# GEOMETRICAL CHARACTERIZATION OF SENSOR PLACEMENT FOR CONE-INVARIANT AND MULTI-AGENT SYSTEMS AGAINST UNDETECTABLE ZERO-DYNAMICS ATTACKS* 


#### Abstract

Undetectable attacks are an important class of malicious attacks threatening the security of cyber-physical systems, which can modify a system's state but leave the system output measurements unaffected, and hence cannot be detected from the output. This paper studies undetectable attacks on cone-invariant systems and multi-agent systems. We first provide a general characterization of zero-dynamics attacks, which characterizes fully undetectable attacks targeting the non-minimum phase zeros of a system. This geometrical characterization makes it possible to develop a defense strategy seeking to place a minimal number of sensors to detect and counter the zero-dynamics attacks on the system's actuators. The detect and defense scheme amounts to computing a set containing potentially vulnerable actuator locations and nodes, and a defense union for feasible placement of sensors based on the geometrical properties of the cones under consideration.


Key words. Undetectable attacks; sensor placement; defense strategy; cone-invariant systems; multi-agent systems.

AMS subject classifications. 15B48, 47L07, 93A16, 93B70

1. Introduction. Cyber-physical systems (CPS), networked and enabled by today's ubiquitous information technology (IT) infrastructure, have been widely regarded as new-generation engineered systems integrated by physical plants, control and communication networks, and computational units. The technological, economic, and societal impact of such systems is vastly expanding due to rapidly increasing network interconnections of different components, such as sensors, actuators, control processing units, and communication devices. Much to our chagrin, however, interconnected network and computing devices expose the vulnerability of CPS and open the door to potential cyber threats. Malicious attackers can gain access to sensing and actuating devices to launch attacks through IT components, which is likely to compromise the safe and reliable operation of a CPS and, in an extreme scenario, lead to catastrophic consequences [1].

In light of the ever-increasing presence and ever-expanding scope of CPS, there have been growing needs to address challenges in detecting attack threats, mitigating the impact of attacks, and designing effective defense strategies. Of particular relevance to this paper are a class of attacks that can be broadly referred to as undetectable attacks. In [2], false data injection attacks were considered for static systems, which inject false, ill-attempted data into the system. Stealthy deception attacks, which alter sensor readings to avoid detection and cause damage, were studied

[^0]in [3]. Replay attacks [4] record the system's operating data and fake the past data as the current operating signal to drive the system. Zero-dynamics attacks target the transmission zeros of the system and hide in the output [5]. Covert attacks exploit decoupling structures to deceive the controller by interrupting the input and the output simultaneously [6].

While as alluded to above various undetectable attacks have been designed, the work on the detection and prevention of such attacks has been scarce. Generally, the detection of an attack can be done by inserting additional signals so as to perturb the system's operating data, or by adding additional sensors, so as to change the system's structure and dynamics. For example, it was shown in [4] that adding a Gaussian signal unknown to the attacker into communication channels can make replay attacks detectable, and in [7], that by placing sensors on all neighbors of the potential attacked nodes, zero-dynamics attacks can be prevented. Nevertheless, one should note that adding a large number of sensors may neither be effective nor feasible in many situations, especially in a distributed setting, because of environmental constraints and cost-efficient considerations. As such, where possible, more efficient defense strategies with a reasonable number of sensors are called for.

In this paper we study zero-dynamics attacks and their detection, in both a single-loop cone-invariant system and multi-agent system configuration. We provide a general mathematical view of such attacks on cone-invariant linear time-invariant (LTI) systems. Zero-dynamics attacks have been previously investigated in [5, 7, 8, $9,10]$. These attacks typically attack the system actuators while hidden from the sensors of the system. As an example, when the system contains a real non-minimum phase zero $s_{0}$, a zero-dynamics attack can be designed in the form [5] of

$$
d(t)=-e^{s_{0} t} d_{0}
$$

This attack signal can alter the system's state response but is unobserved from the system output. In this paper, we present a general zero-dynamics attack configuration, where we show that more generally, non-minimum phase complex zeros can also be subjected to zero attacks.

Our more significant developments are devoted to the detection and prevention of zero-dynamics attacks. Via a geometrical framework, we show that for LTI coneinvariant systems, zero-dynamics attacks can be detected, or a system can be made resistive to such attacks, by developing an efficient sensor placement strategy. While seemingly a mathematical notion, cone-invariant systems encompass many systems of practical interest. One particular instance corresponds to positive systems, whose states form an invariant positive cone. Positive systems have many applications, e.g., in modeling growth behaviors of economical systems, ecological systems, population dynamics, and more generally, dynamic systems involving positivity constraints $[11,12]$. Recently, positive systems have been used to model power grids, traffic flow, communication/computation networks, as well as production planning and logistics [13]. Another case of interest is Lyapunov and Riccati differential equations, which constitute a positive semi-definite cone in the space of Hermitian matrices [14]. Finally, cone-invariance with respect to polyhedral cones and ellipsoidal cones are also of interest and have been considered in [15, 16].

In this paper, for cone-invariant systems and multi-agent systems [17, 18], we develop a unified defense framework against zero-dynamics attacks. This is accomplished by characterizing a vulnerable set containing potential positions or nodes where attacks may get into the system, and meanwhile by constructing a defensive union including feasible sensor placements based on the geometric properties of the cones
under consideration. Our contributions can be summarized as follows. In Section 2, we provide a mathematical background on convex cones and systems that are coneinvariant. In Section 3, we provide a general characterization of zero-dynamics attacks on LTI systems, which enables us to fully characterize all possible attacks utilizing the knowledge of a system's non-minimum phase zeros. Based on this characterization, we develop in Section 4 defense strategies for cone-invariant systems, which amount to placing a minimal number of sensors to counter zero-dynamics attacks on the system's actuators. Section 5 then extends the results to multi-agent systems, in which undetectable attacks are launched on certain nodes of the multi-agent systems and the defense is implemented at other nodes. Section 6 discusses generalizations to multi-input, multi-output (MIMO) systems that are subject to multiple attacks. Section 7 presents numerical studies on a microgrid power network under attack, to illustrate our defense strategies. This paper concludes in Section 8.

## 2. Mathematical Preliminaries.

### 2.1. Convex Cones.

Definition 2.1. A nonempty set $\mathcal{K} \subset \mathbb{R}^{n}$ is said to be a convex cone if $\mathcal{K}+\mathcal{K} \subset \mathcal{K}$ and $a \mathcal{K} \subset \mathcal{K}$ for all $a \geq 0$.

We define the set $-\mathcal{K}=\{x:-x \in \mathcal{K}\}$. A convex cone $\mathcal{K}$ is said to be pointed if $\mathcal{K} \cap-\mathcal{K}=\{0\}$ and solid if int $\mathcal{K}$, i.e., the interior of $\mathcal{K}$, is nonempty. The boundary of $\mathcal{K}$ is denoted as $\partial \mathcal{K}$. Moreover, a closed, pointed, and solid convex cone is called a proper cone. A proper cone induces a partial order in $\mathbb{R}^{n}$, that is, $\forall x, y \in \mathbb{R}^{n}, y \preceq x$ if and only if $x-y \in \mathcal{K}$. Hereafter, unless otherwise specified, all cones we mention will be proper cones.

Herein, we briefly survey some typical and representative cones [11, 19, 20].

- A polyhedral cone represents the intersection of finitely many closed half spaces, each containing the origin on its boundary. Every polyhedral cone intersected by $k$ closed half spaces can be described mathematically as

$$
\mathcal{K}=\left\{x \in \mathbb{R}^{n}: A x \geq_{+} 0, A \in \mathbb{R}^{k \times n}\right\}
$$

where $\geq_{+}$represents the element-wise order. When $k=n$ and $A=I_{n}$, then $\mathcal{K}=\mathbb{R}_{+}^{n}$ denotes the nonnegative orthant, which is also called positive cone.

- A positive semi-definite ( $P S D$ ) cone is one that consists of positive semidefinite matrices, i.e.,

$$
\mathcal{S}_{+}^{n}=\left\{X \in \mathcal{S}^{n}: v^{\top} X v \geq 0, \forall v \in \mathbb{R}^{n}\right\}
$$

where $\mathcal{S}^{n} \in \mathbb{R}^{n \times n}$ denotes the set of $n \times n$ real symmetric matrices. A PSD cone $\mathcal{S}_{+}^{n}$ can be equivalently described in the $n(n+1) / 2$-dimensional vector space, consisting of the elements of all positive semi-definite matrices on the main diagonal and above it.

- An ellipsoidal cone is defined as

$$
\mathcal{K}=\left\{x \in \mathbb{R}^{n}: x^{\top} Q x \leq 0, l_{n}^{\top} x \geq 0\right\}
$$

where $Q \in \mathcal{S}^{n}$ has inertia ${ }^{1}(n-1,0,1)$ and $l_{n}$ is a unit eigenvector of $Q$ corresponding to the negative eigenvalue. When $Q=Q_{n}=\operatorname{diag}\left\{-1, I_{n-1}\right\}$

[^1]and $l_{n}=e_{1}$, the first Euclidean coordinate, then $\mathcal{K}$ becomes a Lorentz cone (also called ice-cream cone or second-order cone), denoted as
$$
\mathcal{L}_{+}^{n}=\left\{x \in \mathbb{R}^{n}: x_{1} \geq \sqrt{x_{2}^{2}+\cdots+x_{n}^{2}}\right\}
$$

Illustrations of the positive cone $\mathbb{R}_{+}^{3}$, the PSD cone $\mathcal{S}_{+}^{2}$ with

$$
\left[\begin{array}{ll}
x_{1} & x_{2} \\
x_{2} & x_{3}
\end{array}\right] \in \mathcal{S}_{+}^{2}
$$

and the Lorentz cone $\mathcal{L}_{+}^{3}$ are given in Fig. 1.


Fig. 1: Positive cone, PSD cone and Lorentz cone.

Given a cone $\mathcal{K}$ in $\mathbb{R}^{n}$, its dual cone is defined as

$$
\mathcal{K}^{*}=\left\{y \in \mathbb{R}^{n}:\langle x, y\rangle \geq 0, \forall x \in \mathcal{K}\right\}
$$

where $\langle$,$\rangle denotes a pre-defined inner product. A cone \mathcal{K}$ is said to be self-dual if $\mathcal{K}=\mathcal{K}^{*}$. For instance, $\mathbb{R}_{+}^{n}$ and $\mathcal{L}_{+}^{n}$ are self-dual cones endowed with the standard inner product $\langle x, y\rangle=x^{\top} y$, and $\mathcal{S}_{+}^{n}$ is a self-dual cone endowed with the inner product $\langle X, Y\rangle=\operatorname{Tr}(X Y)$. One useful lemma reveals the geometric relationship between each entry in $\mathcal{K}$ and its corresponding $\mathcal{K}^{*}$ [11].

Lemma 2.2. For a cone $\mathcal{K}$ and its dual cone $\mathcal{K}^{*}$, it holds that $\langle y, x\rangle>0, \forall y \in$ $\operatorname{int} \mathcal{K}^{*}, x \in \mathcal{K}$.
2.2. Matrices. In the sequel, matrices characterizing the invariance with respect to given cones will be studied.

Definition 2.3. For a cone $\mathcal{K}$, a matrix $A \in \mathbb{R}^{n \times n}$ is called a $\mathcal{K}$-invariant matrix if $A \mathcal{K} \subset \mathcal{K}$.

