Robust market-adjusted systemic risk measures

Matteo Burzoni, Marco Frittelli, Federico Zorzi*

August 19, 2021

Abstract

In this note we consider a system of financial institutions and study systemic risk measures in the presence of a financial market and in a robust setting, namely, where no reference probability is assigned. We obtain a dual representation for convex robust systemic risk measures adjusted to the financial market and show its relation to some appropriate no-arbitrage conditions.

1 Introduction

In a system composed of N financial institutions, a traditional approach to evaluate the risk of each institution $j \in \{1, \ldots, N\}$ is to apply a *univariate monetary risk measure* η^j to the single financial position X^j. Once the risk $\eta^j(X^j)$ of each institution has been determined, a naive assessment of the risk of the entire system $X = (X^1, \ldots, X^N)$ could be given as the sum of the individual risks. However, such a procedure would probably not capture the risk of complex systems and the urge for more satisfactory measures of systemic risk originated, in the recent years, a vast literature. [8] and [13] studied under which conditions a systemic risk measure ρ could be written in the form

$$\rho(X) = \eta(\Lambda(X)) = \inf\{m \in \mathbb{R} \mid \Lambda(X) + m \in \mathcal{A}\}, \tag{1}$$

for some univariate monetary risk measure η with acceptance set $\mathcal{A}$ and some aggregation rule $\Lambda : \mathbb{R}^N \to \mathbb{R}$ that transforms the N-dimensional risk factors into a univariate risk factor. In this case, $\rho(X)$ is the minimal cash amount that secures the system when it is added to the total aggregated loss $\Lambda(X)$. Note that in (1) such a minimal capital is added *after aggregating individual risks.* An alternative approach, see [3, 5, 10], proposes to add capital into the single institutions *before aggregating their individual risks* leading to risk measures of the form:

$$\rho(X) := \inf\left\{\sum_{j=1}^{N} m^j \mid m = [m^1, \cdots, m^N] \in \mathbb{R}^N,\, \Lambda(X+m) \in \mathcal{A}\right\}. \tag{2}$$

As one can see from (2), the difference to (1) is that each $m^j \in \mathbb{R}$ is added to the financial position X^j of institution $j \in \{1, \cdots, N\}$ before the corresponding aggregated loss $\Lambda(X + m)$ is calculated. We refer the interested reader to [5] for more references on systemic risk measures. We point

*email: matteo.burzoni@unimi.it, marco.frittelli@unimi.it, federicozorzi@outlook.it

out that in the above literature $X = (X^1, \dots, X^N)$ is a vector of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consequently the acceptance set $\mathcal{A}$ is a subset of $L^0(\Omega, \mathcal{F}, \mathbb{P})$.

In this paper we depart from this literature in two respects. First, the agents are allowed to operate in a financial market composed of $J+1$ assets $S^0, S^1, \dots, S^{\mathrm{J}}$ and finitely many trading periods $t = 0, \dots, T-1$. We make use of an abstract set $\mathcal{G}$ to describe all possible positions that are achievable by self-financing trading strategies with zero initial cost. Second, no assumptions are made on the probabilities of future events. The sample space is a non-empty subset $\Omega \subseteq ((0, +\infty) \times \mathbb{R}^J)^T$, endowed with the usual Euclidean metric. This approach is robust in the sense that we do not impose a priori any statistical/historical probability measure on Ω but we rather work in a pointwise manner.

The financial position of the $N \in \mathbb{N}$ agents or financial institutions is represented by $X = (X^1, \dots, X^N) \in \mathcal{B}$, where $\mathcal{B} := \mathcal{B}(\mathbb{R}^N)$ is the set of all Borel measurable functions $\Omega \to \mathbb{R}^N$. Note that we also assume that $\mathcal{G}$ is contained in $\mathcal{B}$, namely $\mathcal{G}$ is a set of *vectors*. This allows us to model the case where the agents cannot achieve the same class of terminal payoffs or the case where they even trade in different markets. We are interested in the risk of the entire system that we evaluate in terms of an aggregate univariate position. To achieve this we consider an acceptance set $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R})$ and an aggregation function $\Lambda : \mathbb{R}^N \to \mathbb{R}$. We then evaluate the risk of the financial system by means of the following functional $\rho : \mathcal{B} \to [-\infty, +\infty]$

$$\rho(X) := \inf \left\{ \sum_{i=1}^N m^i \mid m \in \mathbb{R}^N,\ \exists g \in \mathcal{G} : \Lambda(m + X + g) \in \mathcal{A} \right\}, \tag{3}$$

with $\rho(X) = \infty$ if the set on the right hand side is empty. This risk measure (3) is *market-adjusted*, meaning that every agent is allowed to trade in the underlying market, in a self-financing way and according to achievable payoffs, in order to obtain an acceptable aggregate terminal position. We observe that this measure of risk, that we label of the type *first allocate and adjust, then aggregate*, is in the same spirit of the risk measures (2), even though it has the additional market adjustment feature and is specified in a robust framework.

Our aim is to prove a dual representation for the systemic risk measure (3) *and to understand its interplay with possible notions of arbitrage (see Theorem 2.5).*

To develop this theory, we will follow the same approach that [9] adopted for the analysis of the robust pricing-hedging duality in one dimension and extend it to the present multivariate (systemic) setting. We address this problem and make precise statements in Section 2. We refer the interested reader to [9] for more references on robustness in a non systemic framework.

An alternative way of measuring market-adjusted systemic risk employs the use of a second aggregation function $\Gamma : \mathbb{R}^N \to \mathbb{R}$ for the payoffs of the trading strategies, i.e., by means of the following functional $\rho_\Gamma : \mathcal{B} \to [-\infty, +\infty]$

$$\rho_\Gamma(X) := \inf \left\{ \sum_{i=1}^N m^i \mid m \in \mathbb{R}^N,\ \exists g \in \mathcal{G} : \Lambda(m + X) + \Gamma(g) \in \mathcal{A} \right\}. \tag{4}$$

The interpretation is similar to the one above but it is different in spirit. In (3) the agents are operating as N different units both in terms of the financial position X and market payoff g. In (4) the acceptability of the aggregate position $\Lambda(m + X)$ can be influenced by an aggregate market payoff $\Gamma(g)$. We can think of (4) as the risk metric of a single firm composed of N different units and a trading desk operating independently of the N units[1]. The total risk of the firm is assessed by aggregating the static financial position of the firm and the market position separately.

From a mathematical point of view there is very little difference in treating the two cases and we present analogous results in Section 2.1.

