WELL-POSEDNESS OF THE FREE BOUNDARY PROBLEM IN ELASTICDYNAMIC WITH MIXED STABILITY CONDITION

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ABSTRACT. In this paper, we prove the local well-posedness of the free boundary problem in incompressible elastodynamics under a mixed type stability condition, i.e., for each point of the free boundary, at least one of the Taylor sign condition $-\partial_n p > 0$ and the non-collinearity condition holds. This gives an affirmative answer to a problem raised by Trakhinin in [30].

1. INTRODUCTION

1.1. Presentation of the problem.

In this paper, we consider the incompressible inviscid flow in 3-D elastodynamics:

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathrm{div}(\mathbf{F}\mathbf{F}^\top), \\ \partial_t \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{F} = \nabla \mathbf{u}\mathbf{F}, \\ \mathrm{div}\,\mathbf{u} = 0, \end{cases}$$

where $\mathbf{u}(t, x) = (u_1, u_2, u_3)$ denotes the fluid velocity, $p(t, x)$ is the pressure, $\mathbf{F}(t, x) = (F_{ij})_{3\times3}$ is the deformation tensor, $\mathbf{F}^\top = (F_{ji})_{3\times3}$ denotes the transpose of the matrix $\mathbf{F}$, $\mathbf{F}\mathbf{F}^\top$ is the Cauchy-Green tensor in the case of neo-Hookean elastic materials, $(\nabla \mathbf{u})_{ij} = \partial_j u_i$, $(\nabla \mathbf{u}\mathbf{F})_{ij} = \sum_{k=1}^{3} \mathbf{F}_{kj}\partial_k u_i$, $(\mathrm{div}\,\mathbf{F}^\top)_i = \sum_{j=1}^{3} \partial_j F_{ji}$, $(\mathrm{div}(\mathbf{F}\mathbf{F}^\top))_i = \sum_{j,k=1}^{3} \partial_j(F_{ik}F_{jk})$.

We will study the solution of (1.1) defined in a time-dependent domain. Precisely, we let

$$\Omega = \mathbb{T}^2 \times [-1, 1] \subset \mathbb{R}^3, \quad \Gamma_f = \{x \in \Omega | x_3 = f(t, x'), x' = (x_1, x_2) \in \mathbb{T}^2\},$$

$$\Omega_f = \{x \in \Omega | x_3 \le f(t, x'), x' = (x_1, x_2) \in \mathbb{T}^2\}, \qquad Q_T = \bigcup_{t\in(0,T)} \{t\} \times \Omega_f,$$

where Γ_f is the free boundary and is assumed to be a graph. The system reads as

$$(1.2) \quad \begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \sum_{j=1}^{3} (\mathbf{F}_j \cdot \nabla)\mathbf{F}_j & \text{in} \quad Q_T, \\ \mathrm{div}\,\mathbf{u} = 0,\ \mathrm{div}\,\mathbf{F}^\top = 0 & \text{in} \quad Q_T, \\ \partial_t \mathbf{F}_j + \mathbf{u} \cdot \nabla \mathbf{F}_j = (\mathbf{F}_j \cdot \nabla)\mathbf{u} & \text{in} \quad Q_T, \end{cases}$$

with the boundary conditions on the moving interface Γ_f:

$$(1.3) \qquad\qquad \mathbf{u} \cdot \mathbf{N}_f = \partial_t f, \qquad \mathbf{F}_j \cdot \mathbf{N}_f = 0,$$

$$(1.4) \qquad\qquad p = 0.$$

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Here $\mathbf{F}_j = (F_{1j}, F_{2j}, F_{3j})$, $\mathbf{N}_f = (-\partial_1 f, -\partial_2 f, 1)$ and $\mathbf{n}_f = \mathbf{N}_f/|\mathbf{N}_f|$ is the outward unit normal vector. On the artificial boundary $\Gamma^- = \mathbb{T}^2 \times \{-1\}$, we impose the following boundary conditions on $(\mathbf{u}, \mathbf{F})$:

$$(1.5) \qquad u_3 = 0, \qquad F_{3j} = 0 \qquad \text{on } \Gamma^-.$$

The system (1.2) is supplemented with the initial data

$$(1.6) \qquad \mathbf{u}(0, x) = \mathbf{u}_0(x), \qquad \mathbf{F}(0, x) = \mathbf{F}_0(x) \qquad \text{in } \Omega_{f_0},$$

with

$$(1.7) \qquad \begin{cases} \operatorname{div} \mathbf{u}_0 = 0, \ \operatorname{div} \mathbf{F}_{0j} = 0 & x \in \Omega_{f_0}, \\ \mathbf{F}_{0j} \cdot \mathbf{N}_{f_0} = 0 & x \in \Gamma_{f_0}, \\ u_{03} = 0, \ F_{03j} = 0 & x \in \Gamma^-. \end{cases}$$

Let us remark that the divergence free restriction on $\mathbf{F}_j$ is automatically satisfied if $\operatorname{div} \mathbf{F}_{0j} = 0$. Indeed, if we apply the divergence operator to the third equation of (1.2), we will deduce the following transport equation

$$\partial_t \operatorname{div} \mathbf{F}_j + \mathbf{u} \cdot \nabla \operatorname{div} \mathbf{F}_j = 0.$$

Similar argument can be also applied to yield that $\mathbf{F}_j \cdot \mathbf{N}_f = 0$ on Γ_f if $\mathbf{F}_{0j} \cdot \mathbf{N}_{f_0} = 0$ on Γ_{f_0}.

The main goal of this paper is to study the local well-posedness of the system (1.2)-(1.7) under some suitable stability conditions imposed on the initial data.

1.2. **Backgrounds.** The free boundary problems for incompressible inviscid flow have received a lot of attention in the past decades. It is well-known that, under the Taylor sign condition

$$\mathbf{n} \cdot \nabla p \le -\varepsilon < 0 \quad \text{on } \Gamma_f,$$

the water wave problem for the incompressible Euler flow is well-posed [9, 31, 32, 24, 34]. Otherwise, the system could be ill-posed [10], which is known as the Rayleigh-Taylor instability. In addition, the vortex sheet problem for the incompressible Euler flow is always ill-posed, which is called the Kelvin-Helmholtz instability[17]. However, the surface tension has been proved that it could stabilize the Kelvin-Helmholtz and Rayleigh-Taylor instability, see [3, 4, 25].

Syrovatskij [23, 2] observed that the presence of strong tangential magnetic fields can stabilize the Kelvin-Helmholtz instability for magnetohydrodynamics. There are many important works devoted to the rigorous mathematical justification, see [26, 27, 7, 33] for the compressible case and [18, 28, 8, 21] for the incompressible case. We also refer to [19, 22, 29, 20] for the plasma-vacuum problem in magnetohydrodynamics. The effect of the Taylor sign condition in the plasma-vacuum problem has been studied in [13, 12, 11].

There are also several progresses on the free boundary problems for inviscid elastodynamics. Chen-Hu-Wang [5] analyzed the 2-D linearized stability and proved the stabilization effect of elasticity on compressible vortex sheets. Recently, Chen-Huang-Wang-Yuan [6] extended the results to the 3-D nonlinear compressible case. In [30], Trakhinin proved the well-posedness of the fluid-vacuum free boundary problem in compressible elastodynamics under the condition that there are two columns

of the 3×3 deformation tensor which are non-collinear at each point of the initial surface. For the incompressible case, Hao-Wang [14] proved a priori estimates for solutions in Sobolev spaces under the Taylor sign condition. Li-Wang-Zhang [16] proved the stabilization effect of elasticity on both the vortex sheets and fluid-vacuum problem. Gu-Wang [11] proved the local well-posedness in a domain with two disconnected free boundaries, where the Taylor sign condition and non-collinear condition hold on each free boundary.

The aim of this paper is to show the local well-posedness for the fluid-vacuum free boundary problem in incompressible elastodynamics under a mixed type stability condition, i.e., for each point of the free boundary, one of the Taylor sign condition and the non-collinear condition is satisfied. The most important contribution of this paper is that we derived a special evolution equation for the free interface, in which both effects of those two stability conditions can be reflected, and the combination of these two conditions will ensure this evolution equation to be strictly hyperbolic.

1.3. **Main result.** We define

$$\Lambda(\mathbf{F})(x') \overset{\text{def}}{=} \inf_{\phi_1^2 + \phi_2^2 = 1} \sum_{j=1}^{3} (\underline{F_{1j}}\phi_1 + \underline{F_{2j}}\phi_2)^2(x').$$

If $\Lambda(\mathbf{F})(x') \geq c_0 > 0$, we say that $\mathbf{F}$ is non-collinear at x'. Here $\underline{F}$ denotes the restriction of F on Γ_f and x' denotes (x_1, x_2) the first two components of x.

We assume that there exists $c_0 > 0$ such that

$$(1.8) \qquad \begin{cases} \mathbf{N}_f \cdot \underline{\nabla p} \leq -c_0 < 0 & x' \in \Gamma^1, \\ \Lambda(\mathbf{F})\underline{(x')} > c_0 & x' \in \Gamma^2, \end{cases}$$

where Γ^1 and Γ^2 are open sets on $\mathbb{T}^2$ satisfying $\mathbb{T}^2/\Gamma^1 \Subset \Gamma^2$. We call (1.8) the mixed type stability condition.

Let $D_{F_j} = \underline{F_{ij}}\partial_i'$. Our main result is stated as follows.

Theorem 1.1. *Let $s \geq 4$ be an integer and assume that*

$$D_{F_{0k}} f_0 \in H^{s-1/2}(\mathbb{T}^2), \quad f_0 \in H^s(\mathbb{T}^2), \quad \mathbf{u}_0, \mathbf{F}_0 \in H^s(\Omega_{f_0}).$$

Furthermore, for two open sets $\Gamma^1, \Gamma^2 \subset \mathbb{T}^2$ satisfying $\mathbb{T}^2/\Gamma^1 \Subset \Gamma^2$, we assume that there exists $c_0 > 0$ so that

1. $-(1 - 2c_0) \leq f_0 \leq (1 - 2c_0)$;
2. $-\mathbf{N}_{f_0} \cdot \underline{\nabla p_0} \geq 2c_0$ *on* Γ^1;
3. $\Lambda(\mathbf{F}_0)\underline{(x')} \geq 2c_0$ *on* Γ^2.

Then there exists $T > 0$ such that the system (1.2)-(1.7) admits a unique solution $(f, \mathbf{u}, \mathbf{F})$ in $[0, T]$ satisfying

1. $f \in L^\infty([0, T), H^s(\mathbb{T}^2))$, $D_{F_k} f \in L^\infty([0, T), H^{s-1/2}(\mathbb{T}^2))$;
2. $\mathbf{u}, \mathbf{F} \in L^\infty(0, T; H^s(\Omega_f))$;
3. $-(1 - c_0) \leq f \leq (1 - c_0)$;
4. $-\mathbf{N}_f \cdot \underline{\nabla p} \geq c_0$ *on* Γ^1;
5. $\Lambda(\mathbf{F})\underline{(x')} \geq c_0$ *on* Γ^2.

The rest of this paper is organized as follows. In Section 2, we introduce the harmonic coordinate, Dirichlet-Neumann operator and some basic estimates related to the theorem. In Section 3, we derive an evolution equation for the free interface in which both effects of those two stability conditions can be reflected. In Section 4, we construct an ε-regularized system and construct an approximation sequence to the solution of the original system. In Section 5, we prove the existence and uniqueness of the solution.

2. Harmonic coordinate and Dirichlet-Neumann operator

In this section, we recall some facts and well-known results on the harmonic coordinate and Dirichlet-Neumann operators.

We first introduce some notations used throughout this paper. We use $x = (x_1, x_2, x_3)$ or $y = (y_1, y_2, y_3)$ to denote the coordinates in the fluid region, and use $x' = (x_1, x_2)$ or $y' = (y_1, y_2)$ to denote the natural coordinates on the interface. In addition, we will use the Einstein summation notation where a summation from 1 to 2 is implied over repeated index, while a summation from 1 to 3 over repeated index will be explicitly figured out by the symbol $\sum$ (i.e. $a_i b_i = a_1 b_1 + a_2 b_2$, $\sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$).

For a function $g : \Omega_f \to \mathbb{R}$, we denote $\nabla g = (\partial_1 g, \partial_2 g, \partial_3 g)$, and for a function $\eta : \mathbb{T}^2 \to \mathbb{R}$, $\nabla' \eta = (\partial_1 \eta, \partial_2 \eta)$, it is the same for the operator Δ and Δ'. For a function $g : \Omega_f \to \mathbb{R}$, we can define its trace on Γ_f, which are denoted by $\underline{g}(x')$. Thus, for $i = 1, 2$,

$$\partial_i \underline{g}(x') = \partial_i g(x', f(x')) + \partial_3 g(x', f(x')) \partial_i f(x').$$

In this paper we do not distinguish $D_t = \partial_t + u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$ on Ω_f and $D_t = \partial_t + \underline{u_1} \partial'_1 + \underline{u_2} \partial'_2$ on Γ_f. Recalling that $\underline{\mathbf{u}} \cdot \mathbf{N}_f = \partial_t f$, for any function v defined on Ω_f, we have

$$\underline{D_t v} = D_t \underline{v}.$$

We denote by $\| \cdot \|_{H^s(\Omega_f)}$, $|\cdot|_{H^s}$ the Sobolev norm on Ω_f and $\mathbb{T}^2$ respectively.

In the free boundary problem, the functions $(\mathbf{u}, \mathbf{F})$ are defined in a domain changing with time t. To overcome this difficulty, we pull them back to the fixed reference domain Ω_* [16]. Let Γ_* be a fixed graph surface given by

$$\Gamma_* = \{(y_1, y_2, y_3) : y_3 = f_*(y_1, y_2)\},$$

where f_* satisfies $\int_{\mathbb{T}^2} f_*(y') dy' = 0$. The reference domain Ω_* is given by

$$\Omega_* = \{y \in \Omega | y_3 \le f_*(y_1, y_2)\}.$$

We will look for the free boundary which lies in a neighborhood of f_*. As a result, we define

$$\Upsilon(\delta, k) \stackrel{\text{def}}{=} \Big\{ f \in H^k(\mathbb{T}^2) : |f - f_*|_{H^k(\mathbb{T}^2)} \le \delta \Big\}.$$

For $f \in \Upsilon(\delta, k)$, we can define the graph Γ_f by

$$\Gamma_f \stackrel{\text{def}}{=} \Big\{ x \in \Omega_t | x_3 = f(t, x'), \int_{\mathbb{T}^2} f(t, x') dx' = 0 \Big\}.$$

Now we introduce the harmonic coordinate. Given $f \in \Upsilon(\delta, k)$, we define a map Φ_f from Ω_* to Ω_f by the harmonic extension:

$$(2.1) \qquad \begin{cases} \Delta_y \Phi_f = 0, & y \in \Omega_*, \\ \Phi_f(y', f_*(y')) = (y', f(y')), & y' \in \mathbb{T}^2, \\ \Phi_f(y', -1) = (y', -1), & y' \in \mathbb{T}^2. \end{cases}$$

Given Γ_*, there exists $\delta_0 = \delta_0(|f_*|_{W^{1,\infty}}) > 0$ so that Φ_f is a bijection when $\delta \leq \delta_0$. Then we can define an inverse map $\Phi_f^{-1} : \Omega_f \to \Omega_*$ such that

$$\Phi_f^{-1} \circ \Phi_f = \Phi_f \circ \Phi_f^{-1} = \mathrm{Id}.$$

The following properties come from [21].

