (6.5) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3_+}\left((\partial_t\mathcal{T}^\ell\phi)^2+\rho_0|\nabla\mathcal{T}^\ell\phi|^2-(\nabla\varphi_0\cdot\nabla\mathcal{T}^\ell\phi)^2\right)\,dx\\
&\lesssim\big(\|\partial_t\rho_0\|_{L^\infty_+}+\|\nabla\varphi_0\|_{L^\infty_+}\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\big)\|\nabla\mathcal{T}^\ell\phi\|^2_{L^2_+}\\
&\quad+\big(\|\nabla\partial_t\varphi_0\|_{L^\infty_+}\|\nabla\mathcal{T}^\ell\phi\|_{L^2_+}+\|[\mathcal{T}^\ell;P(\varphi_0,D)]\phi\|_{L^2_+}+\|\mathcal{T}^\ell F\|_{L^2_+}\big)\|\partial_t\mathcal{T}^\ell\phi\|_{L^2_+}.
\end{aligned} \tag{6.8}$$

It follows from (4.19) that

$$\begin{aligned}
&\sum_{\ell=1}^{s-1}\|\operatorname{div}\big([\mathcal{T}^\ell;\rho_0]\nabla\phi\big)\|_{L^2_+}\lesssim\|\rho_0-1\|_{W^s}\|\nabla\phi\|_{W^{s-1}},\\
&\sum_{\ell=1}^{s-1}\|\operatorname{div}\big([\mathcal{T}^\ell;\partial_k\varphi_0\nabla\varphi_0]\partial_k\phi\big)\|_{L^2_+}\lesssim\|\nabla\varphi_0\|^2_{W^s}\|\nabla\phi\|_{W^{s-1}}.
\end{aligned}$$

For $s \geq 4$, we get, by applying Lemma 4.1, that

$$\sum_{\ell=1}^{s-1} \|[\mathcal{T}^\ell; \nabla\varphi_0] \cdot \nabla\partial_t\phi\|_{L^2_+} \lesssim \|\nabla\varphi_0\|_{W^{s-1}} \|\nabla\partial_t\phi\|_{W^{s-2}},$$

$$\sum_{\ell=1}^{s-1} \|[\mathcal{T}^\ell; \nabla\partial_t\varphi_0]\nabla\phi\|_{L^2_+} \lesssim \|\nabla\partial_t\varphi_0\|_{W^{s-1}} \|\nabla\phi\|_{W^{s-2}},$$

$$\sum_{\ell=1}^{s-1} \|[\mathcal{T}^\ell, \Delta\varphi_0]\partial_t\phi\|_{L^2_+} \lesssim \|\Delta\varphi_0\|_{W^{s-1}} \|\partial_t\phi\|_{W^{s-2}}.$$

This gives rise to

$$\sum_{\ell=1}^{s-1} \|[\mathcal{T}^\ell; P(\varphi_0, D)]\phi\|_{L^2_+} \lesssim E^{\frac{1}{2}}_{1+s}(\varphi_0(t))(1 + E^{\frac{1}{2}}_{1+s}(\varphi_0(t)))E^{\frac{1}{2}}_s(\phi(t)).$$

Inserting the above estimate into (6.8) gives (6.6). This completes the proof of the lemma. $\square$

Lemma 6.3 (Full energy estimates)**.** *Let ϕ be a smooth enough solution of* (6.2) *on* $[0, T]$. *Then for $t \leq T$, one has*

$$\frac{d}{dt}\widetilde{E}_{s,tan}(\phi(t)) \lesssim \big(1 + E_{s+1}(\varphi_0(t))\big)E_{s,tan}(\phi(t)) + \|F\|^2_{W^{s-1}}. \tag{6.9}$$

Proof. Let $E_s(\phi)$, $E_{s,\tan}(\phi)$, and $\widetilde{E}_{s,\tan}(\phi)$ be determined by (4.22). Along the same line to the proof of (4.26), we claim that

$$E_\ell(\phi(t)) \leq C_\ell\big(E_{\ell,\tan}(\phi(t)) + \|F(t)\|^2_{W^{\ell-2}}\big) \quad \text{for} \quad t \leq T \quad \text{and} \quad \ell = 2, \cdots, s. \tag{6.10}$$

In what follows, we just outline the proof.

We first observe from (6.2) that

$$\begin{aligned}(\rho_0 - (\partial_3\varphi_0)^2)\partial_3^2\phi =& \partial_t^2\phi - \rho_0\Delta_{\rm h}\phi - \nabla\rho_0 \cdot \nabla\phi + 2\nabla\varphi_0 \cdot \nabla\partial_t\phi + (\nabla\varphi_0 \cdot \nabla\phi)\Delta\varphi_0 \\ &+ \Delta\varphi_0\partial_t\phi + \sum_{j=1}^{3}(\nabla\partial_j\varphi_0 \cdot \nabla\phi)\partial_j\varphi_0 + (\nabla_{\rm h}\varphi_0 \cdot \partial_3\nabla_{\rm h}\phi)\partial_3\varphi_0 \\ &+ (\nabla\varphi_0 \cdot \nabla_{\rm h}\nabla\phi)\nabla_{\rm h}\varphi_0 + \nabla\partial_t\varphi_0 \cdot \nabla\phi - F,\end{aligned} \tag{6.11}$$

from which, we infer

$$E^{\frac{1}{2}}_2(\phi(t)) \lesssim E^{\frac{1}{2}}_{2,\tan}(\phi(t)) + E^{\frac{1}{2}}_4(\varphi_0(t))\big(1 + E^{\frac{1}{2}}_4(\varphi_0(t))\big)E^{\frac{1}{2}}_2(\phi(t)) + \|F\|_{L^2_+}.$$

This together with (3.4) ensures that

$$E_2(\phi(t)) \lesssim E_{2,\tan}(\phi(t)) + \|F\|^2_{L^2_+}. \tag{6.12}$$

This proves (6.10) for $\ell = 2$.

Now we assume that (6.10) holds for $\ell = k$, we are going to prove that (6.10) holds for $\ell = k + 1 \leq s$. We first notice that

$$E_{k+1}(\phi(t)) \leq \sum_{j=0}^{k}\big(\|\partial_t^{j+1}\phi(t)\|^2_{H^{k-j}} + \|\nabla\partial_t^j\phi(t)\|^2_{H^{k-j}}\big).$$

• <u>When $j = k$.</u>

We observe from (4.22) that

$$\|\partial_t^{k+1}\phi(t)\|^2_{L^2_+} + \|\nabla\partial_t^k\phi(t)\|^2_{L^2_+} \leq E_{k+1,\tan}(\phi(t)).$$

• <u>When $j = k - 1$.</u>

It follows from (4.22) that

$$\|\partial_t^k\phi\|_{H^1}^2=\|\partial_t^k\phi\|_{L^2_+}^2+\|\nabla\partial_t^k\phi\|_{L^2_+}^2\leq E_{k+1,\tan}(\phi(t)).$$

Whereas notice that

$$\begin{aligned}\|\nabla\partial_t^{k-1}\phi(t)\|_{H^1}^2=&\|\nabla\partial_t^{k-1}\phi(t)\|_{L^2_+}^2+\|\nabla^2\partial_t^{k-1}\phi(t)\|_{L^2_+}^2\\ \lesssim&\|\partial_t\phi\|_{W^{k-1}}^2+\|\nabla\nabla_{\mathrm{h}}\partial_t^{k-1}\phi(t)\|_{L^2_+}^2+\|\partial_3^2\partial_t^{k-1}\phi(t)\|_{L^2_+}^2.\end{aligned}$$

Similar to the proofs of (4.29) and (4.30), we deduce from (6.11) that

$$\|\partial_3^2\partial_t^{k-1}\phi\|_{L^2_+}\leq C\big(E^{\frac12}_{k+1,\tan}(\phi(t))+\big(1+E^{\frac12}_{k+1}(\varphi_0(t))\big)E^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t))\big)+\|\partial_t^{k-1}F\|_{L^2_+}^2.$$

$\|\nabla\partial_t^{k-1}\phi(t)\|_{H^1}$ shares the same estimate.

As a result, it comes out

$$\|\partial_t^k\phi\|_{H^1}+\|\nabla\partial_t^{k-1}\phi(t)\|_{H^1}\leq C\big(E^{\frac12}_{k+1,\tan}(\phi(t))+E^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t))\big)+\|\partial_t^{k-1}F\|_{L^2_+}^2.$$

- When $k-j\geq 2$.

We have

$$\begin{aligned}\|\partial_t^{j+1}\phi\|_{H^{k-j}}^2=&\|\partial_t^{j+1}\phi\|_{H^{k-j-1}}^2+\|\nabla^2\partial_t^{j+1}\phi\|_{H^{k-j-2}}^2\\ \lesssim&\|\partial_t\phi\|_{W^{k-1}}^2+\|\nabla\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-2}}^2+\|\partial_3^2\partial_t^{j+1}\phi\|_{H^{k-j-2}}^2.\end{aligned}$$

By virtue of (6.11), we find

$$\begin{aligned}&\|\partial_3^2\partial_t^{j+1}\phi\|_{H^{k-j-2}}\lesssim\|\partial_t^{j+3}\phi\|_{H^{k-j-2}}+\|\partial_t\phi\|_{W^{k-1}}+\|\nabla\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-2}}\\ &+\|\partial_t^{j+1}\Delta_{\mathrm{h}}\phi\|_{H^{k-j-2}}+\big(1+E^{\frac12}_{k+1}(\varphi_0(t))\big)E^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t))+\|\partial_t^{j+1}F\|_{H^{k-j-2}}.\end{aligned}\tag{6.13}$$

The same estimate holds for $\|\partial_t^{j+1}\phi\|_{H^{k-j}}$.

In the case when $k-j\geq 3$, we have

$$\begin{aligned}\|\nabla\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-2}}^2=&\|\nabla\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-3}}^2+\|\nabla^2\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-3}}^2\\ \lesssim&\|\partial_t\phi\|_{W^{k-1}}^2+\|\nabla\nabla_{\mathrm{h}}^2\partial_t^{j+1}\phi\|_{H^{k-j-3}}^2+\|\partial_3^2\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-3}}^2.\end{aligned}$$

Yet it follows from (6.11) and Lemma 4.1 that

$$\begin{aligned}\|\partial_3^2\nabla_{\mathrm{h}}\partial_t^{j+1}\phi\|_{H^{k-j-3}}\lesssim&\|\nabla_{\mathrm{h}}\partial_t^{j+3}\phi\|_{H^{k-j-3}}+\|\partial_t^{j+1}\nabla_{\mathrm{h}}^3\phi\|_{H^{k-j-3}}\\ &+\big(1+E^{\frac12}_{k+1}(\varphi_0(t))\big)E^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t))+\|\nabla_{\mathrm{h}}\partial_t^{j+1}F\|_{H^{k-j-3}}.\end{aligned}$$

Inserting the above estimate into (6.13) gives rise to

$$\begin{aligned}\|\partial_t^{j+1}\phi\|_{H^{k-j}}\lesssim&\|\partial_t\phi\|_{W^{k-1}}+\|\partial_t^{j+3}\phi\|_{H^{k-j-2}}+\|\partial_t^{j+1}\Delta_{\mathrm{h}}\phi\|_{H^{k-j-2}}\\ &+\|\nabla\nabla_{\mathrm{h}}^2\partial_t^{j+1}\phi\|_{H^{k-j-3}}+\|\nabla_{\mathrm{h}}\partial_t^{j+3}\phi\|_{H^{k-j-3}}+\|\partial_t^{j+1}\nabla_{\mathrm{h}}^3\phi\|_{H^{k-j-3}}\\ &+C\big(1+E^{\frac12}_{k+1}(\varphi_0(t))\big)E^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t))+\|F\|_{W^{k-1}}.\end{aligned}$$

By finite steps of iteration and using the inductive assumption for $\ell=k$, we deduce that

$$\|\partial_t^{j+1}\phi\|_{H^{k-j}}\leq C_k\big(E^{\frac12}_{k+1,\tan}(\phi(t))+\|F\|_{W^{k-1}}\big)+CE^{\frac12}_{k+1}(\varphi_0(t))E^{\frac12}_{k+1}(\phi(t)).$$

The same estimate holds for $\|\nabla\partial_t^j\phi(t)\|_{H^{k-j}}$.

Therefore, by virtue of (3.4), we conclude that

$$E_{k+1}(t)\leq C_k\big(E_{k+1,\tan}(\phi(t))+\|F\|_{W^{k-1}}^2\big).$$

This proves (6.10) for $\ell=k+1$. By combining (6.6) with (6.10), we achieve (6.9). This completes the proof of Lemma 6.3. $\square$

Proof of Theorem 6.1. By applying Gronwall's inequality to (6.9) and using (6.10), we deduce (6.3). This completes the proof of Theorem 6.1. □

7. The existence of solutions to the other asymptotic equations

Let us first present the proof of Proposition 3.3.

Proof of Proposition 3.3. It follows from (2.21) and Proposition 3.2 that

$$\varphi_1|_{z=0} = -\Phi_1|_{Z=0} \in W_{T_0}^{s_0-\frac{3}{2}}(\mathbb{R}^2).$$

Then we deduce from Theorem 3.1 that the wave equation (3.9) with boundary condition (2.21) and initial condition (3.11) has a unique solution φ_1 on $[0, T_0]$. Furthermore, we have

$$\begin{aligned} \|(\partial_t\varphi_1, \nabla\varphi_1)\|_{W_{T_0}^{s_0-3}} \leq & C\big(\|(\nabla\varphi_{1,0}^{\rm in}, \varphi_{1,0}^{\rm in},)\|_{H^{s_0-3}} + \|\Phi_1|_{z=0}\|_{W_T^{s_0-\frac{3}{2}}}\big) \\ \leq & C\big(\|(a_{1,0}^{\rm in}, \nabla\varphi_{1,0}^{\rm in})\|_{H^{s_0-3}} + \|(a_{0,0}^{\rm in}, \nabla\varphi_{0,0}^{\rm in})\|_{H^{s_0-1}}\big). \end{aligned} \tag{7.1}$$

Note from the φ_1 equation of (2.6) that

$$a_1 = -\frac{1}{2a_0}\left(\partial_t\varphi_1 + \nabla\varphi_0\cdot\nabla\varphi_1\right),$$

from which, (7.1) and Lemma 4.1, we deduce (3.12). This completes the proof of the proposition. □

Next let us present the proof of Proposition 3.4.

Proof of Proposition 3.4. Inductively, we assume that we already have

$$\begin{aligned} & (a_0 - 1, \nabla\varphi_0) \in W_{T_0}^{s_0-1}, \quad (a_{j+1}, \nabla\varphi_{j+1}) \in W_{T_0}^{s_0-1-2(j+1)} \quad \text{and} \\ & (A_j, \Phi_{j+1}) \in W_{1,T_0}^{s_0-2(j+1)+\frac{1}{2}} \quad \text{for} \quad j = 0, \cdots, k, \end{aligned} \tag{7.2}$$

we consider the boundary layer problem (2.22-2.23) with boundary condition (2.24). We first get, by inserting (5.1) into (2.23), that

$$\begin{aligned} \frac{1}{2}\partial_Z^2 A_{k+1} = & \frac{\overline{a}_0^2\overline{\partial_z\varphi_0}}{A_0+\overline{a}_0}\partial_Z\Phi_{k+2} + G_k \\ & + \big(3A_0^2 + 6\overline{a}_0A_0 + 2\overline{a}_0^2 + \partial_Z\Phi_1\overline{\partial_z\varphi_0} + \frac{1}{2}|\partial_Z\Phi_1|^2\big)A_{k+1}. \end{aligned} \tag{7.3}$$

Here according to (7.2) and (A.2) in the Appendix A, $G_k \in W_{1,T_0}^{s_{0,k}}$, where and in what follows, we always denote

$$s_{0,k} \stackrel{\text{def}}{=} s_0 - 2(k+2) + \frac{1}{2}.$$

Whereas by multiplying (2.22) by $(A_0 + \overline{a}_0)$, we find

$$\begin{aligned} \frac{1}{2}\partial_Z\big((A_0+\overline{a}_0)^2\partial_Z\Phi_{k+2}\big) + (A_0+\overline{a}_0)(\partial_Z\Phi_1 + \overline{\partial_z\varphi_0})\partial_Z A_{k+1} \\ + \frac{1}{2}A_{k+1}(A_0+\overline{a}_0)\partial_Z^2\Phi_1 = (A_0+\overline{a}_0)\,F_k, \end{aligned}$$

where $F_k \in W_{1,T_0}^{s_{0,k}}$ according to (7.2) and (A.1) in the Appendix A,

Then we deduce from (2.18) that

$$\begin{aligned} \frac{1}{2}\partial_Z\big((A_0+\overline{a}_0)^2\partial_Z\Phi_{k+2}\big) + (A_0+\overline{a}_0)(\partial_Z\Phi_1 + \overline{\partial_z\varphi_0})\partial_Z A_{k+1} \\ - A_{k+1}(\partial_Z\Phi_1 + \overline{\partial_z\varphi_0})\partial_Z A_0 = (A_0+\overline{a}_0)\,F_k. \end{aligned}$$