2 The Main Results

Notation 2.1. *We let $\mathbf{1} := (1, \dots, 1)$ be the N-dimensional vector with entries all equal to 1, so that, if x is a univariate variable, $\mathbf{x} := x\mathbf{1} = (x, \dots, x)$ is the N-dimensional vector with all components equals to x. When comparing multivariate positions, all the inequalities are to be intended componentwise, in particular, $\mathcal{B}^+ = \mathcal{B}^+(\mathbb{R}^N)$ is the set of functions in $\mathcal{B} = \mathcal{B}(\mathbb{R}^N)$ with values in $[0, +\infty)^N$. A set $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R})$ is called monotone if $x \geq y \in \mathcal{A} \Rightarrow x \in \mathcal{A}$.*

Unless otherwise specified, in the remainder of the paper the following assumption holds true.

Assumption 2.2.

1. *$\mathcal{A} \subseteq \mathcal{B}(\mathbb{R})$ is monotone and $0 \in \mathcal{A}$; $\mathcal{G} \subseteq \mathcal{B}$ with $\mathbf{0} \in \mathcal{G}$.*
2. *The aggregation function $\Lambda : \mathbb{R}^N \to \mathbb{R}$ is increasing, with respect to the componentwise order, concave and $\Lambda(\mathbf{0}) = 0$.*
3. *The set $\Lambda^{-1}(\mathcal{A}) - \mathcal{G} = \{X \in \mathcal{B} \mid \Lambda(X + g) \in \mathcal{A}$ for some $g \in \mathcal{G}\}$ is convex.*

The monotonicity of the acceptance set $\mathcal{A}$ is standard in the context of univariate risk measures and the conditions on Λ are also typical in the theory of multivariate risk measures. Notice that the aggregation function is not required to be strictly increasing nor strictly concave. Given the first two, the third condition holds when both $\mathcal{A}$ and $\mathcal{G}$ are convex.

Example 2.3. *Consider concave increasing functions $u, u_i : \mathbb{R} \to \mathbb{R}$ satisfying $u(0) = u_i(0) = 0 \ \forall i \in \{1, \dots, N\}$. The aggregation functions*

$$\Lambda(x) \quad : \quad = \alpha u\left(\sum_{i=1}^N x^i\right) + \sum_{i=1}^N \alpha_i u_i(x^i), \ \text{for } \alpha, \alpha_i \geq 0 \ \forall i \in \{1, \dots, N\}, \tag{5}$$

$$\Lambda(x) \quad : \quad = -\sum_{i=1}^N \alpha_i (x^i)^-, \ \text{for } \alpha_i \geq 0 \ \forall i \in \{1, \dots, N\}, \tag{6}$$

$$\Lambda(x) \quad : \quad = \max_{y \in \mathbb{R}^N_-, b \in \mathbb{R}^N_- \ : \ x_i \geq b_i + y_i - \sum_{j=1}^N \Pi_{ji} y_j} \left\{\sum_{i=1}^N y_i + \gamma \sum_{i=1}^N b_i\right\}, \ \text{for } \gamma > 1 \tag{7}$$

satisfy Assumption 2.2. The function in (5) has been frequently used in the literature on systemic risk measures with either $\alpha = 0$ or $\alpha_i = 0 \ \forall i \in \{1, \dots, N\}$, see e.g. [3, 5]. The function in (6) corresponds to considering the aggregate position as the sum of the debts of the single units, if $\alpha_i = 1 \ \forall i \in \{1, \dots, N\}$. Finally, the function in (7) is derived from a network model where Π_{ji} is the fraction of the total debt of firm j owed to firm i and $\gamma > 1$ is a parameter balancing the trade off between capital injection and reduction of mutual debts, see [8] for more details and examples.

[1] For ease of notation we continue to assume that g and X have the same dimension but, in principle, they could now be different.

We now introduce the functional analytic setting that allows us to prove a robust dual representation for ρ. Let $Z : \Omega \to [1, +\infty)$ be a continuous function with compact sublevel sets $\{\omega \in \Omega : Z(\omega) \leq z\}$ for all $z \in \mathbb{R}$. Let $\mathcal{B}_Z$ be the set of functions $X = (X^1, \dots, X^N) \in \mathcal{B}$ such that $\frac{X^i}{Z}$ is bounded for all $i = 1, \dots, N$. The set of continuous functions in $\mathcal{B}_Z$ is called C_Z, while U_Z is the set of upper semicontinuous functions in $\mathcal{B}_Z$. Their univariate counterparts are $\mathcal{B}_Z(\mathbb{R}), C_Z(\mathbb{R}), U_Z(\mathbb{R})$. We let ca_Z be the space of N-dimensional vectors of Borel measures $\mu = (\mu^1, \dots, \mu^N)$ such that $\int_\Omega Z d\mu^i < +\infty$, for every $i = 1, \dots, N$. We finally form a dual pair $(\mathcal{B}_Z, ca_Z, \langle \ , \ \rangle)$ with

$$\langle X, \mu \rangle := \sum_{i=1}^{N} \int_\Omega X^i d\mu^i, \qquad X \in \mathcal{B}_Z,\ \mu \in ca_Z.$$

We let ca_Z^+ be the positive cone in ca_Z and observe that ca_Z^+ contains the subset $\mathcal{P}_Z$ of N-dimensional vectors of probability measures $\mathbb{P} = (\mathbb{P}^1, \dots, \mathbb{P}^N)$ such that $\mathbb{E}^{\mathbb{P}^i} Z < +\infty$ for $i = 1 \dots, N$. For a functional f on C_Z we define

$$f^*(\mu) := \sup_{X \in C_Z} \{\langle X, \mu \rangle - f(X)\}, \qquad \mu \in ca_Z, \tag{8}$$

which is the convex conjugate of f with respect to the dual system (C_Z, ca_Z).

In Theorem 2.5 below, we prove that ρ admits a dual representation if and only if a certain no arbitrage condition holds. The theorem holds under the following assumption, which is essentially requiring that the set of achievable market payoffs $\mathcal{G}$ is rich enough.

Assumption 2.4. *There exists $\gamma \leq 0$ such that*

$$\forall n \in \mathbb{N},\ \exists z \in [0, +\infty),\ \exists g \in \mathcal{G} \text{ such that } \Lambda\left(\left[\frac{\gamma}{N} + \frac{1}{n} - n(Z - z)^+\right] \mathbf{1} + g\right) \in \mathcal{A}. \tag{A}$$

This condition is a multivariate version of condition (2.1) in [9]; it is satisfied for example when Ω is compact or when, in the market described by $\mathcal{G}$, options which are sufficiently out-of-the-money are available at a sufficiently small price. We give some other sufficient conditions in Proposition 2.8 below. By definition of ρ, condition (A) implies that for all $n \in \mathbb{N}$ there exists $z_n \in \mathbb{R}_+$ such that $\rho(-n(Z - z_n)^+ \mathbf{1}) \leq \frac{N}{n} + \gamma$.