Lemma 2.1. *Let $f \in \Upsilon(\delta_0, s - \frac{1}{2})$ for $s \geq 3$. Then there exists a constant C depending only on δ_0 and $\|f_*\|_{H^{s-\frac{1}{2}}}$ so that*

1. *If $u \in H^\sigma(\Omega_f)$ for $\sigma \in [0, s]$, then*

$$\|u \circ \Phi_f\|_{H^\sigma(\Omega_*)} \leq C\|u\|_{H^\sigma(\Omega_f)}.$$

2. *If $u \in H^\sigma(\Omega_*)$ for $\sigma \in [0, s]$, then*

$$\|u \circ \Phi_f^{-1}\|_{H^\sigma(\Omega_f)} \leq C\|u\|_{H^\sigma(\Omega_*)}.$$

3. *If $u, v \in H^\sigma(\Omega_*)$ for $\sigma \in [2, s]$, then*

$$\|uv\|_{H^\sigma(\Omega_f)} \leq C\|u\|_{H^\sigma(\Omega_f)}\|v\|_{H^\sigma(\Omega_f)}.$$

We will use the Dirichlet-Neumann operator, which maps the Dirichlet boundary value of a harmonic function to its Neumann boundary value. That is to say, for any $g(x') = g(x_1, x_2) \in H^k(\mathbb{T}^2)$, we denote by $\mathcal{H}_f g$ the harmonic extension to Ω_f:

$$(2.2) \qquad \begin{cases} \Delta \mathcal{H}_f g = 0, & x \in \Omega_f, \\ (\mathcal{H}_f g)(x', f(x')) = g(x'), & x' \in \mathbb{T}^2, \\ \mathcal{H}_f g(x', -1) = 0, & x' \in \mathbb{T}^2. \end{cases}$$

We define another harmonic extension for different use:

$$(2.3) \qquad \begin{cases} \Delta \bar{\mathcal{H}}_f g = 0, & x \in \Omega_f, \\ (\bar{\mathcal{H}}_f g)(x', f(x')) = g(x'), & x' \in \mathbb{T}^2, \\ \partial_3 \bar{\mathcal{H}}_f g(x', -1) = 0, & x' \in \mathbb{T}^2. \end{cases}$$

Then, we define two kinds of Dirichlet-Neumann operators:

$$(2.4) \qquad \mathcal{N}_f g \stackrel{def}{=} \mathbf{N}_f \cdot (\nabla \mathcal{H}_f g)\big|_{\Gamma_f}, \quad \bar{\mathcal{N}}_f g \stackrel{def}{=} \mathbf{N}_f \cdot (\nabla \bar{\mathcal{H}}_f g)\big|_{\Gamma_f}.$$

We will use the following properties from [1, 21].

Lemma 2.2. *It holds that*

1. *$\mathcal{N}_f$ is a self-adjoint operator:*

$$(\mathcal{N}_f \psi, \phi) = (\psi, \mathcal{N}_f \phi), \quad \forall \phi, \psi \in H^{1/2}(\mathbb{T}^2);$$

2. $\mathcal{N}_f$ *is a positive operator:*

$$(\mathcal{N}_f\phi, \phi) = \|\nabla\mathcal{H}_f\phi\|^2_{L^2(\Omega_f)} \geq 0, \quad \forall \phi \in H^{1/2}(\mathbb{T}^2);$$

Especially, if $\int_{\mathbb{T}^2} \phi(x')dx' = 0$, *there exists* $c > 0$ *depending on* $c_0, \|f\|_{W^{1,\infty}}$ *such that*

$$(\mathcal{N}_f\phi, \phi) \geq c\|\mathcal{H}_f\phi\|^2_{H^1(\Omega_f)} \geq c|\phi|^2_{H^{1/2}}.$$

3. $\mathcal{N}_f$ *is a bijection from* $H^{k+1}_0(\mathbb{T}^2)$ *to* $H^k_0(\mathbb{T}^2)$ *for* $k \geq 0$, *where*

$$H^k_0(\mathbb{T}^2) \stackrel{def}{=} H^k(\mathbb{T}^2) \cap \Big\{\phi \in L^2(\mathbb{T}^2) : \int_{\mathbb{T}^2} \phi(x')dx' = 0\Big\}.$$

Lemma 2.3. *If* $f \in H^s(\mathbb{T}^2)$ *for* $s > 3$, *then it holds that for any* $\sigma \in [\frac{1}{2}, s]$,

$$(2.5) \qquad\qquad |\mathcal{N}_f^{-1}\phi|_{H^\sigma} \leq C(|f|_{H^s})|\phi|_{H^{\sigma-1}}.$$

Proof. The proof can be found in [21]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The results in the above two lemmas also hold for $\bar{\mathcal{N}}_f$.

3. Evolution equation for the free interface

In this section, we derive the evolution equation for the interface from the original system (1.2)-(1.7). The key ingredient here is that the evolution equation for $\partial'_i f$ could reflect the stabilization mechanism of both two stability conditions explicitly.

Recalling the boundary conditions for **u** and **F**, we have

$$D_t f = \underline{u_3}.$$

For any function $g = g(t, x')$, we also have:

$$(3.1) \qquad [D_t, \partial'_i]g = -\partial'_i\underline{u_j}\partial'_j g, \quad [D_t, \underline{F_{jk}}\partial'_j]g = 0, \quad [\partial'_i, \underline{F_{jk}}\partial'_j]g = \partial'_i\underline{F_{jk}}\partial'_j g.$$

Then by $D_t^2 f = D_t\underline{u_3}$, it follows that

$$
\begin{aligned}
D_t^2\partial'_i f &= D_t\partial'_i D_t f + D_t[D_t, \partial'_i]f = \partial'_i D_t^2 f + [D_t, \partial'_i]D_t f + D_t[D_t, \partial'_i]f \\
&= \partial'_i D_t\underline{u_3} - (\partial'_i\underline{u_j})\partial'_j D_t f - D_t((\partial'_i\underline{u_j})\partial'_j f) \\
&= \partial'_i D_t\underline{u_3} - (\partial'_i\underline{u_j})\partial'_j D_t f - (\partial'_i\underline{u_j})D_t(\partial'_j f) - (D_t\partial'_i\underline{u_j})\partial'_j f \\
&= \partial'_i D_t\underline{u_3} - (\partial'_i\underline{u_j})(\partial'_j\underline{u_k})\partial'_k f - 2(\partial'_i\underline{u_j})D_t(\partial'_j f) - (\partial'_i D_t\underline{u_j})\partial'_j f + (\partial'_i\underline{u_k})(\partial'_k\underline{u_j})\partial'_j f \\
&= \partial'_i D_t\underline{u_3} - 2(\partial'_i\underline{u_j})D_t\partial'_j f - (\partial'_i D_t\underline{u_j})\partial'_j f.
\end{aligned}
$$

Since $\underline{\mathbf{F}_j} \cdot \mathbf{N}_f = 0$, the first equation of (1.2) implies that

$$
\begin{aligned}
\underline{D_t u_i} &= -\underline{\partial_i p} + \sum_{j=1}^3 \underline{\mathbf{F}_j} \cdot \nabla F_{ij} \\
&= -\underline{\partial_i p} + \sum_{j=1}^3 \underline{F_{sj}}\partial'_s\underline{F_{ij}} + \sum_{j=1}^3 \underline{\mathbf{F}_j} \cdot \mathbf{N}_f\partial_3 F_{ij} = -\underline{\partial_i p} + \sum_{j=1}^3 \underline{F_{sj}}\partial'_s\underline{F_{ij}}.
\end{aligned}
$$

Using the equality above, there holds

$$\partial_i' D_t u_3 - (\partial_i' D_t u_j)\partial_j' f$$

$$= -\partial_i'\partial_3 p + \sum_{k=1}^{3} \partial_i'(F_{sk}\partial_s' F_{3k}) + (\partial_i'\partial_j p)\partial_j' f - \sum_{k=1}^{3}(\partial_i'(F_{sk}\partial_s' F_{jk}))\partial_j' f$$

$$= -\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f + \sum_{k=1}^{3}\left((\partial_i' F_{sk})\partial_s'(F_{jk}\partial_j' f) + F_{sk}\partial_s'((\partial_i' F_{jk})\partial_j' f) + F_{sk}\partial_s'(F_{jk}\partial_i'\partial_j' f)\right)$$

$$\quad - \sum_{k=1}^{3}\left((\partial_i' F_{sk})(\partial_s' F_{jk})\partial_j' f + F_{sk}(\partial_i'\partial_s' F_{jk})\partial_j' f\right)$$

$$= -\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f + \sum_{k=1}^{3}\left((\partial_i' F_{sk})F_{jk}\partial_s'\partial_j' f + F_{sk}(\partial_i' F_{jk})\partial_s'\partial_j' f + F_{sk}\partial_s'(F_{jk}\partial_i'\partial_j' f)\right)$$

$$= -\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f + \sum_{k=1}^{3}\left(2(\partial_i' F_{sk})F_{jk}\partial_s'\partial_j' f + F_{sk}\partial_s'(F_{jk}\partial_i'\partial_j' f)\right),$$

where we use $\mathbf{F}_j \cdot \mathbf{N}_f = 0$ again. Recalling that $D_{F_j} = F_{ij}\partial_i'$, we arrive at

$$D_t^2\partial_i' f = -\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f + \sum_{k=1}^{3}\left(2(\partial_i' F_{sk})F_{jk}\partial_s'\partial_j' f + F_{sk}\partial_s'(F_{jk}\partial_i'\partial_j' f)\right) - 2(\partial_i' u_j)D_t\partial_j' f$$

$$= -\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f + \sum_{k=1}^{3} D_{F_k}^2\partial_i' f + \sum_{k=1}^{3} 2(\partial_i' F_{sk})D_{F_k}\partial_s' f - 2(\partial_i' u_j)D_t\partial_j' f.$$

Now, we focus on $-\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f$. We write

$$(3.2) \qquad\qquad p = p_{\mathbf{u},\mathbf{u}} - \sum_{j=1}^{3} p_{\mathbf{F}_j,\mathbf{F}_j},$$

where $p_{\mathbf{u}_1,\mathbf{u}_2}$ is the solution of the following equation:

$$(3.3) \qquad\qquad \begin{cases} \Delta p_{\mathbf{u}_1,\mathbf{u}_2} = -\mathrm{tr}(\nabla\mathbf{u}_1\nabla\mathbf{u}_2) & x \in \Omega_f, \\ p_{\mathbf{u}_1,\mathbf{u}_2} = 0 & x \in \Gamma_f, \\ \partial_3 p_{\mathbf{u}_1,\mathbf{u}_2} = 0 & x \in \Gamma^-. \end{cases}$$

Define auxiliary functions $q_i \overset{\text{def}}{=} \partial_i p + \partial_3 p \mathcal{H}_f(\partial_i' f)$, which satisfies $q_i|_{\Gamma_f} = \partial_i p + \partial_3 p \partial_i' f = 0$. Then, we have

$$-\partial_i'\partial_3 p + (\partial_i'\partial_j p)\partial_j' f = -\partial_i\partial_3 p - \partial_3^2 p \partial_i' f + (\partial_i\partial_j p)\partial_j' f + \partial_3\partial_j p\partial_i' f\partial_j' f$$

$$= -\partial_3 q_i + \partial_3 p\partial_3\mathcal{H}_f(\partial_i' f) + \partial_j' f(\partial_j q_i - \partial_3 p\partial_j\mathcal{H}_f(\partial_i' f))$$

$$= -\mathbf{N}_f \cdot \nabla q_i + \partial_3 p\mathbf{N}_f \cdot \nabla\mathcal{H}_f(\partial_i' f).$$

Finally, we arrive at

$$(3.4) \qquad D_t^2 \partial_i' f = \underline{\partial_3 p} \mathbf{N}_f \cdot \underline{\nabla \mathcal{H}_f(\partial_i' f)} + \sum_{k=1}^{3} D_{F_k}^2 \partial_i' f$$

$$+ \sum_{k=1}^{3} 2(\partial_i' \underline{F_{sk}}) D_{F_k} \partial_s' f - 2(\partial_i' \underline{u_j}) D_t \partial_j' f - \mathbf{N}_f \cdot \underline{\nabla q_i}.$$

From the fact that $p = 0$ on Γ_f, the Taylor sign condition gives

$$-|\mathbf{N}_f|^2 \underline{\partial_3 p} = -\mathbf{N}_f \cdot \underline{\nabla p} \geq c > 0 \qquad \text{on } \Gamma^1.$$

One can see that the equation (3.4) explicitly shows the stabilization mechanism for both two stability conditions.

As Γ_f is a graph, $|\mathbf{N}_f|_{L^\infty} < \infty$. In the rest of this paper, we write the Taylor sign condition as $-\underline{\partial_3 p} \geq c$ just for convenience.

4. ε-REGULARIZED SYSTEM

For the original system (1.2)-(1.7), we choose the initial data $(f_0, \mathbf{u}_0, \mathbf{F}_0)$ satisfies following conditions:

C1. $D_{F_{0k}} f_0 \in H^{s-1/2}(\mathbb{T}^2)$, $f_0 \in H^s(\mathbb{T}^2)$, $\mathbf{u}_0$, $\mathbf{F}_0 \in H^s(\Omega_{f_0})$, where $s \geq 4$ is an integer;

C2. For two open set $\Gamma^1, \Gamma^2 \subset \mathbb{T}^2$ which satisfies $\mathbb{T}^2 / \Gamma^1 \Subset \Gamma^2$, there exists $c_0 > 0$ so that:
   1. $-(1 - 2c_0) \leq f_0 \leq (1 - 2c_0)$;
   2. $-\mathbf{N}_{f_0} \cdot \nabla p_0 \geq 2c_0$ on Γ^1;
   3. $\Lambda(\mathbf{F}_0)(x') \geq 2c_0$ on Γ^2.

In this section, we introduce a regularized system with a suitable initial data.

Consider the system (1.2)-(1.7) with the boundary condition for p in (1.4) replaced by

$$(4.1) \qquad p = -\varepsilon \bar{\mathcal{N}}_f^{-1} \Delta' f \qquad \text{on } \Gamma_f.$$

This system requires f to have a higher regularity, thus we equip it with the initial data $(f_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)$ as defined below. Let

$$(4.2) \qquad f_0^\varepsilon \overset{\text{def}}{=} \eta_\varepsilon * f_0,$$

where $\eta_\varepsilon = \frac{1}{\varepsilon} \eta(\frac{x'}{\sqrt{\varepsilon}})$ is a mollifier. Apparently, it holds that $\varepsilon |f_0^\varepsilon|_{H^{s+1/2}}^2 \leq C |f_0|_{H^s}^2$ and $-(1 - \frac{3}{2}c_0) \leq f_0^\varepsilon \leq (1 - \frac{3}{2}c_0)$ when ε small enough.

We define harmonic coordinate $\Phi_{f_0^\varepsilon} : \Omega_{f_0} \to \Omega_{f_0^\varepsilon}$ based on f_0^ε. Define

$$(4.3) \qquad \mathbf{u}_0^\varepsilon \overset{\text{def}}{=} P_{f_0^\varepsilon}^{\text{div}}(\mathbf{u}_0 \circ \Phi_{f_0^\varepsilon}^{-1}), \qquad \mathbf{F}_0^\varepsilon \overset{\text{def}}{=} \bar{P}_{f_0^\varepsilon}^{\text{div}}(\mathbf{F} \circ \Phi_{f_0^\varepsilon}^{-1}),$$

where $P_{f_0^\varepsilon}^{\mathrm{div}}$ and $\bar{P}_{f_0^\varepsilon}^{\mathrm{div}}$ are projection operators which map a vector field $\Omega_{f_0^\varepsilon}$ to its divergence-free part. More precisely, $P_{f_0^\varepsilon}^{\mathrm{div}}\mathbf{v} = \mathbf{v} - \nabla\phi$, $\bar{P}_{f_0^\varepsilon}^{\mathrm{div}}\mathbf{v} = \mathbf{v} - \nabla\bar{\phi}$ with

$$\begin{cases} \Delta\phi = \operatorname{div}\mathbf{v} & \text{in} \quad \Omega_{f_0^\varepsilon}, \\ \phi = 0 & \text{on} \quad \Gamma_{f_0^\varepsilon}, \\ \partial_3\phi = 0 & \text{on} \quad \Gamma^-, \end{cases}$$

and

$$\begin{cases} \Delta\bar{\phi} = \operatorname{div}\mathbf{v} & \text{in} \quad \Omega_{f_0^\varepsilon}, \\ \mathbf{N}_{f_0^\varepsilon} \cdot \nabla\bar{\phi} = \mathbf{v} \cdot (\partial_1'f_0 - \partial_1'f_0^\varepsilon, \partial_2'f_0 - \partial_2'f_0^\varepsilon, 0) & \text{on} \quad \Gamma_{f_0^\varepsilon}, \\ \partial_3\bar{\phi} = 0 & \text{on} \quad \Gamma^-. \end{cases}$$

Thus, we have $\mathbf{F}_{0j}^\varepsilon \cdot \mathbf{N}^\varepsilon = 0$ and $(\mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)$ satisfies (1.7). Since $D_{\mathbf{F}_{0k}}f_0 \in H^{s-1/2}(\mathbb{T}^2)$, $(\mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)$ have the same regularity as $(\mathbf{u}_0, \mathbf{F}_0)$. It is straightforward to show that $(\mathbf{u}_0^\varepsilon \circ \Phi_{f_0^\varepsilon}, \mathbf{F}_0^\varepsilon \circ \Phi_{f_0^\varepsilon})$ converges to $(\mathbf{u}_0, \mathbf{F}_0)$ in $H^s(\Omega_{f_0})$ when ε tends to 0 [21]. We define

$$(4.4) \qquad M_0^s = |f_0|_{H^s}^2 + \sum_{k=1}^{3} |D_{\mathbf{F}_{0,k}}f_0|_{H^{s-1/2}}^2 + \|\mathbf{u}_0\|_{H^s(\Omega_{f_0})}^2 + \|\mathbf{F}_0\|_{H^s(\Omega_{f_0})}^2,$$

$$(4.5) \qquad M_\varepsilon^s = \varepsilon|f_0^\varepsilon|_{H^{s+1/2}}^2 + |f_0^\varepsilon|_{H^{s-1/2}}^2 + \|\mathbf{u}_0^\varepsilon\|_{H^s(\Omega_{f_0^\varepsilon})}^2 + \|\mathbf{F}_0^\varepsilon\|_{H^s(\Omega_{f_0^\varepsilon})}^2.$$

We choose $\hat{\varepsilon}$ small enough such that for all $\varepsilon \le \hat{\varepsilon}$ there exists a constant C independent of ε satisfying

$$(4.6) \qquad\qquad\qquad M_\varepsilon^s \le C M_0^s.$$

We call the system (1.2)-(1.3), (4.1) with initial data (4.2)-(4.3) *the ε-regularized system*.

