Planar Turán Number of the 6-Cycle

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Abstract

Let $\exp(n, T, H)$ denote the maximum number of copies of T in an n-vertex planar graph which does not contain H as a subgraph. When $T = K_2$, $\exp(n, T, H)$ is the well studied function, the planar Turán number of H, denoted by $\exp(n, H)$. The topic of extremal planar graphs was initiated by Dowden (2016). He obtained sharp upper bound for both $\exp(n, C_4)$ and $\exp(n, C_5)$. Later on, Y. Lan, et al. continued this topic and proved that $\exp(n, C_6) \leq \frac{18(n-2)}{7}$. In this paper, we give a sharp upper bound $\exp(n, C_6) \leq \frac{5}{2}n - 7$, for all $n \geq 18$, which improves Lan's result. We also pose a conjecture on $\exp(n, C_k)$, for $k \ge 7$.

Keywords Planar Turán number, Extremal planar graph

Introduction and Main Results 1

In this paper, all graphs considered are planar, undirected, finite and contain neither loops nor multiple edges. We use C_k to denote the cycle on k vertices and K_r to denote the complete graph on r vertices.

One of the well-known results in extremal graph theory is the Turán Theorem [5], which gives the maximum number of edges that a graph on n vertices can have without containing a K_r as a subgraph. The Erdős-Stone-Simonovits Theorem [2, 3] then generalized this result and asymptotically determines ex(n, H) for all non-bipartite graphs H: $ex(n, H) = (1 - \frac{1}{\chi(H)-1})\binom{n}{2} + o(n^2)$, where $\chi(H)$ denotes the chromatic number of H. Over the last decade, a considerable amount of research work has been carried out in Turán-type problems, i.e., when host graphs are K_n , k-uniform hypergraphs or k-partite graphs, see [3, 6].

In 2016, Dowden [1] initiated the study of Turán-type problems when host graphs are planar, i.e., how many edges can a planar graph on n vertices have, without containing a given smaller graph? The planar Turán number of a graph H, $\exp(n, H)$, is the maximum number of edges in a planar graph on n vertices which does not contain H as a subgraph. Dowden [1] obtained the tight bounds $\exp(n, C_4) \leq \frac{15(n-2)}{7}$, for all $n \geq 4$ and $\exp(n, C_5) \leq \frac{12n-33}{5}$, for all $n \geq 11$. Later on, Y. Lan, et al. [4] obtained bounds $\exp(n, \Theta_4) \leq \frac{12(n-2)}{5}$, for all $n \geq 4$, $\exp(n, \Theta_5) \leq \frac{5(n-2)}{2}$, for all $n \geq 5$ and $\exp(n, \Theta_6) \leq \frac{18(n-2)}{7}$, for all $n \geq 7$, where Θ_k is obtained from a cycle C_k by adding an additional edge joining any two non-consecutive vertices. They also demonstrated that their bounds for Θ_4 and Θ_5 are tight by showing infinitely many values of n and planar graph on n vertices attaining the stated bounds. As a consequence of the bound for Θ_6 in the same paper, they presented the following corollary.

Corollary 1 (Y. Lan, et al.[4]).

$$\exp(n, C_6) \le \frac{18(n-2)}{7}$$

for all $n \ge 6$, with equality when n = 9.

In this paper we present a tight bound for $\exp(n, C_6)$. In particular, we prove the following two theorems to give the tight bound.

We denote the vertex and the edge sets of a graph G by V(G) and E(G) respectively. We also denote the number of vertices and edges of G by v(G) and e(G) respectively. The minimum degree of G is denoted $\delta(G)$. The main ingredient of the result is as follows:

Theorem 2. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge 3$. Then $e(G) \le \frac{5}{2}n - 7$.

We use Theorem 2, which considers only 2-connected graphs with no degree 2 (or 1) vertices and order at least 6, in order to establish our desired result, which bounds gives the desired bound of $\frac{5}{2}n - 7$ for all C_6 -free plane graphs with at least 18 vertices.

Theorem 3. Let G be a C₆-free plane graph on n $(n \ge 18)$ vertices. Then

$$e(G) \le \frac{5}{2}n - 7$$

Indeed, there are 17-vertex graphs on 17 vertices with 36 edges, but $\frac{5}{2}(17) - 7 = 35.5 < 36$. One such graph can be seen in Figure 1.

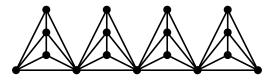


Figure 1: Example of G on 17 vertices such that e(G) > (5/2)v(G) - 7.

We show that, for large graphs, Theorem 3 is tight:

Theorem 4. For every $n \cong 2 \pmod{5}$, there exists a C_6 -free plane graph G with $v(G) = \frac{18n+14}{5}$ and e(G) = 9n, hence $e(G) = \frac{5}{2}v(G) - 7$.

For a vertex v in G, the neighborhood of v, denoted $N_G(v)$, is the set of all vertices in Gwhich are adjacent to v. We denote the degree of v by $d_G(v) = |N_G(v)|$. We may avoid the subscripts if the underlying graph is clear. The minimum degree of G is denoted by $\delta(G)$, the number of components of G is denoted by c(G). For the sake of simplicity, we may use the term k-cycle to mean a cycle of length k and k-face to mean a face bounded by a k-cycle. A k-path is a path with k edges.

2 Proof of Theorem 4: Extremal Graph Construction

First we show that for a plane graph G_0 with n vertices $(n \cong 7 \pmod{10})$, each face having length 7 and each vertex in G_0 having degree either 2 or 3, we can construct G, where G is a C_6 -free plane graph with $v(G) = \frac{18n+14}{5}$ and e(G) = 9n. We then give a construction for such a G_0 as long as $n \cong 7 \pmod{10}$. Using Euler's formula, the fact that every face has length 7 and every degree is 2 or 3, we have $e(G_0) = \frac{7(n-2)}{5}$ and the number of degree 2 and degree 3 vertices in G_0 are $\frac{n+28}{5}$ and $\frac{4n-28}{5}$, respectively.

Given G_0 , we construct first an intermediate graph G' by step (1):

(1) Add halving vertices to each edge of G_0 and join the pair of halving vertices with distance 2, see an example in Figure 2. Let G' denote this new graph, then $v(G') = v(G_0) + e(G_0) = \frac{12n-14}{5}$ and the number of degree 2 and degree 3 vertices in G' is equal to the number of degree 2 and degree 3 vertices in G_0 , respectively.

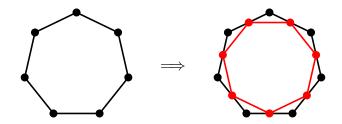


Figure 2: Adding a halving vertex to each edge of G_0 .

To get G, we apply the following steps (2) and (3) on the degree 2 and 3 vertices in G', respectively.

