LONG TIME BEHAVIOR OF STOCHASTIC NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS AND WONG-ZAKAI APPROXIMATIONS*

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Abstract. This paper is devoted to investigating the well-posedness and asymptotic behavior of a class of stochastic nonlocal partial differential equations driven by nonlinear noise. First, the existence of a weak martingale solution is established by using the Faedo-Galerkin approximation and an idea analogous to Da Prato and Zabczyk [12]. Second, we show the uniqueness and continuous dependence on initial values of solutions to the above stochastic nonlocal problem when there exist some variational solutions. Third, the asymptotic local stability of steady-state solutions is analyzed either when the steady-state solutions of the deterministic problem is also solution of the stochastic one, or when this does not happen. Next, to study the global asymptotic behavior, namely, the existence of attracting sets of solutions, we consider an approximation of the noise given by Wong-Zakai's technique using the so called colored noise. For this model, we can use the power of the theory of random dynamical systems and prove the existence of random attractors. Eventually, particularizing in the cases of additive and multiplicative noise, it is proved that the Wong-Zakai approximation models possess random attractors which converge upper-semicontinuously to the respective random attractors of the stochastic equations driven by standard Brownian motions. This fact justifies the use of this colored noise technique to approximate the asymptotic behavior of the models with general nonlinear noises, although the convergence of attractors and solutions is still an open problem.

Key words. Nonlinear stochastic term, colored noise, variational solutions, steady-state solution, attractors, upper semi-continuity.

AMS subject classifications. 60H15, 35B40.

1. Introduction.

Nowadays, a big amount of researchers develop stochastic systems to model phenomena from real world in a more realistic way, as can be seen in the published literature (for instance, [6, 8, 17, 19, 21, 25, 31] and references therein). In this paper, we are concerned with a stochastic version of a nonlocal partial differential equation, which has been well studied by M. Chipot and his collaborators (see [9, 10, 11]), to model the behavior of a migrating population in a bounded habitat or problems with magneto-elastic interactions. Precisely, we are interested in performing a rigorous study of well-posedness and dynamics of the following stochastic nonlocal reaction-diffusion equation,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) + g(t,u)\frac{dW}{dt}, & \text{in } \mathcal{O}\times(\tau,\infty), \\ u = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u(x,\tau) = u_0(x), & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, function $a \in C(\mathbb{R};\mathbb{R}^+)$ and there exist two positive constants m and $\widetilde{m}$, such that

$$m \le a(s) \le \widetilde{m}, \qquad \forall s \in \mathbb{R}. \tag{1.2}$$

Moreover, let $l \in \mathcal{L}(L^2(\mathcal{O});\mathbb{R})$, $f \in C(\mathbb{R})$ and there exist positive constants α_1, α_2, η, κ and $p > 2$, such that

$$(f(s) - f(r))(s-r) \le \eta(s-r)^2, \qquad \forall s, r \in \mathbb{R}, \tag{1.3}$$

$$-\kappa - \alpha_1|s|^p \le f(s)s \le \kappa - \alpha_2|s|^p, \qquad \forall s \in \mathbb{R}. \tag{1.4}$$

From (1.4), we can deduce that there exists $\beta > 0$, such that

$$|f(s)| \le \beta(|s|^{p-1} + 1), \qquad \forall s \in \mathbb{R}. \tag{1.5}$$

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In addition, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis with expectation $\mathbb{E}$, K and U be two separable Hilbert spaces. Let $W(t)$ be a cylindrical Wiener process with values in K defined on the stochastic basis. Denote by $L_2(K,U)$ the set of Hilbert-Schmidt operators from K to U. Eventually, let the initial value $u_0 \in L^2(\mathcal{O})$ and non-autonomous term $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\mathcal{O}))$. The operator l acting on u must be understood as (l,u), but for short we keep the notation $l(u)$.

Now, we impose smoothness condition on the domain, namely, we require $\mathcal{O} \subset \mathbb{R}^N$ to be a bounded open set of class C^k, with $k \geq 2$ such that $k \geq N(p-2)/(2p)$.

Initially, our intention was to prove the well-posedness of problem (1.1) in the sense of Definition 2.6 by following the variational technique which was originally introduced by Pardoux in his thesis [23], and subsequently in many other papers dealing with stochastic partial differential equations in the variational framework (see, e.g. [5, 7, 8, 24]). However, on the one hand, the appearance of the nonlocal term $a(\cdot)$ in our problem makes the analysis more involved, since the main operator, $a(l(u))\Delta u$, does not satisfy the standard assumptions of monotonicity which are required in the aforementioned variational set-up. On the other hand, In the deterministic case (cf. [32]), the compactness method for nonlinear partial differential equations is somehow easier: when L^p bounds on the approximating solutions have been proved, the approximating equations readily give us estimates on the derivatives, and this implies strong convergence of some subsequence, while this strategy does not extend to the stochastic case since the solutions are not differentiable (cf. [14]). Therefore, by carrying out a careful analysis in a satisfactory way, some conclusions are obtained as follows:

- When $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$, we are able to prove the existence of a solution (see Theorem 2.8) in a weaker sense, the so called martingale solution (see Definition 2.7).
- One should expect some positive answers, in some particular cases, about existence of variational solution to problem (1.1). In fact, when l is not a bounded linear operator as in our current case, for instance, when the functional l is given by $l(u) = \|u\|^2_{H^1_0}$, the existence and uniqueness of solution of the following problem

$$\begin{cases} u_t - a(\|u\|^2_{H^1_0})\Delta u = f(u) + h(t), & (t,x) \in (0,\infty) \times \mathcal{O}, \\ u = 0, & (t,x) \in (0,\infty) \times \partial\mathcal{O}, \\ u(0,x) = u_0(x), & x \in \mathcal{O}, \end{cases}$$

were shown in [3]. Moreover, recently, the authors studied in [4] the existence and uniqueness of variational solution to the stochastic version of the above problem,

$$\begin{cases} u_t = a(\|u\|^2_{H^1_0})\Delta u + f(u) + h(t,x) + \sigma(u)dw(t), & (t,x) \in (\tau,\infty) \times \mathcal{O}, \\ u = 0, & (t,x) \in (\tau,\infty) \times \partial\mathcal{O}, \\ u(\tau,x) = u_\tau(x), & x \in \mathcal{O}, \end{cases}$$

by using a monotone iterative approach. Let us point out the key point in the proof is to show that the nonlocal term $-a(\|u\|^2_{H^1_0})\Delta u$ is monotone. This holds true because in [4] it is imposed that

$$s \to a(s^2)s \text{ is non-decreasing.}$$

However, in our case, it is not possible to prove the monotonicity of the operator $-a(l(u))\Delta u$ since $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$.
- If we adopted a Picard scheme as in [18, Chapter 3], defining operator $A(v) := -a(l(u^{n-1}))\Delta v$, we could construct a sequence $\{u^n\}_{n=1}^\infty$, whose limit could be the solution of our problem. In this way, we would overcome the difficulty of proving monotonicity. However, in the last step to prove $\{u^n\}_{n=1}^\infty$ is a Cauchy sequence, we would not have enough regularity to ensure the stopping time

$$t_N^n := \{\tau \leq t \leq T : \|u^n(t)\| \geq N\},$$

is well defined, since $u^n \in L^2(\Omega; L^\infty(\tau,T; L^2(\mathcal{O}))) \cap L^2(\Omega; L^2(\tau,T; H^1_0(\mathcal{O}))) \cap L^p(\Omega; L^p(\tau,T; L^p(\mathcal{O})))$ for $p > 2$ by the Itô formula. As a result, we are not able to use a monotone iterative approach method when $l \in \mathcal{L}(L^2(\mathcal{O}); \mathbb{R})$. As an alternative, we will show the existence of martingale solutions to problem (1.1).

Next, we study the asymptotic local stability when there exist variational solutions to (1.1). Our analysis is intended in two directions: (i) We study the behavior of the solutions to the stochastic problem around steady-state solutions (equilibria) of the deterministic one (i.e. $g \equiv 0$), when the latter are not necessarily equilibria of the stochastic problem. In this case, we prove exponential convergence (in mean square and almost surely) of solutions to (1.1) towards some steady-state solution to the deterministic problem; (ii) When the deterministic and stochastic problems have a common steady-state solution, we prove a sufficient condition ensuring its asymptotic exponential stability in mean square. However, the global asymptotic dynamics cannot be carried out by applying the well-established theory of random dynamical systems in the case of nonlinear noisy terms. This leads us to proceed in a different way as we will describe below.

Notice that, for the particular case in which the noise term is linear (additive or multiplicative), the existence of random attractors of (1.1) has been analyzed in [33] by exploiting the tools of the theory of random dynamical systems. However, when the noise is nonlinear, this theory cannot be applied in a suitable way because it is not proved yet that the stochastic problem (1.1) generates a random dynamical system. Recently, B. X. Wang and his collaborators (see, e.g., [15, 17, 22, 30]) have initiated a new approach to tackle the problem with nonlinear noise. The idea is to replace the noise in (1.1) by a Wong-Zakai approximation, denoted by $\zeta_\delta(\theta_t\omega), \delta \in (0, 1]$ (see details in Section 4), whose integral $\int_0^t \zeta_\delta(\theta_s\omega)\, ds$ converges to the Brownian motion $W_t(\omega)$, uniformly for t in bounded intervals of time, as δ goes to zero. Therefore, we will analyze the following random non-autonomous problem driven by colored noise,

$$\tag{1.6} \begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) + g(t, u)\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O} \times (\tau, \infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x, \tau) = u_0(x), & \text{in } \mathcal{O}. \end{cases}$$

Observe that the above random problem can be analyzed for each fixed ω, therefore it generates a random dynamical system. Hence, the deterministic techniques can be adopted here to state the well-posedness and the existence of a random attractor.

