Capital growth and survival strategies in a market with endogenous prices

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Abstract

We call an investment strategy survival, if an agent who uses it maintains a non-vanishing share of market wealth over the infinite time horizon. In a discretetime multi-agent model with endogenous asset prices determined through a shortrun equilibrium of supply and demand, we show that a survival strategy can be constructed as follows: an agent should assume that only their actions determine the prices and use a growth optimal (log-optimal) strategy with respect to these prices, disregarding the actual prices. Then any survival strategy turns out to be close to this strategy asymptotically. The main results are obtained under the assumption that the assets are short-lived.

Keywords: survival strategies, capital growth, relative growth optimal strategies, endogenous prices, evolutionary finance, martingale convergence.

MSC 2010: 91A25, 91B55. JEL Classification: C73, G11.

1. Introduction

The main object of study of this paper is asymptotic performance of investment strategies in stochastic market models. The mathematical theory of optimal capital growth originated with the works of Kelly (1956), Latané (1959), Breiman (1961), and one of its central results consists in that an agent who maximizes the expected logarithm of wealth achieves the fastest asymptotic growth of wealth over the infinite time horizon (see, e.g., Algoet and Cover (1988)). The standard assumption made in this theory is that an agent has negligible impact on a market, and hence asset prices can be specified by exogenous random processes not depending on agents' strategies. The aim of this paper is to extend these results and describe analogues of growth optimal strategies in a multi-agent market model which may contain assets with endogenously determined prices.

We consider a discrete-time model of a market with two type of assets. Assets of the first type, further called *exogenous*, have prices and dividends represented by exogenous random sequences (without loss of generality, we will assume that the dividends are included in the prices). Agents get profit or loss when the prices of these assets change. Assets of the second type, further called *endogenous*, have exogenous dividends, but their prices are determined endogenously via a short-run equilibrium of supply and demand.

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The supply is exogenous, while the demand is generated by agents' strategies. Typically, an asset with larger dividends is more attractive and therefore has a higher price. An important simplifying assumption that will be made in the paper is that the endogenous assets are short-lived in the sense that they can be though of as financial contracts which can be bought at some moment of time, yield payments at the next time instant, and then expire. For example, they can be derivative securities, loan agreements, contracts for producing goods, etc. It would be interesting to incorporate long-lived assets (e.g. common stock) with endogenous prices into the model, but this is a more difficult task and is left for future research.

We are primarily interested in asymptotic behavior of relative wealth of agents, i.e. their shares in total market wealth. We investigate it from a standpoint of evolutionary dynamics and view a market as a population of different strategies competing for capital. The central concept of the paper is the notion of a *survival* strategy. Such a strategy allows an agent to keep the relative wealth strictly bounded away from zero over the infinite time horizon. Our goal is to construct a survival strategy in an explicit form and to find what effect the presence of this strategy has on the asymptotic distribution of wealth between market agents. In particular, we are interested in conditions under which a strategy is asymptotically *dominating*, i.e. an agent using it becomes the single survivor in a market with the relative wealth converging to 1. In order to find a survival strategy, the notion of a *relative growth optimal* strategy will be useful. This is a strategy with the logarithm of its relative wealth being a submartingale. The fact that a non-positive submartingale converges implies that a relative growth optimal strategy is survival. The convergence of the compensator of this submartingale allows to obtain a sufficient condition for a survival strategy to be also dominating.

Note that, in contrast to the optimal growth theory for markets with exogenous prices, which deals with absolute wealth of agents, we focus on relative wealth, which turns out to be more amenable to asymptotic analysis in the case of endogenous prices. Drokin and Zhitlukhin (2020, Section 6) show that the goals of maximization of relative and absolute wealth in a model with endogenous prices may be incompatible.

Our first main result consists in showing that a relative growth optimal strategy can be constructed as a growth optimal strategy in a market with exogenous prices equal to the endogenous prices induced by this strategy when all the agents in the market use it. We find such a strategy in a tractable form, as a solution of a two-stage optimization problem. On the first stage, an agent determines the portfolio of exogenous assets by maximizing the expected log-return (with some adjustments if it is not integrable); on the second stage the portfolio of endogenous assets is found via a solution of another maximization problem. We show that this strategy is relative growth optimal in any strategy profile, irrespectively of the strategies used by the other agents. Another its feature, which can be attractive for possible applications, is that it needs to know little information about the market: only the current total market wealth and the probability distribution of returns of the exogenous assets and payoffs of the endogenous assets, but does not require the knowledge of the other agents' individual wealth or their strategies. It also does not depend on the spot prices of the endogenous assets, and so is not affected by the impact which an agent may have on the market.

Our second main result shows that the obtained strategy becomes the single surviving strategy in a market if the representative strategy of the other agents is asymptotically different from it in a certain sense. Consequently, if some agent uses this strategy, then any other agent who wants to survive in the market must use an asymptotically similar strategy. As a corollary, we show that this strategy asymptotically determines the prices of the endogenous assets.

The results we obtain are tightly related to and generalize the main results of Amir et al. (2013) and Drokin and Zhitlukhin (2020). Those papers also studied survival and growth optimal strategies in markets with short-lived assets and endogenous prices, however the models were less general. In the former paper it was assumed that there are only assets with endogenous prices; the latter paper also included a risk-free bank account with an exogenous interest rate. Another extension consists in that we allow the model to include constraints on agents' portfolios specified by random convex sets. Among other recent papers related to this setting, let us mention the paper of Belkov et al. (2020), which builds another model that includes assets with endogenous prices and a risk-free asset. A difference with our model is that they assume asset payoffs depend linearly on the amount of money invested in the risk-free asset, which allows to reduce the model to previously known results for models without a risk-free asset.

Let us mention how this paper is related to other results in the literature. In models with exogenous prices, the asymptotic growth optimality of the log-optimal strategy (also called the Kelly strategy, after Kelly (1956)) was proved for a general discrete-time model by Algoet and Cover (1988); a review of other related results in discrete time can be found in, e.g., Cover and Thomas (2012, Chapter 16) or Hakansson and Ziemba (1995). For a treatment of a general model with continuous time and portfolio constraints, and a connection of growth optimal portfolios (numéraire portfolios) with absence of arbitrage, see, e.g., Karatzas and Kardaras (2007).

Among various lines of research on markets with endogenous prices, our paper is most closely related to works in evolutionary finance on stability and survival of investment strategies, which focus on evolutionary dynamics and properties like survival, extinction, dominance, and how they affect the structure of a market. Central to this direction are strategies that perform well irrespectively of competitors' actions. One of the main results consists in that the strategy which splits its investment budget between risky assets proportionally to their expected dividends (often also called the Kelly strategy) survives in a market provided that the agent's beliefs about the dividends are correct. See, for example, the papers of Amir et al. (2005, 2011); Blume and Easley (1992); Evstigneev et al. (2002, 2006); Hens and Schenk-Hoppé (2005), which establish this fact for different models and under different assumptions. Reviews of this direction can be found in Evstigneev et al. (2016) or Amir et al. (2020). Typically, the Kelly strategy turns out to be the only surviving strategy in a market, i.e. it dominates all other asymptotically different strategies. For results on market wealth evolution when agents use strategies different from the Kelly strategy, which may result in survival of several strategies, see, e.g., Bottazzi and Dindo (2014); Bottazzi et al. (2018).

Most of the above mentioned papers (including the present paper) consider agentbased models, where agents' strategies are specified directly as functions of a market state. Another large body of literature consists of results on market selection of investment strategies in the framework of general equilibrium, where agents maximize utility from consumption. Among those results one can mention, for example, Blume and Easley (2006); Borovička (2020); Sandroni (2000); Yan (2008). Holtfort (2019) provides a detailed survey of the literature in evolutionary finance over the last three decades, including also some earlier results.

The paper is organized as follows. Section 2 describes the model. The main results of the paper are stated in the three theorems included in Section 3. Section 4 contains their proofs.

2. The model

2.1. Notation

For vectors $x, y \in \mathbb{R}^N$, we will denote by $\langle x, y \rangle$ their scalar product, and by $|x| = \sum_n |x^n|$, $||x|| = \sqrt{\langle x, x \rangle}$ the L^1 and L^2 norms. If $f : \mathbb{R} \to \mathbb{R}$ is a scalar function and x is a vector, then $f(x) = (f(x^1), \ldots, f(x^N))$ denotes the coordinatewise application of f to x.

By e we will denote the vector consisting of all unit coordinates, e = (1, ..., 1), which may be of different dimensions in different formulas. In particular, $\langle e, x \rangle$ is equal to the sum of coordinates of a vector x.

All equalities and inequalities for random variables are assumed to hold with probability 1 (almost surely), unless else is stated.

2.2. Investors and assets

Let (Ω, \mathcal{F}, P) be a probability space with a discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ on which all random variables will be defined. Without loss of generality, we will assume that \mathcal{F} is P-complete and \mathcal{F}_0 contains all P-null events.

The market in the model consists of M agents (investors) and $N = N_1 + N_2$ assets of two types. The assets of the first type are available in unlimited supply and have exogenous prices; they are treated as in standard models of mathematical finance. The assets of the second type are in limited supply; they yield payoffs which are defined exogenously, but their prices are determined endogenously from an equilibrium of supply and demand in each time period. These assets are short-lived in the sense that they can be purchased by agents at time t, yield payoffs at t + 1, and then get replaced with new assets; agents cannot sell them, and, in particular, short sales are not allowed (adding short sales would lead to conceptual difficulties which we prefer to avoid). We will call the assets of the first and the second type, respectively, exogenous and endogenous.

The prices of the exogenous assets are represented by positive random sequences $(S_t^n)_{t=0}^{\infty}$, $n = 1, \ldots, N_1$, which are \mathbb{F} -adapted (i.e. S_t^n is \mathcal{F}_t -measurable). We assume that dividends, if there are any, are already included in the prices. By $X_t^n = S_t^n / S_{t-1}^n > 0$ we will denote the relative price changes. The payoffs of the endogenous assets (per one unit of an asset) are represented by non-negative adapted sequences $(Y_t^n)_{t=1}^{\infty}$, $n = 1, \ldots, N_2$. Without loss of generality, we assume that the supply of each endogenous asset is equal to 1, so Y_t^n is the total payoff of an asset. Their prices will be defined later, as we first need to define agents' strategies, on which they will depend.