(2) For each degree 2 vertex v in G_0 , let $N(v) = \{v_1, v_2\}$, and so v_1vv_2 forms an induced triangle in G'. Fix v_1 and v_2 , replace v_1vv_2 with a K_5^- by adding vertices v'_1 , v'_2 to V(G') and edges v'_1v , $v'_1v'_2$, v'_1v_1 , v'_1v_2 , v'_2v_1 , v'_2v_2 to E(G'). See Figure 3.

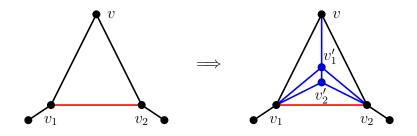


Figure 3: Replacing a degree-2 vertex of G_0 with a K_5^- .

(3) For each degree 3 vertex v in G_0 , such that $N(v) = \{v_1, v_2, v_3\}$, the set of vertices $\{v, v_1, v_2, v_3\}$ then forms an induced K_4 in G'. Fix v_1, v_2 and v_3 , replace this K_4 with a K_5^- by adding a new vertex v' to V(G') and edges $v'v, v'v_1, v'v_2$ to E(G'). See Figure 4.

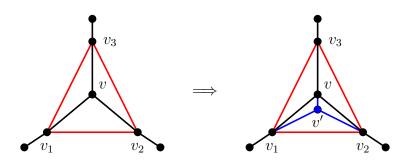


Figure 4: Replacing a degree-3 vertex of G_0 with a K_5^- .

For each integer $k \ge 0$, and n = 10k+7 we present a construction for such a G_0 , call it G_0^k : Let v_i^t and v_i^b $(1 \le i \le k+1)$ be the top and bottom vertices of the heptagonal grids with 3 layers and k columns, respectively (see the red vertices in Figure 5) and v be the extra vertex in G_0^k but not in the heptagonal grid. We join $v_1^t v$, vv_1^b and $v_i^t v_i^b$ $(2 \le i \le k+1)$. Clearly, G_0^k is a (10k+7)-vertex plane graph and each face of G_0^k is a 7-face. Obviously $e(G_0^k) = 14k+7$, and the number of degree 2 and 3 vertices are $2k+7 = \frac{n+28}{5}$ and $8k = \frac{4n-28}{5}$ respectively.

After applying steps (1), (2), and (3) on G_0^k , we get G. It is easy to verify that G is a C_6 -free plane graph with

 $v(G) = v(G_0^k) + e(G_0^k) + 2(2k+7) + 8k = (10k+7) + (14k+7) + 12k + 14 = 36k + 28$ $e(G) = 9v(G_0^k) = 90k + 63.$

Thus, $e(G) = \frac{5}{2}v(G) - 7$.

Remark 1. In fact, for $k \ge 1$ and n = 10k + 2, there exists a graph H_0^k which is obtained from G_0^k by deleting vertices (colored green in Figure 5) x_1 , x_2 , x_3 , x_4 , x_5 and adding the edge $v_1^t y$. Clearly, H_0^k is an 10k + 2-vertex plane graph such that all faces have length 7. Moreover, $e(H_0^k) = 14k$, the number of degree-2 and degree-3 vertices are $2k + 6 = \frac{n+28}{5}$ and

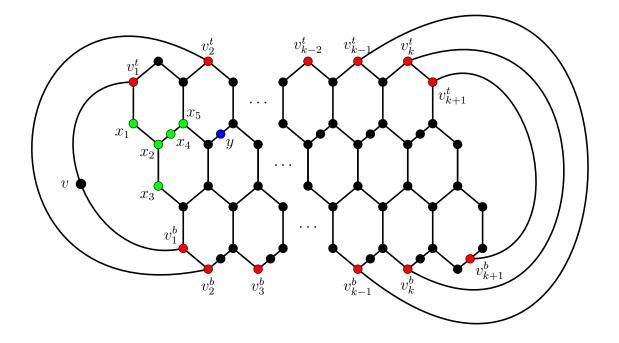


Figure 5: The graph G_0^k , $k \ge 1$, in which each face has length 7. The graph H_0^k (see Remark 1) is obtained by deleting x_1, \ldots, x_5 and adding the edge $v_1^t y$.

 $8k-4 = \frac{4n-28}{5}$, respectively. After applying steps (1), (2), and (3) to H_0^k , we get a graph H that is a C₆-free plane graph with e(H) = (5/2)v(H) - 7.

Thus, for any $k \cong 2 \pmod{5}$, we have the graphs above such that each face is a 7-gon and we get a C_6 -free plane graph on n vertices with (5/2)n - 7 edges for $n \cong 10 \pmod{18}$ if $n \ge 28$.

3 Definitions and Preliminaries

We give some necessary definitions and preliminary results which are needed in the proof of Theorems 2 and 3.

Definition 5. Let G be a plane graph and $e \in E(G)$. If e is not in a 3-face of G, then we call it a **trivial triangular-block**. Otherwise, we recursively construct a **triangular-block** in the following way. Start with H as a subgraph of G, such that $E(H) = \{e\}$.

(1) Add the other edges of the 3-face containing e to E(H).

- (2) Take e' ∈ E(H) and search for a 3-face containing e'. Add these other edge(s) in this 3-face to E(H).
- (3) Repeat step (2) till we cannot find a 3-face for any edge in E(H).

We denote the triangular-block obtained from e as the starting edge, by B(e).

Let G be a plane graph. We have the following three observations:

- (i) If H is a non-trivial triangular-block and $e_1, e_2 \in E(H)$, then $B(e_1) = B(e_2) = H$.
- (ii) Any two triangular-blocks of G are edge disjoint.
- (iii) If B is a triangular-block with the unbounded region being a 3-face, then B is a triangulation graph.

Let \mathcal{B} be the family of triangular-blocks of G. From observation (ii) above, we have

$$e(G) = \sum_{B \in \mathcal{B}} e(B),$$

where e(G) and e(B) are the number of edges of G and B respectively.

Next, we distinguish the types of triangular-blocks that a C_6 -free plane graph may contain. The following lemma gives us the bound on the number of vertices of triangular-blocks.

Lemma 6. Every triangular-block of G contains at most 5 vertices.

Proof. We prove it by contradiction. Let B be a triangular-block of G containing at least 6 vertices. We perform the following operations: delete one vertex from the boundary of the unbounded face of B sequentially until the number of vertices of the new triangular block B' is 6. Next, we show that B' is not a triangular-block in G. Suppose that it is. We consider the following two cases to complete the proof.

Case 1. B' contains a separating triangle.

Let $v_1v_2v_3$ be the separating triangle. Without loss of generality, assume that the inner region of the triangle contains two vertices say, v_4 and v_5 . The outer region of the triangle contains one vertex, say v_6 . Since the unbounded face is a 3-face, the inner structure is a triangulation. Without loss of generality, let the inner structure be as shown in Figure 6(a). Now consider the vertex v_6 . If $v_1, v_2 \in N(v_6)$, then $v_3v_4v_5v_2v_6v_1v_3$ is a 6-cycle in G, a contradiction. Similarly for the cases when $v_1, v_3 \in N(v_6)$ and $v_2, v_3 \in N(v_6)$.