Theorem 2.5. *Under Assumption 2.4 the following are equivalent:*

1. $m \in \mathbb{R}^N$, $\sum m^i < \gamma \Rightarrow \nexists g \in \mathcal{G} : \Lambda(m + g) \in \mathcal{A}$.
2. *There exists* $\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_N) \in \mathcal{P}_Z$ *such that* $\sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[X^i] - \gamma \geq 0$ *for all* $X \in C_Z$ *satisfying* $\Lambda(X + g) \in \mathcal{A}$ *for some* $g \in \mathcal{G}$.
3. ρ *is real valued on* $\mathcal{B}_Z$, $\rho(0) = \gamma$ *and*

$$\rho(X) = \max_{\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_N) \in \mathcal{P}_Z} \left\{ \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}_i}[-X^i] - \rho^*(-\mathbb{Q}) \right\}, \quad X \in C_Z.$$

If in addition one has

$$\rho(X) = \inf_{Y \in C_Z, Y \leq X} \rho(Y) \text{ for all } X \in U_Z, \tag{9}$$

then 1–3 are also equivalent to each one of the following two conditions:

4. *There exists* $\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_N) \in \mathcal{P}_Z$ *such that* $\sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[X^i] - \gamma \geq 0$ *for all* $X \in U_Z$ *satisfying* $\Lambda(X+g) \in \mathcal{A}$ *for some* $g \in \mathcal{G}$.

5. ρ *is real valued on* $\mathcal{B}_Z$, $\rho(0) = \gamma$ *and*

$$\rho(X) = \max_{\mathbb{Q}=(\mathbb{Q}_1,\dots,\mathbb{Q}_N)\in\mathcal{P}_Z} \left\{ \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[-X^i] - \rho^*(-\mathbb{Q}) \right\}, \quad X \in U_Z.$$

Before giving the proof of Theorem 2.5, whose technical parts are postponed to Section 3, we comment on its statement. Recall that $\gamma \leq 0$ is given and fixed.

Condition 1 excludes a situation that we call a *regulatory arbitrage opportunity*, namely a situation where it is possible to make an initial position $m \in \mathbb{R}^N$ such that $\sum_{i=1}^N m^i < \gamma$ acceptable by simply adding an achievable payoff $g \in \mathcal{G}$ that is obtained by trading at zero cost in the financial market. In particular, absence of regulatory arbitrage opportunities implies that none of the N agents can achieve a *(model independent) market arbitrage opportunity*, namely, that the set of achievable payoffs $\mathcal{G}$ cannot contain elements of the form $(0, \cdots, g^i, \cdots, 0)$ with $g^i(\omega) \geq \varepsilon > 0$ for every $\omega \in \Omega$.

Condition 2 provides information regarding the existence of an evaluation measure $\mathbb{Q}$. If it is possible to make a position $X \in C_Z$ acceptable by adding an achievable payoff $g \in \mathcal{G}$, then the evaluation of X given by $\mathbb{Q}$ (and adjusted by γ) must be non negative. In particular, for the case $\gamma = 0$ such an evaluation of X must be non-negative without adjustments. When $\mathcal{G}$ is a linear space of functions in C_Z, we can write $\Lambda(kg + (-kg)) = \Lambda(0) = 0 \in \mathcal{A}$ for any $k \in \mathbb{R}$ and $g \in \mathcal{G} \cap C_Z$. Then, condition 2 implies that

$$\sum \mathbb{E}^{\mathbb{Q}^i}[g^i] = 0 \text{ for all } g \in \mathcal{G} \cap C_Z.$$

Using the terminology of [6], the probability vector $\mathbb{Q}$ is called *fair*. If, in addition, $\mathcal{G}$ contains vectors with only one non-zero components then $\mathbb{Q}$ is a vector of martingale measures, i.e., $\mathbb{E}_{\mathbb{Q}^i}[g^i] = 0$ for every $(0, \cdots, g^i, \cdots, 0) \in \mathcal{G}$.

Finally, condition 3 is the usual dual representation of the Fenchel-Moreau type. Following the original interpretation of [4] each $\mathbb{Q} \in \mathcal{P}_Z$ is a plausible model for the risk X and it is called a *generalized scenario*. The term $\rho^*(-\mathbb{Q})$ has the role to penalize scenarios which are less plausible, possibly by an infinite amount. The value $\rho(X)$ is then the worst-case expectation across all plausible scenarios, suitably penalized. We refer to [11, Chapter 4] and [14, Chapter 8], for more details on the relevance of the dual representation for the univariate case and to [1] and [2] for a comprehensive study of dual representations for systemic risk measures in the non-robust setting.

Remark 2.6. Using an argument similar to [12, Proposition 3.9 and 3.11], the dual representation in item 3 could be reformulated as

$$\rho(X) = \max_{\mathbb{Q}=(\mathbb{Q}_1,\dots,\mathbb{Q}_N)\in\mathcal{P}_Z\cap Bar_{\Lambda^{-1}(\mathcal{A})}\cap(-Bar_{\mathcal{G}})} \left\{ \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[-X^i] - \sigma_{\Lambda^{-1}(\mathcal{A})}(\mathbb{Q}) - \sigma_{\mathcal{G}}(-\mathbb{Q}) \right\}, \quad X \in C_Z,$$

where $Bar_{\mathbb{A}} := \left\{ \mathbb{Q} \in ca_Z^+ \mid \sigma_{\mathbb{A}}(\mathbb{Q}) < +\infty \right\}$ is the domain of finiteness of the support function $\sigma_{\mathbb{A}}(\mathbb{Q}) := \sup_{W\in\mathbb{A}} \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[-W^i]$ of a set $\mathbb{A} \subseteq \mathcal{B}(\mathbb{R}^N)$. Such a formula emphasizes the role of the defining ingredients of ρ in the penalty function ρ^*. A thorough and extensive analysis - in the non robust setting - of such decomposition of ρ^* can be found in [2, Section 3.3]

We next prove Theorem 2.5. On the technical side we need to adapt some results of [9] to the multivariate case and we provide them in Section 3. Note that, differently from [9] we allow $\mathcal{A}$ to contain purely negative acceptable position. This is important in the context of systemic risk in order to include examples of aggregation functions which always yield non-positive random variables, as in Example 2.3 equation (6). We start with an easy observation.

