# MINIMAL PAIRS OF CONVEX SETS WHICH SHARE A RECESSION CONE 

JERZY GRZYBOWSKI AND RYSZARD URBAŃSKI


#### Abstract

Robinson introduced a quotient space of pairs of convex sets which share their recession cone. In this paper minimal pairs of unbounded convex sets, i.e. minimal representations of elements of Robinson's spaces are investigated. The fact that a minimal pair having property of translation is reduced is proved. In the case of pairs of two-dimensional sets a formula for an equivalent minimal pair is given, a criterion of minimality of a pair of sets is presented and reducibility of all minimal pairs is proved. Shephard-Weil-Schneider's criterion for polytopal summand of a compact convex set is generalized to unbounded convex sets. An application of minimal pairs of unbounded convex sets to Hartman's minimal representation of dc-functions is shown. Examples of minimal pairs of three-dimensional sets are given.


## 1. Introduction

For a family $\mathcal{C}\left(\mathbb{R}^{n}\right)$ of all nonempty closed convex subsets of $\mathbb{R}^{n}$ the addition $A+B:=\{a+b \mid a \in A, b \in B\}$ is called a Minkowski or vector or algebraic sum of these sets. For $A, B \in \mathcal{C}\left(\mathbb{R}^{n}\right)$ the modified addition $A \dot{+} B:=\operatorname{cl}(A+B)$ turns the family $\mathcal{C}\left(\mathbb{R}^{n}\right)$ into a commutative semigroup with a neutral element $\{0\}$. Moreover, for all $A, B \in \mathcal{C}\left(\mathbb{R}^{n}\right)$ and all $s, t \geqslant 0$ we have $s(t A)=s(t A), t(A \dot{+} B)=t A \dot{+} t B,(s+t) A=$ $s A \dot{+} t A, 1 A=A$, and $0 A=\{0\}$. A relation $(A, B) \sim(C, D): \Longleftrightarrow$ $A \dot{+} D=B \dot{+} C$ is not transitive because in $\mathcal{C}\left(\mathbb{R}^{n}\right)$ a cancellation law $A \dot{+} B=B \dot{+} C \Longrightarrow A=C$ does not hold true. Therefore, the family $\mathcal{C}\left(\mathbb{R}^{n}\right)$ cannot be embedded into a vector space.

However, the family $\mathcal{B}\left(\mathbb{R}^{n}\right)$ of all nonempty closed bounded convex subsets of $\mathbb{R}^{n}$ can be embedded into a vector space, see Minkowski [24]. In a case of infinitely dimensional topological vector spaces a semigroup of nonempty closed bounded convex sets can be embedded into Minkowski-Rådström-Hörmander space, see Rådström [29], Hörmander [23], Drewnowski [11] and Urbański [36].

[^0]Quotient classes of pairs of convex sets are elements of Minkowski-Rådström-Hörmander spaces. Sets in a given class can be arbitrarily large. The best representation of such a class would be inclusionminimal pair. Inclusion-minimal pairs were studied by Bauer [5], Scholtes [26, 34], Pallaschke [15, 16, 27, 28] and by the authors [12, 13, 19, 20] in connection with quasidifferential calculus. Quasidifferential calculus was developed by Demyanov and Rubinov [8] and studied by many authors including Zhang, Xia, Gao and Wang [38] Basaeva, Kusraev and Kutateladze [4, Antczak [2, Abbasov [1], Dolgopolik [10] and others.

MRH spaces and basic facts about minimal pairs of convex sets are presented in Section 7. An embedding of a semigroup of convex sets is enabled by a cancellation law which was studied for its own sake by the authors [20] and recently generalized to cornets by Molnár and Páles [25].

Robinson [30] proved an order cancellation law

$$
\begin{equation*}
A+B \subset B+C \Longrightarrow A \subset C \tag{olc}
\end{equation*}
$$

for $A, B, C$ from a family of unbounded closed convex sets $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ sharing a common recession cone $V$. Here, $V$ is a closed convex cone in $\mathbb{R}^{n}$ and a recession cone is defined as recc $A:=\left\{x \in \mathbb{R}^{n} \mid x+A \subset A\right\}$. A family $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ with Minkowski addition is a semigroup by Corollary 9.1.1 in [31] and as such can be embedded into a vector space. In this family the closed convex cone $V$ is a neutral element, $A \dot{+} B=$ $A+B$, and multiplication by 0 has to be modified by $0 A:=V$ for $A, B \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. Since a cancellation law holds true, the relation " $\sim$ " is transitive. We put $[A, B]:=[(A, B)]_{\sim}$.
Theorem 1.1. (Robinson, [30]) The family of quotient classes $\widetilde{\mathbb{R}_{V}^{n}}:=$ $\mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right) / \sim$ with with the addition $[A, B]+[C, D]:=[A+C, B+D]$ and the multiplication $t[A, B]:=\left\{\begin{array}{ll}{[t A, t B],} & t \geqslant 0 \\ {[-t B,-t A],} & t<0\end{array}\right.$ is a smallest vector space into which the semigroup $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ can be embedded.
The embedding is defined by $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right) \ni A \longmapsto[A, V] \in \widetilde{\mathbb{R}_{V}^{n}}$. In the vector space $\widetilde{\mathbb{R}_{V}^{n}}$ the neutral element is $[V, V]$ and the opposite element to $[A, B]$ is $-[A, B]=[B, A]$.

If the cone $V$ is trivial, i.e. $V=\{0\}$ the family $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ coincides with a well studied family $\mathcal{B}\left(\mathbb{R}^{n}\right)$ of all nonempty compact convex sets, i.e. of convex bodies.

Robinson's theorem was generalized for closed convex sets in a Banach space by Bielawski and Tabor [6].

Balashov and Polovinkin in their interesting paper [3] extended to unbounded sets the notion of generating sets. In a similar manner this paper extends to unbounded sets the notion of minimal pairs of sets.

In Section 2 we present a definition and a theorem of existence of minimal pairs of sets from $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ and the property of translation of minimal pairs. We give properties of a kernel of minimality $B_{*}$ of a pair $(A, B)$, i.e. a set of all such points $x$ that a pair $(A-x, B-x)$ is minimal. We also give a number of examples.

In Section 3 we prove that a minimal pair of sets is reduced if and only if it has the property of translation.

Properties of minimal pairs of two-dimensional sets are studied in Section 4. We give a criterion for being a summand in Proposition 4.2, a formula for an equivalent minimal pair in Theorem 4.4, a criterion of minimal pair in Theorem 4.5 and prove the reducibility of all minimal pairs in Theorem 4.6.

We generalize Shephard-Weil-Schneider's criterion, i.e. Th. 3.2.11 in [33], to polytopal summands of unbounded convex sets in Theorem 5.2. We also extend Bauer's criterion [5] of reduced pairs of polytopes to $V$-polytopes in Theorem 5.5.

In Section 6 we present an application of minimal pairs of unbounded convex sets to a minimal, according to Hartman [22], representation of dc-functions.

We complete our paper with two appendices. In Section 7 we present selected facts from [16] on minimal pairs of bounded convex sets used in our proofs. In Section 8 we present Minkowski duality between convex sets and sublinear functions needed in the proof of Theorem 5.2.

## 2. Minimal pairs of unbounded convex sets

Let $V$ be a closed convex cone in $\mathbb{R}^{n}$ and $A, B \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. A quotient class $[A, B]$ is ordered in the following way

$$
\left(A_{1}, B_{1}\right) \prec\left(A_{2}, B_{2}\right) \Longleftrightarrow A_{1} \subset A_{2}, B_{1} \subset B_{2}
$$

If a recession cone $V$ is not trivial then a pair $(A+v, B+v), v \in V \backslash\{0\}$ is smaller than $(A, B)$ hence no pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ is minimal. Therefore, we say that a pair $(A, B)$ is 0 -minimal if $(A, B)$ is a minimal element in a subset $\{(C, D) \in[A, B] \mid 0 \in D\}$ of a quotient class $[A, B]$.

The definition of 0 -minimality seems very natural. In the case of a semigroup $\mathcal{B}\left(\mathbb{R}^{n}\right)=\mathcal{C}_{\{0\}}\left(\mathbb{R}^{n}\right)$ of bounded closed convex sets the existence of a minimal pair is guaranteed by the fact that a chain of
compact sets has a nonempty intersection. In the case of $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ an intersection of a chain of sets containing 0 contains the cone $V$.

Every quotient class $[A, B] \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right) / \sim$ contains a 0 -minimal pair. The following theorem was proved by Grzybowski and Przybycien [18] in much more general, possibly infinite dimensional, case.

Theorem 2.1 (existence of a 0-minimal pair). For every pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ with $0 \in B$ there exists an equivalent 0 -minimal pair $\left(A^{\prime}, B^{\prime}\right)$ such that $A^{\prime} \subset A, B^{\prime} \subset B$.