Case 2. B' contains no separating triangle.

Consider a triangular face $v_1v_2v_3v_1$. Let v_4 be a vertex in the triangular-block such that $v_2v_3v_4v_2$ is a 3-face. Notice that $v_1v_4 \notin E(B')$, otherwise we get a separating triangle in B'. Let v_5 be a vertex in B' such that $v_2v_4v_5v_2$ is a 3-face. Notice that v_6 cannot be adjacent to both vertices in any of the pairs $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_3, v_4\}, \text{ or } \{v_4, v_5\}$. Otherwise, $C_6 \subset G$. Also $v_3v_5 \notin E(B')$, otherwise we have a separating triangle. So, let $v_1v_5 \in E(B')$ and $v_1, v_5 \in N(v_6)$ (see Figure 6(b)). In this case $v_1v_6v_5v_2v_4v_3v_1$ results in a 6-cycle, a contradiction.

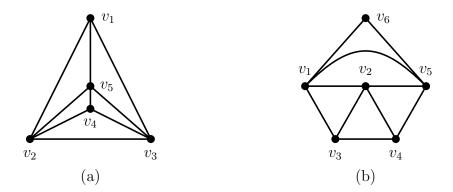


Figure 6: The structure of B' when it contains a separating triangle or not, respectively.

Now we describe all possible triangular-blocks in G based on the number of vertices the block contains. For $k \in \{2, 3, 4, 5\}$, we denote the triangular-blocks on k vertices as B_k .

Triangular-blocks on 5 vertices.

There are four types of triangular-blocks on 5 vertices (see Figure 7). Notice that $B_{5,a}$ is a K_5^- .

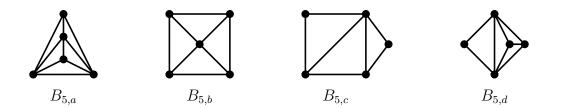


Figure 7: Triangular-blocks on 5 vertices.

Triangular-blocks on 4, 3, and 3 vertices.

There are two types of triangular-blocks on 4 vertices. See Figure 8. Observe that $B_{4,a}$ is a K_4 . The 3-vertex and 2-vertex triangular-blocks are simply K_3 and K_2 (the trivial triangular-block), respectively.

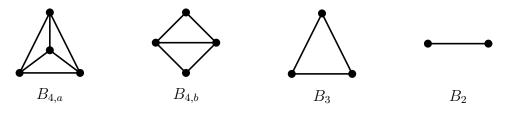


Figure 8: Triangular-blocks on 4,3 and 2 vertices.

Definition 7. Let G be a plane graph.

- (i) A vertex v in G is called a junction vertex if it is in at least two distinct triangularblocks of G.
- (ii) Let B be a triangular-block in G. An edge of B is called an **exterior edge** if it is on a boundary of non-triangular face of G. Otherwise, we call it an **interior edge**. An endvertex of an exterior edge is called an **exterior vertex**. We denote the set of all exterior and interior edges of B by Ext(B) and Int(B) respectively. Let $e \in Ext(B)$, a non-triangular face of G with e on the boundary is called the **exterior face** of e.

Notice that an exterior edge of a non-trivial triangular-block has exactly one exterior face. On the other hand, if G is a 2-connected plane graph, then every trivial triangular-block has two exterior faces. For a non-trivial triangular-block B of a plane graph G, we

call a path $P = v_1 v_2 v_3 \dots v_k$ an *exterior path* of B, if v_1 and v_k are junction vertices and $v_i v_{i+1}$ are exterior edges of B for $i \in \{1, 2, \dots, k-1\}$ and v_j is not junction vertex for all $j \in \{2, 3, \dots, k-1\}$. The corresponding face in G where P is on the boundary of the face is called the *exterior face* of P.

Next, we give the definition of the contribution of a vertex and an edge to the number of vertices and faces of C_6 -free plane graph G. All graphs discussed from now on are C_6 -free plane graph.

Definition 8. Let G be a plane graph, B be a triangular-block in G and $v \in V(B)$. The contribution of v to the vertex number of B is denoted by $n_B(v)$, and is defined as

$$n_B(v) = \frac{1}{\# \ triangular-blocks \ in \ G \ containing \ v}}$$

We define the contribution of B to the number of vertices of G as $n(B) = \sum_{v \in V(B)} n_B(v)$.

Obviously, $v(G) = \sum_{B \in \mathcal{B}} n(B)$, where v(G) is the number of vertices in G and \mathcal{B} is the family of triangular-blocks of G.

Let $B_{K_5^-}$ be a triangular-block of G isomorphic to a $B_{5,a}$ with exterior vertices v_1, v_2, v_3 , where v_1 and v_3 are junction vertices, see Figure 9 for an example. Let F be a face in Gsuch that V(F) contains all exterior vertices $v_{1,1}, \ldots, v_{1,m}, v_{2,1}, \ldots, v_{2,m}, v_{3,1}, \ldots, v_{3,m}$ of m $(m \ge 1)$ copies of $B_{K_5^-}$, such that $v_{1,i}, v_{2,i}, v_{3,i}$ are the exterior vertices of the *i*-th $B_{K_5^-}$ and $v_{1,i}, v_{3,i}$ $(1 \le i \le m)$ are junction vertices. Let C_F denote the cycle associated with the face F. We alter $E(C_F)$ in the following way:

$$E(C'_F) := E(C_F) - \{v_{1,1}v_{2,1}v_{3,1}\} - \dots - \{v_{1,m}v_{2,m}v_{3,m}\} \cup \{v_{1,1}v_{3,1}\} \cup \dots \cup \{v_{1,m}, v_{3,m}\}$$

Hence, the length of F as $|E(C'_F)| = |E(C_F)| - m$. For example, in Figure 9, $|E(C_F)| = 11$ but $|E(C'_F)| = 9$.

Now we are able to define the **contribution** of an "edge" to the number of faces of C_6 -free plane graph G.

Definition 9. Let F be a exterior face of G and $C_F := \{e_1, e_2, \ldots, e_k\}$ be the cycle associated with F. The contribution of an exterior edge e to the face number of the exterior face F, is denoted by $f_F(e)$, and is defined as follows.

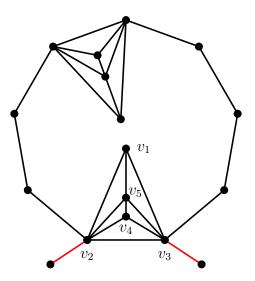


Figure 9: An example of a face containing all the exterior vertices of at least one $B_{K_{\kappa}^{-}}$.

- (i) If e_1 and e_2 are adjacent exterior edges of $B_{K_5^-}$, then $f_F(e_1) + f_F(e_2) = \frac{1}{|C'_F|}$, and $f_F(e_i) = \frac{1}{|C'_F|}$, where $i \in \{3, 4, \dots, k\}$.
- (ii) Otherwise, $f_F(e) = \frac{1}{|C_F|}$.