Naturally, one should expect, at least formally, that the random attractor of (1.6) converges in some sense to a random attractor of the limit problem when δ goes to zero. This is a hard problem, there are answers only in some special cases. Motived by the previous work, for instance [30], we will particularize our study in the cases of additive and multiplicative noise. Indeed, we first study the dynamics of

$$\tag{1.7} \begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \phi \frac{dW}{dt}, & \text{in } \mathcal{O} \times (\tau, \infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x, \tau) = u_0, & \text{in } \mathcal{O}, \end{cases}$$

where, for simplicity, we consider an autonomous version, i.e., $h = 0$ and $g(t, u) = \phi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$. The corresponding approximate problem is

$$\tag{1.8} \begin{cases} \frac{\partial u_\delta}{\partial t} - a(l(u_\delta))\Delta u_\delta = f(u_\delta) + \phi\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O} \times (\tau, \infty), \\ u_\delta = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u_\delta(x, \tau) = u_{0,\delta}, & \text{in } \mathcal{O}, \end{cases}$$

where functions a and f satisfy conditions (1.2)-(1.4) with $p = 2$ and $\beta = C_f$. Then, by using appropriate changes of variable given by Ornstein-Uhlenbeck processes, we prove that both problems generate random dynamical systems which possess random attractors, denoted by $\mathcal{A}$ and $\mathcal{A}_\delta$, respectively. Furthermore, it is shown that $\mathcal{A}_\delta$ converges upper-semicontinuously to $\mathcal{A}$ as δ goes to zero, and the solutions of problem (1.8) converge to solutions of (1.7). More precisely, if $\{\delta_n\}_{n=1}^\infty$ is a sequence satisfying $\delta_n \to 0$ as $n \to +\infty$, u_{δ_n} and u are the solutions of (1.8) and (1.7) with initial values u_{0,δ_n} and u_0, respectively, and if $u_{0,\delta_n} \to u_0$ strongly in $L^2(\mathcal{O})$ as $n \to +\infty$, then for almost all $\omega \in \Omega$ and $t \geq \tau$,

$$u_{\delta_n}(t; \tau, \omega, u_{0,\delta_n}) \to u(t; \tau, \omega, u_0) \qquad \text{strongly in } \ L^2(\mathcal{O}) \ \text{ as } \ n \to +\infty.$$

Finally, we carry out a similar analysis in the case of multiplicative noise, i.e.,

$$
\begin{cases}
\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \sigma u \circ \frac{dW}{dt}, & \text{in } \mathcal{O} \times (\tau, \infty), \\
u = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\
u(x, \tau) = u_0, & \text{in } \mathcal{O},
\end{cases}
\tag{1.9}
$$

and the corresponding approximate problem is

$$
\begin{cases}
\frac{\partial u_\delta}{\partial t} - a(l(u_\delta))\Delta u_\delta = f(u_\delta) + \sigma u \circ \zeta_\delta(\theta_t \omega), & \text{in } \mathcal{O} \times (\tau, \infty), \\
u_\delta = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\
u_\delta(x, \tau) = u_{0,\delta}, & \text{in } \mathcal{O},
\end{cases}
\tag{1.10}
$$

where $\circ$ denotes the Stratonovich sense in stochastic term.

The analysis described above is developed in the following sections. Section 2 is devoted to proving the main theorem about existence and construction of a martingale solution. In Section 3, the local asymptotic behavior of solutions is considered, proving some exponential decay of solutions of the stochastic problem to the steady-state solutions of the deterministic one (i.e., $g \equiv 0$). The global asymptotic behavior of solutions is studied in Section 4 by considering the Wong-Zakai approximate problem of our original one (cf. (1.1)). The theory of random non-autonomous dynamical systems is carried out to prove the existence of a random non-autonomous attractor for the approximate system (cf. (1.6)), which can be considered as a reasonable approximation of the dynamics for our original problem. This claim is justified with the analysis developed in sections 5 and 6, where one can check that the attractors and solutions of the approximate problems converge, in appropriate sense.

2. Existence of martingale solutions to problem (1.1).

In this section, we use the Faedo-Galerkin approximation and an idea analogous to Da Prato and Zabczyk [12] showing the existence of a martingale solution to stochastic nonlocal problem (1.1). This theory has received increasing attention over the last years (see, e.g. [12, 13, 14, 26]). In what follows, we introduce some necessary notations and most of the hypotheses relevant to our analysis.

2.1. Stochastic setting. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all of $\mathbb{P}$-null sets of $\mathcal{F}$. In this manuscript, all stochastic integrals are defined in the sense of Itô and $\mathbb{E}X$ denotes the mathematical expectation of the stochastic process $X = X(t, \omega)$ with respect to $\mathbb{P}$. Given K and U two separable Hilbert spaces, $W(t)$ a cylindrical Wiener process with values in K, we denote by $\mathcal{L}(K, U)$ the space of continuous linear mapping from K to U. By $L_2(K, U)$, which is a subspace of $\mathcal{L}(K, U)$ consisting of Hilbert-Schmidt operators from K to U. It is known that $L_2(K, U)$ is a Hilbert space and its norm is denoted by $\|\cdot\|_{L_2(K,U)}$.

Given $p > 1$, $\alpha \in (0, 1)$, let $W^{\alpha,p}(0, T; U)$ be the Sobolev space of all functions $u \in L^p(0, T; U)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+\alpha p}} dt ds < \infty,$$

endowed with the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;U)} = \int_0^T |u(t)|^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t - s|^{1+\alpha p}} dt ds.$$

For any progressively measurable process $f \in L^2(\Omega \times [0, T]; L_2(K, U))$, we denote by $I(f)$ the Itô integral defined as

$$I(f)(t) = \int_0^t f(s) dW(s), \qquad t \in [0, T].$$

Clearly, $I(f)$ is a progressively measurable process in $L^2(\Omega \times [0, T]; U)$.

LEMMA 2.1. *([14, Lemma 2.1]) Let $p \geq 2$, $0 < \alpha < \frac{1}{2}$. Then, for any progressively measurable process $f \in L^p(\Omega \times [0,T]; L_2(K,U))$, we have*

$$I(f) \in L^p(\Omega; W^{\alpha,p}(0,T;U)),$$

and there exists a constant $C(p,\alpha) > 0$, independent of f, such that

$$\mathbb{E}\|I(f)\|^p_{W^{\alpha,p}(0,T;U)} \leq C(p,\alpha)\mathbb{E}\int_0^T \|f(t)\|^p_{L_2(K,U)}dt.$$

2.2. Notations. We also introduce additional notations frequently used throughout the work, for simplicity, denote by $H = L^2(\mathcal{O})$, $V = H^1_0(\mathcal{O})$ and $V^* = H^{-1}(\mathcal{O})$. Identifying H with its dual, we have the usual chain of dense and compact embeddings $V \subset H \subset V^*$. We denote by $|\cdot|_p$ the norm in $L^p(\mathcal{O})$, $|\cdot|$ and $\|\cdot\|_*$ the norms in H and V^*, by $(\cdot,\cdot)$ and $((\cdot,\cdot))$ the scalar products in H and V, respectively, and by $<\cdot,\cdot>$ the duality product between V and V^*. At last, let $C^\infty_c(\mathcal{O})$ be the space of all functions of class C^∞ with compact supports contained in $\mathcal{O}$.

Given real numbers $a < b$ and $p > 1$, we will denote by $I^p(a,b;H)$ the space of all processes $X \in L^p(\Omega \times (a,b), \mathcal{F} \otimes \mathcal{B}((a,b)), d\mathbb{P} \otimes dt; H)$, where $\mathcal{B}((a,b))$ denotes the Borel σ-algebra on (a,b), such that $X(t)$ is $\mathcal{F}_t$-measurable for a.e. $t \in (a,b)$. Moreover, the space $I^p(a,b;H)$ is a closed subspace of $L^p(\Omega \times (a,b), \mathcal{F} \otimes \mathcal{B}((a,b)), d\mathbb{P} \otimes dt; H)$.

Denote by $A = -\Delta$ with Dirichlet boundary condition in our problem, and let $D(A)$ be the domain of A. In this way, the linear operator $A : D(A) := V \cap H^2(\mathcal{O}) \subset V \to H$ is positive, self-adjoint with compact resolvent. We denote by $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ the eigenvalues of A, and by e_1, $e_2, \cdots$, a corresponding complete orthonormal system in $L^2(\mathcal{O})$ of eigenvectors of A. Recall that for every $v \in V$, the Poincaré inequality

$$\lambda_1(\mathcal{O})|v|^2 \leq \|v\|^2$$

holds. In the sequel, unless otherwise specified, we write λ_1 instead of $\lambda_1(\mathcal{O})$.

2.3. Assumptions on g. Let $g : (\tau,T) \times H \to L_2(H,H)$ satisfy:

g_1) $g(t,0) = 0$ and $\|g(t,u) - g(t,v)\|_{L_2(H,H)} \leq L_g|u-v|$, $\forall u,v \in H$, a.e. $t \in (\tau,T)$;

g_2) For every $\rho \in C^\infty_c(\mathcal{O})$, the mapping $H \ni u \to < g(t,u), \rho > := g(t,u)(\rho) \in H$ is continuous for a.e. $t \in (\tau,T)$.

Remark 2.2. We will show detailedly the proof of existence of martingale solutions to problem (1.1) in the next theorem. To present ideas clearly, we simply do estimations on $g(u)$ instead of $g(t,u)$. Indeed, the idea and procedures to obtain existence of martingale solutions to (1.1) with $g(t,u)$ are similar, we only need to consider for every $t \in (\tau,T]$, $\tilde{u}(t)$ is $\tilde{\mathcal{F}}_t$-measurable, for more details, see [13].

2.4. Preliminaries. We now recall the following results which will be needed to prove the existence of martingale solutions.

LEMMA 2.3. *([14, Theorem 2.1]) Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 be reflexive, with compact embedding of B_0 in B. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$, let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1)$$

endowed with the natural norm. Then the embedding of X in $L^p(0,T;B)$ is compact.

LEMMA 2.4. *([12, Skorohod theorem]) Let X be a complete, separable metric space. For an arbitrary sequence $\{\mu_n\}$, which is tight on $(X, \mathcal{B}(X))$, there exists a subsequence $\{\mu_{n_k}\}$ which converges weakly to a probability measure μ, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with X-valued Borel measurable random variables x_n and x, such that μ_n is the distribution of x_n, μ is the distribution of x and $x_n \to x$, $\mathbb{P}$-a.s.*

LEMMA 2.5. *([26, Vitali's convergence theorem]) Let $p \in [1,\infty)$, $x_n \in L^p(\Omega)$, and x_n converge to x in probability. Then the following statements are equivalent:*

1. $\lim_{n\to\infty} x_n = x$ *in* $L^p(\Omega)$*;*
2. $|x_n|^p$ *is uniformly integrable;*

3. $\lim_{n\to\infty}\mathbb{E}[|x_n|^p]=\mathbb{E}[|x|^p]$.