The agents enter the market at time t = 0 with non-random initial wealth $v_0^m > 0$, $m = 1, \ldots, M$. Actions of an agent at time $t \ge 0$ are described by a pair of vectors $h_t = (\alpha_t, \beta_t)$, where $\alpha_t \in \mathbb{R}^{N_1}, \beta_t \in \mathbb{R}^{N_2}_+$ specify in what proportions this agent allocates the current wealth between the assets of the two types (the wealth sequences are yet to be defined), i.e. the proportion α_t^n (respectively, β_t^n) of wealth is allocated to asset n.¹

Since α_t, β_t are proportions, we require that $\langle e, \alpha_t \rangle + \langle e, \beta_t \rangle = 1$. The components of β_t are non-negative, because short sales of the endogenous assets are not allowed. Additionally, we will assume that it is not possible to buy the endogenous assets on borrowed funds, i.e. $\langle e, \alpha \rangle \in [0, 1]$, and hence $\langle e, \beta \rangle \in [0, 1]$. Consequently, h_t assumes

¹In the literature, time indices are often shifted by 1 forward (so h_t represents actions at time t - 1, and, hence, is a predictable sequence). But in discrete time this is just a matter of notation. For our purposes, it will be more convenient to let h_t specify actions at time t.

values in the set

$$\mathcal{H} = \{ (\alpha, \beta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}_+ : \langle e, \alpha \rangle \in [0, 1], \ \langle e, \beta \rangle = 1 - \langle e, \alpha \rangle \}.$$

In order to emphasize that a pair h_t is selected by agent m we will use the superscript m, e.g. $h_t^m = (\alpha_t^m, \beta_t^m)$.

A strategy of an agent consists of investment proportions h_t^m selected at consecutive moments of time. It may (and, usually, does) depend on a random outcome and market history. In order to specify this dependence, introduce the measurable space (Θ, \mathcal{G}) with

$$\Theta = \Omega \times \mathbb{R}^M_+ \times (\mathcal{H}^M)^{\infty}, \qquad \mathcal{G} = \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^M_+ \times (\mathcal{H}^M)^{\infty}),$$

where an element $\chi = (\omega, v_0, h_0, h_1, \ldots) \in \Theta$ consists of a random outcome ω , a vector of initial wealth $v_0 = (v_0^1, \ldots, v_0^M) \in \mathbb{R}^M_+$, and vectors of investment proportions $h_t = (h_t^1, \ldots, h_t^M)$ selected by the agents at each moment of time. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration on Θ defined by

$$\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^M_+ \times (\mathcal{H}^M)^{t+1}),$$

i.e. \mathcal{G}_t is generated by sets $\Gamma \times V \times H_0 \times \ldots \times H_t \times (\mathcal{H}^M)^\infty$ with $\Gamma \in \mathcal{F}_t$ and Borel sets $V \subseteq \mathbb{R}^M_+$, $H_s \subseteq \mathcal{H}^M$. We define a strategy of an agent as a sequence of \mathcal{G}_t -measurable functions

$$h_t(\chi) \colon \Theta \to \mathcal{H}, \qquad t \ge 0.$$

Basically, h_t can be thought of as a function $h_t(\omega, v_0, h_0, \ldots, h_t)$, but the notation $h_t(\chi)$ will be more convenient for us because we will deal with functions depending on market histories of different length appearing in one formula, see, e.g., (3) below. Note the dependence of h_t on the argument h_t , i.e. an agent may use information (partial or whole) about actions of other agents at the same moment of time t. This information may be available to an agent, for example, from asset prices.

We call a vector of initial wealth v_0 and a strategy profile $(\mathbf{h}^1, \ldots, \mathbf{h}^M)$ feasible if there exists a sequence of \mathcal{F}_t -measurable functions $h_t(\omega) = (h_t^1(\omega), \ldots, h_t^M(\omega)) \in \mathcal{H}^M$ such that for all ω, t, m

$$\boldsymbol{h}_t^m(\boldsymbol{\chi}(\omega)) = h_t^m(\omega), \text{ where } \boldsymbol{\chi}(\omega) = (\omega, v_0, h_0(\omega), h_1(\omega), \ldots).$$
(1)

Such a sequence $h(\omega)$ will be called a *realization* of the agents' strategies corresponding to the given strategy profile and initial wealth. We do not require the uniqueness of a realization, i.e. equation (1) may have several solutions. The main results of the paper will hold for any chosen realization (however, the uniqueness may be desirable for other applications).

Remark 1 (On notation). By the bold font we denote functions which depend on χ , i.e. on a random outcome and market history, while functions which depend only on a random outcome ω (e.g. realizations of strategies) are denoted by the normal font. In particular, if $\boldsymbol{\zeta}$ is a function of χ , then, given a vector of initial wealth and a strategy profile, we denote by $\zeta(\omega)$ its realization $\boldsymbol{\zeta}(\chi(\omega))$, where $\chi(\omega)$ is as in (1).

If ξ is a random variable, i.e. a function of ω only, we will sometimes use the same letter to denote the function $\xi(\chi)$ which just ignores the values of v_0 and h_s , i.e. $\xi(\chi) = \xi(\omega)$ at an element $\chi = (\omega, v_0, h_0, h_1, \ldots)$.

Sufficient conditions for a vector of initial wealth and a strategy profile to be feasible, in general, can be formulated in terms of assumptions of fixed-point theorems, but we do not investigate this question in details – our main goal is to find an optimal strategy, and the strategy which we find will be optimal in any feasible profile. Nevertheless, it is easy to see that a simple sufficient condition for the feasibility is that the functions \boldsymbol{h}_t^m do not depend on the argument h_t , i.e. adapted to the filtration $\mathbb{G}^- = (\mathcal{G}_t^-)_{t\geq 0}$, where

$$\mathcal{G}_t^- = \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^M_+ \times (\mathcal{H}^M)^t).$$

This condition can be interpreted as that at each moment of time the agents decide upon their actions simultaneously and independently of each other.

Now we can define the prices of the endogenous assets and the wealth sequences $\boldsymbol{v}_t^m(\chi)$ inductively in t, beginning with $\boldsymbol{v}_0^m(\chi) = \boldsymbol{v}_0^m$. Denote the prices at time t by $\boldsymbol{p}_t^n(\chi)$, $n = 1, \ldots, N_2$. Suppose for some $\chi \in \Theta$ the wealth sequences are defined up to a moment of time t, and $\boldsymbol{v}_t^m(\chi) \ge 0$ for all m. Then agent m can purchase $\boldsymbol{y}_t^{m,n}(\chi)$ units of asset n at this moment, where

$$\boldsymbol{y}_t^{m,n} = rac{eta_t^{m,n} \boldsymbol{v}_t^m}{\boldsymbol{p}_t^n},$$

and $\beta_t^{m,n}$ (also $\alpha_t^{m,n}$ below) are taken from the component h_t entering χ . In order to clear the market (recall that the supply of each asset is 1), the prices should be equal to

$$\boldsymbol{p}_t^n = \sum_{m=1}^M \beta_t^{m,n} \boldsymbol{v}_t^m.$$
⁽²⁾

Essentially, we employ the principle of moving equilibrium, which operates with economic variables changing with different speeds. In our model, the endogenous asset prices move fast, while the investment proportions selected by the agents move slow; the proportions can be considered fixed while the prices rapidly adjust to clear the market. The mechanics of this adjustment process is not important to us (as long as it does not inflict transaction costs) and it can be modeled by various approaches, e.g. limit order books, auctions, etc. Note that we do not require the agents to agree upon future asset prices at each random outcome. For a discussion of this moving equilibrium approach in a similar model, see Section 4 in Evstigneev et al. (2020).

If $\sum_{m} \beta_t^{m,n}(\chi) = 0$ in formula (2) for some *n*, i.e. no one invests in asset *n*, we put $\boldsymbol{y}_t^{m,n}(\chi) = 0$ for all *m*; in this case the price $\boldsymbol{p}_t^n(\chi)$ can be defined in an arbitrary way with no effect on the agents' wealth, so we will put $\boldsymbol{p}_t^n(\chi) = 0$ in accordance with (2).

Thus, the portfolio of agent m between moments of time t and t+1 consists of $y_t^{m,n}$ units of endogenous asset n, and $x_t^{m,n}$ units of exogenous asset n, where

$$oldsymbol{x}_t^{m,n} = rac{lpha_t^{m,n}oldsymbol{v}_t^m}{S_t^n}.$$

Consequently, the wealth of this agent at t + 1 is determined by the relation

$$\boldsymbol{v}_{t+1}^{m} = \sum_{n=1}^{N_1} \boldsymbol{x}_t^{m,n} S_{t+1}^n + \sum_{n=1}^{N_2} \boldsymbol{y}_t^{m,n} Y_{t+1}^n = \left(\sum_{n=1}^{N_1} \alpha_t^{m,n} X_{t+1}^n + \sum_{n=1}^{N_2} \frac{\beta_t^{m,n} Y_{t+1}^n}{\sum_k \beta_t^{k,n} \boldsymbol{v}_t^k}\right) \boldsymbol{v}_t^m \quad (3)$$

(with 0/0 = 0 in the right-hand side).

Observe that in equation (3) the value of v_{t+1}^m may become negative, which will make the right-hand side of the equation meaningless for the next time period. However, below we will introduce portfolio constraints which prohibit strategies that may lead to negative wealth. In view of this, we will restrict the domain of the functions v_t^m and define them on sets smaller than Θ . Namely, introduce inductively the sets

$$\Theta_t = \{ \chi \in \Theta : \boldsymbol{v}_s^m(\chi) \ge 0 \text{ for all } s \le t, \ m = 1, \dots, M \}, \quad t \ge 0,$$

where $\boldsymbol{v}_s^m(\chi)$ are computed by (3). Note that $\Theta_0 = \Theta$, $\Theta_t \supseteq \Theta_{t+1}$, and $\Theta_t \in \mathcal{G}_t^-$. From now on, we will assume that the functions \boldsymbol{v}_t^m are defined only for $\chi \in \Theta_t$.

It will be also convenient to introduce the sets

$$\Theta'_t = \{ \chi \in \Theta_t : \boldsymbol{v}_t(\chi) \neq 0 \}, \quad t \ge 0.$$

Observe that, essentially, components $(\boldsymbol{\alpha}_t, \boldsymbol{\beta}_t)$ of an agent's strategy need to be defined only on Θ'_t , since elements from $\Omega \setminus \Theta_t$ do not correspond to any realization, and on the set $\{\chi : \boldsymbol{v}_t(\chi) = 0\}$ they can be defined in an arbitrary way without any effect on (zero) wealth.

2.3. Portfolio constraints

Portfolio constraints in the model are specified by a sequence of \mathcal{G}_t^- -measurable random² non-empty closed convex sets $C_t(\chi) \subseteq \mathcal{H}, t \geq 0$. The constraints are the same for each agent.

We say that a strategy h satisfies the portfolio constraints if

$$h_t(\chi) \in C_t(\chi)$$
 for all $t \ge 0$ and $\chi \in \Theta$.