Lemma 2.7. *The map $\rho : \mathcal{B} \to [-\infty, +\infty]$ defined in* (3) *is monotone decreasing, convex and (systemically) cash additive, namely*

$$\rho(X + c) = \rho(X) - \sum_{i=1}^{N} c^i, \text{ for all } X \in \mathcal{B} \text{ and } c \in \mathbb{R}^N.$$

Proof. Since Λ is increasing and $\mathcal{A}$ is monotone, the set $\Lambda^{-1}(\mathcal{A}) - \mathcal{G}$ is monotone and then monotonicity of ρ is easily checked. Regarding convexity, for $m, n \in \mathbb{R}^N$ such that $m + X,\ n + Y \in \Lambda^{-1}(\mathcal{A}) - \mathcal{G}$, one gets $\lambda m + (1-\lambda)n + (\lambda X + (1-\lambda)Y) \in \Lambda^{-1}(\mathcal{A}) - \mathcal{G}$, for all $\lambda \in [0,1]$, by the convexity of $\Lambda^{-1}(\mathcal{A}) - \mathcal{G}$. Hence,

$$\rho\left(\lambda X + (1-\lambda)Y\right) \leq \lambda \sum_{i=1}^{N} m^i + (1-\lambda) \sum_{i=1}^{N} n^i$$

and convexity now follows by taking the infimum over m and n satisfying $m+X, n+Y \in \Lambda^{-1}(\mathcal{A}) - \mathcal{G}$ on the right-hand side. The cash additivity property is trivial. $\square$

Proof of Theorem 2.5. $3 \Longrightarrow 2$. Let $X \in C_Z \cap (\Lambda^{-1}(\mathcal{A}) - \mathcal{G})$. As $\Lambda(X + g) \in \mathcal{A}$ for some $g \in \mathcal{G}$, the definition of ρ implies that $\rho(X) \leq 0$. By the dual formula for ρ in item 3, one obtains

$$\gamma = \rho(0) = \max_{\mathbb{Q} \in \mathcal{P}_Z} -\rho^*(-\mathbb{Q}) = -\min_{\mathbb{Q} \in \mathcal{P}_Z} \rho^*(-\mathbb{Q}),$$

implying the existence of $\hat{\mathbb{Q}} \in \mathcal{P}_Z$ such that $\rho^*(-\hat{\mathbb{Q}}) = -\gamma$. Using the definition of ρ^* and $\rho(X) \leq 0$, we have

$$\begin{aligned} -\gamma = \rho^*(-\hat{\mathbb{Q}}) &\geq \sup_{X \in C_Z \cap (\Lambda^{-1}(\mathcal{A}) - \mathcal{G})} \left\{ \sum_{i=1}^{N} \mathbb{E}^{\hat{\mathbb{Q}}^i}[-X^i] - \rho(X) \right\} \\ &\geq \sup_{X \in C_Z \cap (\Lambda^{-1}(\mathcal{A}) - \mathcal{G})} \left\{ \sum_{i=1}^{N} \mathbb{E}^{\hat{\mathbb{Q}}^i}[-X^i] \right\}, \end{aligned}$$

from which item 2 follows readily.

$2 \Longrightarrow 1$. Let $m \in \mathbb{R}^N$, $g \in \mathcal{G}$ such that $\Lambda(m + g) \in \mathcal{A}$. By item 2, there exists $\mathbb{Q} \in \mathcal{P}_Z$ such that $\sum_{i=1}^N m^i - \gamma = \sum_{i=1}^N E^{\mathbb{Q}_i}[m^i] - \gamma \geq 0$, from which item 1 follows.

$1 \Longrightarrow 3$. Set $\Phi(X) := \rho(-X)$ and notice that Φ is monotone increasing and convex. By the cash additivity property of Lemma 2.7, $\Phi(m) = \Phi(0) + \sum_{i=1}^N m^i$, for all $m \in \mathbb{R}^N$. We now show that $\Phi(0) = \gamma$. By item 1, if $m \in \mathbb{R}^N$ and $\Lambda(m+g) \in \mathcal{A}$ then $\sum m^i \geq \gamma$, which implies $\Phi(0) = \rho(0) \geq \gamma$. Moreover, by Condition (A), for all $n \in \mathbb{N}$ and $z = z(n)$ large enough the following holds

$$\gamma \leq \Phi(0) \leq \Phi\left(n(Z - z)^+ \mathbf{1}\right) \leq \frac{N}{n} + \gamma.$$

This implies $\rho(0) = \Phi(0) = \gamma$. We now show that Φ is real valued on $\mathcal{B}_Z$. Let $X \in \mathcal{B}_Z$ and $k \in \mathbb{N}$ such that $-\frac{1}{2}kZ\mathbf{1} \leq X \leq \frac{1}{2}kZ\mathbf{1}$. Using (A), there exists $z = z(k) \in \mathbb{R}_+$ large enough such that

$\Phi(k(Z-z)^{+}\mathbf{1}) \leq \frac{N}{k} + \gamma < \infty$. By cash additivity, $\Phi(kz\mathbf{1}) = Nkz < \infty$. By monotonicity and convexity of Φ we then deduce

$$\begin{aligned}\Phi(X) &\leq \Phi\left(\frac{k}{2}Z\mathbf{1}\right) = \Phi\left(\frac{1}{2}k(Z-z)\mathbf{1} + \frac{1}{2}kz\mathbf{1}\right) \leq \frac{1}{2}\Phi\left(k(Z-z)\mathbf{1}\right) + \frac{1}{2}\Phi(kz\mathbf{1}) \\ &\leq \frac{1}{2}\Phi\left(k(Z-z)^{+}\mathbf{1}\right) + \frac{1}{2}\Phi(kz\mathbf{1}) < \infty.\end{aligned}$$

By convexity, for any Y we have $\gamma = \Phi(0) \leq \frac{1}{2}\left(\Phi(Y) + \Phi(-Y)\right)$, hence $\Phi(X) \geq \Phi(-\frac{k}{2}Z\mathbf{1}) \geq 2\gamma - \Phi(\frac{k}{2}Z\mathbf{1}) > -\infty$. The conclusion now follows as in the univariate case. Using [9, Lemma **A.2**], Condition (A) implies that also Condition (12) in Theorem 3.1 below holds. Using Theorem 3.1 we obtain the representation $\rho(X) = \Phi(-X) = \max_{\mu \in ca_Z^+} \left\{\langle -X, \mu\rangle - \Phi^*(\mu)\right\}$ for all $X \in C_Z$. It is now sufficient to note that $\Phi(m^{[i]}) = m + \gamma$, where $m^{[i]}$ is the vector with the i-th coordinate equals to $m \in \mathbb{R}$ and all the others are zero, to observe that $\Phi^*(\mu) = +\infty$ for $\mu \in ca_Z^+ \backslash \mathcal{P}_Z$. The desired dual representation in item 3 follows from $\rho^*(-\mu) = \Phi^*(\mu)$, $\mu \in ca_Z^+$.

