# Nonconvex integro-differential sweeping process with applications. 

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#### Abstract

In this paper, we analyze and discuss the well-posedness of a new variant of the so-called sweeping process, introduced by J.J. Moreau in the early 70's 18 with motivation in plasticity theory. In this variant, the normal cone to the (mildly nonconvex) prox-regular moving set $C(t)$, supposed to have an absolutely continuous variation, is perturbed by a sum of a Carathéodory mapping and an integral forcing term. The integrand of the forcing term depends on two time-variables, that is, we study a general integro-differential sweeping process of Volterra type. By setting up an appropriate semi-discretization method combined with a new Gronwall-like inequality (differential inequality), we show that the integro-differential sweeping process has one and only one absolutely continuous solution. We also establish the continuity of the solution with respect to the initial value. The results of the paper are applied to the study of nonlinear integro-differential complementarity systems which are combination of Volterra integro-differential equations with nonlinear complementarity constraints. Another application is concerned with non-regular electrical circuits containing timevarying capacitors and nonsmooth electronic device like diodes. Both applications represent an additional novelty of our paper.


KeywordsMoreau's sweeping process, Volterra integro-differential equation, Differential complementarity systems, Gronwall's inequality, Prox-regular sets, Differential inclusions. AMS subject classifications 49J40,47J20, 47J22, 34G25, 58E35, 74M15, 74M10, 74G25.

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## 1 Introduction

In the seventies, sweeping processes are introduced and deeply studied by J. J. Moreau through a series of papers, in particular [18, 19]. There, it is shown that such processes play an important role in elasto-plasticity, quasi-statics, dynamics, especially in mechanics [18. Roughly speaking, a point is swept by a moving closed convex set $C(t)$ in a Hilbert space $H$, which can be formulated in the form of differential inclusion as follows

$$
\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t)) \quad \text { a.e. } t \in\left[T_{0}, T\right]  \tag{1}\\
x\left(T_{0}\right)=x_{0} \in C\left(T_{0}\right),
\end{array}\right.
$$

where $N_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ in the sense of convex analysis. When the systems are perturbed, it is natural to study the following variant

$$
\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f(t, x(t)) \quad \text { a.e. } t \in\left[T_{0}, T\right]  \tag{2}\\
x\left(T_{0}\right)=x_{0} \in C\left(T_{0}\right)
\end{array}\right.
$$

where $f:\left[T_{0}, T\right] \times H \rightarrow H$ is a Carathéodory mapping.
Actually, diverse approaches for existence of solutions of (1) and (2) are available in the literature: Catching-up method (see, e.g., [19]), regularization procedure (see, e.g., [18, 20]), reduction to unconstrained differential inclusion (see, e.g., [26]). For the first and second methods, existence and uniqueness of solutions follow, to some extent, from the classical Gronwall inequality and the basic relation

$$
\begin{equation*}
\frac{d}{d t}\|x(t)\|^{2}=2\langle\dot{x}(t), x(t)\rangle \tag{3}
\end{equation*}
$$

(whenever meaningful), applied to two solutions or suitable approximate solutions $x_{1}, x_{2}$ of (2), by means of the monotonicity of $N_{C(t)}(\cdot)$ ( hypomonotonicity when $C(t)$ is proxregular). Those features and the Lipschitz property of the forcing term $f$ with respect to the state variable are employed to obtain that the distance between $x_{1}(t)$ and $x_{2}(t)$ is nonincreasing with respect to time $t$. This reasoning allows in general the construction of a Cauchy sequence of approximate solutions, converging to a solution.

Several extensions of the sweeping process in diverse ways (well-posedness and optimal control) have been studied in the literature (see, e.g., [1], [2], [5], 8], [16], [26], [27] and references therein).

The present paper aims to study the following new variant of the sweeping process

$$
\left(P_{\left.f_{1}, f_{2}\right)}\right)\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \quad \text { a.e. } t \in\left[T_{0}, T\right]  \tag{4}\\
x\left(T_{0}\right)=x_{0} \in C\left(T_{0}\right),
\end{array}\right.
$$

where $N_{C(t)}(\cdot)$ denotes the Clarke normal cone to the subset $C(t)$ of $H$.
We will have to use the following assumptions:
$\left(\mathcal{H}_{1}\right)$ For each $t \in\left[T_{0}, T\right], C(t)$ is a nonempty closed subset of $H$ which is r -prox-regular, for some $r \in(0,+\infty]$, and has an absolutely continuous variation, in the sense that there is some absolutely continuous function $v:\left[T_{0}, T\right] \longrightarrow \mathbb{R}$ such that

$$
C(t) \subset C(s)+|v(s)-v(t)| \mathbb{B}_{H}, \quad \forall s, t \in\left[T_{0}, T\right],
$$

where $\mathbb{B}_{H}$ denotes the closed unit ball of $H$ centered at the origin.
$\left(\mathcal{H}_{2}\right) f_{1}:\left[T_{0}, T\right] \times H \longrightarrow H$ is (Lebesgue) measurable in time (i.e., $f(\cdot, x)$ is measurable for each $x \in H$ ), and such that
$\left(\mathcal{H}_{2,1}\right)$ there exist non-negative functions $\beta_{1}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)$ such that

$$
\left\|f_{1}(t, x)\right\| \leq \beta_{1}(t)(1+\|x\|), \text { for all } t \in\left[T_{0}, T\right] \text { and for any } x \in \bigcup_{t \in\left[T_{0}, T\right]} C(t)
$$

$\left(\mathcal{H}_{2,2}\right)$ for each real $\eta>0$ there exists a non-negative function $L_{1}^{\eta}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)$ such that for any $t \in\left[T_{0}, T\right]$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$
\left\|f_{1}(t, x)-f_{1}(t, y)\right\| \leq L_{1}^{\eta}(t)\|x-y\|,
$$

where $B[0, \eta]$ denotes the closed ball centered at the origin with radius $\eta$.
$\left(\mathcal{H}_{3}\right) f_{2}: Q_{\Delta} \times H \longrightarrow H$ is a measurable mapping such that
$\left(\mathcal{H}_{3,1}\right)$ there exists a non-negative function $\beta_{2}(\cdot, \cdot) \in L^{1}\left(Q_{\Delta}, \mathbb{R}\right)$ such that $\left\|f_{2}(t, s, x)\right\| \leq \beta_{2}(t, s)(1+\|x\|)$, for all $(t, s) \in Q_{\Delta}$ and for any $x \in \bigcup_{t \in\left[T_{0}, T\right]} C(t) ;$
$\left(\mathcal{H}_{3,2}\right)$ for each real $\eta>0$ there exists a non-negative function $L_{2}^{\eta}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)$ such that for all $(t, s) \in Q_{\Delta}$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$
\left\|f_{2}(t, s, x)-f_{2}(t, s, y)\right\| \leq L_{2}^{\eta}(t)\|x-y\| .
$$

Above $L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)\left(\right.$ resp. $\left.L^{1}\left(Q_{\Delta}, \mathbb{R}\right)\right)$ stands for the space of Lebesgue integrable functions on $\left[T_{0}, T\right]$ (resp. $Q_{\Delta}$ ), where

$$
Q_{\Delta}:=\left\{(t, s) \in\left[T_{0}, T\right] \times\left[T_{0}, T\right]: s \leq t\right\}
$$

We called the differential inclusion (4) as integro-differential sweeping process because the integral of the state and the velocity are defined in the dynamical system. One can interpret (4) as follows: as long as $x(t)$ is in the interior of the set $C(t)$, we get $N_{C(t)}(x(t))=0$ and (4) reduces to a Volterra integro-differential equation

$$
\left\{\begin{array}{l}
-\dot{x}(t)=f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \quad \text { a.e. } t \in\left[T_{0}, T\right]  \tag{5}\\
x\left(T_{0}\right)=x_{0}
\end{array}\right.
$$

(for at least a small period of time) to satisfy the constraint $x(t) \in C(t)$, until $x(t)$ hits the boundary of the set $C(t)$. At this moment, if the vector field $-\left(f_{1}(t, x(t))+\right.$ $\left.\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right)$ is pointed outside of the set $C(t)$, then any component of this vector field in the direction normal to $C(t)$ at $x(t)$ must be annihilated to maintain the motion of $x$ within the constraint set. So, the system (4) can be considered as a Volterra integro-differential equation (5) under control term $u(t) \in N_{C(t)}(\cdot)$ which guarantees that the trajectory $x(t)$ always belongs to the desired set $C(t)$ for all $t \in\left[T_{0}, T\right]$.

The well-posedness of the classical perturbed sweeping process (2), i.e., $P_{f_{1}, 0}\left(f_{2} \equiv 0\right)$, has been studied by many authors with different assumptions on data, see, e.g., [13, 14, 21] and references therein. Sweeping process involving integral perturbation i.e., $P_{0, f_{2}}\left(f_{1} \equiv 0\right)$ was considered earlier by Brenier, Gangbo and Savare [6] and recently by Colombo and Kozaily [9]. In the latter paper [9] the authors proved the existence and uniqueness of solution with the particular integral $\int_{0}^{t} f_{2}(s, x(s)) d s$, i.e., for the following problem

$$
\left(P_{0, f_{2}}\right)\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+\int_{0}^{t} f_{2}(s, x(s)) d s \quad \text { a.e. } t \in[0, T] \\
x(0)=x_{0} \in C(0) .
\end{array}\right.
$$

It is also worth mentioning that Colombo and Kozaily say in their paper [9]: "of course, existence and uniqueness to ( $P_{0, f_{2}}$ ) is not surprising". Indeed, we observe that with the above integral $\int_{0}^{t} f_{2}(s, x(s)) d s$, the integro-differential sweeping process $\left(P_{f_{1}, f_{2}}\right)$ is equivalent to

$$
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+y(t), \dot{y}(t)=f_{2}(t, x(t)), \quad x(0)=x_{0}, y(0)=0
$$

and so

$$
\overbrace{\binom{-\dot{x}(t)}{-\dot{y}(t)}}^{-\dot{X}(t)} \in N_{C(t) \times H} \overbrace{\binom{x(t)}{y(t)}}^{X(t)}+\overbrace{\binom{f_{1}(t, x(t))+y(t)}{-f_{2}(t, x(t))}}^{f(t, X(t))},
$$

which is a special case of the classical perturbed sweeping process (2); see, e.g., [19, 21] for the situation of unbounded moving sets.
So, in [9], the aim of the authors was not the well-posedness. Their motivation for studying $\left(P_{0, f_{2}}\right)$ was designing a smoother method of penalization, the motivation of which comes from applications to deriving necessary optimality conditions for optimal control problems with sweeping processes.

Now, if the integral involving $f_{2}$ depends on two time-variables as in (4), the reduction of ( $P_{f_{1}, f_{2}}$ ) there to the perturbed sweeping process (2) cannot be applied.

To the best of our knowledge, for the problem under consideration in the case of the function $f_{2}$ depending on two time-variables, that is, in the case of a general integrodifferential sweeping process of Volterra type $\left(P_{f_{1}, f_{2}}\right)$, a well-posedness result, including the existence, uniqueness, and stability of the solution, has not been obtained up to the present time.

In the present paper, we obtain results on the existence and uniqueness of a solution to the Volterra sweeping process $\left(P_{f_{1}, f_{2}}\right)$ in a Hilbert space. This is done with the help of a new Gronwall-like inequality (see Section 3) and of a new scheme corresponding to the existence of absolutely continuous solutions for the quasi-autonomous sweeping processes

$$
\left\{\begin{array}{l}
-\dot{x}_{n}(t) \in N_{C(t)}\left(x_{n}(t)\right)+f_{1}\left(t, x_{n}\left(t_{k}\right)\right)+\sum_{j=0}^{k-1} \int_{t_{j}}^{t_{j+1}} f_{2}\left(t, s, x_{n}\left(t_{j}\right)\right) d s \\
+\int_{t_{k}}^{t} f_{2}\left(t, s, x_{n}\left(t_{k}\right)\right) d s \text { a.e. } t \in\left[t_{k}, t_{k+1}\right] \\
x_{n}\left(T_{0}\right)=x_{0} \in C\left(T_{0}\right)
\end{array}\right.
$$

where $T_{0}=t_{0}<t_{1}<\ldots<t_{n}=T$ is a discretization of the interval $\left[T_{0}, T\right]$.
The outline of the paper is as follows. In Section ??, we recall some preliminary results that we use throughout. In Section 3, we prove a new Gronwall-like inequality ( differential inequality). Then, in Section 4 we present our main existence, uniqueness, and stability result. In Section 5 we use those results in the study of nonlinear integro-differential complementarity systems. This is realized by transforming such systems into integrodifferential sweeping processes of the form (4) where the moving set $C(t)$ is described by a finite number of inequalities. We also provide sufficient verifiable conditions ensuring the absolute continuity of the moving set. Finally, in Section 6, we give a second application of our results to non-regular electrical circuits containing time-varying capacitors and
nonsmooth electronic device like diodes. Both applications represent an additional novelty of our paper.

