# Planar Turán Number of the 6-Cycle 

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#### Abstract

Let $\operatorname{ex}_{\mathcal{P}}(n, T, H)$ denote the maximum number of copies of $T$ in an $n$-vertex planar graph which does not contain $H$ as a subgraph. When $T=K_{2}, \operatorname{ex}_{\mathcal{P}}(n, T, H)$ is the well studied function, the planar Turán number of $H$, denoted by $\operatorname{ex}_{\mathcal{P}}(n, H)$. The topic of extremal planar graphs was initiated by Dowden (2016). He obtained sharp upper bound for both $\operatorname{ex}_{\mathcal{P}}\left(n, C_{4}\right)$ and $\operatorname{ex}_{\mathcal{P}}\left(n, C_{5}\right)$. Later on, Y. Lan, et al. continued this topic and proved that $\operatorname{ex}_{\mathcal{P}}\left(n, C_{6}\right) \leq \frac{18(n-2)}{7}$. In this paper, we give a sharp upper bound $\operatorname{ex}_{\mathcal{P}}\left(n, C_{6}\right) \leq \frac{5}{2} n-7$, for all $n \geq 18$, which improves Lan's result. We also pose a conjecture on $\exp \left(n, C_{k}\right)$, for $k \geq 7$.


Keywords Planar Turán number, Extremal planar graph

## 1 Introduction and Main Results

In this paper, all graphs considered are planar, undirected, finite and contain neither loops nor multiple edges. We use $C_{k}$ to denote the cycle on $k$ vertices and $K_{r}$ to denote the complete graph on $r$ vertices.

One of the well-known results in extremal graph theory is the Turán Theorem [5], which gives the maximum number of edges that a graph on $n$ vertices can have without containing
a $K_{r}$ as a subgraph. The Erdős-Stone-Simonovits Theorem [2, 3] then generalized this result and asymptotically determines $\operatorname{ex}(n, H)$ for all non-bipartite graphs $H: \operatorname{ex}(n, H)=$ $\left(1-\frac{1}{\chi(H)-1}\right)\binom{n}{2}+o\left(n^{2}\right)$, where $\chi(H)$ denotes the chromatic number of $H$. Over the last decade, a considerable amount of research work has been carried out in Turán-type problems, i.e., when host graphs are $K_{n}, k$-uniform hypergraphs or $k$-partite graphs, see [3, 6].

In 2016, Dowden [1 initiated the study of Turán-type problems when host graphs are planar, i.e., how many edges can a planar graph on $n$ vertices have, without containing a given smaller graph? The planar Turán number of a graph $H, \operatorname{ex}_{\mathcal{P}}(n, H)$, is the maximum number of edges in a planar graph on $n$ vertices which does not contain $H$ as a subgraph. Dowden [1] obtained the tight bounds $\operatorname{ex}_{\mathcal{P}}\left(n, C_{4}\right) \leq \frac{15(n-2)}{7}$, for all $n \geq 4$ and $\operatorname{ex}_{\mathcal{P}}\left(n, C_{5}\right) \leq \frac{12 n-33}{5}$, for all $n \geq 11$. Later on, Y. Lan, et al. [4] obtained bounds $\operatorname{ex}_{\mathcal{P}}\left(n, \Theta_{4}\right) \leq \frac{12(n-2)}{5}$, for all $n \geq 4, \operatorname{ex}_{\mathcal{P}}\left(n, \Theta_{5}\right) \leq \frac{5(n-2)}{2}$, for all $n \geq 5$ and $\operatorname{ex}_{\mathcal{P}}\left(n, \Theta_{6}\right) \leq \frac{18(n-2)}{7}$, for all $n \geq 7$, where $\Theta_{k}$ is obtained from a cycle $C_{k}$ by adding an additional edge joining any two non-consecutive vertices. They also demonstrated that their bounds for $\Theta_{4}$ and $\Theta_{5}$ are tight by showing infinitely many values of $n$ and planar graph on $n$ vertices attaining the stated bounds. As a consequence of the bound for $\Theta_{6}$ in the same paper, they presented the following corollary.

Corollary 1 (Y. Lan, et al.[4]).

$$
\operatorname{ex}_{\mathcal{P}}\left(n, C_{6}\right) \leq \frac{18(n-2)}{7}
$$

for all $n \geq 6$, with equality when $n=9$.
In this paper we present a tight bound for $\operatorname{ex}_{\mathcal{P}}\left(n, C_{6}\right)$. In particular, we prove the following two theorems to give the tight bound.

We denote the vertex and the edge sets of a graph $G$ by $V(G)$ and $E(G)$ respectively. We also denote the number of vertices and edges of $G$ by $v(G)$ and $e(G)$ respectively. The minimum degree of $G$ is denoted $\delta(G)$. The main ingredient of the result is as follows:

Theorem 2. Let $G$ be a 2 -connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq$ 3. Then $e(G) \leq \frac{5}{2} n-7$.

We use Theorem 2, which considers only 2 -connected graphs with no degree 2 (or 1) vertices and order at least 6 , in order to establish our desired result, which bounds gives the desired bound of $\frac{5}{2} n-7$ for all $C_{6}$-free plane graphs with at least 18 vertices.

Theorem 3. Let $G$ be a $C_{6}$-free plane graph on $n(n \geq 18)$ vertices. Then

$$
e(G) \leq \frac{5}{2} n-7
$$

Indeed, there are 17-vertex graphs on 17 vertices with 36 edges, but $\frac{5}{2}(17)-7=35.5<36$. One such graph can be seen in Figure 1 .


Figure 1: Example of $G$ on 17 vertices such that $e(G)>(5 / 2) v(G)-7$.

We show that, for large graphs, Theorem 3 is tight:
Theorem 4. For every $n \cong 2(\bmod 5)$, there exists a $C_{6}$-free plane graph $G$ with $v(G)=$ $\frac{18 n+14}{5}$ and $e(G)=9 n$, hence $e(G)=\frac{5}{2} v(G)-7$.

For a vertex $v$ in $G$, the neighborhood of $v$, denoted $N_{G}(v)$, is the set of all vertices in $G$ which are adjacent to $v$. We denote the degree of $v$ by $d_{G}(v)=\left|N_{G}(v)\right|$. We may avoid the subscripts if the underlying graph is clear. The minimum degree of $G$ is denoted by $\delta(G)$, the number of components of $G$ is denoted by $c(G)$. For the sake of simplicity, we may use the term $k$-cycle to mean a cycle of length $k$ and $k$-face to mean a face bounded by a $k$-cycle. A $k$-path is a path with $k$ edges.

## 2 Proof of Theorem 4: Extremal Graph Construction

First we show that for a plane graph $G_{0}$ with $n$ vertices $(n \cong 7(\bmod 10))$, each face having length 7 and each vertex in $G_{0}$ having degree either 2 or 3 , we can construct $G$, where $G$ is a $C_{6}$-free plane graph with $v(G)=\frac{18 n+14}{5}$ and $e(G)=9 n$. We then give a construction for such a $G_{0}$ as long as $n \cong 7(\bmod 10)$.

Using Euler's formula, the fact that every face has length 7 and every degree is 2 or 3 , we have $e\left(G_{0}\right)=\frac{7(n-2)}{5}$ and the number of degree 2 and degree 3 vertices in $G_{0}$ are $\frac{n+28}{5}$ and $\frac{4 n-28}{5}$, respectively.