Unlike in the case of minimal pairs of compact convex sets a pair $(A, B) \in \mathfrak{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ may be 0 -minimal and a translated pair $(A-x, B-x)$ may not. We call a set $B_{*}:=\{x \in B \mid(A-x, B-x)$ is 0 -minimal $\}$ a kernel of minimality of the pair $(A, B)$. Obviously, $B_{*} \subset B$.

By $L_{V}=V \cap(-V)$ we denote the subspace of lineality of the cone $V$. Let us notice that for a pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ we have the following equality

$$
\begin{equation*}
\{b \in B \mid(A-b, B-b) \prec(A, B)\}=B \cap(-V) . \tag{*}
\end{equation*}
$$

The following proposition holds true.
Proposition 2.2. Let $(A, B) \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. If $x \in B_{*}$ then $B \cap(x-V)=$ $x+L_{V}$.

Proof. Let $b \in B \cap(-V)$, then from $(*)$ we have $(A-b, B-b) \prec(A, B)$. Assume that $0 \in B_{*}$. Then the pair $(A, B)$ is 0 -minimal and we get $B-b=B$. Hence $b \in L_{V}$ and $L_{V} \subset B \cap(-V) \subset L_{V}$. If $x \in B_{*}$ then $(A-x, B-x)$ is 0 -minimal and $(B-x) \cap(-V)=L_{V}$.

Proposition 2.2 says that the kernel of minimality is contained in the subset of minimal elements of $B$ with respect to the preorder $\leqslant_{V}$ generated by the cone $V$. Notice also that if the cone $V$ is nontrivial then the set $B_{*}$ is contained in the boundary of $B$.

Lemma $2.3\left(B_{*}\right.$ is an extreme subset of $\left.B\right)$. Let $(A, B) \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. If $x, y \in B$ and $(x+y) / 2 \in B_{*}$ then $x, y \in B_{*}$.

Proof. Denote $z=(x+y) / 2$. By Theorem 2.1 there exists a 0 -minimal pair $\left(A^{\prime}-x, B^{\prime}-x\right) \prec(A-x, B-x)$. Hence $z=(x+y) / 2 \in\left(B^{\prime}+B\right) / 2$ and the pair

$$
\left(\frac{A^{\prime}}{2}+\frac{A}{2}-z, \frac{B^{\prime}}{2}+\frac{B}{2}-z\right) \prec(A-z, B-z) .
$$

Since the pair $(A-z, B-z)$ is 0-minimal, we obtain $B^{\prime} / 2+B / 2=B$. By the cancellation law (olc) we get $B^{\prime} / 2=B / 2$, and $B^{\prime}=B$. Then the pair $(A-x, B-x)$ is 0 -minimal, and $x \in B_{*}$.

Corollary 2.4. Let $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$. If $E \subset B$ is a convex extreme subset of $B$ and the relative interior of $E$ intersects with $B_{*}$ then $E \subset$ $B_{*}$.

Let $A \in \mathcal{C}\left(\mathbb{R}^{n}\right), u, u_{1}, \ldots, u_{k} \in \mathbb{R}^{n}$. Let $A(u)$ be a support set defined by $A(u)=\left\{a \in A \mid\langle a, u\rangle=\max _{x \in A}\langle x, u\rangle\right\}$. Let $A\left(u_{1}, \ldots, u_{k}\right)=$ $A\left(u_{1}, \ldots, u_{k-1}\right)\left(u_{k}\right)$ be an iterated support set. Notice that any subset of $A$ is a convex extreme subset of $A$ if and only if it is an iterated support set of $A$. In particular a singleton consisting of an extreme point of $A$ is an extreme subset of $A$.

If $A, B, C, D \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ and $\bar{u}=\left(u_{1}, \ldots, u_{k}\right) \in\left(\mathbb{R}^{n}\right)^{k}$ and $V\left(u_{1}\right)=L_{V}$ then the following significant facts hold true

$$
(A+B)(\bar{u})=A(\bar{u})+B(\bar{u})
$$

and

$$
(A, B) \sim(C, D) \Longrightarrow(A(\bar{u}), B(\bar{u})) \sim(C(\bar{u}), D(\bar{u}))
$$

The following proposition shows that kernels of minimality of pairs $(A, B)$ and $(B, A)$ "lie on the same side", respectively, of sets $B$ and $A$.

Proposition 2.5. Let $(A, B) \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right), \bar{u}=\left(u_{1}, \ldots, u_{k}\right) \in\left(\mathbb{R}^{n}\right)^{k}$ and $V\left(u_{1}\right)=L_{V}$. If $B(\bar{u}) \subset B_{*}$ then $A(\bar{u}) \subset A_{*}$ where $A_{*}=\{x \in$ $A \mid(B-x, A-x)$ is 0 -minimal $\}$.

Proof. Let $y \in A(\bar{u})$. Then by Theorem 2.1 there exists a 0 -minimal pair $\left(B^{\prime}-y, A^{\prime}-y\right) \prec(B-y, A-y)$. Since $y \in A^{\prime} \subset A$ and $y \in A(\bar{u})$, we obtain $A^{\prime}(\bar{u}) \subset A(\bar{u})$. Since $\left(B^{\prime}, A^{\prime}\right) \sim(B, A)$, we get $B^{\prime}+A=A^{\prime}+B$, and $B^{\prime}(\bar{u})+A(\bar{u})=A^{\prime}(\bar{u})+B(\bar{u}) \subset A(\bar{u})+B(\bar{u})$. Hence by the order law of cancellation $B^{\prime}(\bar{u}) \subset B(\bar{u})$. Consider any $x \in B^{\prime}(\bar{u})$. Since $B(\bar{u}) \subset B_{*}$, the pair $(A-x, B-x)$ is 0 -minimal. Moreover, $x \in B^{\prime}$ and $\left(A^{\prime}-x, B^{\prime}-x\right) \prec(A-x, B-x)$. Then $B^{\prime}=B$, and we have just proved that $y \in A_{*}$

A pair $(A, B)$ or a class $[A, B]$ is said to have a property of translation of 0 -minimal pairs if all equivalent 0 -minimal pairs in $[A, B]$ are connected by translation. This property of translation is distinct from a property of translation of minimal pairs of bounded sets. If the cone $V$ is not trivial we write just 'property of translation' because there is no possibility of misunderstanding.

For $(A, B) \in \mathcal{C}_{\{0\}}^{2}\left(\mathbb{R}^{n}\right)$ a property of translation of 0 -minimal pairs follows from a property of translation of minimal pairs but not the other way around. All pairs of flat compact convex sets from $\mathcal{C}_{\{0\}}^{2}\left(\mathbb{R}^{2}\right)$ satisfy the property of translation of minimal pairs [5, 12, 34], but

Example 2.10(i) presents a number of equivalent 0-minimal pairs not connected by translation.

## Proposition 2.6 (characterization of a kernel of 0-minimal pair).

 Let a 0 -minimal pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ have the property of translation. Then the following assertions hold:(a) The set $\left\{(A-x, B-x) \mid x \in B_{*}\right\}$ is a set of all 0-minimal pairs of the class $[A, B]$.
(b) $x \in B_{*}$ if and only if $B \cap(x-V)=x+L_{V}$.
(c) $B=B_{*}+V$.

Proof. (a) Let $(C, D) \in[A, B]$ be a 0 -minimal pair, then by a property of translation $D=B-z$ for some $z \in \mathbb{R}^{n}$. Since $0 \in D$ we get $z=x$ for a some $x \in B$.
(b) Let $(B-x) \cap(-V)=L_{V}$, we have $0 \in B-x$. By (a) there exist a 0 -minimal pair $(A-z, B-z)$ such that $B-z \subset B-x$. Hence $(B-x)-(z-x)=B-z \subset B-x$. Now, by $(*)$ applied to $(A-x, B-x)$ we get $z-x \in(B-x) \cap(-V)=L_{V}$. Hence $B-z=B-x+V-(z-x) \supset$ $B-x$ and we get $B-z=B-x$.
(c) By (a) for any $b \in B$ there exists $x \in B_{*}$ such that $B-x \subset B-b$. Then $B+b-x \subset B$, and $b-x \in V$. Therefore, $b=x+(b-x) \subset B_{*}+V$, and we get $B_{*}+V \subset B \subset B_{*}+V$.

Remark 2.7. Let us notice that in case of a 0 -minimal pair $(A, B)$ not having the property of translation the equality $B=B_{*}+V$ may hold true, see the pair ( $\widehat{A}_{0}, \widehat{B}_{0}$ ) in Example 2.10(ii), or not, see the pair ( $\widehat{A}_{1}, \widehat{B}_{1}$ ) in Example 2.10(ii).

Obviously, any pair $(A, V)$ has property of translation. Moreover, it is a unique 0 -minimal pair in a quotient class $[A, V]$. The following example gives all 0 -minimal pairs in a quotient class $[V, B]$.