## 2 Notation and preliminaries

Throughout $H$ is a real Hilbert space endowed with the inner product $\langle\cdot, \cdot\rangle$ and associated norm $\|\cdot\|$. As usual, we will denote by $\mathbb{B}_{H}$ or $\mathbb{B}$ the closed unit ball of $H$ and by $B(x, \delta)$ (resp. $B[x, \delta]$ ) the open (resp. closed) ball around $x \in H$ with radius $\delta>0$. For a nonempty subset $S$ of $H$ the associated distance function is denoted by $d_{S}$, that is,

$$
d_{S}(x):=\inf _{y \in S}\|x-y\| \text { for all } x \in H
$$

By $\mathcal{C}\left(\left[T_{0}, T\right], H\right)$ we denote the space of continuous mappings from $\left[T_{0}, T\right]$ into $H$ equipped with the supremum norm $\|\cdot\|_{\infty}$, where we recall that $-\infty<T_{0}<T<+\infty$. As usual $\mathbb{R}$ will denote the set of real numbers, $\mathbb{R}_{+}$the set of non-negative reals, that is, $\mathbb{R}_{+}:=[0,+\infty)$, and $\mathbb{N}$ the set positive integers. In various cases, it will be convenient to use the notation write $I:=\left[T_{0}, T\right]$

The Clarke tangent cone of $S$ at $x \in S$, denoted by $T^{C}(S ; x)$, is the set of $h \in H$ such that, for every sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ of $S$ with $x_{n} \longrightarrow x$ as $n \longrightarrow \infty$ and for every sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ of positive reals with $t_{n} \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a sequence $\left(h_{n}\right)_{n \in \mathbb{N}}$ of $H$ with $h_{n} \longrightarrow h$ as $n \longrightarrow \infty$ satisfying $x_{n}+t_{n} h_{n} \in S$ for all $n \in \mathbb{N}$. This set is obviously a cone containing zero and it is known to be closed and convex. The polar cone of $T^{C}(S ; x)$ is the Clarke normal cone $N^{C}(S ; x)$ of $S$ at $x$, that is

$$
N_{S}^{C}(x):=\left\{v \in H:\langle v, h\rangle \leq 0, \forall h \in T^{C}(S ; x)\right\} .
$$

If $x \notin S$, by convention $T^{C}(S ; x)$ and $N^{C(S ; x)}$ are empty.
For a function $f: H \rightarrow \mathbb{R} \cup\{+\infty\}$, The Clarke subdifferential $\partial_{C} f(x)$ of $f$ at $x$ ( see [23] ) with $f(x)<+\infty$ is defined by

$$
\partial_{C} f(x):=\left\{v \in H:(v,-1) \in N^{C}(\text { epi } f ;(x, f(x)))\right\}
$$

where $H \times \mathbb{R}$ is endowed with the usual product structure and epi $f$ is the epigraph of $f$ , that is,

$$
\text { epi } f:=\{(x, r) \in H \times \mathbb{R}: f(x) \leq r\} .
$$

If $f$ is not finite at $x$, we see that $\partial_{C} f(x)=\emptyset$. In addition to the latter definition, there is another link between the Clarke normal cone and the Clarke subdifferential, given by $\partial_{C} \psi_{S}(x)=N^{C}(S ; x)$, where $\psi_{S}$ denotes the indicator function of the subset $S$ of $H$, i.e. $\psi(y)=0$ if $y \in S$ and $\psi(y)=+\infty$ if $y \notin S$.

A vector $v \in H$ is a proximal subgradient of $f$ at a point $x$ with $f(x)<+\infty$ (see, e.g., [12, 17, 24]) if there exist some reals $\sigma \geq 0$ and $\delta>0$ such that

$$
\langle v, y-x\rangle \leq f(y)-f(x)+\sigma\|y-x\|^{2} \quad \text { for all } y \in B(x, \delta)
$$

The set $\partial_{P} f(x)$ of all proximal subgradients of $f$ at $x$ is the proximal subdifferential of $f$ at $x$. Of course, $\partial_{P} f(x)=\emptyset$ if $f(x)=+\infty$. Note that one always has $\partial_{P} f(x) \subset \partial_{C} f(x)$.

Taking the proximal subdifferential $\partial_{P} \psi_{S}(x)$ of the indicator function $\psi_{S}$ we obtain the proximal normal cone $N_{S}^{P}(x)$ of $S$ at $x$. So, a vector $v \in H$ is a proximal normal vector of $S$ at $x \in S$ if and only if there are reals $\sigma \geq 0$ and $\delta>0$ such that

$$
\begin{equation*}
\langle v, y-x\rangle \leq \sigma\|y-x\|^{2} \quad \text { for all } y \in S \cap B(x, \delta) . \tag{6}
\end{equation*}
$$

So, we always have the inclusion $N_{S}^{P}(x) \subset N_{S}^{C}(x)$ for all $x \in S$.
The proximal normal cone can be described in the following geometrical way (see, e.g., [12])

$$
\begin{equation*}
N_{S}^{P}(x)=\left\{v \in H: \exists r>0 \text { such that } x \in \operatorname{Proj}_{S}(x+r v)\right\}, \tag{7}
\end{equation*}
$$

where

$$
\operatorname{Proj}_{S}(u):=\left\{y \in S: d_{S}(u)=\|u-y\|\right\} .
$$

The proximal normal cone is also connected with the distance function to $S$ through the equalities (see, e.g., [12])

$$
\partial_{P} d_{S}(x)=N_{S}^{P}(x) \cap \mathbb{B}_{H} \text { and } N_{S}^{P}(x)=\mathbb{R}_{+} \partial_{P} d_{S}(x)
$$

where $\mathbb{R}_{+}:=[0,+\infty[$.
In many cases one has to require in (7) that the constant $r$ be uniform for all the unit proximal normal vectors of $S$. The sets which satisfy that property are known as (uniformly) prox-regular sets. Given $r \in] 0,+\infty]$ the closed subset $S$ is (uniformly) $r$-proxregular (see [22]) (called also $r$-positively reached (see [15]) or $r$-proximally smooth (see [11]) provided that, for every $x \in S$ and every unit vector $v \in N_{S}^{P}(x)$ one has $x \in \operatorname{Proj}_{S}(x+r v)$. The latter is equivalent to

$$
S \cap B_{H}(x+r v, r)=\emptyset \text { or equivalently }\left\langle v, x^{\prime}-x\right\rangle \leq \frac{1}{2 r}\left\|x^{\prime}-x\right\|^{2},
$$

for all $x^{\prime} \in S$. Of course, in the latter inequality, $\frac{1}{r}=0$ for $r=+\infty$ (as usual). It is worth pointing out that for $r=+\infty$, the uniform $r$-prox-regularity of the closed set $S$ amounts to its convexity.

Definition 2.1 For a given $r \in(0, \infty]$, a subset $S$ of the Hilbert space $H$ is uniformly $r$ -prox-regular, or r-prox-regular for short, if and only if for all $x \in S$ and all $0 \neq \varsigma \in N_{S}^{P}(x)$ one has

$$
\left\langle\frac{\varsigma}{\|\varsigma\|}, y-x\right\rangle \leq \frac{\|y-x\|^{2}}{2 r}, \quad \forall y \in S
$$

The following propositions summarize some important consequences of prox-regularity needed in the paper. For the proof of these results, we refer the reader to [22, 10].

Proposition 2.2 Let $S$ be a nonempty closed set in $H$ which is uniformly r-prox-regular for some $r \in[0,+\infty]$. Then for any $x_{i} \in S, \varsigma_{i} \in N_{S}^{P}\left(x_{i}\right)$ with $i=1,2$ one has :

$$
\left\langle\varsigma_{2}-\varsigma_{1}, x_{2}-x_{1}\right\rangle \geq-\frac{1}{2}\left(\frac{\left\|\varsigma_{2}\right\|+\left\|\varsigma_{1}\right\|}{r}\right)\left\|x_{2}-x_{1}\right\|^{2} .
$$

Proposition 2.3 Let $S$ be a nonempty closed subset in $H$ and let $r \in(0, \infty]$. If the subset $S$ is uniformly r-prox-regular then the following hold:
(a) The proximal and Clarke normal cones of $S$ coincide.
(b) for all $x \in H$ with $d_{S}(x)<r, \operatorname{Proj}_{S}(x)$ is nonempty and is a singleton set.
(c) the Clarke and the proximal subdifferentials of $d_{S}$ coincide at all points $x \in H$ with $d_{S}(x)<r$.

The assertion (a) in Proposition 2.3 leads us to put $N_{S}(x):=N_{S}^{C}(x)=N_{S}^{P}(x)$ whenever the set $S$ is $r$-prox-regular.

Proposition 2.4 Let $S$ be a nonempty closed subset in $H$ and let $r \in(0, \infty]$. If the subset $S$ is uniformly r-prox-regular then the following hold:
(a) For any $x \in S$ and any $\varsigma \in \partial^{P} d_{S}(x)$ one has for any $y \in H$ such that $d_{S}(y)<r$

$$
\langle\varsigma, y-x\rangle \leq \frac{2}{r}\|y-x\|^{2}+d_{S}(y) .
$$

(b) For any $x \in H$ with $d_{S}(x)<r$, the proximal subdifferential $\partial^{P} d_{S}(x)$ is a nonempty closed convex subset in $H$.

## 3 Gronwall-like differential inequality

We start this section with the following continuous Gronwall's inequality [25].
Lemma 3.1 (Gronwall's inequality) Let $T>T_{0}$ be given reals and $a(\cdot), b(\cdot) \in L^{1}\left(\left[T_{0}, T\right] ; \mathbb{R}\right)$ with $b(t) \geq 0$ for almost all $t \in\left[T_{0}, T\right]$. Let the absolutely continuous function $w$ : $\left[T_{0}, T\right] \longrightarrow \mathbb{R}_{+}$satisfy

$$
(1-\alpha) w^{\prime}(t) \leq a(t) w(t)+b(t) w^{\alpha}(t), \quad \text { a.e. } t \in\left[T_{0}, T\right],
$$

where $0 \leq \alpha<1$. Then for all $t \in\left[T_{0}, T\right]$, one has

$$
w^{1-\alpha}(t) \leq w^{1-\alpha}\left(T_{0}\right) \exp \left(\int_{T_{0}}^{t} a(\tau) d \tau\right)+\int_{T_{0}}^{t} \exp \left(\int_{s}^{t} a(\tau) d \tau\right) b(s) d s
$$

We will need the following lemma which is a straightforward consequence of Gronwall's lemma.

Lemma 3.2 Let $\rho:\left[T_{0}, T\right] \longrightarrow \mathbb{R}$ be a nonnegative absolutely continuous function and let $b_{1}, b_{2}, a:\left[T_{0}, T\right] \longrightarrow \mathbb{R}_{+}$be non-negative Lebesgue integrable functions. Assume that

$$
\begin{equation*}
\dot{\rho}(t) \leq a(t)+b_{1}(t) \rho(t)+b_{2}(t) \int_{T_{0}}^{t} \rho(s) d s, \quad \text { a.e. } t \in\left[T_{0}, T\right] . \tag{8}
\end{equation*}
$$

Then for all $t \in\left[T_{0}, T\right]$, one has

$$
\rho(t) \leq \rho\left(T_{0}\right) \exp \left(\int_{T_{0}}^{t}(b(\tau)+1) d \tau\right)+\int_{T_{0}}^{t} a(s) \exp \left(\int_{s}^{t}(b(\tau)+1) d \tau\right) d s
$$

where $b(t):=\max \left\{b_{1}(t), b_{2}(t)\right\}$, a.e. $t \in\left[T_{0}, T\right]$.
Proof. Put $b(t)=\max \left\{b_{1}(t), b_{2}(t)\right\}$, a.e. $t \in\left[T_{0}, T\right]$. Setting $z(t)=\int_{T_{0}}^{t} \rho(s) d s \Rightarrow$ $\dot{z}(t)=\rho(t), \ddot{z}(t)=\dot{\rho}(t)$. Then from (8) we see that

$$
\ddot{z}(t) \leq a(t)+b_{1}(t) \dot{z}(t)+b_{2}(t) z(t) \leq a(t)+b(t) w(t),
$$

where $w(t)=\dot{z}(t)+z(t)$, for all $t \in\left[T_{0}, T\right]$. Hence, for a.e. $t \in\left[T_{0}, T\right]$

$$
\dot{w}(t)=\ddot{z}(t)+\dot{z}(t) \quad \text { and } \quad \dot{w}(t) \leq a(t)+(b(t)+1) w(t) .
$$

Applying the Gronwall Lemma 3.1 with $w$, one obtains for all $t \in\left[T_{0}, T\right]$

$$
w(t) \leq w\left(T_{0}\right) \exp \left(\int_{T_{0}}^{t}(b(\tau)+1) d \tau\right)+\int_{T_{0}}^{t} a(s) \exp \left(\int_{s}^{t}(b(\tau)+1) d \tau\right) d s
$$

which gives
$\rho(t) \leq \dot{z}(t)+z(t)=w(t) \leq \rho\left(T_{0}\right) \exp \left(\int_{T_{0}}^{t}(b(\tau)+1) d \tau\right)+\int_{T_{0}}^{t} a(s) \exp \left(\int_{s}^{t}(b(\tau)+1) d \tau\right) d s$.