Given $G_{0}$, we construct first an intermediate graph $G^{\prime}$ by step (1).
(1) Add halving vertices to each edge of $G_{0}$ and join the pair of halving vertices with distance 2, see an example in Figure 2, Let $G^{\prime}$ denote this new graph, then $v\left(G^{\prime}\right)=$ $v\left(G_{0}\right)+e\left(G_{0}\right)=\frac{12 n-14}{5}$ and the number of degree 2 and degree 3 vertices in $G^{\prime}$ is equal to the number of degree 2 and degree 3 vertices in $G_{0}$, respectively.


Figure 2: Adding a halving vertex to each edge of $G_{0}$.

To get $G$, we apply the following steps (2) and (3) on the degree 2 and 3 vertices in $G^{\prime}$, respectively.
(2) For each degree 2 vertex $v$ in $G_{0}$, let $N(v)=\left\{v_{1}, v_{2}\right\}$, and so $v_{1} v v_{2}$ forms an induced triangle in $G^{\prime}$. Fix $v_{1}$ and $v_{2}$, replace $v_{1} v v_{2}$ with a $K_{5}^{-}$by adding vertices $v_{1}^{\prime}, v_{2}^{\prime}$ to $V\left(G^{\prime}\right)$ and edges $v_{1}^{\prime} v, v_{1}^{\prime} v_{2}^{\prime}, v_{1}^{\prime} v_{1}, v_{1}^{\prime} v_{2}, v_{2}^{\prime} v_{1}, v_{2}^{\prime} v_{2}$ to $E\left(G^{\prime}\right)$. See Figure 3,


Figure 3: Replacing a degree-2 vertex of $G_{0}$ with a $K_{5}^{-}$.
(3) For each degree 3 vertex $v$ in $G_{0}$, such that $N(v)=\left\{v_{1}, v_{2}, v_{3}\right\}$, the set of vertices $\left\{v, v_{1}, v_{2}, v_{3}\right\}$ then forms an induced $K_{4}$ in $G^{\prime}$. Fix $v_{1}, v_{2}$ and $v_{3}$, replace this $K_{4}$ with a $K_{5}^{-}$by adding a new vertex $v^{\prime}$ to $V\left(G^{\prime}\right)$ and edges $v^{\prime} v, v^{\prime} v_{1}, v^{\prime} v_{2}$ to $E\left(G^{\prime}\right)$. See Figure 4 .


Figure 4: Replacing a degree-3 vertex of $G_{0}$ with a $K_{5}^{-}$.

For each integer $k \geq 0$, and $n=10 k+7$ we present a construction for such a $G_{0}$, call it $G_{0}^{k}$ : Let $v_{i}^{t}$ and $v_{i}^{b}(1 \leq i \leq k+1)$ be the top and bottom vertices of the heptagonal grids with 3 layers and $k$ columns, respectively (see the red vertices in Figure 5) and $v$ be the extra vertex in $G_{0}^{k}$ but not in the heptagonal grid. We join $v_{1}^{t} v, v v_{1}^{b}$ and $v_{i}^{t} v_{i}^{b}(2 \leq i \leq k+1)$. Clearly, $G_{0}^{k}$ is a $(10 k+7)$-vertex plane graph and each face of $G_{0}^{k}$ is a 7 -face. Obviously e $\left(G_{0}^{k}\right)=14 k+7$, and the number of degree 2 and 3 vertices are $2 k+7=\frac{n+28}{5}$ and $8 k=\frac{4 n-28}{5}$ respectively.

After applying steps (1), (2), and (3) on $G_{0}^{k}$, we get $G$. It is easy to verify that $G$ is a $C_{6}$-free plane graph with

$$
\begin{aligned}
& v(G)=v\left(G_{0}^{k}\right)+e\left(G_{0}^{k}\right)+2(2 k+7)+8 k=(10 k+7)+(14 k+7)+12 k+14=36 k+28 \\
& e(G)=9 v\left(G_{0}^{k}\right)=90 k+63
\end{aligned}
$$

Thus, $e(G)=\frac{5}{2} v(G)-7$.
Remark 1. In fact, for $k \geq 1$ and $n=10 k+2$, there exists a graph $H_{0}^{k}$ which is obtained from $G_{0}^{k}$ by deleting vertices (colored green in Figure (5) $x_{1}, x_{2}, x_{3}, x_{4}, x_{5}$ and adding the edge $v_{1}^{t} y$. Clearly, $H_{0}^{k}$ is an $10 k+2$-vertex plane graph such that all faces have length 7 . Moreover, $e\left(H_{0}^{k}\right)=14 k$, the number of degree-2 and degree-3 vertices are $2 k+6=\frac{n+28}{5}$ and


Figure 5: The graph $G_{0}^{k}, k \geq 1$, in which each face has length 7. The graph $H_{0}^{k}$ (see Remark (1) is obtained by deleting $x_{1}, \ldots, x_{5}$ and adding the edge $v_{1}^{t} y$.
$8 k-4=\frac{4 n-28}{5}$, respectively. After applying steps (1), (2), and (3) to $H_{0}^{k}$, we get a graph $H$ that is a $C_{6}$-free plane graph with $e(H)=(5 / 2) v(H)-7$.

Thus, for any $k \cong 2(\bmod 5)$, we have the graphs above such that each face is a 7-gon and we get a $C_{6}$-free plane graph on $n$ vertices with $(5 / 2) n-7$ edges for $n \cong 10(\bmod 18)$ if $n \geq 28$.

## 3 Definitions and Preliminaries

We give some necessary definitions and preliminary results which are needed in the proof of Theorems 2 and 3 .

Definition 5. Let $G$ be a plane graph and $e \in E(G)$. If e is not in a 3-face of $G$, then we call it a trivial triangular-block. Otherwise, we recursively construct a triangular-block in the following way. Start with $H$ as a subgraph of $G$, such that $E(H)=\{e\}$.
(1) Add the other edges of the 3-face containing e to $E(H)$.
(2) Take $e^{\prime} \in E(H)$ and search for a 3-face containing $e^{\prime}$. Add these other edge(s) in this 3-face to $E(H)$.
(3) Repeat step (2) till we cannot find a 3-face for any edge in $E(H)$.

We denote the triangular-block obtained from $e$ as the starting edge, by $B(e)$.
Let $G$ be a plane graph. We have the following three observations:
(i) If $H$ is a non-trivial triangular-block and $e_{1}, e_{2} \in E(H)$, then $B\left(e_{1}\right)=B\left(e_{2}\right)=H$.
(ii) Any two triangular-blocks of $G$ are edge disjoint.
(iii) If $B$ is a triangular-block with the unbounded region being a 3 -face, then $B$ is a triangulation graph.

Let $\mathcal{B}$ be the family of triangular-blocks of $G$. From observation (ii) above, we have

$$
e(G)=\sum_{B \in \mathcal{B}} e(B),
$$

where $e(G)$ and $e(B)$ are the number of edges of $G$ and $B$ respectively.
Next, we distinguish the types of triangular-blocks that a $C_{6}$-free plane graph may contain. The following lemma gives us the bound on the number of vertices of triangular-blocks.

Lemma 6. Every triangular-block of $G$ contains at most 5 vertices.

Proof. We prove it by contradiction. Let $B$ be a triangular-block of $G$ containing at least 6 vertices. We perform the following operations: delete one vertex from the boundary of the unbounded face of $B$ sequentially until the number of vertices of the new triangular block $B^{\prime}$ is 6 . Next, we show that $B^{\prime}$ is not a triangular-block in $G$. Suppose that it is. We consider the following two cases to complete the proof.