From now on, when writing "a strategy", we will always mean a strategy satisfying the portfolio constraints.

Notice that the sets C_t are essentially needed to be defined only for elements $\chi \in \Theta'_t$. Thus it may be convenient to put, for example, $C_t = \mathbb{R}^N_+$ on $\Theta \setminus \Theta'_t$, without any effect on realizations of the agents' wealth in the model.

We will consider portfolio constraints only of the following particular form: they are imposed on the exogenous and endogenous assets separately, and an agent can freely choose what proportion of wealth to invest in the assets of each of the two types. Namely, it will be assumed that

$$\boldsymbol{C}_t = (\boldsymbol{A}_t \times \boldsymbol{B}_t) \cap \mathcal{H},\tag{4}$$

where A_t and B_t are \mathcal{G}_t^- -measurable closed convex sets in \mathbb{R}^{N_1} and $\mathbb{R}^{N_2}_+$ such that $\langle e, \alpha \rangle \in [0, 1], \langle e, \beta \rangle \in [0, 1]$ for any $\alpha \in A_t(\chi), \beta \in B_t(\chi)$. We also require that

if
$$\alpha \in \mathbf{A}_t(\chi)$$
, then $\lambda \alpha \in \mathbf{A}_t(\chi)$ for any $\lambda \in [0, 1/\langle e, \alpha \rangle]$, (5)

if
$$\beta \in \boldsymbol{B}_t(\chi)$$
, then $\lambda \beta \in \boldsymbol{B}_t(\chi)$ for any $\lambda \in [0, 1/\langle e, \beta \rangle]$ (6)

(or $\lambda \in [0, \infty)$ if $\langle e, \alpha \rangle = 0$ or $\langle e, \beta \rangle = 0$); i.e. A_t and B_t can be represented as intersections of some convex cones with the sets $\{\alpha \in \mathbb{R}^{N_1} : \langle e, \alpha \rangle \in [0, 1]\}$ and $\{\beta \in \mathbb{R}^{N_2}_+ : \langle e, \beta \rangle \in [0, 1]\}$ respectively. Note that relation (4) implies that the sets A_t , B_t cannot simultaneously (for the same t, χ) consist of only elements α or, respectively, β with zero sum of coordinates, since then the set C_t would be empty.

²See Section 4.1 for details on random sets.

We will need to further restrict the class of portfolio constraints by introducing several assumptions on the structure of the sets A_t, B_t . In what follows, let $K_t(\omega, d\tilde{\omega})$ denote some fixed version of the regular conditional distribution with respect to \mathcal{F}_t . By P_t and E_t we will denote, respectively, the regular probability and expectation computed with respect to K_t , i.e. for a random event $\Gamma \in \mathcal{F}$ and a random variable ξ we put

$$P_t(\Gamma)(\omega) = K_t(\omega, \Gamma), \qquad E_t(\xi)(\omega) = \int_{\Omega} \xi(\widetilde{\omega}) K_t(\omega, d\widetilde{\omega}).$$

When $\boldsymbol{\xi}$ depends also on market history, i.e. $\boldsymbol{\xi} = \boldsymbol{\xi}(\chi)$ is \mathcal{G} -measurable, we put

$$\mathbf{E}_t(\boldsymbol{\xi})(\chi) = \int_{\Omega} \boldsymbol{\xi}(\widetilde{\omega}, v_0, h_0, \ldots) K_t(\omega, d\widetilde{\omega}), \qquad \chi = (\omega, v_0, h_0, \ldots)$$

provided that the integral is well-defined.

Let us introduce several random sets which will be needed to formulate the assumptions on the sets A_t, B_t :

• the sets of portfolios of exogenous assets which have non-negative values at the next moment of time:

$$D_t(\omega) = \{ \alpha \in \mathbb{R}^{N_1} : P_t(\langle \alpha, X_{t+1} \rangle \ge 0)(\omega) = 1 \};$$

• the linear spaces of *null investments* (portfolios of exogenous assets with zero current and next value):

$$L_t(\omega) = \{ \alpha \in \mathbb{R}^{N_1} : \langle e, \alpha \rangle = 0, \ \mathcal{P}_t(\langle \alpha, X_{t+1} \rangle = 0)(\omega) = 1 \};$$

• the projection of A_t on the orthogonal space L_t^{\perp} :

$$\boldsymbol{A}_t^{\mathrm{p}}(\chi) = \{ \alpha \in L_t^{\perp}(\omega) : \exists u \in L_t(\omega) \text{ such that } \alpha + u \in \boldsymbol{A}_t(\chi) \}.$$

Observe that the sets D_t , L_t are \mathcal{F}_t -measurable, and \mathbf{A}_t^p are \mathcal{G}_t^- -measurable. Indeed, we can represent $D_t(\omega) = \{\alpha : f(\omega, \alpha) = 0\}$ with the function $f(\omega, \alpha) = E_t(\langle \alpha, X_t \rangle^- \land 1)(\omega)$, which is a Carathéodory function, so D_t is measurable by Filippov's theorem (see Proposition 4 in Section 4.1). The set L_t is measurable since it is the intersection of D_t , $-D_t$ and $\{\alpha : \langle \alpha, e \rangle = 0\}$. The measurability of \mathbf{A}_t^p follows from Proposition 6.

Now we are ready to formulate the assumptions on the portfolio constraints. In the remaining part of the paper we always assume that they are satisfied.

Assumptions. For all $t \ge 1$ and $\chi = (\omega, v_0, h_0, ...)$ it holds that (A.1) $A_t(\chi) \subseteq D_t(\omega)$;

- (A.2) there exists $(\alpha, \beta) \in C_t(\chi)$ such that $P_t(\langle \alpha, X_{t+1} \rangle + \langle \beta, Y_{t+1} \rangle > 0)(\omega) = 1;$
- (A.3) $A_t^{p}(\chi) \subseteq A_t(\chi);$
- (A.4) $A_t^{\rm p}(\chi)$ is a compact set.

Let us comment on interpretation of these assumptions. (A.1) is imposed to ensure that any strategy which satisfies the portfolio constraints generates a non-negative wealth sequence. As a consequence, for the realization of any profile of strategies satisfying the portfolio constraints we have

$$\chi(\omega) = (\omega, v_0, h_0(\omega), h_1(\omega), \ldots) \in \Theta_t$$
 a.s. for all $t \ge 0$.

Since the underlying probability space and the filtration are complete, we can assume that the above inclusion holds for all $\omega \in \Omega$, if necessary modifying the strategies on a set of zero probability.

Assumption (A.2) implies that there exists a strategy with a strictly positive wealth sequence. Such a strategy can be found via a standard measurable selection argument, using that C_t are measurable sets. Observe that (A.2) is a very mild assumption. For example, it holds if there is a non-zero vector $\alpha \in A_t$ with all non-negative coordinates (recall that $X_t^n > 0$ for all n), since then $(\alpha/|\alpha|, 0) \in C_t$ by (5).

Assumption (A.3) means that the agents can remove null investments from their portfolios. Note that in the literature it is sometimes required that $L_t \subseteq A_t$ (i.e. any investment that leads to no profit or loss is allowed). It is not difficult to see that in our model this requirement implies (A.3).

Assumption (A.4) will allow to reduce the optimal strategy selection problem to an optimization problem on a compact set. Actually, it is equivalent to the no arbitrage condition for the exogenous assets – or, more precisely, no *unbounded* arbitrage condition – as we show in the next section.

2.4. Absence of unbounded arbitrage opportunities

Let $U_t(\omega)$ denote the cone of arbitrage opportunities in the exogenous assets at time $t \geq 0$, which consists of all $u \in \mathbb{R}^{N_1}$ such that

$$\langle e, u \rangle = 0,$$
 $P_t(\langle u, X_{t+1} \rangle \ge 0)(\omega) = 1,$ $P_t(\langle u, X_{t+1} \rangle > 0)(\omega) > 0.$

We say that there are no unbounded arbitrage opportunities in the model if for all $\chi = (\omega, v_0, h_0, \ldots) \in \Theta$ and $t \ge 0$ the following assumption holds:

(A.5) there is no $u \in U_t(\omega)$ such that $\lambda u \in A_t(\chi)$ for any $\lambda > 0$.

In other words, an agent cannot infinitely multiply the profit from an arbitrage opportunity, but the set A_t may contain some of them. This condition is analogous to the no unbounded increasing profit condition (NUIP), known in connection with numéraire portfolios, see Karatzas and Kardaras (2007, Proposition 3.10). If there are no constraints on the exogenous assets (i.e. $A_t = \{\alpha \in \mathbb{R}^{N_1} : \langle e, \alpha \rangle \in [0, 1]\}$), then (A.5) is equivalent to the usual no-arbitrage condition $U_t = \emptyset$.

Proposition 1. Suppose the model satisfies assumptions (A.1), (A.3). Then assumptions (A.4) and (A.5) are equivalent.

Proof. It is easy to see that (A.4) implies (A.5). Let us prove the converse implication. Suppose (A.5) holds. The closedness of A_t^p follows from that A_t is closed and assumption (A.3).

To prove that A_t^p is bounded, fix $\chi = (\omega, v_0, h_0, ...)$ and suppose, by way of contradiction, that there is a sequence $u_n \in A_t^p(\chi)$ such that $|u_n| \to \infty$. The sequence $u_n/|u_n|$ is bounded, so there exists a convergent subsequence $u_{n_k}/|u_{n_k}| \to u$. It is easy to see that $\langle e, u \rangle = 0$ (because $\langle e, u_n \rangle \in [0, 1]$), and $|u| = 1, u \in L_t^{\perp}(\omega)$. The last two properties imply that $u \notin L_t(\omega)$. Moreover, since $A_t \subseteq D_t$, we have $P_t(\langle u_n, X_{t+1} \rangle \ge 0)(\omega) = 1$, and hence $P_t(\langle u, X_{t+1} \rangle \ge 0)(\omega) = 1$. Consequently, $u \in U_t(\omega)$.

However, for any $\lambda > 0$ and k such that $|u_{n_k}| \ge \lambda$, we have

$$\frac{\lambda}{|u_{n_k}|} u_{n_k} \in \boldsymbol{A}_t(\chi)$$

and, in the limit, $\lambda u \in A_t(\chi)$, so u is an unbounded arbitrage opportunity, which is a contradiction.