We establish now the following new Gronwall-like lemma.
Lemma 3.3 (Gronwall-like differential inequality) Let $\rho:\left[T_{0}, T\right] \longrightarrow \mathbb{R}$ be a nonnegative absolutely continuous function and let $K_{1}, K_{2}, \varepsilon:\left[T_{0}, T\right] \longrightarrow \mathbb{R}_{+}$be non-negative Lebesgue integrable functions. Suppose for some $\epsilon>0$

$$
\begin{equation*}
\dot{\rho}(t) \leq \varepsilon(t)+\epsilon+K_{1}(t) \rho(t)+K_{2}(t) \sqrt{\rho(t)} \int_{T_{0}}^{t} \sqrt{\rho(s)} d s, \quad \text { a.e. } t \in\left[T_{0}, T\right] . \tag{9}
\end{equation*}
$$

Then for all $t \in\left[T_{0}, T\right]$, one has

$$
\begin{aligned}
\sqrt{\rho(t)} & \leq \sqrt{\rho\left(T_{0}\right)+\epsilon} \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right)+\frac{\sqrt{\epsilon}}{2} \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) d s \\
& +2\left(\sqrt{\left.\int_{T_{0}}^{t} \varepsilon(s) d s+\epsilon-\sqrt{\epsilon} \exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right)\right)}\right. \\
& +2 \int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \sqrt{\int_{T_{0}}^{s} \varepsilon(\tau) d \tau+\epsilon} d s
\end{aligned}
$$

where $K(t):=\max \left\{\frac{K_{1}(t)}{2}, \frac{K_{2}(t)}{2}\right\}$, a.e. $t \in\left[T_{0}, T\right]$.
Proof. Set $\lambda(t)=\sqrt{\int_{T_{0}}^{t} \varepsilon(s) d s+\epsilon}$ and $z_{\varepsilon}(t)=\sqrt{\rho(t)+\lambda^{2}(t)}$ for all $t \in\left[T_{0}, T\right]$.
From (9) we have for a.e. $t \in\left[T_{0}, T\right]$

$$
\begin{equation*}
\dot{\rho}(t) \leq \varepsilon(t)+\epsilon+K_{1}(t)\left(\rho(t)+\lambda^{2}(t)\right)+K_{2}(t) \sqrt{\rho(t)+\lambda^{2}(t)} \int_{T_{0}}^{t} \sqrt{\rho(s)+\lambda^{2}(s)} d s \tag{10}
\end{equation*}
$$

and

$$
\dot{z}_{\varepsilon}(t)=\frac{\dot{\rho}(t)+2 \dot{\lambda}(t) \lambda(t)}{2 \sqrt{\rho(t)+\lambda^{2}(t)}}=\frac{\dot{\rho}(t)+\varepsilon(t)}{2 z_{\varepsilon}(t)}, \quad \text { or equivalently } \dot{\rho}(t)=2 z_{\varepsilon}(t) \dot{z}_{\varepsilon}(t)-\varepsilon(t)
$$

hence from (10)

$$
2 z_{\varepsilon}(t) \dot{z}_{\varepsilon}(t) \leq 2 \varepsilon(t)+\epsilon+K_{1}(t) z_{\varepsilon}(t)^{2}+K_{2}(t) z_{\varepsilon}(t) \int_{T_{0}}^{t} z_{\varepsilon}(s) d s
$$

Therefore,

$$
\dot{z}_{\varepsilon}(t) \leq \frac{\varepsilon(t)}{z_{\varepsilon}(t)}+\frac{\epsilon}{2 z_{\varepsilon}(t)}+\frac{K_{1}(t)}{2} z_{\varepsilon}(t)+\frac{K_{2}(t)}{2} \int_{T_{0}}^{t} z_{\varepsilon}(s) d s \leq 2 \dot{\lambda}(t)+\frac{\sqrt{\epsilon}}{2}+\frac{1}{2}\left(K_{1}(t) z_{\varepsilon}(t)+K_{2}(t) \int_{T_{0}}^{t} z_{\varepsilon}(s) d s\right)
$$

Since

$$
\lambda(t)=\sqrt{\int_{T_{0}}^{t} \varepsilon(s) d s+\epsilon} \leq \sqrt{\rho(t)+\int_{T_{0}}^{t} \varepsilon(s) d s+\epsilon}=\sqrt{\rho(t)+\lambda^{2}(t)}=z_{\varepsilon}(t)
$$

then

$$
\frac{1}{z_{\varepsilon}(t)} \leq \frac{1}{\lambda(t)}, \quad \text { or equivalently } \frac{\varepsilon(t)}{z_{\varepsilon}(t)} \leq \frac{\varepsilon(t)}{\lambda(t)}
$$

Also we have $\dot{\lambda}(t)=\frac{\varepsilon(t)}{2 \lambda(t)}$. Then $\frac{\varepsilon(t)}{z_{\varepsilon}(t)} \leq 2 \dot{\lambda}(t)$, and $\sqrt{\epsilon} \leq \sqrt{\epsilon+\int_{T_{0}}^{t} \varepsilon(s) d s}=\lambda(t) \leq z_{\varepsilon}(t)$, hence $\frac{\epsilon}{2 z_{\varepsilon}(t)} \leq \frac{\sqrt{\epsilon}}{2}$. Letting $K(t):=\max \left\{\frac{K_{1}(t)}{2}, \frac{K_{2}(t)}{2}\right\}$ and applying the Gronwall Lemma 3.2 with $z_{\varepsilon}$, one obtains for all $t \in\left[T_{0}, T\right]$

$$
\begin{aligned}
z_{\varepsilon}(t) & \leq z_{\varepsilon}\left(T_{0}\right) \exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right)+\int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right)\left(2 \dot{\lambda}(s)+\frac{\sqrt{\epsilon}}{2}\right) d s \\
& =\sqrt{\rho\left(T_{0}\right)+\epsilon} \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right)+\int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right)\left(2 \dot{\lambda}(s)+\frac{\sqrt{\epsilon}}{2}\right) d s
\end{aligned}
$$

or equivalently

$$
z_{\varepsilon}(t) \leq \sqrt{\rho\left(T_{0}\right)+\epsilon} \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right)+2 \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \dot{\lambda}(s) d s+\frac{\sqrt{\epsilon}}{2} \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right.
$$

On the other hand, from integration by parts, we note that

$$
\begin{aligned}
\int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \dot{\lambda}(s) d s & =\left[\exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \lambda(s)\right]_{T_{0}}^{t}+\int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \lambda(s) \\
& =\lambda(t)-\exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right) \sqrt{\epsilon}+\int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \lambda(
\end{aligned}
$$

which combined with what precedes gives

$$
\begin{aligned}
z_{\varepsilon}(t) & \leq \sqrt{\rho\left(T_{0}\right)+\epsilon} \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right)+\frac{\sqrt{\epsilon}}{2} \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) d s \\
& +2 \lambda(t)-2 \exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right) \sqrt{\epsilon}+2 \int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \lambda(s) d s
\end{aligned}
$$

Consequently, observing that $\sqrt{\rho(t)} \leq \sqrt{\rho(t)+\lambda^{2}(t)}=z_{\varepsilon}(t)$ we obtain

$$
\begin{aligned}
\sqrt{\rho(t)} & \leq \sqrt{\rho\left(T_{0}\right)+\epsilon} \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right)+\frac{\sqrt{\epsilon}}{2} \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) d s+2 \lambda(t) \\
& -2 \exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right) \sqrt{\epsilon}+2 \int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \lambda(s) d s,
\end{aligned}
$$

which completes the proof of the lemma.

## 4 Main results

In this section, we give and prove our main results in the study of the integro-differential sweeping process $\left(P_{f_{1}, f_{2}}\right)$. They concern the existence, uniqueness, and continuous dependence of the solution with respect to the initial data. We state first in the next proposition a result which will be utilized in our development. Clearly, when the sets $C(t)$ are bounded, the hypothesis $\left(\mathcal{H}_{1}\right)$ is ensured by the usual Hausdorff variation hypothesis

$$
\operatorname{haus}(C(s), C(t)) \leq|v(s)-v(t)|, \quad \forall t, s \in\left[T_{0}, T\right],
$$

which according to the equality $\operatorname{haus}\left(S, S^{\prime}\right)=\sup _{y \in H}\left|d_{S}(y)-d_{S^{\prime}}(y)\right|$ for bounded sets $S, S^{\prime}$, amounts to requiring for the bounded sets $C(s), C(t)$ that

$$
\begin{equation*}
\left|d_{C(s)}(y)-d_{C(t)}(y)\right| \leq|v(s)-v(t)|, \quad \forall t, s \in\left[T_{0}, T\right], \quad \forall y \in H . \tag{11}
\end{equation*}
$$

The result is proved in [7, [13] under the hypothesis (11) but the proof in [7, 13] is valid with the hypothesis $\left(\mathcal{H}_{1}\right)$. One of the advantages of $\left(\mathcal{H}_{1}\right)$ is that the sets $C(t)$ there need not be bounded.

Proposition 4.1 Let $H$ be a real Hilbert space, suppose that $C(\cdot)$ satisfies $\left(\mathcal{H}_{1}\right)$. Let $h$ : $\left[T_{0}, T\right] \longrightarrow H$ be a single-valued mapping in $L^{1}\left(\left[T_{0}, T\right], H\right)$. Then for any $x_{0} \in C\left(T_{0}\right)$ there exists a unique absolutely continuous solution $x(\cdot)$ for the following differential inclusion

$$
\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+h(t) \text { a.e } t \in\left[T_{0}, T\right], \\
x\left(T_{0}\right)=x_{0} .
\end{array}\right.
$$

Moreover $x(\cdot)$ satisfies the following inequality

$$
\begin{equation*}
\|\dot{x}(t)+h(t)\| \leq\|h(t)\|+|\dot{v}(t)| \text { a.e. } t \in\left[T_{0}, T\right] . \tag{12}
\end{equation*}
$$

Theorem 4.2 Let $H$ be a real Hilbert space and assume that $\left(\mathcal{H}_{1}\right),\left(\mathcal{H}_{2}\right)$ and $\left(\mathcal{H}_{3}\right)$ are satisfied. Then for any initial point $x_{0} \in H$, with $x_{0} \in C\left(T_{0}\right)$ there exists a unique absolutely continuous solution $x:\left[T_{0}, T\right] \longrightarrow H$ of the differential inclusion $\left(P_{f_{1}, f_{2}}\right)$. This solution satisfies:

1. For a.e. $t \in\left[T_{0}, T\right]$

$$
\begin{equation*}
\left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\| \leq|\dot{v}(t)|+\left\|f_{1}(t, x(t))\right\|+\int_{T_{0}}^{t}\left\|f_{2}(t, s, x(s))\right\| d s \tag{13}
\end{equation*}
$$

2. If $\int_{T_{0}}^{T}\left[\beta_{1}(\tau)+\int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s\right] d \tau<\frac{1}{4}$, one has

$$
\begin{align*}
\left\|f_{1}(t, x(t))\right\| & \leq(1+M) \beta_{1}(t), \text { for all } t \in\left[T_{0}, T\right]  \tag{14}\\
\left\|f_{2}(t, s, x(s))\right\| & \leq(1+M) \beta_{2}(t, s), \text { for all }(t, s) \in Q_{\Delta}, \tag{15}
\end{align*}
$$

and for almost all $t \in\left[T_{0}, T\right]$

$$
\begin{equation*}
\left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\| \leq(1+M)\left(\beta_{1}(t)+\int_{T_{0}}^{t} \beta_{2}(t, s) d s\right)+|\dot{v}(t)| \tag{16}
\end{equation*}
$$

where $M:=2\left(\left\|x_{0}\right\|+\int_{T_{0}}^{T}|\dot{v}(\tau)| d \tau+\frac{1}{2}\right)$.
3. Assume the following strengthened form of assumption $\left(\mathcal{H}_{3,1}\right)$ on the function $f_{2}$ holds:
$\left(\mathcal{H}_{3,1}^{\prime}\right)$ : there exist non-negative functions $\alpha(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)$ and $g(\cdot) \in L^{1}\left(P_{\Delta}, \mathbb{R}\right)$ such that

$$
\left\|f_{2}(t, s, x)\right\| \leq g(t, s)+\alpha(t)\|x\|, \text { for any }(t, s) \in Q_{\Delta} \text { and any } x \in \bigcup_{t \in\left[T_{0}, T\right]} C(t) \text {. }
$$

Then we have

$$
\begin{gather*}
\left\|f_{1}(t, x(t))\right\| \leq(1+\widetilde{M}) \beta_{1}(t), \text { for all } t \in\left[T_{0}, T\right]  \tag{17}\\
\left\|f_{2}(t, s, x(s))\right\| \leq g(t, s)+\alpha(t) \widetilde{M}, \text { a.e. }(t, s) \in Q_{\Delta} \tag{18}
\end{gather*}
$$

and for almost all $t \in\left[T_{0}, T\right]$

$$
\begin{equation*}
\left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\| \leq|\dot{v}(t)|+(1+\widetilde{M}) \beta_{1}(t)+\int_{T_{0}}^{t} g(t, s) d s+T \alpha(t) \widetilde{M}, \tag{19}
\end{equation*}
$$

where

$$
\begin{aligned}
\widetilde{M}:=\left\|x_{0}\right\| \exp ( & \left.\int_{T_{0}}^{T}(b(\tau)+1) d \tau\right)+\exp \left(\int_{T_{0}}^{T}(b(\tau)+1) d \tau\right) \int_{T_{0}}^{T}\left(|\dot{v}(s)|+2 \beta_{1}(s)+2 \int_{T_{0}}^{T} g(s, \tau) d \tau\right) d s \\
& \text { and } b(t):=2 \max \left\{\beta_{1}(t), \alpha(t)\right\} \text { for all } t \in\left[T_{0}, T\right]
\end{aligned}
$$

Proof. The proof of existence of solution is divided in several steps.
Step 1. Discretization of the interval $I=\left[T_{0}, T\right]$.
For each $n \in \mathbb{N}$, divide the interval $I$ into $n$ intervals of length $h=\frac{T-T_{0}}{n}$ and define, for all $i \in\{0, \cdots, n-1\}$

$$
\left\{\begin{array}{l}
t_{i+1}^{n}:=t_{i}^{n}+h=T_{0}+i h,  \tag{20}\\
I_{i}^{n}:=\left[t_{i}^{n}, t_{i+1}^{n}\right],
\end{array}\right.
$$

so that

$$
\begin{equation*}
T_{0}=t_{0}^{n}<t_{1}^{n}<\cdots<t_{i}^{n}<t_{i+1}^{n}<\cdots<t_{n}^{n}=T . \tag{21}
\end{equation*}
$$

Step 2. Construction of the sequence $x_{n}(\cdot)$.
We construct a sequence of mappings $\left(x_{n}(\cdot)\right)_{n \in \mathbb{N}}$ in $\mathcal{C}(I, H)$ which converges uniformly to a solution $x(\cdot)$ of $(P)$.
Our method consists in establishing a sequence of discrete solutions $\left(x_{k}^{n}(\cdot)\right)_{n \in \mathbb{N}}$ in each interval $I_{k}^{n}:=\left[t_{k}^{n}, t_{k+1}^{n}\right](0 \leq k \leq n-1)$ by using Proposition 4.1. Indeed, we proceed as follows.
Given the following problem

$$
\left(P_{0}\right):\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}\left(t, x_{0}\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{0}\right) d s \text { a.e. } t \in\left[T_{0}, t_{1}^{n}\right],  \tag{22}\\
x\left(T_{0}\right)=x_{0} .
\end{array}\right.
$$

Then $\left(P_{0}\right)$ is a perturbed sweeping process with the perturbation depending only on time. Let $h_{0}:\left[T_{0}, t_{1}^{n}\right] \rightarrow H$ be defined by $h_{0}(t):=f_{1}\left(t, x_{0}\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{0}\right) d s$ for all $t \in\left[T_{0}, t_{1}^{n}\right]$. We see by integrable linear growth condition that

$$
\int_{T_{0}}^{T}\left\|h_{0}(t)\right\| d t \leq\left(1+\left\|x_{0}\right\|\right) \int_{T_{0}}^{T} \beta_{1}(t) d t+\left(1+\left\|x_{0}\right\|\right) \int_{T_{0}}^{T} \int_{T_{0}}^{t} \beta_{2}(t, s) d s d t
$$

and since $\beta_{1}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}_{+}\right)$and $\beta_{2}(\cdot) \in L^{1}\left(Q_{\Delta}, \mathbb{R}_{+}\right)$, then $h_{0}(\cdot)$ is an integrable function. Therefore, by Proposition 4.1 the differential inclusion $\left(P_{0}\right)$ has a unique absolutely continuous solution denoted by

$$
\begin{equation*}
x_{0}^{n}(\cdot):\left[T_{0}, t_{1}^{n}\right] \longrightarrow H, \tag{23}
\end{equation*}
$$

satisfying the following inequality

$$
\begin{equation*}
\left\|\dot{x}_{0}^{n}(t)+f_{1}\left(t, x_{0}\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{0}\right) d s\right\| \leq\left\|f_{1}\left(t, x_{0}\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{0}\right) d s\right\|+|\dot{v}(t)| \tag{24}
\end{equation*}
$$

a.e. $t \in\left[T_{0}, t_{1}^{n}\right]$.