By virtue of (5.1), we write

$$\frac{1}{2}\partial_Z\big((A_0+\overline{a}_0)^2\partial_Z\Phi_{k+2}\big)+(\overline{a}_0)^2\overline{\partial_z\varphi_0}\Big(\frac{\partial_Z A_{k+1}}{A_0+\overline{a}_0}-\frac{A_{k+1}\partial_Z A_0}{(A_0+\overline{a}_0)^2}\Big)=(A_0+\overline{a}_0)\,F_k,$$

that is,

$$\partial_Z\Big(\frac{1}{2}(A_0+\overline{a}_0)^2\partial_Z\Phi_{k+2}+(\overline{a}_0)^2\overline{\partial_z\varphi_0}\frac{A_{k+1}}{A_0+\overline{a}_0}\Big)=(A_0+\overline{a}_0)\,F_k. \tag{7.4}$$

Integrating (7.4) over $[Z,\infty[$ gives rise to

$$\partial_Z\Phi_{k+2}=-2(\overline{a}_0)^2\overline{\partial_z\varphi_0}\frac{A_{k+1}}{(A_0+\overline{a}_0)^3}+\frac{2}{(A_0+\overline{a}_0)^2}\int_Z^\infty(A_0+\overline{a}_0)\,F_k\,dZ'. \tag{7.5}$$

Plugging (7.5) into (7.3) leads to

$$\partial_Z^2A_{k+1}=\mathfrak{g}A_{k+1}+\widetilde{G}_k, \tag{7.6}$$

Here and in all that follows, we always denote

$$\begin{aligned}\mathfrak{g}&\stackrel{\text{def}}{=}6A_0^2+12\overline{a}_0A_0+4\overline{a}_0^2+2\partial_Z\Phi_1\overline{\partial_z\varphi_0}+|\partial_Z\Phi_1|^2-4\frac{(\overline{a}_0)^4(\overline{\partial_z\varphi_0})^2}{(A_0+\overline{a}_0)^4},\\ \widetilde{G}_k&\stackrel{\text{def}}{=}2G_k+\frac{4\overline{a}_0^2\overline{\partial_z\varphi_0}}{(A_0+\overline{a}_0)^3}\int_Z^\infty(A_0+\overline{a}_0)\,F_k\,dZ'\in W^{s_{0,k}}_{1,T_0}.\end{aligned} \tag{7.7}$$

Recalling the notation from (2.11) that $\bar{a}_{k+1}(y)=a_{k+1}(y,0)$, we reduce the resolution of the problem (2.22), (2.23) and (2.24) to the following system

$$\begin{cases}\partial_Z^2A_{k+1}=\mathfrak{g}A_{k+1}+\widetilde{G}_k,\\ \partial_Z\Phi_{k+2}=-\frac{2(\overline{a}_0)^2\overline{\partial_z\varphi_0}}{(A_0+\overline{a}_0)^3}A_{k+1}+\frac{2}{(A_0+\overline{a}_0)^2}\int_Z^\infty(A_0+\overline{a}_0)\,F_k\,dZ',\\ A_{k+1}|_{Z=0}=-\bar{a}_{k+1},\quad A_{k+1}|_{Z=+\infty}=0,\quad \Phi_{k+2}|_{Z=+\infty}=0.\end{cases} \tag{7.8}$$

Let's now handle the system (7.8). In order to do it, let $\widetilde{A}_{k+1}\stackrel{\text{def}}{=}A_{k+1}+e^{-3Z}\bar{a}_{k+1}$. Then $\widetilde{A}_{k+1}$ verifies

$$\begin{cases}\partial_Z^2\widetilde{A}_{k+1}=\mathfrak{g}\,\widetilde{A}_{k+1}+\widetilde{G}_k-(9+\mathfrak{g})\,e^{-3Z}\bar{a}_{k+1},\\ \widetilde{A}_{k+1}|_{Z=0}=0,\quad \widetilde{A}_{k+1}|_{Z=+\infty}=0.\end{cases} \tag{7.9}$$

By applying the operator $\mathcal{T}^\ell$ (recalling that $\mathcal{T}=(\partial_t,\nabla_{\rm h})$) with $\ell\in\big[0,[s_{0,k}]\big]$ to the equation (7.9), and then taking the L^2 inner product of the resulting equation with $-e^{2Z}\mathcal{T}^\ell\widetilde{A}_{k+1}$, we have

$$\begin{aligned}\big\|e^Z\mathcal{T}^\ell\partial_Z\widetilde{A}_{k+1}\big\|_{L^2_+}^2-2\big\|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\big\|_{L^2_+}^2=&-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\mathfrak{g}\,\widetilde{A}_{k+1})|e^{2Z}\mathcal{T}^\ell\widetilde{A}_{k+1}\,dy\,dZ\\ &-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell\big(\widetilde{G}_k-(9+\mathfrak{g})\,e^{-3Z}\bar{a}_{k+1}\big)|e^{2Z}\mathcal{T}^\ell\widetilde{A}_{k+1}\,dy\,dZ.\end{aligned}$$

Notice that

$$\begin{aligned}-\int_{\mathbb{R}^3_+}\mathcal{T}^\ell(\mathfrak{g}\,\widetilde{A}_{k+1})|e^{2Z}\mathcal{T}^\ell\widetilde{A}_{k+1}\,dy\,dZ=&-\int_{\mathbb{R}^3_+}\mathfrak{g}\,|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}|^2\,dy\,dZ\\ &-\int_{\mathbb{R}^3_+}e^Z\big[\mathcal{T}^\ell;\mathfrak{g}\big]\,\widetilde{A}_{k+1}|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\,dy\,dZ,\end{aligned}$$

we infer

$$
\begin{aligned}
&\big\|e^Z\mathcal{T}^\ell\partial_Z\widetilde{A}_{k+1}\big\|_{L^2_+}^2+\int_{\mathbb{R}^3_+}(\mathfrak{g}-2)\,|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}|^2\,dy\,dZ\\
&\quad\leq\big\|e^Z\left[\mathcal{T}^\ell;\mathfrak{g}\right]\widetilde{A}_{k+1}\|_{L^2_+}\big\|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\big\|_{L^2_+}+C\Big(\|\widetilde{G}_k\|_{W^{s_{0,k}}_{1,T_0}}\\
&\qquad+\big\|e^{-2Z}\mathcal{T}^\ell(\bar{a}_{k+1})\big\|_{L^2_+}+\big\|e^{-2Z}\mathcal{T}^\ell(\mathfrak{g}\bar{a}_{k+1})\big\|_{L^2+}\Big)\big\|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\big\|_{L^2_+}.
\end{aligned}
$$

Whereas it follows from (3.4), (3.13) and (7.7) that $\mathfrak{g}-2\geq 2-\|\mathfrak{g}-4\|_{L^\infty_+}\geq\frac{3}{2}$ as long as c is small enough in (3.3). So that we obtain

$$
\begin{aligned}
\|e^Z\mathcal{T}^\ell\partial_Z\widetilde{A}_{k+1}\|_{L^2_+}^2+\|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\|_{L^2_+}^2&\lesssim\big\|e^Z[\mathcal{T}^\ell,\mathfrak{g}]\,\widetilde{A}_{k+1}\big\|_{L^2_+}^2\\
&\quad+\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|e^{-2Z}\mathcal{T}^\ell\bar{a}_{k+1})\|_{L^2_+}^2+\|e^{-2Z}\mathcal{T}^\ell(\mathfrak{g}\bar{a}_{k+1})\|_{L^2_+}^2.
\end{aligned}
$$

We deduce from trace theorem and the proof of Lemma 4.1 that

$$
\begin{aligned}
&\big\|e^Z\left[\mathcal{T}^\ell;\mathfrak{g}\right]\widetilde{A}_{k+1}\|_{L^2_+}\lesssim\|\mathcal{T}\mathfrak{g}\|_{L^\infty_{\rm v}(W^{[s_{0,k}]-1}_{T_0})_{\rm h}}\|e^Z\widetilde{A}_{k+1}\|_{L^2_{\rm v}((W^{[s_{0,k}]}_{T_0})_{\rm h}},\\
&\big\|e^{-2Z}\mathcal{T}^\ell(\mathfrak{g}\bar{a}_{k+1})\big\|_{L^2+}\lesssim\|\mathfrak{g}\|_{L^2_{\rm v}(W^{[s_{0,k}]}_{T_0})_{\rm h}}\|a_{k+1}\|_{W^{[s_{0,k}]+\frac{1}{2}}_{T_0}}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\|e^Z\mathcal{T}^\ell\partial_Z\widetilde{A}_{k+1}\|_{L^2_+}^2+\frac{5}{4}\|e^Z\mathcal{T}^\ell\widetilde{A}_{k+1}\|_{L^2_+}^2\\
&\lesssim\Big(\big\|(a_0-1,\nabla\varphi_0)\|^2_{W^{s_0-1}_{T_0}}+\|(A_0,\Phi_1)\|^2_{W^{s_0-\frac{3}{2}}_{1,T_0}}\Big)\|e^Z\widetilde{A}_{k+1}\|_{L^2_{\rm v}((W^{[s_{0,k}]}_{T_0})_{\rm h}}\\
&\quad+\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|a_{k+1}\|^2_{W^{s_{0,k}}_{T_0}}\Big(1+\big\|(a_0-1,\nabla\varphi_0)\|^2_{W^{s_0-1}_{T_0}}+\|(A_0,\Phi_1)\|^2_{W^{s_0-\frac{3}{2}}_{1,T_0}}\Big).
\end{aligned}
$$

In view of (3.4) and (3.13), we get, by summing the above inequality for ℓ from 0 to $[s_{0,k}]$, that

$$
\|e^Z\partial_Z\widetilde{A}_{k+1}\|^2_{L^2_{\rm v}((W^{[s_{0,k}]}_{T_0})_{\rm h}}+\|e^Z\widetilde{A}_{k+1}\|^2_{L^2_{\rm v}((W^{[s_{0,k}]}_{T_0})_{\rm h}}\lesssim\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|a_{k+1}\|^2_{W^{s_{0,k}}_{T_0}},
$$

which in particular implies that

$$
\|e^Z\widetilde{A}_{k+1}\|^2_{L^\infty_{\rm v}((W^{[s_{0,k}]}_{T_0})_{\rm h}}\lesssim\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|a_{k+1}\|^2_{W^{s_{0,k}}_{T_0}}. \tag{7.10}
$$

In general by apply ∂_Z^k for $k\leq[s_{0,k}]$ and performing the above energy estimate, we achieve

$$
\|\widetilde{A}_{k+1}\|^2_{W^{s_{0,k}}_{1,T_0}}\lesssim\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|a_{k+1}\|^2_{W^{s_{0,k}}_{T_0}}. \tag{7.11}
$$

With the above estimate, we deduce from the second equation of (7.8) that

$$
\|\Phi_{k+2}\|^2_{W^{s_{0,k}}_{1,T_0}}\lesssim\|\widetilde{G}_k\|^2_{W^{s_{0,k}}_{1,T_0}}+\|F_k\|^2_{W^{s_{0,k}}_{1,T_0}}. \tag{7.12}
$$

Thanks to the definitions of$\widetilde{A}_{k+1}$, $\widetilde{G}_k$, and F_k, G_k in Appendix A, we deduce (3.13). This ends the proof of Proposition 3.4. □

Finally let us present the proof of Proposition 3.5.

Proof of Proposition 3.5. We first observe from the φ_{k+2} equation of (2.7) that

$$\begin{aligned}\partial_t\varphi_{k+2}\big|_{t=0}=&-\nabla\varphi_0\cdot\nabla\varphi_{k+2}\big|_{t=0}-2a_0a_{k+2}\big|_{t=0}+\frac{1}{2a_0}\left(\Delta a_k+g^{\varphi}_{k+1}\right)\big|_{t=0}\\ =&-\nabla\varphi^{\rm in}_{0,0}\cdot\nabla\varphi^{\rm in}_{k+2,0}-2a^{\rm in}_{0,0}a^{\rm in}_{k+2,0}+\frac{1}{2a^{\rm in}_{0,0}}\left(\Delta a^{\rm in}_{k,0}+g^{\varphi}_{k+1}\big|_{t=0}\right)\\ \stackrel{\rm def}{=}&\varphi^{\rm in}_{k+2,1}\in W^{\bar s_{0,k}}_{T_0}\quad\text{with}\quad \bar s_{0,k}\stackrel{\rm def}{=}s_0-1-2(k+2)\geq 2,\end{aligned}\tag{7.13}$$

we are going to inductively solve the linear equations (2.7) with the initial-boundary conditions

$$\varphi_{k+2}|_{z=0}=-\Phi_{k+2}|_{Z=0},\quad \varphi_{k+2}|_{t=0}=\varphi^{\rm in}_{k+2,0}\quad\text{and}\quad \partial_t\varphi_{k+2}|_{t=0}=\varphi^{\rm in}_{k+2,1}.\tag{7.14}$$

In fact, according to the first equation in (2.4) and (2.7), we write

$$\partial_t(a_0a_{k+2})+\operatorname{div}(a_0a_{k+2}\nabla\varphi_0)+\frac{1}{2}\operatorname{div}(\rho_0\nabla\varphi_{k+2})=a_0f^a_{k+1}.$$

By taking ∂_t to the the φ_{k+1} equation of (2.7) and inserting the above equation into the resulting one, we obtain

$$\begin{aligned}&\partial_t^2\varphi_{k+2}-\operatorname{div}\big(\rho_0\nabla\varphi_{k+2}\big)+\partial_t(\nabla\varphi_0\cdot\nabla\varphi_{k+2})\\ &\quad+\operatorname{div}(\partial_t\varphi_{k+2}\nabla\varphi_0)+\operatorname{div}\big((\nabla\varphi_0\cdot\nabla\varphi_k)\nabla\varphi_0\big)\\ &=-2a_0\,f^a_{k+1}+\partial_t\Big(\frac{1}{2a_0}\big(\Delta a_k+g^{\varphi}_{k+1}\big)\Big)+\operatorname{div}\Big(\frac{1}{2a_0}\big(\Delta a_k+g^{\varphi}_{k+1}\big)\nabla\varphi_0\Big)\stackrel{\rm def}{=}F_{k+2}.\end{aligned}\tag{7.15}$$

Note that F_{k+2} belongs to $W^{\bar s_{0,k}}_{T_0}$ according to the definitions of f^a_{k+1} and g^{φ}_{k+1} in (2.8). Then we deduce from Theorem 3.1 that the system (7.15) with the boundary condition (2.25) has a unique solution φ_{k+2} so that

$$\begin{aligned}\big\|(\partial_t\varphi_{k+2},\nabla\varphi_{k+2})\|_{W^{\bar s_{0,k}}_{T_0}}\leq& C\big(\big\|\big(\nabla\varphi^{\rm in}_{k+2,0},\varphi^{\rm in}_{k+2,1}\big)\big\|_{H^{\bar s_{0,k}}}+\big\|\Phi_{k+2}|_{Z=0}\big\|_{W^{s_{0,k}}_{T}}+\|F_{k+2}\|_{W^{\bar s_{0,k}}_{T_0}}\big)\\ \leq& C\big(\|(a^{\rm in}_{0,0}-1,\nabla\varphi^{\rm in}_{0,0})\|_{H^{s_0-1}}+\sum_{j=1}^{k+2}\|(a^{\rm in}_{j,0},\nabla\varphi^{\rm in}_{j,0})\|_{H^{s_0-2j-1}}\big).\end{aligned}\tag{7.16}$$

With φ_{k+2} thus obtained, it follows from the a_{k+2} equation of (2.7) and Lemma 4.1 that

$$a_{k+2}=\frac{1}{2a_0}\Big(\frac{1}{2a_0}\left(\Delta a_k+g^{\varphi}_{k+1}\right)-\partial_t\varphi_{k+2}-\nabla\varphi_0\cdot\nabla\varphi_{k+2}\Big)\in W^{\bar s_{0,k}}_{T_0},$$

in case $\bar s_{0,k}\geq 2$. Moreover, there holds

$$\|\|a_{k+2}\|_{W^{\bar s_{0,k}}_{T_0}}\leq C\Big(\|(a^{\rm in}_{0,0}-1,\nabla\varphi^{\rm in}_{0,0})\|_{H^{s_0-1}}+\sum_{j=1}^{k+2}\|(a^{\rm in}_{j,0},\nabla\varphi^{\rm in}_{j,0})\|_{H^{s_0-2j-1}}\Big).$$

This completes the proof of (3.14). □

8. Some technical lemmas

Let $(a^{{\rm int},\varepsilon,m},\varphi^{{\rm int},\varepsilon,m}))$ and $(a^{{\rm b},\varepsilon,m},\varphi^{{\rm b},\varepsilon,m}))$ be determined by (3.15), we denote

$$\begin{aligned}&\mathcal{E}_{m,T}\stackrel{\rm def}{=}\big\|(a^{{\rm int},\varepsilon,m}-1,\,\partial_t\varphi^{{\rm int},\varepsilon,m},\nabla\varphi^{{\rm int},\varepsilon,m})\big\|^2_{W^{s_0-2m-5}_T}+\big\|(a^{{\rm b},\varepsilon,m},\,\varphi^{{\rm b},\varepsilon,m})\big\|^2_{W^{s_0-2m-\frac{7}{2}}_{1,T}},\\ &u^{\varepsilon,m}\stackrel{\rm def}{=}\nabla\varphi^{\varepsilon,m},\quad \mathcal{S}_f(g)\stackrel{\rm def}{=}f\cdot\nabla g+\frac{1}{2}g\,\nabla\cdot f,\quad \mathcal{T}\stackrel{\rm def}{=}(\partial_t,\,\nabla_{\rm h}).\end{aligned}\tag{8.1}$$

Then thanks to Propositions 3.1-3.5, we have

$$\mathcal{E}_{m,T}\leq C\,\mathcal{E}_0,\tag{8.2}$$

for $\mathcal{E}_0$ being given by (3.16).