Note that $\sum_{e \in E(F)} f_F(e) = 1$. For a triangular-block B, the total face contribution of B is denoted by f_B and defined as $f_B = (\# \text{ interior faces of } B) + \sum_{e \in Ext(B)} f_F(e)$, where F is the exterior face of B with respective to e. Obviously, $f(G) = \sum_{B \in \mathcal{B}} f(B)$, where f(G) is the number of faces of G.

4 Proof of Theorem 2

We begin by outlining our proof. Let f, n, and e be the number of faces, vertices, and edges of G respectively. Let \mathcal{B} be the family of all triangular-blocks of G.

The main target of the proof is to show that

$$7f + 2n - 5e \le 0.$$
 (1)

Once we show (1), then by using Euler's Formula, e = f + n - 2, we can finish the proof of Theorem 2. To prove (1), we show the existence of a partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of \mathcal{B} such that

$$7 \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{B \in \mathcal{P}_i} e(B) \le 0, \text{ for all } i \in \{1, 2, 3, \dots, m\}. \text{ Since } f = \sum_{B \in \mathcal{B}} f(B)$$
$$n = \sum_{B \in \mathcal{B}} n(B) \text{ and } e = \sum_{B \in \mathcal{B}} e(B) \text{ we have}$$
$$7f + 2n - 5e = 7 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{i}^{m} \sum_{B \in \mathcal{P}_i} e(B)$$
$$= \sum_{i}^{m} \left(7 \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{B \in \mathcal{P}_i} e(B)\right) \le 0.$$

The following proposition will be useful in many lemmas.

Proposition 10. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge 3$.

- (i) If B is a nontrivial triangular-block (that is, not B₂), then none of the exterior faces can have length 5.
- (ii) If B is in $\{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\}$, then none of the exterior faces can have length 4.
- (iii) If B is in $\{B_{5,d}, B_{4,b}\}$ and an exterior face of B has length 4, then that 4-face must share a 2-path with B (shown in blue in Figures 13 and 14) and the other edges of the face must be in trivial triangular-blocks.
- (iv) No two 4-faces can be adjacent to each other.
- *Proof.* (i) Observe that any pair of consecutive exterior vertices of a nontrivial triangularblock has a path of length 2 (counted by the number of edges) between them and any pair of nonconsecutive exterior vertices has a path of length 3 between them. So having a face of length 5 incident to this triangular-block would yield a C_6 , a contradiction.
 - (ii) If B is in $\{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\}$, then any pair of consecutive exterior vertices of the listed triangular-blocks has a path of length 3 between them. It remains to consider nonconsecutive vertices for $\{B_{5,b}, B_{5,c}\}$. For $B_{5,b}$ each pair of nonconsecutive exterior vertices has a path of length 3 between them. In the case where B is $B_{5,c}$, this is true for all pairs without an edge between them. As for the other pairs, if they are in the

same 4-face, then at least one of the degree-2 vertices in B must have degree 2 in G, a contradiction.

(iii) In both $B_{5,d}$ and $B_{4,b}$, any pair of consecutive exterior vertices has a path of length 3 between them. For $B_{5,d}$, in Figure 13, we see that there is a path of length 4 between v_2 and v_4 and so the only way a 4-face can be adjacent to B is via a 2-path with endvertices v_1 and v_3 . In fact, because there is no vertex of degree 2, the path must be $v_1v_4v_3$. For $B_{4,b}$, in Figure 13, we see that because B cannot have a vertex of degree 2, the 4-face and B cannot share the path $v_2v_1v_4$ or the path $v_2v_3v_4$. Thus the only paths that can share a boundary with a 4-face are $v_1v_4v_3$ and $v_1v_2v_3$.

As to the other blocks that form edges of such a 4-face. In Figure 10, we see that if, say, v_1u is in a nontrivial triangular-block, then there is a vertex w in that block, in which case $wv_1xv_4v_3uw$ forms a 6-cycle, a contradiction.

(iv) If two 4-faces share an edge, then there is a 6-cycle formed by deleting that edge. If two 4-faces share a 2-path, then the midpoint of that path is a vertex of degree 2 in G. In both cases, a contradiction.

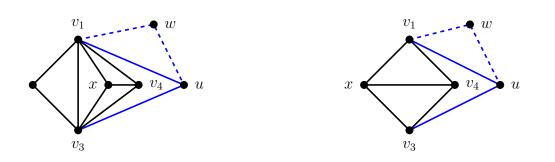


Figure 10: Proposition 10(iii): The blocks defined by blue edges must be trivial.

To show the existence of such a partition we need the following lemmas.

Lemma 11. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge 3$. If B is a triangular-block in G such that $B \notin \{B_{5,d}, B_{4,b}\}$, then $7f(B) + 2n(B) - 5e(B) \le 0$. Proof. We separate the proof into several cases. Case 1: B is $B_{5,a}$.

Let v_1 , v_2 and v_3 be the exterior vertices of K_5^- . At least two of them must be junction vertices, otherwise G contains a cut vertex. We consider 2 possibilities to justify this case.

- (a) Let *B* be $B_{5,a}$ with 3 junction vertices (see Figure 11(a)). By Proposition 10, every exterior edge in *B* is contained in an exterior face with length at least 7. Thus, $f(B) = (\# \text{ interior faces of } B) + \sum_{e \in Ext(B)} f_F(e) \le 5 + 3/7$. Moreover, every junction vertex is contained in at least 2 triangular-blocks, so we have $n(B) \le 2 + 3/2$. With e(B) = 9, we obtain $7f(B) + 2n(B) - 5e(B) \le 0$.
- (b) Let *B* be $B_{5,a}$ with 2 junction vertices, say v_2 and v_3 (see Figure 11(b)). Let *F* and F_1 are exterior faces of the exterior edge v_2v_3 and exterior path $v_2v_1v_3$ of the triangularblock respectively. Notice that v_1v_2 and v_2v_3 are the adjacent exterior edges in the same face F_1 , hence $|C(F_1)| \ge 8$. By Definition 9, we have $f_{F_1}(v_1v_2) + f_{F_1}(v_1v_3) \le 1/7$. Because there can be no C_6 , one can see that regardless of the configuration of the $B_{K_5^-}$, it is the case that $f_F(v_2v_3) \le 1/7$. Thus, $f(B) \le 5 + 2/7$. Moreover, since v_1 and v_3 are contained in at least 2 triangular-blocks, we have $n(B) \le 3 + 2/2$. With e(B) = 9, we obtain $7f(B) + 2n(B) - 5e(B) \le 0$.

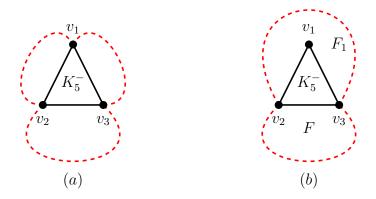


Figure 11: A $B_{5,a}$ triangular-block with 3 and 2 junction vertices, respectively.