Next, let us consider the following problem

$$
\left(P_{1}\right):\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}\left(t, x_{0}^{n}\left(t_{1}^{n}\right)\right)+\int_{T_{0}}^{t_{1}^{n}} f_{2}\left(t, s, x_{0}\right) d s \int_{t_{1}^{n}}^{t} f_{2}\left(t, s, x_{0}^{n}\left(t_{1}^{n}\right)\right) d s \text { a.e. } t \in\left[t_{1}^{n}, t_{2}^{n}\right]  \tag{25}\\
x\left(t_{1}^{n}\right)=x_{0}^{n}\left(t_{1}^{n}\right) .
\end{array}\right.
$$

Let $h_{1}:\left[t_{1}^{n}, t_{2}^{n}\right] \rightarrow H$ be defined by

$$
h_{1}(t):=f_{1}\left(t, x_{0}^{n}\left(t_{1}^{n}\right)\right)+\int_{T_{0}}^{t_{1}^{n}} f_{2}\left(t, s, x_{0}\right) d s+\int_{t_{1}^{n}}^{t} f_{2}\left(t, s, x_{0}^{n}\left(t_{1}^{n}\right)\right) d s \text { for all } t \in\left[t_{1}^{n}, t_{2}^{n}\right]
$$

We can see by integrable linear growth conditions that

$$
\begin{aligned}
\int_{T_{0}}^{T}\left\|h_{1}(t)\right\| d t & \leq\left(1+\left\|x_{0}^{n}\left(t_{1}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \beta_{1}(t) d t+\left(1+\left\|x_{0}\right\|\right) \int_{T_{0}}^{T} \int_{T_{0}}^{t_{1}^{n}} \beta_{2}(t, s) d s d t+\left(1+\left\|x_{0}^{n}\left(t_{1}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \int_{t_{1}^{n}}^{t} \beta_{2}(t, s) d s d t \\
& \leq\left(1+\max \left\{\left\|x_{0}^{n}\left(t_{1}^{n}\right)\right\|,\left\|x_{0}\right\|\right\}\right)\left(\int_{T_{0}}^{T} \beta_{1}(t) d t+\int_{T_{0}}^{T} \int_{T_{0}}^{t} \beta_{2}(t, s) d s d t\right) .
\end{aligned}
$$

We know from the above problem $\left(P_{0}\right)$ that the mapping $x_{0}^{n}(\cdot)$ is absolutely continuous, then in particular bounded on $\left[T_{0}, T\right]$. Further, since $\beta_{1}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}_{+}\right)$and $\beta_{2}(\cdot) \in$ $L^{1}\left(Q_{\Delta}, \mathbb{R}_{+}\right)$, then $h_{1}(\cdot)$ is an integrable mapping. The same arguments as above show that $\left(P_{1}\right)$ has a unique absolutely continuous solution denoted by

$$
\begin{equation*}
x_{1}^{n}(\cdot):\left[t_{1}^{n}, t_{2}^{n}\right] \longrightarrow H, \tag{26}
\end{equation*}
$$

and this solution satisfies the following inequality

$$
\begin{align*}
& \left\|\dot{x}_{1}^{n}(t)+f_{1}\left(t, x_{0}^{n}\left(t_{1}^{n}\right)\right)+\int_{T_{0}}^{t_{1}^{n}} f_{2}\left(t, s, x_{0}\right) d s+\int_{t_{1}^{n}}^{t} f_{2}\left(t, s, x_{0}^{n}\left(t_{1}^{n}\right)\right) d s\right\| \\
& \leq\left\|f_{1}\left(t, x_{0}^{n}\left(t_{1}^{n}\right)\right)+\int_{T_{0}}^{t_{1}^{n}} f_{2}\left(t, s, x_{0}\right) d s+\int_{t_{1}^{n}}^{t} f_{2}\left(t, s, x_{0}^{n}\left(t_{1}^{n}\right)\right) d s\right\|+|\dot{v}(t)| \quad \text { a.e. } t \in\left[t_{1}^{n}, t_{2}^{n}\right] . \tag{27}
\end{align*}
$$

Successively, for each $n$, we have a finite sequence of absolutely continuous mappings $\left(x_{k}^{n}(\cdot)\right)_{0 \leq k \leq n-1}$ with for each $k \in\{0, \cdots, n-1\}$

$$
\begin{equation*}
x_{k}^{n}(\cdot):\left[t_{k}^{n}, t_{k+1}^{n}\right] \longrightarrow H \tag{28}
\end{equation*}
$$

such that

$$
\left(P_{k-1}\right):\left\{\begin{array}{l}
-\dot{x}_{k}^{n}(t) \in N_{C(t)}\left(x_{k}^{n}(t)\right)+f_{1}\left(t, x_{k-1}^{n}\left(t_{k}^{n}\right)\right)+\sum_{j=0}^{k-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{j-1}^{n}\left(t_{j}^{n}\right)\right) d s  \tag{29}\\
+\int_{\substack{t_{k}^{n} \\
t_{2}}}^{t} f_{2}\left(t, s, x_{k-1}^{n}\left(t_{k}^{n}\right)\right) d s \quad \text { a.e. } t \in\left[t_{k}^{n}, t_{k+1}^{n}\right] . \\
x_{k}^{n}\left(t_{k}^{n}\right)=x_{k-1}^{n}\left(t_{k}^{n}\right),
\end{array}\right.
$$

where for $k=0$ we put $x_{-1}^{n}\left(T_{0}\right):=x_{0}$. Moreover

$$
\begin{align*}
& \left\|\dot{x}_{k}^{n}(t)+f_{1}\left(t, x_{k-1}^{n}\left(t_{k}^{n}\right)\right)+\sum_{j=0}^{k-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{j-1}^{n}\left(t_{j}^{n}\right)\right) d s+\int_{t_{k}^{n}}^{t} f_{2}\left(t, s, x_{k-1}^{n}\left(t_{k}^{n}\right)\right) d s,\right\| \\
& \leq\left\|f_{1}\left(t, x_{k-1}^{n}\left(t_{k}^{n}\right)\right)+\sum_{j=0}^{k-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{j-1}^{n}\left(t_{j}^{n}\right)\right) d s+\int_{t_{k}^{n}}^{t} f_{2}\left(t, s, x_{k-1}^{n}\left(t_{k}^{n}\right)\right) d s,\right\|+|\dot{v}(t)|, \tag{30}
\end{align*}
$$

a.e. $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$.

Defining for each $k \in\{0,1, \cdots, n-1\}$ the mapping $h_{k}:\left[t_{k}^{n}, t_{k+1}^{n}\right] \rightarrow H$ by

$$
h_{k}(t):=f_{1}\left(t, x_{k-1}^{n}\left(t_{k}^{n}\right)\right)+\sum_{j=0}^{k-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{j-1}^{n}\left(t_{j}^{n}\right)\right) d s+\int_{t_{k}^{n}}^{t} f_{2}\left(t, s, x_{k-1}^{n}\left(t_{k}^{n}\right)\right) d s
$$

for all $t \in\left[t_{k}^{n}, t_{k+1}^{n}\right]$. Clearly, by integrable linear growth conditions we have

$$
\begin{aligned}
\int_{T_{0}}^{T}\left\|h_{k}(t)\right\| d t & \leq\left(1+\left\|x_{k-1}^{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \beta_{1}(t) d t+\sum_{j=0}^{k-1}\left(1+\left\|x_{j-1}^{n}\left(t_{j}^{n}\right)\right\|\right) \int_{t_{j}^{n}}^{t_{j+1}^{n}} \beta_{2}(t, s) d s d t \\
& +\left(1+\left\|x_{k-1}^{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \int_{t_{k}^{n}}^{t} \beta_{2}(t, s) d s d t \\
& \leq\left(1+\max _{0 \leq j \leq k-1}\left\|x_{j-1}^{n}\left(t_{j}^{n}\right)\right\|\right)\left(\int_{T_{0}}^{T} \beta_{1}(t) d t+\int_{T_{0}}^{T} \int_{T_{0}}^{t} \beta_{2}(t, s) d s d t\right) .
\end{aligned}
$$

We know from the above problems $\left(P_{j}\right)_{0 \leq j \leq k-1}$ that the mapping $x_{k-1}^{n}(\cdot)$ is absolutely continuous, then in particular bounded on $\left[T_{0}, T\right]$. Further, since $\beta_{1}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}_{+}\right)$ and $\beta_{2}(\cdot) \in L^{1}\left(P_{\Delta}, \mathbb{R}_{+}\right)$; the mapping $h_{k}(\cdot)$ is integrable on $\left[t_{k}^{n}, t_{k+1}^{n}\right]$.

Now, we define the sequence $\left(x_{n}(\cdot)\right)_{n}$ from the discrete sequences $\left(x_{k}^{n}().\right)$ as follows. For each $n \in \mathbb{N}$, let $x_{n}(\cdot):\left[T_{0}, T\right] \longrightarrow H$ such that

$$
\begin{equation*}
x_{n}(t):=x_{k}^{n}(t), \text { if } t \in\left[t_{k}^{n}, t_{k+1}^{n}\right] \tag{31}
\end{equation*}
$$

It is obvious from this definition that $x_{n}(\cdot)$ is absolutely continuous.
Let $\theta_{n}(\cdot):\left[T_{0}, T\right] \longrightarrow\left[T_{0}, T\right]$ be defined by

$$
\left\{\begin{array}{l}
\theta_{n}\left(T_{0}\right):=T_{0}  \tag{32}\\
\left.\left.\theta_{n}(t):=t_{k}^{n}, \text { if } t \in\right] t_{k}^{n}, t_{k+1}^{n}\right]
\end{array}\right.
$$

We obtain from (29), (30), (31), (32), that

$$
\left\{\begin{array}{l}
-\dot{x}_{n}(t) \in N_{C(t)}\left(x_{n}(t)\right)+f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s \text { a.e. } t \in\left[T_{0}, T\right]  \tag{33}\\
x_{n}\left(T_{0}\right)=x_{0}
\end{array}\right.
$$

and a.e. $t \in\left[T_{0}, T\right]$ we have

$$
\begin{align*}
& \left\|\dot{x}_{n}(t)+f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s\right\| \\
& \leq\left\|f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s\right\|+|\dot{v}(t)| . \tag{34}
\end{align*}
$$

Step 3. We show that the sequence $\left(\dot{x}_{n}(\cdot)\right)$ is uniformly dominated by an integrable function.
Since $\beta_{1}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}_{+}\right)$and $\beta_{2}(\cdot, \cdot) \in L^{1}\left(P_{\Delta}, \mathbb{R}_{+}\right)$we suppose without loss of generality that

$$
\begin{equation*}
\int_{T_{0}}^{T}\left[\beta_{1}(\tau)+\int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s\right] d \tau<\frac{1}{4} \tag{35}
\end{equation*}
$$

By construction we have for each $i \in\{0, \cdots, n-1\}$ and for a.e. $t \in\left[t_{i}^{n}, t_{i+1}^{n}\right]$

$$
\begin{aligned}
& \left\|\dot{x}_{n}(t)+f_{1}\left(t, x_{n}\left(t_{i}^{n}\right)\right)+\sum_{j=0}^{i-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{n}\left(t_{j}^{n}\right)\right) d s+\int_{t_{i}^{n}}^{t} f_{2}\left(t, s, x_{n}\left(t_{i}^{n}\right)\right) d s\right\| \\
& \leq\left\|f_{1}\left(t, x_{n}\left(t_{i}^{n}\right)\right)+\sum_{j=0}^{i-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} f_{2}\left(t, s, x_{n}\left(t_{j}^{n}\right)\right) d s+\int_{t_{i}^{n}}^{t} f_{2}\left(t, s, x_{n}\left(t_{i}^{n}\right)\right) d s\right\|+|\dot{v}(t)| .
\end{aligned}
$$