Examples. Arbitrage opportunities may be eliminated by imposing appropriate portfolio constraints, even if the unconstrained model with the same exogenous prices S_t has arbitrage. As an example, observe that assumption (A.5) automatically holds when portfolio constraints limit portfolio leverage in the sense that

$$\boldsymbol{A}_t \subseteq \{ \alpha \in \mathbb{R}^{N_1} : c_t | a^+ | \ge |a^-| \},$$

$$\tag{7}$$

where $0 \leq c_t < 1$ is a random variable (or, in particular, a constant), and $\alpha^{\pm} = ((\alpha^1)^{\pm}, \ldots, (\alpha^{N_1})^{\pm})$ are the vectors consisting of the positive and negative parts of the coordinates of α . In this case, if $\langle e, \alpha \rangle = 0$ for $\alpha \in \mathbf{A}_t$, then $\alpha = 0$, so $U_t \cap \mathbf{A}_t = \emptyset$ and (A.5) holds.

Constraint (7) means that the long positions of a portfolio should cover the short positions with some margin, which is determined by the constant c_t . If $\alpha \neq 0$, this is equivalent to that

$$|\alpha^+| \ge |\alpha^-|$$
 and $\frac{|\alpha^-|}{|\alpha^+| - |\alpha^-|} \le c'_t$,

where $c'_t = c_t/(1 - c_t)$, which can be interpreted as that the ratio of the debt to the value of a portfolio (the leverage) is bounded by c'_t . If $c_t = 0$, then (7) prohibits short sales of the exogenous assets. For details on this leverage constraint and how it can be used in problems of hedging and optimal growth, see e.g. Babaei et al. (2020a,b); Evstigneev and Zhitlukhin (2013).

Constraint (7) can be relaxed if one requires

$$\boldsymbol{A}_t \subseteq \{ \alpha \in \mathbb{R}^{N_1} : d_t + c_t | a^+ | \ge |a^-| \},\$$

where $d_t \geq 0$. In this case, A_t may include some portfolios with $\langle e, \alpha \rangle = 0$ (besides $\alpha = 0$), in particular arbitrage opportunities, but the set $A_t \cap \{\alpha : \langle e, \alpha \rangle = 0\}$ remains bounded, so there are still no unbounded arbitrage opportunities.

3. Main results

3.1. The notion of optimality

We will be interested in long-run behavior of relative wealth of agents, i.e. their shares in total market wealth. We define the total market wealth and the relative wealth of agent m as, respectively,

$$oldsymbol{W}_t = \sum_{m=1}^M oldsymbol{v}_t^m, \qquad oldsymbol{r}_t^m = rac{oldsymbol{v}_t^m}{oldsymbol{W}_t}$$

where $\mathbf{r}_t^m = 0$ if $\mathbf{W}_t = 0$. Recall that \mathbf{v}_t is defined on the set Θ_t , hence we will assume that \mathbf{W}_t and \mathbf{r}_t^m are defined only on this set as well.

For a given feasible strategy profile and a vector of initial wealth, by $W_t(\omega) = W_t(\chi(\omega)), r_t^m(\omega) = r_t^m(\chi(\omega))$ we will denote the corresponding realizations defined as in Remark 1. The realizations of the agents' wealth sequences will be denoted by $v_t^m(\omega)$.

Definition 1. In a feasible strategy profile (h^1, \ldots, h^M) with initial wealth $v_0 \in \mathbb{R}^M_+$ such that $v_0^m > 0$, we call a strategy h^m survival³ if

$$\inf_{t \ge 0} r_t^m > 0 \text{ a.s.},$$
$$\lim_{t \ge 0} r_t^m = 1 \text{ a s}$$

and call it *dominating* if

$$\lim_{t \to \infty} r_t^m = 1 \text{ a.s.}$$

Our main goal will be to show that the strategy \hat{h} which we construct in the next section is survival in any strategy profile and dominating in a strategy profile if the strategies of the other agents are, in a certain sense, different from it asymptotically. Consequently, if some agents use \hat{h} , then any other survival strategy should be asymptotically close to it.

Note that any survival strategy is asymptotically unbeatable in the following sense: if agent m uses a survival strategy then there exists a (finite-valued) random variable γ such that

$$r_t^k \leq \gamma r_t^m, \qquad k = 1, \dots, M, \ t \geq 0,$$

which expresses the fact that the wealth of any other agent cannot grow asymptotically faster than the wealth of an agent who uses a survival strategy. For a discussion of unbeatable strategies as a game solution concept in related evolutionary finance models, see e.g. Amir et al. (2013).

At the same time, we would like to emphasize that we do not insist on that all agents should use only survival strategies, as they may have other economic goals or make systematic errors. We only investigate what happens with a market *if* some agents use such strategies.

For construction of a survival strategy, the following notion will be useful.

Definition 2. For a given feasible strategy profile and initial wealth, we call a strategy h^m relative growth optimal if

 $v_t^m > 0$ for all $t \ge 0$ and $\ln r_t^m$ is a submartingale.

Since any non-positive submartingale has a finite limit with probability 1 (see, e.g., Shiryaev (2019, Chapter 7.4)), for a relative growth optimal strategy we have $\lim_{t\to\infty} \ln r_t^m > -\infty$, and therefore $r_{\infty}^m = \lim_{t\to\infty} r_t^m > 0$. This implies the following result.

Proposition 2. A relative growth optimal strategy is survival.

Note that if the relative wealth of an agent is "infinitesimal" (so the strategy of this agent does not affect the prices of the endogenous assets), then a relative growth optimal strategy for this agent, which depends on the current endogenous prices p_t , can be found as a growth optimal portfolio in a market with $N = N_1 + N_2$ exogenous assets, considering p_t as exogenous prices. In particular, if the asset returns are sufficiently integrable, then such a strategy maximizes the one-period expected logarithmic return, see, e.g., Algoet and Cover (1988) or Cover and Thomas (2012, Chapter 16). The important feature of the strategy that we construct in the next section is that it essentially depends⁴ only on the current total market wealth W_t , but not on the current endogenous prices, and hence will be a survival strategy for an agent with any relative wealth.

³We use the terminology of Amir et al. (2013). Note that often a strategy is called survival if $\limsup r_t^m > 0$, see, e.g., Blume and Easley (1992).

⁴Strictly speaking, this strategy may also depend on some additional information contained in the market history χ , but only through the dependence of the portfolio constraints on such information.

3.2. Construction of a relative growth optimal strategy

In this section we find one relative growth optimal strategy in an explicit form. The idea behind the construction of this strategy consists in that we find it as a growth optimal portfolio in a market with endogenous prices induced by it (see Theorem 2). We begin with a lemma which defines the components $\hat{\alpha}, \hat{\beta}$ of this strategy. Its statement is somewhat involved, but clarifying comments will be provided in Remark 2 below.

Recall that we need to define $\widehat{\alpha}_t$, $\widehat{\beta}_t$ only on the set Θ'_t , while on its complement these functions can be defined in an arbitrary way (respecting the \mathcal{G}_t -measurability and the portfolio constraints), since this will not have any effect on realizations of wealth sequences.

In the statement of the lemma and subsequent results, we will use the following agreement to treat indeterminacies: 0/0 = 0, $0 \cdot \ln 0 = 0$, $a \cdot \ln 0 = -\infty$ if a > 0.

Lemma 1. The following statements hold true for each $t \ge 0$.

(a) Consider the \mathcal{G}_{t+1}^- -measurable vectors \mathbf{Y}_{t+1} in \mathbb{R}^{N_2} with the components

$$\widetilde{\boldsymbol{Y}}_{t+1}^{n}(\chi) = Y_{t+1}^{n}(\omega) \operatorname{I}(\exists \beta \in \boldsymbol{B}_{t}(\chi) : \beta^{n} > 0),$$

and the functions

$$g_i(x) = \frac{1}{i} + i \arctan\left(\frac{x}{i}\right), \qquad x \in \mathbb{R}_+, \ i = 1, 2, \dots$$

Then there exist \mathcal{G}_t^- -measurable functions $\widehat{\alpha}_{t,i}$ such that for all $\chi \in \Theta_t'$

$$\widehat{\boldsymbol{\alpha}}_{t,i} \in \operatorname*{argmax}_{\alpha \in \boldsymbol{A}_{t}^{\mathrm{p}}} \big\{ \mathrm{E}_{t} \ln g_{i}(\langle \alpha, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|) - \langle e, \alpha \rangle \big\}.$$
(8)

(b) There exists an increasing sequence of \mathcal{G}_t^- -measurable functions $i_j(\chi)$, $j \geq 1$, with positive integer values, and a \mathcal{G}_t^- -measurable function $\widehat{\alpha}_t$ with values in $\mathbf{A}_t^{\mathrm{p}}$, such that on the set Θ_t'

$$\widehat{oldsymbol{lpha}}_t = \lim_{j o \infty} \widehat{oldsymbol{lpha}}_{t,i_j}.$$

(c) The set $\widetilde{\mathbf{B}}_t = \{\beta \in \mathbf{B}_t(\chi) : |\beta| = 1 - \langle e, \widehat{\alpha}_t(\chi) \rangle\}$ is non-empty for $\chi \in \Theta'_t$ and there exists a \mathcal{G}_t^- -measurable function $\widehat{\boldsymbol{\beta}}_t$ with values in \mathbf{B}_t such that for any $\chi \in \Theta'_t$

$$\widehat{\boldsymbol{\beta}}_{t} \in \operatorname*{argmax}_{\boldsymbol{\beta} \in \widetilde{\boldsymbol{B}}_{t}} \bigg\{ \mathrm{E}_{t} \, \frac{\langle \ln \boldsymbol{\beta}, \boldsymbol{Y}_{t+1} \rangle}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|} \bigg\}.$$
(9)

Theorem 1. In every feasible strategy profile, any strategy $\hat{h} = (\hat{\alpha}, \hat{\beta})$ constructed as in Lemma 1 is relative growth optimal.

Note that Lemma 1 defines $\hat{\alpha}_t$, $\hat{\beta}_t$ not necessarily in a unique way (hence, we write "any strategy \hat{h} " in the theorem). This may be so if, for example, some of the vectors X_t have linearly dependent components.

Let us show that the strategy \hat{h} can be found as an equilibrium strategy of the representative agent who holds a growth optimal portfolio in a market with $N_1 + N_2$ exogenous assets, where the first N_1 assets are the same as in the original market, and the remaining N_2 assets are treated as exogenous with the prices being equal to the prices of the endogenous assets induced by \hat{h} in the original market. This notion of equilibrium

is conceptually similar to the one in the Lucas model of an exchange economy (Lucas, 1978) with the logarithmic utility, though we do not consider consumption.