Denote by $\pi(\mathcal{K})$ the set of cone-invariant matrices over $\mathcal{K}$. For example, any nonnegative matrix $A \in \mathbb{R}^{n \times n}$ is an $\mathbb{R}_{+}^{n}$-invariant matrix. Any matrix $A=B \otimes B, \forall B \in$ $\mathbb{R}^{n \times n}$, is an $\mathcal{S}_{+}^{n}$-invariant matrix, where $\otimes$ denotes the Kronecker product. The matrix $A \in \mathbb{R}^{n \times n}$ is an $\mathcal{L}_{+}^{n}$-invariant matrix if it satisfies the relation

$$
a_{1}^{\top} \in \mathcal{L}_{+}^{n}, \quad A^{\top} Q_{n} A-\delta Q_{n} \preceq 0
$$

for some $\delta \in \mathbb{R}$, where $a_{1}$ is the first row of $A$ and $\preceq$ is defined on $\mathcal{S}_{+}^{n}$ [21].
Definition 2.4. A matrix $A \in \pi(\mathcal{K})$ is said to be $\mathcal{K}$-positive if $A(\mathcal{K} \backslash\{0\}) \subset \operatorname{int} \mathcal{K}$.
Denote by $\pi^{+}(\mathcal{K})$ the set of $\mathcal{K}$-positive matrices.
Except for cone-invariant matrices, cross-positive matrices (exponential coneinvariant matrices) are also introduced herein [22].

Definition 2.5. For a cone $\mathcal{K}$ and its dual cone $\mathcal{K}^{*}$, a matrix $A \in \mathbb{R}^{n \times n}$ is called cross-positive over $\mathcal{K}$ if for all $x \in \mathcal{K}, y \in \mathcal{K}^{*}$ such that $\langle y, x\rangle=0$, it holds that $\langle y, A x\rangle \geq 0$.

Denote by $\varpi(\mathcal{K})$ the set of cross-positive matrices over $\mathcal{K}$. It is straightforward that $\pi(\mathcal{K}) \subset \varpi(\mathcal{K})$ for a given cone $\mathcal{K}$. For instance, $\varpi\left(\mathbb{R}_{+}^{n}\right)$ is the set of Metzler matrices [11]. Any matrix $A=B \oplus B, \forall B \in \mathbb{R}^{n \times n}$, is cross-positive over $\mathcal{S}_{+}^{n}$, where $\oplus$ denotes the Kronecker sum. The matrix $A \in \mathbb{R}^{n \times n}$ is cross-positive over $\mathcal{L}_{+}^{n}$ if

$$
A^{\top} Q_{n}+Q_{n} A-\xi Q_{n} \preceq 0
$$

for some $\xi \in \mathbb{R}[19]$, where $\preceq$ is defined on $\mathcal{S}_{+}^{n}$.
Different from the conventional irreducibility ${ }^{2}$ of a matrix, the cone-irreducibility of cross-positive matrices is defined below [11, 22].

Definition 2.6. A matrix $A \in \varpi(\mathcal{K})$ is said to be $\mathcal{K}$-irreducible if no eigenvector of $A$ lies on the boundary of $\mathcal{K}$.
Denote by $\varpi^{\prime}(\mathcal{K})$ the set of $\mathcal{K}$-irreducible cross-positive matrices.
Remark 2.7. From [23], for any matrix $A \in \varpi\left(\mathbb{R}_{+}^{n}\right)$, the $\mathbb{R}_{+}^{n}$-irreducibility and the conventional matrix irreducibility are equivalent.

Moreover, when $A \in \pi(\mathcal{K})$, there exists another equivalent definition of matrix cone-irreducibility in terms of the face of the cone.

Definition 2.8. Let $\mathcal{F} \subseteq \mathcal{K}$ be a pointed closed cone. Then $\mathcal{F}$ is called a face of $\mathcal{K}$ if $\forall x \in \mathcal{F}$,

$$
0 \preceq y \preceq x \Rightarrow y \in \mathcal{F}
$$

where $\preceq$ is defined on $\mathcal{K}$. The face $\mathcal{F}$ is said to be nontrivial if $\mathcal{F} \neq\{0\}$ and $\mathcal{F} \neq \mathcal{K}$.
Definition 2.9. A matrix $A \in \pi(\mathcal{K})$ is said to be $\mathcal{K}$-irreducible if the only faces of $\mathcal{K}$ leaving invariant with $A$ are $\{0\}$ or $\mathcal{K}$ itself.
Denote by $\pi^{\prime}(\mathcal{K})$ the set of $\mathcal{K}$-irreducible cone-invariant matrices. Then the following relations are known [22].

[^2]Lemma 2.10.

$$
\pi^{+}(\mathcal{K}) \subseteq \pi^{\prime}(\mathcal{K}) \subseteq \pi(\mathcal{K})
$$

The following lemma, adapted from [11], on cross-positive matrices will be useful in the sequel.

Lemma 2.11. Let $\operatorname{spectrum}(A)$ be the set of eigenvalues of $A$.
If $A \in \varpi(\mathcal{K})$, then

1) $\mu=\max \{\operatorname{Re}(\delta): \delta \in \operatorname{spectrum}(A)\}$ is an eigenvalue of $A$,
2) $\mathcal{K}$ and $\mathcal{K}^{*}$ contain right- and left-eigenvectors of $A$ associated with the eigenvalue $\mu$.
If $A \in \varpi^{\prime}(\mathcal{K})$, then
3) $\mu=\max \{\operatorname{Re}(\delta): \delta \in \operatorname{spectrum}(A)\}$ is a simple eigenvalue of $A$,
4) int $\mathcal{K}$ and int $\mathcal{K}^{*}$ contain right- and left-eigenvectors of $A$ associated with the eigenvalue $\mu$.

Note that Lemma 2.11 reduces to the Krein-Rutman theorem if $A \in \pi(\mathcal{K})$, and furthermore, to the well-known Perron-Frobenius theorem if $A \in \pi\left(\mathbb{R}_{+}^{n}\right)$ [11].
2.3. Cone-Invariant Systems. Consider a linear time-invariant (LTI) system with input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, and state $x(t) \in \mathbb{R}^{n}$, with realization

$$
\begin{align*}
\dot{x}(t) & =A x(t)+b u(t) \\
y(t) & =c^{\top} x(t) \tag{2.1}
\end{align*}
$$

We say that this system is cone-invariant over a cone $\mathcal{K}$ if $x(t) \in \mathcal{K}$ and $y(t) \geq 0$ whenever $x(0) \in \mathcal{K}$ and $u(t) \geq 0$ for $t \geq 0$. Equivalently, the following lemma reveals the relations of the system's cone-invariance to the triple $\left(A, b, c^{\top}\right)$ [24, 25].

Lemma 2.12. The LTI system (2.1) is a cone-invariant system over $\mathcal{K}$ if and only if

$$
A \in \varpi(\mathcal{K}), b \in \mathcal{K}, \text { and } c \in \mathcal{K}^{*}
$$

When $\mathcal{K}=\mathbb{R}_{+}^{n}, \mathcal{K}=\mathcal{S}_{+}^{n}$, or $\mathcal{K}=\mathcal{L}_{+}^{n}$, the related cone-invariant system reduces to a positive system, a dynamic covariance system, or a Lorentz cone-invariant system, respectively, which all possess rich theoretical features and broad applications [13, 14, 26].

In the remainder of the paper, we study cone-invariant systems over a general cone $\mathcal{K}$. We examine the behavior of such systems under undetectable attacks and construct accordingly a defense mechanism to counter the attacks. We consider coneinvariant systems that are Hurwitz stable; in other words, the systems are stable in the absence of attacks.
3. Undetectable Attacks. Consider a MIMO LTI system

$$
\begin{align*}
\dot{x}(t) & =A x(t)+B d(t) \\
y(t) & =C x(t) \tag{3.1}
\end{align*}
$$

with output measurement $y(t) \in \mathbb{R}^{l}$ excited by the initial state $x(0) \in \mathbb{R}^{n}$ and a malicious attack $d(t) \in \mathbb{R}^{m}$. Throughout this paper, from (3.1), we address the scenario where the attacks are launched on the system's actuators. We denote the system output as $y(t)=y(x(0), d(t), t)$. An intuitive interpretation of undetectable attacks is that such an attack will result in the same output as that in the absence of attacks [5], and hence cannot be detected from output measurement.

Definition 3.1. An attack $d(t) \in \mathbb{R}^{m}$ on the system (3.1) is undetectable if the system output satisfies the relation

$$
\begin{equation*}
y(x(0)-\tilde{x}(0), d(t), t)=0, \forall t \geq 0 \tag{3.2}
\end{equation*}
$$

for some $x(0)$ and $\tilde{x}(0)$.
Here $x(0)$ and $\tilde{x}(0)$ are understood to be the system's actual initial state and an attack initial condition. Hence, the design of an undetectable attack amounts to finding a input $d(t)$ that, when coordinating with the initial state $x(0)-\tilde{x}(0)$, yields a zero output.

Lemma 3.2. For the system (3.1), the attack signal

$$
\begin{equation*}
d(t)=-e^{s_{0} t} d_{0}, s_{0} \in \mathbb{R}, d_{0} \in \mathbb{R}^{m} \tag{3.3}
\end{equation*}
$$

is undetectable if and only if there exists a vector $\zeta \neq 0$ such that

$$
\left[\begin{array}{cc}
s_{0} I-A & B  \tag{3.4}\\
C & 0
\end{array}\right]\left[\begin{array}{c}
\zeta \\
d_{0}
\end{array}\right]=0
$$

Here the attack initial condition can be obtained by $\tilde{x}(0)=x(0)-\zeta$. It is rather clear that the attack (3.3) is undetectable if and only if $s_{0}$ is a real transmission zero of the system, and hence is also referred to as a zero-dynamics attack. Then a natural question is how to construct a real attack signal cooperated with a real attack initial condition to attack the system with complex transmission zeros? This is answered in the following theorem.