Case 2: B is in $\{B_{4,a}, B_{5,b}, B_{5,c}\}$.

- (a) Let B be a B_{4,a}. By Proposition 10, each face incident to this triangular-block has length at least 7. So, f(B) ≤ 3 + 3/7. Because there is no cut-vertex, this triangular-block must have at least two junction vertices, hence n(B) ≤ 2 + 2/2. With e(B) = 6, we obtain 7f(B) + 2n(B) 5e(B) ≤ 0.
- (b) Let B be a B_{5,b}. There are 4 faces inside the triangular-block and each face incident to this triangular-block has length at least 7. So, f(B) ≤ 4 + 4/7. Because there is no cut-vertex, this triangular-block must have at least two junction vertices, hence n(B) ≤ 3 + 2/2. With e(B) = 8, we obtain 7f(B) + 2n(B) 5e(B) ≤ 0, as seen in Table 2.
- (c) Let B be a $B_{5,c}$. Similarly, $f(B) \le 3 + 5/7$ and because there are at least two junction vertices, $n(B) \le 3 + 2/2$. With e(B) = 7, we obtain $7f(B) + 2n(B) 5e(B) \le -1$.

Case 3: B is B_3 .

Let v_1 , v_2 and v_3 be the exterior vertices of triangular-block B. Each of these three must be junction vertices since there is no degree 2 vertex in G, which implies that each is contained in at least 2 triangular-blocks. We consider two possibilities:

(a) Let the three exterior vertices be contained in exactly 2 triangular-blocks. By Proposition 10(i), the length of each exterior face is either 4 or at least 7. We want to show that at most one exterior face has length 4.

If not, then let x_1 be a vertex that is in two such faces. Consider the triangular-block incident to B at x_1 , call it B'. By Proposition 10, B' is not in $\{B_{5,a}, B_{5,b}, B_{5,c}, B_{4,a}\}$.

If B' is in $\{B_{5,d}, B_{4,b}, B_3\}$, then the triangular-block has vertices ℓ_2, ℓ_3 , each adjacent to x_1 and the length-4 faces consist of $\{v_1, \ell_2, m_2, v_2\}$ and $\{v_1, \ell_3, m_3, v_3\}$. Either $\ell_2 \sim \ell_3$ (in which case $\ell_2 m_2 v_2 v_3 m_3 \ell_3 \ell_2$ is a 6-cycle, see Figure 12(a)) or there is a ℓ' distinct from v_1 that is adjacent to both ℓ_2 and ℓ_3 (in which case $\ell' \ell_2 m_2 v_2 v_1 \ell_3 \ell_2$ is a 6-cycle, see Figure 12(b)).

If B' is B_2 , then the trivial triangular-block is $\{v_1, \ell\}$, in which case $\{\ell, m_2, v_2, v_1, v_3, m_3\}$ is a C_6 , see Figure 12(c). Thus, we may conclude that if each of the three exterior vertices are in exactly 2 triangular-blocks, then $f(B) \leq 1 + 2/7 + 1/4$ and $n(B) \leq 3/2$. With e(B) = 3, we obtain $7f(B) + 2n(B) - 5e(B) \leq -5/4$.

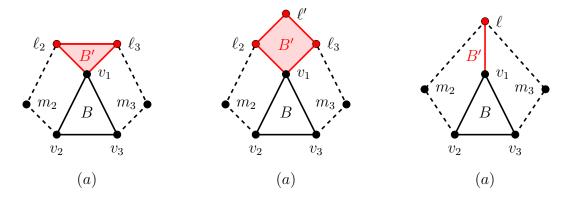


Figure 12: A B_3 triangular-block, B and the various cases of what must occur if B is incident to two 4-faces.

(b) Let at least one exterior vertex be contained in at least 3 triangular-blocks and the others be contained at least 2 triangular-blocks. In this case, we have $f(B) \le 1 + 3/4$ and $n(B) \le 2/2 + 2/3$. With e(B) = 3, we obtain $7f(B) + 2n(B) - 5e(B) \le -1/12$.

Case 4: B is B_2 .

Note that the fact that there is no vertex of degree 2 gives that if an endvertex is in exactly two triangular-blocks, then the other one cannot be a B_2 . We consider three possibilities:

- (a) Let each endvertex be contained in exactly 2 triangular-blocks. Since neither of the triangular-blocks incident to B can be trivial, they cannot be incident to a face of length 5 by Proposition 10(i). Thus, B cannot be incident to a face of length 5. Moreover, the two faces incident to B cannot both be of length 4, again by Proposition 10(iv). Hence, f(B) ≤ 1/4 + 1/7. Clearly n(B) ≤ 2/2 and with e(B) = 1, we obtain 7f(B) + 2n(B) 5e(B) ≤ -1/4.
- (b) Let one endvertex be contained in exactly 2 triangular-blocks and the other endvertex be contained in at least 3 triangular-blocks. This is similar to case (a) in that neither

face can have length 5 and they cannot both have length 4. The only difference is that $n(B) \leq 1/2 + 1/3$ and so $7f(B) + 2n(B) - 5e(B) \leq -7/12$.

(c) Let each endvertex be contained in at least 3 triangular-blocks. The two faces cannot both be of length 4 by Proposition 10(iv). Hence, f(B) ≤ 1/4 + 1/5 and n(B) ≤ 2/3. With e(B) = 1, we obtain 7f(B) + 2n(B) - 5e(B) ≤ -31/60.

Lemma 12. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge 3$. If B is $B_{5,d}$, then $7f(B) + 2n(B) - 5e(B) \le 1/2$. Moreover, $7f(B) + 2n(B) - 5e(B) \le 0$ unless B shares a 2-path with a 4-face.

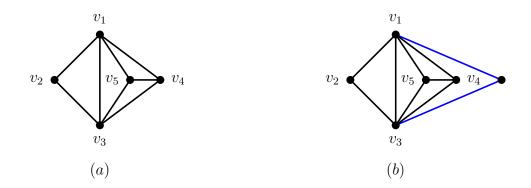


Figure 13: A $B_{5,d}$ triangular-block and how a 4-face must be incident to it.

Proof. Let *B* be $B_{5,d}$ with vertices v_1 , v_2 , v_3 , v_4 , and v_5 , as shown in Figure 13(a). By Proposition 10(i), no exterior face of *B* can have length 5. By Proposition 10(iii), if there is an exterior face of *B* that has length 4, this 4-face must contain the path $v_1v_4v_3$.

Moreover, since there is no vertex of degree 2, v_2 is a junction vertex. Because G has no cut-vertex, there is at least one other junction vertex. We may consider the following cases:

(a) Let v_4 be a junction vertex. This prevents an exterior face of length 4. Thus, each exterior face has length at least 7. Hence, $f(B) \le 4 + 4/7$ and $n(B) \le 3 + 2/2$. With e(B) = 8, we obtain $7f(B) + 2n(B) - 5e(B) \le 0$.