Recall that in a market with exogenous prices a strategy with value $\hat{v}_t > 0$ is called a growth optimal portfolio (or a numéraire portfolio) if for any other strategy with value $v_t \geq 0$ it holds that v_t/\hat{v}_t is a supermartingale. If $z_t = (v_t - v_{t-1})/v_{t-1}$ and $\hat{z}_t = (\hat{v}_t - \hat{v}_{t-1})/\hat{v}_{t-1}$ denote the one-period returns on the strategies' portfolios, then this supermartingality condition is equivalent to that for each $t \geq 0$

$$E_t \frac{1+z_{t+1}}{1+\hat{z}_{t+1}} \le 1.$$
(10)

Let \hat{p}_t denote the endogenous prices that would clear the market if all the agents used the strategy \hat{h} , i.e.

$$\widehat{\boldsymbol{p}}_t^n = \widehat{\boldsymbol{\beta}}_t^n \boldsymbol{W}_t.$$

Let Z_t denote the returns on the endogenous assets in this case,

$$\boldsymbol{Z}_{t+1}^n = \frac{Y_{t+1}^n}{\widehat{\boldsymbol{p}}_t^n}.$$

Consequently, the return on a portfolio (α_t, β_t) would be

$$\langle \boldsymbol{\alpha}_t, X_{t+1} \rangle - 1 + \langle \boldsymbol{\beta}_t, \boldsymbol{Z}_{t+1} \rangle.$$
 (11)

Theorem 2. For any $t \ge 0$, $\chi \in \Theta'_t$, and $(\alpha, \beta) \in C_t(\chi)$, we have (cf. (10), (11))

$$E_{t} \frac{\langle \alpha, X_{t+1} \rangle + \langle \beta, \mathbf{Z}_{t+1} \rangle}{\langle \widehat{\alpha}_{t}, X_{t+1} \rangle + \langle \widehat{\beta}_{t}, \mathbf{Z}_{t+1} \rangle} \le 1.$$
(12)

If $\mathbf{E}_t |\ln(\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle + \langle \widehat{\boldsymbol{\beta}}_t, \boldsymbol{Z}_{t+1} \rangle)| < \infty$, then

$$(\widehat{\boldsymbol{\alpha}}_t, \widehat{\boldsymbol{\beta}}_t) \in \operatorname*{argmax}_{(\alpha,\beta)\in \boldsymbol{C}_t} \operatorname{E}_t \ln(\langle \alpha, X_{t+1} \rangle + \langle \beta, \boldsymbol{Z}_{t+1} \rangle).$$
(13)

Relation (12) expresses the above-mentioned idea of equilibrium, and relation (13) is an analogue of the well-known fact that a numéraire portfolio maximizes one-period expected log-returns, under the respective integrability condition.

Remark 2. Let us comment on technical aspects of the above results. Why in Lemma 1 do we introduce the functions g_i and consider maximization problem (8)? Actually, we would like to find a strategy $(\hat{\alpha}, \hat{\beta})$ such that

$$\widehat{\boldsymbol{\alpha}}_t \text{ maximizes } E_t \ln(\langle \alpha, X_{t+1} \rangle \boldsymbol{W}_t + |Y_{t+1}|) - \langle e, \alpha \rangle \text{ over } \alpha \in \boldsymbol{A}_t,$$
(14)

and, for this $\hat{\alpha}_t$, to define the component $\hat{\beta}_t$ as in (9). This strategy would satisfy inequalities (19) and (26), which play the key role in the proofs.

But it may be not possible to define $\hat{\alpha}_t$ in this way, since problem (14) may have no solution. For this reason, we find the solutions $\hat{\alpha}_{t,i}$ of the maximization problems truncated by the functions g_i and select a convergent subsequence. Then inequalities (19), (26) still remain satisfied. To ensure that such a subsequence exists, we use the observation that it is possible to maximize not over the whole set A_t but over its compact subset A_t^p . We also replace Y_t with \tilde{Y}_t to avoid the situation when an asset yields a positive payoff with positive conditional probability, but it is not possible to invest in it. Note that when no portfolio constraints are imposed on the endogenous assets, i.e. $B_t = \{\beta \in \mathbb{R}^{N_2}_+ : |\beta| \le 1\}$ and hence $\widetilde{Y}_{t+1} = Y_{t+1}$, we can find $\widehat{\beta}_t$ explicitly:

$$\widehat{\boldsymbol{\beta}}_{t}^{n} = \operatorname{E}_{t} \frac{Y_{t+1}^{n}}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |Y_{t+1}|}$$
(15)

(this formula will be used in Section 3.3). Indeed, for $\hat{\beta}_t$ defined by (15), we have $|\hat{\beta}_t| = 1 - \langle e, \hat{\alpha}_t \rangle$ as follows from equality (20) below, and for any $\beta \in \mathbb{R}^{N_2}_+$ with $|\beta| = 1 - \langle e, \hat{\alpha}_t \rangle$ we have

$$E_t \frac{\langle \ln \beta, Y_{t+1} \rangle}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |Y_{t+1}|} = \langle \ln \beta, \widehat{\boldsymbol{\beta}}_t \rangle \le \langle \ln \widehat{\boldsymbol{\beta}}_t, \widehat{\boldsymbol{\beta}}_t \rangle,$$

so $\hat{\boldsymbol{\beta}}_t$ indeed delivers the maximum in (9). The inequality here follows from Gibb's inequality (see Proposition 8 below).

To conclude this section, let us show how the above theorems generalize known results on asymptotically optimal strategies. An immediate corollary from Theorem 2 is that in a market with only exogenous assets the strategy $\hat{\alpha}_t$ is a numéraire portfolio. Note that in this case $\hat{\alpha}_t$ depends only on ω , but not on market history (assuming that the constraints set C_t also depend only on ω).

In a market with only endogenous assets and no portfolio constraints, as follows from (15), the optimal strategy is given by

$$\widehat{\beta}_t^n = \mathcal{E}_t \frac{Y_{t+1}^n}{|Y_{t+1}|}$$

(note that again we have the dependence on ω only). This strategy was obtained by Amir et al. (2013); see also the earlier results of Amir et al. (2005); Evstigneev et al. (2002); Hens and Schenk-Hoppé (2005) for models with short-lived assets which impose additional assumptions on admissible strategies or on asset payoffs.

Finally, suppose that there is only one exogenous asset, short sales of this asset are not allowed, and there are no other portfolio constraints, i.e. $C_t = \mathbb{R}_+ \times \mathbb{R}_+^{N_2}$. Then $\widehat{\alpha}_t(\chi)$ is defined as follows: if $\mathbf{E}_t(X_{t+1}\mathbf{W}_t/|Y_{t+1}|) \leq 1$, then $\widehat{\alpha}_t = 0$; otherwise $\widehat{\alpha}_t$ is the unique solution of the equation

$$\mathcal{E}_t \frac{X_{t+1} \boldsymbol{W}_t}{\alpha + |Y_{t+1}|} = 1.$$

This can be seen from relations (19)–(20) below. Indeed, if $E_t(X_{t+1}\boldsymbol{W}_t/|Y_{t+1}|) \leq 1$, then equality (20) can be true only if $\hat{\boldsymbol{\alpha}}_t = 0$. In the case $E_t(X_{t+1}\boldsymbol{W}_t/|Y_{t+1}|) > 1$, equality (20) has two solutions, the zero one and a non-zero one. But if $\hat{\boldsymbol{\alpha}}_t = 0$, then (19) cannot hold true for $\alpha > 0$, hence we are left only with the non-zero solution.

After $\hat{\alpha}_t$ has been defined as above, the component $\hat{\beta}_t$ can be found from (15), which gives

$$\widehat{\boldsymbol{\beta}}_t^n = \mathrm{E}_t \, \frac{Y_{t+1}^n}{\widehat{\boldsymbol{\alpha}}_t X_{t+1} \boldsymbol{W}_t + |Y_{t+1}|}.$$

This strategy was obtained by Drokin and Zhitlukhin (2020) in the case when the sequence X_t is predictable (e.g. the exogenous asset is a risk-free bond or cash); see Zhitlukhin (2019, 2020) for its extensions to continuous time.

3.3. Asymptotic proximity of survival strategies

In this section we investigate evolution of relative wealth of strategies different from h. The theorem below will be stated for the case when there are no portfolio constraints on the endogenous assets ($B_t = \{\beta \in \mathbb{R}^{N_2}_+ : |\beta| \leq 1\}$). This assumption is necessary because the proof relies on the explicit form of $\hat{\beta}_t$ given by (15).

Given a feasible strategy profile and a vector of initial wealth, by $\bar{h} = (\bar{\alpha}, \bar{\beta})$ we will denote the realization of the representative strategy of all the agents, which we define as the weighted sum of their strategies with r_t^m as the weights:

$$\bar{\alpha}_t = \sum_{m=1}^M r_t^m \alpha_t^m, \qquad \bar{\beta}_t = \sum_{m=1}^M r_t^m \beta_t^m,$$

where α_t, β_t, r_t are the corresponding realizations. In a similar way, by $\tilde{h} = (\tilde{\alpha}, \tilde{\beta})$ we will denote the realization of the representative strategy of agents $m = 2, \ldots, M$ weighted with their relative wealths excluding agent 1:

$$\widetilde{\alpha}_t = \sum_{m=2}^M \frac{r_t^m}{1 - r_t^1} \alpha_t^m, \qquad \widetilde{\beta}_t = \sum_{m=2}^M \frac{r_t^m}{1 - r_t^1} \beta_t^m,$$

where 0/0 = 0.

Theorem 3. Suppose $B_t = \{\beta \in \mathbb{R}^{N_2}_+ : |\beta| \leq 1\}$, and agent 1 uses the strategy $h^1 = \hat{h}$. Considering the realizations of the strategies, the wealth sequences, and the constraints sets, let

$$Q_{t+1}(\omega) = \max_{\alpha \in A_t(\omega)} \langle \alpha, X_{t+1}(\omega) \rangle + \frac{|Y_{t+1}(\omega)|}{W_t(\omega)}.$$

Then, with probability 1,

$$\sum_{t=0}^{\infty} \left(\frac{\langle \alpha_t^1 - \bar{\alpha}_t, X_{t+1} \rangle}{Q_{t+1}} \right)^2 + \|\beta_t^1 - \bar{\beta}_t\|^2 < \infty, \tag{16}$$

and

$$\lim_{t \to \infty} r_t^1 = 1 \text{ on the set } \left\{ \omega : \sum_{t=0}^{\infty} \left(\frac{\langle \alpha_t^1 - \widetilde{\alpha}_t, X_{t+1} \rangle}{Q_{t+1}} \right)^2 + \|\beta_t^1 - \widetilde{\beta}_t\|^2 = \infty \right\}.$$
(17)

Note that the maximum in the definition of Q_{t+1} is attained because, according to Proposition 1, it can be taken over the compact set $A_t^{p}(\omega)$. Furthermore, $Q_{t+1} > 0$ by assumption (A.2).