Theorem 3.3. For the system (3.1), the attack signal

$$
\begin{equation*}
d(t)=-l_{1} \operatorname{Re}\left(e^{s_{0} t} d_{0}\right)-l_{2} \operatorname{Im}\left(e^{s_{0} t} d_{0}\right), s_{0} \in \mathbb{C}, d_{0} \in \mathbb{C}^{m}, \forall l_{1}, l_{2} \in \mathbb{R} \tag{3.5}
\end{equation*}
$$

coordinated with

$$
\begin{equation*}
\tilde{x}(0)=x(0)-\left(l_{1} \operatorname{Re}(\zeta)+l_{2} \operatorname{Im}(\zeta)\right) \tag{3.6}
\end{equation*}
$$

is undetectable if and only if there exists a vector $\zeta \neq 0$ such that the equation (3.4) is satisfied.
Note that for a zero $s_{0} \in \mathbb{R}$, the undetectable attack (3.5) reduces to that in (3.3).
Proof. Denote $d_{0}=d_{1}+d_{2} j, s_{0}=\alpha+\beta j$ and $\zeta=x_{1}+x_{2} j$. The attack signal (3.5) can thus be rewritten as

$$
\begin{aligned}
d(t) & =e^{\alpha t}\left(\sin (\beta t) l_{1} d_{2}-\cos (\beta t) l_{1} d_{1}-\sin (\beta t) l_{2} d_{1}-\cos (\beta t) l_{2} d_{2}\right) \\
& =e^{\alpha t}\left(\sin (\beta t)\left(l_{1} d_{2}-l_{2} d_{1}\right)-\cos (\beta t)\left(l_{1} d_{1}+l_{2} d_{2}\right)\right)
\end{aligned}
$$

For the initial state, it holds that $x(0)-\tilde{x}(0)=l_{1} x_{1}+l_{2} x_{2}$. Also, the real and imaginary parts of the equation (3.4) are obtained as

$$
\begin{align*}
& (\alpha I-A) x_{1}-\beta x_{2}+B d_{1}=0 \\
& (\alpha I-A) x_{2}+\beta x_{1}+B d_{2}=0 \tag{3.7}
\end{align*}
$$

where $C x_{1}=0$, and $C x_{2}=0$. Denote the Laplace transforms of $d(t)$ and $y(t)$ by $D(s)$ and $Y(s)$. Therefore,

$$
D(s)=\frac{\beta}{(s-\alpha)^{2}+\beta^{2}}\left(l_{1} d_{2}-l_{2} d_{1}\right)-\frac{s-\alpha}{(s-\alpha)^{2}+\beta^{2}}\left(l_{1} d_{1}+l_{2} d_{2}\right)
$$

and

$$
\begin{aligned}
Y(s)= & C(s I-A)^{-1} B D(s)+C(s I-A)^{-1}(x(0)-\tilde{x}(0)) \\
= & C(s I-A)^{-1} B \frac{\left(\beta l_{1}-(s-\alpha) l_{2}\right)}{(s-\alpha)^{2}+\beta^{2}} d_{2}-C(s I-A)^{-1} B \frac{\left(\beta l_{2}+(s-\alpha) l_{1}\right)}{(s-\alpha)^{2}+\beta^{2}} d_{1} \\
& +C(s I-A)^{-1}\left(l_{1} x_{1}+l_{2} x_{2}\right) .
\end{aligned}
$$

In addition, together with (3.7), it follows that

$$
\begin{aligned}
Y(s)=- & C(s I-A)^{-1} \frac{\left(\beta l_{1}-(s-\alpha) l_{2}\right)}{(s-\alpha)^{2}+\beta^{2}}\left((\alpha I-A) x_{2}+\beta x_{1}\right) \\
& +C(s I-A)^{-1} \frac{\left(\beta l_{2}+(s-\alpha) l_{1}\right)}{(s-\alpha)^{2}+\beta^{2}}\left((\alpha I-A) x_{1}-\beta x_{2}\right) \\
& +C(s I-A)^{-1} \frac{(s-\alpha)^{2}+\beta^{2}}{(s-\alpha)^{2}+\beta^{2}}\left(l_{1} x_{1}+l_{2} x_{2}\right) \\
= & C(s I-A)^{-1} \frac{\left((s-\alpha) l_{1}+\beta l_{2}\right)}{(s-\alpha)^{2}+\beta^{2}}(s I-A) x_{1} \\
& \quad+C(s I-A)^{-1} \frac{\left((s-\alpha) l_{2}-\beta l_{1}\right)}{(s-\alpha)^{2}+\beta^{2}}(s I-A) x_{2} \\
= & \frac{\left((s-\alpha) l_{1}+\beta l_{2}\right)}{(s-\alpha)^{2}+\beta^{2}} C x_{1}+\frac{\left((s-\alpha) l_{2}-\beta l_{1}\right)}{(s-\alpha)^{2}+\beta^{2}} C x_{2}=0
\end{aligned}
$$

We conclude that $y(x(0)-\tilde{x}(0), d(t), t)=0$, and the proof is completed.
We conclude this section by commenting on zero-dynamics attacks on coneinvariant systems. Note that when restricting general LTI systems to cone-invariant systems over a cone $\mathcal{K}$, because of the cone-invariance additionally required, the definition of undetectable attacks, in comparison to Definition 3.1, should be modified accordingly as follows. Specifically, we say that an attack $d(t) \in \mathbb{R}$ is undetectable on the cone-invariant system (2.1) over a cone $\mathcal{K}$ if Definition 3.1 is satisfied, and additionally the state in (2.1) satisfies the invariance condition $x(t) \in \mathcal{K}, \forall t \geq 0$. In other words, for cone-invariant systems, not only the attack cannot be observed at the output of the system, but also the cone-invariance property of the system must be preserved. Clearly, both the attacks $d(t)$ in (3.3) and (3.5) are unobserved at the output. To ensure that the condition $x(t) \in \mathcal{K}$ is valid, it is necessary that $d(t) \geq 0, \forall t \geq 0$, which can be satisfied for the attack $d(t)$ in (3.3) by letting $d_{0}<0$. On the other hand, the attack $d(t)$ in (3.5) cannot be kept nonnegative since it is a linear combination of sinusoidal signals. Consequently, for a stable cone-invariant system (2.1), only the attack $d(t)$ in (3.3) is still undetectable.
4. Sensor Placement. In light of the discussion at the end of section 3, in this section, we develop a detection strategy to cope with undetectable attacks $d(t)=$ $-e^{s_{0} t} d_{0}$, on the non-minimum phase zeros $s_{0} \geq 0$ of stable cone-invariant systems.
4.1. Protection via Sensor Placement. We consider a single zero-dynamics attack, that is, the attack enters the system through one actuator or one node. Here without loss of generality, we take $b \in\left\{e_{i}, i=1, \ldots, n\right\}$ to represent the accessibility of the attack to the system, where $e_{i}$ is the $i$ th Euclidean coordinate. To characterize the available attacked positions in cone-invariant systems, we make the following assumption.

Assumption 1. The set $\Lambda_{\mathcal{K}}=\left\{e_{i}: e_{i} \in \mathcal{K}\right\}$ is nonempty.
The set $\Lambda_{\mathcal{K}}$ characterizes all accessible positions for potential attackers. The cardinality of $\Lambda_{\mathcal{K}}$ varies with the choice of $\mathcal{K}$. For example,

$$
\begin{gather*}
\Lambda_{\mathbb{R}_{+}^{n}}=\left\{e_{i}, i=1, \ldots, n\right\} \\
\Lambda_{\mathcal{S}_{+}^{n}}=\left\{\operatorname{vec}\left(\operatorname{diag}\left(e_{i}\right)\right), i=1, \ldots, n\right\}, \text { and } \Lambda_{\mathcal{L}_{+}^{n}}=\left\{e_{1}\right\} . \tag{4.1}
\end{gather*}
$$

Clearly, for these cones, $\Lambda_{\mathcal{K}}$ is not empty. Analogously, we also assume that the set $\left\{e_{j}: e_{j} \in \mathcal{K}^{*}\right\}$ is nonempty.

In the rest of this paper, we seek defense approaches by designing $c$ to prevent undetectable attacks. In general, adding a large number of sensors implies an effective defensive strategy. More precisely, by the PBH test [27], the system's observability implies that $x(0)=\tilde{x}(0)$, and hence $b d(t)=0, \forall t \geq 0$, that is, the undetectable attack is non-existent. Nevertheless, this approach is clearly not a cost-efficient way, especially for large-scale and distributed systems. Recalling from Lemma 3.2 the condition for the attack $d(t)=-e^{s_{0} t} d_{0}$, it is clear that a defensive strategy is successful if it can block all non-minimum phase zeros of the given system.

Definition 4.1. We say that a defense strategy, namely a design of $c$, is (almost) successful if one of the following conditions holds:
(i) $\mathbf{R}\left(\left[\begin{array}{cc}s_{0} I-A & b \\ c^{\top} & 0\end{array}\right]\right)=n+1$,
(ii) $\mathbf{R}\left(\left[\begin{array}{cc}s_{0} I-A & b \\ c^{\top} & 0\end{array}\right]\right)<n+1$ and $d_{0}=0$,
for each $s_{0}$ with $\left(s_{0}>0\right) s_{0} \geq 0$. Here $\boldsymbol{R}$ denotes the rank of a matrix.
For $s_{0} \geq 0$, the condition (i) indicates that by designing an appropriate $c, s_{0}$ is not a non-minimum phase zero of system, while the condition (ii), via PBH test, means that $s_{0}$ as a non-minimum phase zero is also an unobservable mode of the system. For $s_{0}=0$ with $d_{0} \neq 0$, the corresponding attack $d(t)=d_{0}$ as a constant signal cannot make the states of a stable system divergent. Hence we call a strategy is almost successful if the above conditions can be satisfied only for $s_{0}>0$.
4.2. Sensor Placement Strategy. In this section, we concentrate on the case with a single measurement, i.e., $c \in\left\{e_{i}, i=1, \ldots, n\right\}$, as a defense strategy to achieve, apparently, the minimal cost. Then we obtain a collective set containing all $c$ that can enable a successful defense.

Theorem 4.2. Given a stable cone-invariant system (2.1) over a cone $\mathcal{K}$, for an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathcal{K}}$, the defense strategy with $\forall c \in \Pi$ is successful, where $\Pi=\left\{e_{j}: e_{j} \in \operatorname{int} \mathcal{K}^{*}\right\}$. Moreover, if $A \in \varpi^{\prime}(\mathcal{K})$, then $\Pi=\left\{e_{j}: e_{j} \in \mathcal{K}^{*}\right\}$.

To establish Theorem 4.2, we need the following lemma.
Lemma 4.3. Given a Hurwitz stable matrix $A$ and a cone $\mathcal{K}$, for each $s_{0} \geq 0$, $\left(s_{0} I-A\right)^{-1}$ is a cone-invariant matrix, i.e., $\left(s_{0} I-A\right)^{-1} \in \pi(\mathcal{K})$ if and only if $A$ is a cross-positive matrix over $\mathcal{K}$, i.e., $A \in \varpi(\mathcal{K})$.

Proof. Sufficiency. Consider a stable system $\dot{x}(t)=A x(t)$ with $A \in \varpi(\mathcal{K})$. If $x(0) \in \mathcal{K}$, through the cone-invariance, we have $x(t) \in \mathcal{K}, \forall t \geq 0$. After taking the Laplace transform, it follows that $X(s)=(s I-A)^{-1} x(0)$. For each $s_{0} \geq 0$, since $A$ is stable, it is larger than the radius of convergence of the Laplace transform, which implies that $X(s)$ does converge at point $s_{0}$. From $X\left(s_{0}\right)=\int_{0}^{\infty} x(t) e^{-s_{0} t} \mathrm{~d} t$, together with the facts that $x(t) \in \mathcal{K}$ and $e^{-s_{0} t}>0$, it is easy to see that $X\left(s_{0}\right) \in \mathcal{K}, \forall s_{0} \geq 0$.

Therefore, for $\forall x(0) \in \mathcal{K}$, we have $X\left(s_{0}\right)=\left(s_{0} I-A\right)^{-1} x(0) \in \mathcal{K}$, which points out that $\left(s_{0} I-A\right)^{-1} \in \pi(\mathcal{K})$.