- (b) Let v_4 fail to be a junction vertex and exactly one of v_1, v_3 be a junction vertex. Without loss of generality let it be v_3 . In this case, again, each exterior face has length¹ at least 7. Again, $f(B) \le 4 + 4/7$ and $n(B) \le 3 + 2/2$. With e(B) = 8, we obtain $7f(B) + 2n(B) - 5e(B) \le 0$.
- (c) Let v₄ fail to be a junction vertex and both v₁ and v₃ be junction vertices. Here either the exterior path v₁v₄v₃ is part of an exterior face of length at least 4 or each edge must be in a face of length at least 7. If the exterior face is of length at least 7, then f(B) ≤ 4 + 4/7, otherwise f(B) ≤ 4 + 2/4 + 2/7. In both cases, n(B) ≤ 2 + 3/2 and e(B) = 8. Hence we obtain 7f(B) + 2n(B) - 5e(B) ≤ -1 in the first instance and 7f(B) + 2n(B) - 5e(B) ≤ 1/2 in the case where B is incident to a 4-face.

Lemma 13. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge 3$. If B is $B_{4,b}$, then $7f(B) + 2n(B) - 5e(B) \le 4/3$. Moreover, $7f(B) + 2n(B) - 5e(B) \le 1/6$ if B shares a 2-path with exactly one 4-face and $7f(B) + 2n(B) - 5e(B) \le 0$ if B fails to share a 2-path with any 4-face.

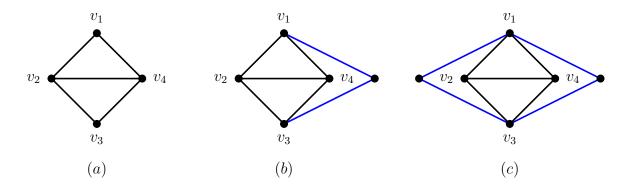


Figure 14: A $B_{4,b}$ triangular-block and how a 4-face must be incident to it.

Proof. Let B be with vertices v_1 , v_2 , v_3 , and v_4 , as shown in Figure 14(a). By Proposition 10(i), no exterior face of B can have length 5. If there is an exterior face of B that has

¹In fact, it can be shown that the length of the exterior face containing the path $v_2v_1v_4v_3$ is at least 9. This yields $f(B) \le 4 + 1/7 + 3/9$ and $7f(B) + 2n(B) - 5e(B) \le -2/3$. However, this precision is unnecessary.

length 4, it is easy to verify that being C_6 -free and having no vertex of degree 2 means that the junction vertices must be v_1 and v_3 . We may consider the following cases.

- (a) Let either v₂ or v₄ be a junction vertex and, without loss of generality, let it be v₂. All the exterior faces have length at least 7 except for the possibility that the path v₁v₄v₃ may form two sides of a 4-face. Hence, f(B) ≤ 2 + 2/4 + 2/7 and n(B) ≤ 1 + 3/2. With e(B) = 5, we obtain 7f(B) + 2n(B) 5e(B) ≤ -1/2.
- (b) Let neither v_2 nor v_4 be a junction vertex. Because there is no cut-vertex, this requires both v_1 and v_3 to be junction vertices. Hence, there are two exterior faces: One that shares the exterior path $v_1v_4v_3$ and the other shares the exterior path $v_1v_2v_3$. Each exterior face has length either 4 or at least 7. We consider several subcases:
 - (i) If both faces are of length at least 7, then $f(B) \le 2 + 4/7$, and $n(B) \le 2 + 2/2$. With e(B) = 5, we obtain $7f(B) + 2n(B) - 5e(B) \le -1$.
 - (ii) If only one of the exterior faces is of length 4, then f(B) ≤ 2 + 2/7 + 2/4.
 Moreover, at least one of v₁, v₃ must be a junction vertex for more than two triangular-blocks, otherwise either v(G) = 5 or the vertex incident to two blue edges in Figure 14(b) is a cut-vertex. Hence, n(B) ≤ 2 + 1/3 + 1/2 and with e(B) = 5, we have 7f(B) + 2n(B) 5e(B) ≤ 1/6.
 - (iii) Both exterior faces are of length 4. Thus f(B) ≤ 2 + 4/4. By Proposition 10(iii), the blocks represented by the blue edges in Figure 14(c) are each trivial. Hence n(B) ≤ 2 + 2/3. With e(B) = 5, we get 7f(B) + 2n(B) 5e(B) ≤ 4/3.

Tables 2 and 3 in Appendix A give a summary of Lemmas 11, 12, and 13.

Lemma 14. Let G be a 2-connected, C_6 -free plane graph on $n \ (n \ge 6)$ vertices with $\delta(G) \ge$ 3. Then the triangular-blocks of G can be partitioned into sets, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that $7 \sum_{B \in \mathcal{P}_i} f(B) + 2 \sum_{B \in \mathcal{P}_i} n(B) - 5 \sum_{B \in \mathcal{P}_i} e(B) \le 0$ for all $i \in [m]$. *Proof.* As it can be seen from Tables 2 and 3 in Appendix A, there are three possible cases where 7f(B) + 2n(B) - 5e(B) assumes a positive value. We deal with each of these blocks as follows.

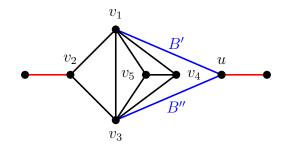


Figure 15: Structure of a $B_{5,d}$ if it is incident to a 4-face, as in Lemma 14. The triangularblocks B' and B'' are trivial.

(1) Let B be a $B_{5,d}$ triangular-block as described in the proof of Lemma 12(c). See Figure 15.

By Proposition 10(iii), the edges v_1u and v_3u are trivial triangular-blocks. Denote these triangular-blocks as B' and B''. Consider B'. One of the exterior faces of B' has length 4 whereas by Proposition 10(iv), the other has length at least 5. It must have length at least 7 because if it had length 5, then the path v_1v_3u would complete it to a 6-cycle. Thus, $f(B') \leq 1/4 + 1/7$. Since the vertex u cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Thus, $n(B') \leq 1/2 + 1/3$. With e(B') = 1, we obtain $7f(B') + 2n(B') - 5e(B') \leq -7/12$ and similarly, $7f(B'') + 2n(B'') - 5e(B'') \leq -7/12$. Define $\mathcal{P}' = \{B, B', B''\}$. Thus, $7\sum_{B^* \in \mathcal{P}'} f(B^*) + 2\sum_{B^* \in \mathcal{P}'} n(B^*) - 5\sum_{B^* \in \mathcal{P}'} e(B^*) \leq 1/2 + 2(-7/12) = -2/3$.

Therefore, for each triangular-block in G as described in Lemma 12(c), it belongs to a set \mathcal{P}' of three triangular-blocks such that $7 \sum_{B^* \in \mathcal{P}'} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{m_1}$ if they exist.