According to $\left(\mathcal{H}_{2,1}\right)$ and $\left(\mathcal{H}_{3,1}\right)$ we have

$$
\begin{aligned}
\left\|\dot{x}_{n}(t)\right\| & \leq 2\left\|f_{1}\left(t, x_{n}\left(t_{i}^{n}\right)\right)\right\|+2 \sum_{j=0}^{i-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}}\left\|f_{2}\left(t, s, x_{n}\left(t_{j}^{n}\right)\right)\right\| d s+2 \int_{t_{i}^{n}}^{t}\left\|f_{2}\left(t, s, x_{n}\left(t_{i}^{n}\right)\right)\right\| d s+|\dot{v}(t)| \\
& \leq 2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \beta_{1}(t)+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \sum_{j=0}^{i-1} \int_{t_{j}^{n}}^{t_{j+1}^{n}} \beta_{2}(t, s) d s \\
& +2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{t_{i}^{n}}^{t} \beta_{2}(t, s) d s+|\dot{v}(t)| \\
& =|\dot{v}(t)|+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \beta_{1}(t)+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{t} \beta_{2}(t, s) d s,
\end{aligned}
$$

and then

$$
\begin{aligned}
\left\|x_{n}\left(t_{i+1}^{n}\right)\right\| & \leq\left\|x_{n}\left(t_{i}^{n}\right)\right\|+\int_{t_{i}^{n}}^{t_{i+1}^{n}}|\dot{v}(\tau)| d \tau+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{t_{i}^{n}}^{t_{t_{+1}^{n}}^{n}} \beta_{1}(\tau) d \tau \\
& +2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{t_{i}^{n}}^{t_{i+1}^{n}} \int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s d \tau .
\end{aligned}
$$

Iterating, it follows that

$$
\begin{aligned}
&\left\|x_{n}\left(t_{i+1}^{n}\right)\right\| \leq\left\|x_{0}\right\|+\sum_{k=0}^{i} \int_{t_{k}^{n}}^{t_{k+1}^{n}}|\dot{v}(\tau)| d \tau+2\left(1+\max _{0 \leq j \leq n}\left\|x_{n}\left(t_{j}^{n}\right)\right\|\right) \sum_{k=0}^{i} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \beta_{1}(\tau) d \tau \\
&+2\left(1+\max _{0 \leq j \leq n}\left\|x_{n}\left(t_{j}^{n}\right)\right\|\right) \sum_{k=0}^{i} \int_{t_{k}^{n}}^{t_{k+1}^{n}} \int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s d \tau
\end{aligned}
$$

This yields the following inequality

$$
\begin{align*}
\left\|x_{n}\left(t_{i+1}^{n}\right)\right\| & \leq\left\|x_{0}\right\|+\int_{T_{0}}^{t_{i+1}^{n}}|\dot{v}(\tau)| d \tau+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{t_{i+1}^{n}} \beta_{1}(\tau) d \tau \\
& +2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{t_{i+1}^{n}} \int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s d \tau . \tag{36}
\end{align*}
$$

The inequality (36) being true for all $i \in\{0, \cdots, n-1\}$, we have

$$
\begin{aligned}
\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\| & \leq\left\|x_{0}\right\|+\int_{T_{0}}^{T}|\dot{v}(\tau)| d \tau+2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \beta_{1}(\tau) d \tau \\
& +2\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right) \int_{T_{0}}^{T} \int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s d \tau
\end{aligned}
$$

which gives by (35)

$$
\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\| \leq\left\|x_{0}\right\|+\int_{T_{0}}^{T}|\dot{v}(\tau)| d \tau+\frac{1}{2}\left(1+\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\|\right)
$$

This can be rewritten as

$$
\begin{equation*}
\max _{0 \leq k \leq n}\left\|x_{n}\left(t_{k}^{n}\right)\right\| \leq M \tag{37}
\end{equation*}
$$

where $M:=2\left(\left\|x_{0}\right\|+\int_{T_{0}}^{T}|\dot{v}(\tau)| d \tau+\frac{1}{2}\right)$.
On one hand, from the growth condition of $f_{1}, f_{2}$ and (37) we have, for almost all $t$ and for all $n$,

$$
\begin{gather*}
\left\|f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)\right\| \leq \beta_{1}(t)\left(1+\left\|x_{n}\left(\theta_{n}(t)\right)\right\|\right) \leq(1+M) \beta_{1}(t) .  \tag{38}\\
\left\|f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right)\right\| \leq \beta_{2}(t, s)\left(1+\left\|x_{n}\left(\theta_{n}(s)\right)\right\|\right) \leq(1+M) \beta_{2}(t, s) . \tag{39}
\end{gather*}
$$

Hence, (34) implies for almost all $t$ and for all $n$

$$
\begin{equation*}
\left\|\dot{x}_{n}(t)+f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s\right\| \leq(1+M)\left(\beta_{1}(t)+\int_{T_{0}}^{t} \beta_{2}(t, s) d s\right)+|\dot{v}(t)|, \tag{40}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\left\|\dot{x}_{n}(t)\right\| \leq 2(1+M)\left(\beta_{1}(t)+\int_{T_{0}}^{t} \beta_{2}(t, s) d s\right)+|\dot{v}(t)| \tag{41}
\end{equation*}
$$

## Step 4. We show that $x_{n}(\cdot)$ converges.

It suffices to show that $x_{n}(\cdot)$ is a Cauchy sequence in the Banach space $\left(C(I, H),\|\cdot\|_{\infty}\right)$. Let $m, n \in \mathbb{N}$. For almost all $t \in\left[T_{0}, T\right]$, we have

$$
\left\{\begin{array}{l}
-\dot{x}_{n}(t)-f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s \in N_{C(t)}\left(x_{n}(t)\right)  \tag{42}\\
-\dot{x}_{m}(t)-f_{1}\left(t, x_{m}\left(\theta_{m}(t)\right)\right)-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{m}\left(\theta_{m}(s)\right)\right) d s \in N_{C(t)}\left(x_{m}(t)\right)
\end{array}\right.
$$

Let us set

$$
\begin{aligned}
& \alpha(t):=(1+M)\left(\beta_{1}(t)+\int_{T_{0}}^{t} \beta_{2}(t, s) d s\right)+|\dot{v}(t)| \\
& \gamma(t):=2(1+M)\left(\beta_{1}(t)+\int_{T_{0}}^{t} \beta_{2}(t, s) d s\right)+|\dot{v}(t)|
\end{aligned}
$$

The absolute continuity of $x_{n}(\cdot)$ gives by (41)

$$
\begin{equation*}
\left\|x_{n}(t)\right\| \leq \eta \text { for all } t \in\left[T_{0}, T\right] \tag{43}
\end{equation*}
$$

with

$$
\eta:=\left\|x_{0}\right\|+\int_{T_{0}}^{T} \gamma(s) d s
$$

Using (40) and the hypomonotonicity of the normal cone $N(C(t) ; \cdot)$, we get that

$$
\begin{aligned}
& \left\langle\dot{x}_{n}(t)+f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s-\dot{x}_{m}(t)-f_{1}\left(t, x_{m}\left(\theta_{m}(t)\right)\right)\right. \\
& \left.-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{m}\left(\theta_{m}(s)\right)\right) d s, x_{n}(t)-x_{m}(t)\right\rangle \leq \frac{\alpha(t)}{r}\left\|x_{n}(t)-x_{m}(t)\right\|^{2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \left\langle\dot{x}_{n}(t)-\dot{x}_{m}(t), x_{n}(t)-x_{m}(t)\right\rangle \leq \frac{\alpha(t)}{r}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \\
& +\left\langle f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)-f_{1}\left(t, x_{m}\left(\theta_{m}(t)\right)\right), x_{m}(t)-x_{n}(t)\right\rangle \\
& +\left\langle\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{m}\left(\theta_{m}(s)\right)\right) d s, x_{m}(t)-x_{n}(t)\right\rangle
\end{aligned}
$$

Applying the Lipschitz continuity of $f_{1}(t, \cdot)$ and $f_{2}(t, s, \cdot)$ with Lipschitz radius $L_{1}^{\eta}(\cdot), L_{2}^{\eta}(\cdot) \in$ $L^{1}\left(I, \mathbb{R}_{+}\right)$on the bounded subset $B[0, \eta]$, it follows that

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \leq \frac{\alpha(t)}{r}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \\
& +L_{1}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\|\left(\left\|x_{n}\left(\theta_{n}(t)\right)-x_{n}(t)\right\|+\left\|x_{n}(t)-x_{m}(t)\right\|+\left\|x_{m}(t)-x_{m}\left(\theta_{m}(t)\right)\right\|\right) \\
& +L_{2}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\|\left(\int_{T_{0}}^{t}\left\|x_{n}\left(\theta_{n}(s)\right)-x_{n}(s)\right\| d s+\int_{T_{0}}^{t}\left\|x_{n}(s)-x_{m}(s)\right\| d s+\int_{T_{0}}^{t}\left\|x_{m}(t)-x_{m}\left(\theta_{m}(t)\right)\right\| d s\right)
\end{aligned}
$$

By (41), we have for each $n \in \mathbb{N}$ and for all $t$,

$$
\left\|x_{n}(t)-x_{n}\left(\theta_{n}(t)\right)\right\|=\left\|\int_{\theta_{n}(t)}^{t} \dot{x}_{n}(\tau) d \tau\right\| \leq \int_{\theta_{n}(t)}^{t}\left\|\dot{x}_{n}(\tau)\right\| d \tau \leq \int_{\theta_{n}(t)}^{t} \gamma(\tau) d \tau
$$

Therefore

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \leq \frac{\alpha(t)}{r}\left\|x_{n}(t)-x_{m}(t)\right\|^{2}+L_{1}^{r}(t)\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \\
& +L_{1}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\|\left(\int_{\theta_{n}(t)}^{t} \gamma(\tau) d \tau+\int_{\theta_{m}(t)}^{t} \gamma(\tau) d \tau\right) \\
& +L_{2}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\|\left(\int_{T_{0}}^{t} \int_{\theta_{n}(s)}^{s} \gamma(\tau) d \tau d s+\int_{T_{0}}^{t} \int_{\theta_{m}(s)}^{s} \gamma(\tau) d \tau d s\right) \\
& +L_{2}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\| \int_{T_{0}}^{t}\left\|x_{n}(s)-x_{m}(s)\right\| d s .
\end{aligned}
$$

Moreover, noting by (43) that

$$
\left\|x_{n}(t)-x_{m}(t)\right\| \leq\left\|x_{n}(t)\right\|+\left\|x_{m}(t)\right\| \leq 2 \eta
$$

we deduce that

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \leq \frac{\alpha(t)}{r}\left\|x_{n}(t)-x_{m}(t)\right\|^{2}+L_{1}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\|^{2} \\
& +2 \eta \mathrm{E}_{1}^{\eta}(t)\left(\int_{\theta_{n}(t)}^{t} \gamma(\tau) d \tau+\int_{\theta_{m}(t)}^{t} \gamma(\tau) d \tau\right)+2 \eta L_{2}^{\eta}(t)\left(\int_{T_{0}}^{t}\left[\int_{\theta_{n}(s)}^{s} \gamma(\tau) d \tau+\int_{\theta_{m}(s)}^{s} \gamma(\tau) d \tau\right] d s\right) \\
& +L_{2}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\| \int_{T_{0}}^{t}\left\|x_{n}(s)-x_{m}(s)\right\| d s \tag{44}
\end{align*}
$$

Let us put

$$
\begin{aligned}
G_{n, m}(t) & :=2 \eta L_{1}^{r}(t)\left(\int_{\theta_{n}(t)}^{t} \gamma(\tau) d \tau+\int_{\theta_{m}(t)}^{t} \gamma(\tau) d \tau\right), \\
\widetilde{G}_{n, m}(s) & :=\int_{\theta_{n}(s)}^{s} \gamma(\tau) d \tau+\int_{\theta_{m}(s)}^{s} \gamma(\tau) d \tau
\end{aligned}
$$

Since $\gamma(\cdot) \in L^{1}\left(I, \mathbb{R}_{+}\right)$and for each $t \in I$, we have $\theta_{n}(t), \theta_{m}(t) \longrightarrow t$, then

$$
\begin{equation*}
\lim _{n, m \rightarrow+\infty} G_{n, m}(t)=0 \quad \text { and } \quad \lim _{n, m \rightarrow+\infty} \widetilde{G}_{n, m}(t)=0 \tag{45}
\end{equation*}
$$

On the other hand, for each $n \in \mathbb{N}$ writing

$$
\int_{\theta_{n}(t)}^{t} \gamma(s) d s \leq \int_{T_{0}}^{T} \gamma(s) d s
$$

we see that

$$
\left|G_{n, m}(t)\right| \leq 4 \eta L_{1}^{\eta}(t)\left(\int_{T_{0}}^{T} \gamma(s) d s\right) \quad \text { and } \quad\left|\tilde{G}_{n, m}(s)\right| \leq 4\left(\int_{T_{0}}^{T} \gamma(s) d s\right) .
$$

Therefore, for all $t \in\left[T_{0}, T\right]$ by (45) and the dominated convergence theorem, we obtain

$$
\begin{align*}
\lim _{n, m \rightarrow+\infty} \int_{T_{0}}^{T} G_{n, m}(t) d t & =0 .  \tag{46}\\
\lim _{n, m \rightarrow+\infty} \int_{T_{0}}^{T} \tilde{G}_{n, m}(s) d s & =0 . \tag{47}
\end{align*}
$$

Note also by (44) that

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t}\left\|x_{n}(t)-x_{m}(t)\right\|^{2} & \leq\left(\frac{\alpha(t)}{r}+L_{1}^{\eta}(t)\right)\left\|x_{n}(t)-x_{m}(t)\right\|^{2}+G_{n, m}(t)+2 \eta L_{2}^{\eta}(t)\left(\int_{T_{0}}^{T} \widetilde{G}_{n, m}(s) d s\right) \\
& +L_{2}^{\eta}(t)\left\|x_{n}(t)-x_{m}(t)\right\| \int_{T_{0}}^{t}\left\|x_{n}(s)-x_{m}(s)\right\| d s
\end{aligned}
$$

Applying Lemma 3.3 with

$$
\begin{gathered}
\rho(t)=\left\|x_{n}(t)-x_{m}(t)\right\|^{2}, K_{1}(t)=2\left(\frac{\alpha(t)}{r}+L_{1}^{\eta}(t)\right), K_{2}(t)=2 L_{2}^{\eta}(t) \\
\varepsilon(t):=\varepsilon_{n, m}(t)=2 G_{n, m}(t)+4 \eta L_{2}^{\eta}(t)\left(\int_{T_{0}}^{T} \widetilde{G}_{n, m}(s) d s\right), \epsilon>0
\end{gathered}
$$

we then see that

$$
\begin{aligned}
\left\|x_{n}(t)-x_{m}(t)\right\| \leq & \sqrt{\left\|x_{n}\left(T_{0}\right)-x_{m}\left(T_{0}\right)\right\|^{2}+\epsilon} \exp \left(\int_{0}^{t}(K(s)+1) d s\right) \\
& +\frac{\sqrt{\epsilon}}{2} \int_{T_{0}}^{t} \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) d s \\
& +2\left(\sqrt{\left.\int_{T_{0}}^{t} \varepsilon_{n, m}(s) d s+\epsilon-\exp \left(\int_{T_{0}}^{t}(K(\tau)+1) d \tau\right) \sqrt{\epsilon}\right)}\right. \\
& +2 \int_{T_{0}}^{t}(K(s)+1) \exp \left(\int_{s}^{t}(K(\tau)+1) d \tau\right) \sqrt{\int_{T_{0}}^{s} \varepsilon_{n, m}(\tau) d \tau+\epsilon d s}
\end{aligned}
$$

where $K(t):=\max \left\{\frac{\alpha(t)}{\eta}+L_{1}^{r}(t), L_{2}^{\eta}(t)\right\}$, for almost all $t \in\left[T_{0}, T\right]$.
This, along with the fact that $\left\|x_{n}\left(T_{0}\right)-x_{m}\left(T_{0}\right)\right\|=0$ and taking $\epsilon \rightarrow 0$, we get

$$
\lim _{n, m \rightarrow+\infty}\left\|x_{n}(\cdot)-x_{m}(\cdot)\right\|_{\infty}=0
$$

Therefore, the sequence $\left(x_{n}(\cdot)\right)$ is a Cauchy sequence in $\left(\mathcal{C}\left(\left[T_{0}, T\right], H\right),\|\cdot\|_{\infty}\right)$ and therefore converges uniformly to a function $x(\cdot) \in \mathcal{C}\left(\left[T_{0}, T\right], H\right)$.