Example 2.8. Let $n=2, A=V=\{0\} \times \mathbb{R}_{+}$be a ray and $B=$ $\left\{\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2} \mid x_{2} \geqslant x_{1}^{2}\right\}$ be an epigraph of a quadratic function. A pair ( $A, B$ ) is obviously 0 -minimal. By Proposition 2.6 a pair $(A-x, B-x)$ is 0 -minimal if and only if $B \cap\left(\left(x_{1}, x_{2}\right)-V\right)=\left\{\left(x_{1}, x_{2}\right)\right\}$, where $L_{V}=$ $\{(0,0)\}$. This equality holds true exactly when $x_{2}=x_{1}^{2}, x_{1} \in \mathbb{R}$. The set $B_{*}$ is equal to the boundary of the set $B$. Notice that $A_{*}=\{(0,0)\}$.

In $\mathbb{R}^{3}$ there exist equivalent minimal pairs not connected by translation. The following example was given as Example 4.1 in [18]. In that example a pair $(C, D)$ - not showed here - was incorrectly presented as 0 -minimal.

Example 2.9. Let $V=\left\{x \in \mathbb{R}^{3} \mid x_{1}=x_{2}=0, x_{3} \leqslant 0\right\}$, $B=\operatorname{conv}\{(-1,-1,0),(-1,1,-1),(1,1,0),(1,-1,-1)\}+V$, $A=\operatorname{conv}(B \cup\{(-2,0,-1),(2,0,-1)\})+V$,
$F=\operatorname{conv}\{(0,-1,-1),(0,0,0),(0,1,-1)\}+V$ and
$E=\operatorname{conv}(F \cup\{(-1,-1,-2),(-1,1,-1),(1,1,-2),(1,-1,-1)\})+V$.
In Figure 2.1 we can see upper faces of sets $A, B, E, F \in \mathcal{C}_{V}\left(\mathbb{R}^{3}\right)$, where $V=\left\{x \in \mathbb{R}^{3} \mid x_{1}=x_{2}=0, x_{3} \leqslant 0\right\}$, large dots represent the origin, and numbers denote the third coordinate of vertices. It can be checked that $A+F=B+E$ and that both pairs $(A, B)$ and $(E, F)$ are 0 -minimal.



Figure 2.1. Two equivalent minimal pairs of unbounded convex sets not connected by translation from Example 2.9.

Let us notice that if a given pair $(A, B)$ does not have a property of translation and $(B-x) \cap(-V)=L_{V}$ then a pair $(A-x, B-x)$ may or may not be 0 -minimal. The following example shows such possibility.

Example 2.10 (i). Let $V=\{(0,0)\} \subset \mathbb{R}^{2}, A, B \in \mathcal{C}_{V}\left(\mathbb{R}^{2}\right), B=\mathrm{conv}$ $\{(0,0),(2,0)\}$ and $A=\operatorname{conv}(B \cup\{(1,1)\})$. Let $p_{0} \in B, p_{1} \in\{x \in$ $\left.\mathbb{R}^{2} \mid x_{2} \leqslant \min \left(1-\left|x_{1}-1\right|, 0\right)\right\}, p_{2} \in\left\{x \in \mathbb{R}^{2}\left|1-\left|x_{1}-1\right|<x_{2}<0\right\}\right.$. Denote $B_{i}=\operatorname{conv}\left\{\left(\left(B-p_{i}\right) \cup\{(0,0)\}\right), i=0,1,2, A_{0}=A-p_{0}, A_{1}=\right.$ conv $\left\{\left(\left(A-p_{1}\right) \cup\{(0,0)\}\right), A_{2}=\right.$ conv $\left\{\left(\left(A-p_{2}\right) \cup\{(0,0),(1,1)\}\right)\right.$ if $\left(p_{2}\right)_{1}<0$ and $A_{2}=\mathrm{conv}\left\{\left(\left(A-p_{2}\right) \cup\{(0,0),(-1,1)\}\right)\right.$ if $\left(p_{2}\right)_{1}>2$. The sets $A_{i}, B_{i}, i=0,1,2$ are represented in Figure 2.2.


Figure 2.2. Pairs of sets described in Example 2.10(i).
All pairs of sets $\left(A_{i}, B_{i}\right), i=0,1,2$ are equivalent to $(A, B)$. We are going to prove that each pair $\left(A_{i}, B_{i}\right)$ is 0 -minimal. Assume that $(C, D) \prec\left(A_{i}, B_{i}\right)$ and $0 \in D \subset B_{i}$. By Theorem 7.4 a pair of polygons is minimal if and only if they have at most one pair of parallel edges that lie on the sam side of polygons. Then the pair of a triangle and a segment is minimal. Since the segment $B$ contains 0 , the pairs $(A, B),\left(A_{0}, B_{0}\right)$ are minimal and 0 -minimal. By Theorem 7.1, i.e. existence of equivalent minimal pair contained in a given pair, and by Theorem 7.2, i.e. uniqueness-up-to-translation of equivalent minimal pairs of flat sets, the set $D$ contains a translate of $B$, namely $B-p_{i}$.

Then obviously $D=B_{i}$. Hence $A_{i}+B_{i}=A_{i}+D=B_{i}+C$, and by the law of cancellation $C=A_{i}$. Therefore, the pairs $\left(A_{i}, B_{i}\right), i=0,1,2$ are 0 -minimal.

It can be proved that there are no other 0-minimal pairs in the quotient class $[A, B]$. Notice that $\left(B_{0}\right)_{*}=B_{0},\left(B_{1}\right)_{*}=\left(B_{2}\right)_{*}=\{(0,0)\}$. By Proposition 2.5 we obtain $\left(A_{0}\right)_{*}=A_{0},\left(A_{1}\right)_{*}=\{(0,0)\}$ and $\left(A_{2}\right)_{*}=$ conv $\{(0,0),(1,1)\}$ if $\left(p_{2}\right)_{1}<0$ or $\left(A_{2}\right)_{*}=\operatorname{conv}\{(0,0),(-1,1)\}$ if $\left(p_{2}\right)_{1}>2$.
(ii) Let $V \subset \mathbb{R}^{3}$ be a cone such that $\left\{x \in V \mid x_{3} \geqslant 0\right\}=\{(0,0,0)\}$. Denote $\widehat{A}=(A \times\{0\})+V$ and $\widehat{B}=(B \times\{0\})+V$. It can be proved that all 0 -minimal pairs in $[\widehat{A}, \widehat{B}]$ are $\left(\widehat{A}_{i}, \widehat{B}_{i}\right), i=0,1,2$ where $\widehat{A}_{i}=\left(A_{i} \times\{0\}\right)+V$ and $\widehat{B}_{i}=\left(B_{i} \times\{0\}\right)+V$ and $A_{i}, B_{i}$ are sets from (i). Notice that $\left(\widehat{B}_{0}\right)_{*}=B_{0} \times\{0\}$ and $\widehat{B}_{0}=\left(\widehat{B}_{0}\right)_{*}+V$, but $\left(\widehat{B}_{1}\right)_{*}=\{(0,0,0)\}$ and $\widehat{B}_{1} \neq V=\left(\widehat{B}_{0}\right)_{*}+V$.

## 3. Reduced pairs of unbounded convex sets

Let us extend a notion of reduced pair of bounded sets from $\mathcal{B}^{2}\left(\mathbb{R}^{n}\right)$ introduced by Bauer [5]. A pair $(A, B) \in \mathfrak{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ is reduced if $[A, B]=$ $\left\{(A+M, B+M) \mid M \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)\right\}$.

In this section we show a relationship between reduced pairs and the property of translation of 0-minimal pairs.

Proposition 3.1. Let $V$ be a closed convex cone. If a pair $(A, B) \in$ $\mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ is reduced then it has the property of translation.

Proof. Let a pair $(C, D) \in[A, B]$ be 0-minimal. Then $(C, D)=(A+$ $M, B+M)$ for some $M \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. Since $0 \in D=B+M$, there exists $b \in B$ such that $-b \in M$. Then $A-b \subset A+M=C, B-b \subset B+M=D$. Since ( $C, D$ ) is 0-minimal, we obtain $C=A-b, D=B-b$.

Proposition 3.2. Let $V \subset \mathbb{R}^{n}$ be a closed convex cone. If a pair $(A, B) \in \mathfrak{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ has the property of translation, then every 0 -minimal pair $(C, D) \in[A, B]$ is reduced.

Proof. Let a pair $(C, D) \in[A, B]$ be 0 -minimal. Let $b \in B$. By Proposition 2.6(a) there exists 0-minimal pair $(C-x, D-x), x \in D_{*}$ such that $C-x \subset A-b, D-x \subset B-b$. We obtain $D+b-x \subset B$, and $b-x \in B \dot{-} D:=\{y \mid D+y \subset B\}$. Then $b=x+(b-x) \in D+(B \dot{-} D) \subset$ $B$, and $B \subset D+(B \dot{-} D) \subset B$. Hence $B=D+(B \dot{-} D)$. Then $A+D=D+(B \dot{-} D)+C$, and by the cancellation law $A=C+(B \dot{-} D)$.