(2) Let B be a $B_{4,b}$ triangular-block as described in the proof of Lemma 13(b)(ii). See Figure 16(a).

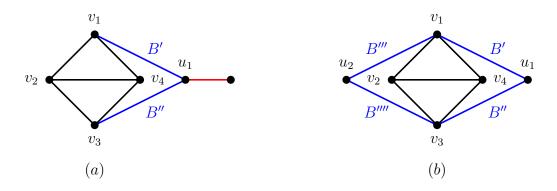


Figure 16: Structure of a $B_{4,b}$ triangular-block if it is incident to a 4-face, as in Lemma 14. The triangular-blocks B', B'', B''', and B'''' are all trivial.

By Proposition 10(iii), the edges v_1u_1 and v_3u_1 are trivial triangular-blocks. Denote them as B' and B'', respectively. Consider B'. One of the exterior faces of B' has length 4 and by Proposition 10(iv), the other has length at least 5. Thus, $f(B') \leq$ 1/4 + 1/5. Since the vertex u_1 cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Thus, $n(B') \leq 1/2 + 1/3$. With e(B') = 1, we obtain $7f(B')+2n(B')-5e(B') \leq -11/60$ and similarly, $7f(B'')+2n(B'')-5e(B'') \leq -11/60$. Define $\mathcal{P}'' = \{B, B', B''\}$. Thus, $7\sum_{B^* \in \mathcal{P}''} f(B^*)+2\sum_{B^* \in \mathcal{P}''} n(B^*)-5\sum_{B^* \in \mathcal{P}''} e(B^*) \leq 1/6 + 2(-11/60) = -1/5$.

Therefore, for each triangular-block in G as described in Lemma 13(b)(ii), it belongs to a set \mathcal{P}'' of three triangular-blocks such that $7 \sum_{B^* \in \mathcal{P}''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_1+1}, \mathcal{P}_{m_1+2}, \ldots, \mathcal{P}_{m_2}$ if they exist.

(3) Let B be a B_{4,b} triangular-block as described in the proof of Lemma 13(b)(iii). See Figure 16(b).

By Proposition 10(iii), the edges v_1u_1 , v_3u_1 , v_1u_2 , and v_3u_2 are trivial triangular-blocks. Denote them as B', B'', B''' and B'''' respectively. Consider B'. One of the exterior faces of B' has length 4 whereas the other has length at least 5. Thus, $f(B') \leq 1/4 + 1/5$. Since the vertex u_1 cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Clearly v_1 is in at least three triangular-blocks. Thus, $n(B') \leq 2/3$. With e(B') = 1, we obtain $7f(B') + 2n(B') - 5e(B') \leq -31/60$ and the same inequality holds for B'', B''', and B''''.

Define
$$\mathcal{P}''' = \{B, B', B'', B''', B''''\}$$
. Thus, $7 \sum_{B^* \in \mathcal{P}''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}''} e(B^*) \le 4/3 + 4(-31/60) = -11/15$.

Therefore, for each triangular-block in G as described in Lemma 13(b)(iii), it belongs to a set \mathcal{P}' of three triangular-blocks such that $7 \sum_{B^* \in \mathcal{P}'''} f(B^*) + 2 \sum_{B^* \in \mathcal{P}'''} n(B^*) - 5 \sum_{B^* \in \mathcal{P}'''} e(B^*) \leq 0$. Denote such sets as $\mathcal{P}_{m_2+1}, \mathcal{P}_{m_2+2}, \ldots, \mathcal{P}_{m_3}$ if they exist.

Now define $\mathcal{P}_{m_3+1} = \mathcal{B} - \bigcup_{i=1}^{m_3} \mathcal{P}_i$, where \mathcal{B} is the set of all blocks of G. Clearly, for each block $B \in \mathcal{P}_{m_3+1}$, $7f(B) + 2n(B) - 5e(B) \leq 0$. Thus, $7\sum_{B \in \mathcal{P}_{m_3+1}} f(B) + 2\sum_{B \in \mathcal{P}_{m_3+1}} n(B) - 5\sum_{B \in \mathcal{P}_{m_3+1}} e(B) \leq 0$. Putting $m := m_3 + 1$ we got the partition $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ of \mathcal{B} meeting the condition of the lemma.

This completes the proof of Theorem 2.

5 Proof of Theorem 3

Let G be a C₆-free plane graph. We will show that either $5v(G) - 2e(G) \ge 14$ or $v(G) \le 17$. If we delete a vertex x from G, then

$$5v(G - x) - 2e(G - x) = 5(v(G) - 1) - 2(e(G) - \deg(x))$$
$$= 5v(G) - 2e(G) - 5 + 2\deg(x)$$
$$\ge 5v(G) - 2e(G) - 1.$$

So, graph G has an induced subgraph G' with $\delta(G) \geq 3$ with

$$5v(G) - 2e(G) \ge 5v(G') - 2e(G') + (v(G) - v(G'))$$
⁽²⁾

In line with usual graph theoretic terminology, we call a maximal 2-connected subgraph a **block**. Let \mathcal{B}' denote the set of blocks of G' with the i^{th} block having n_i vertices and e_i edges. Let b be the total number of blocks of G'. Specifically, let b_2 , b_3 , b_4 , and b_5 denote

	min of $5n - 2e - 5$	
$n \ge 6$	$14-5 \geq 9$	Theorem 2
n = 5	$5(5) - 2(9) - 5 \ge 2$	$B_{5,a}$, Figure 7
n = 4	$5(4) - 2(6) - 5 \ge 3$	$B_{4,a}$, Figure 8
n = 3	$5(3) - 2(3) - 5 \ge 4$	B_3 , Figure 8
n=2	$5(2) - 2(2) - 5 \ge 3$	B_2 , Figure 8

Table 1: Estimates of 5n - 2e - 5 for various block sizes.

the number of blocks of size 2, 3, 4, and 5, respectively. Let b_6 denote the number of blocks of size at least 6. Then we have $b = b_6 + b_5 + b_4 + b_3 + b_2$ and, using Table 1:

$$5v(G') - 2e(G') = 5\left(\sum_{i=1}^{b} n_i - (b-1)\right) - 2\sum_{i=1}^{b} e_i$$
$$= \sum_{i=1}^{b} (5n_i - 2e_i - 5) + 5$$
$$\ge 9b_6 + 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5 \tag{3}$$

Combining (2) and (3), we obtain

$$5v(G) - 2e(G) \ge 9b_6 + 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G'))$$
(4)

If $b_6 \ge 1$, then the right-hand side of (4) is at least 14, as desired. So, let us assume that $b_6 = 0$ and $b = b_5 + b_4 + b_3 + b_2$. Furthermore,

$$v(G') = 5b_5 + 4b_4 + 3b_3 + 2b_2 - (b - 1)$$

= 4b_5 + 3b_4 + 2b_3 + b_2 + 1. (5)

So, substituting $2b_5$ from (5) into (4), we have

$$5v(G) - 2e(G) \ge 2b_5 + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G'))$$

= $\left(\frac{1}{2}v(G') - \frac{3}{2}b_4 - b_3 - \frac{1}{2}b_2 - \frac{1}{2}\right) + 3b_4 + 4b_3 + 3b_2 + 5 + (v(G) - v(G'))$
= $v(G) - \frac{1}{2}v(G') + \frac{3}{2}b_4 + 3b_3 + \frac{5}{2}v_2 + \frac{9}{2}$
 $\ge \frac{1}{2}v(G) + \frac{9}{2},$

which is strictly larger than 13 if $v(G) \ge 18$. Since 5v(G) - 2e(G) is an integer, it is at least 14 and this completes the proof of Theorem 3.