For this regularized system, one can obtain (the derivation is similar to (3.4) but more simple; one can also see [16] for the detailed derivation)

$$(4.7) \qquad \partial_t f^\varepsilon = \theta^\varepsilon,$$

$$(4.8) \qquad \partial_t\theta^\varepsilon = -2(\underline{u_1^\varepsilon}\partial_1'\theta^\varepsilon + \underline{u_2^\varepsilon}\partial_2'\theta^\varepsilon) - \sum_{s,r=1,2} u_s^\varepsilon u_r^\varepsilon \partial_s'\partial_r'f^\varepsilon$$

$$- \mathbf{N}_{f^\varepsilon} \cdot \nabla(\underline{p_{\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon} - \sum_j p_{\mathbf{F}_j^\varepsilon,\mathbf{F}_j^\varepsilon}}) + \sum_j \sum_{s,r=1,2} \underline{F_{sj}^\varepsilon F_{rj}^\varepsilon \partial_s'\partial_r'f^\varepsilon} + \varepsilon\Delta'f^\varepsilon.$$

Therefore, we can write

$$D_t^2 f^\varepsilon = \sum_j \sum_{s,r=1,2} \underline{F_{sj}^\varepsilon F_{rj}^\varepsilon \partial_s'\partial_r'f^\varepsilon} + \varepsilon\Delta'f^\varepsilon + L(\theta^\varepsilon, f^\varepsilon),$$

where $L(\theta^\varepsilon, f^\varepsilon)$ represents lower order terms.

The following proposition is the main results of this section.

Proposition 4.1. *Assume that $(f_0, \mathbf{u}_0, \mathbf{F}_0)$ satisfies C1 and C2, then there exists constants $(\bar{\varepsilon}, \bar{T})$, such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $T \in [0, \bar{T}]$ the system (1.2)-(1.3), (4.1) with initial data (4.2)-(4.3) admits a unique solution $(f^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$ in $[0, T]$ satisfying*

1. $\varepsilon|f^\varepsilon|_{H^{s+1/2}}^2 + |f^\varepsilon|_{H^{s-1/2}}^2 + \|\mathbf{u}^\varepsilon\|_{H^s(\Omega_{f^\varepsilon})}^2 + \|\mathbf{F}^\varepsilon\|_{H^s(\Omega_{f^\varepsilon})}^2 \le C M_0^s$;
3. $-(1 - \frac{1}{2}c_0) \le f^\varepsilon \le (1 - \frac{1}{2}c_0)$;

4. $-\mathbf{N}_{f^\varepsilon} \cdot \nabla p^\varepsilon \geq \frac{1}{2} c_0$ on Γ^1;

5. $\Lambda(\mathbf{F}^\varepsilon)(x') \geq \frac{1}{2} c_0$ on Γ^2.

Here the index s is given in C1, and the constant C is independent of ε and T.

Before presenting the proof, we give some results which will be used. Firstly, from [16, Theorem 1.2], the ε-regularized system is locally well-posed. More precisely, it holds

Proposition 4.2. *Let $s \geq 4$ be an integer. Assume that*

$$f_0^\varepsilon \in H^{s+1/2}(\mathbb{T}^2), \quad \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon \in H^s(\Omega_{f_0^\varepsilon}).$$

Furthermore, assume that there exists $\bar{c}_0 \leq 0$ so that

$$-(1 - 2\bar{c}_0) \leq f_0^\varepsilon \leq (1 - 2\bar{c}_0).$$

Then there exists $T = T(\varepsilon, M_\varepsilon^s) > 0$ such that, the system (1.2)-(1.3) and (4.1) with initial data (4.2)-(4.3) admits a unique solution $(f^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$ on $[0, T]$ satisfying

1. $f^\varepsilon \in L^\infty([0, T), H^{s+1/2}(\mathbb{T}^2))$;
2. $\mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon \in L^\infty(0, T; H^s(\Omega_{f^\varepsilon}))$;
3. $-(1 - \bar{c}_0) \leq f^\varepsilon \leq (1 - \bar{c}_0)$.

Moreover, for $t \in [0, T]$ it holds

$$\varepsilon |f^\varepsilon|^2_{H^{s+1/2}} + |f^\varepsilon|^2_{H^{s-1/2}} + \|\mathbf{u}^\varepsilon\|^2_{H^s(\Omega_{f^\varepsilon})} + \|\mathbf{F}^\varepsilon\|^2_{H^s(\Omega_{f^\varepsilon})} \leq 2M_\varepsilon^s.$$

To extend the solution to a time interval independent of ε, we have to derive a uniform (in ε) a priori estimate for the solution $(f^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$. We will drop the superscript ε of $(f^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon, p^\varepsilon)$ in the rest of this section for convenience.

The following estimates will be frequently used in this section. In these estimates, we assume $f \in C^\infty(\mathbb{T}^2)$ and $\mathbf{u} \in C^\infty(\Omega_f)$. The proofs of these lemmas can be found in Appendix. We also remark that the commutator estimates below are inspired from [24].

Lemma 4.1. *For any function $a \in H^s(\mathbb{T}^2)$ with $s > 2$, we have*

(4.9) $$\left| [a, \langle \nabla' \rangle^s] f \right|_{L^2} \leq C |a|_{H^s} |f|_{H^{s-1}}.$$

Here $\langle \nabla' \rangle^s$ is the s-order derivatives on $\mathbb{T}^2$ which defined as follows

$$\widehat{\langle \nabla' \rangle^s} f(\mathbf{k}) = \left(1 + |\mathbf{k}|^2\right)^{s/2} \hat{f}(\mathbf{k}), \quad \mathbf{k} = (k_1, k_2), \quad k_1, k_2 \in \mathbb{Z}.$$

Corollary 4.1. *For $s > 2$, one has*

$$\left| [D_t, \langle \nabla' \rangle^s] f \right|_{L^2} \leq C \|\mathbf{u}\|_{H^{s+1/2}(\Omega_f)} |f|_{H^s}.$$

Lemma 4.2. *For any function $g \in H^{s+1}(\mathbb{T}^2)$ with $s \geq \frac{3}{2}$, it holds that*

$$|[\bar{\mathcal{N}}_f, D_t] g|_{H^s} \lesssim |f|^3_{H^{s+1}} \|\mathbf{u}\|_{H^{s+3/2}(\Omega_f)} |g|_{H^{s+1}}.$$

Lemma 4.3. *For any functions $a \in H^{3/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, it holds that*

$$|[\mathcal{N}_f, a] g|_{L^2} \lesssim |f|_{H^3} |a|_{H^{3/2}} |g|_{H^{1/2}}.$$

Lemma 4.4. *For any functions $a \in H^{3/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, it holds that*

$$\int_{\mathbb{T}^2} a(\mathcal{N}_f g)gdx' \lesssim |f|_{H^3}|a|_{H^{3/2}}|g|^2_{H^{1/2}}.$$

Lemma 4.5. *For any function $a \in H^{5/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, it holds that*

$$\int_{\mathbb{T}^2} a([D_t, \mathcal{N}_f]g)gdx' \leq C\|a\|_{H^{5/2}}|f|^2_{H^4}\|\mathbf{u}\|_{H^4(\Omega_f)}|g|^2_{H^{1/2}}.$$

Lemma 4.6. [15] *For any function $g \in H^{s-3/2}(\mathbb{T}^2)$ with $s \geq 3$, it holds that*

$$|[\langle\nabla'\rangle^{s-3/2}, \mathcal{N}_f]g|_{L^2} \lesssim |f|^2_{H^{s-1/2}}|g|_{H^{s-3/2}}.$$

4.1. **Stability condition of the ε-regularized system.** This subsection is devoted to showing that the mixed type stability condition is valid for the initial data of the ε-regularized system and it can be preserved in a uniform time.

Lemma 4.7. *Giving $(f_0, \mathbf{u}_0, \mathbf{F}_0)$ satisfies C1 and C2, we assume that $(f, \mathbf{u}, \mathbf{F})$ is the solution of system (1.2)-(1.3), (4.1) with initial data $(f_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)$ defined in (4.2)-(4.3), and satisfies*

$$(4.10) \qquad \varepsilon|f|^2_{H^{s+1/2}} + |f|^2_{H^{s-1/2}} + \|\mathbf{u}\|^2_{H^s(\Omega_f)} + \|\mathbf{F}\|^2_{H^s(\Omega_f)} \leq CM_0^s$$

on $[0, T]$. Then, there exist constants $\widetilde{\varepsilon}$ and $\widetilde{T} \leq T$ such that for each $\varepsilon \in (0, \widetilde{\varepsilon}]$ and $t \in [0, \widetilde{T}]$,

1. *$-\partial_3 \nabla p(t, x') \geq \frac{1}{2}c_0$ on Γ^1;*
2. *$\Lambda(\mathbf{F})(t, x') \geq \frac{1}{2}c_0$ on Γ^2.*

Proof. For the ε-regularized system, the pressure can be written as

$$p = \bar{p} + \mathring{p},$$

where

$$\bar{p} \overset{\text{def}}{=} -\varepsilon\bar{\mathcal{H}}_f\bar{\mathcal{N}}_f^{-1}\Delta'f, \qquad \mathring{p} \overset{\text{def}}{=} p_{\mathbf{u},\mathbf{u}} - \sum_{j=1}^{3} p_{\mathbf{F}_j,\mathbf{F}_j}.$$

Recalling that $p_0 = p_{\mathbf{u}_0,\mathbf{u}_0} - \sum_{j=1}^{3} p_{\mathbf{F}_{0j},\mathbf{F}_{0j}}$ satisfies the Taylor sign condition on Γ_{f_0} with $x' \in \Gamma^1$, we choose ε small enough such that

$$-\partial_3(\underline{p_{\mathbf{u}_0^\varepsilon,\mathbf{u}_0^\varepsilon} - \sum_{j=1}^{3} p_{\mathbf{F}_{0j}^\varepsilon,\mathbf{F}_{0j}^\varepsilon}}) > \frac{3}{2}c_0 \text{ on } \Gamma^1, \quad \Lambda(\mathbf{F}_0^\varepsilon) > \frac{3}{2}c_0 \text{ on } \Gamma^2.$$

From Lemma 2.3, we know that

$$|\partial_3\underline{\bar{p}}(0, \cdot)|_{L^\infty} \leq \|\bar{p}(0, \cdot)\|_{H^4(\Omega_{f_0^2})} \leq C\varepsilon|f_0^\varepsilon|_{H^{7/2}}|f_0^\varepsilon|_{H^{9/2}} \leq \sqrt{\varepsilon}CM_\varepsilon^s.$$

Taking ε small enough, we have

$$-\partial_3\underline{p}(0, x') > c_0 \quad \text{on } \Gamma^1.$$

Similar to [16], it is direct to show that the non-collinear condition $\Lambda(\mathbf{F}) > \frac{1}{2}c_0$ will hold in a short time independent of ε. So, we only need to focus on the Taylor sign condition.

Firstly, we give an estimate of $\partial_t p$. Applying D_t to the first equation of (1.2), with the help of the third equation of (1.2) we have

$$D_t^2 u_i + D_t \partial_i p = \sum_{j=1}^3 (D_t \mathbf{F}_j \cdot \nabla F_{ij} + \mathbf{F}_j \cdot \nabla D_t F_{ij} - (\mathbf{F}_j \cdot \nabla \mathbf{u}) \cdot \nabla F_{ij})$$

$$= \sum_{j=1}^3 \mathbf{F}_j \cdot \nabla (\mathbf{F}_j \cdot \nabla u_i).$$

Taking divergence on both side of the above equation, we have

$$(4.11) \quad \partial_i^2 D_t p = \sum_{s=1}^3 ((\partial_i^2 u_s) \partial_s p + (\partial_i u_s) \partial_i \partial_s p)$$

$$- \sum_{k=1}^3 \left((\partial_i D_t u_k) \partial_k u_i + 2(\partial_i u_k) \partial_k (D_t u_i) \right) + \sum_{s,k=1}^3 2(\partial_i u_s)(\partial_s u_k)(\partial_k u_i)$$

$$+ \sum_{s,j,k=1}^3 \left((\partial_i F_{kj})(\partial_k F_{sj}) \partial_s u_i + F_{kj}(\partial_i \partial_k F_{sj}) \partial_s u_i + 2(\partial_i F_{kj}) F_{sj} \partial_k \partial_s u_i \right).$$

Here we used the result that div $\mathbf{u} = 0$.

Recalling that

$$\partial_3 p = 0 \quad \text{on } \Gamma^-, \quad p = -\varepsilon \bar{\mathcal{N}}_f^{-1} \Delta' f \quad \text{on } \Gamma_f,$$

the equation (4.11) is equipped with the following boundary conditions:

$$D_t p = -\varepsilon D_t \bar{\mathcal{N}}_f^{-1} \Delta' f = -\varepsilon \bar{\mathcal{N}}_f^{-1} D_t \Delta' f - \varepsilon \bar{\mathcal{N}}_f^{-1} [\bar{\mathcal{N}}_f, D_t] \bar{\mathcal{N}}_f^{-1} \Delta' f \quad \text{on } \Gamma_f,$$

$$\partial_3 D_t p = \partial_3 u_1 \partial_1 p + \partial_3 u_2 \partial_2 p \quad \text{on } \Gamma^-.$$

When $s \geq 1$, we have the following estimate

$$\|D_t p\|_{H^{s+2}(\Omega_f)} \lesssim \|\Delta D_t p\|_{H^s(\Omega_f)} + |f|_{H^{s+3/2}} |D_t p|_{H^{s+3/2}(\mathbb{T}^2)} + |\partial_3 D_t p|_{H^{s+1/2}(\Gamma^-)}.$$

Substituting (4.11) into it, one can obtain that

$$(4.12) \quad \|D_t p\|_{H^{s+2}(\Omega_f)}$$

$$\lesssim \|\mathbf{u}\|_{H^{s+3}(\Omega_f)} (\|p\|_{H^{s+3}(\Omega_f)} + \|\mathbf{F}\|_{H^{s+3}(\Omega_f)}^2) + \|D_t \mathbf{u}\|_{H^{s+2}(\Omega_f)} \|\mathbf{u}\|_{H^{s+2}(\Omega_f)}$$

$$+ \|\mathbf{u}\|_{H^{s+2}(\Omega_f)}^3 + \varepsilon |f|_{H^{s+2}}^2 |D_t \Delta' f|_{H^{s+1/2}} + \varepsilon |f|_{H^{s+2}}^2 |[\bar{\mathcal{N}}_f, D_t] \bar{\mathcal{N}}_f^{-1} \Delta' f|_{H^{s+1/2}}$$

$$\lesssim (1 + \|\mathbf{u}\|_{H^{s+3}(\Omega_f)} + \|\mathbf{F}\|_{H^{s+3}(\Omega_f)} + \|p\|_{H^{s+3}(\Omega_f)} + |f|_{H^{s+5/2}}^4 + \varepsilon |f|_{H^{s+7/2}})^3,$$

where in the last inequality, we used Lemma 2.3 and Lemma 4.2.

