A note on seminormality of cut polytopes

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ABSTRACT. We prove that seminormality of cut polytopes is equivalent to normality. This settles two conjectures regarding seminormality of cut polytopes.

A cut A|B in a graph G(V, E) is an unordered partition (A|B = B|A) of its vertices, i.e. $A \sqcup B = V$. A cut A|B determines a point in $\mathbb{R}^{|E|}$, denoted by $\delta_{A|B}$, with value 1 on edges e separated by the cut (i.e. $|e \cap A| = 1$ and $|e \cap B| = 1$) and value 0 on edges e within cut parts (i.e. $e \subset A$ or $e \subset B$). The cut polytope $\operatorname{Cut}^{\Box}(G)$ corresponding to a graph G is the convex hull of points $\delta_{A|B}$ over all cuts A|B in G, see e.g. [10].

A polytope P is normal if for any $k \in \mathbb{N}$ every lattice point (a point that belongs to the lattice spanned by the lattice points of P) in kP is a sum of k lattice points from P. A slightly weaker property of a polytope is 'very ampleness', see e.g. [6]. A polytope P is very ample if it has only finitely many gaps – lattice points in kP (for some k) which are not a sum of k lattice points from P. This is equivalent to the fact that for every vertex $v \in P$ the monoid of lattice points in the real cone generated by P - v is generated by lattice points of P - v [1, Def. 2.2.7], [7, Ex. 4.9]. A polytope P is seminormal [5] if for every lattice point x, if 2x and 3x are not gaps, then nor is x.

The most well-known conjecture in this area is the following.

CONJECTURE 1 ([10]). The cut polytope $\operatorname{Cut}^{\square}(G)$ is normal if and only if the graph G has no K_5 minor.

The implication from the left to the right is known, as $\operatorname{Cut}^{\square}(K_5)$ is not normal [10] and normality of cut polytopes is a minor closed property [8]. The difficulty of the conjecture lies in proving that for graphs with no K_5 minor the cut polytope is normal. We start with a proof of a weaker property – very ampleness, for which we did not find an explicit reference and which is a corollary of [4, Corollary 1.3].

THEOREM 2. Suppose a graph G has no K_5 minor. Then the cut polytope $\operatorname{Cut}^{\square}(G)$ is very ample.

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PROOF. First, we show that the cut polytope is transitive. That is, for any two vertices v_1, v_2 of $\operatorname{Cut}^{\square}(G)$ there exists an affine isomorphism φ of $\mathbb{R}^{|E|}$ such that $\varphi(\operatorname{Cut}^{\square}(G)) = \operatorname{Cut}^{\square}(G)$ and $\varphi(v_1) = v_2$. It is enough to show that when $v_1 = \delta_{\emptyset|E}$ and $v_2 = \delta_{A|B}$ is arbitrary. Map $\varphi_{A|B}$ is defined by: $x_e \to x_e$ when e is contained in A or B, and $x_e \to 1 - x_e$ when e is separated by the cut A|B. Observe that

$$\varphi_{A|B}(\delta_{C|D}) = \delta_{(A\cap D)\cup(B\cap C)|(A\cap C)\cup(B\cap D)}$$

In particular, $\varphi_{A|B}(\delta_{\emptyset|E}) = \delta_{A|B}$.

Next, we note that for every vertex $\delta_{A|B} \in \operatorname{Cut}^{\square}(G)$ the monoid of lattice points in the real cone generated by $\operatorname{Cut}^{\square}(G) - \delta_{A|B}$ is isomorphic, via $(\varphi_{A|B} - \delta_{A|B})^{-1}$, to the monoid of lattice points in the real cone generated by $\operatorname{Cut}^{\square}(G) - \delta_{\emptyset|E} =$ $\operatorname{Cut}^{\square}(G)$. Thus, in order to show that $\operatorname{Cut}^{\square}(G)$ is very ample it is enough to check the second characterization of very ample polytopes for a single vertex $\delta_{\emptyset|E}$. Therefore, very ample property of the cut polytope coincides with the class \mathscr{H} in [**3**] of graphs whose set of cuts is a Hilbert basis in $\mathbb{R}^{|E|}$.

The statement that remains is proved in [4, Corollary 1.3] and for planar graphs already in [9]. Since both rely on the four color theorem, the theorem also does. \Box

Remark that the cut polytope of K_5 is very ample [2]. Moreover, very ampleness of cut polytopes is not a minor closed property [3]. In particular, Theorem 2 does not give a characterization of graphs with very ample cut polytopes.

Using Theorem 2 we settle Conjectures 1.2 and 4.5 from [5].