Necessity. For $s_{0}=0$, it can be easily verified that $A \in \varpi(\mathcal{K})$ if $-A^{-1} \in \pi(\mathcal{K})$. For $s_{0}>0$, since $\left(s_{0} I-A\right)^{-1} \in \pi(\mathcal{K})$, it follows that $\left(I-\frac{1}{s_{0}} A\right)^{-1} \in \pi(\mathcal{K})$. According to Definition 2.5, we define $f(k)=\left\langle y,(I-k A)^{-1} x\right\rangle$, where $x \in \mathcal{K}, y \in \mathcal{K}^{*}$ such that $\langle y, x\rangle=0$. Clearly, it holds that $f(0)=0$. In addition, since $(I-k A)^{-1} x \in \mathcal{K}$ and $y \in \mathcal{K}^{*}$, it follows that $f(k) \geq 0, \forall k>0$. Hence, denote $\dot{f}(k)$ as the first derivative of $f(k)$. We have $\dot{f}(0) \geq 0$. On the other hand, after calculating $\dot{f}(k)$, we obtain

$$
\dot{f}(k)=\left\langle y,(I-k A)^{-1} A(I-k A)^{-1} x\right\rangle
$$

Thus, we have $\dot{f}(0)=\langle y, A x\rangle \geq 0$. The matrix $A$ is cross-positive over $\mathcal{K}$, i.e., $A \in$ $\varpi(\mathcal{K})$.

Now, we are ready to prove Theorem 4.2.
Proof. For $s_{0} \geq 0$, since $A$ is stable, the matrix $s_{0} I-A$ has full rank and is invertible. Then,

$$
\mathbf{R}\left(\left[\begin{array}{cc}
s_{0} I-A & b \\
c^{\top} & 0
\end{array}\right]\right)=\mathbf{R}\left(\left[\begin{array}{cc}
s_{0} I-A & 0 \\
0 & -c^{\top}\left(s_{0} I-A\right)^{-1} b
\end{array}\right]\right)
$$

The matrix above is full rank if and only if $c^{\top}\left(s_{0} I-A\right)^{-1} b \neq 0$ with $\forall s_{0} \geq 0, \forall b \in \Lambda$, and $\forall c \in \Pi$.

At first, by Lemma 4.3, we have $\left(s_{0} I-A\right)^{-1} \in \pi(\mathcal{K})$. Since $b \in \Lambda_{\mathcal{K}} \subseteq \mathcal{K}$, it follows that $\left(s_{0} I-A\right)^{-1} b \in \mathcal{K}$. In light of Lemma 2.2, we claim that $c^{\top}\left(s_{0} I-A\right)^{-1} b>0$ with $\forall c \in\left\{e_{j}: e_{j} \in \operatorname{int} \mathcal{K}^{*}\right\}$.

Consider the case $A \in \varpi^{\prime}(\mathcal{K})$ and $c \in\left\{e_{j}: e_{j} \in \mathcal{K}^{*}\right\}$. From Definition 2.4, if $\left(s_{0} I-A\right)^{-1}$ is a $\mathcal{K}$-positive matrix, i.e., $\left(s_{0} I-A\right)^{-1} \in \pi^{+}(\mathcal{K})$, we have $\left(s_{0} I-A\right)^{-1} b \in$ $\operatorname{int} \mathcal{K}$, and thus $c^{\top}\left(s_{0} I-A\right)^{-1} b>0$ with the help of Lemma 2.2. Hence, we need to verify that $\left(s_{0} I-A\right)^{-1} \in \pi^{+}(\mathcal{K})$ with $A \in \varpi^{\prime}(\mathcal{K})$ and $\forall s_{0} \geq 0$.

Assume that there exists $\bar{s}_{0} \geq 0$ such that $\left(\bar{s}_{0} I-A\right)^{-1} \notin \pi^{+}(\mathcal{K})$, then there exist $l \in \mathcal{K}-\{0\}$ and $p \in \partial \mathcal{K}$ such that $p=\left(\bar{s}_{0} I-A\right)^{-1} l$, i.e., $l=\left(\bar{s}_{0} I-A\right) p$. Choose $q \in \mathcal{K}^{*}$ such that $\langle q, p\rangle=0$. It follows that $\langle q, A p\rangle \geq 0$. Therefore, we have

$$
0 \leq\langle q, l\rangle=\left\langle q,\left(\bar{s}_{0} I-A\right) p\right\rangle=\bar{s}_{0}\langle q, p\rangle-\langle q, A p\rangle \leq 0
$$

indicating $\langle q, l\rangle=0$, and then $l \in \partial \mathcal{K}$. Given $\bar{s}_{1}>\bar{s}_{0}$, denote $w=\left(\bar{s}_{1} I-A\right) p=$ $\left(\bar{s}_{1}-\bar{s}_{0}\right) p+l$, we have $w \in \mathcal{K}$. Furthermore, it follows that $w \in \partial \mathcal{K}$ since

$$
\langle q, w\rangle=\left(\bar{s}_{1}-\bar{s}_{0}\right)\langle q, p\rangle+\langle q, l\rangle=0
$$

In light of [23], we can construct the face $\mathcal{F}_{w}$ of $\mathcal{K}$ generated by $w$ with

$$
\mathcal{F}_{\omega}=\{x: a x \preceq w, \exists a>0\}
$$

where $\preceq$ is defined on $\mathcal{K}$. Since $w=\left(\bar{s}_{1}-\bar{s}_{0}\right) p+l \neq 0, \forall x \in \mathcal{F}_{w} \subseteq \mathcal{K}$, we have

$$
\begin{aligned}
0 & \leq\langle q, x\rangle=\left\langle q, \frac{1}{a}(a x-w)+\frac{1}{a} w\right\rangle \\
& =\frac{1}{a}\langle q,(a x-w)\rangle=-\frac{1}{a}\langle q,(w-a x)\rangle \leq 0
\end{aligned}
$$

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We obtain that $\langle q, x\rangle=0$ and $x \in \partial \mathcal{K}$. According to Definition 2.8, the face $\mathcal{F}_{w}$ is nontrivial. Next, we can obtain that

$$
\begin{array}{r}
0 \preceq\left(\bar{s}_{1} I-A\right)^{-1} z \preceq\left(\bar{s}_{1} I-A\right)^{-1} \frac{w}{a}=\frac{p}{a} \\
=\frac{w-l}{\left(\bar{s}_{1}-\bar{s}_{0}\right) a} \preceq \frac{w}{\left(\bar{s}_{1}-\bar{s}_{0}\right) a},
\end{array}
$$

which yields that $\mathcal{F}_{w}$ is invariant with respect to $\left(\bar{s}_{1} I-A\right)^{-1}$, so is to $\left(\bar{s}_{0} I-A\right)^{-1}$. According to Definition 2.8 and 2.9 , it can be shown that $\left(\bar{s}_{0} I-A\right)^{-1} \notin \pi^{\prime}(\mathcal{K})$. In addition, by Definition 2.6, we can verify the existence of an eigenvector of $\left(\bar{s}_{0} I-A\right)^{-1}$ located on $\partial \mathcal{K}$, so is to $A$ since $\left(\bar{s}_{0} I-A\right)^{-1}$ and $A$ share the same eigenvectors, which therefore leads to contradiction. Hence, if $A$ is a $\mathcal{K}$-irreducible matrix, $\left(s_{0} I-A\right)^{-1}$ is a $\mathcal{K}$-positive matrix for any $s_{0} \geq 0$.

The following corollaries specialize the result in Theorem 4.2 to positive systems, dynamic covariance systems, and Lorentz cone-invariant systems. Note that all related cones $\mathbb{R}_{+}^{n}, \mathcal{S}_{+}^{n}$ and $\mathcal{L}_{+}^{n}$ are self-dual and the corresponding $\Lambda_{\mathbb{R}_{+}^{n}}, \Lambda_{\mathcal{S}_{+}^{n}}, \Lambda_{\mathcal{L}_{+}^{n}}$ are specified in (4.1).

Corollary 4.4. Consider a stable positive system in (2.1) such that $A \in \varpi^{\prime}\left(\mathbb{R}_{+}^{n}\right)$. For an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathbb{R}_{+}^{n}}$, the defense strategy with $\forall c \in \Pi=$ $\Lambda_{\mathbb{R}_{+}^{n}}$ is successful.
For the positive cone $\mathbb{R}_{+}^{n}$, we have $\forall e_{i} \in \partial \mathbb{R}_{+}^{n}$. Since the matrix $A$ is $\mathbb{R}_{+}^{n}$-irreducible, i.e., $A \in \varpi^{\prime}\left(\mathbb{R}_{+}^{n}\right)$, in light of Theorem 4.2, we have $\Pi=\Lambda_{\mathbb{R}_{+}^{n}}$. Therefore, for the stable positive system with an $\mathbb{R}_{+}^{n}$-irreducible matrix $A$, all the positions (states) are allowed to be attacked and an arbitrary single measurement can be successful.

Corollary 4.5. Consider a stable dynamic covariance system in (2.1) such that $A \in \varpi^{\prime}\left(\mathcal{S}_{+}^{n}\right)$. For an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathcal{S}_{+}^{n}}$, the defense strategy with $\forall c \in \Pi=\Lambda_{\mathcal{S}_{+}^{n}}$ is successful.

Not all Euclidean coordinates in $\mathbb{R}^{\frac{n(n+1)}{2}}$ belong to the PSD cone $\mathcal{S}_{+}^{n}$. We have $\Lambda_{\mathcal{S}_{+}^{n}}=$ $\left\{\operatorname{vec}\left(\operatorname{diag}\left(e_{i}\right)\right), i=1, \ldots, n\right\}$ with $\operatorname{vec}\left(\operatorname{diag}\left(e_{i}\right)\right) \in \partial \mathcal{S}_{+}^{n}$. Hence, since $A \in \varpi^{\prime}\left(\mathcal{S}_{+}^{n}\right)$, it follows that $\Pi=\Lambda_{\mathcal{S}_{+}^{n}}$.

Corollary 4.6. Consider a stable Lorentz cone-invariant system in (2.1) such that $A \in \varpi\left(\mathcal{L}_{+}^{n}\right)$. For an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathcal{L}_{+}^{n}}$, the defense strategy with $\forall c \in \Pi=\Lambda_{\mathcal{L}_{+}^{n}}$ is successful.
For the Lorentz cone $\mathcal{L}_{+}^{n}$, there exists only $e_{1} \in \mathcal{L}_{+}^{n}$, moreover $e_{1} \in \operatorname{int} \mathcal{L}_{+}^{n}$. Then we have $\Pi=\Lambda_{\mathcal{L}_{+}^{n}}=\left\{e_{1}\right\}$. For Lorentz cone-invariant systems, only the position labeled by $e_{1}$ can be attacked and the same position should also be measured to achieve the protection.

It is worth emphasizing that existing works mainly focus on the protection of static systems, like power systems, in which undetectable attacks should hide in the kernel space of the studied matrix, projecting attack signals into the measurement residual. This work extends this protection scheme into dynamic systems and prevents undetectable attacks corresponding to non-minimum phase zeros.
4.3. Marginally Stable Cone-Invariant System. An LTI system is marginally stable if and only if the real part of every eigenvalue of the matrix $A$ is non-positive and at least one simple eigenvalue is located on the imaginary axis. In this section, we extend the preceding defensive strategies to marginally stable systems.

ThEOREM 4.7. Given a marginally stable cone-invariant system (2.1) over a cone $\mathcal{K}$, if $A \in \varpi^{\prime}(\mathcal{K})$, for an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathcal{K}}$, the defense strategy with $\forall c \in \Pi$ is successful, where $\Pi=\left\{e_{j}: e_{j} \in \mathcal{K}^{*}\right\}$.