Recalling that

$$p = -\varepsilon \bar{\mathcal{H}}_f \bar{\mathcal{N}}_f^{-1} \Delta' f + p_{\mathbf{u},\mathbf{u}} - \sum_{j=1}^3 p_{\mathbf{F}_j, \mathbf{F}_j},$$

and using Lemma 2.3, we get

$$(4.13) \quad \|p\|_{H^s(\Omega_f)} \lesssim \|\mathbf{u}\|_{H^{s-1}(\Omega_f)}^2 + \|\mathbf{F}\|_{H^{s-1}(\Omega_f)}^2 + \varepsilon |f|_{H^{s+1/2}}^2.$$

This means that $\|D_t p\|_{H^3(\Omega_f)}$ can be controlled by the initial energy M_ε^s. Using (4.6), when $\varepsilon \le \hat{\varepsilon}$, we have $|\partial_3 D_t p|_{L^\infty} \le (1 + CM_0^{s2})^3$ and immediately

$$|D_t \partial_3 p|_{L^\infty} \le (1 + CM_0^{s2})^3.$$

As a conclusion, it holds for any $x \in \Gamma_f$ that

$$\partial_3 p(t, x) > \partial_3 p(0, x) - t(1 + CM_0^{s2})^3.$$

Then there exists a constant $\widetilde{T}$, which is independent of ε, such that $-\partial_3 p(t, x') > \frac{1}{2}c_0$ for any $x' \in \Gamma_1$ and $t \in [0, \widetilde{T}]$. This finishes the proof of Lemma 4.7.   □

4.2. ε-independent energy estimate.

Now we derive the ε-independent energy estimate for the regularized system.

Similar to (3.4), considering that

$$\begin{aligned}
-\partial_i' \partial_3 \bar{p} + (\partial_i' \partial_j \bar{p}) \partial_j' f &= -\mathbf{N}_f \cdot \partial_i'(\nabla \bar{p}) \\
&= -\partial_i'(\mathbf{N}_f \cdot \nabla \bar{p}) + \partial_i'(\mathbf{N}_f) \cdot \nabla \bar{p} \\
&= \varepsilon \partial_i' \Delta' f - \partial_1' \partial_i' f \partial_1 \bar{p} - \partial_2' \partial_i' f \partial_2 \bar{p},
\end{aligned}$$

we can derived for the ε-regularized system that

$$\begin{aligned}
(4.14) \quad D_t^2 \partial_i' f =& \partial_3 p \mathbf{N}_f \cdot \nabla \mathcal{H}_f(\partial_i' f) + \sum_{k=1}^{3} D_{F_k}^2 \partial_i' f + \varepsilon \Delta' \partial_i' f + \sum_{k=1}^{3} 2(\partial_i' F_{sk}) D_{F_k} \partial_s' f \\
& - 2(\partial_i' u_j) D_t \partial_j' f - \mathbf{N}_f \cdot \nabla q_i - \partial_1' \partial_i' f \partial_1 \bar{p} - \partial_2' \partial_i' f \partial_2 \bar{p},
\end{aligned}$$

where $q_i = \partial_i \mathring{p} + \partial_3 \mathring{p} \mathcal{H}_f(\partial_i' f)$. Motivated by the above equation, we introduce

$$\begin{aligned}
E_\varepsilon^s(t) =& |D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 + \sum_{k=1}^{3} |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 + \varepsilon |\partial_i' f|_{H^{s-1/2}}^2 \\
& + \int_{\Omega_f} \tilde{\mathfrak{a}} (\nabla \mathcal{H}_f(\langle \nabla' \rangle^{s-3/2} \partial_i' f))^2 \, dx + |f|_{L^2}^2 + |\partial_t f|_{L^2}^2 + \|\mathbf{u}\|_{H^s(\Omega_f)}^2 + \|\mathbf{F}\|_{H^s(\Omega_f)}^2,
\end{aligned}$$

where $\tilde{\mathfrak{a}}$ is a suitably chosen function satisfying

$$\begin{cases} 0 < c_0 \le \tilde{\mathfrak{a}} \le C & x \in \Omega_f, \\ \tilde{\mathfrak{a}} = -\partial_3 p & x' \in \Gamma^1. \end{cases}$$

Therefore we have

$$C\|\partial_i' f\|_{\dot{H}^{1/2}}^2 \le \int_{\Omega_f} \tilde{\mathfrak{a}} (\nabla \mathcal{H}_f(\partial_i' f))^2 \, dx \le C\|\partial_i' f\|_{\dot{H}^{1/2}}^2.$$

The function $\tilde{\mathfrak{a}}$ is constructed in the following way. Define $\mathfrak{a} = -\partial_3 p$. Recalling the proof of Lemma 4.7, (4.12) and (4.13) shows that for the solutions constructed in Proposition 4.2, $-\partial_3 p$ have a uniform bound when $(t, x') \in ([0, \widetilde{T}], \mathbb{T}^2)$. Therefore,

we can choose constant $\tilde{c} = \tilde{c}(M_0^s)$ such that $\tilde{c} - \underline{\partial_3 p} \geq c_0$ for all $x' \in \mathbb{T}^2$. Then we let $\bar{\mathfrak{a}} = \mathfrak{a} + \phi\tilde{c}$, where $\phi \in C^\infty(\mathbb{T}^2)$ is a cutoff function satisfying

(4.15)
$$\begin{cases} 0 \leq \phi \leq 1 & x' \in \mathbb{T}^2, \\ \phi = 0 & x' \in \mathbb{T}^2/\Gamma^2, \\ \phi = 1 & x' \in \mathbb{T}^2/\Gamma^1. \end{cases}$$

If the Taylor sign condition holds on Γ^1, it follows that $\bar{\mathfrak{a}} \geq c_0$ for $(t, x') \in [0, \tilde{T}] \times \mathbb{T}^2$. We choose $\tilde{\mathfrak{a}}$ as the solution of following equation:

(4.16)
$$\begin{cases} \Delta\tilde{\mathfrak{a}} = 0 & x \in \Omega_f, \\ \tilde{\mathfrak{a}} = \bar{\mathfrak{a}} & x \in \Gamma_f, \\ \tilde{\mathfrak{a}} = c_0 & x \in \Gamma^-. \end{cases}$$

The maximum principle yields that

$$c_0 \leq \tilde{\mathfrak{a}} \leq \tilde{c} \qquad \text{on } \Omega_f.$$

Thus, the constructed $\tilde{\mathfrak{a}}$ meets our requirement.

The uniform a priori estimate is stated as follows:

Proposition 4.3. *Giving* $(f_0, \mathbf{u}_0, \mathbf{F}_0)$ *satisfies C1 and C2, we assume that* $(f, \mathbf{u}, \mathbf{F})$ *is the solution of system* (1.2)-(1.3), (4.1) *with initial data* $(f_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{F}_0^\varepsilon)$ *defined in* (4.2)-(4.3), *and satisfying stability condition* (1.8) *on* $[0, T]$, *then it holds that*

$$\sup_{t\in[0,T]} E_\varepsilon^s(t) \leq CE_\varepsilon^s(0)e^{CT},$$

where C is a constant independent of ε.

We first introduce an elliptic estimate for a vector field.

Lemma 4.8. *For any vector field* $\mathbf{v}$ *defined on* Ω_f, *we have*

$$\begin{aligned} \|\mathbf{v}\|_{H^s(\Omega_f)} \leq & C(|f|_{H^{s-1/2}})\big(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)} \\ & + \sum_{i=1,2} |\partial_i' \mathbf{v} \cdot \mathbf{N}_f|_{H^{s-3/2}} + \|\mathbf{v}\|_{H^{s-1}(\Omega_f)}\big). \end{aligned}$$

Proof. The proof can be found in [25]. We represent it here for completeness. From the fact that

$$\Delta\mathbf{v} = -\nabla \times (\nabla \times \mathbf{v}) + \nabla(\operatorname{div} \mathbf{v}),$$

it suffices to prove (let $L_0 = |f|_{H^{s-1/2}}$)

(4.17) $\ |\mathbf{N}_f \cdot \underline{\nabla\mathbf{v}}|_{H^{s-3/2}} \leq C(L_0)(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \sum_{i=1,2} |\partial_i' \underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}}).$

Define

$$\mathbf{w}(\mathbf{v}) = (\underline{\partial_1 \mathbf{v}} \cdot \mathbf{N}_f, \underline{\partial_2 \mathbf{v}} \cdot \mathbf{N}_f, \underline{\partial_3 \mathbf{v}} \cdot \mathbf{N}_f).$$

Then (4.17) is equivalent to

(4.18) $\quad |\mathbf{w}|_{H^{s-3/2}} \leq C(L_0)(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \sum_{i=1,2} |\partial_i' \underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}}),$

since one has

$$|\mathbf{N}_f \cdot \underline{\nabla \mathbf{v}} - \mathbf{w}|_{H^{s-3/2}} = |\mathbf{N}_f \times (\underline{\nabla \times \mathbf{v}})|_{H^{s-3/2}} \le C(L_0)(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)}).$$

Direct calculations give that

(4.19) $$\tau_i \cdot \mathbf{w} = \underline{\partial_i \mathbf{v}} \cdot \mathbf{N}_f + \underline{\partial_3 \mathbf{v}} \cdot \mathbf{N}_f \partial_i' f = \partial_i' \underline{\mathbf{v}} \cdot \mathbf{N}_f,$$

where $\tau_1 = (1, 0, \partial_1 f)$ and $\tau_2 = (0, 1, \partial_2 f)$.

To estimate $\mathbf{N}_f \cdot \mathbf{w}$, we write

(4.20) $$(1 + |\nabla' f|^2)\mathbf{I} = (1 + (\partial_2' f)^2)\tau_1 \otimes \tau_1 - \partial_1' f \partial_2' f(\tau_1 \otimes \tau_2 + \tau_2 \otimes \tau_1)$$
$$+ (1 + (\partial_1' f)^2)\tau_2 \otimes \tau_2 + \mathbf{N}_f \otimes \mathbf{N}_f.$$

Since

$$\partial_i' \partial_j' \underline{\mathbf{v}} = \partial_i'(\partial_j \mathbf{v} + \underline{\partial_3 \mathbf{v}}\partial_j' f) = \underline{((\tau_i \otimes \tau_j) : \nabla^2)\mathbf{v}} + \underline{\partial_3 \mathbf{v}}\partial_i'\partial_j' f,$$

one has

$$\underline{((\tau_i \otimes \tau_j) : \nabla^2)\mathbf{v}} \cdot \mathbf{N}_f = \partial_i'\partial_j'\underline{\mathbf{v}} \cdot \mathbf{N}_f - \underline{\partial_3 \mathbf{v}} \cdot \mathbf{N}_f \partial_i'\partial_j' f$$
$$= \partial_i'(\partial_j'\underline{\mathbf{v}} \cdot \mathbf{N}_f) - \partial_j'\underline{\mathbf{v}} \cdot \partial_i' \mathbf{N}_f - \underline{\partial_3 \mathbf{v}} \cdot \mathbf{N}_f \partial_i'\partial_j' f.$$

Therefore, together with (4.20), one can get

$$\left|\left(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla^2\right)\mathbf{v} \cdot \mathbf{N}_f - \underline{\left((1 + |\nabla' f|^2)\mathbf{I} : \nabla^2\right)\mathbf{v} \cdot \mathbf{N}_f}\right|_{H^{s-5/2}}$$
$$\le C(L_0)(|\partial_j'\underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}} + \|\mathbf{v}\|_{H^{s-1}(\Omega_f)}).$$

On the other hand

$$\left|\underline{\left((1 + |\nabla' f|^2)\mathbf{I} : \nabla^2\right)\mathbf{v} \cdot \mathbf{N}_f}\right|_{H^{s-5/2}} = \left|(1 + |\nabla' f|^2)\underline{\Delta \mathbf{v}} \cdot \mathbf{N}_f\right|_{H^{s-5/2}}$$
$$\le C(L_0)(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)}),$$

which implies

$$\left|\underline{\left(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla^2\right)\mathbf{v} \cdot \mathbf{N}_f}\right|_{H^{s-5/2}} \le C(L_0)\Big(|\partial_j'\underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}} + \|\mathbf{v}\|_{H^{s-1}(\Omega_f)}$$
$$+ \|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)}\Big).$$

We rewrite

$$\left(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla^2\right)\mathbf{v} \cdot \mathbf{N}_f = \mathbf{N}_f \cdot \underline{\nabla(\mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}} : \nabla \mathbf{v})} - \underline{(\mathbf{N}_f \cdot \nabla)(\mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}}) : \underline{\nabla \mathbf{v}}}$$
$$= \mathbf{N}_f \cdot \nabla \mathcal{H}_f(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla \mathbf{v}) - \underline{(\mathbf{N}_f \cdot \nabla)(\mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}}) : \underline{\nabla \mathbf{v}}}$$
$$+ \mathbf{N}_f \cdot \underline{\nabla(\mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}} : \nabla \mathbf{v} - \mathcal{H}_f(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla \mathbf{v}))},$$

where $\mathbf{N}_{f\mathcal{H}}$ denotes the harmonic extension of $\mathbf{N}_f$, then we have

$$\left|\underline{(\mathbf{N}_f \cdot \nabla)(\mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}}) : \nabla \mathbf{v}}\right|_{H^{s-5/2}} \le C(L_0)\|\mathbf{v}\|_{H^{s-1}(\Omega_f)}.$$

For $Q = \mathbf{N}_{f\mathcal{H}} \otimes \mathbf{N}_{f\mathcal{H}} : \nabla \mathbf{v} - \mathcal{H}_f(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla \mathbf{v})$, one has

$$Q|_{\Gamma_f} = Q|_{\Gamma_-} = 0, \quad \|\Delta Q\|_{H^{s-3}(\Omega_f)} \le C(L_0)\Big(\|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)}\Big),$$

which implies

$$|\mathbf{N}_f \cdot \nabla \mathcal{H}_f(\mathbf{N}_f \otimes \mathbf{N}_f : \nabla \mathbf{v})|_{H^{s-5/2}} \leq C(L_0)\Big(|\partial_j' \underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}} + \|\mathbf{v}\|_{H^{s-1}(\Omega_f)}$$
$$+ \|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)}\Big).$$

Noticing that $\mathbf{N}_f \otimes \mathbf{N}_f : \underline{\nabla \mathbf{v}} = \mathbf{N}_f \cdot \mathbf{w}$, we have

$$|\mathbf{N}_f \cdot \mathbf{w}|_{H^{s-3/2}} \leq C(L_0)\Big(|\partial_j' \underline{\mathbf{v}} \cdot \mathbf{N}_f|_{H^{s-3/2}} + \|\mathbf{v}\|_{H^{s-1}(\Omega_f)}$$
$$+ \|\nabla \times \mathbf{v}\|_{H^{s-1}(\Omega_f)} + \|\operatorname{div} \mathbf{v}\|_{H^{s-1}(\Omega_f)}\Big).$$

Together with (4.19), we obtain (4.18), which finishes the proof of this lemma.   □

Now, we are in the position to prove Proposition 4.3.