Relation (16) essentially shows that if one agent uses the strategy \hat{h} then this agent asymptotically determines the representative strategy of the market so that \bar{h}_t becomes close to \hat{h}_t in the sense that the series in (16) converges, and, consequently,

$$\frac{\langle \alpha_t^1 - \bar{\alpha}_t, X_{t+1} \rangle}{Q_{t+1}} \to 0, \quad \beta_t^1 - \bar{\beta}_t \to 0 \quad \text{as } t \to \infty.$$

Relation (17) provides a sufficient condition for an agent using the strategy \hat{h} to dominate in the market, which happens when the realization of the representative strategy of the other agents is asymptotically different from the realization of \hat{h} in the sense

that the series in (17) diverges. From here, we also get a necessary condition for a strategy to be survival. Indeed, a survival strategy must survive against \hat{h} , so if agents $m = 1, \ldots, M - 1$ use the strategy \hat{h} , and hence can be considered as a single agent, the remaining agent m = M has to use a strategy with a realization close to \hat{h} in the sense of (17).

Another corollary from Theorem 3 is that the presence of an agent who uses the strategy \hat{h} asymptotically determines the relative prices $\rho_t^n = p_t^n/W_t$ of the endogenous assets. It is not difficult to see that $\rho_t^n = \bar{\beta}_t^n$, and hence (16) implies that for each n and $t \to \infty$ we have $\hat{\beta}_t^n - \rho_t^n \to 0$.

4. Proofs of the main results

4.1. Auxiliary results on random sets

In this section we provide several results from the theory of random sets which will be used in the proofs for dealing with portfolio constraints.

By a random set (or a measurable correspondence) in \mathbb{R}^N defined on a measurable space (S, S) we call a set-valued function $\phi: S \to 2^{\mathbb{R}^N}$ such that for any open set $A \subseteq \mathbb{R}^N$ it holds that $\phi^{-1}(A) \in S$, where $\phi^{-1}(A) = \{s : \phi(s) \cap A \neq \emptyset\}$ is the lower inverse of A. An equivalent definition is that the distance function $d(x, \phi(s))$ is S-measurable for any $x \in \mathbb{R}^N$ (where $d(x, \emptyset) = \infty$). In what follows, the role of (S, S) will be played by $(\Omega, \mathcal{F}_t), (\Theta, \mathcal{G}_t), \text{ or } (\Theta, \mathcal{G}_t^-)$.

A random set is called closed (respectively, compact, non-empty) if $\phi(s)$ is closed (compact, non-empty) for any $s \in S$. A measurable selector is an S-measurable function ξ such that $\xi(s) \in \phi(s)$ for any s. A function $f(s, x) \colon S \times \mathbb{R}^N \to \mathbb{R}$ is called a Carathéodory function if it is measurable in s and continuous in x.

The following results are known for random sets in \mathbb{R}^N .

Proposition 3. If ϕ_n , n = 1, 2, ..., are random sets, then $\cup_n \phi_n$ is a random set; if ϕ_n are also closed, then $\cap_n \phi_n$ is a closed random set.

Proposition 4 (Filippov's theorem). Suppose ϕ is a non-empty compact random set, f is a Carathéodory function, and π is a measurable function. Then the correspondence

$$\psi(s) = \{ x \in \phi(s) : f(s, x) = \pi(s) \}$$

is measurable and compact. Moreover, if ψ is non-empty, then it has a measurable selector ξ , and hence $f(s,\xi(s)) = \pi(s)$.

Proposition 5 (Measurable maximum theorem). For a non-empty compact random set ϕ and a Carathéodory function f, let μ be the maximum function and ψ be the argmax correspondence defined by

$$\mu(s) = \max_{x \in \phi(s)} f(s, x), \qquad \psi(s) = \operatorname*{argmax}_{x \in \phi(s)} f(s, x).$$

Then μ is measurable, and ψ is non-empty, compact, measurable, and has a measurable selector.

Proofs of the above results can be found in the book of Aliprantis and Border (2006, Chapter 18) for random sets in general metric spaces, except the result about $\bigcap_n \phi_n$, which holds (in a metric space) if ϕ_n are compact. For \mathbb{R}^N , it can be extended to closed sets using that \mathbb{R}^N is σ -compact.

For the reader's convenience, the following results are provided with proofs (they are not included in the above-mentioned book).

Proposition 6. Let L be a random linear subspace of \mathbb{R}^N (i.e. for each s the set L(s) is a linear space and the correspondence L is measurable), L^{\perp} be the orthogonal space, and ϕ be a closed random set in \mathbb{R}^N . Then the projection correspondence

$$\operatorname{pr}_L \phi(s) = \{ x \in L(s) : \exists y \in L^{\perp}(s) \text{ such that } x + y \in \phi(s) \}$$

is measurable.

Proof. By Castaing's theorem (see Corollary 18.14 in Aliprantis and Border (2006)), a non-empty closed correspondence is measurable if and only if it can be represented as the closure of a countable family of measurable selectors from it. Hence, we can find measurable ξ_i such that $\phi(s) = \operatorname{cl}\{\xi_i(s), i \ge 1\}$ on the set $\{s : \phi(s) \neq \emptyset\}$. Using that

$$\operatorname{cl}(\operatorname{pr}_L \phi(s)) = \begin{cases} \operatorname{cl}\{\operatorname{pr}_L \xi_i(s), i \ge 1\}, & \text{if } \phi(s) \neq \emptyset, \\ \emptyset, & \text{if } \phi(s) = \emptyset, \end{cases}$$

one can see that $cl(pr_L \phi)$ is measurable. Since the measurability of a correspondence is equivalent to the measurability of its closure (Aliprantis and Border, 2006, Lemma 18.3), $pr_L \phi$ is measurable.

Proposition 7. Let ϕ be a non-empty compact random set and ξ_n be a sequence of measurable selectors from it. Then there exists a measurable selector ξ from ϕ and a sequence of measurable functions $1 \leq i_1(s) < i_2(s) < \ldots$ with integer values such that $\lim_{j\to\infty} \xi_{i_j(s)}(s) = \xi(s)$ for all s.

Proof. The set $\psi(s) = \bigcap_n \operatorname{cl}\{\xi_k(s), k \ge n\}$ is measurable, non-empty, and closed, so there exists a measurable selector $\xi \in \psi$ (by Castaing's theorem mentioned above). Then the sequence i_j can be constructed by induction as follows. Put $i_1 = 1$. If i_j is defined, consider the random set $\eta_j(s) = \{k > i_j(s) : |\xi_k(s) - \xi(s)| \le j^{-1}\} \subset \mathbb{N}$, which is measurable, non-empty, and closed. Let i_{j+1} be a measurable selector from η_j . Then $|\xi_{i_{j+1}} - \xi| < j^{-1}$, which gives the desired convergence.

4.2. Proof of Lemma 1

Proof of claim (a). Fix any $t \ge 0$. Let $f_i(\chi, \alpha)$ be the function which is maximized in the definition of $\widehat{\alpha}_{t,i}$, i.e. on the set Θ'_t put

$$f_i = \mathcal{E}_t \ln g_i(\langle \alpha, X_{t+1} \rangle \boldsymbol{W}_t + | \boldsymbol{Y}_{t+1} |) - \langle e, \alpha \rangle,$$

while on the set $\Theta \setminus \Theta'_t$ put $f_i = 0$. The function f_i is a Carathéodory function, and the set $A^{\rm p}_t$, over which it is maximized, is compact by Proposition 1. Hence the measurable maximum theorem implies the existence of a measurable selector $\hat{\alpha}_{t,i}$ from the argmax in (8).

Proof of claim (b) readily follows from Proposition 7. Before we continue with the proof of claim (c), let us show that $\hat{\alpha}_t$ satisfies a number of relations that will be used in its proof, as well as in the proof of Theorem 1.

Lemma 2. For any $t \ge 0$, $\chi \in \Theta'_t$, and $\alpha \in A_t(\chi)$ we have

$$P_t(\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + | \widetilde{\boldsymbol{Y}}_{t+1} | > 0) = 1,$$
(18)

$$E_t\left(\frac{\langle \widehat{\boldsymbol{\alpha}}_t - \alpha, X_{t+1} \rangle \boldsymbol{W}_t}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|}\right) \ge \langle e, \widehat{\boldsymbol{\alpha}}_t - \alpha \rangle,$$
(19)

$$E_t \left(\frac{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + | \widetilde{\boldsymbol{Y}}_{t+1} |} \right) = \langle e, \widehat{\boldsymbol{\alpha}}_t \rangle.$$
(20)

Proof. Fix t, χ, α, i , and consider the function $u(\varepsilon) = f_i((1 - \varepsilon)\widehat{\alpha}_t + \varepsilon \alpha), \ \varepsilon \in [0, 1]$, i.e.

$$u(\varepsilon) = \mathcal{E}_t \ln g_i \big(\langle (1-\varepsilon) \widehat{\boldsymbol{\alpha}}_{t,i} + \varepsilon \alpha, X_{t+1} \rangle \boldsymbol{W}_t + | \widetilde{\boldsymbol{Y}}_{t+1} | \big) (\chi) - \langle e, (1-\varepsilon) \widehat{\boldsymbol{\alpha}}_{t,i}(\chi) + \varepsilon \alpha \rangle.$$
(21)

Since $\ln g_i(x)$ is concave for $x \ge 0$, the function $u(\varepsilon)$ is also concave. As it attains the maximum value at $\varepsilon = 0$, the right derivative $u'(0) \le 0$. We want to interchange the order of differentiation and taking the expectation. The expectation in (21) can be written as $E_t \ln g_i(q(\widetilde{\omega}, \varepsilon)) = \int_{\Omega} \ln g_i(q(\widetilde{\omega}, \varepsilon)) K_t(\omega, d\widetilde{\omega})$ with

$$q(\widetilde{\omega},\varepsilon) = \langle (1-\varepsilon)\widehat{\boldsymbol{\alpha}}_{t,i}(\widetilde{\chi}) + \varepsilon\alpha, X_{t+1}(\widetilde{\omega}) \rangle \boldsymbol{W}_t(\widetilde{\chi}) + |\widetilde{\boldsymbol{Y}}_{t+1}(\widetilde{\chi})|$$

where $\tilde{\chi} = (\tilde{\omega}, v_0, h_0, ...)$ and $v_0, (h_s)_{s \ge 0}$ are taken from $\chi = (\omega, v_0, h_0, ...)$. By applying Fatou's lemma, we obtain ($\tilde{\omega}$ is omitted for brevity)

$$\left(\mathrm{E}_{t} \ln g_{i}(q(\varepsilon))\right)_{\varepsilon=0}^{\prime} \geq \mathrm{E}_{t} \frac{g_{i}^{\prime}(q(0))}{g_{i}(q(0))} \langle \alpha - \widehat{\boldsymbol{\alpha}}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_{t}.$$
(22)

Fatou's lemma can be applied since for $\varepsilon \in [0, 1)$ we have the lower bound (P_t-a.s. in $\widetilde{\omega}$)

$$\frac{\ln g_i(q(\varepsilon)) - \ln g_i(q(0))}{\varepsilon} \ge (\ln g_i(q(\varepsilon)))' = \frac{g_i'(q(\varepsilon))}{g_i(q(\varepsilon))} \langle \alpha - \widehat{\alpha}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_t$$
$$\ge -ig_i'((1-\varepsilon) \langle \widehat{\alpha}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_t) \langle \widehat{\alpha}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_t \ge -\frac{i^3}{1-\varepsilon}$$

Here in the first inequality we used the concavity of $\ln g_i(q(\varepsilon))$. In the second inequality we used the relation $P_t(\langle \alpha, X_{t+1} \rangle \ge 0) = 1$, the bound $g_i(x) \ge 1/i$, and that $g'_i(x)$ is non-increasing for $x \ge 0$. The last last inequality holds because $g'_i(x)x \le i^2$.