Case 1. $B^{\prime}$ contains a separating triangle.
Let $v_{1} v_{2} v_{3}$ be the separating triangle. Without loss of generality, assume that the inner region of the triangle contains two vertices say, $v_{4}$ and $v_{5}$. The outer region of the triangle
contains one vertex, say $v_{6}$. Since the unbounded face is a 3 -face, the inner structure is a triangulation. Without loss of generality, let the inner structure be as shown in Figure 6(a). Now consider the vertex $v_{6}$. If $v_{1}, v_{2} \in N\left(v_{6}\right)$, then $v_{3} v_{4} v_{5} v_{2} v_{6} v_{1} v_{3}$ is a 6 -cycle in $G$, a contradiction. Similarly for the cases when $v_{1}, v_{3} \in N\left(v_{6}\right)$ and $v_{2}, v_{3} \in N\left(v_{6}\right)$.

Case 2. $B^{\prime}$ contains no separating triangle.
Consider a triangular face $v_{1} v_{2} v_{3} v_{1}$. Let $v_{4}$ be a vertex in the triangular-block such that $v_{2} v_{3} v_{4} v_{2}$ is a 3 -face. Notice that $v_{1} v_{4} \notin E\left(B^{\prime}\right)$, otherwise we get a separating triangle in $B^{\prime}$. Let $v_{5}$ be a vertex in $B^{\prime}$ such that $v_{2} v_{4} v_{5} v_{2}$ is a 3 -face. Notice that $v_{6}$ cannot be adjacent to both vertices in any of the pairs $\left\{v_{1}, v_{2}\right\},\left\{v_{1}, v_{3}\right\},\left\{v_{2}, v_{5}\right\},\left\{v_{3}, v_{4}\right\}$, or $\left\{v_{4}, v_{5}\right\}$. Otherwise, $C_{6} \subset G$. Also $v_{3} v_{5} \notin E\left(B^{\prime}\right)$, otherwise we have a separating triangle. So, let $v_{1} v_{5} \in E\left(B^{\prime}\right)$ and $v_{1}, v_{5} \in N\left(v_{6}\right)$ (see Figure 6(b)). In this case $v_{1} v_{6} v_{5} v_{2} v_{4} v_{3} v_{1}$ results in a 6 -cycle, a contradiction.

(a)

(b)

Figure 6: The structure of $B^{\prime}$ when it contains a separating triangle or not, respectively.

Now we describe all possible triangular-blocks in $G$ based on the number of vertices the block contains. For $k \in\{2,3,4,5\}$, we denote the triangular-blocks on $k$ vertices as $B_{k}$.

## Triangular-blocks on 5 vertices.

There are four types of triangular-blocks on 5 vertices (see Figure (7). Notice that $B_{5, a}$ is a $K_{5}^{-}$.


Figure 7: Triangular-blocks on 5 vertices.

## Triangular-blocks on 4, 3, and 3 vertices.

There are two types of triangular-blocks on 4 vertices. See Figure 8. Observe that $B_{4, a}$ is a $K_{4}$. The 3-vertex and 2-vertex triangular-blocks are simply $K_{3}$ and $K_{2}$ (the trivial triangular-block), respectively.

$B_{2}$
Figure 8: Triangular-blocks on 4,3 and 2 vertices.

Definition 7. Let $G$ be a plane graph.
(i) A vertex $v$ in $G$ is called a junction vertex if it in at least two distinct triangularblocks of $G$.
(ii) Let $B$ be a triangular-block in $G$. An edge of $B$ is called an exterior edge if it is on a boundary of non-triangular face of $G$. Otherwise, we call it an interior edge. An endvertex of an exterior edge is called an exterior vertex. We denote the set of all exterior and interior edges of $B$ by $\operatorname{Ext}(B)$ and $\operatorname{Int}(B)$ respectively. Let $e \in \operatorname{Ext}(B)$, a non-triangular face of $G$ with $e$ on the boundary is called the exterior face of $e$.

Notice that an exterior edge of a non-trivial triangular-block has exactly one exterior face. On the other hand, if $G$ is a 2-connected plane graph, then every trivial triangularblock has two exterior faces. For a non-trivial triangular-block $B$ of a plane graph $G$, we
call a path $P=v_{1} v_{2} v_{3} \ldots v_{k}$ an exterior path of $B$, if $v_{1}$ and $v_{k}$ are junction vertices and $v_{i} v_{i+1}$ are exterior edges of $B$ for $i \in\{1,2, \ldots, k-1\}$ and $v_{j}$ is not junction vertex for all $j \in\{2,3, \ldots, k-1\}$. The corresponding face in $G$ where $P$ is on the boundary of the face is called the exterior face of $P$.

Next, we give the definition of the contribution of a vertex and an edge to the number of vertices and faces of $C_{6}$-free plane graph $G$. All graphs discussed from now on are $C_{6}$-free plane graph.

Definition 8. Let $G$ be a plane graph, $B$ be a triangular-block in $G$ and $v \in V(B)$. The contribution of $v$ to the vertex number of $B$ is denoted by $n_{B}(v)$, and is defined as

$$
n_{B}(v)=\frac{1}{\# \text { triangular-blocks in } G \text { containing } v} .
$$

We define the contribution of $B$ to the number of vertices of $G$ as $n(B)=\sum_{v \in V(B)} n_{B}(v)$.
Obviously, $v(G)=\sum_{B \in \mathcal{B}} n(B)$, where $v(G)$ is the number of vertices in $G$ and $\mathcal{B}$ is the family of triangular-blocks of $G$.

Let $B_{K_{5}^{-}}$be a triangular-block of $G$ isomorphic to a $B_{5, a}$ with exterior vertices $v_{1}, v_{2}, v_{3}$, where $v_{1}$ and $v_{3}$ are junction vertices, see Figure 9 for an example. Let $F$ be a face in $G$ such that $V(F)$ contains all exterior vertices $v_{1,1}, \ldots, v_{1, m}, v_{2,1}, \ldots, v_{2, m}, v_{3,1}, \ldots, v_{3, m}$ of $m$ $(m \geq 1)$ copies of $B_{K_{5}^{-}}$, such that $v_{1, i}, v_{2, i}, v_{3, i}$ are the exterior vertices of the $i$-th $B_{K_{5}^{-}}$and $v_{1, i}, v_{3, i}(1 \leq i \leq m)$ are junction vertices. Let $C_{F}$ denote the cycle associated with the face $F$. We alter $E\left(C_{F}\right)$ in the following way:

$$
E\left(C_{F}^{\prime}\right):=E\left(C_{F}\right)-\left\{v_{1,1} v_{2,1} v_{3,1}\right\}-\cdots-\left\{v_{1, m} v_{2, m} v_{3, m}\right\} \cup\left\{v_{1,1} v_{3,1}\right\} \cup \ldots \cup\left\{v_{1, m}, v_{3, m}\right\}
$$

Hence, the length of $F$ as $\left|E\left(C_{F}^{\prime}\right)\right|=\left|E\left(C_{F}\right)\right|-m$. For example, in Figure 9, $\left|E\left(C_{F}\right)\right|=11$ but $\left|E\left(C_{F}^{\prime}\right)\right|=9$.

Now we are able to define the contribution of an "edge" to the number of faces of $C_{6}$-free plane graph $G$.

Definition 9. Let $F$ be a exterior face of $G$ and $C_{F}:=\left\{e_{1}, e_{2}, \ldots, e_{k}\right\}$ be the cycle associated with $F$. The contribution of an exterior edge e to the face number of the exterior face $F$, is denoted by $f_{F}(e)$, and is defined as follows.