Proof. Estimate of f. Using (4.14), direct calculation shows that

$$\frac{d}{dt}\frac{1}{2}|D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}^2$$
$$= \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(D_t^2\langle\nabla'\rangle^{s-3/2}\partial_i'f) - \frac{1}{2}\underline{u_j}\partial_j'(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 dx'$$
$$= \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}D_t^2\partial_i'f)dx'$$
$$+ \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)([D_t, \langle\nabla'\rangle^{s-3/2}]D_t\partial_i'f + D_t[D_t, \langle\nabla'\rangle^{s-3/2}]\partial_i'f)dx'$$
$$+ \int_{\mathbb{T}^2}\frac{1}{2}\partial_j'\underline{u_j}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 dx'$$
$$= \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}D_t^2\partial_i'f)dx'$$
$$+ \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)([D_t, \langle\nabla'\rangle^{s-3/2}]D_t\partial_i'f + [\underline{u_j}, \langle\nabla'\rangle^{s-3/2}]D_t\partial_j'\partial_i'f)dx'$$
$$+ \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)([\underline{D_t u_j}, \langle\nabla'\rangle^{s-3/2}]\partial_j'\partial_i'f) + \frac{1}{2}\partial_j'\underline{u_j}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 dx'$$
$$\leq \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}D_t^2\partial_i'f)dx'$$
$$+ C\|\mathbf{u}\|_{H^{s-1}(\Omega_f)}^2|D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}^2 + C\|D_t\mathbf{u}\|_{H^{s-1}(\Omega_f)}|D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|\partial_i'f|_{H^{s-3/2}}.$$

From the first equation of (1.2), using (4.12) and (4.13), we immediately have $\|D_t\mathbf{u}\|_{H^{s-1}(\Omega_f)} \leq CE_\varepsilon^s$. Next we consider the estimate of

$$\int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}D_t^2\partial_i'f)dx'$$
$$= \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\underline{\partial_3 p}\mathbf{N}_f \cdot \nabla\mathcal{H}_f(\partial_i'f))dx'$$
$$+ \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\sum_{k=1}^{3}D_{\mathbf{F}_k}^2\partial_s'f)dx'$$

$$- \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} (\mathbf{N}_f \cdot \underline{\nabla q_i})) dx'$$

$$+ \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f) \langle \nabla' \rangle^{s-3/2} ((\partial_i' \underline{F_{sk}}) D_{\mathbf{F}_k} \partial_s' f - 2(\partial_i' \underline{u_j}) D_t \partial_j' f) dx'$$

$$+ \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f) \langle \nabla' \rangle^{s-3/2} (\varepsilon \Delta' \partial_i' f - \partial_1' \partial_i' f \partial_1 \underline{\bar{p}} - \partial_2' \partial_i' f \partial_2 \underline{\bar{p}}) dx'$$

$$\triangleq I_1 + I_2 + I_3 + I_4 + I_5.$$

Estimate of I_1. Recalling the definition of $\mathfrak{a}$, we rewrite

$$I_1 = - \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \mathfrak{a} \mathcal{N}_f (\partial_i' f)) dx'$$

$$= - \int_{\mathbb{T}^2} \mathfrak{a} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} (\partial_i' f)) dx'$$

$$- \int_{\mathbb{T}^2} \mathfrak{a} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)([\langle \nabla' \rangle^{s-3/2}, \mathcal{N}_f] (\partial_i' f)) dx'$$

$$- \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)([\langle \nabla' \rangle^{s-3/2}, \mathfrak{a}] \mathcal{N}_f (\partial_i' f)) dx'.$$

The first term is the principle one. To estimate it, we calculate

$$\frac{d}{dt} \int_{\mathbb{T}^2} \mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$= \int_{\mathbb{T}^2} D_t [\mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f)] - \underline{u_j} \partial_j' [\mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f)] dx'$$

$$= \int_{\mathbb{T}^2} D_t (\mathfrak{a}) (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) + \mathfrak{a} (D_t \mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$+ \int_{\mathbb{T}^2} \mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f) D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f dx' + \int_{\mathbb{T}^2} (\partial_j' \underline{u_j}) \mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$= 2 \int_{\mathbb{T}^2} \mathfrak{a} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$+ \int_{\mathbb{T}^2} D_t (\langle \nabla' \rangle^{s-3/2} \partial_i' f)([\mathcal{N}_f, \mathfrak{a}] \langle \nabla' \rangle^{s-3/2} \partial_i' f) + \mathfrak{a} ([D_t, \mathcal{N}_f] \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$+ \int_{\mathbb{T}^2} D_t (\mathfrak{a}) (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) + (\partial_j' \underline{u_j}) \mathfrak{a} (\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'.$$

Where we use the following result. Since $\mathcal{N}_f$ is a self-adjoint operator, we have

$$\int_{\mathbb{T}^2} \mathfrak{a} (D_t \mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$= \int_{\mathbb{T}^2} \mathfrak{a} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\mathcal{N}_f \langle \nabla' \rangle^{s-3/2} \partial_i' f) dx' + \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)([\mathcal{N}_f, \mathfrak{a}] \langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$+ \int_{\mathbb{T}^2} \mathfrak{a} ([D_t, \mathcal{N}_f] \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'.$$

We write

$$\int_{\mathbb{T}^2} \mathfrak{a}(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx' = \int_{\mathbb{T}^2} \bar{\mathfrak{a}}(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$- \int_{\mathbb{T}^2} (\bar{\mathfrak{a}} - \mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'.$$

Recalling the definition of $\tilde{\mathfrak{a}}$ and using integration by parts, one can get

$$\int_{\mathbb{T}^2} \bar{\mathfrak{a}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$= \int_{\Omega_f} \tilde{\mathfrak{a}}(\nabla\mathcal{H}_f(\langle\nabla'\rangle^{s-3/2}\partial_i'f))^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} \mathbf{N}_f \cdot \underline{\nabla\tilde{\mathfrak{a}}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 dx'.$$

Summarizing the above results gives that

$$I_1 = -\frac{d}{dt}\frac{1}{2}\int_{\Omega_f} \tilde{\mathfrak{a}}(\nabla\mathcal{H}(\langle\nabla'\rangle^{s-3/2}\partial_i'f))^2 dx$$
$$- \frac{d}{dt}\frac{1}{4}\int_{\mathbb{T}^2} \mathbf{N}_f \cdot \underline{\nabla\tilde{\mathfrak{a}}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 dx'$$
$$+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} (\bar{\mathfrak{a}} - \mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$+ \frac{1}{2}\int_{\mathbb{T}^2} D_t(\langle\nabla'\rangle^{s-3/2}\partial_i'f)([\mathcal{N}_f, \mathfrak{a}]\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$+ \frac{1}{2}\int_{\mathbb{T}^2} \mathfrak{a}([D_t, \mathcal{N}_f]\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$+ \frac{1}{2}\int_{\mathbb{T}^2} D_t(\mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$+ \frac{1}{2}\int_{\mathbb{T}^2} (\partial_j'\underline{u_j})\mathfrak{a}(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$
$$- \int_{\mathbb{T}^2} \mathfrak{a}(\langle\nabla'\rangle^{s-3/2}D_t\partial_i'f)([\langle\nabla'\rangle^{s-3/2}, \mathcal{N}_f](\partial_i'f))dx'$$
$$- \int_{\mathbb{T}^2} (\langle\nabla'\rangle^{s-3/2}D_t\partial_i'f)([\langle\nabla'\rangle^{s-3/2}, \mathfrak{a}]\mathcal{N}_f(\partial_i'f))dx'.$$

We will control all the terms on the right hand side except the first one by using the energy.

From Lemma 4.3, Lemma 4.5 and Lemma 4.6 and the commutator estimates for the Dirichlet-Neumann operators, we have

$$\int_{\mathbb{T}^2} D_t(\langle\nabla'\rangle^{s-3/2}\partial_i'f)([\mathcal{N}_f, \mathfrak{a}]\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx' \lesssim |D_t(\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|f|_{H^3}|\mathfrak{a}|_{H^{3/2}}|\partial_i'f|_{H^{s-1}},$$

$$\int_{\mathbb{T}^2} \mathfrak{a}([D_t, \mathcal{N}_f]\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx' \lesssim |\mathfrak{a}|_{H^{5/2}}|f|_{H^{s-1/2}}^3\|\mathbf{u}\|_{H^4(\Omega_f)}|\partial_i'f|_{H^{s-1}}^2,$$

$$\int_{\mathbb{T}^2} \mathfrak{a}(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)([\langle\nabla'\rangle^{s-3/2}, \mathcal{N}_f](\partial_i'f))dx' \lesssim |\mathfrak{a}|_{L^\infty}|f|_{H^{s-1/2}}^2|D_t(\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|\partial_i'f|_{H^s}.$$

Moreover, by Lemma 4.4 we can get

$$\int_{\mathbb{T}^2} D_t(\mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx' \lesssim |f|_{H^3}|D_t\mathfrak{a}|_{H^{3/2}}|\partial_i'f|^2_{H^{s-1}},$$

$$\int_{\mathbb{T}^2} (\partial_j'u_j)\mathfrak{a}(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx' \lesssim |f|_{H^3}|\mathfrak{a}|_{H^{3/2}}\|\mathbf{u}\|_{H^3(\Omega_f)}|\partial_i'f|^2_{H^{s-1}}.$$

Likewise, using Lemma 2.2 and Lemma 4.1, one has

$$\int_{\mathbb{T}^2} (D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)([\langle\nabla'\rangle^{s-3/2}, \mathfrak{a}]\mathcal{N}_f(\partial_i'f))dx' \lesssim |D_t(\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|\mathfrak{a}|_{H^{s-3/2}}|\partial_i'f|_{H^{s-3/2}}.$$

Recalling the definition of $\bar{\mathfrak{a}}$, $\bar{\mathfrak{a}} - \mathfrak{a}$ is a function independent of t with support on Γ^2, we can use $|D_{F_k}\partial_i'f|^2_{H^{s-3/2}}$ to control

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^2}(\bar{\mathfrak{a}}-\mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$=\frac{1}{2}\int_{\mathbb{T}^2}(\bar{\mathfrak{a}}-\mathfrak{a})(D_t\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)$$

$$+ (\bar{\mathfrak{a}}-\mathfrak{a})(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$-\frac{1}{2}\int_{\mathbb{T}^2}(\bar{\mathfrak{a}}-\mathfrak{a})\underline{u_j}\partial_j'((\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f))dx'$$

$$=\frac{1}{2}\int_{\mathbb{T}^2}(\bar{\mathfrak{a}}-\mathfrak{a})([D_t, \mathcal{N}_f]\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)$$

$$+ (D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)\mathcal{N}_f((\bar{\mathfrak{a}}-\mathfrak{a})\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$+\frac{1}{2}\int_{\mathbb{T}^2}([(\bar{\mathfrak{a}}-\mathfrak{a}), \mathcal{N}_f]\langle\nabla'\rangle^{s-3/2}\partial_i'f)(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)$$

$$+ \mathcal{N}_f((\bar{\mathfrak{a}}-\mathfrak{a})\langle\nabla'\rangle^{s-3/2}\partial_i'f)(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$+\frac{1}{2}\int_{\mathbb{T}^2}(\partial_j'(\underline{u_j}(\bar{\mathfrak{a}}-\mathfrak{a})))(\mathcal{N}_f\langle\nabla'\rangle^{s-3/2}\partial_i'f)(\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$\lesssim|\bar{\mathfrak{a}}-\mathfrak{a}|_{H^{3/2}}|f|_{H^4}\|\mathbf{u}\|_{H^4(\Omega_f)}|\partial_i'f|^2_{H^{s-1}} + |D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|(\bar{\mathfrak{a}}-\mathfrak{a})\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{H^1}$$

$$+ |\bar{\mathfrak{a}}-\mathfrak{a}|_{H^{3/2}}|f|_{H^3}|\partial_i'f|_{H^{s-1}}|D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2} + |\partial_j'(\underline{u_j}(\bar{\mathfrak{a}}-\mathfrak{a}))|_{H^{3/2}}|\partial_i'f|^2_{H^{s-1}}$$

$$\lesssim(1 + |\bar{\mathfrak{a}}-\mathfrak{a}|_{H^{3/2}} + \|\mathbf{u}\|_{H^4(\Omega_f)} + |f|_{H^4})^3(|\partial_i'f|_{H^{s-1}} + |D_{F_k}\partial_i'f|_{H^{s-3/2}} + |D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2})^2,$$

where we have used Lemma 2.2 and Lemma 4.3-4.5. In a similar way, we have

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{T}^2}\mathbf{N}_f\cdot\underline{\nabla\tilde{\mathfrak{a}}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2dx'$$

$$=\frac{1}{2}\int_{\mathbb{T}^2}D_t(\mathbf{N}_f\cdot\underline{\nabla\tilde{\mathfrak{a}}})(\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2 + 2\mathbf{N}_f\cdot\underline{\nabla\tilde{\mathfrak{a}}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)(D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f)dx'$$

$$+\frac{1}{2}\int_{\mathbb{T}^2}(\partial_j'\underline{u_j})\mathbf{N}\cdot\underline{\nabla\tilde{\mathfrak{a}}}(\langle\nabla'\rangle^{s-3/2}\partial_i'f)^2dx'$$

$$\lesssim|D_t(\mathbf{N}_f\cdot\underline{\nabla\tilde{\mathfrak{a}}})|_{L^2}|\langle\nabla'\rangle^{s-3/2}\partial_i'f|^2_{L^4}$$

$$+ |\mathbf{N}_f\cdot\underline{\nabla\tilde{\mathfrak{a}}}|_{L^\infty}|D_t\langle\nabla'\rangle^{s-3/2}\partial_i'f|_{L^2}|\partial_i'f|_{H^{s-3/2}} + |(\partial_j'\underline{u_j})\mathbf{N}\cdot\underline{\nabla\tilde{\mathfrak{a}}}|_{L^\infty}|\partial_i'f|^2_{H^{s-3/2}}$$

$$\lesssim \|\mathbf{u}\|_{H^3}^2 |f|_{H^4}^2 (|D_t \bar{\mathfrak{a}}|_{H^{3/2}(\Omega_f)} + |\bar{\mathfrak{a}}|_{H^{5/2}})(|\partial_i' f|_{H^{s-1}} + |D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2})^2.$$

Recalling the definition of $\bar{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}$, (4.12) and (4.13) ensure that all the norms related to these functions in our proof can be controlled by energy $E_\varepsilon^s(t)$.

Combining above estimates, it follows that

$$I_1 + \frac{d}{dt} \frac{1}{2} \int_{\Omega_f} \tilde{\mathfrak{a}} (\nabla \mathcal{H}(\langle \nabla' \rangle^{s-3/2} \partial_i' f))^2 dx \le P(E_\varepsilon^s(t)),$$

where P is a polynomial.

Estimate of I_2. Using the fact that $[D_t, D_{\mathbf{F}_k}] = 0$, we have

$$I_2 = \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} \sum_{k=1}^3 D_{\mathbf{F}_k}^2 \partial_s' f) dx'$$

$$= \sum_{k=1}^3 \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)([\langle \nabla' \rangle^{s-3/2}, D_{\mathbf{F}_k}] D_{\mathbf{F}_k} \partial_s' f + D_{\mathbf{F}_k} [\langle \nabla' \rangle^{s-3/2}, D_{\mathbf{F}_k}] \partial_s' f) dx'$$

$$+ \sum_{k=1}^3 \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(D_{\mathbf{F}_k} D_{\mathbf{F}_k} \langle \nabla' \rangle^{s-3/2} \partial_s' f) dx'$$

$$= - \sum_{k=1}^3 \int_{\mathbb{T}^2} D_t (D_{\mathbf{F}_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f)(\langle \nabla' \rangle^{s-3/2} D_{\mathbf{F}_k} \partial_s' f) dx'$$

$$- \sum_{k=1}^3 \int_{\mathbb{T}^2} (\partial_r' \underline{F_{rk}})(D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)(D_{\mathbf{F}_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f) dx'$$

$$+ \sum_{k=1}^3 \int_{\mathbb{T}^2} (D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f)([\langle \nabla' \rangle^{s-3/2}, D_{\mathbf{F}_k}] D_{\mathbf{F}_k} \partial_s' f + D_{\mathbf{F}_k} [\langle \nabla' \rangle^{s-3/2}, D_{\mathbf{F}_k}] \partial_s' f) dx'$$

$$\triangleq I_{21} + I_{22} + I_{23}.$$

Lemma 4.1 gives that

$$I_{22} + I_{23} \lesssim |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 |\mathbf{F}|_{H^s(\Omega_f)} |D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2.$$

Since

$$I_{21} + \frac{d}{dt} \frac{1}{2} |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 = - \int_{\mathbb{T}^2} \underline{u_j} \partial_j' (\langle \nabla' \rangle^{s-3/2} D_{F_k} \partial_i' f)^2 dx'$$

$$\le |\nabla u|_{L^\infty(\Omega_f)} |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2,$$

we obtain the estimate of I_2:

$$I_2 + \frac{d}{dt} \frac{1}{2} |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 \lesssim P(E_\varepsilon^s(t)).$$

Estimate of I_3, I_4 and I_5. Recalling that $q_i = \partial_i \mathring{p} + \partial_3 \mathring{p} \mathcal{H}_f(\partial_i' f)$, $q_i|_{\Gamma_f} = \partial_i \mathring{p} + \partial_3 \mathring{p} \partial_i' f = 0$, from the definition of $\mathring{p}$ and $\mathcal{H}_f$, we have

$$\partial_3 q_i|_{\Gamma^-} = (\partial_i \partial_3 \mathring{p} + \partial_{33} \mathring{p} \mathcal{H}_f(\partial_i' f) + \partial_3 \mathring{p} \partial_3 \mathcal{H}_f(\partial_i' f))_{\Gamma^-} = 0,$$

$$\Delta q_i = \partial_i \Delta \mathring{p} + \partial_3 \Delta \mathring{p} \mathcal{H}_f(\partial_i' f) + 2 \partial_3 \nabla \mathring{p} \cdot \nabla \mathcal{H}_f(\partial_i' f),$$

which implies

$$\|q_i\|_{H^s(\Omega_f)} \leq \|\partial_i \Delta \mathring{p} + \partial_3 \Delta \mathring{p} \mathcal{H}_f(\partial'_i f) + 2\partial_3 \nabla \mathring{p} \cdot \nabla \mathcal{H}_f(\partial'_i f)\|_{H^{s-2}(\Omega_f)}$$
$$\lesssim \|\mathring{p}\|_{H^{s+2}(\Omega_f)} + \|\mathring{p}\|_{H^{s+1}(\Omega_f)}|\partial'_i f|_{H^{s-5/2}} + \|\mathring{p}\|_{H^{s+1}(\Omega_f)}|\partial'_i f|_{H^{s-3/2}}.$$