Propositions 3.1 and 3.2 can be summed up in the following theorem.

Theorem 3.3 (equivalence of reducibility and property of translation). Let $V$ be a closed convex cone. A pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ is reduced if and only if this pair has the property of translation and is a translate of some 0-minimal pair.

Corollary 3.4. Let $V \subset \mathbb{R}^{n}$ be a closed convex cone. If a pair $(A, B) \in$ $\mathcal{C}_{V}^{2}\left(\mathbb{R}^{n}\right)$ has the property of translation, then $(B, A)$ has the property of translation.

Proof. If $(A, B)$ has the property of translation then by Proposition 3.2 some equivalent pair $(C, D)$ is reduced, i.e. $[A, B]=\{(C+M, D+$ $\left.M) \mid M \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)\right\}$. Hence the pair $(D, C)$ is reduced and by Proposition 3.1 the class $[B, A]$ has the property of translation.

Theorem 3.3 shows a difference between the property of translation of minimal pairs and the property of translation of 0-minimal pairs. There exists a broad class of minimal pairs of compact convex sets that satisfy the property of translation not being reduced pairs. For example all pairs of convex polygons $(A, B)$ with exactly one pair $(A(u), B(u))$ of parallel edges (see Theorem 7.4). The authors can prove that if in a pair $(A, B) \in \mathcal{B}^{2}\left(\mathbb{R}^{3}\right)$ a set $A$ is a tetrahedron and for all triangular faces $A(u)$ the pairs $(A(u), B(u))$ are minimal then $(A, B)$ is minimal and has the property of translation. Such pair $(A, B)$ may be a pair of convex polyhedra and possess one or more pairs $(A(v), B(v))$ of parallel edges. By Theorem 5.5, i.e. Bauer's criterion for reduced polytopes the pair $(A, B)$ is not reduced. Sufficient and necessary condition for having the property of translation in $\mathbb{R}^{3}$ is not known. On the other hand every 0 -minimal pair having the property of translation of 0 -minimal pairs is inevitably reduced.

## 4. Minimal pairs of unbounded planar convex sets

In order to prove propositions and theorems of this section we need to present the notion and properties of an arc-length function $f_{A}$ corresponding to a planar set $A$. Let us consider a nonempty unbounded closed convex set $A \subset \mathbb{R}^{2}$. Let a recession cone $V$ of $A$ be pointed and unbounded . Obviously, $V$ is a planar convex angle of a measure $\pi-2 \vartheta$ with $\vartheta \in(0, \pi / 2]$. Assume that the negative part of the $x$-axis bisects the angle $V$. We construct an arc-length function $f_{A}$ following the approach from [18].

Let $u \in \mathbb{R}^{2}$ and a support set of $A$ in the direction of $u$ be a set $A(u):=\left\{a \in A \mid\langle u, a\rangle=\max _{b \in A}\langle u, b\rangle\right\}$. Obviously, the support set $A(u)$ is a singleton, a segment, a ray or an empty set. Let $H_{A}$ : $(-\vartheta, \vartheta) \longrightarrow \mathrm{bd} A$ be a boundary function, where $H_{A}(t)$ is the center of the set $A(\cos t, \sin t)$, which is either a segment or a singleton. We also denote by $f_{A}:(-\vartheta, \vartheta) \longrightarrow \mathbb{R}$ an arc length function of $A$, with a value
$f_{A}(t), t \geqslant 0$ equal to the length of the arc contained in the boundary $\mathrm{bd} A$ joining points $H_{A}(0)$ and $H_{A}(t)$. If $t<0$ let a value $f_{A}(t)$ be opposite to the length of the arc joining $H_{A}(0)$ and $H_{A}(t)$. The function $f_{A}$ is non-decreasing, $f_{A}(0)=0$ and $f_{A}(t)=\frac{1}{2}\left(f_{A}\left(t^{+}\right)+f_{A}\left(t^{-}\right)\right)$where $f\left(t^{+}\right)=\lim _{s \rightarrow t^{+}} f(s), f\left(t^{-}\right)=\lim _{s \rightarrow t^{-}} f(s)$, for $t \in(-\vartheta, \vartheta)$.

On the other hand, let $f$ be any non-decreasing real function defined on an open interval $(-\vartheta, \vartheta)$, such that

$$
\begin{equation*}
f(0)=0 \text { and } f(t)=\frac{1}{2}\left(f\left(t^{+}\right)+f\left(t^{-}\right)\right), t \in(-\vartheta, \vartheta) . \tag{**}
\end{equation*}
$$

We define the function $H_{f}:(-\vartheta, \vartheta) \longrightarrow \mathbb{R}^{2}$ with the help of Stieltjes integral

$$
H_{f}(t):=\left\{\begin{array}{c}
\int_{0}^{t}(-\sin s, \cos s) d f(s), \quad t \geqslant 0 \\
-\int_{t}^{0}(-\sin s, \cos s) d f(s), \quad t<0
\end{array}\right.
$$

We denote $A_{f}:=\operatorname{cl} \operatorname{conv}\left(\operatorname{im} H_{f}\right)+V$. Then we have $A_{f_{A}}=A-H_{A}(0)$ and $f_{A_{f}}=f$. The following proposition summarizes properties of the correspondence between non-decreasing functions and convex sets.

Proposition 4.1. Let $A, B \in \mathcal{C}_{V}\left(\mathbb{R}^{2}\right)$ and $f, g$ be non-decreasing functions satisfying $(* *)$. The following formulas hold true:
$f_{A+B}=f_{A}+f_{B}, f_{t A}=t f_{A}$ for $t \geqslant 0$,
$A_{f+g}=A_{f}+A_{g}, A_{t f}=t A_{f}$ for $t \geqslant 0$,
$f_{A_{g}}=g, A_{f_{B}}=B-H_{B}(0)=B-\operatorname{midpoint} B(u), u=(1,0)$,
$f_{V} \equiv 0, A_{f}=V$ for $f \equiv 0$.
Proposition 4.2 (criterion of planar summands). $A$ set $A \in$ $\mathcal{C}_{V}\left(\mathbb{R}^{2}\right)$ is a summand of $B \in \mathcal{C}_{V}\left(\mathbb{R}^{2}\right)$ if and only if a function $f_{B}-f_{A}$ is non-decreasing.

Proof. If $B=A+C$ then $f_{B}-f_{A}=f_{C}$. On the other hand if a function $g=f_{B}-f_{A}$ is non-decreasing then $B-H_{B}(0)=A-H_{A}(0)+A_{g}$.
The following proposition is needed in the proof of Theorem 5.2, i.e. a criterion for polytopal summands.

Proposition 4.3 (criterion of a polygonal summand). Let $P$ be a convex polygon, $K \in \mathcal{C}\left(\mathbb{R}^{2}\right)$. Assume that the recession cone of $K$ is not a straight line. Then $P$ is a summand of $K$ if and only if for all $u \in S^{1}$ the support set $K(u)$ is empty or contains a translate of $P(u)$.
Proof. $\Longrightarrow$ ) If $K=P+L$ for some closed convex set $L$, and a face $K(u)$ is nonempty then $K(u)=P(u)+L(u)$, and $K(u)$ contains a translate of $P(u)$.
$\Longleftarrow)$ If the cone $V=\operatorname{recc} K=K-K$ is a plane or a half-plane than the theorem obviously holds true. Otherwise $V$ is an angle of a measure $\pi-2 \vartheta, 0<\vartheta \leqslant \pi / 2$. We may assume that the $x$-axis is bisecting the cone $V$ and that the negative part of the $x$-axis is contained in $V$.

It is enough to show that $P+V$ is a summand of $K$. Arc-length function $f_{P+V}$ is locally constant and noncontinuous only at $t \in(-\vartheta, \vartheta)$ such that a support set $(P+V)(u), u=(\cos t, \sin t)$ is a side of $P$. Since every segment $(P+V)(u)$, having length equal to $f_{P+V}^{+}(t)-f_{P+V}^{-}(t)$, is contained in some translate of a segment $K(u)$, of the lengh equal to $f_{K}^{+}(t)-f_{K}^{-}(t)$, the difference of arc-length functions $g=f_{K}-f_{P+V}$ is non-decreasing. By Proposition 4.2 the set $P+V$ is a summand of $K$.

Let us define the ordering of non-decreasing functions taking value 0 at 0 . For two functions $f, g$ we say that $f$ precedes $g$ if and only if $g-f$ is nondecreasing. Next two theorems on 0-minimal pairs in a plane correspond to Theorem 3.1 and Corollary 3.2 from [18].