Proof. For $s_{0}>0$, the proof is technically similar to that for Theorem 4.2, thus, omitted. For $s_{0}=0$, in light of Lemma 2.11, there exists a zero eigenvalue of $A \in$ $\varpi^{\prime}(\mathcal{K})$ and $A$ is not invertible. Consider the matrix $\left[s_{0}-A b ; c^{\top} 0\right]=\left[-A e_{i} ; e_{j}^{\top} 0\right]$. If $e_{i} \notin \operatorname{Im}(A) \cap \Lambda_{\mathcal{K}}$, we can easily verify that either the matrix $\left[-A e_{i} ; e_{j}^{\top} 0\right]$ has full rank or it is not full rank but $d_{0}=0$, satisfying the conditions (i) and (ii) in Definition 4.1, respectively. Assume that there exists $e_{i} \in \operatorname{Im}(A) \cap \Lambda_{\mathcal{K}}$, which also indicates a vector $\xi$ such that $e_{i}=A \xi$. From Lemma 2.11, there exists a vector $y \in \operatorname{int} \mathcal{K}^{*}$ satisfying $y^{\top} A=0$. Therefore, it follows that

$$
\left\langle y, e_{i}\right\rangle=\langle y, A \xi\rangle=\left\langle y A^{\top}, \xi\right\rangle=0
$$

On the other hand, since $y \in \operatorname{int} \mathcal{K}^{*}$ and $e_{i} \in \Lambda_{\mathcal{K}} \in \mathcal{K}$, by Lemma 2.2, we have $\left\langle y, e_{i}\right\rangle>0$, which leads to contradiction.
5. Multi-Agent Systems. In this section, we consider sensor placement strategies for multi-agent systems against undetectable attacks. We show that first-order multi-agent systems can be formulated as a special case of marginally stable positive systems, whereas second-order multi-agent systems are not. Nevertheless, the features of positive cone $\mathbb{R}_{+}^{n}$ still shed useful lights into the defense strategy of second-order systems.
5.1. Algebraic Graph Theory. Let $\mathcal{G}=(\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote a weighted digraph with the set of agents (vertices or nodes) $\mathcal{V}=\left\{v_{1}, \ldots, v_{n}\right\}$, the set of edges (links) $\mathcal{E}=\mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $A=\left(a_{i j}\right)$ with nonnegative elements $a_{i j}$. An edge of $\mathcal{G}$ is denoted by $\left(v_{i}, v_{j}\right) \in \mathcal{E}$ if there exists a directed link from agent $v_{i}$ to agent $v_{j}$ with weight $a_{i j}>0$, i.e., agent $v_{j}$ can receive information from agent $v_{i}$. Denote $\mathcal{N}_{i}=\left\{j \in \mathcal{V}:\left(v_{j}, v_{i}\right) \in \mathcal{E}\right\}$ as the neighborhood set of node $i$. A path from node $v_{1}$ to node $v_{k}$ is a sequence of nodes $v_{1}, \ldots, v_{k}$, such that for each $i, 1 \leq i \leq k-1$, $\left(v_{i}, v_{i+1}\right)$ is an edge. The in-degree of agent $v_{i}$ is defined as $\operatorname{deg}_{i}^{\text {in }}=\sum_{j=1}^{n} a_{i j}$. Define for the graph $\mathcal{G}$ the Laplacian matrix $L=\mathcal{D}-A$, where $\mathcal{D} \triangleq \operatorname{diag}\left(\operatorname{deg}_{1}^{\mathrm{in}}, \ldots, \operatorname{deg}_{n}^{\mathrm{in}}\right)$ is the in-degree matrix. Its row sums equal to zero. Note that any negative Laplacian matrix $-L$ is a Metzler matrix, which is cross-positive over the positive cone $\mathbb{R}_{+}^{n}$. And it follows that $L$ can be decomposed as $L=\alpha I-\bar{L}$, where $\alpha=\rho(\bar{L})$ is the spectral radius of $\bar{L}$, and $\bar{L} \in \pi\left(\mathbb{R}_{+}^{n}\right)$.

Connectivity is one of the basic concepts in graph theory. A digraph is strongly connected if it contains a directed path for every pair of vertices. Meanwhile, the Laplacian matrix $L$ is irreducible if and only if its corresponding digraph is strongly connected. Different from the strong connectivity, a weaker connectivity requires a digraph to have a directed spanning tree, by which we mean that a certain agent $v_{i}$ exists such that for any other agent, there is a directed path from $v_{i}$ to that agent. The agent $v_{i}$ is called a root of the spanning tree. A digraph can be broken down into strongly connected components and every component is a strongly connected subgraph. Consider a digraph with $M$ strongly connected components. The $i$-th strongly connected component is denoted as $S C C_{i}, i=1, \ldots, M$ with $n_{i}$ nodes. The Laplacian matrix in this case can be represented by a block triangular matrix, with a
proper arrangement of the nodes:

$$
L=\left[\begin{array}{cccc}
L_{1} & * & * & * \\
0 & L_{2} & * & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & L_{M}
\end{array}\right]
$$

In this form, $L_{i} \in \mathbb{R}^{n_{i} \times n_{i}}$. Clearly, if the digraph contains spanning trees, all feasible roots are located in the first component $S C C_{1}$ associated with the submatrix $L_{1}$ above. For example, Fig. 2 shows a digraph of 7 agents, divided into two strongly connected components. The corresponding Laplacian matrix is given by

$$
L=\left[\begin{array}{ccc|ccc}
1.5 & -1.5 & 0 & 0 & 0 & 0  \tag{5.1}\\
0 & 2.8 & -0.3 & -2.5 & 0 & 0 \\
-2 & 0 & 3.5 & -1.5 & 0 & 0 \\
\hline 0 & 0 & 0 & 0.1 & -0.1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -2.7 & 0 & 2.7
\end{array}\right] .
$$



Fig. 2: A digraph contains two strongly connected components
5.2. Single-Integrator Agent Dynamics. Consider the single-integrator agents with state-space equations given by

$$
\begin{equation*}
\dot{x}_{i}(t)=u_{i}(t) \tag{5.2}
\end{equation*}
$$

together with the distributed consensus protocol

$$
\begin{equation*}
u_{i}(t)=-\sum_{j \in \mathcal{N}_{i}}\left(x_{i}(t)-x_{j}(t)\right) \tag{5.3}
\end{equation*}
$$

where $x_{i} \in \mathbb{R}$ is the state and $u_{i} \in \mathbb{R}$ is the control law. Then the closed-loop system combining (5.2) and (5.3) under an undetectable attack can be written as:

$$
\begin{align*}
\dot{x}(t) & =-L x(t)+b d(t) \\
y(t) & =c^{\top} x(t) \tag{5.4}
\end{align*}
$$

where $x=\left[x_{1}, \ldots, x_{n}\right]^{\top}$, and $L$ is the Laplacian matrix of the corresponding digraph. Note that this first-order multi-agent system is a marginally stable cone-invariant system over the positive cone $\mathbb{R}_{+}^{n}$. When the digraph is strongly connected, in view of Remark 2.7, $L$ is irreducible and thus is $\mathbb{R}_{+}^{n}$-irreducible. According to Theorem 4.7, we have the following result immediately.

Corollary 5.1. Consider the first-order multi-agent system (5.4) defined on a strongly connected digraph. For an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathbb{R}_{+}^{n}}$, the defense strategy with $\forall c \in \Pi$ is successful, where $\Pi=\Lambda_{\mathbb{R}_{+}^{n}}$.

Moreover, going beyond Theorem 4.7, we can generalize Corollary 5.1 by weakening the strong connectivity requirement of the digraph in the following theorem.

THEOREM 5.2. Consider the first-order multi-agent system (5.4) defined on a digraph.
(1) For any $S C C_{i}$, define the set

$$
\begin{equation*}
\Lambda_{i}=\left\{e_{j}: \sum_{l=1}^{i-1} n_{l}<j \leq \sum_{l=1}^{i} n_{l}\right\} \tag{5.5}
\end{equation*}
$$

in which $\Lambda_{1}=\left\{e_{j}: 0<j \leq n_{1}\right\}$. For an undetectable attack (3.3) with $\forall b \in \Lambda_{i}$, the defense strategy with $\forall c \in \Pi_{i}$ is almost successful, where $\Pi_{i}=\Lambda_{i}$.
(2) If the digraph contains spanning trees, for an undetectable attack (3.3) with $\forall b \in$ $\Lambda_{\mathbb{R}_{+}^{n}}$, the defense strategy with $\forall c \in \Pi$ is almost successful, where $\Pi=\Lambda_{1}$.

Proof. Note that, for a general digraph, the corresponding Laplacian matrix $L$ may be reducible. For $s_{0}>0$, in light of Lemma 4.3, we have $\left(s_{0} I+L\right)^{-1} \in \pi\left(\mathbb{R}_{+}^{n}\right)$. We decompose $L$ as $L=\alpha I-\bar{L}$ with $\alpha=\rho(\bar{L})$ and $\bar{L} \in \pi\left(\mathbb{R}_{+}^{n}\right)$. Then it follows that

$$
\left(s_{0} I+L\right)^{-1}=\left(\left(s_{0}+\alpha\right) I-\bar{L}\right)^{-1}=\frac{1}{s_{0}+\alpha}\left(I-\frac{1}{s_{0}+\alpha} \bar{L}\right)^{-1}
$$

From $\rho\left(\frac{1}{s_{0}+\alpha} \bar{L}\right)<1$, we can refer to Taylor series and get

$$
\frac{1}{s_{0}+\alpha}\left(I-\frac{1}{s_{0}+\alpha} \bar{L}\right)^{-1}=\frac{1}{s_{0}+\alpha} \sum_{k=0}^{\infty}\left(\frac{1}{s_{0}+\alpha} \bar{L}\right)^{k} .
$$

For $\bar{L} \in \pi\left(\mathbb{R}_{+}^{n}\right)$, if $\bar{L}_{x, y}>0$, the $(x, y)$-th entry of $\bar{L}$, it implies that there exists an edge $\left(v_{x}, v_{y}\right)$ in the digraph. Thus, one can claim that $\bar{L}_{x, y}^{k}>0$ if and only if there exists at least one path from node $v_{x}$ to node $v_{y}$ within $k$ steps.

In the case (1), in view of the strong connectivity of $S C C_{i}$, then for any two nodes $v_{x}, v_{y}$ in $S C C_{i}$, there exists at least one directed path with not more than $n_{i}$ steps, yielding that $\sum_{k=0}^{\infty} \bar{L}_{x, y}^{k}>0$, so is to $\frac{1}{s_{0}+\alpha}\left(I-\frac{1}{s_{0}+\alpha} \bar{L}\right)_{x, y}^{-1}$. Consequently, we have $c^{\top}\left(s_{0} I+L\right)^{-1} b>0$ with $\forall s_{0}>0, \forall b \in \Lambda_{i}$, and $\forall c \in \Pi_{i}$.

In the case (2), the sets $\Lambda_{\mathbb{R}_{+}^{n}}$ and $\Pi=\Lambda_{1}$ can be determined in a similar manner by noting that $S C C_{1}$ contains all roots of spanning trees in the digraph.
5.3. Double-Integrator Agent Dynamics. Consider the double-integrator agents with state-space equations described by

$$
\begin{align*}
& \dot{\xi}_{i}(t)=\zeta_{i}(t)  \tag{5.6}\\
& \dot{\zeta}_{i}(t)=u_{i}(t)
\end{align*}
$$

where $\xi_{i} \in \mathbb{R}$ and $\zeta_{i} \in \mathbb{R}$ are the corresponding states and $u_{i} \in \mathbb{R}$ denotes the distributed control law. Next, we extend our preceding sensor placement strategies into second-order multi-agent systems which are not invariant with the positive cone $\mathbb{R}_{+}^{n}$. Before showing main results of this section, we provide the following lemma at first.