Lemma 8.1. *Let $s_0 \geq 2m+9$ be an integer and $a^{\varepsilon,m}$ be defined by (3.15). Let f and g be smooth enough functions satisfying the homogeneous boundary conditions $f|_{z=0} = g|_{z=0} = 0$. Then one has*

$$\varepsilon^2 \Big| \int_{\mathbb{R}^3_+} \frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} f\, g\, dx \Big| \lesssim \mathcal{E}_0^{\frac12} \, \|\varepsilon f\|_{H^1} \, \|\varepsilon g\|_{H^1} \quad \text{and} \quad \varepsilon^2 \Big\| \frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} f \Big\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12} \, \|\varepsilon f\|_{H^1}, \tag{8.3}$$

if $j \in [0, s_0 - 2m - 9]$, we also have

$$\begin{aligned} &\varepsilon^2 \Big| \int_{\mathbb{R}^3_+} \mathcal{T}^j \Big(\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} \Big) f\, g\, dx \Big| \lesssim \mathcal{E}_0^{\frac12} \, \|\varepsilon f\|_{H^1} \, \|\varepsilon g\|_{H^1}, \\ &\varepsilon \Big\| \big[\mathcal{T}^j ; \, \frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} \big] f \Big\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12} \sum_{k=0}^{j-1} \|\mathcal{T}^k f\|_{H^1}. \end{aligned} \tag{8.4}$$

Proof. In view of (3.15), $a^{\varepsilon,m} = a^{\mathrm{int},\varepsilon,m} + [a^{\mathrm{b},\varepsilon,m}]_\varepsilon$, we write

$$\varepsilon^2 \int_{\mathbb{R}^3_+} \frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} f\, g\, dx = \varepsilon^2 \int_{\mathbb{R}^3_+} \frac{\Delta a^{\mathrm{int},\varepsilon,m} + [\Delta_{\mathrm{h}} a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f\, g\, dx + \int_{\mathbb{R}^3_+} \frac{[\partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f\, g\, dx.$$

Note that $f|_{z=0} = 0$, Hardy's inequality ensures that $\|z^{-1} f\|_{L^2(\mathbb{R}^3_+)} \leq C \|\partial_z f\|_{L^2(\mathbb{R}^3_+)}$, so that we infer

$$\begin{aligned} \varepsilon^2 \Big| \int_{\mathbb{R}^3_+} \frac{\Delta a^{\mathrm{int},\varepsilon,m} + [\Delta_{\mathrm{h}} a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f\, g\, dx \Big| &\lesssim \Big\| \frac{\Delta a^{\mathrm{int},\varepsilon,m} + [\Delta_{\mathrm{h}} a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} \Big\|_{L^\infty_+} \|\varepsilon f\|_{L^2_+} \|\varepsilon g\|_{L^2_+} \\ &\lesssim \mathcal{E}_0^{\frac12} \, \|\varepsilon\, f\|_{L^2_+} \, \|\varepsilon\, g\|_{L^2_+}, \end{aligned}$$

and

$$\begin{aligned} \Big| \int_{\mathbb{R}^3_+} \frac{[\partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f\, g\, dx \Big| &= \Big| \int_\Omega \frac{[Z^2 \partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} (\varepsilon\, z^{-1} f)\, (\varepsilon\, z^{-1} g)\, dx \Big| \\ &\lesssim \Big\| \frac{Z^2 \partial_Z^2 a^{\mathrm{b},\varepsilon,m}}{a^{\varepsilon,m}} \Big\|_{L^\infty_+} \|\varepsilon\, z^{-1} f\|_{L^2_+} \, \|\varepsilon\, z^{-1} g\|_{L^2_+} \\ &\lesssim \Big\| \frac{Z^2 \partial_Z^2 a^{\mathrm{b},\varepsilon,m}}{a^{\varepsilon,m}} \Big\|_{L^\infty_+} \|\varepsilon\, \partial_z f\|_{L^2_+} \|\varepsilon\, \partial_z g\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12} \|\varepsilon\, \partial_z f\|_{L^2_+} \|\varepsilon\, \partial_z g\|_{L^2_+}, \end{aligned}$$

where we used the fact that $\|\frac{Z^2 \partial_Z^2 a^{\mathrm{b},\varepsilon,m}}{a^{\varepsilon,m}}\|_{L^\infty_+} \lesssim \mathcal{E}_0^{\frac12}$ due to $\partial_Z^2 a^{\mathrm{b},\varepsilon,m} \in \mathcal{W}_{1,T_0}^{s_0 - 2m - \frac{11}{2}}$. This leads to the first inequality of (8.3).

Along the same line, we observe that

$$\begin{aligned} \varepsilon^2 \Big\| \frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} f \Big\|_{L^2_+} \leq& \varepsilon^2 \Big\| \frac{\Delta a^{\mathrm{int},\varepsilon,m} + [\Delta_{\mathrm{h}} a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} \Big\|_{L^\infty_+} \|f\|_{L^2_+} + \Big\| \frac{[\partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f \Big\|_{L^2_+} \\ \lesssim& \varepsilon\, \mathcal{E}_0^{\frac12} \, \|\varepsilon\, f\|_{L^2_+} + \Big\| \frac{[\partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f \Big\|_{L^2_+}. \end{aligned} \tag{8.5}$$

Whereas it follows from inequality, $\|z^{-1} f\|_{L^2(\mathbb{R}^3_+)} \leq C \|\partial_z f\|_{L^2(\mathbb{R}^3_+)}$, that we

$$\begin{aligned} \Big\| \frac{[\partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} f \Big\|_{L^2_+} =& \Big\| \frac{[Z \partial_Z^2 a^{\mathrm{b},\varepsilon,m}]_\varepsilon}{a^{\varepsilon,m}} (\varepsilon\, z^{-1} f) \Big\|_{L^2_+} \\ \lesssim& \Big\| \frac{Z \partial_Z^2 a^{\mathrm{b},\varepsilon,m}}{a^{\varepsilon,m}} \Big\|_{L^\infty_+} \|\varepsilon\, z^{-1} f\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12} \|\varepsilon\, \partial_z f\|_{L^2_+}, \end{aligned}$$

which together with (8.5) ensures the second inequality of (8.3).

The inequalities of (8.4) can be proved along the same line, we omit the details here. This completes the proof of Lemma 8.1. $\square$

Lemma 8.2. *Let f and g be smooth enough functions which satisfy the homogenous boundary condition $f|_{z=0} = g|_{z=0} = 0$. Then one has*

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} \mathcal{S}_{u^{\varepsilon,m}}(f)\, g\, dx = -\int_{\mathbb{R}^3_+} \mathcal{S}_{u^{\varepsilon,m}}(g)\ f\, dx, \quad \int_{\mathbb{R}^3_+} \mathcal{S}_{u^{\varepsilon,m}}(f)\ f\, dx = 0,\\
&\int_{\mathbb{R}^3_+} \big(\mathcal{S}_{u^{\varepsilon,m}}(g)\partial_t f - \mathcal{S}_{u^{\varepsilon,m}}(f)\partial_t g\big)\, dx = \frac{d}{dt}\int_{\mathbb{R}^3_+} \mathcal{S}_{u^{\varepsilon,m}}(g)\, f\, dx - \int_{\mathbb{R}^3_+} \mathcal{S}_{\partial_t u^{\varepsilon,m}}(g)\, f\, dx.
\end{aligned}
\tag{8.6}
$$

Moreover, for $j \in [0, s_0 - 2m - 6]$, there holds

$$
\|\mathcal{S}_{\mathcal{T}^j u^{\varepsilon,m}}(f)\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12}\|f\|_{H^1} \quad \text{and} \quad \|[\mathcal{T}^j; \mathcal{S}_{u^{\varepsilon,m}}](f)\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac12}\textstyle\sum_{k=0}^{j-1}\|\mathcal{T}^k f\|_{H^1}.
\tag{8.7}
$$

Proof. The first two equalities in (8.6) can be obtained by using integration by parts. Whereas observing that

$$
\begin{aligned}
&\int_{\mathbb{R}^3_+} \big(\mathcal{S}_{u^{\varepsilon,m}}(g)\, \partial_t f - \mathcal{S}_{u^{\varepsilon,m}}(f)\, \partial_t g\big)\, dx\\
&\qquad = \frac{d}{dt}\int_{\mathbb{R}^3_+} \mathcal{S}_{u^{\varepsilon,m}}(g)\, f\, dx - \int_{\mathbb{R}^3_+} \mathrm{div}\big(u^{\varepsilon,m} f \partial_t g\big) - \int_{\mathbb{R}^3_+} \mathcal{S}_{\partial_t u^{\varepsilon,m}}(g)\, f\, dx.
\end{aligned}
$$

Then the second equation in (8.6) follows from the homogeneous boundary condition of f.

Next we just prove the first inequality of (8.7) for the case $j = 0$. We observe that

$$
\begin{aligned}
\|f\, \nabla \cdot u^{\varepsilon,m}\|_{L^2_+} =& \big\|f\big(\mathrm{div}\, u^{\mathrm{int},\varepsilon,m} + [\mathrm{div}_{\mathrm{h}}\, u^{\mathrm{b},\varepsilon,m}]_\varepsilon + \varepsilon^{-1}[\partial_Z u^{\mathrm{b},\varepsilon,m}]_\varepsilon\big)\big\|_{L^2_+}\\
\lesssim& \|f\|_{L^6_+}\|\,\mathrm{div}\, u^{\mathrm{int},\varepsilon,m}\|_{L^3_+} + \|f\|_{L^2_+}\|\,\mathrm{div}_{\mathrm{h}}\, u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_+}\\
&+ \|z^{-1} f\|_{L^2_+}\|Z\, \partial_Z u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_+}\\
\lesssim& \|f\|_{H^1}\big(\|u^{\mathrm{int},\varepsilon,m}\|_{H^2} + \|u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_{\mathrm{v}}(H^{\frac52}_{\mathrm{h}})}\big) + \|\partial_z f\|_{L^2_+}\|Z\, \partial_Z u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_{\mathrm{v}}(H^{\frac32}_{\mathrm{h}})}.
\end{aligned}
$$

As a result, we achieve

$$
\begin{aligned}
\|\mathcal{S}_{u^{\varepsilon,m}}(f)\|_{L^2_+} \lesssim& \|u^{\varepsilon,m} \cdot \nabla f\|_{L^2_+} + \|f\, \nabla \cdot u^{\varepsilon,m}\|_{L^2_+}\\
\lesssim& \|f\|_{H^1}\Big(\|u^{\varepsilon,m}\|_{L^\infty_+} + \|u^{\mathrm{int},\varepsilon,m}\|_{H^2} + \|u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_{\mathrm{v}}(H^{\frac52}_{\mathrm{h}})} + \|Z\, \partial_Z u^{\mathrm{b},\varepsilon,m}\|_{L^\infty_{\mathrm{v}}(H^{\frac32}_{\mathrm{h}})}\Big)\\
\lesssim& \mathcal{E}_0^{\frac12}\|f\|_{H^1},
\end{aligned}
$$

which leads to the first inequality of (8.7) for $j = 0$.

The second inequality of (8.7) follows from the first one. This ends the proof of Lemma 8.2. $\square$

Lemma 8.3. *Let $(\mathfrak{A}_1, \mathfrak{A}_2)$ be smooth enough solution to the following system:*

$$
\begin{cases}
\varepsilon\big(\partial_t + \mathcal{S}_{u^{\varepsilon,m}}(\cdot)\big)\mathfrak{A}_1 + \frac{\varepsilon^2}{2}\Delta\mathfrak{A}_2 = \Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\mathfrak{A}_2 - \varepsilon^{m+2}\hbar_2 + f_1 - f_2,\\
\varepsilon\big(\partial_t + \mathcal{S}_{u^{\varepsilon,m}}(\cdot)\big)\mathfrak{A}_2 - \frac{\varepsilon^2}{2}\Delta\mathfrak{A}_1 + 2(a^{\varepsilon,m})^2\mathfrak{A}_1 = -\chi_1\mathfrak{A}_1 + \varepsilon^{m+1}\chi_2 - g_1 - g_2,\\
\mathfrak{A}_1|_{z=0} = 0, \quad \mathfrak{A}_2|_{z=0} = 0.
\end{cases}
\tag{8.8}
$$

Then if $s_0 - 2m - 10 \geq 0$, *one has*

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{\varepsilon^2}{4}\|(\nabla\mathfrak{A}_1,\, \nabla\mathfrak{A}_2)\|_{L^2_+}^2 + \int_{\mathbb{R}^3_+}\Big(\big((a^{\varepsilon,m})^2 + \frac{1}{2}\chi_1\big)\mathfrak{A}_1^2 + \mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)\mathfrak{A}_1\Big)\, dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^3_+}\Big(\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\mathfrak{A}_2^2 - 2\varepsilon^{m+2}\hbar_2\mathfrak{A}_2 - 2\varepsilon^{m+1}\chi_2\mathfrak{A}_1\Big)\, dx\Big\} \\
&+ \int_{\mathbb{R}^3_+}\big(g_2\partial_t\mathfrak{A}_1 - f_2\partial_t\mathfrak{A}_2\big)\, dx \lesssim \big(1 + \|\hbar_1\|_{L^\infty_+} + \|\partial_t\hbar_1\|_{L^\infty_+}\big)\|\varepsilon\,(\mathfrak{A}_1, \mathfrak{A}_2)\,\|_{H^1}^2 \\
&+ \big(1 + \|\chi_1\|_{L^\infty_+} + \|\partial_t\chi_1\|_{L^\infty_+}\big)\|\mathfrak{A}_1\|_{L^2_+}^2 + \|(\varepsilon^{-1}f_1,\, \varepsilon^{-1}g_1,\, g_2,\, f_2)\|_{L^2_+}^2 \\
&+ \|\nabla\,(f_1,\, g_1)\|_{L^2_+}^2 + \varepsilon^{2(m+1)}\big(\|(\chi_2, \varepsilon\hbar_2)\|_{L^2_+}^2 + \|(\partial_t\chi_2, \partial_t\hbar_2)\|_{L^2_+}^2\big).
\end{aligned}
\tag{8.9}
$$

Proof. We first get, by taking the L^2 inner product of the $\mathfrak{A}_1$ equation of (8.8) with $-\partial_t\mathfrak{A}_2$, that

$$
\begin{aligned}
&-\varepsilon\int_{\mathbb{R}^3_+}\big(\partial_t\mathfrak{A}_1 + \mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_1)\big)\partial_t\mathfrak{A}_2\, dx + \frac{\varepsilon^2}{4}\frac{d}{dt}\|\nabla\,\mathfrak{A}_2\|_{L^2_+}^2 \\
&= -\int_{\mathbb{R}^3_+}\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\,\mathfrak{A}_2\,\partial_t\mathfrak{A}_2\, dx - \int_{\mathbb{R}^3_+}\big(f_1 - \varepsilon^{m+2}\hbar_2 - f_2\big)\,\partial_t\mathfrak{A}_2\, dx.
\end{aligned}
\tag{8.10}
$$

While by substituting the $\mathfrak{A}_2$ equation of (8.8) into $-\int_{\mathbb{R}^3_+} f_1\,\partial_t\mathfrak{A}_2\, dx$, one has

$$
\begin{aligned}
&-\int_{\mathbb{R}^3_+} f_1\,\partial_t\mathfrak{A}_2\, dx = \frac{\varepsilon}{2}\int_{\mathbb{R}^3_+}\nabla\, f_1\cdot\nabla\mathfrak{A}_1\, dx \\
&+ \varepsilon^{-1}\int_{\mathbb{R}^3_+} f_1\,\big(\varepsilon\mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_2) + \big(2(a^{\varepsilon,m})^2 + \chi_1\big)\mathfrak{A}_1 - \varepsilon^{m+1}\chi_2 + g_1 + g_2\big)\, dx.
\end{aligned}
$$

By inserting the above equality into (8.10) and using integrating by parts, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\frac{\varepsilon^2}{4}\|\nabla\mathfrak{A}_2\|_{L^2_+}^2 + \frac{1}{2}\int_{\mathbb{R}^3_+}\Big(\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\,\mathfrak{A}_2^2 - 2\varepsilon^{m+2}\hbar_2\,\mathfrak{A}_2\Big)\, dx\Big\} \\
&-\varepsilon\int_{\mathbb{R}^3_+}\big(\partial_t\mathfrak{A}_1 + \mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_1)\big)\,\partial_t\mathfrak{A}_2\, dx - \int_{\mathbb{R}^3_+} f_2\,\partial_t\mathfrak{A}_2\, dx = -\varepsilon^{m+2}\int_{\mathbb{R}^3_+}\partial_t\hbar_2\,\mathfrak{A}_2\, dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^3_+}\partial_t\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\,\mathfrak{A}_2^2\, dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^3_+}\nabla\, f_1\cdot\nabla\mathfrak{A}_1\, dx \\
&+ \varepsilon^{-1}\int_{\mathbb{R}^3_+} f_1\,\big(\varepsilon\mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_2) + \big(2(a^{\varepsilon,m})^2 + \chi_1\big)\mathfrak{A}_1 - \varepsilon^{m+1}\chi_2 + g_1 + g_2\big)\, dx.
\end{aligned}
\tag{8.11}
$$