THEOREM 3. The cut polytope $\operatorname{Cut}^{\square}(G)$ of a graph G is seminormal if and only if it is normal. In particular, the class of graphs G for which $\operatorname{Cut}^{\square}(G)$ is seminormal is minor closed.

PROOF. If $\operatorname{Cut}^{\square}(G)$ is normal, then clearly it is seminormal.

Let G be a graph such that $\operatorname{Cut}^{\square}(G)$ is seminormal. Then by [5, Corollary 4.4] graph G has no K_5 minor. By Theorem 2 the cut polytope $\operatorname{Cut}^{\square}(G)$ is very ample. Suppose contrary, that $\operatorname{Cut}^{\square}(G)$ is not normal – it has gaps. Since $\operatorname{Cut}^{\square}(G)$ is very ample, it has only finitely many gaps. Let x be a largest gap, i.e. a gap that belongs to the largest dilation k. Then 2x and 3x belong to larger dilations, so they are not gaps. Since $\operatorname{Cut}^{\square}(G)$ is seminormal, x is also not a gap. A contradiction.

We show how a part of Conjecture 1 is equivalent to the four color theorem.

THEOREM 4. The fact that every lattice point in $3 \operatorname{Cut}^{\square}(G)$ is a sum of 3 lattice points from $\operatorname{Cut}^{\square}(G)$ for a planar graph G is equivalent to the four color theorem.

PROOF. One implication, proving the four color theorem, is presented in [7, Proposition 9.4], but originally the idea is due to David Speyer. We note that this implication only uses a decomposition of one specific point in $3 \operatorname{Cut}^{\square}(G)$.

For the other implication we extend the assertion to loopless multigraphs and proceed by induction on the number of edges. Let $p \in 3 \operatorname{Cut}^{\square}(G)$ be a lattice point. Let E_0 be the set of edges $e \in E(G)$ such that p(e) = 0. Consider the contraction $G' := G/E_0$. Notice that G' may have multiple edges, but it is loopless. Indeed, if $e \in E(G)$ became a loop in G', then there was a path between endpoints x, y of econsisting of edges from E_0 . This is impossible, as since p(e) > 0 and p is a convex combination of cuts, points x, y were separated by some cut. Now, we may identify p with a point in $3 \operatorname{Cut}^{\Box}(G')$ and conclude by induction. Similarly, if there is an edge $e \in G$ such that p(e) = 3 we may take a cut A|B that separates e and apply the isomorphism $\varphi_{A|B}$. We have $\varphi_{A|B}(p)(e) = 0$. Hence, by induction $\varphi_{A|B}(p)$ is a sum of three cuts and so is p.

We are left with the case when for each edge e we have p(e) = 1 or p(e) = 2. We claim that the set E_1 of edges e for which p(e) = 1 forms precisely the edges of some cut A|B. This is equivalent to the fact that on the dual graph G^* the set E_1 is a cycle – by which we mean an edge disjoint union of circuits, in the language of matroid theory, or simple cycles, in the language of graph theory. As p is a lattice point, for any cut of G^* the sum of values p(e) over the cut is even. The same must be true for E_1 . In particular, every vertex $v \in G^*$ must be adjacent to an even number of edges of E_1 . By the Euler cycle argument, we know that E_1 is a cycle, which finishes the proof of the claim.

Let $V = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4$ be the four coloring of G. Since G is loopless

$$(2, 2, \dots, 2) = \delta_{V_1 \sqcup V_2 \mid V_3 \sqcup V_4} + \delta_{V_1 \sqcup V_3 \mid V_2 \sqcup V_4} + \delta_{V_1 \sqcup V_4 \mid V_2 \sqcup V_3}.$$

Therefore, after applying $\varphi_{A|B}$ we get the assertion of the theorem

$$p = \varphi_{A|B}(\delta_{V_1 \sqcup V_2|V_3 \sqcup V_4}) + \varphi_{A|B}(\delta_{V_1 \sqcup V_3|V_2 \sqcup V_4}) + \varphi_{A|B}(\delta_{V_1 \sqcup V_4|V_2 \sqcup V_3}).$$

COROLLARY 5. Let G be a planar graph. For k = 1, 2, 3 every lattice point of $k \operatorname{Cut}^{\square}(G)$ is a sum of k lattice points of $\operatorname{Cut}^{\square}(G)$.

PROOF. The case k = 1 is obvious. The case k = 3 follows from the four color theorem by Theorem 4. For k = 2, as in the proof of Theorem 4, we may reduce it to the case where on every edge the point has value one. As in the proof of Theorem 4, this must be a cut point, and as a consequence the graph must be bipartite. We add $\delta_{\emptyset|V} = 0$ to obtain a sum of precisely two lattice points of $\operatorname{Cut}^{\square}(G)$. \square

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