Figure 9: An example of a face containing all the exterior vertices of at least one $B_{K_{5}^{-}}$.
(i) If $e_{1}$ and $e_{2}$ are adjacent exterior edges of $B_{K_{5}^{-}}$, then $f_{F}\left(e_{1}\right)+f_{F}\left(e_{2}\right)=\frac{1}{\left|C_{F}^{\prime}\right|}$, and $f_{F}\left(e_{i}\right)=\frac{1}{\left|C_{F}^{\prime}\right|}$, where $i \in\{3,4, \ldots, k\}$.
(ii) Otherwise, $f_{F}(e)=\frac{1}{\left|C_{F}\right|}$.

Note that $\sum_{e \in E(F)} f_{F}(e)=1$. For a triangular-block $B$, the total face contribution of $B$ is denoted by $f_{B}$ and defined as $f_{B}=(\#$ interior faces of $B)+\sum_{e \in \operatorname{Ext}(B)} f_{F}(e)$, where $F$ is the exterior face of $B$ with respective to $e$. Obviously, $f(G)=\sum_{B \in \mathcal{B}} f(B)$, where $f(G)$ is the number of faces of $G$.

## 4 Proof of Theorem 2

We begin by outlining our proof. Let $f, n$, and $e$ be the number of faces, vertices, and edges of $G$ respectively. Let $\mathcal{B}$ be the family of all triangular-blocks of $G$.

The main target of the proof is to show that

$$
\begin{equation*}
7 f+2 n-5 e \leq 0 \tag{1}
\end{equation*}
$$

Once we show (1), then by using Euler's Formula, $e=f+n-2$, we can finish the proof of Theorem 2. To prove (1), we show the existence of a partition $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{m}$ of $\mathcal{B}$ such that
$7 \sum_{B \in \mathcal{P}_{i}} f(B)+2 \sum_{B \in \mathcal{P}_{i}} n(B)-5 \sum_{B \in \mathcal{P}_{i}} e(B) \leq 0$, for all $i \in\{1,2,3 \ldots, m\}$. Since $f=\sum_{B \in \mathcal{B}} f(B)$, $n=\sum_{B \in \mathcal{B}} n(B)$ and $e=\sum_{B \in \mathcal{B}} e(B)$ we have

$$
\begin{aligned}
7 f+2 n-5 e & =7 \sum_{i}^{m} \sum_{B \in \mathcal{P}_{i}} f(B)+2 \sum_{i}^{m} \sum_{B \in \mathcal{P}_{i}} n(B)-5 \sum_{i}^{m} \sum_{B \in \mathcal{P}_{i}} e(B) \\
& =\sum_{i}^{m}\left(7 \sum_{B \in \mathcal{P}_{i}} f(B)+2 \sum_{B \in \mathcal{P}_{i}} n(B)-5 \sum_{B \in \mathcal{P}_{i}} e(B)\right) \leq 0 .
\end{aligned}
$$

The following proposition will be useful in many lemmas.

Proposition 10. Let $G$ be a 2-connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq 3$.
(i) If $B$ is a nontrivial triangular-block (that is, not $B_{2}$ ), then none of the exterior faces can have length 5.
(ii) If $B$ is in $\left\{B_{5, a}, B_{5, b}, B_{5, c}, B_{4, a}\right\}$, then none of the exterior faces can have length 4 .
(iii) If $B$ is in $\left\{B_{5, d}, B_{4, b}\right\}$ and an exterior face of $B$ has length 4 , then that 4 -face must share a 2-path with $B$ (shown in blue in Figures 13 and 14) and the other edges of the face must be in trivial triangular-blocks.
(iv) No two 4-faces can be adjacent to each other.

Proof. (i) Observe that any pair of consecutive exterior vertices of a nontrivial triangularblock has a path of length 2 (counted by the number of edges) between them and any pair of nonconsecutive exterior vertices has a path of length 3 between them. So having a face of length 5 incident to this triangular-block would yield a $C_{6}$, a contradiction.
(ii) If $B$ is in $\left\{B_{5, a}, B_{5, b}, B_{5, c}, B_{4, a}\right\}$, then any pair of consecutive exterior vertices of the listed triangular-blocks has a path of length 3 between them. It remains to consider nonconsecutive vertices for $\left\{B_{5, b}, B_{5, c}\right\}$. For $B_{5, b}$ each pair of nonconsecutive exterior vertices has a path of length 3 between them. In the case where $B$ is $B_{5, c}$, this is true for all pairs without an edge between them. As for the other pairs, if they are in the
same 4 -face, then at least one of the degree- 2 vertices in $B$ must have degree 2 in $G$, a contradiction.
(iii) In both $B_{5, d}$ and $B_{4, b}$, any pair of consecutive exterior vertices has a path of length 3 between them. For $B_{5, d}$, in Figure 13, we see that there is a path of length 4 between $v_{2}$ and $v_{4}$ and so the only way a 4 -face can be adjacent to $B$ is via a 2 -path with endvertices $v_{1}$ and $v_{3}$. In fact, because there is no vertex of degree 2 , the path must be $v_{1} v_{4} v_{3}$. For $B_{4, b}$, in Figure [13, we see that because $B$ cannot have a vertex of degree 2 , the 4 -face and $B$ cannot share the path $v_{2} v_{1} v_{4}$ or the path $v_{2} v_{3} v_{4}$. Thus the only paths that can share a boundary with a 4 -face are $v_{1} v_{4} v_{3}$ and $v_{1} v_{2} v_{3}$.

As to the other blocks that form edges of such a 4 -face. In Figure 10, we see that if, say, $v_{1} u$ is in a nontrivial triangular-block, then there is a vertex $w$ in that block, in which case $w v_{1} x v_{4} v_{3} u w$ forms a 6-cycle, a contradiction.
(iv) If two 4 -faces share an edge, then there is a 6 -cycle formed by deleting that edge. If two 4 -faces share a 2 -path, then the midpoint of that path is a vertex of degree 2 in $G$. In both cases, a contradiction.


Figure 10: Proposition 1q(iii): The blocks defined by blue edges must be trivial.

To show the existence of such a partition we need the following lemmas.
Lemma 11. Let $G$ be a 2 -connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq 3$. If $B$ is a triangular-block in $G$ such that $B \notin\left\{B_{5, d}, B_{4, b}\right\}$, then $7 f(B)+2 n(B)-5 e(B) \leq 0$.

Proof. We separate the proof into several cases.

## Case 1: $B$ is $B_{5, a}$.

Let $v_{1}, v_{2}$ and $v_{3}$ be the exterior vertices of $K_{5}^{-}$. At least two of them must be junction vertices, otherwise $G$ contains a cut vertex. We consider 2 possibilities to justify this case.
(a) Let $B$ be $B_{5, a}$ with 3 junction vertices (see Figure 11(a)). By Proposition 10, every exterior edge in $B$ is contained in an exterior face with length at least 7. Thus, $f(B)=(\#$ interior faces of $B)+\sum_{e \in E x t(B)} f_{F}(e) \leq 5+3 / 7$. Moreover, every junction vertex is contained in at least 2 triangular-blocks, so we have $n(B) \leq 2+3 / 2$. With $e(B)=9$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$.
(b) Let $B$ be $B_{5, a}$ with 2 junction vertices, say $v_{2}$ and $v_{3}$ (see Figure 11(b)). Let $F$ and $F_{1}$ are exterior faces of the exterior edge $v_{2} v_{3}$ and exterior path $v_{2} v_{1} v_{3}$ of the triangularblock respectively. Notice that $v_{1} v_{2}$ and $v_{2} v_{3}$ are the adjacent exterior edges in the same face $F_{1}$, hence $\left|C\left(F_{1}\right)\right| \geq 8$. By Definition 9, we have $f_{F_{1}}\left(v_{1} v_{2}\right)+f_{F_{1}}\left(v_{1} v_{3}\right) \leq 1 / 7$. Because there can be no $C_{6}$, one can see that regardless of the configuration of the $B_{K_{5}^{-}}$, it is the case that $f_{F}\left(v_{2} v_{3}\right) \leq 1 / 7$. Thus, $f(B) \leq 5+2 / 7$. Moreover, since $v_{1}$ and $v_{3}$ are contained in at least 2 triangular-blocks, we have $n(B) \leq 3+2 / 2$. With $e(B)=9$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$.