## Step 5 . We show that $x(\cdot)$ is absolutely continuous.

We have for almost all $t \in I$ and for any $n$,

$$
\left\|\dot{x}_{n}(t)\right\| \leq \gamma(t)
$$

So we can extract a subsequence of $\left(\dot{x}_{n}(\cdot)\right)$ (that, without loss of generality, we do not relabel) which converges weakly in $L^{1}(I, H)$ to a function $g(\cdot) \in L^{1}(I, H)$. This means that

$$
\int_{T_{0}}^{T}\left\langle\dot{x}_{n}(s), h(s)\right\rangle d s \longrightarrow \int_{T_{0}}^{T}\langle g(s), h(s)\rangle d s, \forall h \in L^{\infty}(I, H) .
$$

Now observe that for all $z \in H$

$$
\int_{T_{0}}^{T}\left\langle\dot{x}_{n}(s), z \cdot 1_{\left[T_{0}, t\right]}(s)\right\rangle d s=\int_{T_{0}}^{t}\left\langle\dot{x}_{n}(s), z\right\rangle d s=\left\langle z, \int_{T_{0}}^{t} \dot{x}_{n}(s) d s\right\rangle .
$$

and

$$
\int_{T_{0}}^{T}\left\langle g(s), z \cdot 1_{\left[T_{0}, t\right]}(s)\right\rangle d s=\int_{T_{0}}^{t}\langle g(s), z\rangle d s=\left\langle z, \int_{T_{0}}^{t} g(s) d s\right\rangle .
$$

So from the weak convergence we deduce that

$$
\int_{T_{0}}^{t} \dot{x}_{n}(s) d s \longrightarrow \int_{T_{0}}^{t} g(s) d s \text { weakly in } H
$$

This implies that

$$
x_{n}\left(T_{0}\right)+\int_{T_{0}}^{t} \dot{x}_{n}(s) d s \longrightarrow x\left(T_{0}\right)+\int_{T_{0}}^{t} g(s) d s \text { weakly in } H
$$

But $x_{n}(\cdot)$ is absolutely continuous, so

$$
x_{n}(t)=x_{n}\left(T_{0}\right)+\int_{T_{0}}^{t} \dot{x}_{n}(s) d s \longrightarrow x\left(T_{0}\right)+\int_{T_{0}}^{t} g(s) d s \text { weakly in } H .
$$

On the other hand, we have for all $t \in\left[T_{0}, T\right]$

$$
x_{n}(t) \longrightarrow x(t) \text { strongly in } H,
$$

hence we get

$$
x(t)=x\left(T_{0}\right)+\int_{T_{0}}^{t} g(s) d s
$$

Therefore, $x(\cdot)$ is absolutely continuous and $\dot{x}(t)=g(t)$ a.e. $t \in\left[T_{0}, T\right]$, so in particular

$$
\begin{equation*}
\|x(t)\| \leq \tilde{\eta} \text { for all } t \in\left[T_{0}, T\right], \tag{48}
\end{equation*}
$$

with

$$
\tilde{\eta}:=\left\|x_{0}\right\|+\int_{T_{0}}^{T} g(s) d s
$$

Step 6. We show that $x(\cdot)$ is a solution of $\left(P_{f_{1}, f_{2}}\right)$.
For each $t \in I$, since $\theta_{n}(t) \longrightarrow t$ for all $t \in I$ and $x_{n}(\cdot)$ converges uniformly to $x(\cdot)$, we have $x_{n}\left(\theta_{n}(t)\right) \longrightarrow x(t)$.
Let us set for each $t \in I$

$$
y_{n}(t):=\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s, \text { and } y(t):=\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s .
$$

We have shown in the above step that $\dot{x}_{n}(\cdot)$ converges weakly to $\dot{x}(\cdot)$ in $L^{1}(I, H)$. Moreover, by (43) and (48) we can choose some real $c>0$ such that, for each $n$, $\left\|x_{n}\left(\theta_{n}(t)\right)\right\|,\|x(t)\| \leq c$ for all $t \in\left[T_{0}, T\right]$. Therefore, by assumption, there exists $L_{1}^{c}(\cdot), L_{2}^{c}(\cdot) \in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}_{+}\right)$such that $f_{1}(t, \cdot)$ and $f_{2}(t, s, \cdot)$ are $L_{1}^{c}(t)$-Lipschitz and $L_{2}^{c}(t)$ Lipschitz respectively on $B[0, c]$. It follows that

$$
\begin{gather*}
\int_{T_{0}}^{T}\left\|f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)-f_{1}(t, x(t))\right\| d t \leq \int_{T_{0}}^{T} L_{1}^{c}(t)\left\|x_{n}\left(\theta_{n}(t)\right)-x(t)\right\| d t  \tag{49}\\
\int_{T_{0}}^{T}\left\|y_{n}(t)-y(t)\right\| d t \leq \int_{T_{0}}^{T} L_{2}^{c}(t) \int_{T_{0}}^{t}\left\|x_{n}\left(\theta_{n}(s)\right)-x(s)\right\| d s d t . \tag{50}
\end{gather*}
$$

Note that for every $(t, s) \in Q_{\Delta}$

$$
\begin{aligned}
L_{1}^{c}(t)\left\|x_{n}\left(\theta_{n}(t)\right)-x(t)\right\| & \leq 2 c L_{1}^{c}(t), \\
L_{2}^{c}(t) \int_{T_{0}}^{t}\left\|x_{n}\left(\theta_{n}(s)\right)-x(s)\right\| d s & \leq 2 c\left(T-T_{0}\right) L_{2}^{c}(t) .
\end{aligned}
$$

Then by (49), 50) and by the Lebesgue dominated convergence theorem

$$
\begin{gathered}
f_{1}\left(\cdot, x_{n}\left(\theta_{n}(\cdot)\right)\right) \longrightarrow f_{1}(\cdot, x(\cdot)) \text { strongly in } L^{1}(I, H) . \\
y_{n}(\cdot) \longrightarrow y(\cdot) \text { strongly in } L^{1}(I, H) .
\end{gathered}
$$

This implies that

$$
\zeta_{n}(\cdot):=\dot{x}_{n}(\cdot)+f_{1}\left(\cdot, x_{n}\left(\theta_{n}(\cdot)\right)\right)+y_{n}(\cdot) \longrightarrow \zeta(\cdot):=\dot{x}(\cdot)+f_{1}(\cdot, x(\cdot))+y(\cdot)
$$

weakly in $L^{1}(I, H)$.
By Mazur's lemma we can find a convex combination $\sum_{k=n}^{r(n)} S_{k, n} \zeta_{k}(\cdot)$, with $\sum_{k=n}^{r(n)} S_{k, n}=1$ and $S_{k, n} \in[0,1]$ for all $k, n$, which converges strongly in $L^{1}(I, H)$ to $\zeta(\cdot)$. Extracting a subsequence, we may suppose that $\sum_{k=n}^{r(n)} S_{k, n} \zeta_{k}(\cdot)$ converges almost everywhere on $I$ to some mapping $\zeta(\cdot)$.
Further, we know that there is a negligible set $N \subset I$ such that for each $t \in I \backslash N$ one has for all $n \in \mathbb{N}$

$$
-\zeta_{n}(t):=-\dot{x}_{n}(t)-f_{1}\left(t, x_{n}\left(\theta_{n}(t)\right)\right)-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{n}\left(\theta_{n}(s)\right)\right) d s \in N_{C(t)}\left(x_{n}(t)\right)
$$

Fix any $t \in I \backslash N$ and any $n \in \mathbb{N}$. From Definition 2.1 of the normal cone, one has for every $z \in C(t)$

$$
\left\langle-\zeta_{n}(t), z-x_{n}(t)\right\rangle \leq \frac{\gamma(t)}{2 r}\left\|z-x_{n}(t)\right\|^{2} \text { for all } z \in C(t)
$$

hence

$$
\begin{equation*}
\left\langle-\zeta_{n}(t), z-x_{n}(t)\right\rangle \leq \frac{\gamma(t)}{2 r}\left(\|z-x(t)\|+\left\|x(t)-x_{n}(t)\right\|\right)^{2}:=\lambda_{n}(t) \tag{51}
\end{equation*}
$$

with $\lim _{n \longrightarrow \infty} \lambda_{n}(t)=\frac{\gamma(t)}{2 r}\|z-x(t)\|^{2}$. Therefore,

$$
\begin{aligned}
\langle-\zeta(t), z-x(t)\rangle & =\left\langle-\zeta(t)+\sum_{k=n}^{r(n)} S_{k, n} \zeta_{k}(t), z-x(t)\right\rangle+\sum_{k=n}^{r(n)} S_{k, n}\left\langle-\zeta_{k}(t), z-x_{k}(t)\right\rangle \\
& +\sum_{k=n}^{r(n)} S_{k, n}\left\langle-\zeta_{k}(t),-x(t)+x_{k}(t)\right\rangle .
\end{aligned}
$$

The first expression of the second member of the latter equality tends to zero by what precedes, and keeping in mind that $\left|\zeta_{k}(t)\right| \leq \gamma(t)$, we also see that the third expression tends to zero. Concerning the second expression, thanks to (51), it satisfies the estimate

$$
\sum_{k=n}^{r(n)} S_{k, n}\left\langle-\zeta_{k}(t), z-x_{k}(t)\right\rangle \leq \sum_{k=n}^{r(n)} S_{k, n} \lambda_{k}(t) .
$$

Thus, passing to the limit we obtain

$$
\langle-\zeta(t), z-x(t)\rangle \leq \frac{\gamma(t)}{2 r}\|z-x(t)\|^{2}, \quad \forall z \in C(t)
$$

This proves that

$$
-\dot{x}(t)-f_{1}(t, x(t))-\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \in N_{C(t)}(x(t)) \text {, a.e. } t \in\left[T_{0}, T\right],
$$

and thus

$$
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \text {, a.e. } t \in\left[T_{0}, T\right] .
$$

Now consider the situation when

$$
\int_{T_{0}}^{T}\left[\beta_{1}(\tau)+\int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s\right] d \tau \geq \frac{1}{4}
$$

We fix a subdivision of $\left[T_{0}, T\right]$ given by $T_{0}, T_{1}, \ldots, T_{k}=T$ such that, for any $0 \leq i \leq k-1$,

$$
\int_{T_{i}}^{T_{i+1}}\left[\beta_{1}(\tau)+\int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s\right] d \tau<\frac{1}{4}
$$

Then, by what precedes, there exists an absolutely continuous map $x_{0}:\left[T_{0}, T_{1}\right] \longrightarrow H$ such that $x_{0}\left(T_{0}\right)=x_{0}, x_{0}(t) \in C(t)$ for all $t \in\left[T_{0}, T_{1}\right]$, and

$$
-\dot{x}_{0}(t) \in N_{C(t)}\left(x_{0}(t)\right)+f_{1}\left(t, x_{0}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{0}(s)\right) d s \text {, a.e. } t \in\left[T_{0}, T_{1}\right] .
$$

Similarly, there is an absolutely continuous map $x_{1}:\left[T_{1}, T_{2}\right] \longrightarrow H$ such that $x_{1}\left(T_{1}\right)=x_{0}\left(T_{1}\right), x_{1}(t) \in C(t)$ for all $t \in\left[T_{1}, T_{2}\right]$, and

$$
-\dot{x}_{1}(t) \in N_{C(t)}\left(x_{1}(t)\right)+f_{1}\left(t, x_{1}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{1}(s)\right) d s, \text { a.e. } t \in\left[T_{1}, T_{2}\right] .
$$

By induction, we obtain for each $0 \leq i \leq k-1$ a finite sequence of absolutely continuous maps $x_{i}:\left[T_{i}, T_{i+1}\right] \longrightarrow H$ such that for each $0 \leq i \leq k-1, x_{i}\left(T_{i}\right)=x_{i-1}\left(T_{i}\right)$ and $x_{i}(t) \in C(t)$ for all $t \in\left[T_{i}, T_{i+1}\right]$, and

$$
-\dot{x}_{i}(t) \in N_{C(t)}\left(x_{i}(t)\right)+f_{1}\left(t, x_{i}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{i}(s)\right) d s \text {, a.e. } t \in\left[T_{i}, T_{i+1}\right] .
$$

We set $x_{-1}(0)=x_{0}$ and define the mapping $x:\left[T_{0}, T\right] \longrightarrow H$ given by

$$
x(t)=x_{i}(t) \text {, textif } t \in\left[T_{i}, T_{i+1}\right], \quad 0 \leq i \leq k-1 .
$$

Obviously, $x(\cdot)$ is an absolutely continuous mapping satisfying $x\left(T_{0}\right)=x_{0}, x(t) \in C(t)$ for all $t \in\left[T_{0}, T\right]$ and

$$
\begin{equation*}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s, \text { a.e. } t \in\left[T_{0}, T\right] \tag{52}
\end{equation*}
$$

## Step 7. We prove the estimations.

Let $x(\cdot)$ be a solution of $\left(P_{f_{1}, f_{2}}\right)$.
Take $N \subset\left[T_{0}, T\right]$ such that $\lambda(N)=0$ and the inclusion (52) holds for every $t \in\left[T_{0}, T\right] \backslash N$. Fix any $t \in\left[T_{0}, T\right] \backslash N$.