Accordingly, it holds that

$$I_3 = -\int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)(\langle\nabla'\rangle^{s-3/2}(\mathbf{N}_f \cdot \underline{\nabla q_i}))dx'$$
$$\leq |D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f|_{L^2}\|\mathbf{N} \cdot \underline{\nabla q_i}|_{H^{s-3/2}} \leq P(E^s_\varepsilon(t)).$$

The term I_4 can be estimated in a similar way. For I_5, we have

$$I_5 = \int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)\langle\nabla'\rangle^{s-3/2}(\varepsilon\Delta'\partial'_i f - \partial'_1\partial'_i f\underline{\partial_1\bar{p}} - \partial'_2\partial'_i f\underline{\partial_2\bar{p}})dx'.$$

Recalling the definition of $\bar{p}$, we have

$$\int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)\langle\nabla'\rangle^{s-3/2}(\varepsilon\Delta'\partial'_i f)dx'$$
$$=\varepsilon\int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)(\Delta'\langle\nabla'\rangle^{s-3/2}\partial'_i f)dx'$$
$$=-\varepsilon\int_{\mathbb{T}^2}\nabla'(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)\cdot(\nabla'\langle\nabla'\rangle^{s-3/2}\partial'_i f)dx'$$
$$=-\frac{\varepsilon}{2}\partial_t\int_{\mathbb{T}^2}(\nabla'\langle\nabla'\rangle^{s-3/2}\partial'_i f)\cdot(\nabla'\langle\nabla'\rangle^{s-3/2}\partial'_i f)dx'$$
$$-\varepsilon\int_{\mathbb{T}^2}([\nabla',\underline{u_j}])\langle\nabla'\rangle^{s-3/2}\partial'_j\partial'_i f\cdot(\nabla'\langle\nabla'\rangle^{s-3/2}\partial'_i f)dx'$$
$$+\frac{\varepsilon}{2}\int_{\mathbb{T}^2}(\partial'_j\underline{u_j})(\langle\nabla'\rangle^{s-3/2}\nabla'\partial'_i f)^2dx'.$$

Using Lemma 2.3, we have

$$\int_{\mathbb{T}^2}(D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f)\langle\nabla'\rangle^{s-3/2}(\partial'_j\partial'_i f\partial_j\bar{p})dx' \lesssim |D_t\langle\nabla'\rangle^{s-3/2}\partial'_i f|_{L^2}|\partial'_j\partial'_i f|_{H^{s-3/2}}|\underline{\partial_j\bar{p}}|_{H^{s-3/2}}$$
$$\lesssim |D_t\partial'_i f|_{H^{s-3/2}}|\partial'_i f|_{H^{s-1/2}}|\varepsilon\bar{\mathcal{N}}^{-1}\Delta' f|_{H^{s-3/2}}$$
$$\lesssim |D_t\partial'_i f|_{H^{s-3/2}}|f|_{H^{s-3/2}}\varepsilon|f|^2_{H^{s+1/2}}.$$

Combining these estimates, we have

$$I_5 + \frac{d}{dt}\frac{1}{2}\varepsilon|\partial'_i f|^2_{H^{s+1}} \leq P(E^s_\varepsilon(t)).$$

Estimate of $(\mathbf{u}, \mathbf{F})$. Recalling the boundary condition on Γ_f

$$\mathbf{u} \cdot \mathbf{N}_f = \partial_t f, \quad \mathbf{F}_j \cdot \mathbf{N}_f = 0,$$

we have

$$D_t\partial'_i f = \partial'_i D_t f - \partial'_i u_j\partial'_j f = \partial'_i\underline{u_3} - \partial'_i u_j\partial'_j f = \partial'_i\underline{\mathbf{u}} \cdot \mathbf{N}_f,$$
$$\underline{F_{sk}}\partial'_i\partial'_s f = \partial'_i(\underline{\mathbf{F}_k} \cdot \mathbf{N}_f) - \underline{\mathbf{F}_k} \cdot \partial'_i\mathbf{N}_f = \partial'_i\underline{\mathbf{F}_k} \cdot \mathbf{N}_f.$$

From Lemma 4.8, it holds that

$$C\|\mathbf{u}\|_{H^s(\Omega_f)} \leq (\|\nabla \times \mathbf{u}\|_{H^{s-1}(\Omega_f)} + \sum_{i=1,2} |D_t\partial_i' f|_{H^{s-3/2}(\mathbb{T}^2)} + \|\mathbf{u}\|_{H^{s-1}(\Omega_f)}) \leq CE_\varepsilon^s(t),$$

$$C\|\mathbf{F}_k\|_{H^s(\Omega_f)} \leq (\|\nabla \times \mathbf{F}_k\|_{H^{s-1}(\Omega_f)} + \sum_{i=1,2} |D_{\mathbf{F}_k}\partial_i' f|_{H^{s-3/2}(\mathbb{T}^2)} + \|\mathbf{F}_k\|_{H^{s-1}(\Omega)}) \leq CE_\varepsilon^s(t).$$

Then it suffices to prove that

$$\|\nabla \times \mathbf{u}\|_{H^{s-1}(\Omega_f)}^2 + \|\mathbf{u}\|_{H^{s-1}(\Omega_f)}^2 + \|\nabla \times \mathbf{F}_k\|_{H^{s-1}(\Omega_f)}^2 + \|\mathbf{F}\|_{H^{s-1}(\Omega_f)}^2 \leq \int_0^T P(E_\varepsilon^s(t))dt.$$

Similar to [16, Proposition 4.4], it is direct to obtain that

$$\frac{d}{dt}\left(\|\nabla \times \mathbf{u}\|_{H^{s-1}(\Omega_f)}^2 + \|\nabla \times \mathbf{F}_k\|_{H^{s-1}(\Omega_f)}^2\right) \lesssim P(E_\varepsilon^s(t))(\|\nabla \times \mathbf{u}\|_{H^{s-1}(\Omega_f)}^2 + \|\nabla \times \mathbf{F}_k\|_{H^{s-1}(\Omega_f)}^2).$$

With the help of evolution equations of $\mathbf{u}$ and $\mathbf{F}$ in (1.2), we have

$$\frac{d}{dt}(\|\nabla^{s-1}\mathbf{u}\|_{L^2(\Omega_f)}^2 + \sum_{j=1}^3 \|\nabla^{s-1}\mathbf{F}_j\|_{L^2(\Omega_f)}^2)$$

$$\lesssim \int_{\Omega_f} \nabla^{s-1}(\sum_{j=1}^3 \mathbf{F}_j \cdot \nabla\mathbf{F}_j - \nabla p) \cdot \nabla^{s-1}\mathbf{u}dx + \sum_{j=1}^3 \int_{\Omega_f} \nabla^{s-1}(\mathbf{F}_j \cdot \nabla\mathbf{u}) \cdot \nabla^{s-1}\mathbf{F}_jdx$$

$$+ \|\mathbf{u}\|_{H^{s-1}(\Omega_f)}^3 + \sum_{j=1}^3 \|\mathbf{F}_j\|_{H^{s-1}(\Omega_f)}^2\|\mathbf{u}\|_{H^{s-1}(\Omega_f)}$$

$$\lesssim -\sum_{j=1}^3 \int_{\Omega_f} (\nabla \cdot \mathbf{F}_j)\nabla^{s-1}\mathbf{u} \cdot \nabla^{s-1}\mathbf{F}_jdx$$

$$+ \|p\|_{H^s(\Omega_f)}\|\mathbf{u}\|_{H^{s-1}(\Omega_f)} + \|\mathbf{u}\|_{H^{s-1}(\Omega_f)}^3 + \sum_{j=1}^3 \|\mathbf{F}_j\|_{H^{s-1}(\Omega_f)}^2\|\mathbf{u}\|_{H^{s-1}(\Omega_f)}$$

$$\lesssim \left(\sum_{j=1}^3 \|\mathbf{F}_j\|_{H^{s-1}(\Omega_f)} + \|\mathbf{u}\|_{H^{s-1}(\Omega_f)} + \|p\|_{H^s(\Omega_f)}\right)^3.$$

From (4.13) we have $\|p\|_{H^s(\Omega_f)} \lesssim E_\varepsilon^s(t)$, which implies

$$\frac{d}{dt}(\|\mathbf{u}\|_{H^s(\Omega_f)} + \|\mathbf{F}_j\|_{H^s(\Omega_f)}) \leq P(E_\varepsilon^s(t)).$$

Completing the proof. Combining all these estimates above, we have

$$\frac{d}{dt}E_\varepsilon^s(t) \leq P(E_\varepsilon^s(t)).$$

The proposition follows from Gronwall's inequality.   □

4.3. Approximation sequence to the solution of the original system. Now, we give the proof of Proposition 4.1.

Proof. Choose $\bar{\varepsilon} = \min(\hat{\varepsilon}, \tilde{\varepsilon})$, where $\hat{\varepsilon}$ and $\tilde{\varepsilon}$ are given in (4.6) and Lemma 4.7 separately. For each $\varepsilon \leq \bar{\varepsilon}$, we have

$$\varepsilon|f_0^\varepsilon|_{H^{s+1/2}}^2 + |f_0^\varepsilon|_{H^{s-1/2}}^2 + \|\mathbf{u}_0^\varepsilon\|_{H^s(\Omega_{f_0^\varepsilon})}^2 + \|\mathbf{F}_0^\varepsilon\|_{H^s(\Omega_{f_0^\varepsilon})}^2 \leq CM_0^s.$$

Using Proposition 4.2, we solve $(f, \mathbf{u}, \mathbf{F})$ on $[0, T(\varepsilon, (2C + \bar{C})M_0^s)]$ which satisfies

$$\sup_{t \in [0,T]} \left(\varepsilon|f|_{H^{s+1/2}}^2 + |f|_{H^{s-1/2}}^2 + \|\mathbf{u}\|_{H^s(\Omega_f)}^2 + \|\mathbf{F}\|_{H^s(\Omega_f)}^2\right) \leq 2(2C + \bar{C})M_0^s,$$

where $\bar{C}$ is given in Proposition 4.3. Lemma 4.7 shows that for $0 \leq t \leq \hat{T}$, $(f, \mathbf{u}, \mathbf{F})$ satisfies the mixed type stability condition (1.8), where $\hat{T} = \min(\widetilde{T}, T)$. Proposition 4.3 shows that there exists a constant $\bar{C}$ such that

$$\sup_{t \in [0,\hat{T}]} E_\varepsilon^s(t) \leq \bar{C}E_\varepsilon^s(0)e^{\bar{C}\hat{T}}.$$

If $\bar{C}E_\varepsilon^s(0)e^{\bar{C}\hat{T}} \leq (2C + \bar{C})M_0^s$, then $(f, \mathbf{u}, \mathbf{F})(\hat{T})$ still satisfies the assumption of Proposition 4.2. Therefore, we can extend $(f, \mathbf{u}, \mathbf{F})$ to $[0, \hat{T} + T]$. The above steps can be repeated until the energy is larger than $(2C + \bar{C})M_0^s$. Since $\bar{C}$ is independent of ε, we can extend $(f, \mathbf{u}, \mathbf{F})$ to a lifespan $\bar{T}$ independent of ε. This finishes the proof of Proposition 4.1. $\qquad\square$

5. WELL-POSEDNESS OF THE ORIGINAL SYSTEM

5.1. Taking the limit $\varepsilon \to 0$. To prove existences of solutions of the original system (1.2)-(1.7), we consider the limit $\varepsilon \to 0$. For the solution $(f^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$ of each ε-regularized system given in Proposition 4.1, we pull $(\mathbf{u}^\varepsilon, \mathbf{F}^\varepsilon)$ back to a fixed domain.
We define

$$\tilde{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \circ \Phi_{f^\varepsilon}, \quad \tilde{\mathbf{F}}^\varepsilon = \mathbf{F}^\varepsilon \circ \Phi_{f^\varepsilon},$$

where $\Phi_{f^\varepsilon} : \Omega_{f_0} \to \Omega_{f^\varepsilon}$ is the harmonic coordinate defined in Section 2. Then by Lemma 2.1, the following energy

$$\tilde{E}_\varepsilon^s(t) = |D_t\langle\nabla'\rangle^{s-3/2}\partial_i' f^\varepsilon|_{L^2}^2 + \sum_{k=1}^3 |D_{F_k^\varepsilon}\langle\nabla'\rangle^{s-3/2}\partial_i' f^\varepsilon|_{L^2}^2 + \varepsilon|\partial_i' f^\varepsilon|_{H^{s-1/2}}^2 + |\partial_i' f^\varepsilon|_{H^{s-1}}^2$$

$$+ |f^\varepsilon|_{L^2(\mathbb{T}^2)}^2 + |\partial_t f^\varepsilon|_{L^2(\mathbb{T}^2)}^2 + \|\tilde{\mathbf{u}}^\varepsilon\|_{H^s(\Omega_{f_0})}^2 + \|\tilde{\mathbf{F}}^\varepsilon\|_{H^s(\Omega_{f_0})}^2$$

also have an ε-independent upper bound on $[0, T]$.

Taking $\varepsilon \to 0$, there exists a subsequence of $(f^\varepsilon, \tilde{\mathbf{u}}^\varepsilon, \tilde{\mathbf{F}}^\varepsilon)$ which converges weakly to a limit which we denote by $(f, \tilde{\mathbf{u}}, \tilde{\mathbf{F}})$ satisfying

$$D_t f \in L^2([0, T], H^{s-1/2}(\mathbb{T}^2)), f \in L^2([0, T], H^s(\mathbb{T}^2)), D_{\tilde{\mathbf{F}}_k} f \in L^2([0, T], H^{s-1/2}(\mathbb{T}^2)),$$

$$\int_{\mathbb{T}^2} f dx' = 0, \ \tilde{\mathbf{u}} \in L^2([0, T], H^s(\Omega_{f_0}), \tilde{\mathbf{F}} \in L^2([0, T], H^s(\Omega_{f_0}).$$

In addition, this convergence will be strong in spaces with low regularity. For

$$\mathbf{u} = \tilde{\mathbf{u}} \circ \Phi_f^{-1}, \quad \mathbf{F} = \tilde{\mathbf{F}} \circ \Phi_f^{-1},$$

standard arguments show that $(f, \mathbf{u}, \mathbf{F})$ solves (1.2)-(1.7).

Define the energy

$$E^s(t) = |D_t \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 + \sum_{k=1}^{3} |D_{F_k} \langle \nabla' \rangle^{s-3/2} \partial_i' f|_{L^2}^2 + \int_{\Omega_f} \tilde{\mathfrak{a}} (\nabla \mathcal{H}_f (\langle \nabla' \rangle^{s-3/2} \partial_i' f))^2 dx$$

$$+ |f|_{L^2}^2 + |\partial_t f|_{L^2}^2 + \|\mathbf{u}\|_{H^s(\Omega_f)}^2 + \|\mathbf{F}\|_{H^s(\Omega_f)}^2.$$

Similar to Proposition 4.3, one can also have

$$\sup_{t \in [0,T]} E^s(f, \mathbf{u}, \mathbf{F})(t) \leq 2C M_0^s,$$

where T is the ε-independent lifespan obtained in Section 4.

5.2. **Uniqueness.** Assume that $(f^A, \mathbf{u}^A, \mathbf{F}^A)$ and $(f^B, \mathbf{u}^B, \mathbf{F}^B)$ are two solutions to (1.2)-(1.7) on $[0, T]$. We denote the difference $(f^A - f^B, \tilde{\mathbf{u}}^A - \tilde{\mathbf{u}}^B, \tilde{\mathbf{F}}^A - \tilde{\mathbf{F}}^B)$ by $(f^D, \tilde{\mathbf{u}}^D, \tilde{\mathbf{F}}^D)$.