(a)

(b)

Figure 11: A $B_{5, a}$ triangular-block with 3 and 2 junction vertices, respectively.

Case 2: $B$ is in $\left\{B_{4, a}, B_{5, b}, B_{5, c}\right\}$.
(a) Let $B$ be a $B_{4, a}$. By Proposition 10, each face incident to this triangular-block has length at least 7. So, $f(B) \leq 3+3 / 7$. Because there is no cut-vertex, this triangularblock must have at least two junction vertices, hence $n(B) \leq 2+2 / 2$. With $e(B)=6$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$.
(b) Let $B$ be a $B_{5, b}$. There are 4 faces inside the triangular-block and each face incident to this triangular-block has length at least 7. So, $f(B) \leq 4+4 / 7$. Because there is no cut-vertex, this triangular-block must have at least two junction vertices, hence $n(B) \leq 3+2 / 2$. With $e(B)=8$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$, as seen in Table 2.
(c) Let $B$ be a $B_{5, c}$. Similarly, $f(B) \leq 3+5 / 7$ and because there are at least two junction vertices, $n(B) \leq 3+2 / 2$. With $e(B)=7$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-1$.

Case 3: $B$ is $B_{3}$.
Let $v_{1}, v_{2}$ and $v_{3}$ be the exterior vertices of triangular-block $B$. Each of these three must be junction vertices since there is no degree 2 vertex in $G$, which implies that each is contained in at least 2 triangular-blocks. We consider two possibilities:
(a) Let the three exterior vertices be contained in exactly 2 triangular-blocks. By Proposition 1q(i), the length of each exterior face is either 4 or at least 7. We want to show that at most one exterior face has length 4.

If not, then let $x_{1}$ be a vertex that is in two such faces. Consider the triangular-block incident to $B$ at $x_{1}$, call it $B^{\prime}$. By Proposition 10, $B^{\prime}$ is not in $\left\{B_{5, a}, B_{5, b}, B_{5, c}, B_{4, a}\right\}$.

If $B^{\prime}$ is in $\left\{B_{5, d}, B_{4, b}, B_{3}\right\}$, then the triangular-block has vertices $\ell_{2}, \ell_{3}$, each adjacent to $x_{1}$ and the length- 4 faces consist of $\left\{v_{1}, \ell_{2}, m_{2}, v_{2}\right\}$ and $\left\{v_{1}, \ell_{3}, m_{3}, v_{3}\right\}$. Either $\ell_{2} \sim \ell_{3}$ (in which case $\ell_{2} m_{2} v_{2} v_{3} m_{3} \ell_{3} \ell_{2}$ is a 6 -cycle, see Figure 12(a)) or there is a $\ell^{\prime}$ distinct from $v_{1}$ that is adjacent to both $\ell_{2}$ and $\ell_{3}$ (in which case $\ell^{\prime} \ell_{2} m_{2} v_{2} v_{1} \ell_{3} \ell_{2}$ is a 6 -cycle, see Figure 12(b)).

If $B^{\prime}$ is $B_{2}$, then the trivial triangular-block is $\left\{v_{1}, \ell\right\}$, in which case $\left\{\ell, m_{2}, v_{2}, v_{1}, v_{3}, m_{3}\right\}$ is a $C_{6}$, see Figure 12(c). Thus, we may conclude that if each of the three exterior vertices are in exactly 2 triangular-blocks, then $f(B) \leq 1+2 / 7+1 / 4$ and $n(B) \leq 3 / 2$. With $e(B)=3$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-5 / 4$.

(a)

(a)

(a)

Figure 12: A $B_{3}$ triangular-block, $B$ and the various cases of what must occur if $B$ is incident to two 4 -faces.
(b) Let at least one exterior vertex be contained in at least 3 triangular-blocks and the others be contained at least 2 triangular-blocks. In this case, we have $f(B) \leq 1+3 / 4$ and $n(B) \leq 2 / 2+2 / 3$. With $e(B)=3$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-1 / 12$.

Case 4: $B$ is $B_{2}$.
Note that the fact that there is no vertex of degree 2 gives that if an endvertex is in exactly two triangular-blocks, then the other one cannot be a $B_{2}$. We consider three possibilities:
(a) Let each endvertex be contained in exactly 2 triangular-blocks. Since neither of the triangular-blocks incident to $B$ can be trivial, they cannot be incident to a face of length 5 by Proposition 1q(i). Thus, $B$ cannot be incident to a face of length 5. Moreover, the two faces incident to $B$ cannot both be of length 4 , again by Proposition 1q)(iv). Hence, $f(B) \leq 1 / 4+1 / 7$. Clearly $n(B) \leq 2 / 2$ and with $e(B)=1$, we obtain $7 f(B)+$ $2 n(B)-5 e(B) \leq-1 / 4$.
(b) Let one endvertex be contained in exactly 2 triangular-blocks and the other endvertex be contained in at least 3 triangular-blocks. This is similar to case (a) in that neither
face can have length 5 and they cannot both have length 4 . The only difference is that $n(B) \leq 1 / 2+1 / 3$ and so $7 f(B)+2 n(B)-5 e(B) \leq-7 / 12$.
(c) Let each endvertex be contained in at least 3 triangular-blocks. The two faces cannot both be of length 4 by Proposition 10.(iv). Hence, $f(B) \leq 1 / 4+1 / 5$ and $n(B) \leq 2 / 3$. With $e(B)=1$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-31 / 60$.

Lemma 12. Let $G$ be a 2 -connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq 3$. If $B$ is $B_{5, d}$, then $7 f(B)+2 n(B)-5 e(B) \leq 1 / 2$. Moreover, $7 f(B)+2 n(B)-5 e(B) \leq 0$ unless $B$ shares a 2-path with a 4-face.

(a)

(b)

Figure 13: A $B_{5, d}$ triangular-block and how a 4 -face must be incident to it.

Proof. Let $B$ be $B_{5, d}$ with vertices $v_{1}, v_{2}, v_{3}, v_{4}$, and $v_{5}$, as shown in Figure 13(a). By Proposition 10[(i), no exterior face of $B$ can have length 5. By Proposition 1q[(iii), if there is an exterior face of $B$ that has length 4 , this 4 -face must contain the path $v_{1} v_{4} v_{3}$.

Moreover, since there is no vertex of degree $2, v_{2}$ is a junction vertex. Because $G$ has no cut-vertex, there is at least one other junction vertex. We may consider the following cases:
(a) Let $v_{4}$ be a junction vertex. This prevents an exterior face of length 4 . Thus, each exterior face has length at least 7. Hence, $f(B) \leq 4+4 / 7$ and $n(B) \leq 3+2 / 2$. With $e(B)=8$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$.
(b) Let $v_{4}$ fail to be a junction vertex and exactly one of $v_{1}, v_{3}$ be a junction vertex. Without loss of generality let it be $v_{3}$. In this case, again, each exterior face has length ${ }^{1}$ at least 7. Again, $f(B) \leq 4+4 / 7$ and $n(B) \leq 3+2 / 2$. With $e(B)=8$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq 0$.
(c) Let $v_{4}$ fail to be a junction vertex and both $v_{1}$ and $v_{3}$ be junction vertices. Here either the exterior path $v_{1} v_{4} v_{3}$ is part of an exterior face of length at least 4 or each edge must be in a face of length at least 7. If the exterior face is of length at least 7, then $f(B) \leq 4+4 / 7$, otherwise $f(B) \leq 4+2 / 4+2 / 7$. In both cases, $n(B) \leq 2+3 / 2$ and $e(B)=8$. Hence we obtain $7 f(B)+2 n(B)-5 e(B) \leq-1$ in the first instance and $7 f(B)+2 n(B)-5 e(B) \leq 1 / 2$ in the case where $B$ is incident to a 4 -face.