Theorem 4.4 (formula for an equivalent 0 -minimal pair). Let $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{2}\right)$. Denote $g_{A}:=f_{A}-\inf \left(f_{A}, f_{B}\right)$ and $g_{B}:=f_{B}-$ $\inf \left(f_{A}, f_{B}\right)$. Then the pair $\left(A_{g_{A}}+H_{A}(0)-H_{B}(0), A_{g_{B}}\right)$ is 0-minimal and belongs to $[A, B]$.

Theorem 4.5 (criterion of 0-minimality). Let $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{2}\right)$. The pair $(A, B)$ is minimal if and only if $\inf \left(f_{A}, f_{B}\right) \equiv 0$ and $0 \in$ $B($ cost, sint $)$ for some $t \in(-\theta, \theta)$.

Theorem 4.6 (0-minimal pair is reduced). Let $V$ be a pointed unbounded convex cone in $\mathbb{R}^{2}$. Then every 0 -minimal pair $(A, B) \in$ $\mathcal{Q}_{V}^{2}\left(\mathbb{R}^{2}\right)$ is reduced.

Proof. Let $(C, D) \in[A, B]$. Then $A+D=B+C, f_{A}+f_{D}=f_{B}+f_{C}$ and $H_{A}(0)+H_{D}(0)=H_{B}(0)+H_{C}(0)$. We have $f_{C}+\inf \left(f_{A}, f_{B}\right) \prec$ $\inf \left(f_{C}+f_{A}, f_{C}+f_{B}\right)$ and $\inf \left(f_{C}+f_{A}, f_{C}+f_{B}\right)-f_{C} \prec \inf \left(f_{A}, f_{B}\right)$. Then $f_{C}+\inf \left(f_{A}, f_{B}\right)=\inf \left(f_{C}+f_{A}, f_{C}+f_{B}\right)=\inf \left(f_{C}+f_{A}, f_{D}+f_{A}\right)=$ $f_{A}+\inf \left(f_{C}, f_{D}\right)$. Hence $g_{C}:=f_{C}-\inf \left(f_{C}, f_{D}\right)=f_{A}-\inf \left(f_{A}, f_{B}\right)=f_{A}$. In a similar way $g_{D}:=f_{D}-\inf \left(f_{C}, f_{D}\right)=f_{B}-\inf \left(f_{A}, f_{B}\right)=f_{B}$. Thus $(C, D)=\left(A_{f_{C}}+H_{C}(0), A_{f_{D}}+H_{D}(0)\right)=\left(A_{g_{C}}+A_{\text {inf }\left(f_{C}, f_{D}\right)}+H_{C}(0), A_{g_{D}}+\right.$ $\left.A_{\inf \left(f_{C}, f_{D}\right)}+H_{D}(0)\right)=\left(A_{f_{A}}+A_{\inf \left(f_{C}, f_{D}\right)}+H_{C}(0), A_{f_{B}}+A_{\inf \left(f_{C}, f_{D}\right)}+\right.$ $\left.H_{D}(0)\right)=\left(A-H_{A}(0)+A_{\inf \left(f_{C}, f_{D}\right)}+H_{C}(0), B-H_{B}(0)+A_{\inf \left(f_{C}, f_{D}\right)}+\right.$ $\left.H_{D}(0)\right)=\left(A+A_{\inf \left(f_{C}, f_{D}\right)}+H_{D}(0)-H_{B}(0), B+A_{\inf \left(f_{C}, f_{D}\right)}+H_{D}(0)-\right.$ $\left.H_{B}(0)\right)$.

## 5. Criterion for polytopal summands

In this section we generalize Shephard-Weil-Schneider criterion for a polytope being a summand of compact convex subset of $\mathbb{R}^{n}$. The following Theorem 5.1 (Theorem 3.2.11. in [33) was proved by Shephard [35] in the case of a polytope $K$ and by Weil [37] in the case of compact convex $K$. A strengthening of the theorem appeared in Grzybowski, Urbański and Wiernowolski [21].

Theorem. 5.1 (Shephard-Weil-Schneider criterion). Let $P, K \in$ $\mathcal{B}\left(\mathbb{R}^{n}\right), n \geqslant 2$, $P$ be a polytope. Then $P$ is a summand of $K$ if and only if the support set $K(u)$ contains a translate of $P(u)$, whenever $P(u)$ is an edge of $P, u \in S^{n-1}$.

The next theorem, a generalization of Theorem 5.1 to an unbounded convex set $K$, is based on Schneider's proof from [32] presented in Encyclopedia of Mathematics and its Applications 151 [33].

Theorem 5.2 (criterion for a polytopal summand). Let $K \in$ $\mathcal{C}\left(\mathbb{R}^{n}\right), n \geqslant 2$, a recession cone $V$ of $K$ be pointed and $P \subset \mathbb{R}^{n}$ be a polytope. Then $P$ is a summand of $K$ if and only if every nonempty bounded support set $K(u)$ contains a translate of $P(u)$, whenever $P(u)$ is an edge of $P, u \in S^{n-1}$.

Proof. $\Longrightarrow)$ If a polytope $P$ is a summand of $K$ then there exists a set $A \in \mathcal{C}\left(\mathbb{R}^{n}\right)$ such that $K=P+A$. If a support set $K(u)$ is nonempty then it is a Minkowski sum of respective support sets $K(u)=$ $P(u)+A(u)$. Hence $K(u)$ contains a translate of $P(u)$, whether $P(u)$ is an edge or not.
$\Longleftarrow)$ We are going to apply Minkowski duality between convex sets and sublinear functions. Basic facts on Minkowski duality are presented in Section 8. Since the cone $V:=\operatorname{recc} K$ is pointed, the effective domain dom $h_{K}$ has a nonempty interior. If a difference of support functions $g:=h_{K}-h_{P}$ is convex in the interior of $\operatorname{dom} h_{K}$ then a function $g=h_{K}-h_{P}$ is sublinear and lower semicontinuous. Then $K=P+\left.\partial g\right|_{0}$, and $P$ is a summand of $K$. Hence we need to prove that the function $g$ is convex over int dom $h_{K}$. Notice that $\operatorname{int} V^{\circ} \subset \operatorname{dom} h_{K} \subset V^{\circ}$, where $V^{\circ}$ is a polar of the cone $V$.

Let $x, y \in \operatorname{int} \operatorname{dom} h_{K}$. If 0 lies between $x$ and $y$ then $0 \in \operatorname{int} \operatorname{dom} h_{K}$ and $\operatorname{dom} h_{K}=\mathbb{R}^{n}$. Hence $V=\{0\}$. This is true only if $K$ is bounded. In this case the polytope $P$ is a summand of $K$ by Theorem 5.1.

Otherwise, $\operatorname{lin}\{x, y\}$ is a two-dimensional subspace of $\mathbb{R}^{n}$. Let pr: $\mathbb{R}^{n} \longrightarrow \operatorname{lin}\{x, y\}$ be a perpendicular projection. Images $\operatorname{pr} K$ and $\operatorname{pr} P$ of $K$ and $P$ by projection pr are two-dimensional convex sets. For any $z \in \operatorname{lin}\{x, y\}$ equalities $h_{K}(z)=h_{\mathrm{pr} K}(z)$ and $h_{P}(z)=h_{\mathrm{pr} P}(z)$ hold true for respective support functions. Assume that every side of the convex polygon $\operatorname{pr} P$, that is $(\operatorname{pr} P)(u), u \in \operatorname{lin}\{x, y\}$ is equal to an image
$\operatorname{pr}(P(u))$ of a single edge $P(u)$ of the polytope $P$. It simply means that the support set $P(u)$ is an edge of $P$. Then if a set $(\operatorname{pr} K)(u)=$ $\operatorname{pr}(K(u)), u \in \operatorname{lin}\{x, y\}$ is nonempty then $(\operatorname{pr} K)(u)$ contains a translate of $\operatorname{pr}(P(u))=(\operatorname{pr} P)(u)$ since $K(u)$ contains a translate of $P(u)$. Hence by Proposition 4.3 the set $\operatorname{pr} P$ is a summand of $\operatorname{pr} K$. Thus $h_{\mathrm{pr} K}-h_{\mathrm{pr} P}$ is a convex function, and the function $g=h_{K}-h_{P}$ restricted to $\operatorname{lin}\{x, y\}$ is also convex. Therefore, $g\left(\frac{x+y}{2}\right) \leqslant \frac{g(x)+g(y)}{2}$.
If not every side of $\mathrm{pr} P$ is equal to a projection of a single edge of $P$ then still there exists a sequence $\left(y_{n}\right)$ tending to $y$ such that any side of polygon $\operatorname{pr}_{n} P$, where $\operatorname{pr}_{n}$ is a perpendicular projection onto the subspace $\operatorname{lin}\left\{x, y_{n}\right\}$, is equal to a projection of single edge $P(u)$ of $P$.