On the other hand, we get, by taking the L^2 inner product of the $\mathfrak{A}_2$ equation of (8.8) with $\partial_t\mathfrak{A}_1$, that

$$
\begin{aligned}
&\varepsilon\int_{\mathbb{R}^3_+}\big(\partial_t\mathfrak{A}_2 + \mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_2)\big)\,\partial_t\mathfrak{A}_1\, dx + \frac{\varepsilon^2}{4}\frac{d}{dt}\|\nabla\,\mathfrak{A}_1\|_{L^2_+}^2 + \frac{d}{dt}\int_{\mathbb{R}^3_+}\big((a^{\varepsilon,m})^2 + \frac{1}{2}\chi_1\big)\mathfrak{A}_1^2\, dx \\
&= \int_{\mathbb{R}^3_+}\big(\varepsilon^{m+1}\chi_2 - g_1 - g_2\big)\partial_t\mathfrak{A}_1\, dx + \int_{\mathbb{R}^3_+}\big(2\, a^{\varepsilon,m}\partial_t a^{\varepsilon,m} + \partial_t\chi_1\big)\mathfrak{A}_1^2\, dx.
\end{aligned}
\tag{8.12}
$$

By using the $\mathfrak{A}_1$ equation of (8.8), we write

$$-\int_{\mathbb{R}^3_+} g_1\,\partial_t\mathfrak{A}_1\,dx = -\frac{\varepsilon}{2}\int_{\mathbb{R}^3_+}\nabla g_1\cdot\nabla\mathfrak{A}_2\,dx - \varepsilon^{-1}\int_{\mathbb{R}^3_+} g_1\,(f_1-f_2)\,dx$$
$$+\varepsilon^{m+1}\int_{\mathbb{R}^3_+} g_1\,\hbar_2\,dx + \varepsilon^{-1}\int_{\mathbb{R}^3_+} g_1\Big(\varepsilon\mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_1) - \big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\big)\mathfrak{A}_2\Big)\,dx.$$

By inserting the above equality into (8.12) and using integrating by parts, we find

$$\begin{aligned}
&\frac{\varepsilon^2}{4}\frac{d}{dt}\|\nabla\mathfrak{A}_1\|_{L^2_+}^2 + \frac{d}{dt}\int_{\mathbb{R}^3_+}\big(\big((a^{\varepsilon,m})^2+\frac{1}{2}\chi_1\big)\mathfrak{A}_1^2 - \varepsilon^{m+1}\chi_2\mathfrak{A}_1\big)\,dx\\
&\quad+\varepsilon\int_{\mathbb{R}^3_+}\big(\partial_t\mathfrak{A}_2+\mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_2)\big)\,\partial_t\mathfrak{A}_1\,dx + \int_{\mathbb{R}^3_+} g_2\,\partial_t\mathfrak{A}_1\,dx = -\varepsilon^{m+1}\int_{\mathbb{R}^3_+}\partial_t\chi_2\,\mathfrak{A}_1\,dx\\
&\quad+\int_{\mathbb{R}^3_+}\big(2\,a^{\varepsilon,m}\partial_t a^{\varepsilon,m}+\partial_t\chi_1\big)\mathfrak{A}_1^2\,dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^3_+}\nabla g_1\cdot\nabla\mathfrak{A}_2\,dx + \varepsilon^{m-1}\int_{\mathbb{R}^3_+} g_1\,\hbar_2\,dx\\
&\quad+\varepsilon^{-1}\int_{\mathbb{R}^3_+} g_1\Big(f_2-f_1+\varepsilon\mathcal{S}_{u^{\varepsilon,m}}(\mathfrak{A}_1) - \big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\big)\Big)\mathfrak{A}_2\Big)\,dx.
\end{aligned}\tag{8.13}$$

Thanks to (8.6), we get, by summing up (8.11) and (8.13), that

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{\varepsilon^2}{4}\|(\nabla\mathfrak{A}_1,\,\nabla\mathfrak{A}_2)\|_{L^2_+}^2 + \frac{1}{2}\int_{\mathbb{R}^3_+}\Big(\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\mathfrak{A}_2^2 - 2\varepsilon^{m+2}\hbar_2\,\mathfrak{A}_2\Big)\,dx\\
&\quad+\int_{\mathbb{R}^3_+}\big((a^{\varepsilon,m})^2+\frac{1}{2}\chi_1\big)\mathfrak{A}_1^2 - \varepsilon^{m+1}\chi_2\mathfrak{A}_1 + \mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)\,\mathfrak{A}_1\,\big)dx\Big\}\\
&\quad+\int_{\mathbb{R}^3_+}(g_2\,\partial_t\mathfrak{A}_1 - f_2\,\partial_t\mathfrak{A}_2)\,dx = \mathfrak{R},
\end{aligned}\tag{8.14}$$

where

$$\begin{aligned}
\mathfrak{R} \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3_+}\big(\mathcal{S}_{\partial_t u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)\,\mathfrak{A}_1 + \big(2\,a^{\varepsilon,m}\partial_t a^{\varepsilon,m}+\partial_t\chi_1\big)\mathfrak{A}_1^2\big)\,dx\\
&+\frac{1}{2}\int_{\mathbb{R}^3_+}\partial_t\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\mathfrak{A}_2^2\,dx - \varepsilon^{m+1}\int_{\mathbb{R}^3_+}\big(\partial_t\chi_2\mathfrak{A}_1 + \varepsilon\partial_t\hbar_2\,\mathfrak{A}_2\big)\,dx\\
&+\frac{\varepsilon}{2}\int_{\mathbb{R}^3_+}\big(\nabla f_1\cdot\nabla\mathfrak{A}_1 - \nabla g_1\cdot\nabla\mathfrak{A}_2\big)\,dx + \varepsilon^{-1}\int_{\mathbb{R}^3_+}\big(f_1\,\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2) + g_1\,\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_1)\big)\,dx\\
&+\varepsilon^{-1}\int_{\mathbb{R}^3_+}\Big(f_1\,\big(2(a^{\varepsilon,m})^2+\chi_1\big)\mathfrak{A}_1 - g_1\,\big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} - \varepsilon^{m+2}\hbar_1\Big)\mathfrak{A}_2\\
&\qquad\qquad\qquad + f_1\,g_2 + g_1\,f_2 - \varepsilon^{m+1}(f_1\chi_2 - \varepsilon g_1\,\hbar_2)\Big)\,dx.
\end{aligned}$$

Notice that $s_0-2m\geq 10$, we get, by applying Lemmas 8.1 and 8.2, that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3_+}\mathcal{S}_{\partial_t u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)\,\mathfrak{A}_1\,dx\Big| \lesssim \|\mathcal{S}_{\partial_t u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)\|_{L^2_+}\,\|\mathfrak{A}_1\|_{L^2_+} \lesssim \mathcal{E}_0^{\frac{1}{2}}\|\varepsilon\mathfrak{A}_2\|_{H^1}\,\|\mathfrak{A}_1\|_{L^2_+},\\
&\frac{\varepsilon^2}{4}\Big|\int_{\mathbb{R}^3_+}\partial_t\Big(\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big)\mathfrak{A}_1^2\,dx\Big| \lesssim \mathcal{E}_0^{\frac{1}{2}}\|\varepsilon\mathfrak{A}_1\|_{H^1}^2,
\end{aligned}$$

and

$$\varepsilon^{-1}\Big|\int_{\mathbb{R}^3_+}\big(f_1\,\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_2)+g_1\,\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathfrak{A}_1)\big)\,dx\Big|\lesssim\mathcal{E}_0^{\frac12}\varepsilon^{-1}\|(f_1,\,g_1)\|_{L^2_+}\|\varepsilon(\mathfrak{A}_1,\,\mathfrak{A}_2)\|_{H^1},$$

$$\varepsilon^{-1}\Big|\int_{\mathbb{R}^3_+}g_1\,\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\mathfrak{A}_2\,dx\Big|\lesssim\varepsilon^{-1}\|g_1\|_{L^2_+}\Big\|\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\mathfrak{A}_2\Big\|_{L^2_+}\lesssim\mathcal{E}_0^{\frac12}\varepsilon^{-1}\|g_1\|_{L^2_+}\|\varepsilon\mathfrak{A}_2\|_{H^1},$$

and

$$\Big|\int_{\mathbb{R}^3_+}(2\,a^{\varepsilon,m}\partial_t a^{\varepsilon,m}+\partial_t\chi_1)\mathfrak{A}_1^2\,dx\Big|\lesssim\big(\mathcal{E}_0+\|\partial_t\chi_1\|_{L^\infty_+}\big)\|\mathfrak{A}_1^2\|^2_{L^2_+},$$

$$\frac{\varepsilon^{m+2}}{2}\Big|\int_{\mathbb{R}^3_+}\partial_t\hbar_1\,\mathfrak{A}_2^2\,dx\Big|\lesssim\varepsilon^m\|\partial_t\hbar_1\|_{L^\infty_+}\|\varepsilon\mathfrak{A}_2\|^2_{L^2_+},$$

$$\varepsilon^{m+1}\Big|\int_{\mathbb{R}^3_+}\big(\partial_t\chi_2\mathfrak{A}_1+\varepsilon\partial_t\hbar_2\,\mathfrak{A}_2\big)\,dx\Big|\lesssim\varepsilon^{m+1}\|(\partial_t\chi_2,\partial_t\hbar_2)\|_{L^2_+}\|(\mathfrak{A}_1,\,\varepsilon\mathfrak{A}_2)\|_{L^2_+},$$

and

$$\varepsilon^{-1}\Big|\int_{\mathbb{R}^3_+}f_1\,\big(2(a^{\varepsilon,m})^2+\chi_1\big)\mathfrak{A}_1\,dx\Big|\lesssim\varepsilon^{-1}\big(\mathcal{E}_0+\|\chi_1\|_{L^\infty_+}\big)\|f_1\|_{L^2_+}\|\mathfrak{A}_1\|_{L^2_+},$$

$$\varepsilon^{m+1}\Big|\int_{\mathbb{R}^3_+}g_1\hbar_1\mathfrak{A}_2\,dx\Big|\lesssim\varepsilon^m\|\hbar_1\|_{L^\infty_+}\|g_1\|_{L^2_+}\|\varepsilon\mathfrak{A}_2\|_{L^2_+},$$

$$\varepsilon^m\Big|\int_{\mathbb{R}^3_+}\big(-f_1\chi_2+\varepsilon g_1\,\hbar_2\big)\,dx\Big|\lesssim\varepsilon^m\big(\|f_1\|_{L^2_+}\,\|\chi_2\|_{L^2_+}+\varepsilon\|g_1\|_{L^2_+}\,\|\hbar_2\|_{L^2_+}\big),$$

and

$$\frac{\varepsilon}{2}\Big|\int_{\mathbb{R}^3_+}\big(\nabla f_1\cdot\nabla\mathfrak{A}_1-\nabla g_1\cdot\nabla\mathfrak{A}_2\big)\,dx\Big|\lesssim\|(\nabla f_1,\nabla g_1)\|_{L^2_+}\|\varepsilon(\nabla\mathfrak{A}_1,\,\nabla\mathfrak{A}_2)\|_{L^2_+},$$

$$\varepsilon^{-1}\Big|\int_{\mathbb{R}^3_+}(f_1\,g_2+g_1\,f_2)\,dx\Big|\lesssim\varepsilon^{-1}\big(\|f_1\|_{L^2_+}\,\|g_2\|_{L^2_+}+\|g_1\|_{L^2_+}\,\|f_2\|_{L^2_+}\big).$$

By substituting the above inequalities into (8.14), we obtain (8.9). This completes the proof of Lemma 8.3. □

9. Validity of the WKB expansion

The goal of this section is to present the proof of Theorem 3.2.

Proposition 9.1. *Let $\Psi^{a,m}$ be given by (3.15). Then one has*

$$a^{\varepsilon,m}|_{z=0}=1,\quad\varphi^{\varepsilon,m}|_{z=0}=0\tag{9.1}$$

and

$$\mathrm{GP}(\Psi^{a,m})\overset{\text{def}}{=}i\varepsilon\partial_t\Psi^{a,m}+\frac{\varepsilon^2}{2}\Delta\Psi^{a,m}-\Psi^{a,m}(|\Psi^{a,m}|^2-1)=R^{\varepsilon,m}e^{\frac{i}{\varepsilon}\varphi^{\varepsilon,m}}\tag{9.2}$$

where $R^{\varepsilon,m}$ is of the form:

$$R^{\varepsilon,m}=-\,\varepsilon^{m+1}\,a^{\varepsilon,m}\big(\varepsilon\,R_a^{\mathrm{int},m}+[R_a^{b,m}]_\varepsilon\big)+i\varepsilon^{m+2}\,\big(\varepsilon\,R_\varphi^{\mathrm{int},m}+[R_\varphi^{b,m}]_\varepsilon\big).\tag{9.3}$$

Moreover, there holds

$$\big\|(R_a^{\mathrm{int},m},\;R_\varphi^{\mathrm{int},m})\big\|_{W^{s_0-2(m+3)}_{T_0}}+\big\|(R_a^{b,m},\;R_\varphi^{b,m})\big\|_{\mathcal{W}^{s_0-2(m+3)+\frac12}_{1,T_0}}\lesssim\mathcal{E}_0.\tag{9.4}$$

Proof. Let $a^{\varepsilon,m}$ and $\varphi^{\varepsilon,m}$ be given by (3.15). Then we observe from the computations presented in Section 2 that

$$\begin{aligned}
&\partial_t a^{\varepsilon,m} + \nabla(\varphi^{\varepsilon,m} - \varepsilon^{m+2}\varphi_{m+2})\cdot\nabla a^{\varepsilon,m} + \frac{1}{2}a^{\varepsilon,m}\Delta(\varphi^{\varepsilon,m} - \varepsilon^{m+2}\varphi_{m+2})\\
&\qquad\qquad = \varepsilon^{m+2}\big(R_a^{\mathrm{int},m} + \varepsilon^{-1}[R_a^{\mathrm{b},m}]_\varepsilon\big),\\
&\partial_t(\varphi^{\varepsilon,m} - \varepsilon^{m+2}\varphi_{m+2}) + \frac{1}{2}|\nabla(\varphi^{\varepsilon,m} - \varepsilon^{m+2}\varphi_{m+2})|^2 + (a^{\varepsilon,m})^2 - 1 - \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\\
&\qquad\qquad = \varepsilon^{m+2}\big(R_\varphi^{\mathrm{int},m} + \varepsilon^{-1}[R_\varphi^{\mathrm{b},m}]_\varepsilon\big),
\end{aligned}$$

where $R_a^{\mathrm{int},m}$, $R_\varphi^{\mathrm{int},m}$, $R_a^{\mathrm{b},m}$ and $R_\varphi^{\mathrm{b},m}$ satisfy

$$\big\|(R_a^{\mathrm{int},m},\ R_\varphi^{\mathrm{int},m})\big\|_{W_{T_0}^{s_0-2(m+3)}} + \big\|(R_\varphi^{\mathrm{b},m},\ R_\varphi^{\mathrm{b},m})\big\|_{\mathcal{W}_{1,T_0}^{s_0-2(m+3)+\frac{1}{2}}} \lesssim \mathcal{E}_0. \tag{9.5}$$

The above equations can also be written as

$$\begin{aligned}
&\partial_t a^{\varepsilon,m} + \nabla\varphi^{\varepsilon,m}\cdot\nabla a^{\varepsilon,m} + \frac{1}{2}a^{\varepsilon,m}\Delta\varphi^{\varepsilon,m}\\
&\quad = \varepsilon^{m+2}\big(\nabla\varphi_{m+2}\cdot\nabla a^{\varepsilon,m} + \frac{1}{2}a^{\varepsilon,m}\Delta\varphi_{m+2} + R_a^{\mathrm{int},m} + \varepsilon^{-1}[R_a^{\mathrm{b},m}]_\varepsilon\big),\\
&\partial_t\varphi^{\varepsilon,m} + \frac{1}{2}|\nabla\varphi^{\varepsilon,m}|^2 + (a^{\varepsilon,m})^2 - 1 - \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\\
&\quad = \varepsilon^{m+2}\Big(\partial_t\varphi_{m+2} + \frac{1}{2}(2\nabla\varphi^{\varepsilon,m}\cdot\nabla\varphi_{m+2} - \varepsilon^{m+2}|\nabla\varphi_{m+2}|^2) + R_\varphi^{\mathrm{int},m} + \varepsilon^{-1}[R_\varphi^{\mathrm{b},m}]_\varepsilon\Big),
\end{aligned}$$

from which, (9.5) and (8.2), we infer

$$\begin{aligned}
&\partial_t a^{\varepsilon,m} + \nabla\varphi^{\varepsilon,m}\cdot\nabla a^{\varepsilon,m} + \frac{1}{2}a^{\varepsilon,m}\Delta\varphi^{\varepsilon,m} = \varepsilon^{m+2}\big(R_a^{\mathrm{int},m} + \varepsilon^{-1}[R_a^{\mathrm{b},m}]_\varepsilon\big),\\
&\partial_t\varphi^{\varepsilon,m} + \frac{1}{2}|\nabla\varphi^{\varepsilon,m}|^2 + (a^{\varepsilon,m})^2 - 1 - \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} = \varepsilon^{m+2}\big(R_\varphi^{\mathrm{int},m} + \varepsilon^{-1}[R_\varphi^{\mathrm{b},m}]_\varepsilon\big),
\end{aligned}\tag{9.6}$$

where $R_a^{\mathrm{int},m}$, $R_\varphi^{\mathrm{int},m}$, $R_a^{\mathrm{b},m}$ and $R_\varphi^{\mathrm{b},m}$ satisfy (9.3).