Particularly, if $\sup_n \mathbb{E}[|x_n|^q]<\infty$ *for some* $p<q<\infty$*, or if there exists a* $y\in L^p(\Omega)$ *such that* $|x_n|<y$ *for all* n*, then the above properties hold true.*

2.5. Definitions of solutions. We introduce the concepts of solution of problem (1.1).

Definition 2.6. *(Variational solution) A solution of* (1.1) *is a stochastic process* $u\in I^2(\tau,T;V)\cap L^2(\Omega;C(\tau,T;H))\cap I^p(\tau,T;L^p(\mathcal{O}))$ *for all* $T\geq\tau$*, with the initial value* $u(\tau)=u_0\in L^2(\Omega;H)$*, such that*

$$\begin{aligned} u(t)=u_0+\int_\tau^t a(l(u(s)))\Delta u(s)ds+\int_\tau^t f(u(s))ds+\int_\tau^t h(s)ds\\ +\int_\tau^t g(u(s))dW(s),\quad \mathbb{P}\text{-}a.s.\quad \forall t\in(\tau,T],\end{aligned}$$

where the above integro-equality should be understood in $V^*+L^q(\mathcal{O})$*, and* q *is the conjugate exponent of* p*.*

Definition 2.7. *(Martingale solution) We say there exists a martingale solution of equation* (1.1) *if there exist*

- *a stochastic basis* $(\tilde\Omega,\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\geq0},\tilde{\mathbb{P}})$*;*
- *a cylindrical Wiener process* $\tilde W$ *on the space* H*;*
- *a progressively measurable process* $\tilde u:[\tau,T]\times\tilde\Omega\to H$ *with* $\tilde{\mathbb{P}}$*-a.e. paths*

$$\tilde u(\cdot,\omega)\in L^2(\tau,T;V)\cap L^\infty(\tau,T;H)\cap L^p(\tau,T;L^p(\mathcal{O})),$$

such that for all $t\in[\tau,T]$ *and* $v\in V\cap L^p(\mathcal{O})$*,*

$$\begin{aligned}(\tilde u(t),v)+\int_\tau^t a(l(\tilde u(s)))<A\tilde u(s),v>ds=(\tilde u_0,v)+\int_\tau^t(f(\tilde u(s)),v)ds\\ +\int_\tau^t<h(s),v>ds+\left(\int_\tau^t g(\tilde u(s))d\tilde W(s),v\right),\end{aligned}\tag{2.1}$$

where the identity holds $\tilde{\mathbb{P}}$*-a.s.*

2.6. Main results. We now prove the existence of martingale solutions to problem (1.1) after presenting all the required conditions, lemmas and techniques.

Theorem 2.8. *Assume that* $a\in C(\mathbb{R};\mathbb{R}^+)$ *satisfies (1.2),* $f\in C(\mathbb{R})$ *fulfills (1.3)-(1.4),* $g:H\to L_2(H,H)$ *satisfies* $g_1)$*-*$g_2)$*. Moreover,* $h\in L^2_{loc}(\mathbb{R};V^*)$ *and* $l\in\mathcal{L}(L^2(\mathcal{O});\mathbb{R})$*. Then, for every initial datum* $u_0\in H$*, there exists at least one martingale solution to problem* (1.1)*.*

Proof. We split the proof into several steps.

Step 1. Faedo-Galerkin approximation. Making use of spectral theory, we recall that $\{e_i\}_{i=1}^\infty$ is the orthonormal basis of H consisting of the eigenfunctions of A in V. Observe that, thanks to the regularity imposed on the domain $\mathcal{O}$, each eigenfunction $e_i\in L^p(\mathcal{O})$.

Before going further, we first define two projection operators related to

$$\begin{aligned} P_n:\quad H&\longrightarrow V_n:=\mathrm{span}[e_1,\cdots,e_n],\\ \phi&\longrightarrow\sum_{i=1}^n(\phi,e_i)e_i.\end{aligned}$$

The first one is given by

$$\begin{aligned} P_n^1:\qquad V^*&\longrightarrow V^*,\\ v&\longrightarrow\ [\phi\in V\to<P_n^1v,\phi>:=<v,P_n\phi>].\end{aligned}$$

To define the second one, we recall that $A = -\Delta$ with homogeneous Dirichlet boundary condition, i.e. the isomorphism from V into V^*, which can be also seen as an unbounded operator in H. Let us consider the domains of fractional powers of A,

$$D(A^{k/2}) = \{u \in H : \sum_{i\geq 1} \lambda_i^k (u, e_i)^2 < \infty\}.$$

Now we are ready to define the second projection operator, which is given by

$$\begin{aligned} P_n^2 : \quad & L^q(\mathcal{O}) \longrightarrow D(A^{-k/2}), \\ & v \longrightarrow [\phi \in D(A^{k/2}) \to < P_n^2 v, \phi >_{D(A^{-k/2}),D(A^{k/2})} := (v, P_n\phi)]. \end{aligned}$$

Observe that P_n^1 and P_n^2 are the continuous extensions in V^* and $L^q(\mathcal{O})$ of P_n, respectively. Therefore, from now on we will denote both projections by P_n making an abuse of notation.

Let us consider the classical Faedo-Galerkin approximation in the space V_n,

$$\begin{cases} du_n(t) = [-a(l(u_n(t)))Au_n(t) + P_n f(u_n(t)) + P_n h(t)]\, dt + P_n g(u_n(t))dW(t), \quad t \in (\tau, T], \\ u_n(\tau) = P_n u_0. \end{cases} \tag{2.2}$$

In what follows, we will show for all $n \in \mathbb{N}$, there exist three positive constants C_1, C_2 and C_3, such that

$$\mathbb{E}\left[\sup_{\tau\leq t\leq T} |u_n(t)|^2\right] \leq C_1, \tag{2.3}$$

$$\mathbb{E}\int_\tau^T \|u_n(t)\|^2 dt \leq C_2, \tag{2.4}$$

and

$$\mathbb{E}\int_\tau^T |u_n(t)|_p^p dt \leq C_3. \tag{2.5}$$

Applying the Itô formula to $|u_n|^2$ $(n \geq 1)$ and integrating from τ to T, we have

$$\begin{aligned} |u_n(t)|^2 = |P_n u_0|^2 &+ 2\int_\tau^t a(l(u_n(s))) < -Au_n(s), u_n(s) > ds + 2\int_\tau^t (P_n f(u_n(s)), u_n(s))ds \\ &+ 2\int_\tau^t < P_n h(s), u_n(s) > ds + 2\int_\tau^t (u_n(s), P_n g(u_n(s))dW(s)) \\ &+ \int_\tau^t \|P_n g(u_n(s))\|^2_{L_2(H,H)} ds, \qquad \text{a.e. } t \in (\tau, T]. \end{aligned}$$

Making use of (1.2) and (1.4), we obtain

$$\begin{aligned} |u_n(t)|^2 + 2m\int_\tau^t \|u_n(s)\|^2 ds + 2\alpha_2 \int_\tau^t |u_n(s)|_p^p ds &\leq |u_0|^2 + 2\kappa|\mathcal{O}|(T-\tau) \\ &+ 2\int_\tau^t \|h(s)\|_* \|u_n(s)\| ds + 2\int_\tau^t (u_n(s), P_n g(u_n(s))dW(s)) \\ &+ \int_\tau^t \|P_n g(u_n(s))\|^2_{L_2(H,H)} ds, \qquad \text{a.e. } t \in (\tau, T]. \end{aligned}$$

Applying the Young inequality and taking into account of g_1) to the above inequality, we arrive at

$$\begin{aligned} |u_n(t)|^2 + m\int_\tau^t \|u_n(s)\|^2 ds + 2\alpha_2\int_\tau^t |u_n(s)|_p^p ds &\leq |u_0|^2 + 2\kappa|\mathcal{O}|(T-\tau) + \frac{1}{m}\int_\tau^t \|h(s)\|_*^2 ds \\ &+ L_g \int_\tau^t |u_n(s)|^2 ds + 2\int_\tau^t (u_n(s), P_n g(u_n(s))dW(s)) \, . \end{aligned} \tag{2.6}$$

Taking supremum and expectation on both sides of (2.6), by means of the Burkholder-Davis-Gundy inequality, we derive

$$\begin{aligned}\mathbb{E}\left[\sup_{\tau\le s\le t}|u_n(s)|^2\right] &\le 2\mathbb{E}|u_0|^2+4\kappa|\mathcal{O}|(T-\tau)+\frac{2}{m}\mathbb{E}\int_\tau^t\|h(s)\|_*^2ds\\ &\quad+2\left(1+2C_b^2\right)L_g\int_\tau^t\mathbb{E}\left[\sup_{\tau\le r\le s}|u_n(r)|^2\right]ds,\end{aligned}$$

where C_b is the constant derived from Burkholder-Davis-Gundy estimate. By iterating the preceding inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{\tau\le s\le t}|u_n(s)|^2\right] &\le \left(2\mathbb{E}|u_0|^2+4\kappa|\mathcal{O}|(T-\tau)+\frac{2}{m}\mathbb{E}\int_\tau^t\|h(s)\|_*^2ds\right)\\ &\quad\times\sum_{i=0}^{n-1}\frac{(2(1+2C_b^2)L_g)^i(t-\tau)^i}{i!}\le e^{(2+4C_b^2)L_g(T-\tau)}\le const.\end{aligned}$$

Moreover, it follows from (2.6) that

$$m\mathbb{E}\int_\tau^t\|u_n(s)\|^2ds\le\mathbb{E}|u_0|^2+2\kappa|\mathcal{O}|(T-\tau)+\frac{1}{m}\mathbb{E}\int_\tau^t\|h(s)\|_*^2ds+L_g\int_\tau^t\mathbb{E}\left[\sup_{\tau\le r\le s}|u_n(r)|^2\right]ds,$$

and

$$2\alpha_2\mathbb{E}\int_\tau^t|u_n(s)|_p^pds\le\mathbb{E}|u_0|^2+2\kappa|\mathcal{O}|(T-\tau)+\frac{1}{m}\mathbb{E}\int_\tau^t\|h(s)\|_*^2ds+L_g\int_\tau^t\mathbb{E}\left[\sup_{\tau\le r\le s}|u_n(r)|^2\right]ds.$$

Thus, the desired results (2.3)-(2.5) are proved.