Lemma 13. Let $G$ be a 2 -connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq 3$. If $B$ is $B_{4, b}$, then $7 f(B)+2 n(B)-5 e(B) \leq 4 / 3$. Moreover, $7 f(B)+2 n(B)-5 e(B) \leq 1 / 6$ if $B$ shares a 2-path with exactly one 4-face and $7 f(B)+2 n(B)-5 e(B) \leq 0$ if $B$ fails to share a 2-path with any 4-face.


Figure 14: A $B_{4, b}$ triangular-block and how a 4 -face must be incident to it.

Proof. Let $B$ be with vertices $v_{1}, v_{2}, v_{3}$, and $v_{4}$, as shown in Figure 14(a). By Proposition 1q(i), no exterior face of $B$ can have length 5 . If there is an exterior face of $B$ that has

[^0]length 4 , it is easy to verify that being $C_{6}$-free and having no vertex of degree 2 means that the junction vertices must be $v_{1}$ and $v_{3}$. We may consider the following cases.
(a) Let either $v_{2}$ or $v_{4}$ be a junction vertex and, without loss of generality, let it be $v_{2}$. All the exterior faces have length at least 7 except for the possibility that the path $v_{1} v_{4} v_{3}$ may form two sides of a 4 -face. Hence, $f(B) \leq 2+2 / 4+2 / 7$ and $n(B) \leq 1+3 / 2$. With $e(B)=5$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-1 / 2$.
(b) Let neither $v_{2}$ nor $v_{4}$ be a junction vertex. Because there is no cut-vertex, this requires both $v_{1}$ and $v_{3}$ to be junction vertices. Hence, there are two exterior faces: One that shares the exterior path $v_{1} v_{4} v_{3}$ and the other shares the exterior path $v_{1} v_{2} v_{3}$. Each exterior face has length either 4 or at least 7 . We consider several subcases:
(i) If both faces are of length at least 7 , then $f(B) \leq 2+4 / 7$, and $n(B) \leq 2+2 / 2$. With $e(B)=5$, we obtain $7 f(B)+2 n(B)-5 e(B) \leq-1$.
(ii) If only one of the exterior faces is of length 4 , then $f(B) \leq 2+2 / 7+2 / 4$. Moreover, at least one of $v_{1}, v_{3}$ must be a junction vertex for more than two triangular-blocks, otherwise either $v(G)=5$ or the vertex incident to two blue edges in Figure 14(b) is a cut-vertex. Hence, $n(B) \leq 2+1 / 3+1 / 2$ and with $e(B)=5$, we have $7 f(B)+2 n(B)-5 e(B) \leq 1 / 6$.
(iii) Both exterior faces are of length 4 . Thus $f(B) \leq 2+4 / 4$. By Proposition 1d(iii), the blocks represented by the blue edges in Figure 14(c) are each trivial. Hence $n(B) \leq 2+2 / 3$. With $e(B)=5$, we get $7 f(B)+2 n(B)-5 e(B) \leq 4 / 3$.

Tables 2 and 3 in Appendix $A$ give a summary of Lemmas 11, 12, and 13,
Lemma 14. Let $G$ be a 2-connected, $C_{6}$-free plane graph on $n(n \geq 6)$ vertices with $\delta(G) \geq$ 3. Then the triangular-blocks of $G$ can be partitioned into sets, $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{m}$ such that $7 \sum_{B \in \mathcal{P}_{i}} f(B)+2 \sum_{B \in \mathcal{P}_{i}} n(B)-5 \sum_{B \in \mathcal{P}_{i}} e(B) \leq 0$ for all $i \in[m]$.

Proof. As it can be seen from Tables 2 and 3 in Appendix A, there are three possible cases where $7 f(B)+2 n(B)-5 e(B)$ assumes a positive value. We deal with each of these blocks as follows.


Figure 15: Structure of a $B_{5, d}$ if it is incident to a 4 -face, as in Lemma 14. The triangularblocks $B^{\prime}$ and $B^{\prime \prime}$ are trivial.
(1) Let $B$ be a $B_{5, d}$ triangular-block as described in the proof of Lemma 12(c). See Figure 15.

By Proposition (iii), the edges $v_{1} u$ and $v_{3} u$ are trivial triangular-blocks. Denote these triangular-blocks as $B^{\prime}$ and $B^{\prime \prime}$. Consider $B^{\prime}$. One of the exterior faces of $B^{\prime}$ has length 4 whereas by Proposition 10(iv), the other has length at least 5. It must have length at least 7 because if it had length 5 , then the path $v_{1} v_{3} u$ would complete it to a 6 -cycle. Thus, $f\left(B^{\prime}\right) \leq 1 / 4+1 / 7$. Since the vertex $u$ cannot be of degree 2 , then this vertex is shared in at least three triangular-blocks. Thus, $n\left(B^{\prime}\right) \leq 1 / 2+1 / 3$. With $e\left(B^{\prime}\right)=1$, we obtain $7 f\left(B^{\prime}\right)+2 n\left(B^{\prime}\right)-5 e\left(B^{\prime}\right) \leq-7 / 12$ and similarly, $7 f\left(B^{\prime \prime}\right)+2 n\left(B^{\prime \prime}\right)-5 e\left(B^{\prime \prime}\right) \leq$ $-7 / 12$. Define $\mathcal{P}^{\prime}=\left\{B, B^{\prime}, B^{\prime \prime}\right\}$. Thus, $7 \sum_{B^{*} \in \mathcal{P}^{\prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime}} n\left(B^{*}\right)-5 \sum_{B^{*} \in \mathcal{P}^{\prime}} e\left(B^{*}\right) \leq$ $1 / 2+2(-7 / 12)=-2 / 3$.

Therefore, for each triangular-block in $G$ as described in Lemma 12)(c), it belongs to a set $\mathcal{P}^{\prime}$ of three triangular-blocks such that $7 \sum_{B^{*} \in \mathcal{P}^{\prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime}} n\left(B^{*}\right)-5 \sum_{B^{*} \in \mathcal{P}^{\prime}} e\left(B^{*}\right) \leq$ 0 . Denote such sets as $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{m_{1}}$ if they exist.
(2) Let $B$ be a $B_{4, b}$ triangular-block as described in the proof of Lemma 13)(b) (ii). See Figure 16(a).


Figure 16: Structure of a $B_{4, b}$ triangular-block if it is incident to a 4-face, as in Lemma 14. The triangular-blocks $B^{\prime}, B^{\prime \prime}, B^{\prime \prime \prime}$, and $B^{\prime \prime \prime \prime}$ are all trivial.