On the other hand, it is easy to observe that

$$GP(\Psi^{a,m}) = R^{\varepsilon,m}e^{\frac{i}{\varepsilon}\varphi^{\varepsilon,m}} \quad \text{with} \quad R^{\varepsilon,m} = \big(-a^{\varepsilon,m}R_\varphi^m + \frac{\varepsilon^2}{2}\Delta a^{\varepsilon,m}\big) + i\varepsilon R_a^m,$$

where

$$\begin{aligned}
R_\varphi^m &= \partial_t\varphi^{\varepsilon,m} + \frac{1}{2}|\nabla\varphi^{\varepsilon,m}|^2 + (|a^{\varepsilon,m}|^2 - 1),\\
R_a^m &= \partial_t a^{\varepsilon,m} + \nabla\varphi^{\varepsilon,m}\cdot\nabla a^{\varepsilon,m} + \frac{1}{2}a^{\varepsilon,m}\Delta\varphi^{\varepsilon,m},
\end{aligned}\tag{9.7}$$

which along with (9.6) implies (9.3). This ends the proof of Proposition 9.1. □

Let us now turn to the proof of Proposition 3.6.

Proof of Proposition 3.6. Once again we shall only present the *a priori* estimates. Let w and ϕ be real-valued functions, we are going to seek the true solution of (1.1) with the form (3.17). In view of (1.2), (w,ϕ) satisfies the following initial condition

$$w|_{t=0} = \varepsilon^{m+2}\big(a_{m+2,0}^{\mathrm{in}} + R_{a,0}^\varepsilon\big), \quad \phi|_{t=0} = \varepsilon^{m+2}R_{\varphi,0}^\varepsilon \quad \text{with} \quad \lim_{\varepsilon\to 0}\big\|(R_{a,0}^\varepsilon, \nabla R_{\varphi,0}^\varepsilon)\big\|_{H^{s_0-2m-5}} = 0.$$

We shall divide the proof of Proposition 3.6 into the following steps:

Step 1. The derivation of the error equation

Substituting (3.18) into (1.1) yields

$$
\begin{aligned}
(9.8)\qquad & i\varepsilon\big(\partial_t \mathfrak{w} + (u^{\varepsilon,m}\cdot\nabla)\mathfrak{w} + \frac{1}{2}\mathfrak{w}\nabla\cdot u^{\varepsilon,m}\big) + \frac{\varepsilon^2}{2}\Delta\mathfrak{w} - 2w_{\mathrm{R}}(a^{\varepsilon,m})^2 \\
&\qquad\qquad = R_\varphi^m \mathfrak{w} - R^{\varepsilon,m} + Q^\varepsilon(\mathfrak{w}),
\end{aligned}
$$

where R_φ^m and $R^{\varepsilon,m}$ are defined in (9.7) and (9.3), and

$$
\begin{aligned}
(9.9)\qquad Q^\varepsilon(\mathfrak{w}) &\overset{\text{def}}{=} \big(a^{\varepsilon,m} + \mathfrak{w}\big)\big(|a^{\varepsilon,m} + \mathfrak{w}|^2 - |a^{\varepsilon,m}|^2\big) - 2w_{\mathrm{R}}(a^{\varepsilon,m})^2 \\
&= a^{\varepsilon,m}\big(w_{\mathrm{R}}^2 + w_{\mathrm{I}}^2\big) + \mathfrak{w}\big(w_{\mathrm{R}}^2 + w_{\mathrm{I}}^2 + 2\,a^{\varepsilon,m}\,w_{\mathrm{R}}\big).
\end{aligned}
$$

Notice that $w_{\mathrm{R}} = w\cos\phi + a^{\varepsilon,m}(\cos\phi - 1)$, $w_{\mathrm{I}} = (a^{\varepsilon,m} + w)\sin\phi$, we have the following initial boundary condition for $(w_{\mathrm{R}}, w_{\mathrm{I}})$:

$$
\begin{aligned}
& w_{\mathrm{R}}|_{z=0} = 0, \quad w_{\mathrm{I}}|_{z=0} = 0, \\
& w_{\mathrm{R}}|_{t=0} = \varepsilon^{m+2}\big(a_{m+2,0}^{\mathrm{in}} + R_{a,0}^\varepsilon\big)\cos\big(\varepsilon^{m+2}R_{\varphi,0}^\varepsilon\big) \\
(9.10)\qquad & \qquad\qquad + a^{\varepsilon,m}(0)\big(\cos\big(\varepsilon^{m+2}R_{\varphi,0}^\varepsilon\big) - 1\big) \overset{\text{def}}{=} w_{\mathrm{R},0}^\varepsilon, \\
& w_{\mathrm{I}}|_{t=0} = \big(a^{\varepsilon,m}(0) + \varepsilon^{m+2}\big(a_{m+2,0}^{\mathrm{in}} + R_{a,0}^\varepsilon\big)\big)\sin\big(\varepsilon^{m+2}R_{\varphi,0}^\varepsilon\big) \overset{\text{def}}{=} w_{\mathrm{I},0}^\varepsilon.
\end{aligned}
$$

Then by taking separating the imaginary and real parts of (9.8), we derive the system (3.19) for $(w_{\mathrm{R}}, w_{\mathrm{I}})$ with

$$
(9.11)\qquad r_a^m \overset{\text{def}}{=} \varepsilon\,R_a^{\mathrm{int},m} + [R_a^{\mathrm{b},m}]_\varepsilon \quad \text{and} \quad r_\varphi^m \overset{\text{def}}{=} \varepsilon\,R_\varphi^{\mathrm{int},m} + [R_\varphi^{\mathrm{b},m}]_\varepsilon.
$$

Step 2. The estimate of $\|\varepsilon(w_{\mathrm{R}}, w_{\mathrm{I}})\|_{L_T^\infty(H^1)}$

In view of (8.6) and (3.19), we get, by using L_+^2 energy estimate, that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+^3}\big(|\varepsilon w_{\mathrm{I}}|^2 + |\varepsilon w_{\mathrm{R}}|^2\big)\,dx &= \int_{\mathbb{R}_+^3}\Big(2(a^{\varepsilon,m})^2 + \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2}r_\varphi^m\Big)w_{\mathrm{R}}|\varepsilon w_{\mathrm{I}}\,dx \\
(9.12)\qquad &+ \int_{\mathbb{R}_+^3}\big(\varepsilon^{m+1}a^{\varepsilon,m}r_a^m - \mathrm{Re}Q^\varepsilon(\mathfrak{w})\big)|\varepsilon w_{\mathrm{I}}\,dx \\
&+ \int_{\mathbb{R}_+^3}\Big(\big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2}r_\varphi^m\big)\,w_{\mathrm{I}} + \mathrm{Im}Q^\varepsilon(\mathfrak{w}) - \varepsilon^{m+2}r_\varphi^m\Big)|\varepsilon w_{\mathrm{R}}\,dx.
\end{aligned}
$$

If $s_0 - 2m \geq 9$, we deduce from (8.2) and (9.4) that

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}_+^3}\Big(2(a^{\varepsilon,m})^2 + \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2}r_\varphi^m\Big)w_{\mathrm{R}}|\varepsilon w_{\mathrm{I}}\,dx\Big| \\
&\lesssim \Big(\|a^{\varepsilon,m}\|_{L_+^\infty}^2 + \varepsilon^{m+2}\|r_\varphi^m\|_{L_+^\infty} + \Big\|\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big\|_{L_+^\infty}\Big)\|w_{\mathrm{R}}\|_{L_+^2}\,\|\varepsilon w_{\mathrm{I}}\|_{L_+^2} \\
&\lesssim \|w_{\mathrm{R}}\|_{L_+^2}^2 + \|\varepsilon w_{\mathrm{I}}\|_{L_+^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}_+^3}\big(\varepsilon^{m+1}a^{\varepsilon,m}r_a^m - \mathrm{Re}Q^\varepsilon(\mathfrak{w})\big)|\varepsilon w_{\mathrm{I}}\,dx\Big| \\
&\lesssim \big(\varepsilon^{m+1}\|a^{\varepsilon,m}\|_{L_+^\infty}\|r_\varphi^m\|_{L_+^2} + \|\mathrm{Re}Q^\varepsilon(\mathfrak{w})\|_{L_+^2}\big)\|\varepsilon w_{\mathrm{I}}\|_{L_+^2} \\
&\lesssim \big(\varepsilon^{m+1}\mathcal{E}_0 + \|\mathrm{Re}Q^\varepsilon(\mathfrak{w})\|_{L_+^2}\big)\|\varepsilon w_{\mathrm{I}}\|_{L_+^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3_+}\Big(\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r^m_\varphi\Big)\,w_{\rm I}+{\rm Im}Q^\varepsilon(\mathfrak{w})-\varepsilon^{m+2}r^m_\varphi\Big)\big|\varepsilon w_{\rm R}\,dx\Big|\\
&\lesssim\Big(\Big(\frac{\varepsilon^2}{2}\Big\|\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big\|_{L^\infty_+}+\varepsilon^{m+2}\|r^m_\varphi\|_{L^\infty_+}\Big)\,\|\varepsilon w_{\rm I}\|_{L^2_+}+\|\varepsilon\,{\rm Im}Q^\varepsilon(\mathfrak{w})\|_{L^2_+}+\varepsilon^{m+3}\|r^m_\varphi\|_{L^2_+}\Big)\|w_{\rm R}\|_{L^2_+}\\
&\lesssim\big(\mathcal{E}_0^{\frac12}\,\|\varepsilon w_{\rm I}\|_{L^2_+}+\|\varepsilon\,{\rm Im}Q^\varepsilon(\mathfrak{w})\|_{L^2_+}+\varepsilon^{m+2}\mathcal{E}_0^{\frac12}\big)\|w_{\rm R}\|_{L^2_+}.
\end{aligned}
$$

By inserting the above inequalities into (9.12), we get

$$
\frac{d}{dt}\|\varepsilon(w_{\rm R},\,w_{\rm I})\|^2_{L^2_+}\lesssim\|(w_{\rm R},\varepsilon w_{\rm I}\|^2_{L^2_+}+\|{\rm Re}Q^\varepsilon(\mathfrak{w})\|^2_{L^2_+}+\|\varepsilon\,{\rm Im}Q^\varepsilon(\mathfrak{w})\|^2_{L^2_+}+\varepsilon^{2m+2}\mathcal{E}_0. \tag{9.13}
$$

On the other hand, by applying Lemma 8.3 with $\hbar_1=\hbar_2=r^m_\varphi,\ f_1={\rm Im}Q^\varepsilon(\mathfrak{w}),\ f_2=0$, $\chi_1=\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r^m_\varphi$, $\chi_2=a^{\varepsilon,m}r^m_a$, $g_1={\rm Re}Q^\varepsilon(\mathfrak{w})$ and $g_2=0$, we achieve

$$
\frac{d}{dt}\widetilde{\mathfrak{E}}_1\lesssim\mathfrak{E}_1+\varepsilon^{-2}\big(\|Q^\varepsilon(\mathfrak{w})\|^2_{L^2_+}+\|\varepsilon\nabla Q^\varepsilon(\mathfrak{w})\|^2_{L^2_+}\big), \tag{9.14}
$$

where

$$
\begin{aligned}
&\mathfrak{E}_1\stackrel{\rm def}{=}\varepsilon^{2m+2}\mathcal{E}_0+\|\varepsilon(\,w_{\rm I},\,w_{\rm R})\|^2_{H^1}+\|w_{\rm R}\|^2_{L^2_+}\quad\text{and}\\
&\widetilde{\mathfrak{E}}_1\stackrel{\rm def}{=}C_0\,\varepsilon^{2m+2}\mathcal{E}_0+\frac{\varepsilon^2}{4}\big\|(\nabla\,w_{\rm I},\,\nabla\,w_{\rm R})\big\|^2_{L^2_+}+\int_{\mathbb{R}^3_+}\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon w_{\rm I})|w_{\rm R}\,dx\\
&\qquad+\frac12\int_{\mathbb{R}^3_+}\Big(2(a^{\varepsilon,m})^2+\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r^m_\varphi\Big)w_{\rm R}^2\,dx\\
&\qquad+\frac12\int_{\mathbb{R}^3_+}\Big(\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}-\varepsilon^{m+2}r^m_\varphi\Big)w_{\rm I}^2-2\varepsilon^{m+2}r^m_\varphi\,w_{\rm I}-2\varepsilon^{m+1}a^{\varepsilon,m}r^m_a\,w_{\rm R}\Big)\,dx.
\end{aligned} \tag{9.15}
$$

Step 3. High-order tangential derivatives estimates

The main result states as follow, the proof of which will be postponed after the proof of Proposition 3.6.

Lemma 9.1. *Let $s_0\geq 2m+9+N$ be an integer, we denote*

$$
\begin{aligned}
&\mathfrak{E}_N\stackrel{\rm def}{=}\varepsilon^{2m+2}\mathcal{E}_0+\sum_{j=0}^{N-1}\big(\|\mathcal{T}^jw_{\rm R}\|^2_{L^2_+}+\|\varepsilon\mathcal{T}^j(w_{\rm R},\,w_{\rm I})\|^2_{H^1}\big)\quad\textit{and}\\
&\widetilde{\mathfrak{E}}_N\stackrel{\rm def}{=}C_N\varepsilon^{2m+2}\mathcal{E}_0+\sum_{j=0}^{N-1}\dot{E}_j\quad\textit{with}\\
&\dot{E}_j\stackrel{\rm def}{=}\Big\{\frac14\big\|\varepsilon\,\big(\nabla\,\mathcal{T}^j\,w_{\rm R},\,\nabla\,\mathcal{T}^j\,w_{\rm I}\big)\big\|^2_{L^2_+}+\int_{\mathbb{R}^3_+}(a^{\varepsilon,m})^2|\mathcal{T}^j\,w_{\rm R}|^2\,dx\\
&\qquad+\frac12\int_{\mathbb{R}^3_+}\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r^m_\varphi\Big)\big(|\mathcal{T}^j\,w_{\rm I}|^2+|\mathcal{T}^j\,w_{\rm R}|^2\big)\,dx\\
&\qquad-\int_{\mathbb{R}^3_+}\big(\varepsilon^{m+2}\mathcal{T}^jr^m_a+\frac12R_{\partial_t,1,j}\big)|\mathcal{T}^jw_{\rm I}\,dx\\
&\qquad+\int_{\mathbb{R}^3_+}\big(\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathcal{T}^j\,w_{\rm I})-\varepsilon^{m+1}\mathcal{T}^j(a^{\varepsilon,m}r^m_a)+\frac12R_{\partial_t,2,j}\big)|\mathcal{T}^j\,w_{\rm R}\,dx,
\end{aligned} \tag{9.16}
$$

where

$$
R_{\partial_t,1,j} \stackrel{\text{def}}{=} [\mathcal{T}^j;\, \mathcal{S}_{u^{\varepsilon,m}}](\varepsilon\, w_{\mathrm{R}}) - \Big[\mathcal{T}^j;\, \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2} r_\varphi^m\Big] w_{\mathrm{I}},
$$

(9.17)

$$
R_{\partial_t,2,j} \stackrel{\text{def}}{=} [\mathcal{T}^j;\, \mathcal{S}_{u^{\varepsilon,m}}](\varepsilon w_{\mathrm{I}}) + \Big[\mathcal{T}^j;\, 2(a^{\varepsilon,m})^2 + \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2} r_\varphi^m\Big] w_{\mathrm{R}}.
$$

Then we have

$$
\frac{d}{dt}\widetilde{\mathfrak{E}}_N \leq C\,\mathfrak{E}_N + \varepsilon^{-2}\sum_{j=0}^{N-1}\big(\|\mathcal{T}^j Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2 + \|\varepsilon\nabla\,\mathcal{T}^j Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big). \tag{9.18}
$$

Step 4. Estimates of nonlinear terms

Lemma 9.2. *Let $N \geq 4$ and $s_0 \geq 2m+6+N$ be integers. Then one has*

$$
\varepsilon^{-2}\sum_{j=0}^{N-1}\big(\|\mathcal{T}^j Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2 + \|\varepsilon\nabla\,\mathcal{T}^j Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big) \lesssim \varepsilon^{-8}\mathfrak{E}_N\big(1+\varepsilon^{-2}\mathfrak{E}_N\big)\,\mathfrak{E}_N. \tag{9.19}
$$

The proof of this lemma will be postponed below.

Next, we claim that

$$
\widetilde{\mathfrak{E}}_N \sim \mathfrak{E}_N, \quad \text{i.e.} \quad C_1^{-1}\mathfrak{E}_N \leq \widetilde{\mathfrak{E}}_N \leq C_1\mathfrak{E}_N \tag{9.20}
$$

for some positive constant C_1.