Remark 2. Observe that for $n \ge 17$, the only graphs on n vertices with e edges such that e > (5/2)n - 7 have blocks of order 5 or less and by (4), there are at most 4 such triangular blocks. A bit of analysis shows that the maximum number of edges is achieved when the number of blocks of order 5 is as large as possible.

6 Conclusions

We note that the proof of Theorem 2, particularly Lemma 14, can be rephrased in terms of a discharging argument.

We believe that our construction in Theorem 4 can be generalized to prove $\exp(n, C_{\ell})$ for ℓ sufficiently large. That is, for certain values of n, we try to construct G_0 , a plane graph with all faces of length $\ell + 1$ with all vertices having degree 3 or degree 2.

If such a G_0 exists, then the number of degree-2 and degree-3 vertices are $\frac{(\ell-5)n+4(\ell+1)}{\ell-1}$ and $\frac{4(n-\ell-1)}{\ell-1}$, respectively. We could then apply steps similar to (1), (2), and (3) in the proof of Theorem 4 in that we add halving vertices and insert a graph $B_{\ell-1}$ (see Figure 17) in place of vertices of degree 2 and 3. For the resulting graph G,

$$v(G) = v(G_0) + e(G_0) + (\ell - 4) \frac{(\ell - 5)n + 4(\ell + 1)}{\ell - 1} + (\ell - 5) \frac{4(n - \ell - 1)}{\ell - 1}$$
$$= n + \frac{\ell + 1}{\ell - 1}(n - 2) + \frac{(\ell^2 - 5\ell)n + 2(\ell + 1)}{\ell - 1}$$
$$= \frac{\ell^2 - 3\ell}{\ell - 1}n + \frac{2(\ell + 1)}{\ell}$$
$$e(G) = (3\ell - 9)v(G_0) = (3\ell - 9)n$$

Therefore, $e(G) = \frac{3(\ell-1)}{\ell}v(G) - \frac{6(\ell+1)}{\ell}$. We conjecture that this is the maximum number of edges in a C_{ℓ} -free planar graph.

Conjecture 15. Let G be an n-vertex C_{ℓ} -free plane graph $(\ell \geq 7)$, then there exists an integer $N_0 > 0$, such that when $n \geq N_0$, $e(G) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$.

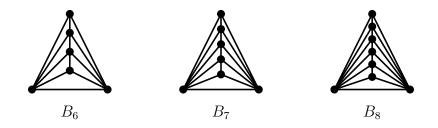


Figure 17: $B_{\ell-1}$ is used in the construction of a C_{ℓ} -free graph.

7 Acknowledgements

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A Tables

The following tables give a summary of the results from Lemmas 11, 12, and 13.

A red edge incident to a vertex of a triangular-block indicates the corresponding vertex is a junction vertex. Moreover, if a vertex has only one red edge, it is to indicate the vertex is shared in at least two triangular-blocks. Whereas if a vertex has two red edges, it means that the vertex is shared in at least three blocks.

Case	В	Diagram	$f(B) \leq$	$n(B) \leq$	e(B) =	$7f + 2n - 5e \leq$
Lemma 11 1(a)	$B_{5,a}$		$5 + \frac{3}{7}$	$2 + \frac{3}{2}$	9	0
Lemma 11 1(b)	$B_{5,a}$		$5 + \frac{2}{7}$	$3 + \frac{2}{2}$	9	0
Lemma 11 2(b)	$B_{5,b}$	\square	$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 11 2(c)	$B_{5,c}$		$3 + \frac{5}{7}$	$3 + \frac{2}{2}$	7	-1
Lemma 12 (a)	$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 12 (b)	$B_{5,d}$		$4 + \frac{4}{7}$	$3 + \frac{2}{2}$	8	0
Lemma 12 (c)	$B_{5,d}$		$4 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{3}{2}$	8	$\frac{1}{2}$ *

A pair of blue edges indicates the boundary of a 4-face.

Table 2: All possible B_5 blocks in G and the estimation of 7f(B) + 2n(B) - 5e(B).

Case	В	Diagram	$f(B) \leq$	$n(B) \leq$	e(B) =	$7f + 2n - 5e \le$
Lemma 11 2(a)	$B_{4,a}$		$3 + \frac{3}{7}$	$2 + \frac{2}{2}$	6	0
Lemma 13 (a)	$B_{4,b}$	${\longleftarrow}$	$2 + \frac{2}{4} + \frac{2}{7}$	$1 + \frac{3}{2}$	5	$-\frac{1}{2}$
Lemma 13 (b)(i)	$B_{4,b}$		$2 + \frac{4}{7}$	$2 + \frac{2}{2}$	5	-1
Lemma 13 (b)(ii)	$B_{4,b}$	\bigwedge	$2 + \frac{2}{4} + \frac{2}{7}$	$2 + \frac{1}{3} + \frac{1}{2}$	5	$\frac{1}{6}$ \star
Lemma 13 (b)(iii)	$B_{4,b}$	${\longleftrightarrow}$	$2 + \frac{2}{4} + \frac{2}{4}$	$2 + \frac{2}{3}$	5	$\frac{4}{3}$ \star
Lemma 13 3(a)	B_3		$1 + \frac{2}{7} + \frac{1}{4}$	$\frac{3}{2}$	3	$-\frac{5}{4}$
Lemma 13 3(b)	B_3		$1 + \frac{3}{4}$	$\frac{2}{2} + \frac{1}{3}$	3	$-\frac{1}{12}$
Lemma 13 4(a)	B_2	••••	$\frac{1}{4} + \frac{1}{7}$	$\frac{2}{2}$	1	$-\frac{1}{4}$
Lemma 13 4(b)	B_2	••<	$\frac{1}{4} + \frac{1}{7}$	$\frac{1}{2} + \frac{1}{3}$	1	$-\frac{7}{12}$
Lemma 13 4(c)	B_2	\succ	$\frac{1}{4} + \frac{1}{5}$	$\frac{2}{3}$	1	$-\frac{31}{60}$

Table 3: All possible B_4, B_3 and B_2 blocks in G and the estimate of 7f(B) + 2n(B) - 5e(B).