By Proposition 10(iii), the edges $v_{1} u_{1}$ and $v_{3} u_{1}$ are trivial triangular-blocks. Denote them as $B^{\prime}$ and $B^{\prime \prime}$, respectively. Consider $B^{\prime}$. One of the exterior faces of $B^{\prime}$ has length 4 and by Proposition 1q(iv), the other has length at least 5. Thus, $f\left(B^{\prime}\right) \leq$ $1 / 4+1 / 5$. Since the vertex $u_{1}$ cannot be of degree 2 , then this vertex is shared in at least three triangular-blocks. Thus, $n\left(B^{\prime}\right) \leq 1 / 2+1 / 3$. With $e\left(B^{\prime}\right)=1$, we obtain $7 f\left(B^{\prime}\right)+2 n\left(B^{\prime}\right)-5 e\left(B^{\prime}\right) \leq-11 / 60$ and similarly, $7 f\left(B^{\prime \prime}\right)+2 n\left(B^{\prime \prime}\right)-5 e\left(B^{\prime \prime}\right) \leq-11 / 60$. Define $\mathcal{P}^{\prime \prime}=\left\{B, B^{\prime}, B^{\prime \prime}\right\}$. Thus, $7 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} n\left(B^{*}\right)-5 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} e\left(B^{*}\right) \leq 1 / 6+$ $2(-11 / 60)=-1 / 5$.

Therefore, for each triangular-block in $G$ as described in Lemma (13)(b)(ii), it belongs to a set $\mathcal{P}^{\prime \prime}$ of three triangular-blocks such that $7 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} n\left(B^{*}\right)-$ $5 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} e\left(B^{*}\right) \leq 0$. Denote such sets as $\mathcal{P}_{m_{1}+1}, \mathcal{P}_{m_{1}+2}, \ldots, \mathcal{P}_{m_{2}}$ if they exist.
(3) Let $B$ be a $B_{4, b}$ triangular-block as described in the proof of Lemma (b) (iii). See Figure 16(b).

By Proposition 1q(iii), the edges $v_{1} u_{1}, v_{3} u_{1}, v_{1} u_{2}$, and $v_{3} u_{2}$ are trivial triangular-blocks. Denote them as $B^{\prime}, B^{\prime \prime}, B^{\prime \prime \prime}$ and $B^{\prime \prime \prime \prime}$ respectively. Consider $B^{\prime}$. One of the exterior faces of $B^{\prime}$ has length 4 whereas the other has length at least 5 . Thus, $f\left(B^{\prime}\right) \leq 1 / 4+1 / 5$. Since the vertex $u_{1}$ cannot be of degree 2, then this vertex is shared in at least three triangular-blocks. Clearly $v_{1}$ is in at least three triangular-blocks. Thus, $n\left(B^{\prime}\right) \leq 2 / 3$. With $e\left(B^{\prime}\right)=1$, we obtain $7 f\left(B^{\prime}\right)+2 n\left(B^{\prime}\right)-5 e\left(B^{\prime}\right) \leq-31 / 60$ and the same inequality
holds for $B^{\prime \prime}, B^{\prime \prime \prime}$, and $B^{\prime \prime \prime \prime}$.
Define $\mathcal{P}^{\prime \prime \prime}=\left\{B, B^{\prime}, B^{\prime \prime}, B^{\prime \prime \prime}, B^{\prime \prime \prime \prime}\right\}$. Thus, $7 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} n\left(B^{*}\right)-5 \sum_{B^{*} \in \mathcal{P}^{\prime \prime}} e\left(B^{*}\right) \leq$ $4 / 3+4(-31 / 60)=-11 / 15$.

Therefore, for each triangular-block in $G$ as described in Lemma 13)(b)((iii), it belongs to a set $\mathcal{P}^{\prime}$ of three triangular-blocks such that $7 \sum_{B^{*} \in \mathcal{P}^{\prime \prime \prime}} f\left(B^{*}\right)+2 \sum_{B^{*} \in \mathcal{P}^{\prime \prime \prime}} n\left(B^{*}\right)-$ $5 \sum_{B^{*} \in \mathcal{P}^{\prime \prime \prime}} e\left(B^{*}\right) \leq 0$. Denote such sets as $\mathcal{P}_{m_{2}+1}, \mathcal{P}_{m_{2}+2}, \ldots, \mathcal{P}_{m_{3}}$ if they exist.

Now define $\mathcal{P}_{m_{3}+1}=\mathcal{B}-\bigcup_{i=1}^{m_{3}} \mathcal{P}_{i}$, where $\mathcal{B}$ is the set of all blocks of $G$. Clearly, for each block $B \in \mathcal{P}_{m_{3}+1}, 7 f(B)+2 n(B)-5 e(B) \leq 0$. Thus, $7 \sum_{B \in \mathcal{P}_{m_{3}+1}} f(B)+2 \sum_{B \in \mathcal{P}_{m_{3}+1}} n(B)-$ $5 \sum_{B \in \mathcal{P}_{m_{3}+1}} e(B) \leq 0$. Putting $m:=m_{3}+1$ we got the partition $\mathcal{P}_{1}, \mathcal{P}_{2}, \ldots, \mathcal{P}_{m}$ of $\mathcal{B}$ meeting the condition of the lemma.

This completes the proof of Theorem 2.

## 5 Proof of Theorem 3

Let $G$ be a $C_{6}$-free plane graph. We will show that either $5 v(G)-2 e(G) \geq 14$ or $v(G) \leq 17$.
If we delete a vertex $x$ from $G$, then

$$
\begin{aligned}
5 v(G-x)-2 e(G-x) & =5(v(G)-1)-2(e(G)-\operatorname{deg}(x)) \\
& =5 v(G)-2 e(G)-5+2 \operatorname{deg}(x) \\
& \geq 5 v(G)-2 e(G)-1 .
\end{aligned}
$$

So, graph $G$ has an induced subgraph $G^{\prime}$ with $\delta(G) \geq 3$ with

$$
\begin{equation*}
5 v(G)-2 e(G) \geq 5 v\left(G^{\prime}\right)-2 e\left(G^{\prime}\right)+\left(v(G)-v\left(G^{\prime}\right)\right) \tag{2}
\end{equation*}
$$

In line with usual graph theoretic terminology, we call a maximal 2-connected subgraph a block. Let $\mathcal{B}^{\prime}$ denote the set of blocks of $G^{\prime}$ with the $i^{\text {th }}$ block having $n_{i}$ vertices and $e_{i}$ edges. Let $b$ be the total number of blocks of $G^{\prime}$. Specifically, let $b_{2}, b_{3}, b_{4}$, and $b_{5}$ denote

|  | min of $5 n-2 e-5$ |  |
| :--- | :---: | :--- |
| $n \geq 6$ | $14-5 \geq 9$ | Theorem 2] |
| $n=5$ | $5(5)-2(9)-5 \geq 2$ | $B_{5, a}$, Figure 7 |
| $n=4$ | $5(4)-2(6)-5 \geq 3$ | $B_{4, a}$, Figure 8$]$ |
| $n=3$ | $5(3)-2(3)-5 \geq 4$ | $B_{3}$, Figure 8 |
| $n=2$ | $5(2)-2(2)-5 \geq 3$ | $B_{2}$, Figure 8 |

Table 1: Estimates of $5 n-2 e-5$ for various block sizes.
the number of blocks of size $2,3,4$, and 5 , respectively. Let $b_{6}$ denote the number of blocks of size at least 6 . Then we have $b=b_{6}+b_{5}+b_{4}+b_{3}+b_{2}$ and, using Table 1:

$$
\begin{align*}
5 v\left(G^{\prime}\right)-2 e\left(G^{\prime}\right) & =5\left(\sum_{i=1}^{b} n_{i}-(b-1)\right)-2 \sum_{i=1}^{b} e_{i} \\
& =\sum_{i=1}^{b}\left(5 n_{i}-2 e_{i}-5\right)+5 \\
& \geq 9 b_{6}+2 b_{5}+3 b_{4}+4 b_{3}+3 b_{2}+5 \tag{3}
\end{align*}
$$