We first get, by a similar the proof of (9.27) and (9.29), that

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3_+}\big(R_{\partial_t,2,j}|\mathcal{T}^j\, w_{\mathrm{R}} - R_{\partial_t,1,j}|\mathcal{T}^j\, w_{\mathrm{I}}\big)\,dx\Big| \lesssim&\, \mathcal{E}_0^{\frac12}\|\mathcal{T}^j\,\varepsilon\, w_{\mathrm{I}}\|_{L^2_+}\sum_{k=0}^{j-1}\|\varepsilon\,\mathcal{T}^k\,(w_{\mathrm{R}},\, w_{\mathrm{I}})\|_{H^1}\\
&+ \mathcal{E}_0^{\frac12}\|\mathcal{T}^j\, w_{\mathrm{R}}\|_{L^2_+}\sum_{k=0}^{j}\big(\|\varepsilon\,\mathcal{T}^k\, w_{\mathrm{I}}\|_{H^1} + \|\mathcal{T}^k\, w_{\mathrm{R}}\|_{L^2_+}\big).
\end{aligned}
$$

Whereas it follows from Lemmas 8.1 and 8.2 that

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3_+}\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2} r_\varphi^m\Big)\big(|\mathcal{T}^j\, w_{\mathrm{I}}|^2 + |\mathcal{T}^j\, w_{\mathrm{R}}|^2\big)\,dx\Big|\\
&\lesssim \big(\mathcal{E}_0^{\frac12} + \varepsilon^m\|r_\varphi^m\|_{L^\infty_+}\big)\big\|\varepsilon\,\mathcal{T}^j(w_{\mathrm{R}}, w_{\mathrm{I}})\big\|_{H^1}^2 \lesssim \mathcal{E}_0^{\frac12}\big\|\varepsilon\,\mathcal{T}^j(w_{\mathrm{R}}, w_{\mathrm{I}})\big\|_{H^1}^2,
\end{aligned}
$$

and

$$
\Big|\int_{\mathbb{R}^3_+}\mathcal{S}_{u^{\varepsilon,m}}(\varepsilon\mathcal{T}^j\, w_{\mathrm{I}})|\mathcal{T}^j\, w_{\mathrm{R}}\,dx\Big| \lesssim \mathcal{E}_0^{\frac12}\|\varepsilon\mathcal{T}^j\, w_{\mathrm{I}}\|_{H^1}\|\mathcal{T}^j\, w_{\mathrm{R}}\|_{L^2_+},
$$

and

$$
\varepsilon^{m+1}\Big|\int_{\mathbb{R}^3_+}\big(\varepsilon\mathcal{T}^j r_a^m|\mathcal{T}^j w_{\mathrm{I}} - \mathcal{T}^j(a^{\varepsilon,m} r_a^m)|\mathcal{T}^j\, w_{\mathrm{R}}\big)\,dx\Big| \lesssim \mathcal{E}_0^{\frac12}\varepsilon^{m+1}\big(\|\varepsilon\mathcal{T}^j\, w_{\mathrm{I}}\|_{L^2} + \|\mathcal{T}^j\, w_{\mathrm{R}}\|_{L^2_+}\big).
$$

Finally, by virtue of (8.2), we have $\|a^{\varepsilon,m}-1\|_{L^\infty_+} \lesssim \mathcal{E}_0^{\frac12}$, from which and (9.16), we deduce that

$$
\begin{aligned}
\dot{E}_j \geq&\frac14\,\Big(\big\|\varepsilon\,(\nabla\,\mathcal{T}^j\, w_{\mathrm{R}},\, \nabla\,\mathcal{T}^j\, w_{\mathrm{I}})\big\|_{L^2_+}^2 + \|\mathcal{T}^j\, w_{\mathrm{R}}\|_{L^2_+}^2\Big)\\
&- C\mathcal{E}_0^{\frac12}\big(\|\varepsilon\,(\mathcal{T}^j\, w_{\mathrm{R}},\, \mathcal{T}^j\, w_{\mathrm{I}})\big\|_{H^1}^2 + \|\mathcal{T}^j\, w_{\mathrm{R}}\|_{L^2_+}^2\big) - C\mathcal{E}_0\varepsilon^{2m+2}.
\end{aligned}
$$

This ensures (9.20) as long as c in (3.3) and ε are small enough and C_N in (9.16) satisfies $C_N \geq C+1$.

Now we are in a position to complete the proof of Proposition 3.6. Indeed by inserting (9.19) into (9.18), we find

$$\frac{d}{dt}\widetilde{\mathfrak{E}}_N \le C\big(\mathfrak{E}_N + \varepsilon^{-8}\mathfrak{E}_N\big(1+\varepsilon^{-2}\mathfrak{E}_N\big)\,\mathfrak{E}_N\big). \tag{9.21}$$

Let T_0 be determined by Proposition 3.1 and $N \ge 4$, we define

$$T_2^\star \stackrel{\text{def}}{=} \sup\big\{T' \in (0,\,T],\quad \mathfrak{E}_N(t) \le \mathfrak{C}\mathcal{E}_0\varepsilon^{2m+2} \quad \forall \quad t \in [0,T']\,\big\} \tag{9.22}$$

for some positive constant $\mathfrak{C}$ to be determined later on. We are going to prove that $T_2^\star = T_0$ provided that c in (3.3) and ε are sufficiently small.

Indeed for $m \ge 4$ and $\varepsilon \le \Big(\frac{1}{4\mathfrak{C}\mathcal{E}_0}\Big)^{\frac{1}{2(m-3)}}$, one has

$$\varepsilon^{-8}\mathfrak{E}_N(t) \le \mathfrak{C}\mathcal{E}_0\varepsilon^{2m-6} \le \frac{1}{4} \quad \forall\, t \in [0, T_2^\star],$$

from which, (9.20) and (9.21), we infer

$$\frac{d}{dt}\widetilde{\mathfrak{E}}_N \le 2C\,\mathfrak{E}_N \le 2CC_1\widetilde{\mathfrak{E}}_N. \tag{9.23}$$

Thanks to (9.10), we get, by applying Gronwall's inequality to (9.23), that

$$\widetilde{\mathfrak{E}}_N(t) \le Ce^{2CC_1T_0}\,\mathcal{E}_0\varepsilon^{2m+2} \quad \forall t \le T_2^\star. \tag{9.24}$$

Then by taking $\mathfrak{C} = 2CC_1e^{2CC_1T_0}$ in (9.22), we deduce from (9.20) that

$$\mathfrak{E}_N(t) \le C_1\widetilde{\mathfrak{E}}_N(t) \le \frac{1}{2}\mathfrak{C}\mathcal{E}_0\varepsilon^{2m+2} \quad \forall t \le T_2^\star.$$

This contradicts with (9.22), and this in turn shows that $T_2^\star = T_0$, moreover, there holds (3.20). This completes the proof of Proposition 3.6. $\square$

Proposition 3.6 has been proved provided that we present the proof of Lemmas 9.1 and 9.2.

Proof of Lemma 9.1. By applying $\mathcal{T}^j$ with $j \in \{1,2,...,N-1\}$ to (3.19), we find

$$\begin{cases} \varepsilon\big(\partial_t + \mathcal{S}_{u^{\varepsilon,m}}(\cdot)\big)(\mathcal{T}^j\,w_{\mathrm{R}}) + \frac{\varepsilon^2}{2}\Delta\left(\mathcal{T}^j\,w_{\mathrm{I}}\right) \\ \quad = \Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2}r_\varphi^m\Big)\mathcal{T}^j w_{\mathrm{I}} - \varepsilon^{m+2}\mathcal{T}^j r_\varphi^m + \mathcal{T}^j\mathrm{Im}Q^\varepsilon(\mathfrak{w}) - R_{\partial_t,1,j}, \\ \varepsilon\big(\partial_t + \mathcal{S}_{u^{\varepsilon,m}}(\cdot)\big)(\mathcal{T}^j w_{\mathrm{I}}) - \frac{\varepsilon^2}{2}\Delta(\mathcal{T}^j w_{\mathrm{R}}) + \Big(2(a^{\varepsilon,m})^2 + \frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}} + \varepsilon^{m+2}r_\varphi^m\Big)\mathcal{T}^j\,w_{\mathrm{R}} \\ \quad = \varepsilon^{m+1}\mathcal{T}^j(a^{\varepsilon,m}r_a^m) - \mathcal{T}^j\mathrm{Re}Q^\varepsilon(\mathfrak{w}) - R_{\partial_t,2,j}, \\ \mathcal{T}^j\,w_{\mathrm{R}}|_{z=0} = 0, \quad \mathcal{T}^j w_{\mathrm{I}}|_{z=0} = 0, \end{cases} \tag{9.25}$$

with $R_{\partial_t,1,j}$ and $R_{\partial_t,2,j}$ being given by (9.17).

Notice that

$$\begin{aligned} \int_{\mathbb{R}^3_+}\big(R_{\partial_t,2,j}|\mathcal{T}^j\partial_t\,w_{\mathrm{R}} - R_{\partial_t,1,j}|\mathcal{T}^j\partial_t w_{\mathrm{I}}\big)\,dx =& \frac{d}{dt}\int_{\mathbb{R}^3_+}\big(R_{\partial_t,2,j}|\mathcal{T}^j\ w_{\mathrm{R}} - R_{\partial_t,1,j}|\mathcal{T}^j\,w_{\mathrm{I}}\big)\,dx \\ &- \int_{\mathbb{R}^3_+}\big(\partial_t\,R_{\partial_t,2,j}|\mathcal{T}^j\,w_{\mathrm{R}} - \partial_t\,R_{\partial_t,1,j}|\mathcal{T}^j\,w_{\mathrm{I}}\big)\,dx. \end{aligned}$$

Then for $\dot{E}_j$ given by (9.16), we get, by applying Lemma 8.3, that

$$\begin{aligned} &\frac{d}{dt}\dot{E}_j(t) - \int_{\mathbb{R}^3_+}\big(\partial_t\,R_{\partial_t,2,j}\,\mathcal{T}^j\,w_{\mathrm{R}} - \partial_t\,R_{\partial_t,1,j}\,\mathcal{T}^j\,w_{\mathrm{I}}\big)\,dx \\ &\lesssim \mathcal{E}_0\,\varepsilon^{2m+2} + \|\varepsilon\mathcal{T}^j(w_{\mathrm{R}},w_{\mathrm{I}})\|_{H^1}^2 + \|\mathcal{T}^j\,w_{\mathrm{R}}\|_{L^2_+}^2 \\ &\quad + \|(R_{\partial_t,1,j},R_{\partial_t,2,j})\|_{L^2_+}^2 + \varepsilon^{-2}\big(\|\mathcal{T}^jQ^\varepsilon(\mathfrak{w})\|_{L^2_+}^2 + \|\varepsilon\nabla\,\mathcal{T}^jQ^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big). \end{aligned} \tag{9.26}$$

Observing that

$$\begin{aligned}\partial_t\big[\mathcal{T}^j;\mathcal{S}_u\big]f =&\partial_t\mathcal{T}^j\mathcal{S}_u(f)-\mathcal{S}_u(\mathcal{T}^j\partial_t f)-\mathcal{S}_{\partial_t u}(\mathcal{T}^j f)\\ =&\big[\partial_t\mathcal{T}^j;\mathcal{S}_u\big]f-\mathcal{S}_{\partial_t u}(\mathcal{T}^j f),\end{aligned}$$

and

$$\partial_t\big[\mathcal{T}^j;g\big]f =\partial_t\mathcal{T}^j(gf)-g\mathcal{T}^j\partial_t f-\partial_t g\mathcal{T}^j f=\big[\partial_t\mathcal{T}^j;g\big]f-\partial_t g\mathcal{T}^j f.$$

In view of (9.17), we write

$$\begin{aligned}\partial_t R_{\partial_t,2,j} =&\big[\partial_t\mathcal{T}^j;\,\mathcal{S}_{u^{\varepsilon,m}}\big](\varepsilon w_{\rm I})+\Big[\partial_t\mathcal{T}^j;\,2(a^{\varepsilon,m})^2+\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r_\varphi^m\Big]w_{\rm R}\\ &-\mathcal{S}_{\partial_t u^{\varepsilon,m}}\big(\varepsilon\mathcal{T}^j w_{\rm I}\big)-\partial_t\Big(2(a^{\varepsilon,m})^2+\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r_\varphi^m\Big)\mathcal{T}^j w_{\rm R}.\end{aligned}$$

It follows from Lemmas 8.1 and 8.2 that for $s_0-2m-9\geq j+1$,

$$|\int_{\mathbb{R}^3_+}\partial_t\,R_{\partial_t,2,j}|\mathcal{T}^j\,w_{\rm R}\,dx\big|\lesssim\mathcal{E}_0^{\frac12}\sum_{k=0}^j\big(\|\mathcal{T}^k\varepsilon w_{\rm I}\|_{H^1}+\|\mathcal{T}^k w_{\rm R}\|_{L^2_+}\big)\,\|\mathcal{T}^j\,w_{\rm R}\|_{L^2_+}. \tag{9.27}$$

Along the same line, we write

$$\begin{aligned}\partial_t R_{\partial_t,1,j}\stackrel{\rm def}{=}&\big[\partial_t\mathcal{T}^j;\,\mathcal{S}_{u^{\varepsilon,m}}\big](\varepsilon\,w_{\rm R})-\mathcal{S}_{\partial_t u^{\varepsilon,m}}\big(\varepsilon\mathcal{T}^j w_{\rm R}\big)\\ &-\frac{\varepsilon^2}{2}\Big[\partial_t\mathcal{T}^j;\,\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big]w_{\rm I}+\frac{\varepsilon^2}{2}\partial_t\Big(\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big)\mathcal{T}^j w_{\rm I}+\varepsilon^{m-1}\partial_t\big[\mathcal{T}^j;r_\varphi^m\big]w_{\rm I}.\end{aligned} \tag{9.28}$$

Notice that

$$\big[\mathcal{T}^{j+1};\,\mathcal{S}_{u^{\varepsilon,m}}\big](\varepsilon\,w_{\rm R})=\varepsilon\sum_{k=1}^{j+1}C_j^k\mathcal{S}_{\mathcal{T}^k u^{\varepsilon,m}}(\mathcal{T}^{j+1-k}w_{\rm R}).$$

In view of (8.6), we write

$$\begin{aligned}\int_{\mathbb{R}^3_+}\big[\mathcal{T}^{j+1};\,\mathcal{S}_{u^{\varepsilon,m}}\big](\varepsilon\,w_{\rm R})|\mathcal{T}^j\,w_{\rm I}\,dx =&\varepsilon\sum_{k=1}^{j+1}C_j^k\int_{\mathbb{R}^3_+}\mathcal{S}_{\mathcal{T}^k u^{\varepsilon,m}}(\mathcal{T}^{j+1-k}w_{\rm R})|\mathcal{T}^j\,w_{\rm I}\,dx\\ =&-\sum_{k=1}^{j+1}C_j^k\int_{\mathbb{R}^3_+}\mathcal{S}_{\mathcal{T}^k u^{\varepsilon,m}}(\varepsilon\mathcal{T}^j\,w_{\rm I})|\mathcal{T}^{j+1-k}w_{\rm R}\,dx,\end{aligned}$$

from which and (8.7), we infer

$$|\int_{\mathbb{R}^3_+}\big[\mathcal{T}^{j+1};\,\mathcal{S}_{u^{\varepsilon,m}}\big](\varepsilon\,w_{\rm R})|\mathcal{T}^j\,w_{\rm I}\,dx\big|\lesssim\mathcal{E}_0^{\frac12}\|\varepsilon\mathcal{T}^j\,w_{\rm I}\|_{H^1}\sum_{k=0}^j\|\mathcal{T}^k w_{\rm R}\|_{L^2_+}.$$

The same estimate holds for $\int_{\mathbb{R}^3_+}\mathcal{S}_{\partial_t u^{\varepsilon,m}}\big(\varepsilon\mathcal{T}^j w_{\rm R}\big)\,|\mathcal{T}^j\,w_{\rm I}\,dx$.