Therefore, from (21) and (22) we obtain

$$0 \ge u'(0) \ge \mathcal{E}_t(\boldsymbol{\xi}_i \langle \alpha, X_{t+1} \rangle \boldsymbol{W}_t) - \mathcal{E}_t(\boldsymbol{\xi}_i \langle \widehat{\boldsymbol{\alpha}}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_t) - \langle e, \alpha - \widehat{\boldsymbol{\alpha}}_{t,i} \rangle,$$
(23)

where

$$\boldsymbol{\xi}_{i} = \frac{g_{i}'(q(0))}{g_{i}(q(0))} = \frac{g_{i}'(\langle \widehat{\boldsymbol{\alpha}}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_{t} + | \boldsymbol{Y}_{t+1} |)}{g_{i}(\langle \widehat{\boldsymbol{\alpha}}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_{t} + | \boldsymbol{\widetilde{Y}}_{t+1} |)}$$

One can see that for all $x \ge 0$

$$0 \le \frac{xg'_i(x)}{g_i(x)} \le 2, \qquad \lim_{i \to \infty} \frac{xg'_i(x)}{g_i(x)} = I(x > 0).$$
(24)

The above inequality can be obtained by using that $\arctan(x/i) \ge x/(2i)$ if $x \le i$ and $\arctan(x/i) \ge \pi/4$ if $x \ge i$; the computation of the limit is straightforward. Relations

(24) allow to apply the dominated convergence theorem to the second expectation in (23), which gives

$$\lim_{i \to \infty} \mathcal{E}_t(\boldsymbol{\xi}_i \langle \widehat{\boldsymbol{\alpha}}_{t,i}, X_{t+1} \rangle \boldsymbol{W}_t) = \mathcal{E}_t \left(\frac{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + | \widetilde{\boldsymbol{Y}}_{t+1} |} \right),$$
(25)

where at this point we assume 0/0 = 0 in the right-hand side (according with the indicator in (24)). However, as follows from assumption (A.2), there exists $\tilde{\alpha}$ such that $P_t(\langle \tilde{\alpha}, X_{t+1} \rangle \boldsymbol{W}_t + | \boldsymbol{\tilde{Y}}_{t+1} | > 0) = 1$. Applying Fatou's lemma to the first expectation in (23) with $\alpha = \tilde{\alpha}$, we find that (18) must hold, since otherwise we would have

$$\liminf_{i \to \infty} \mathcal{E}_t(\boldsymbol{\xi}_i \langle \widetilde{\alpha}, X_{t+1} \rangle \boldsymbol{W}_t) = +\infty,$$

which contradicts (23). Consequently, for any $\alpha \in \mathbf{A}_t$ we have

$$\liminf_{i\to\infty} \mathcal{E}_t(\boldsymbol{\xi}_i \langle \alpha, X_{t+1} \rangle \boldsymbol{W}_t) \geq \mathcal{E}_t\left(\frac{\langle \alpha, X_{t+1} \rangle \boldsymbol{W}_t}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|}\right),$$

which together with (23) and (25) implies (19).

Let us prove (20). If $\langle e, \hat{\alpha}_t(\chi) \rangle = 1$, it clearly follows from (19) with $\alpha = 0$. If $\langle e, \hat{\alpha}_t(\chi) \rangle < 1$, we can consider small $\varepsilon > 0$ and take as α in (19)

$$\alpha^{(\pm\varepsilon)} := (1\pm\varepsilon)\widehat{\alpha}_t(\chi) \in A_t(\chi),$$

which gives (20) after simple transformations.

Proof of claim (c) of Lemma 1. If $B_t(\chi) \neq \{0\}$, then $\tilde{B}_t(\chi) \neq \emptyset$ in view of (6). If $B_t(\chi) = \{0\}$, then $\tilde{Y}_t(\chi) = 0$, and (20) implies that $\langle e, \hat{\alpha}_t(\chi) \rangle = 1$, so $\tilde{B}_t(\chi) = \{0\}$ is non-empty again.

Let $f(\chi,\beta)$ denote the function being maximized in (9):

$$f(\chi,\beta) = \sum_{n=1}^{N_2} \ln \beta^n \operatorname{E}_t \left(\frac{\widetilde{\boldsymbol{Y}}_{t+1}^n}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) (\chi).$$

The function f may be discontinuous in β . In order to apply the measurable maximum theorem, let us take \mathcal{G}_{t-1} -measurable $\tilde{\boldsymbol{\beta}}(\chi) \in \boldsymbol{B}_t(\chi)$ such that $|\boldsymbol{\tilde{\beta}}(\chi)| = 1 - \hat{\boldsymbol{\alpha}}_t(\chi)$ and $\boldsymbol{\tilde{\beta}}^n(\chi) > 0$ if $P_t(\boldsymbol{\tilde{Y}}_{t+1}^n > 0)(\omega) > 0$. Then we can consider the function

$$\widetilde{f}(\chi,\beta) = \max(f(\chi,\beta), f(\chi,\widetilde{\boldsymbol{\beta}}(\chi))),$$

which is a Carathéodory function and satisfies the relation

$$\operatorname*{argmax}_{\beta \in \widetilde{\boldsymbol{B}}_t} f(\chi, \beta) = \operatorname*{argmax}_{\beta \in \widetilde{\boldsymbol{B}}_t} f(\chi, \beta).$$

Hence the measurable maximum theorem can be applied to \tilde{f} , giving $\hat{\beta}_t$ which also maximizes f.

4.3. Proofs of Theorems 1 and 2

Let us prove two more inequalities which together with (19) will be used in the proofs. Lemma 3. For any $t \ge 0$, $\chi \in \Theta'_t$, and $\beta \in B_t(\chi)$ we have

$$E_t \left(\frac{\langle \ln \widehat{\boldsymbol{\beta}}_t - \ln \beta, \widetilde{\boldsymbol{Y}}_{t+1} \rangle}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) \ge |\widehat{\boldsymbol{\beta}}_t| - |\beta|,$$
(26)

$$E_{t} \frac{|\widetilde{\boldsymbol{Y}}_{t+1}| - \sum_{n} \beta^{n} \widetilde{\boldsymbol{Y}}_{t+1}^{n} / \widehat{\boldsymbol{\beta}}_{t}^{n}}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|} \ge |\widehat{\boldsymbol{\beta}}_{t}| - |\boldsymbol{\beta}|,$$
(27)

where in (27) we let $\beta^n \widetilde{\boldsymbol{Y}}_{t+1}^n(\chi) / \widehat{\boldsymbol{\beta}}_t^n(\chi) = 0$ if $\widehat{\boldsymbol{\beta}}_t^n(\chi) = 0$ (then $P_t(\widetilde{\boldsymbol{Y}}_{t+1}^n = 0)(\chi) = 1$ as follows from (9)).

Proof. Clearly, (26) holds if $|\beta| = |\widehat{\beta}_t(\chi)|$, as follows from the definition of $\widehat{\beta}_t$. If $|\beta| \neq |\widehat{\beta}_t(\chi)|$, we have

$$\mathbf{E}_{t} \left(\frac{\langle \ln \widehat{\boldsymbol{\beta}}_{t} - \ln \beta, \widetilde{\boldsymbol{Y}}_{t+1} \rangle}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) \geq \mathbf{E}_{t} \left(\frac{|\widetilde{\boldsymbol{Y}}_{t+1}| \ln(|\widehat{\boldsymbol{\beta}}_{t}|/|\beta|)}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) \\ \geq \mathbf{E}_{t} \left(\frac{|\widetilde{\boldsymbol{Y}}_{t+1}|}{\langle \widehat{\boldsymbol{\alpha}}_{t}, X_{t+1} \rangle \boldsymbol{W}_{t} + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) \frac{|\widehat{\boldsymbol{\beta}}_{t}| - |\beta|}{|\widehat{\boldsymbol{\beta}}_{t}|} = |\widehat{\boldsymbol{\beta}}_{t}| - |\beta|,$$

where in the first inequality we represented $\ln \beta = \ln(\beta |\hat{\beta}_t|/|\beta|) - \ln(|\hat{\beta}_t|/|\beta|)$ and applied (26) to $\beta |\hat{\beta}_t|/|\beta|$ instead of β ; in the second inequality we used the estimate $\ln a \ge 1-a^{-1}$; and in the equality applied (20). This proves (26).

To prove (27), observe that the function

$$f(\varepsilon) = \mathcal{E}_t \left(\frac{\langle \ln((1-\varepsilon)\widehat{\boldsymbol{\beta}}_t + \varepsilon\beta), \widetilde{\boldsymbol{Y}}_{t+1} \rangle}{\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|} \right) - |(1-\varepsilon)\widehat{\boldsymbol{\beta}}_t + \varepsilon\beta_t|, \qquad \varepsilon \in [0,1],$$

attains its maximum at $\varepsilon = 0$ and is differentiable on [0, 1), so its derivative at zero

$$f'(0) = \sum_{n=1}^{N_2} \frac{(\beta^n - \widehat{\boldsymbol{\beta}}_t^n) \widetilde{\boldsymbol{Y}}_{t+1}^n}{\widehat{\boldsymbol{\beta}}_t^n (\langle \widehat{\boldsymbol{\alpha}}_t, X_{t+1} \rangle \boldsymbol{W}_t + |\widetilde{\boldsymbol{Y}}_{t+1}|)} + |\widehat{\boldsymbol{\beta}}_t| - |\beta|$$

should be non-positive, which gives (27) (here, the *n*-th term in the sum is treated as zero when $\hat{\beta}_t^n(\chi) = 0$, and hence $\tilde{Y}_{t+1}^n = 0$).