For the difference functions, we introduce the following energy

$$E_D^s(t) = |D_t^A \langle \nabla' \rangle^{s-5/2} \partial_i' f^D|_{L^2}^2 + \sum_{k=1}^{3} |D_{F_k^A} \langle \nabla' \rangle^{s-5/2} \partial_i' f^D|_{L^2}^2 + \int_{\Omega_{fA}} \tilde{\mathfrak{a}}^A (\nabla \mathcal{H}_{f^A} (\langle \nabla' \rangle^{s-5/2} \partial_i' f^D))^2 dx$$

$$+ |f^D|_{L^2}^2 + |\partial_t f^D|_{L^2}^2 + \|\tilde{\mathbf{u}}^D\|_{H^{s-1}(\Omega_{f_0})}^2 + \|\tilde{\mathbf{F}}^D\|_{H^{s-1}(\Omega_{f_0})}^2,$$

where D_t^A means material derivation generated from $\mathbf{u}^A$, and $\tilde{\mathfrak{a}}^A$ is defined by $(f^A, \mathbf{u}^A, \mathbf{F}^A)$ in the same way as $\tilde{\mathfrak{a}}$. Apparently $E_D^s(0) = 0$. We will prove that $E_D^s(t) \equiv 0$.

First of all, by elliptic estimates we know that

$$\|\Phi_{f^A} - \Phi_{f^B}\|_{H^{s-1/2}(\Omega_{f_0})} \lesssim |f^A - f^B|_{H^{s-1}} \lesssim E_D^s(t).$$

Recall that

(5.1) $$\partial_t^2 \partial_i' f + \underline{u_j \partial_t \partial_j' \partial_i' f} + D_u^2 \partial_i' f - \sum_{k=1}^{3} D_{F_k}^2 \partial_i' f - \underline{\partial_3 p} \mathcal{N}_f (\partial_i' f) = g,$$

where g is the lower order term

$$g = -\partial_t \underline{u_j} \partial_j' \partial_i' f + \sum_{k=1}^{3} 2(\partial_i' F_{sk}) D_{\mathbf{F}_k} \partial_s' f - 2(\partial_i' u_j) D_t \partial_j' f + 2(\partial_i' u_j)(\partial_j' u_s) \partial_s' f - \mathbf{N}_f \cdot \nabla q_i.$$

For the two evolution equations of f^A and f^B, we use g^A and g^B to denote their lower order terms separately. Subtracting these two equations, we have

$$\partial_t^2 \partial_i' (f^D) + \underline{u_j^A \partial_t \partial_j' \partial_i' f^D} + D_{u^A}^2 \partial_i' f^D - \sum_{k=1}^{3} D_{F_k^A}^2 \partial_i' f^D - \underline{\partial_3 p^A} \mathcal{N}_{f^A} (\partial_i' f^D)$$

$$= -\underline{(u_j^A - u_j^B)} \partial_t \partial_j' \partial_i' f^B - (D_{u^A}^2 - D_{u^B}^2) \partial_i' f^B + \sum_{k=1}^{3} (D_{F_k^A}^2 - D_{F_k^B}^2) \partial_i' f^B$$

(5.2) $$+ (\underline{\partial_3 p^A} \mathcal{N}_{f^A} \partial_i' f^B - \underline{\partial_3 p^B} \mathcal{N}_{f^B} \partial_i' f^B) + g^A - g^B.$$

The most difficult term in (5.2) is $\partial_3 p^A \mathcal{N}_{f^A} \partial'_i f^B - \partial_3 p^B \mathcal{N}_{f^B} \partial'_i f^B$, which can be written as

$$\partial_3 p^A \mathcal{N}_{f^A} \partial'_i f^B - \partial_3 p^B \mathcal{N}_{f^B} \partial'_i f^B$$
$$= (\partial_3 p^A - \partial_3 p^B) \mathbf{N}_{f^A} \mathcal{H}_{f^A} \partial'_i f^B + (\partial_3 p^B (\mathbf{N}_{f^A} - \mathbf{N}_{f^B}) \mathcal{H}_{f^A} \partial'_i f^B)$$
$$+ (\partial_3 p^B \mathbf{N}_{f^B} (\mathcal{H}_{f^A} - \mathcal{H}_{f^B}) \partial'_i f^B).$$

The first two terms can be estimated in standard methods, while the last term can be treat in the following way.

For each function ψ defined on $\mathbb{T}^2$, we define

$$\widehat{\mathcal{H}}_{f^B} \psi = \mathcal{H}_{f^B} \psi \circ \Phi_{f^B} \circ \Phi_{f^A}^{-1}.$$

Then we have

(5.3)
$$\begin{cases} \Delta(\mathcal{H}_{f^A} \psi - \widehat{\mathcal{H}}_{f^B} \psi) = -\Delta \widehat{\mathcal{H}}_{f^B} \psi & x \in \Omega_{f^A}, \\ \mathcal{H}_{f^A} \psi - \widehat{\mathcal{H}}_{f^B} \psi(x', f^A(x')) = 0 & x' \in \mathbb{T}^2, \\ \mathcal{H}_{f^A} \psi - \widehat{\mathcal{H}}_{f^B} \psi(x', -1) = 0 & x' \in \mathbb{T}^2. \end{cases}$$

By the definition of harmonic coordinate, we know that

$$\partial_3^2 \widehat{\mathcal{H}}_{f^B} \psi = (\partial_3^2 \mathcal{H}_{f^B} \psi) \circ \Phi_{f^B} \circ \Phi_{f^A}^{-1} \left((\partial_3 \Phi_{f^B}^{(3)}) \circ \Phi_{f^A}^{-1} \frac{1}{(\partial_3 \Phi_{f^A}^{(3)}) \circ \Phi_{f^A}^{-1}} \right)^2$$

$$+ (\partial_3 \mathcal{H}_{f^B} \psi) \circ \Phi_{f^B} \circ \Phi_{f^A}^{-1} \frac{(\partial_3^2 \Phi_{f^B}^{(3)}) \circ \Phi_{f^A}^{-1} (\partial_3 \Phi_{f^A}^{(3)}) \circ \Phi_{f^A}^{-1} - (\partial_3^2 \Phi_{f^A}^{(3)}) \circ \Phi_{f^A}^{-1} (\partial_3 \Phi_{f^B}^{(3)}) \circ \Phi_{f^A}^{-1}}{((\partial_3 \Phi_{f^A}^{(3)}) \circ \Phi_{f^A}^{-1})^3}.$$

The equations for $\partial_1^2 \widehat{\mathcal{H}}_{f^B} \psi$ and $\partial_2^2 \widehat{\mathcal{H}}_{f^B} \psi$ are similar. From $(\Delta \mathcal{H}_{f^B} \psi) \circ \Phi_{f^B} \circ \Phi_{f^A}^{-1} = 0$, we know that

$$\Delta \widehat{\mathcal{H}}_{f^B} \psi = C(\nabla(\Phi_{f^A} - \Phi_{f^B}), \nabla^2(\Phi_{f^A} - \Phi_{f^B})) \circ \Phi_{f^A}^{-1}.$$

Then the right hand side in (5.2) can be controlled by E_D^s.

Similar to Proposition 4.2, we can prove

$$\frac{d}{dt} \left(|D_t^A \partial'_i f^D|^2_{H^{s-1}(\mathbb{T}^2)} + \sum_{k=1}^{3} |D_{F_k^A} \partial'_i f^D|^2_{H^{s-1}(\mathbb{T}^2)} + \int_{\Omega_{f^A}} \tilde{\mathfrak{a}}^A (\nabla \mathcal{H}_{f^A}(\langle \nabla' \rangle^{s-1} \partial'_i f^D))^2 dx \right) \lesssim E_D^s(t).$$

Now we show that

$$\frac{d}{dt} (\|\tilde{\mathbf{u}}^D\|^2_{H^{s-1}(\Omega_{f_0})} + \|\tilde{\mathbf{F}}^D\|^2_{H^{s-1}(\Omega_{f_0})}) \lesssim E_D^s.$$

For a vector field $\tilde{\mathbf{v}}$ defined on Ω_{f_0}, we define

$$\mathrm{curl}_C \tilde{\mathbf{v}} = (\mathrm{curl}(\tilde{\mathbf{v}} \circ \Phi_{f^C}^{-1})) \circ \Phi_{f^C},$$
$$\mathrm{div}_C \tilde{\mathbf{v}} = (\mathrm{div}(\tilde{\mathbf{v}} \circ \Phi_{f^C}^{-1})) \circ \Phi_{f^C}.$$

It is clear that

$$C\|\mathbf{u}^A - \tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|^2_{H^{s-1}(\Omega_{f^A})} \leq \|\tilde{\mathbf{u}}^D\|^2_{H^{s-1}(\Omega_{f_0})} \leq C\|\mathbf{u}^A - \tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|^2_{H^{s-1}(\Omega_{f^A})}.$$

Thus, we only need to estimate $\|\mathbf{u}^A - \tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|^2_{H^{s+1/2}(\Omega_{f^A})}$. Recalling Lemma 4.8, we know that

$$\|\mathbf{u}^A - \tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-1}(\Omega_{f^A})}$$

$$\leq C(|f^A|_{H^{s-3/2}})\Big(\|\operatorname{curl}\mathbf{u}^A - \operatorname{curl}\tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-2}(\Omega_{f^A})} + \|\operatorname{div}\mathbf{u}^A - \operatorname{div}\tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-2}(\Omega_{f^A})}$$

$$+ \sum_{i=1,2} |\partial_i'\underline{\mathbf{u}}^A \cdot \mathbf{N}_{f^A} - \partial_i'\underline{\tilde{\mathbf{u}}}^B \circ \Phi_{f^A}^{-1} \cdot \mathbf{N}_{f^A}|_{H^{s-5/2}(\mathbb{T}^2)} + \|\mathbf{u}^A - \tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-2}(\Omega_{f^A})}\Big).$$

For the terms on the right hand side, we have

$$\|\operatorname{curl}\mathbf{u}^A - \operatorname{curl}\tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-2}(\Omega_{f^A})}$$

$$\lesssim \|\operatorname{curl}_A \mathbf{u}^A - \operatorname{curl}_B \mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})} + \|(\operatorname{curl}_B - \operatorname{curl}_A)\mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})}$$

$$\lesssim \|\operatorname{curl}_A \mathbf{u}^A - \operatorname{curl}_B \mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})} + \|\Phi_{f^A} - \Phi_{f^B}\|_{H^{s-1}(\Omega_{f_0})},$$

$$\|\operatorname{div}\mathbf{u}^A - \operatorname{div}\tilde{\mathbf{u}}^B \circ \Phi_{f^A}^{-1}\|_{H^{s-2}(\Omega_{f^A})}$$

$$\lesssim \|\operatorname{div}_A \mathbf{u}^A - \operatorname{div}_B \mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})} + \|(\operatorname{div}_B - \operatorname{div}_A)\mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})}$$

$$\lesssim \|\Phi_{f^A} - \Phi_{f^B}\|_{H^{s-1}(\Omega_{f_0})},$$

$$|\partial_i'\underline{\mathbf{u}}^A \cdot \mathbf{N}_{f^A} - \partial_i'\underline{\tilde{\mathbf{u}}}^B \circ \Phi_{f^A}^{-1} \cdot \mathbf{N}_{f^A}|_{H^{s-5/2}(\mathbb{T}^2)}$$

$$\lesssim |\partial_i'\underline{\tilde{\mathbf{u}}}^A \cdot \mathbf{N}_{f^A} - \partial_i'\underline{\tilde{\mathbf{u}}}^B \cdot \mathbf{N}_{f^B}|_{H^{s-5/2}(\mathbb{T}^2)} + |\partial_i'\underline{\tilde{\mathbf{u}}}^B \cdot (\mathbf{N}_{f^A} - \mathbf{N}_{f^B})|_{H^{s-5/2}(\mathbb{T}^2)}$$

$$\lesssim |D_t^A \partial_i' f^D|_{H^{s-5/2}(\mathbb{T}^2)} + |D_{u^D} \partial_i' f^B|_{H^{s-5/2}(\mathbb{T}^2)} + |f^D|_{H^{s-3/2}(\mathbb{T}^2)},$$

where $D_u = \underline{u_1}\partial_1' + \underline{u_2}\partial_2'$.

As a result, it holds that

$$\|\tilde{\mathbf{u}}^D\|^2_{H^{s-1}(\Omega_{f_0})}$$

$$\lesssim |D_t^A \partial_i f^D|_{H^{s-3/2}(\mathbb{T}^2)} + |f^D|_{H^{s-3/2}(\mathbb{T}^2)} + \|\operatorname{curl}_A \mathbf{u}^A - \operatorname{curl}_B \mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})} + \|\tilde{\mathbf{u}}^D\|^2_{H^{s-2}(\Omega_{f_0})}.$$

We have similar estimates for $\|\tilde{\mathbf{F}}^D\|^2_{H^{s-1}(\Omega_{f_0})}$.

Then using the method in Proposition 4.3, it follows that

$$\frac{d}{dt}(\|\operatorname{curl}_A \mathbf{u}^A - \operatorname{curl}_B \mathbf{u}^B\|_{H^{s-2}(\Omega_{f_0})} + \|\tilde{\mathbf{u}}^D\|^2_{H^{s-2}(\Omega_{f_0})}$$

$$+ \sum_{j=1}^3 \|\operatorname{curl}_A \mathbf{F}_j^A - \operatorname{curl}_B \mathbf{F}_j^B\|_{H^{s-2}(\Omega_{f_0})} + \sum_{j=1}^3 \|\tilde{\mathbf{F}}_j^D\|^2_{H^{s-2}(\Omega_{f_0})}) \lesssim E_D^s(t).$$

Thus, finally we have

$$\frac{d}{dt}E_D^s(t) \leq C(M_0^s)E_D^s(t),$$

which finishes the proof of uniqueness.

APPENDIX A. ESTIMATES RELATED TO THE D-N OPERATOR

Lemma A.1. *For any function $a \in H^s(\mathbb{T}^2)$ with $s > 2$, we have*

$$(A.1) \qquad \left|[a, \langle\nabla'\rangle^s]f\right|_{L^2} \leq C|a|_{H^s}|f|_{H^{s-1}}.$$

Here $\langle\nabla'\rangle^s$ is the s-order derivatives on $\mathbb{T}^2$ which is defined as follows

$$\widehat{\langle\nabla'\rangle^s f}(\mathbf{k}) = \left(1 + |\mathbf{k}|^2\right)^{\frac{s}{2}} \hat{f}(\mathbf{k}), \quad \mathbf{k} = (k_1, k_2), \quad k_1, k_2 \in \mathbb{Z}.$$

Corollary A.1. *For $s > 2$, we have*

$$\left|[D_t, \langle\nabla'\rangle^s]f\right|_{L^2} \leq C\|\mathbf{u}\|_{H^{s+1/2}(\Omega_f)}|f|_{H^s}.$$

Lemma A.2. *Let $D_t = \partial_t + \underline{u}_1\partial_1' + \underline{u}_2\partial_2'$, then*

$$D_t\mathbf{N}_f = \frac{-\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial_2'f)^2\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_1'f\partial_2'f\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}\tau_1$$

$$+ \frac{-\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial_1'f)^2\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_1'f\partial_2'f\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}\tau_2$$

$$+ \frac{\partial_1'f\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_2'f\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}\mathbf{N}_f,$$

where

$$\mathbf{N}_f = (-\partial_1'f, -\partial_2'f, 1), \quad \tau_1 = (1, 0, \partial_1'f), \quad \tau_2 = (0, 1, \partial_2'f).$$

Proof. Let $\mathbf{n}_f = \frac{\mathbf{N}_f}{|\mathbf{N}_f|}$ and write $D_t\mathbf{N}_f = A\tau_1 + B\tau_2 + (D_t\mathbf{N}_f \cdot \mathbf{n}_f)\mathbf{n}_f$, where A, B are undetermined coefficients. Direct calculation shows that

$$D_t\mathbf{N}_f \cdot \mathbf{n}_f = \frac{\partial_1'f\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_2'f\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f}{\sqrt{1 + (\partial_1'f)^2 + (\partial_2'f)^2}}.$$

Thanks to $\mathbf{N}_f \cdot \tau_1 = \mathbf{N}_f \cdot \tau_2 = 0$, it follows that

$$A(1 + \partial_1'f^2) + B(\partial_1'f\partial_2'f) = D_t\mathbf{N}_f \cdot \tau_1 = -\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f.$$

Similarly, we have

$$A(\partial_1'f\partial_2'f) + B(1 + \partial_2'f^2) = D_t\mathbf{N}_f \cdot \tau_2 = -\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f.$$