Since the function $g=h_{K}-h_{P}$ is continuous in the interior of $\operatorname{dom} h_{K}$, we obtain

$$
g\left(\frac{x+y}{2}\right)=\lim _{n \longrightarrow \infty} g\left(\frac{x+y_{n}}{2}\right) \leqslant \lim _{n \longrightarrow \infty} \frac{g(x)+g\left(y_{n}\right)}{2}=\frac{g(x)+g(y)}{2} .
$$

Since $g$ is continuous in int dom $h_{K}$ and $x, y$ are arbitrary, we have just proved that $g$ is convex in int dom $h_{K}$. On the other hand $g=h_{K}-h_{P}$ is lower semicontinuous, hence convex in all $\mathbb{R}^{n}$. Therefore, by Theorem 8.1, we obtain $K=P+\left.\partial g\right|_{0}$.

Remark 5.3. Notice that in Theorem 5.2 the assumption of recession cone being pointed is necessary. For example let $K$ be a straight line in $\mathbb{R}^{n}$ and let $P$ be any polytope not contained in a straight line parallel to $K$. Then $P$ is not a summand of $K$. However, if a support set $K(u)$ is not empty then $K(u)=K$, and $K(u)$ is unbounded.

Let us extend a notion of polytope to unbounded sets sharing a pointed recession cone $V$. By $\mathcal{P}_{V}\left(\mathbb{R}^{n}\right):=\left\{P+V \mid P \in \mathcal{P}\left(\mathbb{R}^{n}\right)\right\}$, where $\mathcal{P}\left(\mathbb{R}^{n}\right)$ is a family of all nonempty polytopes in $\mathbb{R}^{n}$, we denote the family of sums of polytopes and the cone $V$. We call elements of the family $\mathcal{P}_{V}\left(\mathbb{R}^{n}\right)$ by $V$-polytopes. $V$-polytope is the smallest convex set with a recession cone $V$ containing a given finite set of points. The following theorem is straightforward corollary from Theorem 5.2.

Theorem 5.4 (criterion for a $V$-polytopal summand). Let $V$ be a pointed convex cone, $K \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right), n \geqslant 2$, and $P \in \mathcal{P}_{V}\left(\mathbb{R}^{n}\right)$. Then $P$ is a summand of $K$ if and only if a nonempty bounded support set $K(u)$ contains a translate of $P(u)$, whenever $P(u)$ is an edge of $P, u \in S^{n-1}$.

Let $A, B \in \mathcal{C}\left(\mathbb{R}^{n}\right)$. We call two bounded support sets $A(u)$ and $B(u)$ equiparallel edges if they are parallel line segments. Bauer in [5] gave the following necessary and sufficient criterion for reduced pairs of polytopes.

Theorem 5.5 (Bauer's criterion for reduced pair of polytopes). A pair $(A, B)$ of polytopes in $\mathbb{R}^{n}$ is reduced if and only if $A$ and $B$ have no equiparallel edges.

The next theorem generalizes Bauer's criterion to reduced pairs of $V$ polytopes.

Theorem 5.6 (criterion for reduced pair of $V$-polytopes). Let $V$ be a pointed convex cone. Then a pair $(A, B) \in \mathcal{P}_{V}^{2}\left(\mathbb{R}^{n}\right)$ is reduced if and only if $A$ and $B$ have no equiparallel edges.
Proof. $\Longleftarrow)$ Let $A$ and $B$ have no equiparallel edges. Assume that $A+D=B+C=: E$ for some $C, D \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. In order to prove that $A+B$ is a summand of $E$, let $(A+B)(u)$ be an edge. Since $A$ and $B$ have no equiparallel edges, $A(u)$ and $B(u)$ cannot be line segments both at the same time. Then one of these, say $B(u)$, is a singleton and $(A+B)(u)$ is a translate of $A(u)$. Hence the set $E(u)=A(u)+D(u)$ contains a translate of $(A+B)(u)$. By Theorem 5.4, the set $A+B$ is a summand of $E=A+D=B+C$. Therefore, $E=A+B+M$ for some $M \in \mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$. By the cancellation law $(C, D)=(A+M, B+M)$.
$\Longrightarrow)$ If $A(u)$ and $B(u)$ are parallel edges then we can construct a pair $\left(A^{\prime}, B^{\prime}\right)$ equivalent to $(A, B)$ such that $A \subset A^{\prime}, B \subset B^{\prime}$ and no translate of $A(u)$ is contained in $A^{\prime}(u)$. This construction was given by Bauer in Theorem 5.3 [5] for a pair of polytopes.
6. Application. Minimal representation of a difference of convex functions

Let $V \subset \mathbb{R}^{n+1}$ be a nontrivial closed convex cone such that $V \cap\{x \in$ $\left.\mathbb{R}^{n+1} \mid x_{n+1} \geqslant 0\right\}=\{0\}$. A pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n+1}\right)$ is H-minimal if $(A, B)$ is a minimal element in the family $\{(C, D) \in[A, B] \mid 0 \in$ $D$ and $\left.\forall x \in D: x_{n+1} \leqslant 0\right\}$. The definition of $H$-minimality corresponds to Hartman's [22] definition of a minimal representation of a dc-function $f=g-h$, i.e. a difference of convex functions $g$ and $h$, defined on the open unit ball in $\mathbb{R}^{n}$.

Let us notice that for two convex and lower semicontinous functions $g, h: \mathbb{R}^{n} \longrightarrow \mathbb{R} \cup\{+\infty\}$ we can find corresponding cosed convex sets $A, B$ such that

$$
g(x)=h_{A}(x, 1), h(x)=h_{B}(x, 1), x \in \mathbb{R}^{n} .
$$

The sets $A, B$ are defined by

$$
\begin{aligned}
A & :=\left\{(x, t) \in \mathbb{R}^{n} \times \mathbb{R} \mid \forall y:\langle(x, t),(y, 1)\rangle \leqslant g(y)\right\} \\
& =\{(x, t) \mid \forall y:\langle x, y\rangle+t \leqslant g(y)\}, \\
B & :=\{(x, t) \mid \forall y:\langle x, y\rangle+t \leqslant h(y)\} .
\end{aligned}
$$

Indeed, by Theorem 8.2 the set $A$ is a subdifferential of such a lower semicontinuous sublinear function $\hat{g}$, that $\hat{g}(x, t)=t g(x / t), t>0$ and $\hat{g}(x, t)=+\infty, t<0$. In fact $A=$ hypo $\left(-g^{*}\right)$ i.e. the convex set $A$ is equal to a hypograph of a function $-g^{*}$ where $g^{*}$ is a convex conjugate of $g$ [31]. We also have $B=\operatorname{hypo}\left(-h^{*}\right)$.

Hartman, defining minimal representation of a dc-function $f=g-h$ in section 6 of [22], requires that $g, h$ are as small as possible under conditions of $h \leqslant 0$ and $h(0)=0$. The function $h$ is non-negative if and only if $0_{\mathbb{R}^{n+1}} \in B$. Besides, $h(0) \leqslant 0$ implies $B \subset \mathbb{R}^{n} \times \mathbb{R}_{-}$. If $B \subset \mathbb{R}^{n} \times \mathbb{R}_{-}$then $h(0)=h_{B}(0,1)=\sup _{(x, t) \in B}\langle(x, t),(0,1)\rangle=\sup _{(x, t) \in B} t \leqslant 0$.

Hartman considers dc-function $f=g-h$ defined on an interior of a unit ball $\mathbb{B}$ in $\mathbb{R}^{n}$. In order to represent convex functions $g, h$ by convex sets we extend them outside of int $\mathbb{B}$ by $g(x)=h(x)=\infty$ for $x \notin \mathbb{B}$ and $g(x)=\liminf _{y \rightarrow x,\|y\|<1} g(y), h(x)=\liminf _{y \rightarrow x,\|y\|<1} h(y)$ for $\|x\|=1$.

Since effective domains of $g$ and $h$ contain an open Euclidean unit ball and are contained in a closed unit ball $\mathbb{B}$, the sets $A$ and $B$ share recession cone $V$ defined by $V=\left\{(x, t) \in \mathbb{R}^{n} \times \mathbb{R} \mid t \leqslant-\|x\|_{2}\right\}$. From previous considerations follows the next theorem.

Theorem 6.1. A representation $f=g-h$ of a dc-function is minimal according to Hartman if and only if a pair of sets $(A, B)$, where $A, B$ are defined by $(* * *)$, is $H$-minimal in $\mathfrak{C}_{V}^{2}\left(\mathbb{R}^{n+1}\right)$.

If we replace in Hartman's definition an open unit ball with an interior of a closed convex set $K$ containing 0 then corresponding sets $A$ and $B$ share a recession cone $V$ defined by $V:=\bigcup_{t \geqslant 0} t\left(K^{\circ} \times\{-1\}\right)$ where $K^{\circ}$ is a polar of $K$.

The following proposition is obvious.
Proposition 6.2. A pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n+1}\right)$ is $H$-minimal if and only if it is 0 -minimal and $B \subset\left\{x \in \mathbb{R}^{n+1} \mid x_{n+1} \leqslant 0\right\}$.