Step 2. Tightness. For each $n\in\mathbb{N}$, the solution u_n of the Galerkin equation defines a measure $\mathcal{L}(u_n)$ on $L^2(\tau,T;V)\cap L^\infty(\tau,T;H)\cap L^p(\tau,T;L^p(\mathcal{O}))$. Using lemmas 2.1 and 2.3, together with estimates (2.3)-(2.5), we will prove the tightness of this set of measures.

Decompose now u_n as

$$\begin{aligned}u_n(t)&=P_nu_0-\int_\tau^t a(l(u_n(s)))Au_n(s)ds+\int_\tau^t P_nf(u_n(s))ds+\int_\tau^t P_nh(s)ds\\ &\quad+\int_\tau^t P_ng(u_n(s))dW(s)=I_n^1+I_n^2+I_n^3+I_n^4+I_n^5.\end{aligned}\tag{2.7}$$

We will estimate each term of (2.7). Since $u_0\in H$, it is easy to check there exists a constant C_4, such that

$$\mathbb{E}|I_n^1|^2\le C_4.$$

For I_n^2, by (1.2), (2.4), the Hölder inequality and Fubini Theorem, there exists a constant C_5, such that

$$\begin{aligned}\mathbb{E}\|I_n^2\|^2_{W^{1,2}(\tau,T;V^*)}&=\mathbb{E}\|I_n^2\|^2_{L^2(\tau,T;V^*)}+\mathbb{E}\|\frac{dI_n^2}{dt}\|^2_{L^2(\tau,T;V^*)}\\ &=\mathbb{E}\int_\tau^T\left\|\int_\tau^t -a(l(u_n(s)))Au_n(s)ds\right\|_*^2dt+\mathbb{E}\int_\tau^T\|-a(l(u_n(s)))Au_n(s)\|_*^2ds\\ &\le\tilde{m}^2(T-\tau)\mathbb{E}\int_\tau^T\int_\tau^t\|-Au_n(s)\|_*^2dsdt+\tilde{m}^2\mathbb{E}\int_\tau^T\|-Au_n(t)\|_*^2dt\\ &\le C\left(\tilde{m}^2(T-\tau)^2+\tilde{m}^2\right)\mathbb{E}\int_\tau^T\|u_n(t)\|^2dt\le C_5.\end{aligned}$$

For I_n^3, let $q=\frac{p}{p-1}\in(1,2)$ be the conjugate of p, we first derive the following estimate by (1.5),

$$
\begin{aligned}
|f(u_n)|_q^q &= \int_{\mathcal{O}}|f(u_n)|^q dx \le \beta^q\int_{\mathcal{O}}(|u_n|^{p-1}+1)^q dx \le 2^{q-1}\beta^q\int_{\mathcal{O}}|u_n|^{q(p-1)}dx + 2^{q-1}\beta^q|\mathcal{O}|\\
&:= 2^{q-1}\beta^q|u_n|_p^p + 2^{q-1}\beta^q|\mathcal{O}|.
\end{aligned}
$$

Observe that $P_nf(u_n)\in L^q(\tau,T;H^{-k}(\mathcal{O}))$ since $f(u_n)\in L^q(\tau,T;L^q(\mathcal{O}))$. By the above inequality, (2.5), the Hölder inequality and Fubini Theorem, there exists a constant C_6, such that

$$
\begin{aligned}
\mathbb{E}\|I_n^3\|^q_{W^{1,q}(\tau,T;H^{-k}(\mathcal{O}))} &= \mathbb{E}\|I_n^3\|^q_{L^q(\tau,T;H^{-k}(\mathcal{O}))} + \mathbb{E}\|\frac{dI_n^3}{dt}\|^q_{L^q(\tau,T;H^{-k}(\mathcal{O}))}\\
&= \mathbb{E}\int_\tau^T\left|\int_\tau^t P_nf(u_n(s))ds\right|^q_{H^{-k}(\mathcal{O})}dt + \mathbb{E}\int_\tau^T|P_nf(u_n(t))|^q_{H^{-k}(\mathcal{O})}dt\\
&\le \mathbb{E}\int_\tau^T\left(\int_\tau^t|P_nf(u_n(s))|_{H^{-k}(\mathcal{O})}ds\right)^q dt + \mathbb{E}\int_\tau^T|f(u_n(t))|_q^q dt\\
&\le \left((T-\tau)^{\frac{1}{p-1}+1}+1\right)\mathbb{E}\int_\tau^T|f(u_n(t))|_q^q dt \le C_6.
\end{aligned}
$$

For I_n^4, by the Hölder inequality and Fubini Theorem, there exists a constant C_7, such that

$$
\begin{aligned}
\mathbb{E}\|I_n^4\|^2_{W^{1,2}(\tau,T;V^*)} &= \mathbb{E}\|I_n^4\|^2_{L^2(\tau,T;V^*)} + \mathbb{E}\|\frac{dI_n^4}{dt}\|^2_{L^2(\tau,T;V^*)}\\
&= \mathbb{E}\int_\tau^T\left\|\int_\tau^t P_nh(s)ds\right\|_*^2 dt + \mathbb{E}\int_\tau^T\|P_nh(t)\|_*^2dt\\
&\le ((T-\tau)^2+1)\mathbb{E}\|h\|^2_{L^2(\tau,T;V^*)} \le C_7.
\end{aligned}
$$

As for the last term I_n^5, by Lemma 2.1, assumption g_1) and (2.3), we know there exists a constant $C_8(\alpha)$, such that for every $\alpha\in(0,\frac12)$, we have

$$\mathbb{E}\|I_n^5\|^2_{W^{\alpha,2}(\tau,T;H)} \le C_8(\alpha).$$

Obviously, for $\alpha\in(0,\frac12)$, the natural continuous embedding $D(A^{k/2})\hookrightarrow H^k(\mathcal{O})\hookrightarrow L^p(\mathcal{O})$ implies

$$
\begin{aligned}
W^{1,2}(\tau,T;V^*)\subset W^{1,q}(\tau,T;V^*)\subset W^{\alpha,q}(\tau,T;V^*)\subset W^{\alpha,q}(\tau,T;D(A^{-k/2})),\\
W^{\alpha,2}(\tau,T;H)\subset W^{\alpha,q}(\tau,T;H)\subset W^{\alpha,q}(\tau,T;V^*)\subset W^{\alpha,q}(\tau,T;D(A^{-k/2})),
\end{aligned}
$$

and

$$W^{1,q}(\tau,T;H^{-k}(\mathcal{O}))\subset W^{\alpha,q}(\tau,T;H^{-k}(\mathcal{O}))\subset W^{\alpha,q}(\tau,T;D(A^{-k/2})).$$

Collecting all the previous estimates for I_n^1-I_n^5, together with the above natural embedding results, we obtain

$$\mathbb{E}\|u_n\|_{W^{\alpha,q}(\tau,T;D(A^{-k/2}))}\le C(\alpha),$$

for all $\alpha\in(0,\frac12)$ and $C(\alpha)>0$. Actually, thanks to (2.4), we deduce that the laws $\mathcal{L}(u_n)$ are bounded in probability in

$$L^2(\tau,T;V)\cap W^{\alpha,q}(\tau,T;D(A^{-k/2})).$$

Additionally, $L^2(\tau,T;V)\subset L^q(\tau,T;V)$, hence, it follows from Lemma 2.3 that $\mathcal{L}(u_n)$ is tight in $L^q(\tau,T;H)$.

Step 3. Pass to limit. By Step 2, we obtain the set of measures $\mathcal{L}(u_n)$ is tight on the space $L^q(\tau,T;H)$. Moreover, Lemma 2.4 implies there exists a stochastic basis $(\tilde\Omega,\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\ge0},\tilde{\mathbb{P}})$, and on this basis, there exist $L^q(\tau,T;H)$-valued random variables $\{\tilde u_{n_k}\}$ $(k\ge1)$ and $\tilde u$, such that

$$\tilde u_{n_k} \text{ has the same law as } u_{n_k} \text{ on } L^q(\tau,T;H) \text{ and } \tilde u_{n_k}\to\tilde u \text{ in } L^q(\tau,T;H),\quad \tilde{\mathbb{P}}\text{-}a.s. \tag{2.8}$$

In the sequel, let us denote the subsequence $\tilde{u}_{n_k}$ again by $\tilde{u}_n$.

Since $u_n \in C(\tau, T; P_nH)$, $\mathbb{P}$-a.s. together with the fact that $\tilde{u}_n$ has the same law as u_n, we derive for each $n \geq 1$,

$$\mathcal{L}(\tilde{u}_n)(C(\tau, T; P_nH)) = 1, \qquad \tilde{\mathbb{P}}\text{-}a.s.$$

By similar arguments as (2.3)-(2.5), we know there exist three positive constants $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$, such that for all $n \geq 1$,

$$\tilde{\mathbb{E}}\left[\sup_{\tau \leq t \leq T} |\tilde{u}_n(t)|^2\right] \leq \tilde{C}_1, \tag{2.9}$$

$$\tilde{\mathbb{E}}\int_\tau^T \|\tilde{u}_n(t)\|^2 dt \leq \tilde{C}_2, \tag{2.10}$$

and

$$\tilde{\mathbb{E}}\int_\tau^T |\tilde{u}_n(t)|_p^p dt \leq \tilde{C}_3. \tag{2.11}$$

Based on the above estimates, it holds that the sequence $\{\tilde{u}_n(\cdot, \omega)\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^p(\tau, T; L^p(\mathcal{O}))$. Also, (2.8) implies that $\tilde{u}_n \to \tilde{u}$ in $L^q(\tau, T; H)$, $\tilde{\mathbb{P}}$-a.s. Therefore, we conclude that

$$\tilde{u}(\cdot, \omega) \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^p(\tau, T; L^p(\mathcal{O})), \qquad \tilde{\mathbb{P}}\text{-}a.s. \tag{2.12}$$