Lemma 5.3. For a Laplacian matrix $L$ defined on a strongly connected digraph, we have $e_{i} \notin \operatorname{Im}(L)$ and $e_{i} \notin \operatorname{Im}\left(L^{\top}\right), \forall e_{i}, i \in\{1, \ldots, n\}$.

Proof. We prove the lemma by contradiction. Assume that there exists $e_{i}$ such that $e_{i} \in \operatorname{Im}(A)$. Then we can obtain $x \neq 0$ satisfying $e_{i}=L x$. In light of Lemma 2.11, there exists a strictly positive vector $v$ such that $v^{\top} L=0$. It follows that $v^{\top} e_{i}=v^{\top} L x=0$, showing that the $i$ th entry of $v$ is zero. This fact contradicts with the strict positivity of $v$. Also, the result $e_{i} \notin \operatorname{Im}\left(L^{\top}\right)$ can be proved in a similar manner.

Consider two general types of distributed control laws, respectively. The first class of distributed control law is designed as

$$
\begin{equation*}
u_{i}(t)=-k_{i} \zeta_{i}(t)+\sum_{j \in N_{i}} v_{i, j}\left(\xi_{j}(t)-\xi_{i}(t)\right) \tag{5.7}
\end{equation*}
$$

where $k_{i}, v_{i, j}>0, \forall i, j=1, \ldots, N[28,29]$. One available physical interpretation of $\xi_{i}(t)$ and $\zeta_{i}(t)$ are the position and the velocity of the agent $i$ in a mechanical system, and $k_{i}$ can be interpreted as the radio of agent's damper to mass. The agents under the control law (5.7) move toward the position of their neighbors while damping their current velocity [30]. The closed-loop system combining (5.6) and (5.7) under an undetectable attack can be written as:

$$
\begin{align*}
{\left[\begin{array}{l}
\dot{\xi}(t) \\
\dot{\zeta}(t)
\end{array}\right] } & =\left[\begin{array}{cc}
0 & I \\
-L & -K
\end{array}\right]\left[\begin{array}{l}
\xi(t) \\
\zeta(t)
\end{array}\right]+b d(t)  \tag{5.8}\\
y(t) & =c^{\top}\left[\begin{array}{l}
\xi(t) \\
\zeta(t)
\end{array}\right]
\end{align*}
$$

where $\xi(t)=\left[\xi_{1}(t), \ldots, \xi_{n}(t)\right]^{\top}, \zeta(t)=\left[\zeta_{1}(t), \ldots, \zeta_{n}(t)\right]^{\top}, L$ is the Laplacian matrix, and $K=\operatorname{diag}\left\{k_{1}, \ldots, k_{n}\right\}$. Denote the canonical basis in $\mathbb{R}^{2 n}$ as $\left\{f_{i}, i=1, \ldots, 2 n\right\}$, i.e., $f_{j}=\left[e_{j}^{\top} \mathbf{0}^{\top}\right]^{\top}$ and $f_{j+n}=\left[\mathbf{0}^{\top} e_{j}^{\top}\right]^{\top}, j=1, \ldots, n$, where $\mathbf{0}$ denotes an $n$-dimensional zero vector.

Theorem 5.4. Consider the second-order multi-agent system (5.8) defined on a strongly connected digraph. For an undetectable attack (3.3) with $\forall b \in \Lambda_{\mathbb{R}_{+}^{2 n}}$, the defense strategy with $\forall c \in \Pi$ is successful, where $\Pi=\Lambda_{\mathbb{R}_{+}^{2 n}}=\left\{f_{i}, i=1, \ldots, 2 n\right\}$.

Proof. We start by the case $s_{0}>0$. Consider all the four possible combinations of $b$ and $c$ with $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ or $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$, and $c=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ or $c=$ $\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]^{\top}, \forall e_{i}, e_{j}:$
(i) If $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]$, we have

$$
\begin{aligned}
& \mathbf{R}\left(\left[\begin{array}{cc}
s_{0} I-A & b \\
c^{\top} & 0
\end{array}\right]\right)=\mathbf{R}\left(\left[\begin{array}{ccc}
s_{0} I & -I & e_{i} \\
L & s_{0} I+K & \mathbf{0} \\
e_{j}^{\top} & \mathbf{0}^{\top} & 0
\end{array}\right]\right) \\
& =\mathbf{R}\left(\begin{array}{cc}
\left.\left[\begin{array}{ccc}
\mathbf{0}_{n \times n} & -I & e_{i} \\
I & \mathbf{0}_{n \times n} & \left(s_{0} I+K\right) e_{i} \\
\mathbf{0}^{\top} & \mathbf{0}^{\top} & -e_{j}^{\top}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right) e_{i}
\end{array}\right]\right)
\end{array} .=\right.\text {, }
\end{aligned}
$$

where $\mathbf{0}_{n \times n} \in \mathbb{R}^{n \times n}$ is the zero matrix. The matrix $s_{0}^{2} I+s_{0} K+L$ is invertible since $s_{0}>0$ and $K=\operatorname{diag}\left(k_{1}, \ldots, k_{n}\right)$ is a strictly positive diagonal matrix. Therefore, it holds that

$$
\mathbf{R}\left(\left[\begin{array}{cc}
s_{0} I-A & b  \tag{5.9}\\
c^{\top} & 0
\end{array}\right]\right)=2 n+1
$$

if and only if

$$
e_{j}^{\top}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right) e_{i} \neq 0, \forall e_{i}, e_{j} .
$$

(ii) If $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]$, we have that (5.9) holds if and only if

$$
e_{j}^{\top}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1} e_{i} \neq 0, \forall e_{i}, e_{j}
$$

(iii) If $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]$, we have that (5.9) holds if and only if

$$
e_{j}^{\top} s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1} e_{i} \neq 0, \forall e_{i}, e_{j}
$$

(iv) If $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]$, we have that (5.9) holds if and only if $e_{j}^{\top}\left(I-s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right)\right) e_{i} \neq 0, \forall e_{i}, e_{j}$.
The cases (i), (ii), and (iii) require us to prove that all entries of the matrix $\left(s_{0}^{2} I+\right.$ $\left.s_{0} K+L\right)^{-1}$ are non-zero, whilst we need to clarify that the matrix $I-s_{0}\left(s_{0}^{2} I+s_{0} K+\right.$ $L)^{-1}\left(s_{0} I+K\right)$ holds no zero entry for the case (iv).

For the cases (i), (ii), and (iii), one can easily obtain that $-K-\left(1 / s_{0}\right) L$ is a stable matrix and we omit the proof. We then prove that all entries of $\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}$ are non-zero via proving that $-K-\left(1 / s_{0}\right) L$ is a Metzler matrix, i.e., $-K-\left(1 / s_{0}\right) L \in$ $\varpi\left(\mathbb{R}_{+}^{n}\right)$. According to Definition 2.5, $\forall x, y \in \mathbb{R}_{+}^{n}$ such that $\langle y, x\rangle=0$, it follows that

$$
\left\langle y,\left(-K-\left(1 / s_{0}\right) L\right) x\right\rangle=\left(1 / s_{0}\right)\langle y,-L x\rangle-\langle y, K x\rangle
$$

Since $-L \in \varpi\left(\mathbb{R}_{+}^{n}\right)$, we have $\langle y,-L x\rangle \geq 0$. On the other hand, we can easily obtain that $\langle y, K x\rangle=0$ on account of the fact that $\langle y, x\rangle=0$ and the strict positiveness of the diagonal matrix $K$. As such, by noting that $\left\langle y,\left(-K-\left(1 / s_{0}\right) L\right) x\right\rangle \geq 0$, we claim that $-K-\left(1 / s_{0}\right) L \in \varpi\left(\mathbb{R}_{+}^{n}\right)$. Additionally, since $L$ is irreducible, we have $-K-\left(1 / s_{0}\right) L \in \varpi^{\prime}\left(\mathbb{R}_{+}^{n}\right)$. Hence, all the entries of $\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}$ are strictly positive by Theorem 4.2.

For the case (iv), we consider the matrix $I-s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right)$. Since all the entries of $\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}$ are strictly positive, the matrix $s_{0}\left(s_{0}^{2} I+s_{0} K+\right.$ $L)^{-1}\left(s_{0} I+K\right)$ is also a strictly positive matrix. One can easily see that

$$
s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right)=\left(I+\left(s_{0}^{2} I+s_{0} K\right)^{-1} L\right)^{-1}
$$

in which the matrix $\left(s_{0}^{2} I+s_{0} K\right)^{-1} L$, denoted as $\tilde{L}$, is also a Laplacian matrix defined on a strongly connected digraph. Together with the fact that $(I+\tilde{L})^{-1}$ is a row stochastic matrix, which is implied by that $(I+\tilde{L})$ is a row stochastic matrix, it follows that the $(i, j)$-th entry $(I+\tilde{L})_{i, j}^{-1} \in(0,1), \forall i, j$, which further leads to

$$
\begin{aligned}
& \left(I-s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right)\right)_{i, i}>0 \\
& \left(I-s_{0}\left(s_{0}^{2} I+s_{0} K+L\right)^{-1}\left(s_{0} I+K\right)\right)_{i, j}<0, \forall i \neq j
\end{aligned}
$$

Finally, we study the case $s_{0}=0$. When $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ or $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$, and $c^{\top}=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]$, the matrices

$$
\left[\begin{array}{ccc}
\mathbf{0}_{n \times n} & -I & e_{i}  \tag{5.10}\\
L & K & \mathbf{0} \\
e_{j}^{\top} & \mathbf{0}^{\top} & 0
\end{array}\right] \quad \text { or } \quad\left[\begin{array}{ccc}
\mathbf{0}_{n \times n} & -I & \mathbf{0} \\
L & K & e_{i} \\
e_{j}^{\top} & \mathbf{0}^{\top} & 0
\end{array}\right]
$$

are both full rank in view of the fact, from Lemma 5.3, that $e_{i} \notin \operatorname{Im}(L), e_{j} \notin$ $\operatorname{Im}\left(L^{\top}\right), \forall e_{i}, e_{j}$. When $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ or $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$, and $c^{\top}=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]$, clearly, the matrices

$$
\left[\begin{array}{ccc}
\mathbf{0}_{n \times n} & -I & e_{i}  \tag{5.11}\\
L & K & \mathbf{0} \\
\mathbf{0}^{\top} & e_{j}^{\top} & 0
\end{array}\right] \quad \text { or } \quad\left[\begin{array}{ccc}
\mathbf{0}_{n \times n} & -I & \mathbf{0} \\
L & K & e_{i} \\
\mathbf{0}^{\top} & e_{j}^{\top} & 0
\end{array}\right]
$$

are not full rank. However, since $e_{i} \notin \operatorname{Im}(L), \forall e_{i}$, recall the equation (3.4), we can arrive at $d_{0}=0$ directly. The proof is now completed by invoking the condition (ii) in Definition 4.1.