From above equations, we can solve

$$A = \frac{(-\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial_2'f)^2\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_1'f\partial_2'f\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f)}{1 + (\partial_1'f)^2 + (\partial_2'f)^2},$$

$$B = \frac{(-\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial_1'f)^2\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial_1'f\partial_2'f\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f)}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}.$$

$\square$

Lemma A.3. *For any function $g \in H^{s+1}(\mathbb{T}^2)$ with $s \geq \frac{3}{2}$, it holds that*

$$|[\bar{\mathcal{N}}_f, D_t]g|_{H^s} \lesssim |f|_{H^{s+1}}^3\|\mathbf{u}\|_{H^{s+3/2}(\Omega_f)}|g|_{H^{s+1}}.$$

Proof. We start to analyze of $[D_t, \bar{\mathcal{H}}_f]g$. Direct calculation shows that

$$\Delta D_t\bar{\mathcal{H}}_f g = D_t\Delta\bar{\mathcal{H}}_f g + 2\nabla\mathbf{u} : \nabla^2\bar{\mathcal{H}}_f g + \Delta\mathbf{u} \cdot \nabla\bar{\mathcal{H}}_f g$$

$$= 2\nabla\mathbf{u} : \nabla^2\bar{\mathcal{H}}_f g + \Delta\mathbf{u} \cdot \nabla\bar{\mathcal{H}}_f g.$$

So we have

$$[D_t, \bar{\mathcal{H}}_f]g = \bar{\Delta}^{-1}(2\nabla \mathbf{u} : \nabla^2 \bar{\mathcal{H}}g + \Delta \mathbf{u} \cdot \nabla \bar{\mathcal{H}}_f gg) + \bar{\bar{\mathcal{H}}}_f g,$$

where $\bar{\Delta}^{-1}$ means

(A.2)
$$\begin{cases} \Delta \bar{\Delta}^{-1}g = g & x \in \Omega_f; \\ \bar{\Delta}^{-1}g = 0 & x \in \Gamma_f; \\ \partial_3 \bar{\Delta}^{-1}g = 0 & x \in \Gamma^-, \end{cases}$$

and $\bar{\bar{\mathcal{H}}}_f$ means

(A.3)
$$\begin{cases} \Delta \bar{\bar{\mathcal{H}}}_f g = 0 & x \in \Omega_f; \\ \bar{\bar{\mathcal{H}}}_f g = 0 & x \in \Gamma_f; \\ \partial_3 \bar{\bar{\mathcal{H}}}_f g = \partial_3 u_1 \partial_1 \bar{\mathcal{H}}_f g + \partial_3 u_2 \partial_2 \bar{\mathcal{H}}_f g & x \in \Gamma^-. \end{cases}$$

Next we consider $\bar{\mathcal{N}}_f$. Using Lemma A.2 we get

$$\begin{aligned}
[D_t, \bar{\mathcal{N}}_f]g &= D_t(\mathbf{N}_f \cdot \nabla \bar{\mathcal{H}}_f g) - \mathbf{N}_f \cdot \nabla \bar{\mathcal{H}}_f(D_t g) \\
&= \mathbf{N}_f \cdot \nabla D_t \bar{\mathcal{H}}_f g - \nabla \bar{\mathcal{H}}_f g \cdot (\mathbf{N}_f \cdot \nabla \mathbf{u}) + \nabla \bar{\mathcal{H}}_f g \cdot (D_t \mathbf{N}_f) - \mathbf{N}_f \cdot \nabla \bar{\mathcal{H}}_f(D_t g) \\
&= \mathbf{N}_f \cdot \nabla \bar{\Delta}^{-1}(2\nabla \mathbf{u} : \nabla^2 \bar{\mathcal{H}}_f g + \Delta \mathbf{u} \cdot \nabla \bar{\mathcal{H}}_f g) + \mathbf{N}_f \cdot \nabla \bar{\bar{\mathcal{H}}}_f g - \nabla \bar{\mathcal{H}}_f g \cdot (\mathbf{N}_f \cdot \nabla \mathbf{u}) \\
&\quad + \frac{(-\partial_1' \mathbf{u} \cdot \mathbf{N}_f - (\partial_2' f)^2 \partial_1' \mathbf{u} \cdot \mathbf{N}_f + \partial_1' f \partial_2' f \partial_2' \mathbf{u} \cdot \mathbf{N}_f)\partial_1' g}{1 + (\partial_1' f)^2 + (\partial_2' f)^2} \\
&\quad + \frac{(-\partial_2' \mathbf{u} \cdot \mathbf{N}_f - (\partial_1' f)^2 \partial_2' \mathbf{u} \cdot \mathbf{N}_f + \partial_1' f \partial_2' f \partial_1' \mathbf{u} \cdot \mathbf{N}_f)\partial_2' g}{1 + (\partial_1' f)^2 + (\partial_2' f)^2} \\
&\quad + \frac{\partial_1' f \partial_1' \mathbf{u} \cdot \mathbf{N}_f + \partial_2' f \partial_2' \mathbf{u} \cdot \mathbf{N}_f}{1 + (\partial_1' f)^2 + (\partial_2' f)^2} \bar{\mathcal{N}}_f g.
\end{aligned}$$

As a conclusion, it is easy for us to get

$$|[\bar{\mathcal{N}}_f, D_t]g|_{H^s} \lesssim |f|_{H^{s+1}}^3 \|\mathbf{u}\|_{H^{s+3/2}(\Omega_f)} |g|_{H^{s+1}}.$$

$\square$

Lemma A.4. *For any functions* $a \in H^{3/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, *it holds that*

$$|[\mathcal{N}_f, a]g|_{L^2} \lesssim |f|_{H^3} |a|_{H^{3/2}} |g|_{H^{1/2}}.$$

Proof. Similar to Lemma A.3, we have that

$$[\mathcal{N}_f, a]g = g\mathcal{N}_f a - 2\mathbf{N}_f \cdot \nabla \Delta^{-1}(\nabla \mathcal{H}_f a \cdot \nabla \mathcal{H}_f g).$$

It holds that

$$|g\mathcal{N}_f a|_{L^2} \leq |g|_{L^4}^{1/2} |\mathcal{N}_f a|_{L^4}^{1/2} \lesssim |g|_{H^{1/2}}^{1/2} |\mathcal{N}_f a|_{H^{1/2}}^{1/2}.$$

On the other hand, one has

$$|\mathbf{N}_f \cdot \nabla \Delta^{-1}(\nabla \mathcal{H}_f a \cdot \nabla \mathcal{H}_f g)|_{L^2} \lesssim |f|_{H^{5/2}} \|\nabla \mathcal{H}_f a \cdot \nabla \mathcal{H}_f g\|_{H^{-1/2}(\Omega_f)}.$$

For each test function ϕ with $\|\phi\|_{H^{1/2}(\Omega_f)} = 1$, we have

$$\int_{\Omega_f} \nabla\mathcal{H}_f a \cdot \nabla\mathcal{H}_f g \phi dx$$

$$\leq \|\phi\|_{L^3(\Omega_f)} \|\nabla\mathcal{H}_f g\|_{L^2(\Omega_f)} \|\nabla\mathcal{H}_f a\|_{L^6(\Omega_f)}$$

$$\lesssim \|\phi\|_{H^{1/2}(\Omega_f)} |g|_{H^{1/2}} |a|_{H^{3/2}},$$

then the conclusion follows easily. $\qquad\square$

Lemma A.5. *For any functions* $a \in H^{3/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, *it holds that*

$$\int_{\mathbb{T}^2} a(\mathcal{N}_f g)g dx' \lesssim |f|_{H^3} |a|_{H^{3/2}} |g|^2_{H^{1/2}}.$$

Proof. We use $\tilde{a} = \mathcal{H}_f a$ to denote the harmonic extension of a. Then it holds that

$$\int_{\mathbb{T}^2} a(\mathcal{N}_f g)g dx' = \int_{\Omega_f} (\nabla\mathcal{H}_f g) \cdot \nabla(\tilde{a}\mathcal{H}_f g)dx$$

$$= \int_{\Omega_f} (\nabla\mathcal{H}_f g) \cdot (\nabla\tilde{a}\mathcal{H}_f g + \tilde{a}\nabla\mathcal{H}_f g)dx.$$

Using integration by parts, we know that

$$\int_{\Omega_f} \nabla\mathcal{H}_f g \cdot (\nabla\tilde{a}\mathcal{H}_f g)dx = \int_{\mathbb{T}^2} \mathbf{N}_f \cdot \underline{\nabla\tilde{a}}g g dx' - \int_{\Omega_f} \mathcal{H}_f g \nabla\tilde{a} \cdot \nabla\mathcal{H}_f g dx.$$

Thus we have

$$\int_{\mathbb{T}^2} a\mathcal{N}_f g g dx' = \int_{\Omega_f} \tilde{a}(\nabla\mathcal{H}_f g)^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} \mathcal{N}_f a g g dx'$$

$$\lesssim |a|_{L^\infty} |g|^2_{H^{1/2}} + |f|_{H^3} |a|_{H^1} |g|^2_{L^4}$$

$$\lesssim |f|_{H^3} |a|_{H^{3/2}} |g|^2_{H^{1/2}}.$$

$$\square$$

Lemma A.6. *For any function* $a \in H^{5/2}(\mathbb{T}^2)$, $g \in H^{1/2}(\mathbb{T}^2)$, *it holds that*

$$\int_{\mathbb{T}^2} a([D_t, \mathcal{N}_f]g)g dx' \leq C|a|_{H^{5/2}} |f|^2_{H^4} \|\mathbf{u}\|_{H^4(\Omega_f)} |g|^2_{H^{1/2}}.$$

Proof. Similar to Lemma A.3, we have

$$[D_t, \mathcal{N}_f]g = \mathbf{N}_f \cdot \underline{\nabla\Delta^{-1}(2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g + \Delta\mathbf{u} \cdot \nabla\mathcal{H}_f g)} - \nabla\mathcal{H}_f g \cdot (\mathbf{N}_f \cdot \underline{\nabla\mathbf{u}})$$

$$+ \frac{(-\partial'_1\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial'_2 f)^2\partial'_1\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial'_1 f\partial'_2 f\partial'_2\underline{\mathbf{u}} \cdot \mathbf{N}_f)\partial'_1 g}{1 + (\partial'_1 f)^2 + (\partial'_2 f)^2}$$

$$+ \frac{(-\partial'_2\underline{\mathbf{u}} \cdot \mathbf{N}_f - (\partial'_1 f)^2\partial'_2\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial'_1 f\partial'_2 f\partial'_1\underline{\mathbf{u}} \cdot \mathbf{N}_f)\partial'_2 g}{1 + (\partial'_1 f)^2 + (\partial'_2 f)^2}$$

$$+ \frac{\partial'_1 f\partial'_1\underline{\mathbf{u}} \cdot \mathbf{N}_f + \partial'_2 f\partial'_2\underline{\mathbf{u}} \cdot \mathbf{N}_f}{1 + (\partial'_1 f)^2 + (\partial'_2 f)^2}\mathcal{N}_f g,$$

where Δ^{-1} means

$$(A.4) \quad \begin{cases} \Delta\Delta^{-1}g = g & x \in \Omega_f; \\ \Delta^{-1}g = 0 & x \in \Gamma_f; \\ \Delta^{-1}g = 0 & x \in \Gamma^-. \end{cases}$$

Using $\tilde{a} = \mathcal{H}_f a$ to denote the harmonic extension of a, we can write

$$\int_{\mathbb{T}^2} a\mathbf{N}_f \cdot \underline{\nabla\Delta^{-1}(2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g + \Delta\mathbf{u} \cdot \nabla\mathcal{H}_f g)}g dx'$$

$$= \int_{\Omega_f} (2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g + \Delta\mathbf{u} \cdot \nabla\mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx$$

$$+ \int_{\Omega_f} \nabla\Delta^{-1}(2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g + \Delta\mathbf{u} \cdot \nabla\mathcal{H}_f g) \cdot (\tilde{a}\nabla\mathcal{H}_f g + \mathcal{H}_f g\nabla\tilde{a}) dx.$$

Since $\mathbf{u}$ is divergence free, one can get

$$\int_{\Omega_f} (2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx$$

$$= 2\int_{\Omega_f} (\partial_r u_s \partial_r \partial_s \mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx$$

$$= \int_{\Omega_f} (\partial_r u_s \partial_r \partial_s \mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx + \int_{\mathbb{T}^2} (\underline{\nabla\mathcal{H}_f g \cdot (\mathbf{N}_f \cdot \underline{\nabla\mathbf{u}})})a g dx'$$

$$- \int_{\Omega_f} (\partial_r^2 u_s \partial_s \mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx - \int_{\Omega_f} \partial_r u_s \partial_s \mathcal{H}_f g \partial_r(\tilde{a}\mathcal{H}_f g) dx$$

$$= \int_{\mathbb{T}^2} (\underline{\nabla\mathcal{H}_f g \cdot (\mathbf{N}_f \cdot \underline{\nabla\mathbf{u}})})a g dx' - \int_{\Omega_f} (\partial_r^2 u_s \partial_s \mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx$$

$$+ \int_{\Omega_f} \partial_r u_s(\partial_s(\partial_r\mathcal{H}_f g\tilde{a}\mathcal{H}_f g) - \partial_r\mathcal{H}_f g\partial_s(\tilde{a}\mathcal{H}_f g) - \partial_s\mathcal{H}_f g\partial_r(\tilde{a}\mathcal{H}_f g)) dx$$

$$= \int_{\mathbb{T}^2} (\underline{\nabla\mathcal{H}_f g \cdot (\mathbf{N}_f \cdot \underline{\nabla\mathbf{u}})})a g dx' - \int_{\Omega_f} (\partial_r^2 u_s \partial_s \mathcal{H}_f g)\tilde{a}\mathcal{H}_f g dx$$

$$+ \int_{\mathbb{T}^2} (\underline{\partial_r\mathbf{u} \cdot \mathbf{N}_f \partial_r\mathcal{H}_f g})a g dx' - \int_{\Omega_f} \partial_r u_s(\partial_r\mathcal{H}_f g\partial_s(\tilde{a}\mathcal{H}_f g) + \partial_s\mathcal{H}_f g\partial_r(\tilde{a}\mathcal{H}_f g)) dx.$$

Analyzing the boundary term carefully, we find that

$$\int_{\mathbb{T}^2} (\underline{\partial_r\mathbf{u} \cdot \mathbf{N}_f \partial_r\mathcal{H}_f g})a g dx'$$

$$= \int_{\mathbb{T}^2} \frac{(\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f + (\partial_2'f)^2\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f - \partial_1'f\partial_2'f\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f)\partial_1'g}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}a g dx'$$

$$+ \int_{\mathbb{T}^2} \frac{(\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f + (\partial_1'f)^2\partial_2'\underline{\mathbf{u}} \cdot \mathbf{N}_f - \partial_1'f\partial_2'f\partial_1'\underline{\mathbf{u}} \cdot \mathbf{N}_f)\partial_2'g}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}a g dx'$$

$$+ \int_{\mathbb{T}^2} \frac{((\mathbf{N}_f \cdot \nabla)\mathbf{u} \cdot \mathbf{N}_f)\mathcal{N}_f g}{1 + (\partial_1'f)^2 + (\partial_2'f)^2}a g dx',$$

Combining above results, we have

$$\int_{\mathbb{T}^2} a([D_t, \mathcal{N}_f]g)g dx'$$

$$= \int_{\Omega_f} \nabla\Delta^{-1}(2\nabla\mathbf{u} : \nabla^2\mathcal{H}_f g + \Delta\mathbf{u} \cdot \nabla\mathcal{H}_f g)(\tilde{a}\nabla g + g\nabla\tilde{a}) dx$$

$$+ \int_{\mathbb{T}^2} \underline{\partial_3\mathbf{u}} \cdot \mathbf{N}_f \mathcal{N}_f g a g dx' - \int_{\Omega_f} (\nabla\mathbf{u} + \nabla\mathbf{u}^\top) : \nabla\mathcal{H}_f g \otimes \nabla(\tilde{a}\mathcal{H}_f g) dx.$$

Using Lemma A.5, we finally arrive at

$$\int_{\mathbb{T}^2} a([D_t, \mathcal{N}_f]g)g dx' \le C|a|_{H^{5/2}}|f|_{H^4}^2 \|\mathbf{u}\|_{H^4(\Omega_f)}|g|_{H^{1/2}}^2.$$

$\square$

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