We will show now that for each $n \geq 1$, the process $\tilde{M}_n$ with trajectories in $C(\tau, T; H)$ defined as

$$\tilde{M}_n(t) = \tilde{u}_n(t) - P_n\tilde{u}_0 + \int_\tau^t a(l(\tilde{u}_n(s)))P_nA\tilde{u}_n(s)ds - \int_\tau^t P_nf(\tilde{u}_n(s))ds - \int_\tau^t P_nh(s)ds, \quad t \in (\tau, T], \tag{2.13}$$

is a square integrable martingale with respect to the filtration $\tilde{\mathcal{F}}_{n,t} = \sigma\{\tilde{u}_n(s), \tau \leq s \leq t\}$, having the following quadratic variation

$$\langle\langle \tilde{M}_n \rangle\rangle_t = \int_\tau^t P_n g(\tilde{u}_n(s)) g(\tilde{u}_n(s))^* P_n ds, \quad t \in (\tau, T]. \tag{2.14}$$

Indeed, both facts (cf. (2.13)-(2.14)) are true since $\tilde{u}_n$ and u_n have the same law. To be more precise, we define

$$M_n(t) = u_n(t) - P_nu_0 + \int_\tau^t a(l(u_n(s)))P_nAu_n(s)ds - \int_\tau^t P_nf(u_n(s))ds - \int_\tau^t P_nh(s)ds, \quad t \in (\tau, T].$$

Obviously, $M_n(t)$ is a square integrable martingale with respect to the filtration $\mathcal{F}_{n,t} = \sigma\{u_n(s), \tau \leq s \leq t\}$, since

$$M_n(t) = \int_\tau^t P_n g(u_n(s)) dW(s), \qquad t \in (\tau, T]. \tag{2.15}$$

It follows from (2.8) that

$$\mathcal{L}(\tilde{M}_n) = \mathcal{L}(M_n), \quad \mathbb{E}|M_n(t)| < \infty \quad \text{and} \quad \tilde{\mathbb{E}}|\tilde{M}_n(t)|^2 < \infty. \tag{2.16}$$

Moreover, let φ be a real valued bounded and continuous function on $L^q(\tau, s; H)$, $\tau \leq s \leq t \leq T$, as $M_n(\cdot)$ is a $\mathcal{F}_{n,t} = \sigma\{u_n(s) : \tau \leq s \leq t\}$ martingale, we obtain for all ψ, $\zeta \in D(A^{k/2})$,

$$\mathbb{E}[\langle M_n(t) - M_n(s), \psi\rangle \varphi(u_{n|[\tau,s]})] = 0,$$

and

$$\mathbb{E}\Big[\Big(\langle M_n(t),\psi\rangle\langle M_n(t),\zeta\rangle-\langle M_n(s),\psi\rangle\langle M_n(s),\zeta\rangle \\ -\int_s^t\left(g(u_n(\sigma))^*P_n\psi,g(u_n(\sigma))^*P_n\zeta\right)d\sigma\Big)\varphi(u_{n|[\tau,s]})\Big]=0.$$

The notation $\langle\cdot,\cdot\rangle$ denotes the duality between $D(A^{k/2})$ and $D(A^{-k/2})$. Thanks to the fact $(2.16)_1$, we have

$$\tilde{\mathbb{E}}[\langle\tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle\varphi(\tilde{u}_{n|[\tau,s]})]=0, \tag{2.17}$$

and

$$\tilde{\mathbb{E}}\Big[\Big(\langle\tilde{M}_n(t),\psi\rangle\langle\tilde{M}_n(t),\zeta\rangle-\langle\tilde{M}_n(s),\psi\rangle\langle\tilde{M}_n(s),\zeta\rangle \\ -\int_s^t\left(g(\tilde{u}_n(\sigma))^*P_n\psi,g(\tilde{u}_n(\sigma))^*P_n\zeta\right)d\sigma\Big)\varphi(\tilde{u}_{n|[\tau,s]})\Big]=0. \tag{2.18}$$

We now will take limits in (2.17) and (2.18), let $\tilde{M}$ be a $D(A^{-k/2})$-valued process defined by,

$$\tilde{M}(t)=\tilde{u}(t)-\tilde{u}_0+\int_\tau^t a(l(\tilde{u}(s)))A\tilde{u}(s)ds-\int_\tau^t f(\tilde{u}(s))ds-\int_\tau^t h(s)ds,\ t\in(\tau,T]. \tag{2.19}$$

To prove the final result, we first show some auxiliary lemmas.

Lemma 2.9. *Suppose the conditions of Theorem 2.8 are true. Then, for all $s,t\in(\tau,T]$ such that $s\le t$ and for all $\psi\in D(A^{k/2})$, we have:*

(a) $\lim_{n\to\infty}(\tilde{u}_n(t),P_n\psi)=(\tilde{u}(t),\psi)$, $\tilde{\mathbb{P}}$*-a.s.*

(b) $\lim_{n\to\infty}\int_s^t<a(l(\tilde{u}_n(\sigma)))A\tilde{u}_n(\sigma),P_n\psi>d\sigma=\int_s^t<a(l(\tilde{u}(\sigma)))A\tilde{u}(\sigma),\psi>d\sigma$, $\tilde{\mathbb{P}}$*-a.s.*

(c) $\lim_{n\to\infty}\int_s^t(f(\tilde{u}_n(\sigma)),P_n\psi)d\sigma=\int_s^t(f(\tilde{u}(\sigma)),\psi)d\sigma$, $\tilde{\mathbb{P}}$*-a.s.*

Proof. Let us fix $s,t\in(\tau,T]$, $s\le t$ and $\psi\in D(A^{k/2})$. By (2.9)-(2.12), we obtain

$$\begin{cases}\tilde{u}_n(\cdot,\omega)\to\tilde{u}(\cdot,\omega) \text{ weakly in } L^2(\tau,T;V),\ \tilde{\mathbb{P}}\text{-a.s.}\\ \tilde{u}_n(\cdot,\omega)\to\tilde{u}(\cdot,\omega) \text{ weak-star in } L^\infty(\tau,T;H),\ \tilde{\mathbb{P}}\text{-a.s.}\\ \tilde{u}_n(\cdot,\omega)\to\tilde{u}(\cdot,\omega) \text{ weakly in } L^p(\tau,T;L^p(\mathcal{O})),\ \tilde{\mathbb{P}}\text{-a.s.}\\ \tilde{u}_n(\cdot,\omega)\to\tilde{u}(\cdot,\omega) \text{ strongly in } L^q(\tau,T;H),\ \tilde{\mathbb{P}}\text{-a.s.}\\ \tilde{u}_n(t,\omega)\to\tilde{u}(t,\omega) \text{ strongly in } H,\ \text{a.e. } t\in(\tau,T],\ \tilde{\mathbb{P}}\text{-a.s.}\\ \tilde{u}_n(t,x,\omega)\to\tilde{u}(t,x,\omega)\ \text{a.e. } (t,x)\in(\tau,T]\times\mathcal{O},\ \tilde{\mathbb{P}}\text{-a.s.}\end{cases} \tag{2.20}$$

Thus, assertion (a) holds true since $P_n\psi\to\psi$ in H as $n\to\infty$, $\tilde{\mathbb{P}}$-a.s.

We now prove (b). On the one hand, since $l\in\mathcal{L}(L^2(\mathcal{O});\mathbb{R})$ and $a\in C(\mathbb{R};\mathbb{R}^+)$, by $(2.20)_5$, we have

$$l(\tilde{u}_n)=(l,\tilde{u}_n)\xrightarrow{n\to\infty}(l,\tilde{u})=l(\tilde{u}),$$

hence, $a(l(\tilde{u}_n))\to a(l(\tilde{u}))$ as $n\to\infty$. On the other hand, with the help of fact $P_n\psi\to\psi$ in V as $n\to\infty$, we infer that $\tilde{\mathbb{P}}$-a.s.

$$\int_s^t<a(l(\tilde{u}_n(\sigma)))A\tilde{u}_n(\sigma),P_n\psi>d\sigma=\int_s^t a(l(\tilde{u}_n(\sigma)))((\tilde{u}_n(\sigma),P_n\psi))d\sigma \\ \xrightarrow{n\to\infty}\int_s^t a(l(\tilde{u}(\sigma)))((\tilde{u}(\sigma),\psi))d\sigma=\int_s^t a(l(\tilde{u}(\sigma)))<A\tilde{u}(\sigma),\psi>d\sigma.$$

Thus, (b) is proved.

We will now move to the last assertion. It follows from (2.20)$_6$ that $\tilde{u}_n(\sigma,x,\omega)\to\tilde{u}(\sigma,x,\omega)$ in $\mathcal{O}$ for a.e. $(\sigma,x)\in(\tau,T]\times\mathcal{O}$ as $n\to\infty$. In addition, $f(\tilde{u}_n)$ is bounded in $L^q(\tau,T;L^q(\mathcal{O}))$, making use of [20, Lemma 1.3], we obtain $f(\tilde{u}_n)\to f(\tilde{u})$ weakly in $L^q(\tau,T;L^q(\mathcal{O}))$. In addition, $P_n\psi\to\psi$ in $L^p(\mathcal{O})$, thus, for almsot all $\omega\in\tilde{\Omega}$, we obtain

$$\int_s^t (f(\tilde{u}_n(\sigma)),P_n\psi)d\sigma \xrightarrow{k\to\infty} \int_s^t (f(\tilde{u}(\sigma)),\psi)d\sigma.$$

The proof of this lemma is complete. ∎

Lemma 2.10. *Suppose the conditions of Theorem 2.8 are true. Then, for all $s,t\in(\tau,T]$, every $s\le t$ and $\psi\in D(A^{k/2})$, we have,*

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\langle\tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle\varphi(\tilde{u}_{n|[\tau,s]})\right]=\tilde{\mathbb{E}}\left[\langle\tilde{M}(t)-\tilde{M}(s),\psi\rangle\varphi(\tilde{u}_{|[\tau,s]})\right].$$

Proof. We will prove this lemma by using Vitali's convergence theorem (cf. Lemma 2.5). Let us fix $s,t\in(\tau,T]$, for every $\psi\in D(A^{k/2})$, by the definition of projection operator P_n defined in Step 1 of Theorem 2.8, we derive

$$\begin{aligned}\langle\tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle=(\tilde{u}_n(t)-\tilde{u}_n(s),P_n\psi)+\int_s^t a(l(\tilde{u}_n(\sigma)))<A\tilde{u}_n(\sigma),P_n\psi>d\sigma\\ -\int_s^t (f(\tilde{u}_n(\sigma)),P_n\psi)d\sigma-\int_s^t <h(\sigma),P_n\psi>d\sigma.\end{aligned}$$