Consider the second type of distributed control law given by

$$
\begin{equation*}
u_{i}(t)=r \sum_{j \in N_{i}} v_{i, j}\left(\zeta_{j}(t)-\zeta_{i}(t)\right)+\sum_{j \in N_{i}} v_{i, j}\left(\xi_{j}(t)-\xi_{i}(t)\right), \tag{5.12}
\end{equation*}
$$

where $r>0$ and $v_{i, j}>0, \forall i, j=1, \ldots, N[18,31]$. One motivation for such distributed control law comes from power systems, where $\xi_{i}(t)$ and $\zeta_{i}(t)$ are interpreted as the phase and the frequency of the agent $i$. Under this control law, the power system reaches consensus while not only the phase differences but also the frequency difference are penalized [32]. The compact form of the closed-loop system combining (5.6) and (5.12) under an undetectable attack is given by

$$
\begin{align*}
{\left[\begin{array}{c}
\dot{\xi}(t) \\
\dot{\zeta}(t)
\end{array}\right] } & =\left[\begin{array}{cc}
0 & I \\
-L & -r L
\end{array}\right]\left[\begin{array}{l}
\xi(t) \\
\zeta(t)
\end{array}\right]+b d(t),  \tag{5.13}\\
y(t) & =c^{\top}\left[\begin{array}{l}
\xi(t) \\
\zeta(t)
\end{array}\right] .
\end{align*}
$$

ThEOREM 5.5. Consider a second-order multi-agent system (5.13) defined on a strongly connected digraph.
(1) For an undetectable attack (3.3) with $\forall b \in \Lambda_{1}=\left\{f_{i}, i=1, \ldots, n\right\}$, the defense strategy with $\forall c \in \Pi$ is almost successful, where $\Pi=\Lambda_{\mathbb{R}_{+}^{2 n}}$.
(2) For an undetectable attack (3.3) with $\forall b \in \Lambda_{2}=\left\{f_{i}, \stackrel{+}{i}=n+1, \ldots, 2 n\right\}$, the defense strategy with $\forall c \in \Pi$ is successful, also $\Pi=\Lambda_{\mathbb{R}_{+}^{2 n}}$.

Proof. Consider the case $s_{0}>0$. Likewise, from the four possible combinations of $b$ and $c$, we have:
(i) If $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]$, we need to verify that $e_{\underline{j}}^{\top}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right) e_{i} \neq 0, \forall e_{i}, e_{j}$.
(ii) If $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}e_{j}^{\top} & \mathbf{0}^{\top}\end{array}\right]$, we need to verify that

$$
e_{j}^{\top}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1} e_{i} \neq 0, \forall e_{i}, e_{j}
$$

(iii) If $b=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{i}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]$, we need to verify that $e_{j}^{\top} s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1} e_{i} \neq 0, \forall e_{i}, e_{j}$,
(iv) If $b=\left[\begin{array}{ll}e_{i}^{\top} & \mathbf{0}^{\top}\end{array}\right]^{\top}$ and $c^{\top}=\left[\begin{array}{ll}\mathbf{0}^{\top} & e_{j}^{\top}\end{array}\right]$, we need to verify that

$$
e_{j}^{\top}\left(I-s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)\right) e_{i} \neq 0, \forall e_{i}, e_{j}
$$

The cases (i) and (iv) represent $\Lambda_{1}$ and the cases (ii) and (iii) point to $\Lambda_{2}$.
For the cases (ii) and (iii), it follows as in the proofs of Theorem 4.2 and 5.4 that all the entries of the matrix $\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}$ are non-zero.

For the case (i), we study the matrix the matrix $\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)=$ $\left(s_{0} I+\left(s_{0} I+r L\right)^{-1} L\right)^{-1}$. We first show that $\left(s_{0} I+r L\right)^{-1} L$ is Laplacian. Clearly, its row sums equal to zero. Let $L=\alpha I-\bar{L}$ with $\alpha=\rho(\bar{L})$ and $\bar{L} \in \pi\left(\mathbb{R}_{+}^{n}\right)$. It follows that

$$
\left(s_{0} I+r L\right)^{-1} L=\frac{1}{s_{0}+r \alpha}\left(I-\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{-1}(\alpha I-\bar{L})
$$

Since $\rho\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)<1$, we have $\left(I-\left(r /\left(s_{0}+r \alpha\right)\right) \bar{L}\right)^{-1}=\sum_{k=0}^{\infty}\left(\left(r /\left(s_{0}+r \alpha\right)\right) \bar{L}\right)^{k}$. This leads to

$$
\begin{aligned}
\left(s_{0} I+r L\right)^{-1} L & =\frac{1}{s_{0}+r \alpha} \sum_{k=0}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k}(\alpha I-\bar{L}) \\
& =\frac{\alpha}{s_{0}+r \alpha} I+\frac{\alpha}{s_{0}+r \alpha} \sum_{k=1}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k}-\frac{1}{r} \sum_{k=1}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k} \\
& =\frac{\alpha}{s_{0}+r \alpha} I-\frac{s_{0}}{r\left(s_{0}+r \alpha\right)} \sum_{k=1}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k} \\
& =\frac{s_{0}}{r\left(s_{0}+r \alpha\right)}\left(\frac{r \alpha}{s_{0}} I-\sum_{k=1}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k}\right)
\end{aligned}
$$

where $\left(\left(r /\left(s_{0}+r \alpha\right)\right) \bar{L}\right)^{k} \in \pi\left(\mathbb{R}_{+}^{n}\right), \forall k$, . Furthermore,

$$
\rho\left(\sum_{k=1}^{\infty}\left(\frac{r}{s_{0}+r \alpha} \bar{L}\right)^{k}\right)=\sum_{k=1}^{\infty}\left(\frac{r \alpha}{s_{0}+r \alpha}\right)^{k}=\frac{r \alpha}{s_{0}}
$$

where the Perron-Frobenius theorem yields the first equality. Hence, $\left(s_{0} I+r L\right)^{-1} L$ is an M-matrix with row sums equal to zero, that is a Laplacian matrix [29]. In addition, it is irreducible. We hence claim that all the entries of $\left(s_{0} I+\left(s_{0} I+r L\right)^{-1} L\right)^{-1}$ are strictly positive.

For the case (iv), we consider the matrix $I-s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)$. We have proved that all entries of $s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)$ are strictly positive. On the other hand, one can easily see that $s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)=$ $\left(I+\left(I+\left(r / s_{0}\right) L\right)^{-1} L\right)^{-1}$ is a row stochastic matrix since $I+\left(I+\left(r / s_{0}\right) L\right)^{-1} L$ itself is row stochastic, which yields that

$$
\begin{aligned}
& \left(I-s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)\right)_{i, i}>0 \\
& \left(I-s_{0}\left(s_{0}^{2} I+\left(s_{0} r+1\right) L\right)^{-1}\left(s_{0} I+r L\right)\right)_{i, j}<0, \forall i \neq j
\end{aligned}
$$

Consider the case $s_{0}=0$. In light of Lemma 5.3, in the case (ii), the associated matrix $\left[-A, b ; c^{\top}, 0\right]$ has full rank. In the case (iii), the matrix is not full rank, but it can be easily verified that $d_{0}$ associated with $s_{0}=0$ always equals to zero. Therefore, for the attack with $\forall b \in \Lambda_{2}\left(\Lambda_{1}\right)$, any defense $c$ located in the set $\Pi$ is (almost) successful.

In Theorem 5.5, it holds that $\Lambda_{1} \cup \Lambda_{2}=\Lambda_{\mathbb{R}_{+}^{2 n}}$. Therefore, together with Theorem 5.4, for both second-order multi-agent systems (5.8) and (5.13), all the potentially vulnerable positions for undetectable attacks have been considered. At the same
time, an arbitrary single measurement in $\Pi=\left\{f_{i}, i=1, \ldots, 2 n\right\}$ is successful to prevent those attacks.
6. Further Discussion on MIMO Case. In the preceding sections, we proposed a single-sensor placement strategy to defend a single attack. This strategy is effective for single-input, single-output (SISO) systems. We now discuss its extension to MIMO systems.

Assume that multiple undetectable attacks enter the system through $m$ channels. Specifically, in the system (3.1), let $B=\left[b_{1}, \ldots, b_{m}\right] \in \mathbb{R}^{n \times m}$, and $C^{\top}=\left[c_{1}, \ldots, c_{l}\right] \in$ $\mathbb{R}^{n \times l}$, where $b_{i}, c_{j}$ are the $i$ th and $j$ th Euclidean coordinates of $\mathbb{R}^{n}$, respectively. According to Definition 4.1, for the multiple attacks, the defense strategy will be successful if and only if

$$
\begin{equation*}
\mathbf{R}\left(C\left(s_{0} I-A\right)^{-1} B\right)=m \tag{6.1}
\end{equation*}
$$

for each $s_{0}$ with $s_{0} \geq 0$. Apparently, for the extreme case such that $m=l=n$ and $B=C=I$, it yields that

$$
\mathbf{R}\left(C\left(s_{0} I-A\right)^{-1} B\right)=\mathbf{R}\left(\left(s_{0} I-A\right)^{-1}\right)=n
$$

That is, if all nodes and channels are placed by sensors, then even they all have been attacked, the defense strategy is still successful. Nevertheless, if $m \ll n$, together with the cost-efficient consideration, the complete sensor placement is indeed impractical. Under this circumstance, we assume that $m=l<n$. The condition (6.1) can reduce to

$$
\begin{equation*}
\operatorname{det}\left(C\left(s_{0} I-A\right)^{-1} B\right) \neq 0, \quad \forall s_{0} \geq 0 \tag{6.2}
\end{equation*}
$$

We use det to represent the matrix determinant. Different from the SISO case concentrating on the entries of $\left(s_{0} I-A\right)^{-1}$, for the MIMO case, it requires us to study the minors of $\left(s_{0} I-A\right)^{-1}$. The cut-down columns and rows are determined by the matrices $B$ and $C$.

As an example, we design defense strategy against multiple attacks over the firstorder multi-agent system (5.4) $\dot{x}(t)=-L x(t)$ to demonstrate the highly increased complexity from the SISO scenario to the MIMO one. From the condition (6.2), it is necessary to examine $\operatorname{det}\left(C\left(s_{0} I+L\right)^{-1} B\right), \quad \forall s_{0}>0$. Denote by $\mathcal{I}_{B}, \mathcal{I}_{C} \subseteq\{1, \ldots, n\}$ the sets of integers with the cardinality $m$ indexed by the matrices $B$ and $C$. Then we use $\left(s_{0} I+L\right)_{\mathcal{I}_{C} \mathcal{I}_{B}}^{-1}$ to denote the submatrix of $\left(s_{0} I+L\right)^{-1}$ restricted to rows ordered by $C$ and columns ordered by $B$. From the Jacobi's complementary minor formula [33], it follows that $\operatorname{det}\left(\left(s_{0} I+L\right)_{\mathcal{I}_{C} \mathcal{I}_{B}}^{-1}\right) \neq 0$ if and only if $\operatorname{det}\left(\left(s_{0} I+L\right)_{\overline{\mathcal{I}}_{B} \overline{\mathcal{I}}_{C}}\right) \neq 0$, where $\overline{\mathcal{I}}_{B}$ and $\overline{\mathcal{I}}_{C}$ are the complementary sets of $\mathcal{I}_{B}$ and $\mathcal{I}_{C}$. We add a fictitious vertex on the underlying digraph indexed by 0 , where all other vertices point to the vertex 0 with the weight $s_{0}$ on each new edge. The enlarged Laplacian matrix is noted as $\hat{L} \in \mathbb{R}^{(n+1) \times(n+1)}$. Then it is straightforward that

$$
\operatorname{det}\left(\left(s_{0} I+L\right)_{\overline{\mathcal{I}}_{B} \overline{\mathcal{I}}_{C}}\right)=\operatorname{det}\left(\hat{L}_{\overline{\mathcal{I}}_{B} \overline{\mathcal{I}}_{C}}\right)
$$

Therefore, the target injection set and the sensor placement set should contain all available sets $\mathcal{I}_{B}$ and $\mathcal{I}_{C}$ such that $\operatorname{det}\left(\hat{L}_{\overline{\mathcal{I}}_{B} \overline{\mathcal{I}}_{C}}\right) \neq 0, \forall s_{0}>0$. They can be solved effectively based on the underlying digraph's structure in light of the all minors matrix tree theorem [34]. Note that if the digraph is strongly connected and $m=1$, we can arrive at Corollary 5.1 directly from the matrix tree theorem.