This proves $1 \iff 2 \iff 3$. Suppose now Condition (9) holds. The implications $5 \implies 4$ and $4 \implies 2$ are easily seen and similar to above. We conclude by proving $3 \implies 5$. Consider once again $\Phi(X) := \rho(-X)$. By (9), Φ satisfies condition 3 of Theorem 3.3 below. From item 3 and Condition (A), Φ is a real-valued, increasing convex functional on $\mathcal{B}_Z$ that satisfies Condition (16) below. The desired representation follows from Theorem 3.3 Condition 1 and the fact that $\Phi^*(\mu) = +\infty$ for $\mu \in ca_Z^+ \backslash \mathcal{P}_Z$, as in $1 \implies 3$. □

We end this section by discussing when the assumptions of Theorem 2.5 are satisfied. A trivial case for (A) is when Ω is a compact subset of $\mathbb{R}^{T(J+1)}$, as Z is a continuous function. On the other hand, Assumption (9) is verified if ρ is continuous from below, since for all $X \in U_Z$, there exists a sequence $(Y_n)_{n\in\mathbb{N}}$ in C_Z such that $Y_n \uparrow X$. We next present another sufficient condition for (A).

Proposition 2.8. *Consider a continuous and strictly increasing aggregation function Λ and $\alpha : \mathcal{P}_Z \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying:*

1. $\inf_{\mathbb{P}\in\mathcal{P}_Z} \alpha(\mathbb{P}) = 0$.
2. $\alpha(\mathbb{P}) \geq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}^i}\beta(Z)$ *for all* $\mathbb{P} \in \mathcal{P}_Z$, *where* $\beta : [1, +\infty) \to \mathbb{R}$ *is an increasing function with the property* $\lim_{x\to+\infty} \frac{\beta(x)}{x} = +\infty$.

Condition (A) holds for the following acceptance set:

$$\mathcal{A} := \left\{X \in \mathcal{B}_Z(\mathbb{R}) : \sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}^i}[X] + \alpha(\mathbb{P}) \geq 0 \text{ for all } \mathbb{P} \in \mathcal{P}_Z\right\} + \mathcal{B}^+(\mathbb{R}). \tag{10}$$

Proof. The proof is similar to that of [9, Proposition **2.2**] so we only present the main differences. By passing to the lower convex hull, β in 2 can be assumed convex and Jensen's inequality yields

$$\alpha(\mathbb{P}) \geq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}^i}\beta(Z) \geq \beta\left(\sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}^i} Z\right).$$

Consider now the sets $T_a := \left\{\mathbb{P} \in \mathcal{P}_Z : \sum_{i=1}^{N} \mathbb{E}^{\mathbb{P}^i}\beta(Z) \leq a\right\}, a \in \mathbb{R}$. By using that β is increasing and $Z \geq 1$, we deduce that, for each $\mathbb{P}^i$ with $\mathbb{P} \in T_a$, it holds $\mathbb{E}^{\mathbb{P}^i}\beta(Z) \leq a - (N-1)\beta(1)$ for all $i = 1, \ldots, N$. We deduce that the one-dimensional projections $\pi^i(T_a)$ are $\sigma(\mathcal{P}_Z, C_Z)$-compact. Hence, by identifying $(C_Z(\mathbb{R}), ca_Z^+(\mathbb{R}))$ with $(C_b(\mathbb{R}), ca^+(\mathbb{R}))$, one can use Prokhorov's theorem to obtain the compactness of $\pi^i(T_a)$. It follows that the sets $\{\mathbb{P} \in \mathcal{P}_Z : \alpha(\mathbb{P}) \leq a\}, a \in \mathbb{R}$ are relatively $\sigma(\mathcal{P}_Z, C_Z)$-compact. Define now $\tau : \mathcal{B}_Z(\mathbb{R}) \to \mathbb{R}$ as

$$\tau(X) := \sup_{\mathbb{P} \in \mathcal{P}_Z} \left(\sum_{i=1}^N \mathbb{E}^{\mathbb{P}^i}[-X] - \alpha(\mathbb{P}) \right),$$

so that one can rewrite $\mathcal{A} = \{X \in \mathcal{B}_Z(\mathbb{R}) : \tau(X) \leq 0\} + \mathcal{B}^+(\mathbb{R})$. Fix $n \in \mathbb{N}$ and take $X_z := (\frac{1}{n} - n(Z-z)^+)\mathbf{1} \in \mathcal{B}_Z$. Using that Λ is continuous and non-decreasing, $-\Lambda(X_z) \downarrow -\Lambda(\frac{1}{n}\mathbf{1})$ as $z \to \infty$. Using the compactness of the sets $\{\mathbb{P} \in \mathcal{P}_Z : \alpha(\mathbb{P}) \leq a\}$, we can now apply [9, Lemma **A**.**4**], which readily extends to the multivariate case, to deduce that $\tau(\Lambda(X_z)) \downarrow \tau(\Lambda(\frac{1}{n}\mathbf{1})) = -N\Lambda(\frac{1}{n}\mathbf{1})$. The last equality follows from 1 and, using that Λ is strictly increasing in this proposition, we have $-N\Lambda(\frac{1}{n}\mathbf{1}) < 0$. We deduce that for z large enough, $\Lambda(X_z)$ is acceptable. Since $n \in \mathbb{N}$ was arbitrary, Condition (A) is satisfied with the choice of $\gamma = 0$. $\square$

2.1 Different aggregation functions

As discussed in the introduction, an alternative way of measuring market-adjusted systemic risk consists in the use of a second aggregation function $\Gamma : \mathbb{R}^N \to \mathbb{R}$ for the payoffs of the trading strategies. In this scenario, ρ_Γ is defined as in (4) and Theorem 2.5 changes consequently. In particular, in this subsection the following Assumption replaces Assumption 2.2:

Assumption 2.9. *1. $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R})$ is monotone and $0 \in \mathcal{A}$; $\mathcal{G} \subseteq \mathcal{B}$ with $\mathbf{0} \in \mathcal{G}$.*

2. The aggregation functions $\Lambda : \mathbb{R}^N \to \mathbb{R}$ and $\Gamma : \mathbb{R}^N \to \mathbb{R}$ are increasing with respect to the componentwise order, concave and $\Lambda(\mathbf{0}) = \Gamma(\mathbf{0}) = 0$.