By means of Lemma 2.9 and $P_n\psi\to\psi$ in V as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\langle\tilde{M}_n(t)-\tilde{M}_n(s),\psi\rangle=\langle\tilde{M}(t)-\tilde{M}(s),\psi\rangle,\quad\tilde{\mathbb{P}}\text{-a.s.}\tag{2.21}$$

Observe that, φ is a real valued bounded and continuous function on $L^q(\tau,s;H)$, hence,

$$\lim_{n\to\infty}\varphi(\tilde{u}_{n|[\tau,s]})=\varphi(\tilde{u}_{|[\tau,s]}),\ \tilde{\mathbb{P}}\text{-a.s.}\quad\text{and}\quad\sup_{n\in\mathbb{N}}\|\varphi(\tilde{u}_{n|[\tau,s]})\|_\infty<\infty,$$

where we have used the notation $\|\cdot\|_\infty:=\|\cdot\|_{L^\infty}$. Let us define

$$X_n(\omega):=\Big(\langle\tilde{M}_n(t,\omega),\psi\rangle-\langle\tilde{M}_n(s,\omega),\psi\rangle\Big)\varphi(\tilde{u}_{n|[s,\tau]}),\quad\omega\in\tilde{\Omega}.$$

According to Vitali's convergence theorem, we need to check the functions $\{X_n(\omega)\}_{n\in\mathbb{N}}$ are uniformly integrable, namely,

$$\sup_{n\ge1}\tilde{\mathbb{E}}|X_n|^2<\infty.\tag{2.22}$$

In fact, for each $n\in\mathbb{N}$, we have

$$\tilde{\mathbb{E}}|X_n|^2\le 2\|\varphi\|_\infty\|\psi\|^2_{D(A^{k/2})}\tilde{\mathbb{E}}\left(|\tilde{M}_n(t)|^2+|\tilde{M}_n(s)|^2\right).\tag{2.23}$$

Since $\tilde{M}_n$ is a continuous martingale with quadratic variation defined in (2.14), by the Burkholder-Davis-Gundy inequality, (2.9) and g_1), we derive

$$\tilde{\mathbb{E}}\left[\sup_{t\in(\tau,T]}|\tilde{M}_n(t)|^2\right]\le c\tilde{\mathbb{E}}\left[\int_\tau^T\|P_ng(\tilde{u}_n(\sigma))\|^2_{L_2(H,H)}d\sigma\right]\le cL_g\tilde{\mathbb{E}}\left[\int_\tau^T|\tilde{u}_n(\sigma)|^2d\sigma\right]<\infty,\tag{2.24}$$

here and in the sequel, c is a positive and finite constant obtained by the Burkholder-Davis-Gundy inequality estimate. It follows from (2.23)-(2.24) that (2.22) holds. Since the sequence $\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable and by (2.21), it is $\tilde{\mathbb{P}}$-a.s. pointwise convergent, application of the Vitali convergence theorem completes the proof of this lemma. ∎

LEMMA 2.11. *Suppose the conditions of Theorem 2.8 are true. Then, for all $s,t\in(\tau,T]$, $s\le t$, every ψ and $\zeta\in D(A^{k/2})$, we have*

$$\begin{aligned}&\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\left(\langle\tilde M_n(t),\psi\rangle\langle\tilde M_n(t),\zeta\rangle-\langle\tilde M_n(s),\psi\rangle\langle\tilde M_n(s),\zeta\rangle\right)\varphi(\tilde u_{n|[\tau,s]})\right]\\&=\tilde{\mathbb{E}}\left[\left(\langle\tilde M(t),\psi\rangle\langle\tilde M(t),\zeta\rangle-\langle\tilde M(s),\psi\rangle\langle\tilde M(s),\zeta\rangle\right)\varphi(\tilde u_{|[\tau,s]})\right].\end{aligned}$$

Proof. Let us fix $s,t\in(\tau,T]$, where $s\le t$, for all ψ, $\zeta\in D(A^{k/2})$, we define

$$X_n(\omega):=\left[\left(\langle\tilde M_n(t),\psi\rangle\langle\tilde M_n(t),\zeta\rangle-\langle\tilde M_n(s),\psi\rangle\langle\tilde M_n(s),\zeta\rangle\right)\varphi(\tilde u_{n|[\tau,s]})\right],\quad\omega\in\tilde\Omega.$$

$$X(\omega):=\left[\left(\langle\tilde M(t),\psi\rangle\langle\tilde M(t),\zeta\rangle-\langle\tilde M(s),\psi\rangle\langle\tilde M(s),\zeta\rangle\right)\varphi(\tilde u_{|[\tau,s]})\right],\quad\omega\in\tilde\Omega.$$

By Lemma 2.9, we derive $\lim_{n\to\infty}X_n(\omega)=X(\omega)$ for $\tilde{\mathbb{P}}$-almost all $\omega\in\tilde\Omega$.

Next, we will prove that the functions $\{X_n\}_{n\in\mathbb{N}}$ are uniformly integrable. To this end, it is enough to check

$$\sup_{n\ge1}\tilde{\mathbb{E}}|X_n|^{p/2}<\infty.\tag{2.25}$$

Notice that,

$$\tilde{\mathbb{E}}|X_n|^{p/2}\le2\|\varphi\|_\infty^{p/2}\|\psi\|_{D(A^{k/2})}^{p/2}\|\zeta\|_{D(A^{k/2})}^{p/2}\tilde{\mathbb{E}}\left(|\tilde M_n(t)|^p+|\tilde M_n(s)|^p\right).\tag{2.26}$$

The same arguments as in Lemma 2.10 deduces that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\sup_{t\in(\tau,T]}|\tilde M_n(t)|^p\right]&\le c\tilde{\mathbb{E}}\left(\int_\tau^T\|P_ng(\tilde u_n(\sigma))\|_{L_2(H,H)}^2d\sigma\right)^{p/2}\\&\le cL_g^{p/2}\tilde{\mathbb{E}}\left(\int_\tau^T|\tilde u_n(\sigma)|^2d\sigma\right)^{p/2}<\infty.\end{aligned}\tag{2.27}$$

By (2.27)-(2.26), the conclusion (2.25) holds true. The Vitali convergence theorem shows

$$\lim_{n\to\infty}\tilde{\mathbb{E}}[X_n(\omega)]=\tilde{\mathbb{E}}[X(\omega)].$$

Thus, the proof of this lemma is finished. ∎

LEMMA 2.12. (*Convergence in quadratic variation*) *Suppose the conditions of Theorem 2.8 are true. Then, for any s, $t\in(\tau,T]$ and $s<t$, every ψ, $\zeta\in D(A^{k/2})$, we have*

$$\begin{aligned}&\lim_{n\to\infty}\tilde{\mathbb{E}}\left[\left(\int_s^t(g(\tilde u_n(\sigma))^*P_n\psi,g(\tilde u_n(\sigma))^*P_n\zeta)d\sigma\right)\varphi(\tilde u_{n|[\tau,s]})\right]\\&=\tilde{\mathbb{E}}\left[\left(\int_s^t(g(\tilde u(\sigma))^*\psi,g(\tilde u(\sigma))^*\zeta)d\sigma\right)\varphi(\tilde u_{|[\tau,s]})\right].\end{aligned}$$

Proof. Let us fix ψ, $\zeta\in D(A^{k/2})$, we denote

$$X_n(\omega):=\left(\int_s^t(g(\tilde u_n(\sigma))^*P_n\psi,g(\tilde u_n(\sigma))^*P_n\zeta)d\sigma\right)\varphi(\tilde u_{n|[\tau,s]}).$$

We will check the functions X_n are uniformly integrable and convergent $\tilde{\mathbb{P}}$-a.s.

Uniform integrability. It is enough to show that

$$\sup_{n\geq 1}\tilde{\mathbb{E}}|X_n|^{p/2}<\infty. \tag{2.28}$$

Since $\psi,\,\zeta\in D(A^{k/2})$, by g_1), for almost all $\omega\in\tilde{\Omega}$, we obtain

$$|g(\tilde{u}_n(\sigma,\omega))^*P_n\psi|\leq\|g(\tilde{u}_n(\sigma,\omega))\|_{L_2(H,H)}|P_n\psi|\leq\sqrt{L_g}|\tilde{u}_n(\sigma,\omega)|\|\psi\|_{D(A^{k/2})}.$$

Thus, by means of the fact that for almost all $\omega\in\tilde{\Omega}$, $\tilde{u}_n(\omega)\in L^p(\tau,T;L^p(\mathcal{O}))$, g_1) and the Young inequality, together with the above estimate, we have

$$\begin{aligned}|X_n|^{p/2}&=\left|\left(\int_s^t(g(\tilde{u}_n(\sigma))^*P_n\psi,g(\tilde{u}_n(\sigma))^*P_n\zeta)d\sigma\right)\varphi(\tilde{u}_{n|[\tau,s]})\right|^{p/2}\\&\leq\|\varphi\|_\infty^{p/2}\left(\int_s^t(g(\tilde{u}_n(\sigma))^*P_n\psi,g(\tilde{u}_n(\sigma))^*P_n\zeta)d\sigma\right)^{p/2}\\&\leq L_g^{p/2}\|\varphi\|_\infty^{p/2}\|\psi\|_{D(A^{k/2})}^{p/2}\|\zeta\|_{D(A^{k/2})}^{p/2}\left(\int_s^t|\tilde{u}_n(\sigma)|^2d\sigma\right)^{p/2}\\&\leq L_g^{p/2}\|\varphi\|_\infty^{p/2}\|\psi\|_{D(A^{k/2})}^{p/2}\|\zeta\|_{D(A^{k/2})}^{p/2}\left(\int_s^t1^{\frac{p}{p-2}}d\sigma\right)^{\frac{p-2}{2}}\int_s^t|\tilde{u}_n(\sigma)|_p^pd\sigma\\&\leq L_g^{p/2}(T-\tau)^{\frac{p-2}{2}}\|\varphi\|_\infty^{p/2}\|\psi\|_{D(A^{k/2})}^{p/2}\|\zeta\|_{D(A^{k/2})}^{p/2}\|\tilde{u}_n\|_{L^p(\tau,T;L^p(\mathcal{O}))}^p.\end{aligned}$$

Consequently, by (2.11), we have

$$\sup_{n\geq1}\tilde{\mathbb{E}}|X_n|^{p/2}\leq L_g^{p/2}(T-\tau)^{\frac{p-2}{2}}\|\varphi\|_\infty^{p/2}\|\psi\|_{D(A^{k/2})}^{p/2}\|\zeta\|_{D(A^{k/2})}^{p/2}\tilde{\mathbb{E}}\|\tilde{u}_n\|_{L^p(\tau,T;L^p(\mathcal{O}))}^p<\infty,$$

which implies (2.28) holds.