In the next step, we will continue to extend our analysis to the multiple attacks over more general cone-invariant systems in a progressive manner.
7. Examples. In this section, we use two power network models as examples to illustrate our results, demonstrating how sensors may be placed to prevent undetectable attacks. We consider the decentralized voltage control schemes in interconnected buses over networks. Each bus can be controlled to regulate its voltage magnitude and phase-angle independently. We refer to [32, 35] for the network models and the terminologies.

Example 7.1. In this example, we verify the result in Theorem 4.2 for positive systems. We start by considering the voltage magnitude dynamics of the power network that consists of a set of interconnected buses. Assume that the power network is outfitted with $n$ buses which are all inverter buses(microgrids). For bus $i$, its voltage magnitude dynamics can be modeled by a single integrator

$$
\tau_{i} \dot{V}_{i}(t)=u_{i}(t)
$$

where $\tau_{i}>0$ is the inverter's time-constant. Consider the voltage quadratic droop controller [36] described by

$$
u_{i}(t)=-\kappa_{i} V_{i}(t)\left(V_{i}(t)-V_{i}^{*}(t)\right)-Q_{i}(t),
$$

where $\kappa_{i}>0, V_{i}^{*}$, and $Q_{i}$ are the controller gain, the nominal voltage, and the reactive power injection at bus $i$, respectively. The reactive power injection $Q_{i}$ is given by:

$$
Q_{i}(t)=-\sum_{j \in \mathcal{N}_{i}} k_{i j} \cos \left(\theta_{i}(t)-\theta_{j}(t)\right)
$$

where $k_{i j}=V_{i} V_{j} b_{i j}, b_{i j}$ is the susceptance of the power line between buses $i$ and $j$, and $\theta_{i}$ is the phase angle of bus $i$. Since all the phase angles are close, under the standard decoupling approximation, it follows that $\cos \left(\theta_{i}-\theta_{j}\right) \approx 1$. Denoting $V=\left[V_{1}, \ldots, V_{n}\right]^{\top}$, $\tau=\left[\tau_{1}, \ldots, \tau_{n}\right]^{\top}$, and $\kappa=\left[\kappa_{1}, \ldots, \kappa_{n}\right]^{\top}$. After the Jacobian linearization around an equilibrium point $\bar{V}$, the state-space form (2.1) of the power network's voltage magnitude dynamics is given by the state $x=V-\bar{V}$ and the system matrix

$$
A=-\operatorname{diag}(\bar{V}) \operatorname{diag}(\tau)^{-1}\left(\operatorname{diag}(\kappa)+B_{\mathcal{G}}\right)
$$

where $B_{\mathcal{G}}$ is the susceptance matrix over the underlying topology.


Fig. 3: The 4-bus power network with a line topology.

Consider the 4-bus power network with a line topology, as depicted in Fig 3 [36]. It can be treated as the simplest model used in studies of the schematic of a parallel
microgrid. For simplicity, we assume that $\bar{V}_{i}=1$ and $\tau_{i}=1$ for all inverter buses. The controller gains $\kappa$ and the corresponding susceptance matrix $B_{\mathcal{G}}$ were arbitrarily generated within reasonable ranges. The voltage magnitude dynamics is characterized by

$$
\begin{aligned}
A & =\left[\begin{array}{cccc}
-4.03 & 1.48 & 0 & 0 \\
1.48 & -3.57 & 1.57 & 0 \\
0 & 1.57 & -3.24 & 0.64 \\
0 & 0 & 0.64 & -1.25
\end{array}\right], \quad b=\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right], \\
c^{\top} & =\left[\begin{array}{ccrc}
0 & 0 & 1 & 0
\end{array}\right],
\end{aligned}
$$

where $A$ is an irreducible and stable Metzler matrix, whilst $b$ and $c$ belong to $\Lambda_{\mathbb{R}_{+}^{4}}=$ $\Pi=\left\{e_{i}, i=1, \ldots, 4\right\}$. Since the unique real zero of the system is -1.25 , one possible solution of (3.4) with $s_{0}=-1.25$ is $x(0)-\tilde{x}(0)=[-5.5800-3.559608 .7322]^{\top}$ and $d_{0}=10.2441$. This implies an attack signal is $d(t)=-10.2441 e^{-1.25 t}$.

In Fig. 4, the solid lines indicate the trajectories initiated from $x(0)=[12.5822$ $10.037513 .444714 .7301]^{\top}$ with attack $d(t)$, while the dot lines are initiated from the possibly fraudulent initial state $\tilde{x}(0)=\left[\begin{array}{lll}18.1621 & 13.5972 & 13.44475 .9979\end{array}\right]^{\top}$. Note that the outputs of these two cases coincide always, which shows the undetectability of the attack $d(t)$. In Fig. 5, there are the trajectories initiated from $x(0)$ with (solid lines) and without (dash-dot lines) the attack $d(t)$. The attack $d(t)$ cannot affect the asymptotic stability of the system. In these figures, $x(t), \tilde{x}(t)$, and $\hat{x}(t)$ represent the normal state trajectories, the fraudulent state trajectories, and the attacked state trajectories, respectively.


Fig. 4: The undetectability of the attack $d(t)$ on the 4 -bus power network defined in Example 7.1.

Example 7.2. In this example, the result in Theorem 5.4 is verified. We turn to consider the phase-angle dynamics of power network with 9 buses, as depicted in Fig. 6 [35]. Assume that all buses are synchronous cases. The phase-angle dynamics of


Fig. 5: The influence of the attack $d(t)$ on the 4 -bus power network defined in Example 7.1.


Fig. 6: The 9-bus power network.
bus $i$ can be described by the so-called swing equation:

$$
m_{i} \ddot{\theta}_{i}(t)+d_{i} \dot{\theta}_{i}(t)-P_{m i}(t)+\sum_{j \in \mathcal{N}_{i}} P_{i j}(t)=0
$$

where $\theta_{i}$ is the phase angle of bus $i, m_{i}$ and $d_{i}$ are the inertia and damping coefficients, respectively, $P_{m i}$ is the mechanical input power, and $P_{i j}$ is the active power flow from bus $i$ to $j$. The active power flow $P_{i j}$ is given by:

$$
P_{i j}(t)=k_{i j} \sin \left(\theta_{i}(t)-\theta_{j}(t)\right)
$$

where $k_{i j}=V_{i} V_{j} b_{i j}, V_{i}$ is the voltage magnitude of bus $i$, and $b_{i j}$ is the susceptance
of the power line between buses $i$ and $j$. Consider the case when the phase angles are close, after the linearization, the phase-angle dynamics of bus $i$ can be rewritten as:

$$
m_{i} \ddot{\theta}_{i}(t)+d_{i} \dot{\theta}_{i}(t)-P_{m i}(t)+\sum_{j \in \mathcal{N}_{i}} k_{i j}\left(\theta_{i}(t)-\theta_{j}(t)\right)=0
$$

The 9-bus power network's topological parameters are noted in Fig. 6. For simplicity, we normalized the voltage magnitudes $V_{i}=1, \forall i$, while the rest dynamic coefficients of the buses were arbitrarily taken from reasonable values. The phase-angle dynamics of the power network can be rewritten in the state-space form (5.8) with the size $A \in \mathbb{R}^{18 \times 18}$. The system's state is denoted by $x=\left[\theta_{1}, \ldots, \theta_{9}, \dot{\theta}_{1}, \ldots, \dot{\theta}_{9}\right]^{\top}$. The attacker's target is specified as $x_{1}$. We choose to measure $x_{10}$ as the defense strategy, which can be readily available through phase measurement units (PMU). The real zeros of the system are 0 and -0.1123 . For $s_{0}=0$, the solution to (3.4) satisfies $d_{0}=0$. For $s_{0}=-0.1123$, one solution to the equation (3.4) is $x(0)-\tilde{x}(0)=$ [6.7189 7.6055 6.43016.7189 5.6482 6.1816 6.05406.8980 $7.15530 .0000-0.8538-0.7218-0.7542-0.6340$
$-0.6939-0.6796-0.7743-0.8032]^{\top}$ and $d_{0}=0.7542$, which implies the attack signal $d(t)=-0.7542 e^{-0.1123 t}$.

For the clarity, only the behaviors of buses 1,3, and 5 are recorded in figures. The solid lines in Fig. 7 are the trajectories initiated from $\left\{x_{i}(0), i=1,3,5,10,12,14\right\}$ with attack $d(t)$, while the dot lines are initiated from $\left\{\tilde{x}_{i}(0), i=1,3,5,10,12,14\right\}$. Note that the outputs of these two cases (the cyan line) are exactly the same and the undetectability is confirmed. In Fig. 8, there are the trajectories initiated from $x_{i}(0)$ with (solid lines) and without (dash lines) the attack $d(t)$. The 9-bus power system can still achieve synchronous with the attack $d(t)$ while the final synchronous points of $\left\{x_{j}(t), j=1,3,5\right\}$ are deflected.


Fig. 7: The undetectability of the attack $d(t)$ on the 9 -bus power network defined in Example 7.2.
8. Conclusion. An explicit and efficient detection and defense framework for cone-invariant systems and multi-agent systems against undetectable attacks has been


Fig. 8: The influence of the attack $d(t)$ on the 9 -bus power network defined in Example 7.2.
developed in this paper. We have shown that for an undetectable attacks getting into the system through any position in $\Lambda_{\mathcal{K}}$, any defense by placing one sensor in the union $\Pi$ is successful or almost successful. The sets $\Lambda_{\mathcal{K}}$ and $\Pi$ can be characterized by the intersection of the canonical basis vectors and the cone $\mathcal{K}$ or its dual cone $\mathcal{K}^{*}$, which are easily computable from a geometric perspective. For multi-agent systems with single- and double-integrator dynamics, we show that if the associated digraph is strongly connected, the corresponding sets $\Lambda_{\mathcal{K}}$ and $\Pi$ will both contain all nodes in the digraph.

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[^1]:    ${ }^{1}$ The inertia of a real symmetric matrix is a triple of the number of positive eigenvalues, zero eigenvalues, and negative eigenvalues.

[^2]:    ${ }^{2}$ The matrix $A$ is irreducible if no permutation matrix $P$ exists such that $P^{\top} A P=\left[\begin{array}{cc}A_{11} & A_{12} \\ 0 & A_{22}\end{array}\right]$, where $A_{11}$ and $A_{22}$ are square submatrices.