By definition of proximal normal cone, there is some real $a_{0}>0$ such that for any $a \in\left(0, a_{0}\right.$ ]

$$
x(t) \in \operatorname{Proj}_{C(t)}\left(x(t)-a \dot{x}(t)-a f_{1}(t, x(t))-a \int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right)
$$

We derive from the latter inclusion that

$$
\begin{aligned}
& a\left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\|=d_{C(t)}\left(x(t)-a \dot{x}(t)-a \varepsilon f_{1}(t, x(t))-a \int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right) \\
& \leq|v(t)-v(\tau)|+\left\|x(t)-x(\tau)-a \dot{x}(t)-a f_{1}(t, x(t))-a \int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\|
\end{aligned}
$$

since $x(\tau) \in C(\tau)$ for all $\tau \in\left[T_{0}, T\right]$. For any $\tau \in T_{0}, t\left[\right.$ with $t-a_{0}<\tau<t$, taking $a=t-\tau$ one obtains

$$
\begin{aligned}
& \left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\| \\
& \leq \frac{|v(t)-v(\tau)|}{t-\tau}+\left\|\frac{x(t)-x(\tau)}{t-\tau}-\dot{x}(t)-f_{1}(t, x(t))-\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\|
\end{aligned}
$$

Making $\tau \uparrow t$ yields

$$
\begin{align*}
\left\|\dot{x}(t)+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s\right\| & \leq\|\dot{v}(t) \mid+\|-f_{1}(t, x(t))-\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \| \\
& \leq|\dot{v}(t)|+\left\|f_{1}(t, x(t))\right\|+\int_{T_{0}}^{t}\left\|f_{2}(t, s, x(s))\right\| d s \tag{53}
\end{align*}
$$

This justifies (13).
Now assume

$$
\int_{T_{0}}^{t}\left[\beta_{1}(\tau)+\int_{T_{0}}^{\tau} \beta_{2}(\tau, s) d s\right] d \tau<\frac{1}{4}
$$

We have from (38), (39) and (40) that the estimates (14), (15) and (16) are obviously fulfilled.
If in addition

$$
\left\|f_{2}(t, s, x)\right\| \leq g(t, s)+\alpha(t)\|x\|
$$

we have from (53) that

$$
\begin{align*}
\|\dot{x}(t)\| & \leq|\dot{v}(t)|+2\left\|f_{1}(t, x(t))\right\|+2 \int_{T_{0}}^{t}\left\|f_{2}(t, s, x(s))\right\| d s \\
& \leq|\dot{v}(t)|+2 \beta_{1}(t)(1+\|x(t)\|)+2 \int_{T_{0}}^{t} g(t, s) d s+2 \alpha(t) \int_{T_{0}}^{t}\|x(s)\| d s \\
& =|\dot{v}(t)|+2 \beta_{1}(t)+2 \int_{T_{0}}^{t} g(t, s) d s+2 \beta_{1}(t)\|x(t)\|+2 \alpha(t) \int_{T_{0}}^{t}\|x(s)\| d s \tag{54}
\end{align*}
$$

Putting $\rho(t):=\left\|x_{0}\right\|+\int_{T_{0}}^{t}\|\dot{x}(s)\| d s$ and noting that $\|x(t)\| \leq \rho(t)$, the inequality (54) ensures that

$$
\dot{\rho}(t) \leq|\dot{v}(t)|+2 \beta_{1}(t)+2 \int_{T_{0}}^{t} g(t, s) d s+2 \beta_{1}(t) \rho(t)+2 \alpha(t) \int_{T_{0}}^{t} \rho(s) d s
$$

Applying Gronwall Lemma 3.2 with $\rho(\cdot)$, one obtains

$$
\begin{aligned}
\|x(t)\| \leq \rho(t) & \leq\left\|x_{0}\right\| \exp \left(\int_{T_{0}}^{t}(b(\tau)+1) d \tau\right) \\
& +\int_{T_{0}}^{t}\left(|\dot{v}(s)|+2 \beta_{1}(s)+2 \int_{T_{0}}^{s} g(s, \tau) d \tau\right) \exp \left(\int_{s}^{t}(b(\tau)+1) d \tau\right) d s
\end{aligned}
$$

where $b(\tau):=2 \max \left\{\beta_{1}(\tau), \alpha(\tau)\right\}$ for almost all $\tau \in\left[T_{0}, T\right]$. This yields the validity of (17), (18) and (19).

Step 8. Uniqueness.

Now, we turn to the uniqueness. If $x_{1}(\cdot), x_{2}(\cdot)$ are two solutions, the hypo-monotonicity property of the normal cone yields for almost all $t \in\left[T_{0}, T\right]$

$$
\begin{aligned}
& \left\langle-\dot{x}_{1}(t)-f_{1}\left(t, x_{1}(t)\right)-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{1}(s)\right) d s+\dot{x}_{2}(t)+f_{1}\left(t, x_{2}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{2}(s)\right) d s,\right. \\
& \left.x_{2}(t)-x_{1}(t)\right\rangle \leq \frac{1}{2 r}\left\|x_{2}(t)-x_{1}(t)\right\|^{2} \sum_{i=1}^{2}\left(\left\|\dot{x}_{i}(t)\right\|+\left\|f_{1}\left(t, x_{i}(t)\right)\right\|+\int_{T_{0}}^{t}\left\|f_{2}\left(s, x_{i}(s)\right)\right\| d s\right),
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
& \left\langle\dot{x}_{2}(t)-\dot{x}_{1}(t), x_{2}(t)-x_{1}(t)\right\rangle \\
& \leq \frac{1}{2 r}\left\|x_{2}(t)-x_{1}(t)\right\|^{2} \sum_{i=1}^{2}\left(\left\|\dot{x}_{i}(t)\right\|+\left\|f_{1}\left(t, x_{i}(t)\right)\right\|+\int_{T_{0}}^{t}\left\|f_{2}\left(t, s, x_{i}(s)\right)\right\| d s\right) \\
& +\left\langle f_{1}\left(t, x_{1}(t)\right)-f_{1}\left(t, x_{2}(t)\right), x_{2}(t)-x_{1}(t)\right\rangle \\
& +\left\langle\int_{T_{0}}^{t} f_{2}\left(t, s, x_{1}(s)\right) d s-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{2}(s)\right) d s, x_{2}(t)-x_{1}(t)\right\rangle .
\end{aligned}
$$

Since the absolutely continuous mappings $x_{1}(\cdot)$ and $x_{2}(\cdot)$ are in particular bounded on $\left[T_{0}, T\right]$, we can choose some real $\eta>0$ such that, for each $i=1,2,\left\|x_{i}(t)\right\| \leq \eta$ for all $t \in\left[T_{0}, T\right]$. The latter inequality assures us that

$$
\begin{aligned}
& \frac{d}{d t} \frac{1}{2}\left\|x_{2}(t)-x_{1}(t)\right\|^{2} \leq L_{2}^{\eta}(t)\left\|x_{2}(t)-x_{1}(t)\right\| \int_{T_{0}}^{t}\left\|x_{2}(s)-x_{1}(s)\right\| d s \\
& +\left(L_{1}^{\eta}(t)+\frac{1}{2 r} \sum_{i=1}^{2}\left(\left\|\dot{x}_{i}(t)\right\|+\left\|f_{1}\left(t, x_{i}(t)\right)\right\|+\int_{T_{0}}^{t}\left\|f_{2}\left(t, s, x_{i}(s)\right)\right\| d s\right)\right)\left\|x_{2}(t)-x_{1}(t)\right\|^{2} .
\end{aligned}
$$

Finally, setting $\rho(t):=\left\|x_{2}(t)-x_{1}(t)\right\|^{2}$ we get

$$
\begin{aligned}
\dot{\rho}(t) & \leq\left(2 L_{1}^{\eta}(t)+\frac{1}{r} \sum_{i=1}^{2}\left(\left\|\dot{x}_{i}(t)\right\|+\left\|f_{1}\left(t, x_{i}(t)\right)\right\|+\int_{T_{0}}^{t}\left\|f_{2}\left(t, s, x_{i}(s)\right)\right\| d s\right)\right) \rho(t) \\
& +2 L_{2}^{\eta}(t) \sqrt{\rho(t)} \int_{T_{0}}^{t} \sqrt{\rho(s)} d s
\end{aligned}
$$

hence it suffices to invole Lemma 3.3 with $\varepsilon(\cdot), \epsilon>0$ arbitrary. Then the proof of the theorem is complete .

Proposition 4.3 Assume that the assumptions of Theorem 4.2 (in case 3 ) holds. For each $a \in C\left(T_{0}\right)$, denote by $x_{a}(\cdot)$ the unique solution of the integro-differential sweeping process

$$
\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s \text { a.e in }\left[T_{0}, T\right] \\
x\left(T_{0}\right)=a \in C\left(T_{0}\right)
\end{array}\right.
$$

Then, the map $\psi: a \longrightarrow x_{a}(\cdot)$ from $C\left(T_{0}\right)$ to the space $C\left(\left[T_{0}, T\right], H\right)$ endowed with the uniform convergence norm is Lipschitz on any bounded subset of $C\left(T_{0}\right)$.

Proof. Let $M$ be any fixed positive real number. We are going to prove that $\psi$ is Lipschitz on $C\left(T_{0}\right) \cap M \mathbb{B}$.
According to Theorem 4.2 ( case 3 ), there exists a real number $M_{1}$ depending only on $M$ such that, for all $z \in C\left(T_{0}\right) \cap M \mathbb{B}$ and for almost all $(t, s) \in Q_{\Delta}$
$\left\|\dot{x}_{z}(t)+f_{1}\left(t, x_{z}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{z}(s)\right) d s\right\| \leq \alpha(t):=|\dot{v}(t)|+\left(1+M_{1}\right) \beta_{1}(t)+\int_{T_{0}}^{t} g(t, s) d s+T \alpha(t) M_{1}$,
Thanks to this last inequality, for some $\eta>0$ depending only on $M$, for all $z \in C\left(T_{0}\right) \cap M \mathbb{B}$ and for all $t \in\left[T_{0}, T\right]$, we have

$$
\begin{equation*}
x_{z}(t) \in B[0, \eta] . \tag{55}
\end{equation*}
$$

Fix any $a, b \in C\left(T_{0}\right) \cap M \mathbb{B}$. By the hypomonotonicity property of the normal cone, we have for almost all $(t, s) \in Q_{\Delta}$

$$
\begin{aligned}
& \left\langle-\dot{x}_{a}(t)-f_{1}\left(t, x_{a}(t)\right)-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{a}(s)\right) d s+\dot{x}_{b}(t)+f_{1}\left(t, x_{b}(t)\right)+\int_{T_{0}}^{t} f_{2}\left(t, s, x_{b}(s)\right) d s, x_{2}(t)-x_{1}(t)\right. \\
& \leq \frac{\alpha(t)}{r}\left\|x_{b}(t)-x_{a}(t)\right\|^{2}
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\left\langle\dot{x}_{b}(t)-\dot{x}_{a}(t), x_{b}(t)-x_{a}(t)\right\rangle & \leq \frac{\alpha(t)}{r}\left\|x_{b}(t)-x_{a}(t)\right\|^{2}+\left\langle f_{1}\left(t, x_{a}(t)\right)-f_{1}\left(t, x_{b}(t)\right), x_{b}(t)-x_{a}(t)\right\rangle \\
& +\left\langle\int_{T_{0}}^{t} f_{2}\left(t, s, x_{a}(s)\right) d s-\int_{T_{0}}^{t} f_{2}\left(t, s, x_{b}(s)\right) d s, x_{b}(t)-x_{a}(t)\right\rangle .
\end{aligned}
$$

Since, by assumption $\left(\mathcal{H}_{2,2}\right)$ and $\left(\mathcal{H}_{3,2}\right)$, there are a non-negative functions $L_{1}^{\eta}(\cdot), L_{2}^{\eta}(\cdot)$ $\in L^{1}\left(\left[T_{0}, T\right], \mathbb{R}\right)$ such that $f_{1}(t, \cdot)$ and $f_{2}(t, s, \cdot)$ are $L_{1}^{\eta}(\cdot)$-Lipschitz, $L_{2}^{\eta}(\cdot)$-Lipschitz (respectively) on $B[0, \eta]$, the above inequality along with (55), entails that for almost all $t \in\left[T_{0}, T\right]$,

$$
\frac{d}{d t}\left\|x_{b}(t)-x_{a}(t)\right\|^{2} \leq 2\left(L_{1}^{\eta}(t)+\frac{\alpha(t)}{r}\right)\left\|x_{b}(t)-x_{a}(t)\right\|^{2}+2 L_{2}^{\eta}(y)\left\|x_{b}(t)-x_{a}(t)\right\| \int_{T_{0}}^{t}\left\|x_{b}(s)-x_{a}(s)\right\| d s
$$

Applying Gronwall-like differential inequality in Lemma 3.3, it results that

$$
\sup _{t \in[0, T]}\left\|x_{b}(t)-x_{a}(t)\right\| \leq\|b-a\| \exp \left(\int_{T_{0}}^{t}(K(s)+1) d s\right),
$$

where $K(t):=\max \left\{L_{1}^{\eta}(t)+\frac{\alpha(t)}{r}, L_{2}^{\eta}(t)\right\}$, for almost all $t \in\left[T_{0}, T\right]$. The proof is then complete.