Combining (2) and (3), we obtain

$$
\begin{equation*}
5 v(G)-2 e(G) \geq 9 b_{6}+2 b_{5}+3 b_{4}+4 b_{3}+3 b_{2}+5+\left(v(G)-v\left(G^{\prime}\right)\right) \tag{4}
\end{equation*}
$$

If $b_{6} \geq 1$, then the right-hand side of (4) is at least 14 , as desired.
So, let us assume that $b_{6}=0$ and $b=b_{5}+b_{4}+b_{3}+b_{2}$. Furthermore,

$$
\begin{align*}
v\left(G^{\prime}\right) & =5 b_{5}+4 b_{4}+3 b_{3}+2 b_{2}-(b-1) \\
& =4 b_{5}+3 b_{4}+2 b_{3}+b_{2}+1 \tag{5}
\end{align*}
$$

So, substituting $2 b_{5}$ from (5) into (4), we have

$$
\begin{aligned}
5 v(G)-2 e(G) & \geq 2 b_{5}+3 b_{4}+4 b_{3}+3 b_{2}+5+\left(v(G)-v\left(G^{\prime}\right)\right) \\
& =\left(\frac{1}{2} v\left(G^{\prime}\right)-\frac{3}{2} b_{4}-b_{3}-\frac{1}{2} b_{2}-\frac{1}{2}\right)+3 b_{4}+4 b_{3}+3 b_{2}+5+\left(v(G)-v\left(G^{\prime}\right)\right) \\
& =v(G)-\frac{1}{2} v\left(G^{\prime}\right)+\frac{3}{2} b_{4}+3 b_{3}+\frac{5}{2} v_{2}+\frac{9}{2} \\
& \geq \frac{1}{2} v(G)+\frac{9}{2}
\end{aligned}
$$

which is strictly larger than 13 if $v(G) \geq 18$. Since $5 v(G)-2 e(G)$ is an integer, it is at least 14 and this completes the proof of Theorem 3.

Remark 2. Observe that for $n \geq 17$, the only graphs on $n$ vertices with $e$ edges such that $e>(5 / 2) n-7$ have blocks of order 5 or less and by (4), there are at most 4 such triangular blocks. A bit of analysis shows that the maximum number of edges is achieved when the number of blocks of order 5 is as large as possible.

## 6 Conclusions

We note that the proof of Theorem 2, particularly Lemma 14, can be rephrased in terms of a discharging argument.

We believe that our construction in Theorem 4 can be generalized to prove $\operatorname{ex}_{\mathcal{P}}\left(n, C_{\ell}\right)$ for $\ell$ sufficiently large. That is, for certain values of $n$, we try to construct $G_{0}$, a plane graph with all faces of length $\ell+1$ with all vertices having degree 3 or degree 2 .

If such a $G_{0}$ exists, then the number of degree-2 and degree-3 vertices are $\frac{(\ell-5) n+4(\ell+1)}{\ell-1}$ and $\frac{4(n-\ell-1)}{\ell-1}$, respectively. We could then apply steps similar to (1), (2), and (3) in the proof of Theorem 4 in that we add halving vertices and insert a graph $B_{\ell-1}$ (see Figure 17) in place of vertices of degree 2 and 3 . For the resulting graph $G$,

$$
\begin{aligned}
v(G) & =v\left(G_{0}\right)+e\left(G_{0}\right)+(\ell-4) \frac{(\ell-5) n+4(\ell+1)}{\ell-1}+(\ell-5) \frac{4(n-\ell-1)}{\ell-1} \\
& =n+\frac{\ell+1}{\ell-1}(n-2)+\frac{\left(\ell^{2}-5 \ell\right) n+2(\ell+1)}{\ell-1} \\
& =\frac{\ell^{2}-3 \ell}{\ell-1} n+\frac{2(\ell+1)}{\ell} \\
e(G) & =(3 \ell-9) v\left(G_{0}\right)=(3 \ell-9) n
\end{aligned}
$$

Therefore, $e(G)=\frac{3(\ell-1)}{\ell} v(G)-\frac{6(\ell+1)}{\ell}$. We conjecture that this is the maximum number of edges in a $C_{\ell}$-free planar graph.

Conjecture 15. Let $G$ be an n-vertex $C_{\ell}$-free plane graph $(\ell \geq 7)$, then there exists an integer $N_{0}>0$, such that when $n \geq N_{0}, e(G) \leq \frac{3(\ell-1)}{\ell} n-\frac{6(\ell+1)}{\ell}$.


Figure 17: $B_{\ell-1}$ is used in the construction of a $C_{\ell}$-free graph.

## 7 Acknowledgements

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## A Tables

The following tables give a summary of the results from Lemmas 11, 12, and 13 ,
A red edge incident to a vertex of a triangular-block indicates the corresponding vertex is a junction vertex. Moreover, if a vertex has only one red edge, it is to indicate the vertex is shared in at least two triangular-blocks. Whereas if a vertex has two red edges, it means that the vertex is shared in at least three blocks.

A pair of blue edges indicates the boundary of a 4 -face.

| Case | $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B)=$ | $7 f+2 n-5 e \leq$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\begin{gathered} \text { Lemma } 11 \\ 1(\mathrm{a}) \end{gathered}$ | $B_{5, a}$ |  | $5+\frac{3}{7}$ | $2+\frac{3}{2}$ | 9 | 0 |
| $\begin{gathered} \text { Lemma } 11 \\ 1(\mathrm{~b}) \end{gathered}$ | $B_{5, a}$ |  | $5+\frac{2}{7}$ | $3+\frac{2}{2}$ | 9 | 0 |
| $\begin{gathered} \text { Lemma } 11 \\ 2(\mathrm{~b}) \end{gathered}$ | $B_{5, b}$ |  | $4+\frac{4}{7}$ | $3+\frac{2}{2}$ | 8 | 0 |
| $\begin{gathered} \text { Lemma } 11 \\ 2(\mathrm{c}) \end{gathered}$ | $B_{5, c}$ |  | $3+\frac{5}{7}$ | $3+\frac{2}{2}$ | 7 | -1 |
| $\begin{gathered} \text { Lemma } 12 \\ \text { (a) } \end{gathered}$ | $B_{5, d}$ |  | $4+\frac{4}{7}$ | $3+\frac{2}{2}$ | 8 | 0 |
| $\begin{gathered} \text { Lemma } 12 \\ \text { (b) } \end{gathered}$ | $B_{5, d}$ |  | $4+\frac{4}{7}$ | $3+\frac{2}{2}$ | 8 | 0 |
| $\begin{gathered} \text { Lemma } 12 \\ (\mathrm{c}) \end{gathered}$ | $B_{5, d}$ |  | $4+\frac{2}{4}+\frac{2}{7}$ | $2+\frac{3}{2}$ | 8 | $\frac{1}{2} \star$ |

Table 2: All possible $B_{5}$ blocks in $G$ and the estimation of $7 f(B)+2 n(B)-5 e(B)$.


Table 3: All possible $B_{4}, B_{3}$ and $B_{2}$ blocks in $G$ and the estimate of $7 f(B)+2 n(B)-5 e(B)$.


[^0]:    ${ }^{1}$ In fact, it can be shown that the length of the exterior face containing the path $v_{2} v_{1} v_{4} v_{3}$ is at least 9 . This yields $f(B) \leq 4+1 / 7+3 / 9$ and $7 f(B)+2 n(B)-5 e(B) \leq-2 / 3$. However, this precision is unnecessary.