Pointwise convergence on $\tilde{\Omega}$. Let us fix $\omega\in\tilde{\Omega}$ such that

$$\tilde{u}_n(\cdot,\omega)\to\tilde{u}(\cdot,\omega)\text{ in }L^q(\tau,T;H).$$

We will show

$$\lim_{n\to\infty}\int_s^t(g(\tilde{u}_n(\sigma,\omega))^*P_n\psi,g(\tilde{u}_n(\sigma,\omega))^*P_n\zeta)d\sigma=\int_s^t(g(\tilde{u}(\sigma,\omega))^*\psi,g(\tilde{u}(\sigma,\omega))^*\zeta)d\sigma.$$

Indeed, it is sufficient to prove

$$g(\tilde{u}_n(\cdot,\omega))^*P_n\psi\xrightarrow{n\to\infty}g(\tilde{u}(\cdot,\omega))^*\psi\quad\text{in }L^2(s,t;H). \tag{2.29}$$

Notice that,

$$\begin{aligned}&\int_s^t\left|g(\tilde{u}_n(\sigma,\omega))^*P_n\psi-g(\tilde{u}(\sigma,\omega))^*\psi\right|^2d\sigma\\&\leq\int_s^t\left(|g(\tilde{u}_n(\sigma,\omega))^*(P_n\psi-\psi)|+|g(\tilde{u}_n(\sigma,\omega))^*\psi-g(\tilde{u}(\sigma,\omega))^*\psi|\right)^2d\sigma\\&\leq2\int_s^t\|g(\tilde{u}_n(\sigma,\omega))^*\|_{L_2(H,H)}^2|P_n\psi-\psi|^2d\sigma+2\int_s^t|g(\tilde{u}_n(\sigma,\omega))^*\psi-g(\tilde{u}(\sigma,\omega))^*\psi|^2d\sigma\\&:=2J_1(n)+2J_2(n).\end{aligned}$$

Let us first consider $J_1(n)$, since $\psi \in D(A^{k/2})$, we have $\lim_{n\to\infty} \|P_n\psi - \psi\| = 0$, by g_1) and the fact that $\tilde{u}_n \in L^\infty(\tau, T; H)$ for almost all $\omega \in \tilde{\Omega}$, we have

$$\int_s^t \|g(\tilde{u}_n(\sigma,\omega))\|^2_{L_2(H,H)} d\sigma \le L_g \int_s^t |\tilde{u}_n(\sigma,\omega)|^2 d\sigma \le L_g(T-\tau) \sup_{t\in(\tau,T]} |\tilde{u}_n(t,\omega)|^2 < \infty.$$

Thus,

$$\lim_{n\to\infty} J_1(n) = \lim_{n\to\infty} \int_s^t \|g(\tilde{u}_n(\sigma,\omega))\|^2_{L_2(H,H)} |P_n\psi - \psi|^2 d\sigma = 0.$$

Now, we will consider the other term $J_2(n)$, it is enough to check for every $\psi \in H$, $J_2(n) \to 0$ as $n \to \infty$. To this end, we first prove the result is true for every $\psi \in C_c^\infty(\mathcal{O})$. Since $\tilde{u}_n(\cdot,\omega) \to \tilde{u}(\cdot,\omega)$ in $L^q(\tau,T;H)$ for almost all $\omega \in \tilde{\Omega}$, there exists a subsequence $\{\tilde{u}_{n_k}(\cdot,\omega)\}_{k\in\mathbb{N}}$, such that

$$\tilde{u}_{n_k}(\sigma,\omega) \to \tilde{u}(\sigma,\omega) \text{ in } H \ \text{ a.e. } \sigma \in (\tau,T], \text{ as } k \to \infty.$$

Hence, by assumption g_2), we have

$$g(\tilde{u}_{n_k}(\sigma,\omega))^*\psi \to g(\tilde{u}(\sigma,\omega))^*\psi \text{ in } H \ \text{ a.e. } \sigma \in (\tau,T], \text{ as } k \to \infty.$$

In conclusion, by the Vitali convergence theorem, we derive

$$\lim_{k\to\infty} \int_s^t |g(\tilde{u}_{n_k}(\sigma,\omega))^*\psi - g(\tilde{u}(\sigma,\omega))^*\psi|^2 d\sigma = 0 \ \text{ for all } \psi \in C_c^\infty(\mathcal{O}).$$

Repeating the above reasoning for all subsequences, we infer that from every subsequence of the sequence $g(\tilde{u}_n(\sigma,\omega))^*\psi$, we can choose the subsequence convergent in $L^2(s,t;H)$ to the same limit. Thus, the whole sequence $g(\tilde{u}_n(\sigma,\omega))^*\psi$ is convergent to $g(\tilde{u}(\sigma,\omega))^*\psi$. At the same time,

$$\lim_{n\to\infty} J_2(n) = 0 \quad \text{for every } \psi \in C_c^\infty(\mathcal{O}).$$

If $\psi \in H$, then for every $\varepsilon > 0$, we can find $\psi_\varepsilon \in C_c^\infty(\mathcal{O})$ such that $|\psi - \psi_\varepsilon| \le \varepsilon$. Thanks to the fact that for almost all $\omega \in \tilde{\Omega}$, $\tilde{u}_n(\cdot,\omega)$, $\tilde{u}(\cdot,\omega) \in L^\infty(\tau,T;H)$, by g_1), we obtain

$$\begin{aligned}
&\int_s^t |g(\tilde{u}_n(\sigma,\omega))^*\psi - g(\tilde{u}(\sigma,\omega))^*\psi|^2 d\sigma \\
&\le 2\int_s^t |\left[g(\tilde{u}_n(\sigma,\omega))^* - g(\tilde{u}(\sigma,\omega))^*\right](\psi - \psi_\varepsilon)|^2 d\sigma + 2\int_s^t |\left[g(\tilde{u}_n(\sigma,\omega))^* - g(\tilde{u}(\sigma,\omega))^*\right]\psi_\varepsilon|^2 d\sigma \\
&\le 4\int_s^t [|g(\tilde{u}_n(\sigma,\omega))|^2_{L_2(H,H)} + |g(\tilde{u}(\sigma,\omega))|^2_{L_2(H,H)}]|\psi - \psi_\varepsilon|^2 d\sigma + 2\int_s^t |\left[g(\tilde{u}_n(\sigma,\omega))^* - g(\tilde{u}(\sigma,\omega))^*\right]\psi_\varepsilon|^2 d\sigma \\
&\le 4L_g\varepsilon^2 \int_s^t \left(|\tilde{u}_n(\sigma,\omega)|^2 + |\tilde{u}(\sigma,\omega)|^2\right) d\sigma + 2\int_s^t |\left[g(\tilde{u}_n(\sigma,\omega))^* - g(\tilde{u}(\omega,\sigma))^*\right]\psi_\varepsilon|^2 d\sigma.
\end{aligned}$$

In conclusion, we proved that

$$\lim_{n\to\infty} \int_s^t |g(\tilde{u}_n(\sigma,\omega))^*\psi - g(\tilde{u}(\sigma,\omega))^*\psi|^2 d\sigma = 0,$$

thus, we finish the proof of (2.29) and this lemma. ∎

Now, we can pass to the limit of (2.17) and (2.18) by using lemmas 2.10 and 2.11-2.12, respectively. Therefore, for all ψ, $\zeta \in D(A^{k/2})$, we obtain

$$\tilde{\mathbb{E}}[\langle \tilde{M}(t) - \tilde{M}(s), \psi\rangle \varphi(\tilde{u}_{|[\tau,s]})] = 0, \tag{2.30}$$

and

$$\tilde{\mathbb{E}}\bigg[\Big(\langle\tilde{M}(t),\psi\rangle\langle\tilde{M}(t),\zeta\rangle-\langle\tilde{M}(s),\psi\rangle\langle\tilde{M}(s),\zeta\rangle \\ -\int_s^t (g(\tilde{u}(\sigma))^*P_n\psi, g(\tilde{u}(\sigma))^*P_n\zeta)\Big)\varphi(\tilde{u}_{|[\tau,s]})\bigg]=0, \tag{2.31}$$

where $\tilde{M}$ is a $D(A^{-k/2})$-valued process defined by (2.19).

Continuation of the proof of Theorem 2.8. Eventually, we apply an idea analogous to the reasoning used by Da Prato and Zabczyk, see [12, Section 8.3]. Consider the operator $A: D(A)\subset V\to H$, the inverse operator $A^{-1}: H\to D(A)\subset V$, which is everywhere well-defined, bounded and compact, and the dual operator $(A^{-1})^*: V^*\to H$. Since V^* is a dense subspace of $D(A^{-k/2})$, we can extend the continuous operator $(A^{-1})^*: D(A^{-k/2})\to H$. By (2.30) and (2.31) with $\psi := A^{-1}\alpha$ and $\zeta := A^{-1}\beta$, where α, $\beta\in H$, we infer that $(A^{-1})^*\tilde{M}(t)$, $t\in(\tau,T]$ is a continuous square integrable martingale in H, whose dual is itself, with respect to the filtration $\tilde{\mathcal{F}}_t := \sigma\{\tilde{u}(s):\tau\le s\le t\}$, having the quadratic variation

$$\langle\langle (A^{-1})^*\tilde{M}\rangle\rangle_t=\int_\tau^t (A^{-1})^*g(\tilde{u}(s))(g(\tilde{u}(s))A^{-1})^*ds,\quad t\in(\tau,T].$$

In particular, the continuity of the process $(A^{-1})^*\tilde{M}$ follows from the fact that $\tilde{u}\in C(\tau,T;H)$. By the representation theorem [12, Theorem 8.2], there exist