In Example 2.10(ii) all equivalent 0-minimal pairs are $H$-minimal. Obviously, 0 -minimal pairs may not be $H$-minimal. See the next example.

Example 6.3. Let $T: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{3}, T\left(x_{1}, x_{2}, x_{3}\right):=\left(x_{1}, x_{2}, s x_{1}+t x_{2}+\right.$ $\left.x_{3}\right), s, t \in \mathbb{R}$. Consider convex sets from Example 2.10(ii). For $i=$ $0,1,2$ the pairs $\left(T\left(\widehat{A}_{i}\right), T\left(\widehat{B}_{i}\right)\right)$ are 0 -minimal. The pair $\left(T\left(\widehat{A}_{i}\right), T\left(\widehat{B}_{i}\right)\right)$ is $H$-minimal if and only if $-s\left(p_{i}\right)_{1}-t\left(p_{i}\right)_{2} \leqslant 0$ and $s\left(2-\left(p_{i}\right)_{1}\right)-$ $t\left(p_{i}\right)_{2} \leqslant 0$. For example the pair $\left(T\left(\widehat{A}_{0}\right), T\left(\widehat{B}_{0}\right)\right)$ where $T\left(x_{1}, x_{2}, x_{3}\right):=$ $\left(x_{1}, x_{2}, x_{1}+x_{3}\right), p_{0}=(0,0)$ is 0 -minimal and not $H$-minimal.

Obviously, any pair which is 0 -minimal and not $H$-minimal does not contain a $H$-minimal pair.

Theorem 6.4 (existence of $H$-minimal pairs). Let $(A, B) \in$ $\mathcal{C}_{V}^{2}\left(\mathbb{R}^{n+1}\right)$. There exists an equivalent $H$-minimal pair $\left(A^{\prime}, B^{\prime}\right)$ such that $A^{\prime} \subset A-b, B^{\prime} \subset B-b$ for some $b \in B$.

Proof. Let $b \in B$ and $b_{n+1}=\max _{x \in B} x_{n+1}$. By Theorem 2.1 there exists a 0 -minimal pair $\left(A^{\prime}, B^{\prime}\right)$ contained in $(A-b, B-b)$. Since $B^{\prime} \subset B-b \subset$ $\left\{x \in \mathbb{R}^{n+1} \mid x_{n+1} \leqslant 0\right\}$, the pair $\left(A^{\prime}, B^{\prime}\right)$ is $H$-minimal.

Remark 6.5. It is possible that among equivalent pairs of sets a $H$ minimal pair is unique even if this pair does not have the property of translation. For example the pair $\left(T\left(\widehat{B}_{0}\right)-(1,1,1), T\left(\widehat{A}_{0}\right)-(1,1,1)\right)$ from Example 6.3, where $T\left(x_{1}, x_{2}, x_{3}\right):=\left(x_{1}, x_{2}, x_{2}+x_{3}\right), p_{0}=(0,0)$, is a unique $H$-minimal pair in the quotient class $\left[T\left(\widehat{B}_{0}\right), T\left(\widehat{A}_{0}\right)\right]$. Notice that $T\left(\widehat{A}_{0}\right)-(1,1,1)=\operatorname{conv}\{(0,0,0),(-1,-1,-1),(1,-1,-1)\}+V$, $T\left(\widehat{B}_{0}\right)-(1,1,1)=\operatorname{conv}\{(-1,-1,-1),(1,-1,-1)\}+V$. Convex functions corresponding to these two sets are $g\left(x_{1}, x_{2}\right):=\left|x_{1}\right|-x_{2}-1$ and $h\left(x_{1}, x_{2}\right):=\max \left(0,\left|x_{1}\right|-x_{2}-1\right)$. They are the unique Hartmanminimal convex functions, such that $f\left(x_{1}, x_{2}\right):=\min \left(0,\left|x_{1}\right|-x_{2}-1\right)=$ $g\left(x_{1}, x_{2}\right)-h\left(x_{1}, x_{2}\right)$.

Proposition 6.6. Let a pair $(A, B) \in \mathcal{C}_{V}^{2}\left(\mathbb{R}^{n+1}\right)$ be reduced and $V \cap$ $\left\{x \in \mathbb{R}^{n+1} \mid x_{n+1} \geqslant 0\right\}=L_{V}$. Then a pair $(A-x, B-x), x \in B$ is $H$-minimal if and only if $x_{n+1}=\sup _{y \in B} y_{n+1}=h_{B}(u)$, where $u=$ $(0, \ldots, 0,1) \in \mathbb{R}^{n+1}$.

Proof. By Theorem 3.3 the pair $(A, B)$ has property of translation. Proposition follows from criterion of 0-minimality in Proposition 2.6(b) and from characterization of $H$-minimality in Proposition 6.2.

The following example shows, that reducibility (property of translation) in the assumptions of Proposition 2.6(b) is essential.

Example 6.7. Consider a pair ( $\widehat{A}_{1}, \widehat{B}_{1}$ ) in Example 2.10(ii). This pair is $H$-minimal but not reduced. If $x \in \widehat{B}_{1}$ then $\left(\widehat{A}_{1}-x, \widehat{B}_{1}-x\right)$ is $H$ minimal if and only if $x=0$. Still $\left(\widehat{A}_{0}-x, \widehat{B}_{0}-x\right)$ is $H$-minimal if and only if $x \in\left(\widehat{B}_{0}\right)_{*}$, i.e. $x=\left(x_{1}, \ldots, x_{n+1}\right) \in \widehat{B}_{0}$ and $x_{n+1}=\sup _{y \in \widehat{B}_{0}} y_{n+1}=0$.

## 7. Appendix. Minimal pairs of closed bounded convex sets

Let $\mathcal{B}\left(\mathbb{R}^{n}\right)$ be a family of all nonempty compact convex sets, i.e. convex bodies. The idea of treating compact convex sets as numbers or, rather, as vectors goes back to Minkowski [24]. A semigroup of nonempty bounded closed convex subsets $\mathcal{B}(X)$ of a vector space $X$ was embedded
into a topological vector space in the case of a normed space $X$ by Rådström [29], a locally convex space $X$ by Hörmander [23] and a topological vector space by Urbański [36]. The embedding was possible thanks to an order cancellation law:

$$
A+B \subset \operatorname{cl}(B+C) \Longrightarrow A \subset C \text { for } A, B, C \in \mathcal{B}(X)
$$

For a concise proof of an order cancellation law in a more general setting we refer the reader to Proposition 5.1 in [14]. Convex sets are embedded into Minkowski-Rådström-Hörmander space $\widetilde{X}=\mathcal{B}^{2}(X) / \sim$ of quotient classes, where a relation of equivalence is defined by $(A, B) \sim$ $(C, D): \Longleftrightarrow \operatorname{cl}(A+D)=\operatorname{cl}(B+C)$.

A new motivation to study pairs of convex sets came from quasidifferential calculus of Demyanov and Rubinov [8, 8], where a quasidifferential $D f\left(x_{0}\right)$ is a pair of convex sets $(A, B)=\left(\left.\underline{\partial} f\right|_{x_{0}},\left.\bar{\partial} f\right|_{x_{0}}\right)$ called sub- and superdifferential. Rather than a pair of sets $(A, B)$ a quasidifferential is a quotient class $[A, B]:=[(A, B)]_{\sim}$.

The best representation of a quotient class $[A, B]$ is a reduced pair, i.e. a pair $(A, B)$ such that $[A, B]=\{(A+C, B+C) \mid C \in \mathcal{B}(X)\}$. Then all translates of $(A, B)$ give all minimal elements of $[A, B]$. Reduced pairs were studied by Bauer [5]. However, not every quotient class $[A, B]$ contains a reduced pair. We say that a pair $(A, B)$, or a quotient class $[A, B]$ has property of translation if all minimal pairs in $[A, B]$ are translates of each other. There exist not reduced minimal pairs that have property of translation. The following theorem holds true.

Theorem 7.1. ([19, 26]) Let $X$ be a reflexive Banach space. For every pair $(A, B) \in \mathcal{B}^{2}(X)$, there exists an inclusion-minimal pair $(C, D) \in$ $[A, B]$ such that $C \subset A, D \subset B$.

Caprari and Penot [7] proved existence of inclusion minimal pairs in a quotient class $[A, B] \in \mathcal{C}(X) \times \mathcal{K}(X) / \sim$ where $\mathcal{K}(X)$ is a family of all nonempty compact convex subsets of a locally convex vector space $X$.

Theorem 7.2. ([5, 12, 34]) Let $(A, B) \in \mathcal{B}^{2}\left(\mathbb{R}^{2}\right)$. A minimal pair in $[A, B]$ is unique up to translation.

Theorem 7.2 basically states that every minimal pair of two-dimensional compact convex sets has property of translation.

Example 7.3. ([12]) In $\mathbb{R}^{3}$ we have equivalent minimal pairs not connected by translation.