3. The set $\Lambda^{-1}(\mathcal{A} - \Gamma(\mathcal{G})) = \{X \in \mathcal{B} \mid \Lambda(X) + \Gamma(g) \in \mathcal{A}$ for some $g \in \mathcal{G}\}$ is convex.

As in the previous case, ρ_Γ is still a monotone decreasing, convex and (systematically) cash additive map, and the same functional analytic setting is considered.

Assumption 2.10. *There exists $\gamma \leq 0$ such that*

$$\forall n \in \mathbb{N},\ \exists z \in [0, +\infty),\ \exists g \in \mathcal{G} \text{ such that } \Lambda\left(\left[\frac{\gamma}{N} + \frac{1}{n} - n(Z-z)^+\right]\mathbf{1}\right) + \Gamma(g) \in \mathcal{A}. \quad \text{(A)}$$

Theorem 2.11. *Under Assumption 2.10 the following are equivalent:*

1. $m \in \mathbb{R}^N$, $\sum m^i < \gamma \Rightarrow \nexists g \in \mathcal{G} : \Lambda(m) + \Gamma(g) \in \mathcal{A}$.

2. There exists $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_N) \in \mathcal{P}_Z$ such that $\sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[X^i] - \gamma \geq 0$ for all $X \in C_Z$ satisfying $\Lambda(X) + \Gamma(g) \in \mathcal{A}$ for some $g \in \mathcal{G}$.

3. ρ_Γ is real valued on $\mathcal{B}_Z$, $\rho_\Gamma(0) = \gamma$ and

$$\rho_\Gamma(X) = \max_{\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_N) \in \mathcal{P}_Z} \left\{ \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[-X^i] - \rho_\Gamma^*(-\mathbb{Q}) \right\}, \quad X \in C_Z.$$

If, in addition, one has

$$\rho_\Gamma(X) = \inf_{Y\in C_Z, Y\le X} \rho_\Gamma(Y) \text{ for all } X \in U_Z, \tag{11}$$

then 1–3 are also equivalent to each one of the following two conditions:

4. *There exists* $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_N) \in \mathcal{P}_Z$ *such that* $\sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[X^i] - \gamma \ge 0$ *for all* $X \in U_Z$ *satisfying* $\Lambda(X) + \Gamma(g) \in \mathcal{A}$ *for some* $g \in \mathcal{G}$.

5. ρ_Γ *is real valued on* $\mathcal{B}_Z$, $\rho_\Gamma(0) = \gamma$ *and*

$$\rho_\Gamma(X) = \max_{\mathbb{Q}=(\mathbb{Q}_1,\ldots,\mathbb{Q}_N)\in\mathcal{P}_Z} \left\{ \sum_{i=1}^N \mathbb{E}^{\mathbb{Q}_i}[-X^i] - \rho_\Gamma^*(-\mathbb{Q}) \right\}, \quad X \in U_Z.$$

The proof of this theorem is analogous to the one of Theorem 2.5 and is omitted.

3 Multivariate analytical results and proofs

This section is dedicated to the extension of the analytical results in Section **A.1** of [9]. The space C_Z of continuous functions $X : \Omega \to \mathbb{R}^N$ such that each component of X/Z is bounded forms a Stone vector lattice, with the partial order given by $X \le Y \iff X^i(\omega) \le Y^i(\omega)\ \forall\omega \in \Omega,\ \forall i = 1, \ldots, N$. In [9] the cone $ca_Z^+(\mathbb{R})$ is endowed with the weak convergences topology $\sigma\left(ca_Z^+(\mathbb{R}), C_Z(\mathbb{R})\right)$, derived by the one described in Chapter **8** of [7]. Since a sequence of multivariate measures $\left(\mu_n = (\mu_n^1, \ldots, \mu_n^N)\right)_n$ weakly converges ($\rightharpoonup$) to a multivariate measure $\mu = (\mu^1, \ldots, \mu^N)$ if and only if $\mu_n^i \rightharpoonup \mu^i$ for all $i = 1, \ldots, N$, it is natural to endow $ca_Z^+ = \left(ca_Z^+(\mathbb{R})\right)^N$ with the product topology. This choice allows for easy extensions of the analytical results to the multivariate case, since compactness and metrizability are preserved for a finite product of topological spaces. In fact $ca_Z^+(\mathbb{R})$ endowed with $\sigma\left(ca_Z^+(\mathbb{R}), C_Z(\mathbb{R})\right)$ is metrizable, as stated in the proof of [9, Lemma **A.4**]: since Ω is a separable metric space, this follows by results in [7, Section **8.3**]. Given a convex increasing functional $\Psi : C_Z \to \mathbb{R}\cup\{+\infty\}$, its convex conjugate function $\Psi^* : ca_Z \to \mathbb{R}\cup\{+\infty\}$ is defined in (8).

We first seek for a dual representation formula for general increasing convex functionals on C_Z.

Theorem 3.1 (Multivariate version of [9, Theorem **A.1**])**.** *Let* $\Psi : C_Z \to \mathbb{R}^N$ *be an increasing convex functional with the property that for every* $X \in C_Z$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\lim_{z\to+\infty} \Psi\big(X + \varepsilon(Z - z)^+\mathbf{1}\big) = \Psi(X). \tag{12}$$

Then

$$\Psi(X) = \max_{\mu\in ca_Z^+} \left\{\langle X, \mu\rangle - \Psi^*(\mu)\right\}. \tag{13}$$

Proof. Fix $X \in C_Z$. By the definition of Ψ^* it is obvious that

$$\Psi(X) \ge \sup_{\mu\in ca_Z^+} \left\{\langle X, \mu\rangle - \Psi^*(\mu)\right\}. \tag{14}$$

We extend the approach of [9, Theorem **A.1**] to this setting. The Hahn-Banach extension theorem applied to the null functional on the trivial subspace $\{0\} \subseteq C_Z$ implies the existence of an increasing linear functional $\zeta_X : C_Z \to \mathbb{R}$ dominated by $\Psi_X(Y) := \Psi(X+Y) - \Psi(X)$. Moreover, one can define N increasing linear functionals $\zeta_X^i : C_Z(\mathbb{R}) \to \mathbb{R}$ as follows: For $\varphi \in C_Z(\mathbb{R})$ and $\varphi^{[i]} := (0,\dots,0,\varphi,0,\dots,0) \in C_Z$, i. e. $\varphi^{[i]}$ is the N-dimensional functional with the i-th component equals to φ and all the others are zero, we set