Proof of Theorem 1. Assume that the strategy \hat{h} is used by agent m = 1. Let us fix the initial wealth and the strategies of the other agents, and pass on to a realization of the strategies $h_t^m = (\alpha_t^m, \beta_t^m)$, wealth v_t^m , and relative wealth r_t^m as functions of ω only. In notation for agent 1, we will also use the hat instead of the superscript "1", i.e. $\hat{\alpha} = \alpha^1$, $\hat{\beta} = \beta^1$, etc.

Introduce the predictable sequence of random vectors $F_t \in \mathbb{R}^{N_2}_+$ with the components

$$F_t^n = \frac{\widehat{\beta}_t^n}{\sum_m r_t^m \beta_t^{m,n}},$$

where 0/0 = 0. From (3), we obtain the relations

$$\widehat{v}_{t+1} = \left(\langle \widehat{\alpha}_t, X_{t+1} \rangle + \frac{\langle F_t, \widetilde{Y}_{t+1} \rangle}{W_t} \right) \widehat{v}_t, \qquad W_{t+1} = \left(\sum_{m=1}^M r_t^m \langle \alpha_t^m, X_{t+1} \rangle + \frac{|\widetilde{Y}_{t+1}|}{W_t} \right) W_t.$$

Consequently, we find $\ln \hat{r}_{t+1} - \ln \hat{r}_t = f_t(X_{t+1}, \tilde{Y}_{t+1})$, where $f_t = f_t(\omega, x, y)$ is the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^N)$ -measurable function

$$f_t(x,y) = \ln\left(\frac{\langle \widehat{\alpha}_t, x \rangle W_t + \langle F_t, y \rangle}{W_t \sum_m r_t^m \langle \alpha_t^m, x \rangle + |y|}\right)$$

(the argument ω is omitted for brevity).

We need to show that $E_t f_t(X_{t+1}, \widetilde{Y}_{t+1}) \ge 0$. Rewrite the function $f_t(x, y)$ as

$$f_t(x,y) = \ln\left(\frac{\langle \widehat{\alpha}_t, x \rangle W_t + |y|}{W_t \sum_m r_t^m \langle \alpha_t^m, x \rangle + |y|}\right) + \ln\left(\frac{\langle \widehat{\alpha}_t, x \rangle W_t + \langle F_t, y \rangle}{\langle \widehat{\alpha}_t, x \rangle W_t + |y|}\right)$$

$$:= f_t^{(1)}(x,y) + f_t^{(2)}(x,y).$$
(28)

For the first term, we can use the inequality $\ln x \ge 1 - x^{-1}$ and apply (19), which gives

$$\operatorname{E}_{t} f_{t}^{(1)}(X_{t+1}, \widetilde{Y}_{t+1}) \geq \operatorname{E}_{t} \frac{\langle \widehat{\alpha}_{t} - \sum_{m} r_{t}^{m} \alpha_{t}^{m}, X_{t+1} \rangle W_{t}}{\langle \widehat{\alpha}_{t}, X_{t+1} \rangle W_{t} + |\widetilde{Y}_{t+1}|} \geq \left\langle e, \ \widehat{\alpha}_{t} - \sum_{m=1}^{M} r_{t}^{m} \alpha_{t}^{m} \right\rangle.$$
 (29)

For the second term in (28), we have

$$\operatorname{E}_{t} f_{t}^{(2)}(X_{t+1}, \widetilde{Y}_{t+1}) \geq \operatorname{E}_{t} \frac{\langle \ln F_{t}, \widetilde{Y}_{t+1} \rangle}{\langle \widehat{\alpha}_{t}, X_{t+1} \rangle W_{t} + |\widetilde{Y}_{t+1}|} \geq |\widehat{\beta}_{t}| - \sum_{m=1}^{M} r_{t}^{m} |\beta_{t}^{m}|, \qquad (30)$$

where the first inequality follows from the concavity of the logarithm, and the second one follows from that $\ln F_t = \ln \hat{\beta}_t - \ln \sum_m r_t^m \beta_t^m$ and inequality (26).

Using that $|\beta_t^m| + \langle e, \alpha_t^m \rangle = 1$, we see that $E_t f_t(X_{t+1}, Y_{t+1}) \ge 0$, hence $E_t \ln \hat{r}_{t+1} \ge \ln \hat{r}_t$. Since $\ln \hat{r}_t$ is a non-positive sequence, this inequality also implies the integrability of $\ln \hat{r}_t$ (by induction, beginning with $\ln \hat{r}_0$), so it is a submartingale.

Proof of Theorem 2. When all the agents use \hat{h} , from (2) we find $p_t^n = \hat{\beta}_t^n W_t$, and hence $\langle \hat{\beta}_t, Z_{t+1} \rangle = \tilde{Y}_{t+1}/W_t$. Adding (19) and (27), we obtain (12). Then (13) follows by Jensen's inequality.

4.4. Proof of Theorem 3

We will need the following proposition which provides two inequalities of a general nature.

Proposition 8. 1) For any $a, b \in (0, 1]$

$$\ln\frac{a+b}{2} - \frac{\ln a + \ln b}{2} \ge \frac{(a-b)^2}{8}.$$
(31)

2) Suppose $x, y \in \mathbb{R}^N_+$ are two vectors such that $|x| \leq 1$, $|y| \leq 1$, and for each n it holds that if $y^n = 0$, then also $x^n = 0$. Then

$$\langle x, \ln x - \ln y \rangle \ge \frac{\|x - y\|^2}{4} + |x| - |y|.$$
 (32)

Proof. 1) Assume $a \leq b$. The inequality clearly holds if a = b. Let f(a) be the difference of its left-hand side and right-hand side, with b fixed. It is enough to show that $f'(a) \leq 0$ for $a \in (0, b]$. After differentiation, this becomes equivalent to $a(a + b) \leq 2$. The latter inequality is clearly true, provided that $a, b \in (0, 1]$.

2) Inequality (32) follows from a known inequality for the Kullback-Leibler divergence if x/|x| and y/|y| are considered as probability distributions on a set of N elements. Its short direct proof can be found in Drokin and Zhitlukhin (2020, Lemma 2).

Proof of Theorem 3. We will use the same notation for realizations of strategies as in the proof of Theorem 1. It was shown that $\ln \hat{r}_t$ is a submartingale. Let c_t be its compensator, i.e. the predictable non-decreasing sequence such that $\ln \hat{r}_t - c_t$ is a martingale; in the explicit form

$$c_t = \sum_{s \le t} (\mathbf{E}_{s-1} \ln \widehat{r}_s - \ln \widehat{r}_{s-1}).$$

As was shown in the proof of Theorem 1,

$$c_{t+1} - c_t = \mathcal{E}_t f_t(X_{t+1}, Y_{t+1}) = \mathcal{E}_t(f_t^{(1)}(X_{t+1}, Y_{t+1}) + f_t^{(2)}(X_{t+1}, Y_{t+1}))$$

with $f^{(1)}, f^{(2)}$ defined in (28). Since $\ln \hat{r}_t$ is non-positive and converges, we have $c_{\infty} < \infty$ with probability 1. Let us consider again inequalities (29)–(30) and strengthen them using Proposition 8. Fix $t \geq 1$ and let

$$a = \frac{\langle \hat{\alpha}_t, X_{t+1} \rangle + |Y_{t+1}| / W_t}{Q_{t+1}}, \qquad b = \frac{\langle \bar{\alpha}_t, X_{t+1} \rangle + |Y_{t+1}| / W_t}{Q_{t+1}}.$$

Note that $a, b \in (0, 1]$. Then

$$E_t f_t^{(1)}(X_{t+1}, Y_{t+1}) = 2 E_t \left(\ln a - \frac{\ln a + \ln b}{2} \right)$$

$$\geq 2 E_t \left(\ln \frac{a+b}{2} - \frac{\ln a + \ln b}{2} \right) + \langle e, \hat{\alpha}_t - \bar{\alpha}_t \rangle \qquad (33)$$

$$\geq \left(\frac{\langle \hat{\alpha} - \bar{\alpha}, X_{t+1} \rangle}{2Q_{t+1}} \right)^2 + \langle e, \hat{\alpha}_t - \bar{\alpha}_t \rangle.$$

Here, in the first inequality we used the estimate

$$\mathbf{E}_t \left(\ln a - \ln \frac{a+b}{2} \right) \ge \frac{1}{2} \mathbf{E}_t \frac{\langle \widehat{\alpha}_t - \overline{\alpha}_t, X_{t+1} \rangle W_t}{\langle \widehat{\alpha}_t, X_{t+1} \rangle W_t + |Y_{t+1}|} \ge \frac{1}{2} \langle e, \widehat{\alpha}_t - \overline{\alpha}_t \rangle,$$

which is obtained similarly to (29). In the second inequality of (33) we applied (31).

For the function $f^{(2)}$, using that there are no portfolio constraints on the endogenous assets, so $\hat{\beta}_t$ is given by (15), we find

$$E_t f_t^{(2)}(X_{t+1}, Y_{t+1}) \ge E_t \frac{\langle \ln F_t, Y_{t+1} \rangle}{\langle \widehat{\alpha}_t, X_{t+1} \rangle W_t + |Y_{t+1}|} = \langle \ln F_t, \widehat{\beta}_t \rangle = \langle \widehat{\beta}_t, \ln \widehat{\beta}_t - \ln \overline{\beta}_t \rangle \\
 \ge \frac{\|\widehat{\beta}_t - \overline{\beta}_t\|^2}{4} + |\widehat{\beta}_t| - |\overline{\beta}_t|,$$
(34)

where the first inequality is obtained similarly to (30), and in the second one we applied (32). Consequently, from (33), (34), we obtain

$$c_{t+1} - c_t \ge \left(\frac{\langle \widehat{\alpha}_t - \overline{\alpha}_t, X_{t+1} \rangle}{2Q_{t+1}}\right)^2 + \frac{\|\widehat{\beta}_t - \overline{\beta}_t\|^2}{4}.$$

From here, using that $c_{\infty} < \infty$, we get (16). Moreover, $\hat{\alpha}_t - \bar{\alpha}_t = (1 - \hat{r}_t)(\hat{\alpha}_t - \tilde{\alpha}_t)$ and $\hat{\beta}_t - \bar{\beta}_t = (1 - \hat{r}_t)(\hat{\beta}_t - \tilde{\beta}_t)$, so on the set (17) we necessarily have $\lim_{t\to\infty} \hat{r}_t = 1$.

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