While applying Lemma 8.1 yields

$$\begin{aligned}&|\int_{\mathbb{R}^3_+}\Big(-\frac{\varepsilon^2}{2}\big[\partial_t\mathcal{T}^j;\,\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\big]w_{\rm I}+\frac{\varepsilon^2}{2}\partial_t\Big(\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}\Big)\mathcal{T}^j w_{\rm I}\Big)|\mathcal{T}^j\,w_{\rm I}\,dx|\\ &\qquad\lesssim\mathcal{E}_0^{\frac12}\|\varepsilon\mathcal{T}^j\,w_{\rm I}\|_{H^1}\sum_{k=0}^j\|\varepsilon\mathcal{T}^k w_{\rm I}\|_{H^1}.\end{aligned}$$

And it follows from (9.4) and (9.11) that

$$\begin{aligned}
&\varepsilon^{m+2}\Big|\int_{\mathbb{R}^3_+}\partial_t\big[\mathcal{T}^j;r_\varphi^m\big]w_{\rm I}|\mathcal{T}^j\,w_{\rm I}\,dx\Big|\\
&\lesssim \varepsilon^m\sum_{k=1}^j\big(\|\mathcal{T}^k\partial_t r_\varphi^m\varepsilon\mathcal{T}^{j-k}w_{\rm I}\|_{L^2_+}+\|\mathcal{T}^k r_\varphi^m\,\varepsilon\mathcal{T}^{j-k}\partial_t w_{\rm I}\|_{L^2_+}\big)\|\varepsilon\mathcal{T}^j\,w_{\rm I}\|_{L^2_+}\\
&\lesssim \varepsilon^m\sum_{k=1}^j\mathcal{E}_0^{\frac12}\big(\|\varepsilon\,\mathcal{T}^{j-k}w_{\rm I}\|_{L^2_+}+\|\varepsilon\mathcal{T}^{j-k}\partial_t w_{\rm I}\|_{L^2_+}\big)\|\varepsilon\mathcal{T}^j\,w_{\rm I}\|_{L^2_+}.
\end{aligned}$$

As a result, thanks to (9.28), we conclude

$$\Big|\int_{\mathbb{R}^3_+}\partial_t\,R_{\partial_t,1,j}|\mathcal{T}^j\,w_{\rm I}\,dx\Big|\lesssim\mathcal{E}_0^{\frac12}\|\varepsilon\,\mathcal{T}^j\,w_{\rm I}\|_{H^1}\sum_{k=0}^j\big(\|\mathcal{T}^k w_{\rm R}\|_{L^2_+}+\|\mathcal{T}^k\varepsilon w_{\rm I}\|_{H^1}\big). \tag{9.29}$$

Finally, it follows from Lemmas 8.1 and 8.2 that

$$\|R_{\partial_t,1,j}\|_{L^2_+}+\|R_{\partial_t,2,j}\|_{L^2_+}\lesssim\sum_{k=0}^{j-1}\big(\|\mathcal{T}^k w_{\rm R}\|_{L^2_+}+\|\mathcal{T}^k\varepsilon(w_{\rm R},w_{\rm I})\|_{H^1}\big). \tag{9.30}$$

By inserting the estimates (9.27), (9.29) and (9.30) into (9.26) gives rise to

$$\begin{aligned}
\frac{d}{dt}\dot E_j(t)\lesssim&\mathcal{E}_0\,\varepsilon^{2m+2}+\sum_{k=0}^j\big(\|\varepsilon\mathcal{T}^k(w_{\rm R},w_{\rm I})\|_{H^1}^2+\|\mathcal{T}^k\,w_{\rm R}\|_{L^2_+}^2\big)\\
&+\varepsilon^{-2}\big(\|\mathcal{T}^jQ^\varepsilon(\mathfrak{w})\|_{L^2_+}^2+\|\varepsilon\nabla\,\mathcal{T}^jQ^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big).
\end{aligned} \tag{9.31}$$

Summing up (9.14) and (9.31) for j from 1 to $N-1$ leads to (9.18). This completes the proof of Lemma 9.1. □

Proof of Lemma 9.2. Let's estimate the nonlinear terms in (9.18). Indeed in view of (9.9), we have

$$\begin{aligned}
&{\rm Re}Q^\varepsilon(\mathfrak{w})=3a^{\varepsilon,m}w_{\rm R}^2+a^{\varepsilon,m}w_{\rm I}^2+w_{\rm R}^3+w_{\rm R}\,w_{\rm I}^2,\\
&{\rm Im}Q^\varepsilon(\mathfrak{w})=w_{\rm I}w_{\rm R}^2+w_{\rm I}^3+2\,a^{\varepsilon,m}\,w_{\rm I}\,w_{\rm R}.
\end{aligned} \tag{9.32}$$

Recalling the Sobolev embedding and the classical interpolation inequality that

$$\|f\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\lesssim\varepsilon^{-1}\|f\|_{L^2_+}\|\varepsilon\partial_z f\|_{L^2},\qquad\|f\|_{L^2_{\rm v}(L^\infty_{\rm h})}^2\lesssim\|f\|_{L^2_+}\|(1+\nabla_{\rm h}^2)f\|_{L^2_+},$$

and

$$\begin{aligned}
\|f\|_{L^\infty_+}^2\lesssim&\|f\|_{L^\infty_{\rm v}(L^2_{\rm h})}\|f\|_{L^\infty_{\rm v}(H^2_{\rm h})}\\
\lesssim&\|f\|_{L^2_+}^{\frac12}\|f\|_{H^1}^{\frac12}\|(1+\nabla_{\rm h}^2)f\|_{L^2_+}^{\frac12}\|(1+\nabla_{\rm h}^2)\,f\|_{H^1}^{\frac12}.
\end{aligned} \tag{9.33}$$

As a result, it comes out

$$\begin{aligned}
&\|w_{\rm R}\|_{L^\infty_+}^2+\|\mathcal{T}w_{\rm R}\|_{L^\infty_+}^2\lesssim\varepsilon^{-1}\,\mathfrak{E}_4,\quad\|\mathcal{T}^{j-\ell}\,w_{\rm R}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\lesssim\varepsilon^{-1}\mathfrak{E}_{j-\ell+1},\\
&\|\mathcal{T}^\ell w_{\rm R}\|_{L^2_{\rm v}(L^\infty_{\rm h})}^2\lesssim\mathfrak{E}_{\ell+2},\quad\|\mathcal{T}^\ell\nabla w_{\rm R}\|_{L^2_{\rm v}(L^\infty_{\rm h})}^2\lesssim\varepsilon^{-2}\mathfrak{E}_{\ell+3},
\end{aligned} \tag{9.34}$$

and

$$\begin{aligned}
&\|w_{\rm I}\|_{L^\infty_+}^2+\|\mathcal{T}w_{\rm I}\|_{L^\infty_+}^2\lesssim\varepsilon^{-2}\,\mathfrak{E}_4,\quad\|\mathcal{T}^{j-\ell}\,w_{\rm I}\|_{L^\infty_{\rm v}(L^2_{\rm h})}^2\lesssim\varepsilon^{-2}\mathfrak{E}_{j-\ell+1},\\
&\|\mathcal{T}^\ell w_{\rm I}\|_{L^2_{\rm v}(L^\infty_{\rm h})}^2\lesssim\varepsilon^{-2}\mathfrak{E}_{\ell+2},\quad\|\mathcal{T}^\ell\nabla w_{\rm I}\|_{L^2_{\rm v}(L^\infty_{\rm h})}^2\lesssim\varepsilon^{-2}\mathfrak{E}_{\ell+3}.
\end{aligned} \tag{9.35}$$

It follows from (9.34) and (9.35) that for $j \le N-1$,

$$
\begin{aligned}
\|\mathcal{T}^j\,(w_{\mathrm{R}}^2)\|_{L^2_+}^2 \lesssim & \sum_{k=0}^{(j-2)_+} \|\mathcal{T}^{j-\ell}\,w_{\mathrm{R}}\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}^2 \|\mathcal{T}^\ell w_{\mathrm{R}}\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}^2 \\
& + \big(\|\mathcal{T}^{j-1}\,w_{\mathrm{R}}\|_{L^2_+}^2 + \|\mathcal{T}^j\,w_{\mathrm{R}}\|_{L^2_+}^2\big)\big(\|w_{\mathrm{R}}\|_{L^\infty_+}^2 + \|\mathcal{T} w_{\mathrm{R}}\|_{L^\infty_+}^2\big) \\
\lesssim & \varepsilon^{-1}\Big(\sum_{\ell=0}^{(j-2)_+} \mathfrak{E}_{j-\ell+1}\mathfrak{E}_{\ell+2} + \mathfrak{E}_4\mathfrak{E}_j\Big) \lesssim \varepsilon^{-1}\mathfrak{E}_N^2 \quad \text{if} \quad N \ge 4.
\end{aligned} \tag{9.36}
$$

Along the same line, we have

$$
\begin{aligned}
&\|\mathcal{T}^j\,(w_{\mathrm{I}}^2)\|_{L^2_+}^2 + \|\mathcal{T}^j\,(w_{\mathrm{I}} w_{\mathrm{R}})\|_{L^2_+}^2 \lesssim \varepsilon^{-4}\mathfrak{E}_N^2, \\
&\|\mathcal{T}^j\,(w_{\mathrm{I}}^3,\, w_{\mathrm{R}}^3,\, w_{\mathrm{I}}^2 w_{\mathrm{R}},\, w_{\mathrm{I}} w_{\mathrm{R}}^2)\|_{L^2_+}^2 \lesssim \varepsilon^{-6}\mathfrak{E}_N^3.
\end{aligned}
$$

Then by virtue of (8.2), for $s_0 - 2m \ge j+7$, we have

$$
\|\mathcal{T}^j\,(a^{\varepsilon,m} w_{\mathrm{R}}^2)\|_{L^2_+}^2 \lesssim \sum_{\ell=0}^{j} \|\mathcal{T}^{j-\ell}\,a^{\varepsilon,m}\|_{L^\infty_+}^2 \|\mathcal{T}^\ell w_{\mathrm{R}}^2\|_{L^2_+}^2 \lesssim \varepsilon^{-1}\sum_{k=0}^{N-1} \mathfrak{E}_{\ell+1}^2 \lesssim \varepsilon^{-1}\mathfrak{E}_N^2.
$$

Similarly, we have

$$
\|\mathcal{T}^j (a^{\varepsilon,m} w_{\mathrm{I}} w_{\mathrm{R}})\|_{L^2_+}^2 + \|\mathcal{T}^j (a^{\varepsilon,m} w_{\mathrm{I}}^2)\|_{L^2_+}^2 \lesssim \varepsilon^{-4}\mathfrak{E}_N^2.
$$

Therefore, we conclude that

$$
\varepsilon^{-2}\sum_{j=0}^{N-1}\big(\|\mathcal{T}^j \mathrm{Re}(Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2 + \|\mathcal{T}^j \mathrm{Im}(Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big) \lesssim \varepsilon^{-6}\mathfrak{E}_N^2 + \varepsilon^{-8}\mathfrak{E}_N^3. \tag{9.37}
$$

On the other hand, we observe that

$$
\begin{aligned}
\|\mathcal{T}^j\,\nabla(a^{\varepsilon,m} w_{\mathrm{R}}^2)\|_{L^2_+}^2 \lesssim & \sum_{\ell=0}^{j}\big(\|\mathcal{T}^{j-\ell}\,\nabla\,a^{\varepsilon,m}\|_{L^\infty_+}^2 \|\mathcal{T}^\ell (w_{\mathrm{R}})^2\|_{L^2_+}^2 \\
& + \|\mathcal{T}^{j-\ell}\,a^{\varepsilon,m}\|_{L^\infty_+}^2 \|\mathcal{T}^\ell (w_{\mathrm{R}}\,\nabla w_{\mathrm{R}})\|_{L^2_+}^2\big) \\
\lesssim & \varepsilon^{-2}\sum_{\ell=0}^{j} \|\mathcal{T}^\ell w_{\mathrm{R}}^2\|_{L^2_+}^2 + \sum_{\ell=0}^{j} \|\mathcal{T}^\ell (w_{\mathrm{R}}\,\nabla w_{\mathrm{R}})\|_{L^2_+}^2.
\end{aligned} \tag{9.38}
$$

Yet notice that

$$
\begin{aligned}
\|\mathcal{T}^\ell\,(w_{\mathrm{R}}\,\nabla\,w_{\mathrm{R}})\|_{L^2_+}^2 \lesssim & \sum_{k=0}^{(\ell-2)_+} \|\mathcal{T}^{\ell-k}\,w_{\mathrm{R}}\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}^2 \|\mathcal{T}^k \nabla w_{\mathrm{R}}\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}^2 \\
& + \|\mathcal{T}^{\ell-1}\,w_{\mathrm{R}}\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}^2 \|\mathcal{T}\nabla w_{\mathrm{R}}\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}^2 + \|\mathcal{T}^\ell\,w_{\mathrm{R}}\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}^2 \|\nabla w_{\mathrm{R}}\|_{L^2_{\mathrm{v}}(L^\infty_{\mathrm{h}})}^2,
\end{aligned}
$$

which together with (9.34) and (9.35) ensures that

$$
\|\mathcal{T}^\ell\,(w_{\mathrm{R}}\,\nabla\,w_{\mathrm{R}})\|_{L^2_+}^2 \lesssim \varepsilon^{-3}\sum_{k=0}^{(\ell-2)_+} \mathfrak{E}_{\ell-k+1}\mathfrak{E}_{k+3} + \varepsilon^{-3}\mathfrak{E}_{\ell+1}\mathfrak{E}_4 \lesssim \varepsilon^{-3}\mathfrak{E}_N^2. \tag{9.39}
$$

By inserting (9.36) and (9.39) into (9.38) gives rise to

$$
\|\mathcal{T}^j\,\nabla(a^{\varepsilon,m} w_{\mathrm{R}}^2)\|_{L^2_+}^2 \lesssim \varepsilon^{-3}\mathfrak{E}_N^2 \quad \text{for} \quad j \le N-1.
$$

Exactly along the same line, we achieve

$$\begin{aligned}&\|\mathcal{T}^{N-1}\nabla(a^{\varepsilon,m}w_{\rm R}w_{\rm I})\|_{L^2_+}^2+\|\mathcal{T}^{N-1}\nabla(a^{\varepsilon,m}w_{\rm I}^2)\|_{L^2_+}^2\lesssim\varepsilon^{-6}\,\mathfrak{E}_N^2,\\&\|\mathcal{T}^{N-1}\nabla(w_{\rm R}^3,\,w_{\rm I}^3,\,w_{\rm R}^2w_{\rm R},\,w_{\rm R}w_{\rm I}^2)\|_{L^2_+}^2\lesssim\ (\varepsilon^{-4}\mathfrak{E}_N)^2\,\mathfrak{E}_N.\end{aligned}$$

As a result, it comes out

$$\varepsilon^{-2}\sum_{j=0}^{N-1}\big(\|\mathcal{T}^j\nabla{\rm Re}Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2+\|\mathcal{T}^j\nabla{\rm Im}Q^\varepsilon(\mathfrak{w})\|_{L^2_+}^2\big)\lesssim\varepsilon^{-8}\mathfrak{E}_N^2+\varepsilon^{-10}\mathfrak{E}_N^3.$$

Along with (9.37), we obtain (9.19). This completes the proof of Lemma 9.2. □

Now we are in a position to complete the proof of Theorem 3.2.

Proof of Theorem 3.2. In order to get the second order full derivatives of $(w_{\rm R},w_{\rm I})$, we may make use of the system (9.25) for $j=0,1,2$. In fact, according to the $w_{\rm R}$ equation of (9.25), we get

$$\begin{aligned}\varepsilon^2\big\|\Delta\left(\mathcal{T}^j\,w_{\rm I}\right)\big\|_{L^2_+}\lesssim&\varepsilon\|\partial_t\mathcal{T}^j\,w_{\rm R}\|_{L^2_+}+\varepsilon\|\mathcal{S}_{u^{\varepsilon,m}}(\mathcal{T}^j\,w_{\rm R})\|_{L^2_+}+\|\mathcal{T}^j{\rm Im}(Q^\varepsilon(\mathfrak{w})\|_{L^2_+}\\&+\Big\|\Big(\frac{\varepsilon^2}{2}\frac{\Delta a^{\varepsilon,m}}{a^{\varepsilon,m}}+\varepsilon^{m+2}r_\varphi^m\Big)\mathcal{T}^jw_{\rm I}\Big\|_{L^2_+}+\varepsilon^{m+2}\|\mathcal{T}^jr_\varphi^m\|_{L^2_+}+\|R_{\partial_t,1,j}\|_{L^2_+}^2.\end{aligned}$$

Thanks to (8.2), (9.30) and (9.37), we get, by applying Lemmas 8.1 and 8.2, that

$$\begin{aligned}\varepsilon^2\big\|\Delta\left(\mathcal{T}^j\,w_{\rm I}\right)\big\|_{L^2_+}\lesssim&\varepsilon\|\partial_t\mathcal{T}^j\,w_{\rm R}\|_{L^2_+}+\varepsilon^{-4}\mathcal{E}_4^2\big(1+\varepsilon^{-2}\mathcal{E}_4\big)\\&+\mathcal{E}_0^{\frac12}\varepsilon^{m+1}+\sum_{\ell=0}^j\big(\|\mathcal{T}^\ell w_{\rm R}\|_{L^2_+}+\varepsilon\big\|\mathcal{T}^\ell(w_{\rm R},w_{\rm I})\big\|_{H^1}\big),\end{aligned}$$

from which and Proposition 3.6, we infer

$$\sum_{j=0}^2\big\|\Delta\left(\mathcal{T}^j\,w_{\rm I}\right)\big\|_{L^\infty_{T_0}(L^2_+)}\lesssim\mathcal{E}_0^{\frac12}\varepsilon^{m-1}.$$

The same estimate holds for $\Delta\left(\mathcal{T}^j\,w_{\rm R}\right)$, and then

$$\sum_{j=0}^2\big\|\Delta\mathcal{T}^j\,(w_{\rm R},\,w_{\rm I})\big\|_{L^\infty_{T_0}(L^2_+)}\lesssim\mathcal{E}_0^{\frac12}\varepsilon^{m-1}.\tag{9.40}$$

While it follows from Proposition 3.6 that

$$\sum_{j=0}^3\|\mathcal{T}^j(w_{\rm R},w_{\rm I})\|_{L^\infty_{T_0}(H^1)}\lesssim\mathcal{E}_0^{\frac12}\varepsilon^m.\tag{9.41}$$

Let us now turn to estimate $\|\mathfrak{w}\|_{W^{1,\infty}}$ for $\mathfrak{w}$ given by (3.18). We first deduce from (9.34) and (9.35) that

$$\|\mathfrak{w}\|_{L^\infty_+}\leq\|w_{\rm R}\|_{L^\infty_+}+\|w_{\rm I}\|_{L^\infty_+}\leq C\varepsilon^{-2}\mathfrak{E}_4^{\frac12}\leq C\mathcal{E}_0^{\frac12}\varepsilon^{m-1}.\tag{9.42}$$