$$\zeta_X^i(\varphi) := \zeta_X(\varphi^{[i]}). \tag{15}$$

Hence, since ζ_X is linear, for all $Y = (Y^1,\dots,Y^N) \in C_Z$ we have $\zeta_X(Y) = \sum_{i=1}^N \zeta_X^i(Y^i)$. It will now be sufficient to prove that, for $(X_n)_n$ in C_Z satisfying $X_n \downarrow 0$, there exists a constant $\eta > 0$ such that $\Psi_X(\eta X_n) \downarrow 0$. This would imply that $\zeta_X(X_n) \downarrow 0$, as ζ_X is linear and dominated by Ψ_X. So, given $(\varphi_n)_n$ in $C_Z(\mathbb{R})$ satisfying $\varphi_n \downarrow 0$, by (15) it follows that $\zeta_X^i(\varphi_n) = \zeta_X(\varphi_n^{[i]}) \downarrow 0$ for all $i = 1,\dots,N$, since $(\varphi_n^{[i]})_n$ is a sequence going to 0 in C_Z. Hence, as in the univariate case, the Daniell-Stone theorem implies that there exists $\mu_X^i \in ca_Z^+(\mathbb{R})$ such that $\zeta_X^i(\varphi) = \langle \varphi, \mu_X^i \rangle$ for all $\varphi \in C_Z(\mathbb{R})$. Defining $\mu_X := (\mu_X^1,\dots,\mu_X^N) \in ca_Z^+$, one has, for all $Y \in C_Z$,

$$\zeta_X(Y) = \sum_{i=1}^N \zeta_X^i(Y^i) = \sum_{i=1}^N \langle Y^i, \mu_X^i \rangle = \langle Y, \mu_X \rangle .$$

Since $\Psi_X(Y) \geq \zeta_X(Y)$ for all $Y \in C_Z$, by the identity $Y = X + Y - X$ it follows that $\langle X, \mu_X \rangle - \Psi(X) \geq \langle X+Y, \mu_X \rangle - \Psi(X+Y)$. Moreover, as for any $W \in C_Z, Y := W - X$ is still in C_Z, it follows that

$$\langle X, \mu_X \rangle - \Psi(X) \geq \langle W, \mu_X \rangle - \Psi(W)$$

for all $W \in C_Z$. Hence, $\Psi^*(\mu_X) = \langle X, \mu_X \rangle - \Psi(X)$. This implies, along with (14), the dual representation in (13) and the maximum is attained by μ_X. From now on the proof is identical to the one of [9, Theorem **A.1**], we provide it for completeness. Fix $(X_n)_n$ in C_Z satisfying $X_n \downarrow 0, \varepsilon > 0$ such that (12) holds and $m > 0$ so that $X_1 \leq mZ\mathbf{1}$. Set $\eta = \frac{\varepsilon}{4m}$ and $\delta > 0$. Assumption 12 implies the existence of a $z > 0$ such that $\Psi_X(\varepsilon(Z-z)^+\mathbf{1}) < \delta$, and the set $\{Z \leq 2z\}$ is compact. It is important to note that each component of X_n is continuous for all n. So, there are no difficulties in the application of Dini's lemma, which implies that

$$x_n := (x_n^1,\dots,x_n^N), \text{ for } x_n^i := \max_{\omega \in \{Z \leq 2z\}} X_n^i(\omega),$$

are the elements of a sequence in $\mathbb{R}^N$ decreasing to 0. Also, for $x \in \mathbb{R}^N$, $x \mapsto \Psi_X(x)$ is a continuous function since it is convex from $\mathbb{R}^N$ to $\mathbb{R}$, and so there exists n_0 such that $\Psi_X(2\eta x_n) \leq \delta$ for all $n \geq n_0$. Now,

$$X_n \leq X_n \mathbb{1}_{\{Z \leq 2z\}} + X_1 \mathbb{1}_{\{Z > 2z\}} \leq x_n \mathbb{1}_{\{Z \leq 2z\}} + mZ\mathbf{1}\mathbb{1}_{\{Z > 2z\}} \leq x_n + 2m(Z-z)^+\mathbf{1}^2$$

implies that $\frac{X_n - x_n}{2m} \leq (Z-z)^+\mathbf{1}$, and therefore, since Ψ_X is increasing,

$$\Psi_X(2\eta(X_n - x_n)) = \Psi_X\left(\varepsilon \frac{X_n - x_n}{2m}\right) \leq \delta \text{ for all } n.$$

This gives, by the convexity of Ψ_X and the fact that $\Psi_X(0) = 0$, that

$$\Psi_X(\eta X_n) \leq \frac{\Psi_X(2\eta x_n) + \Psi_X(2\eta(X_n - x_n))}{2} \leq \delta \text{ for all } n \geq n_0,$$

implying that $\Psi_X(\eta X_n) \downarrow 0$. This completes the proof. □

[2]It is trivial to verify $mZ\mathbb{1}_{\{Z>2z\}} \leq 2m(Z-z)^+$, since if $\omega \notin \{Z > 2z\}$, then $0 \leq 2m(Z-z)^+$ and, if $\omega \in \{Z > 2z\}$, then $mZ \leq 2m(Z-z) \iff Z > 2z$.

A sufficient condition for Condition (12) is the following:

$$\lim_{z\to+\infty}\Psi\left(n(Z-z)^+\mathbf{1}\right)=\Psi(0) \text{ for every } n\in\mathbb{N}. \tag{16}$$

As stated in [9, Lemma **A.2**], that is trivially valid in a multivariate case, Condition (16) implies Condition (12) for an increasing convex functional $\Psi : C_Z \to \mathbb{R}$. We next extend the dual representation to U_Z,

Lemma 3.2 (Multivariate version of [9, Lemma **A.3**]). *Let $\Psi : C_Z \to \mathbb{R}$ be an increasing convex functional. The following hold.*

1. *There exists an increasing convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}\cup\{+\infty\}$ satisfying $\lim_{x\to+\infty}\frac{\varphi(x)}{x} = +\infty$ such that $\Psi^*(\mu)\geq\varphi\left(\langle Z\mathbf{1},\mu\rangle\right)$ for all $\mu\in ca_Z^+$.*
2. *If Ψ satisfies (16), the sublevel sets $T_a := \left\{\mu\in ca_Z^+ : \Psi^*(\mu)\leq a\right\}$, $a\in\mathbb{R}$, are $\sigma(ca_Z^+, C_Z)$-compact.*

Proof. Defining $\varphi(x) := \sup_{y\in\mathbb{R}^+}\{xy-\Psi(yZ\mathbf{1})\}$, item 1 follows as in [9, Lemma **A.3**]. By definition, Ψ^* is a $\sigma(ca_Z^+, C_Z)$-lower semicontinuous function. Hence, the sets T_a are $\sigma(ca_Z^+, C_Z)$-closed. Every $\mu\in T_a$ satisfies

$$m\left\langle (Z-z)^+\mathbf{1},\mu\right\rangle - \Psi\Big(m(Z-z)^+\mathbf{1}\Big)\leq\Psi^*(\mu)\leq a \text{ for all } m,z\in\mathbb{R}^+.$$