## 5 Nonlinear integro-differential complementarity systems

In this section, as a consequence of Theorem 4.2, we obtain the existence and uniqueness of solutions for nonlinear integro-differential complementarity systems. Our results generalize those from [3].

Let $T>T_{0}$ be real numbers, $I=\left[T_{0}, T\right], n, m \in \mathbb{N}, f_{1}: I \times \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}, f_{2}: Q_{\Delta} \times \mathbb{R}^{n} \longrightarrow$ $\mathbb{R}^{n}$ and $g: I \times \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m}$ be given mappings. Assuming that $g(t, \cdot)$ is differentiable for each $t \in I$, the NIDCS (associated with $f_{1}, f_{2}$ and $g$ ) can be described as

$$
(\text { NIDCS }):\left\{\begin{array}{l}
-\dot{x}(t)=f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s+\nabla g(t, \cdot)(x(t))^{T} z(t) \\
0 \leq z(t) \perp g(t, x) \leq 0,
\end{array}\right.
$$

where $z: I \longrightarrow \mathbb{R}^{m}$ is unknown mapping. The term $\nabla g(t, \cdot)(x(t))^{T} z(t)$ can be seen as the generalized reactions due to the constraints in mechanics.
Of course, the behaviour of a solution with respect to $t$ is connected to the variation with respect to $t$ of the set constraint

$$
\begin{equation*}
C(t)=\left\{x \in \mathbb{R}^{n}: g_{1}(t, x) \leq 0, g_{2}(t, x) \leq 0, \ldots, g_{m}(t, x) \leq 0\right\}, \tag{56}
\end{equation*}
$$

where we set $g(t, \cdot)=\left(g_{1}(t, \cdot), g_{2}(t, \cdot), \ldots, g_{m}(t, \cdot)\right)$ for each $t \in I$.
Theorem 5.1 [3] Let $C(t)$ be defined as in (56) and assume that, there exists an extended real $\rho \in] 0, \infty]$ such that

1. for all $t \in I$, for all $k \in\{1, \ldots, m\}, g_{k}(t, \cdot)$ is continuously differentiable on $U_{\rho}(C(t)):=$ $\left\{y \in \mathbb{R}^{n}: d_{C(t)}(y)<\rho\right\}$;
2. there exists a real $\gamma>0$ such that, for all $t \in I$, for all $k \in\{1, \ldots, m\}, g_{k}(t, \cdot)$, for all $x, y \in U_{\rho}(C(t))$

$$
\left\langle\nabla g_{k}(t, \cdot)(x)-\nabla g_{k}(t, \cdot)(y), x-y\right\rangle \geq-\gamma\|x-y\|^{2},
$$

that is, $\nabla g_{k}(t, \cdot)$ is $\gamma$-hypomonotone on $U_{\rho}(C(t))$;
3. there is a real $\delta>0$ such that for all $(t, x) \in I \times \mathbb{R}^{n}$ with $x \in \operatorname{bdry}(C(t))$, there exists $\bar{v} \in \mathbb{B}$ satisfying, for all $k \in\{1, \ldots, m\}$

$$
\begin{equation*}
\langle\nabla g(t, \cdot)(x), \bar{v}\rangle \leq-\delta . \tag{57}
\end{equation*}
$$

Then for all $t \in I$, the set $C(t)$ is $r$-prox-regular with $r=\min \left\{\rho, \frac{\delta}{\gamma}\right\}$.
The nonlinear differential complementarity systems (NDCS) (i.e., (NIDCS) with $f_{2} \equiv 0$ ) was studied in [3], where the authors transform the (NDCS) involving inequality constraints $C(t)$ to a perturbed sweeping process . We extend this approach by transforming (NIDCS) into an integro-differential sweeping process of the form (4). Also, in contrast to [3], we do not assume that the moving set $C(t)$ described by a finite number of inequalities is absolutely continuous with respect to the Hausdorff distance. Rather, we provide sufficient verifiable conditions ensuring this regularity needed on $C(\cdot)$.

Proposition 5.2 Let $C(t)$ be defined as in (56). Assume that there exist an absolutely continuous function $w$, a real $\delta>0$ and a vector $y \in \mathbb{R}^{n}$ with $\|y\|=1$ such that for each $i=1, \ldots m$

$$
\begin{gather*}
g_{i}(t, x) \leq g_{i}(s, x)+|w(t)-w(s)|, \quad \text { for all } x \in U_{r}(C(s)),  \tag{58}\\
\left\langle\nabla g_{i}(t, \cdot)(x), y\right\rangle \leq-\delta, \text { for, all } t \in I, x \in U_{r}(C(t)), \tag{59}
\end{gather*}
$$

where $r$ denotes the prox-regularity constant of all sets $C(t)$. Then $C(\cdot)$ is $v(\cdot)-$ absolutely continuous on I with $v(\cdot):=\delta^{-1} w(\cdot)$.

Proof. Let $s, t \in I$, let $x \in C(s)$ and choose a subdivision $T_{0}<T_{1}<\ldots<T_{p}=T$ such that $\int_{T_{k-1}}^{T_{k}}|\dot{v}(\tau)| d \tau<r$ for every $k=1, \cdots, p$. Fix any $k=1, \ldots, p$ and $s, t \in\left[T_{k-1}, T_{k}\right]$. Take any $i=1, \ldots, m$ and note that

$$
\begin{align*}
g_{i}(t, x+|v(t)-v(s)| y) & =\left(g_{i}(t, x+|v(t)-v(s)| y)-g_{i}(s, x+|v(t)-v(s)| y)\right) \\
& +g_{i}(s, x+|v(t)-v(s)| y) \\
& \leq|w(t)-w(s)|+g_{i}(s, x+|v(t)-v(s)| y) \\
& =|w(t)-w(s)|+g_{i}(s, x) \\
& +\int_{0}^{1}\left\langle\nabla_{2} g_{i}(s, x+\theta y|v(t)-v(s)|), y\right| v(t)-v(s)| \rangle d \theta . \tag{60}
\end{align*}
$$

According to 59) and to the inclusion $x \in C(s)$ it ensues that

$$
g_{i}(t, x+|v(t)-v(s)| y) \leq|w(t)-w(s)|-\delta|v(t)-v(s)| \leq 0 .
$$

This being true for every $i=1, \ldots, m$, it follows that $x+|v(t)-v(s)| y$ belongs to $C(t)$, otherwise stated, $x \in C(t)+|v(t)-v(s)|(-y)$. It results that $C(s) \subset C(t)+|v(t)-v(s)| \mathbb{B}$. Since the variables s and t play symmetric roles, the set-valued mapping $C(\cdot)$ has an absolutely continuous variation on $\left[T_{k-1}, T_{k}\right]$. From this we clearly derive that $C(\cdot)$ has an absolutely continuous variation on $I$.

Example 5.3 Let $m=1, n=2, T=1, g(t, x)=t^{\frac{1}{3}}-x_{1}-x_{2}^{2}$, and define

$$
C(t)=\left\{x \in \mathbb{R}^{2}: g(t, x) \leq 0\right\} .
$$

Clearly that $C(t)$ is $r$-prox-regular, since $g(t, \cdot)$ satisfies all assumptions of theorem 5.1 for all $t \in I$. Now we check 58) and 59). Let $x \in \mathbb{R}^{2}, t, s \in I$. Fix any $\delta \in(0,1]$ and put $y=(1,0)$. Then

$$
\begin{gathered}
g(t, x)-g(s, x)=t^{\frac{1}{3}}-s^{\frac{1}{3}} \leq\left|t^{\frac{1}{3}}-s^{\frac{1}{3}}\right|=|w(t)-w(s)|, \\
\langle\nabla g(t, \cdot)(x), y\rangle=-1 \leq-\delta .
\end{gathered}
$$

We see that $w(t)=t^{1 / 3}$ is not Lipschitz on I but it is absolutely continuous there. Then $C(\cdot)$ has an absolutely continuous variation $v$ on $I$, with $v(t)=t^{1 / 3} / \delta$.

Theorem 5.4 Assume that the assumptions in Theorem 5.1, Proposition 5.2 and conditions $\left(\mathcal{H}_{2}\right)$, $\left(\mathcal{H}_{3}\right)$ are satisfied. Then, for every initial data $x_{0}$ with $g\left(0, x_{0}\right) \leq 0$, problem (NIDCS) has one and only one solution $x(\cdot)$.

Proof. In the same arguments like in [3] one has the equivalent between the problem (NIDCS) and the integro-differential sweeping process

$$
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{T_{0}}^{t} f_{2}(t, s, x(s)) d s
$$

Therefore, all assumptions of Theorem 4.2 are satisfied and the conclusion follows.

## 6 Applications to non-regular electrical circuits

The aim of this section is to illustrate the integro-differential sweeping process in the theory of non-regular electrical circuits. Electrical devices like diodes are described in terms of Ampere-Volt characteristic which is (possibly) a multifunction expressing the difference of potential $v_{D}$ across the device as a function of current $i_{D}$ going through the device 5 .
Let us consider the electrical system shown in Fig. 1 that is composed of three resistors $R_{1} \geq 0, R_{2} \geq 0$ with voltage/current laws $V_{R_{k}}=R_{k} x_{k}(k=1,2)$, two inductors $L_{1} \geq 0$, $L_{2} \geq 0$ with voltage/current laws $V_{L_{k}}=R_{k} \dot{x}_{k}(k=1,2)$, three capacitors with a timevarying capacitances $C_{1}(t) \neq 0, C_{2}(t) \neq 0$ and $C_{3}(t) \neq 0$ with voltage/current laws $V_{C_{k}}=\frac{1}{C_{k}(t)} \int x_{k}(t) d t, k=1,2,3$, two ideal diodes with characteristics $0 \leq-V_{D_{k}} \perp i_{k} \geq 0$ and an absolutely continuous current source $i:[0, T] \rightarrow R$.

Using Kirchhoff's laws, we have

$$
\left\{\begin{array}{l}
V_{R_{1}}+V_{R_{2}}+V_{L_{1}}+V_{C_{1}}+V_{C_{3}}=-V_{D_{1}} \in-N\left(\mathbb{R}_{+} ; x_{1}-i\right) \\
V_{R_{1}}-V_{R_{2}}+V_{L_{2}}+V_{C_{2}}-V_{C_{3}}=-V_{D_{2}} \in-N\left(\mathbb{R}_{+} ; x_{2}\right) .
\end{array}\right.
$$



Figure 1: Electrical circuit with resistors, inductances, time-varying capacitors and ideal diodes.

Therefore the dynamics of this circuit is given by

$$
\begin{align*}
& \overbrace{\binom{-\dot{x}_{1}(t)}{-\dot{x}_{2}(t)}}^{-\dot{x}(t)} \in N_{[i(t),+\infty[\times[0,+\infty[ }(x(t))+\overbrace{\left(\begin{array}{cc}
\frac{R_{1}+R_{2}}{L_{1_{2}}} & -\frac{R_{2}}{L_{k_{2}}} \\
-\frac{R_{2}}{L_{2}} & \frac{R_{1}+R_{2}}{L_{2}}
\end{array}\right)}^{A_{1}} \overbrace{\binom{x_{1}(t)}{x_{2}(t)}}^{x(t)} \\
& +\int_{0}^{t} \overbrace{[\begin{array}{cc}
\frac{1}{L_{1} C_{1}(t)}+\frac{1}{L_{1} C_{3}(t)} & -\frac{1}{L_{2} C_{3}(t)}
\end{array} \overbrace{\frac{1}{L_{1} C_{3}(t)}}^{L_{2} C_{2}(t)}+\frac{1}{L_{2} C_{3}(t)}}{ }^{2} \overbrace{\binom{x_{1}(s)}{x_{2}(s)}}^{A_{2}}+\overbrace{\binom{\frac{1}{L_{1} C_{1}(t)} i(s)}{0}}^{x(s))}] d s . \tag{61}
\end{align*}
$$

Proposition 6.1 Assume that $i:[0, T] \longrightarrow \mathbb{R}$ is an absolutely continuous function and $C_{k}:[0, T] \longrightarrow \mathbb{R}^{*}, k=1,2,3$ are continuous functions. Then for any initial condition $x(0)=x_{0} \in C(0)$, problem (61) has one and only one absolutely continuous solution $x(\cdot)$.

Proof. Put $w(t)=(i(t), 0)^{t}, C(t):=w(t)+\left[0,+\infty\left[\times\left[0,+\infty\left[, f_{1}(t, x)=A_{1} x\right.\right.\right.\right.$, $f_{2}(t, s, x)=A_{2}(t) x+\frac{1}{L_{1} C_{1}(t)} w(s)$. So can be rewritten in the frame of our problem $\left(P_{f_{1}, f_{2}}\right)$ as

$$
\left\{\begin{array}{l}
-\dot{x}(t) \in N_{C(t)}(x(t))+f_{1}(t, x(t))+\int_{0}^{t} f_{2}(t, s, x(s)) d \text { d a.e. in }[0, T] \\
x(0)=x_{0} \in C(0)
\end{array}\right.
$$

Then the above data satisfying all the assumptions of Theorem 4.2 ( precisely case 3 ),
with

$$
v(t)=\int_{0}^{t}\|\dot{w}(s)\| d s, \quad \beta_{1}(t)=\left\|A_{1}\right\|, g(t, s)=\frac{1}{L_{1} C_{1}(t)}\|w(s)\|, \quad \alpha_{2}(t)=\left\|A_{2}(t)\right\| .
$$

This finishes the proof.

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