Figure 7.1. Three equivalent minimal pairs not connected by translation.

In Figure 7.1 the solid $B$ is a regular octahedron, $D$ is an elongated octahedron, $F$ is a hexagon, $A$ is a rhombohedron and $E$ is a cuboctahedron.

In [13, 15, 27] more quotient classes $[A, B]$ with no unique minimal pair were found in $\mathbb{R}^{3}$. However, the set of all equivalent minimal pairs was never effectively described for a quotient class $[A, B]$ with no unique minimal element. All these results enabled calculus of pairs of convex sets in a way analogous to fractional arithmetics [28].

The following theorem states a necessary and sufficient criterion for minimal pairs of convex polygons.

Theorem 7.4 (Theorem 3.5 in [16]). A pair $(A, B)$ of flat polytopes is minimal if and only if $A$ and $B$ have at most one pair of parallel edges that lie on the same side of the polytopes.

## 8. Appendix. Minkowski duality

In this section we present Minkowski duality between closed convex sets and sublinear functions.

A 4-tuple $\left(X, \mathbb{R}_{+},+, \cdot\right)$, where an operation of addition '+' and of multiplication by nonnegative numbers '. ' are defined for elements of the set $X$, is called an abstract convex cone if (1) the pair $(X,+)$ is a commutative group and for all $x, y \in X$ and all $s, t \geqslant 0$ we have (2) $1 x=x$, (3) $0 x=0$, (4) $s(t x)=(s t) x$, (5) $t(x+y)=t x+t y$ and (6) $(s+t) x=s x+t x$.

If a set $A$ belongs to the family $\mathcal{C}\left(\mathbb{R}^{n}\right)$ of all nonempty closed convex subsets of $\mathbb{R}^{n}$ then its support function $h_{A}$ is defined by

$$
h_{A}:=\sup _{a \in A}\langle a, \cdot\rangle .
$$

Minkowski addition $A \dot{+} B$ of sets belonging to $\mathcal{C}\left(\mathbb{R}^{n}\right)$ is defined by $A \dot{+} B:=\operatorname{cl}(A+B)$. Obviously, if one of these sets is bounded then $A \dot{+} B=A+B$.

If a function $h: \mathbb{R}^{n} \longrightarrow \mathbb{R} \cup\{\infty\}$ belongs to the family $\mathcal{S}_{\text {lsc }}^{\infty}\left(\mathbb{R}^{n}\right)$ of all sublinear (positively homogenous and convex) lower semicontinuous
functions then its subdifferential at 0 is a closed convex set defined by $\left.\partial h\right|_{0}:=\left\{x \in \mathbb{R}^{n} \mid\langle x, \cdot\rangle \leqslant h\right\}$.

Both 4-tuples $\left(\mathcal{C}\left(\mathbb{R}^{n}\right), \mathbb{R}_{+}, \dot{+}, \cdot\right)$ and $\left(\mathcal{S}_{l s c}^{\infty}\left(\mathbb{R}^{n}\right), \mathbb{R}_{+},+, \cdot\right)$ are abstract convex cones and Minkowski duality establishes isomorphic relationship between these cones.

Theorem 8.1. The mapping $\left.\mathcal{S}_{\text {lsc }}^{\infty}\left(\mathbb{R}^{n}\right) \ni h \longmapsto \partial h\right|_{0} \in \mathcal{C}\left(\mathbb{R}^{n}\right)$ is an isomorphic bijection from an abstract convex cone $\mathcal{S}_{\text {lsc }}^{\infty}\left(\mathbb{R}^{n}\right)$ onto an abstract convex cone $\mathcal{C}\left(\mathbb{R}^{n}\right)$. The mapping $\mathcal{C}\left(\mathbb{R}^{n}\right) \ni A \longmapsto h_{A} \in \mathcal{S}_{\text {lsc }}^{\infty}\left(\mathbb{R}^{n}\right)$ is an inverse mapping. Moreover, a restriction of the mapping to the subfamily of finite sublinear functions $\mathcal{S}\left(\mathbb{R}^{n}\right)$ is an isomorphic bijection from a subcone $\mathcal{S}\left(\mathbb{R}^{n}\right)$ onto a subcone $\mathcal{B}\left(\mathbb{R}^{n}\right)$ of all nonempty bounded closed convex sets.

Theorem 8.1 is stated in [17] in a general case for a dual pair $(X, Y)$ of linear spaces over $\mathbb{R}$ where $\langle\cdot, \cdot\rangle$ is such a bilinear function, that functions $\{\langle y, \cdot\rangle\}_{y \in Y}$ separate points in $X$ and functions $\{\langle\cdot, x\rangle\}_{x \in X}$ separate points in $Y$.

Let $h: \mathbb{R}^{n} \longrightarrow \mathbb{R} \cup\{\infty\}$ be a sublinear lower semicontinous function. An effective domain dom $h \subset \mathbb{R}^{n}$ is a convex cone, its closure cldom $h$ is a closed convex cone.

Let $V$ be a closed convex cone in $\mathbb{R}^{n}$. A characteristic function $\chi_{V}$ is defined by $\chi_{V}(x):=\left\{\begin{array}{cc}0, & x \in V, \\ \infty, & x \notin V .\end{array}\right.$ A subdifferential of $\chi_{V}$ at 0 coincides with a polar cone $V^{\circ}$. Therefore, $\left.\partial\left(\chi_{V}\right)\right|_{0}=V^{\circ}$, and $h_{V}=\chi_{V^{\circ}}$.

By $\mathcal{S}_{l s c, V}^{\infty}\left(\mathbb{R}^{n}\right)$ we denote a subfamily of sublinear functions with finite values in the relative interior of $V$ and infinite outside of $V$. Values of such a function on the relative boundary of $V$ are determined by its values in the relative interior. A 4-tuple $\left(\mathcal{S}_{l s c, V}^{\infty}\left(\mathbb{R}^{n}\right), \mathbb{R}_{+},+, \cdot\right)$ is an abstract convex cone after modifying multiplication by 0 in the following way $0 h=: \chi_{V}$.

By $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ we denote all closed convex sets $A$ having their recession cone $\operatorname{recc} A:=A \dot{-} A=\{x \mid x+A \subset A\}$ equal to $V$. Again the family $\mathcal{C}_{V}\left(\mathbb{R}^{n}\right)$ is an abstract convex cone after modifying multiplication by 0 with a formula $0 A=V$ [30].

Theorem 8.2. The mapping $\left.\mathcal{S}_{l s c, V}^{\infty}\left(\mathbb{R}^{n}\right) \ni h \longmapsto \partial h\right|_{0} \in \mathcal{C}_{V^{\circ}}\left(\mathbb{R}^{n}\right)$ is an isomorphic bijection from an abstract convex cone $\mathcal{S}_{\text {lsc, },}^{\infty}\left(\mathbb{R}^{n}\right)$ onto an abstract convex cone $\mathcal{C}_{V^{\circ}}\left(\mathbb{R}^{n}\right)$. The mapping $\mathcal{C}_{V^{\circ}}\left(\mathbb{R}^{n}\right) \ni A \longmapsto h_{A} \in$ $\mathcal{S}_{l s c, V}^{\infty}\left(\mathbb{R}^{n}\right)$ an is an inverse mapping.

Proof. In view of Theorem 8.1 it is enough to prove that for any function $h \in \mathcal{S}_{l s c}^{\infty}\left(\mathbb{R}^{n}\right)$ closed convex cones $V_{1}=\operatorname{cl} \operatorname{dom} h$ and $V_{2}=\operatorname{recc}\left(\left.\partial h\right|_{0}\right)$ are mutually polar.

Notice that $h=h+\chi_{V_{1}}$. By Theorem 8.1 we obtain $\left.\partial h\right|_{0}=\left.\partial h\right|_{0}+$ $\left.\partial\left(\chi_{V_{1}}\right)\right|_{0}=\left.\partial h\right|_{0}+V_{1}^{\circ}$. Then $\left.\left.V_{1}^{\circ} \subset \partial h\right|_{0} \dot{-} \partial h\right|_{0}=\operatorname{recc}\left(\left.\partial h\right|_{0}\right)=V_{2}$.

On the other hand $\left.\partial h\right|_{0}=\left.\partial h\right|_{0}+V_{2}$. By Theorem 8.1 we get $h=$ $h+h_{V_{2}}=h+\chi_{V_{2}^{\circ}}$. Hence $\operatorname{dom} h \subset V_{2}^{\circ}$, and $V_{1} \subset V_{2}^{\circ}$. Therefore, $V_{2} \subset V_{1}^{\circ}$.

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Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Uniwersytetu Poznańskiego 4, 87, 61-614 Poznań, Poland Email address: jgrz@amu.edu.pl

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Uniwersytetu Poznańskiego 4, 87, 61-614 Poznań, Poland

Email address: rich@amu.edu.pl


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