On the other hand, we deduce from (9.33) that

$$\begin{aligned}\|\nabla f\|_{L^\infty_+}^2=&\|\nabla_{\rm h}f\|_{L^\infty_+}^2+\|\partial_z f\|_{L^\infty_+}^2\\\lesssim&\|\nabla_{\rm h}f\|_{H^1}\|(1+\nabla_{\rm h}^2)\,\nabla_{\rm h}f\|_{H^1}+\|\partial_zf\|_{H^1}\|(1+\nabla_{\rm h}^2)\,\partial_zf\|_{H^1},\end{aligned}$$

which along with the fact that

$$
\begin{aligned}
\|\partial_z f\|_{H^1} &\lesssim \|\nabla f\|_{L^2_+} + \|\nabla_{\rm h}\,\partial_z f\|_{L^2_+} + \|\partial_z^2 f\|_{L^2_+}\\
&\lesssim \|f\|_{H^1} + \|\nabla_{\rm h} f\|_{H^1} + \|\Delta f\|_{L^2_+} + \|\nabla_{\rm h}^2 f\|_{L^2_+}.
\end{aligned}
$$

ensures that

$$
\begin{aligned}
\|\nabla f\|_{L^\infty_+}^2 &\lesssim \|\nabla_{\rm h} f\|_{H^1}\|(1+\nabla_{\rm h}^2)\,\nabla_{\rm h} f\|_{H^1} + \big(\|f\|_{H^1} + \|\nabla_{\rm h} f\|_{H^1} + \|\Delta f\|_{L^2_+} + \|\nabla_{\rm h}^2 f\|_{L^2_+}\big)\\
&\quad\times\big(\|(1+\nabla_{\rm h}^2)f\|_{H^1} + \|\nabla_{\rm h}(1+\nabla_{\rm h}^2)\,f\|_{H^1} + \|\Delta(1+\nabla_{\rm h}^2)\,f\|_{L^2_+} + \|\nabla_{\rm h}^2(1+\nabla_{\rm h}^2)\,f\|_{L^2_+}\big)\\
&\lesssim \sum_{j=0}^{3}\|\mathcal{T} f\|_{H^1}^2 + \sum_{j=0}^{2}\|\Delta\mathcal{T} f\|_{L^2_+}^2.
\end{aligned}
$$

Therefore, we obtain from (9.40) and (9.41) that for any $t\in[0,T_0]$

$$
\begin{aligned}
\|\nabla\mathfrak{w}(t)\|_{L^\infty} &\leq \|\nabla w_{\rm R}(t)\|_{L^\infty} + \|\nabla w_{\rm I}(t)\|_{L^\infty}\\
&\leq C\Big(\sum_{j=0}^{3}\|\mathcal{T}(w_{\rm R},\, w_{\rm I})\|_{H^1} + \sum_{j=0}^{2}\|\Delta\mathcal{T}(w_{\rm R},\, w_{\rm I})\|_{L^2_+}\Big)\\
&\leq C\mathcal{E}_0^{\frac{1}{2}}\varepsilon^{m-1}.
\end{aligned}
$$

This together with (9.42) ensures (3.21). This ends the proof of Theorem 3.2. $\square$

Appendix A. The source terms F_k in (2.22) and G_k in (2.23)

Indeed we observe from (2.13) that

$$
\begin{aligned}
F_k \overset{\text{def}}{=} &-\partial_t A_k - \sum_{\ell=0}^{k} F_{1,\ell} - \sum_{\ell=2}^{k+1} F_{2,\ell} - \sum_{\ell=1}^{k} F_{3,\ell} - \sum_{\ell=1}^{k+1} F_{4,\ell} + \sum_{\ell_1+\ell_2=k-1} F_{5,\ell_1,\ell_2}\\
&- \sum_{\substack{\ell_1+\ell_2+j=k\\ 2\leq j\leq k}} F_{6,\ell_1,\ell_2,j} - \sum_{\substack{\ell_1+\ell_2+j=k+1\\ 2\leq j\leq k+1}} F_{7,\ell_1,\ell_2,j} - \sum_{\substack{\ell_1+\ell_2+j=k+2\\ 2\leq j\leq k+2}} F_{8,\ell_1,\ell_2,j}
\end{aligned}
\tag{A.1}
$$

where

$$
\begin{aligned}
F_{1,\ell} \overset{\text{def}}{=} &\nabla_{\rm h}\Phi_\ell\cdot\nabla_{\rm h}A_{k-\ell} + \frac{A_\ell}{2}\big(\Delta_{\rm h}\Phi_{k-\ell} + \overline{\Delta\varphi_{k-\ell}}\big) + \overline{\nabla_y\varphi_\ell}\cdot\nabla_{\rm h}A_{k-\ell} + \nabla_{\rm h}\Phi_\ell\cdot\overline{\nabla_{\rm h}a_{k-\ell}}\\
&+\frac{\overline{a_\ell}}{2}\Delta_{\rm h}\Phi_{k-\ell} + Z\big(\overline{\partial_z^2\varphi_\ell}\partial_Z A_{k-\ell} + \partial_Z\Phi_\ell\overline{\partial_z^2 a_{k-\ell}}\big) + \frac{Z^2}{2}\frac{\overline{\partial_z^2 a_\ell}}{2}\partial_Z^2\Phi_{k-\ell},\\
F_{2,\ell} \overset{\text{def}}{=} &\,\partial_Z\Phi_\ell\partial_Z A_{k+2-\ell},\quad F_{3,\ell} \overset{\text{def}}{=} \frac{1}{2}(A_\ell + \overline{a_\ell})\partial_Z^2\Phi_{k+2-\ell},
\end{aligned}
$$

and

$$
\begin{aligned}
F_{4,\ell} \overset{\text{def}}{=} &\overline{\partial_z\varphi_\ell}\partial_Z A_{k+1-\ell} + \partial_Z\Phi_\ell\overline{\partial_z a_{k+1-\ell}} + Z\frac{\overline{\partial_z a_{k+1-\ell}}}{2}\partial_Z^2\Phi_\ell,\\
F_{5,\ell_1,\ell_2} \overset{\text{def}}{=} &Z\Big(\overline{\nabla_y\partial_z\varphi_{\ell_1}}\cdot\nabla_{\rm h}A_{\ell_2} + \nabla_{\rm h}\Phi_{\ell_1}\cdot\overline{\nabla_{\rm h}\partial_z a_{\ell_2}} + \frac{\overline{\partial_z a_{\ell_1}}}{2}\Delta_{\rm h}\Phi_{\ell_2} + \frac{A_{\ell_1}}{2}\overline{\Delta\partial_z\varphi_{\ell_2}}\Big)\\
&+\frac{Z^2}{2}\big(\overline{\partial_z^3\varphi_{\ell_1}}\partial_Z A_{\ell_2} + \partial_Z\Phi_{\ell_1}\overline{\partial_z^3 a_{\ell_2}}\big),
\end{aligned}
$$

and

$$F_{6,\ell_1,\ell_2,j} \stackrel{\text{def}}{=} \frac{Z^j}{j!}\Big(\overline{\nabla_y\partial_z^j\varphi_{\ell_1}}\cdot\nabla_{\rm h}A_{\ell_2}+\nabla_{\rm h}\Phi_{\ell_1}\cdot\overline{\nabla_{\rm h}\partial_z^j a_{\ell_2}}+\frac{\overline{\partial_z^j a_{\ell_1}}}{2}\Delta_{\rm h}\Phi_{\ell_2}+\frac{A_{\ell_1}}{2}\overline{\Delta\partial_z^j\varphi_{\ell_2}}\Big),$$

$$F_{7,\ell_1,\ell_2,j} \stackrel{\text{def}}{=} \frac{Z^j}{j!}\big(\overline{\partial_z^{j+1}\varphi_{\ell_1}}\partial_Z A_{\ell_2}+\partial_Z\Phi_{\ell_1}\overline{\partial_z^{j+1}a_{\ell_2}}\big),\quad F_{8,\ell_1,\ell_2,j} \stackrel{\text{def}}{=} \frac{Z^j}{j!}\frac{\overline{\partial_z^j a_{\ell_1}}}{2}\partial_Z^2\Phi_{\ell_2}.$$

Whereas we observe from (2.14) that

$$\begin{aligned}
G_k \stackrel{\text{def}}{=} &\sum_{\substack{\ell_1+\ell_2+j=k+1\\0\le\ell_1\le k}}\frac{Z^j}{j!}\big(\overline{\partial_z^j a_{\ell_1}}\partial_t\Phi_{\ell_2}+A_{\ell_1}\overline{\partial_t\partial_z^j\varphi_{\ell_2}}\big)+\sum_{\ell=0}^k A_\ell\partial_t\Phi_{k+1-\ell}\\
&+\sum_{\ell_1+\ell_2+\ell_3+j_1+j_2=k+1}\frac{Z^{j_1+j_2}}{j_1!j_2!}\Big(\overline{\partial_z^{j_1}a_{\ell_1}\nabla_{\rm h}\partial_z^{j_2}\varphi_{\ell_2}}\cdot\nabla_{\rm h}\Phi_{\ell_3}\Big)\\
&+\sum_{\substack{\ell_1+\ell_2+\ell_3+j_1+j_2=k+1\\0\le\ell_1\le k}}\frac{Z^{j_1+j_2}}{j_1!j_2!}A_{\ell_1}\Big(\frac12\overline{\nabla\partial_z^{j_1}\varphi_{\ell_2}}\cdot\overline{\nabla\partial_z^{j_2}\varphi_{\ell_3}}+3\overline{\partial_z^{j_1}a_{\ell_2}\partial_z^{j_2}a_{\ell_3}}\Big)\\
&+\sum_{\substack{\ell_1+\ell_2+\ell_3+j_1+j_2=k+2\\1\le\ell_3\le k+1}}\frac{Z^{j_1+j_2}}{j_1!j_2!}\big(\overline{\partial_z^{j_1}a_{\ell_1}\partial_z^{j_2+1}\varphi_{\ell_2}}\partial_Z\Phi_{\ell_3}\big)\\
&+\sum_{\substack{\ell_1+\ell_2+\ell_3+j=k+1\\0\le\ell_1\le k}}\frac{Z^j}{j!}A_{\ell_1}(\overline{\nabla_{\rm h}\partial_z^j\varphi_{\ell_2}}\cdot\nabla_{\rm h}\Phi_{\ell_3})+\sum_{\ell_1+\ell_2+\ell_3+j=k+1}\frac{Z^j}{j!}\frac{\overline{\partial_z^j a_{\ell_1}}}{2}\nabla_{\rm h}\Phi_{\ell_2}\cdot\nabla_h\Phi_{\ell_3}\\
&+\sum_{\substack{\ell_1+\ell_2+\ell_3+j=k+1\\0\le\ell_1,\ell_2\le k}}\frac{Z^j}{j!}3\overline{\partial_z^j a_{\ell_1}}A_{\ell_2}A_{\ell_3}+\sum_{\substack{\ell_1+\ell_2+\ell_3+j=k+2\\0\le\ell_1\le k,\,1\le\ell_3\le k+1}}\frac{Z^j}{j!}A_{\ell_1}\overline{\partial_z^{j+1}\varphi_{\ell_2}}\partial_Z\Phi_{\ell_3}\\
&+\sum_{\substack{\ell_1+\ell_2+\ell_3+j=k+3\\1\le\ell_2+\ell_3\le k+1}}\frac{Z^j}{j!}\frac{\overline{\partial_z^j a_{\ell_1}}}{2}\partial_Z\Phi_{\ell_2}\partial_Z\Phi_{\ell_3}-\frac12\Delta_{\rm h}A_{k-1}.
\end{aligned}\tag{A.2}$$

Appendix B. The proof of Lemma 4.1

Proof of Lemma 4.1. We split the proof of (4.7) into the following cases:

• When $k<s-\frac32$. In this case, $s-k>\frac32$, so that $H^{s-k}(\mathbb{R}^3_+)$ is an algebra. As a result, it comes out

$$\begin{aligned}
\|\partial_t^k(fg)(t)\|_{H^{s-k}}&\lesssim\sum_{\ell=0}^k\|\partial_t^\ell f(t)\partial_t^{k-\ell}g(t)\|_{H^{s-k}}\\
&\lesssim\sum_{\ell=0}^k\|\partial_t^\ell f(t)\|_{H^{s-k}}\|\partial_t^{k-\ell}g(t)\|_{H^{s-k}}\\
&\lesssim\sum_{\ell=0}^k\|f(t)\|_{W^{s+\ell-k}}\|g(t)\|_{W^{s-\ell}}\lesssim\|f(t)\|_{W^s}\|g(t)\|_{W^s}.
\end{aligned}$$

• When $k=[s]-1\ge s-\frac32$. We first observe that

$$\|fg\|_{H^\tau}\lesssim\|f\|_{H^\tau}\|g\|_{H^2}\quad\forall\tau\in[0,2).\tag{B.1}$$

The proof of the above inequality can be obtained by first extend the function to the whole space and then using the law of product in the classical Sobolev space (see [3]). We skip the details here.

When $[s]\geq 3$, we get, by applying (B.1), that

$$\begin{aligned}\|\partial_t^{[s]-1}(fg)(t)\|_{H^{s+1-[s]}}\lesssim&\|\partial_t^{[s]-1}f(t)g(t)\|_{H^{s+1-[s]}}+\|f(t)\partial_t^{[s]-1}g(t)\|_{H^{s+1-[s]}}\\&+\sum_{\ell=1}^{[s]-2}\|\partial_t^\ell f(t)\partial_t^{[s]-1-\ell}g(t)\|_{H^{s+1-[s]}}\\\lesssim&\|\partial_t^{[s]-1}f(t)\|_{H^{s+1-[s]}}\|g(t)\|_{H^2}+\|f(t)\|_{H^2}\|\partial_t^{[s]-1}g(t)\|_{H^{s+1-[s]}}\\&+\sum_{\ell=1}^{[s]-2}\|\partial_t^\ell f(t)\|_{H^2}\|\partial_t^{[s]-1-\ell}g(t)\|_{H^{s+1-[s]}}\\\lesssim&\|f(t)\|_{W^s}\|g(t)\|_{H^2}+\|f(t)\|_{H^2}\|g(t)\|_{W^s}+\sum_{\ell=1}^{[s]-2}\|f(t)\|_{W^{\ell+2}}\|g(t)\|_{W^{s-\ell}}\\\lesssim&\|f(t)\|_{W^s}\|g(t)\|_{W^s}.\end{aligned}$$

The case for $[s]=2$ can be proved along the same line.

• When $k=[s]$. We first recall the following law of product in Sobolev space from [3]

$$\|fg\|_{s_1+s_2-\frac{3}{2}}\leq C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}\quad\text{if}\quad s_1,s_2\in]0,3/2[.\tag{B.2}$$

In the case when $s-[s]\in\left]0,\frac{1}{2}\right[$, we get, by applying (B.2), that

$$\|fg\|_{H^{s-[s]}}\lesssim\|fg\|_{H^{\frac{1}{2}}}\lesssim\|f\|_{H^1}\|g\|_{H^1}.$$

Then applying the above inequality and (B.1) yields

$$\begin{aligned}\|\partial_t^{[s]}(fg)(t)\|_{H^{s-[s]}}\lesssim&\|\partial_t^{[s]}f(t)g(t)\|_{H^{s-[s]}}+\|f(t)\partial_t^{[s]}g(t)\|_{H^{s-[s]}}+\sum_{\ell=1}^{[s]-1}\|\partial_t^\ell f(t)\partial_t^{[s]-\ell}g(t)\|_{H^{s-[s]}}\\\lesssim&\|\partial_t^{[s]}f(t)\|_{H^{s-[s]}}\|g(t)\|_{H^2}+\|f(t)\|_{H^2}\|\partial_t^{[s]}g(t)\|_{H^{s-[s]}}\\&+\sum_{\ell=1}^{[s]-1}\|\partial_t^\ell f(t)\|_{H^1}\|\partial_t^{[s]-\ell}g(t)\|_{H^1}\\\lesssim&\|f(t)\|_{W^s}\|g(t)\|_{H^2}+\|f(t)\|_{H^2}\|g(t)\|_{W^s}+\sum_{\ell=1}^{[s]-1}\|f(t)\|_{W^{\ell+1}}\|g(t)\|_{W^{s-\ell+1}}\\\lesssim&\|f(t)\|_{W^s}\|g(t)\|_{W^s}.\end{aligned}$$

In the case when $s-[s]\in\left]\frac{1}{2},1\right[$, we get, by applying (B.2), that

$$\|fg\|_{H^{s-[s]}}\lesssim\|fg\|_{H^{2(s-[s])-\frac{1}{2}}}\lesssim\|f\|_{H^{s-[s]+\frac{1}{2}}}\|g\|_{H^{s-[s]+\frac{1}{2}}}.$$

Applying the above inequality gives rise to

$$\begin{aligned}\sum_{\ell=1}^{[s]-1}\|\partial_t^\ell f(t)\partial_t^{[s]-\ell}g(t)\|_{H^{s-[s]}}\lesssim&\sum_{\ell=1}^{[s]-1}\|\partial_t^\ell f(t)\|_{H^{s-[s]+\frac{1}{2}}}\|\partial_t^{[s]-\ell}g(t)\|_{H^{s-[s]+\frac{1}{2}}}\\\lesssim&\sum_{\ell=1}^{[s]-1}\|f(t)\|_{W^{s-[s]+\ell+\frac{1}{2}}}\|g(t)\|_{W^{s-\ell+\frac{1}{2}}}\\\lesssim&\|f(t)\|_{W^s}\|g(t)\|_{W^s}.\end{aligned}$$

By summing up the above estimates, we obtain (4.7). This completes the proof of Lemma 4.1. □

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