Hence, by the definition of Ψ^* and Assumption (16) one has that, for every $m\in\mathbb{R}_+$, there exists a $z\in\mathbb{R}_+$ such that

$$\left\langle (Z-z)^+\mathbf{1},\mu\right\rangle\leq\frac{a+\Psi(0)+1}{m} \quad \text{for all } \mu\in T_a.$$

Let us now consider, for $i\in\{1,\ldots,N\}$, the projection map $\pi^i : ca_Z^+\to ca_Z^+(\mathbb{R}), \pi^i : (\mu^1,...,\mu^N)\mapsto\mu^i$ and the set $\pi^i(T_a)$. It obviously holds that $\left\langle (Z-z)^+,\mu^i\right\rangle\leq\langle (Z-z)^+\mathbf{1},\mu\rangle$, leading to the following inequality (that is Condition (**A.5**) in the proof of [9, Lemma **A.3**]):

$$\begin{aligned}\lim_{z\to+\infty}\sup_{\mu^i\in\pi^i(T_a)}\left\langle Z\mathbb{1}_{\{Z<2z\}},\mu^i\right\rangle &\leq \lim_{z\to+\infty}\sup_{\mu\in T_a}\left\langle Z\mathbb{1}_{\{Z<2z\}},\mu\right\rangle\\ &\leq\lim_{z\to+\infty}\sup_{\mu\in T_a}\left\langle 2(Z-z)^+\mathbf{1},\mu\right\rangle = 0.\end{aligned}$$

From item 1, it follows that $\left\langle Z,\mu^i\right\rangle\leq\langle Z\mathbf{1},\mu\rangle\leq\varphi^{-1}(a)<+\infty\ \forall\mu\in T_a$ (and so $\forall\mu^i\in\pi^i(T_a)$) that is condition (**A.6**) in [9] for $\pi^i(T_a)$. Identifying C_Z with the space of N-dimensional vectors of continuous bounded functions C_b by the function $f : X\mapsto\frac{X}{Z}$ and ca_Z^+ with the set of N-dimensional vectors of finite Borel measures ca^+ by the function $g:\mu\mapsto Zd\mu$, conditions (**A.5**) and (**A.6**) imply that $\pi^i(g(T_a)) = h(\pi^i(T_a))$ is tight, where $h : ca_Z^+(\mathbb{R})\to ca^+, \mu^1\mapsto Zd\mu^1$. So one obtains from Prokhorov's theorem that $h(\pi^i(T_a))$ is $\sigma\big(ca^+(\mathbb{R}), C_b(\mathbb{R})\big)$-compact, that is equivalent to $\pi^i(T_a)$ being $\sigma\big(ca_Z^+(\mathbb{R}), C_Z(\mathbb{R})\big)$-compact. Clearly, since T_a is a closed subset of $\pi^1(T_a)\times\cdots\times\pi^N(T_a)$, which is a $\sigma(ca_Z^+, C_Z)$-compact set as product of compact sets, T_a is $\sigma(ca_Z^+, C_Z)$-compact. $\square$

Theorem 3.3 (Multivariate version of [9, Theorem **A.5**]). *Let $\Psi(X) : U_Z\to\mathbb{R}$ be an increasing convex functional satisfying Condition (16). Then the following are equivalent:*

1. $\Psi(X) = \max_{\mu\in ca_Z^+}\{\langle X,\mu\rangle - \Psi^*(\mu)\}$ *for all* $X\in U_Z$

2. $\Psi(X) \downarrow \Psi(X)$ *for all* $X \in U_Z$ *and every sequence* $(X_n)_n$ *in* C_Z *such that* $X_n \downarrow X$

3. $\Psi(X) = \inf_{Y \in C_Z, Y \geq X} \Psi(Y)$ *for all* $X \in U_Z$

4. $\Psi^*(\mu) = \sup_{X \in U_Z} \{\langle X, \mu \rangle - \Psi(X)\}$ *for all* $\mu \in ca_Z^+$.

Proof. The proof follows from Theorem 3.1 and Lemma 3.2 as in the univariate case and we thus omit that. This is because the convergence is intended componentwise and the results can be applied on each coordinate. □

References

[1] C. Ararat and B. Rudloff, *Dual representations for systemic risk measures*, Math. Fin. Econ., 14 (2020), pp. 139–174.

[2] M. Arduca, P. Koch-Medina, and C. Munari, *Dual representations for systemic risk measures based on acceptance sets*, Math. Fin. Econ., 15 (2021), pp. 155–184.

[3] Y. Armenti, S. Crépey, S. Drapeau, and A. Papapantoleon, *Multivariate shortfall risk allocation and systemic risk*, SIAM J. Fin. Math., 9 (2018), pp. 90–126.

[4] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heat, *Coherent measures of risk*, Math. Fin., 9 (1999), pp. 203–228.

[5] F. Biagini, J.-P. Fouque, M. Frittelli, and T. Meyer-Brandis, *A unified approach to systemic risk measures via acceptance sets*, Math. Fin., 29 (2019), pp. 329–367.

[6] ———, *On fairness of systemic risk measures*, Fin. Stoch, 24 (2020), pp. 513–564.

[7] V. Bogachev, *Measure Theory*, no. vol. 2, Springer Berlin Heidelberg, 2007.

[8] C. Chen, G. Iyengar, and C. C. Moallemi, *An axiomatic approach to systemic risk*, Management Science, 59 (2013), pp. 1373–1388.

[9] P. Cheridito, M. Kupper, and L. Tangpi, *Duality formulas for robust pricing and hedging in discrete time*, SIAM J. Fin. Math., 8 (2017), pp. 738–765.

[10] Z. Feinstein, B. Rudloff, and S. Weber, *Measures of systemic risk*, SIAM J. Fin. Math., 8 (2017), pp. 672–708.

[11] H. Föllmer and A. Schied, *Stochastic Finance: An Introduction in Discrete Time*, De Gruyter Textbook, De Gruyter, 2016.

[12] M. Frittelli and G. Scandolo, *Risk Measures And Capital Requirements For Processes*, Math. Fin., 16 (2006), pp. 589–612.

[13] E. Kromer, L. Overbeck, and K. Zilch, *Systemic risk measures on general measurable spaces*, Math. Methods Oper. Res., 84 (2016), pp. 323–357.

[14] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, USA, 2015.