- a stochastic basis $(\tilde{\tilde{\Omega}},\tilde{\tilde{\mathcal{F}}},\{\tilde{\tilde{\mathcal{F}}}_t\}_{t\ge0},\tilde{\tilde{\mathbb{P}}})$;
- a cylindrical Wiener process $\tilde{\tilde{W}}$ defined on this basis;
- a progressively measurable process $\tilde{\tilde{u}}$ such that

$$(A^{-1})^*\tilde{\tilde{u}}(t)-(A^{-1})^*\tilde{\tilde{u}}_0+(A^{-1})^*\int_\tau^t a(l(\tilde{\tilde{u}}(s)))A\tilde{\tilde{u}}(s)ds-(A^{-1})^*\int_\tau^t f(\tilde{\tilde{u}}(s))ds-(A^{-1})^*\int_\tau^t h(s)ds \\ =\int_0^t (A^{-1})^*g(\tilde{\tilde{u}}(s))d\tilde{\tilde{W}}(s).$$

However,

$$\int_\tau^t (A^{-1})^*g(\tilde{\tilde{u}}(s))d\tilde{\tilde{W}}(s)=(A^{-1})^*\int_\tau^t g(\tilde{\tilde{u}}(s))d\tilde{\tilde{W}}(s).$$

Hence, it follows from (2.12) that $\tilde{\tilde{u}}:[\tau,T]\times\tilde{\tilde{\Omega}}\to H$ with $\tilde{\tilde{\mathbb{P}}}$-a.s. paths,

$$\tilde{\tilde{u}}(\cdot,\omega)\in L^2(\tau,T;V)\cap L^\infty(\tau,T;H)\cap L^p(\tau,T;L^p(\mathcal{O})),$$

satisfies for all $t\in[\tau,T]$ and for all $v\in V\cap L^p(\mathcal{O})$,

$$(\tilde{\tilde{u}}(t),v)+\int_\tau^t a(l(\tilde{\tilde{u}}(s)))<A\tilde{\tilde{u}}(s),v>ds=(\tilde{\tilde{u}}_0,v)+\int_\tau^t (f(\tilde{\tilde{u}}(s)),v)ds \\ +\int_\tau^t <h(s),v>ds+\left(\int_\tau^t g(\tilde{\tilde{u}}(s))d\tilde{\tilde{W}}(s),v\right),$$

where the identity holds $\tilde{\tilde{\mathbb{P}}}$-a.s.

The proof of this theorem is finished. □

Although we are not able to prove the existence of variational solutions to problem (1.1), we can show that there exists at most one solution when the coefficient $a(\cdot)$ is locally Lipschitz.

THEOREM 2.13. *Assume $a\in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz and satisfies (1.2), $f\in C(\mathbb{R};\mathbb{R}^+)$ fulfills (1.3)-(1.4), $g: H\to L_2(H,H)$ satisfies g_1) and $l\in L^2(\mathcal{O})$. In addition, let $h\in L^2(\Omega;L^2_{loc}(\mathbb{R}^+;V^*))$ and $u_0\in L^2(\Omega;H)$. Then, there exists at most one solution to problem (1.1) in the sense of Definition 2.6.*

Proof. Suppose there are two solutions u and v of problem (1.1) in the sense of Definition 2.6. Let $\sigma(t) = \exp(-\mu \int_\tau^t ||u(s)||^2 ds)$ for all $\tau \le t \le T$, which is positive and well-defined (cf. Step 1 of Theorem 2.8), where μ is a proper constant to be chosen later. Applying the Itô formula to $\sigma(t)|u(t) - v(t)|^2$, by (1.2) and (1.3), we have

$$\begin{aligned} &\sigma(t)|u(t)-v(t)|^2 + 2m\int_\tau^t \sigma(s)\|u(s)-v(s)\|^2 ds \\ &\le 2\int_\tau^t \sigma(s)|a(l(u(s))) - a(l(v(s)))|\|u(s)\|\|u(s)-v(s)\|ds + 2\eta\int_\tau^t \sigma(s)|u(s)-v(s)|^2 ds \\ &+ 2\int_\tau^t \sigma(s)\left(u(s)-v(s), g(u(s))dW(s) - g(v(s))dW(s)\right) + \int_\tau^t \sigma(s)\|g(u(s)) - g(v(s))\|^2_{L_2(H,H)} ds \\ &- \mu\int_\tau^t \sigma(s)\|u(s)\|^2|u(s)-v(s)|^2 ds \end{aligned} \tag{2.32}$$

Since a is Locally Lipschitz, denote this Lipschitz constant by L_a, by the Young inequality, we have

$$\begin{aligned} &2\sigma(s)|a(l(u(s))) - a(l(v(s)))|\|u(s)\|\|u(s)-v(s)\| \\ &\le 2L_a|l|\sigma(s)|u(s)-v(s)|\|u(s)\|\|u(s)-v(s)\| \\ &\le \mu\sigma(s)\|u(s)\|^2|u(s)-v(s)|^2 + \frac{L_a^2|l|^2\sigma(s)}{\mu}\|u(s)-v(s)\|^2. \end{aligned}$$

Thus, by g_1) and the above inequality, (2.32) becomes

$$\begin{aligned} \sigma(t)|u(t)-v(t)|^2 &+ 2m\int_\tau^t \sigma(s)\|u(s)-v(s)\|^2 ds \\ &\le \frac{L_a^2|l|^2}{\mu}\int_\tau^t \sigma(s)\|u(s)-v(s)\|^2 ds + (2\eta + L_g)\int_\tau^t \sigma(s)|u(s)-v(s)|^2 ds \\ &+ 2\int_\tau^t \sigma(s)\left(u(s)-v(s), g(u(s))dW(s) - g(v(s))dW(s)\right). \end{aligned}$$

Taking the supremum (w.r.t. t) and expectation on both sides of the above inequality, by (1.2), we obtain

$$\begin{aligned} \mathbb{E}\left[\sup_{\tau\le s\le t}\sigma(s)|u(s)-v(s)|^2\right] &\le \frac{L_a^2|l|^2}{\mu}\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^s \sigma(r)\|u(r)-v(r)\|^2 dr\right] \\ &+ (2\eta+L_g)\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^s \sigma(r)|u(r)-v(r)|^2 dr\right] \\ &+ 2\mathbb{E}\left[\sup_{\tau\le s\le t}\left|\int_\tau^s \sigma(r)\left(u(r)-v(r), g(u(r))dW(r) - g(v(r))dW(r)\right)\right|\right], \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} 2m\mathbb{E}\int_\tau^t \sigma(s)\|u(s)-v(s)\|^2 ds &\le \frac{L_a^2|l|^2}{\mu}\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^s \sigma(r)\|u(r)-v(r)\|^2 dr\right] \\ &+ (2\eta+L_g)\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^t \sigma(r)|u(r)-v(r)|^2 dr\right] \\ &+ 2\mathbb{E}\left[\sup_{\tau\le s\le t}\left|\int_\tau^s \sigma(r)\left(u(r)-v(r), g(u(r))dW(r) - g(v(r))dW(r)\right)\right|\right]. \end{aligned} \tag{2.34}$$

For the first term of the right hand side of (2.33), since μ is positive, we have

$$\frac{L_a^2|l|^2}{\mu}\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^s \sigma(r)\|u(r)-v(r)\|^2dr\right] = \frac{L_a^2|l|^2}{\mu}\mathbb{E}\int_\tau^t \sigma(s)\|u(s)-v(s)\|^2ds. \tag{2.35}$$

For the second term of the right hand side of (2.33), by the same arguments as above, we obtain

$$(2\eta+L_g)\mathbb{E}\left[\sup_{\tau\le s\le t}\int_\tau^s \sigma(r)|u(r)-v(r)|^2dr\right] \le (2\eta+L_g)\mathbb{E}\int_\tau^t \sup_{\tau\le r\le s}\sigma(r)|u(r)-v(r)|^2ds. \tag{2.36}$$

Next, assumption g_1), the Burkholder-Davis-Gundy and Young inequalities imply

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{\tau\le s\le t}\left|\int_\tau^s \sigma(r)\left(u(r)-v(r), g(u(r))dW(r)-g(v(r))dW(r)\right)\right|\right]\\
&\le 2c\mathbb{E}\left[\sup_{\tau\le s\le t}\sigma(s)|u(s)-v(s)|^2 \int_\tau^t \sigma(s)\|g(u(s))-g(v(s))\|^2_{L_2(H,H)}ds\right]^{\frac{1}{2}}\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{\tau\le s\le t}\sigma(s)|u(s)-v(s)|^2\right] + 4c^2L_g\mathbb{E}\int_\tau^t \sup_{\tau\le r\le s}\sigma(r)|u(r)-v(r)|^2ds.
\end{aligned} \tag{2.37}$$

Consequently, substituting (2.35)-(2.37) into (2.33)-(2.34), letting $m\mu = L_a^2|l|^2$, we deduce

$$\mathbb{E}\left[\sup_{\tau\le s\le t}\sigma(s)|u(s)-v(s)|^2\right] \le 4\left(2\eta+L_g+4c^2L_g\right)\int_\tau^t \mathbb{E}\left[\sup_{\tau\le r\le s}\sigma(r)|u(r)-v(r)|^2\right]ds.$$

It follows from the Gronwall lemma that

$$\mathbb{E}\left[\sup_{\tau\le s\le t}\sigma(s)|u(s)-v(s)|^2\right] = 0, \qquad \forall t\in(\tau,T].$$

Thus, we have $u(t) = v(t)$ for a.a. $\omega\in\Omega$ and a.e. $t\in(\tau,T]$ since $\sigma(t)$ is positive. The proof of this theorem is complete. □

For the rest of this manuscript, to carry out the analysis of asymptotic behavior of solutions to (1.1) in the sense of Definition 2.6 and their Wong-Zakai approximation, we will assume, for simplicity, $W(t)$ is a standard 1D Brownian motion. Moreover, let $g:(\tau,T)\times H\to H$ be a nonlinear operator, satisfying:

g1) The mapping $t\in(\tau,T)\to g(t,u)\in H$ is Lebesgue measurable, for all $u\in H$;

g2) $g(t,0)=0$, a.e. $t\in(\tau,T)$;

g3) There exists a positive constant L_g (we use the same constant when no confusion is possible), such that
$$|g(t,u)-g(t,v)|^2\le L_g|u-v|^2, \qquad \forall u,v\in H, \quad \text{a.e. } t\in(\tau,T).$$

3. Asymptotic behavior of solutions to problem (1.1) around steady-state solutions of the deterministic problem. In this section, we are interested in analyzing the long time behavior of solutions to problem (1.1) with respect to equilibria of the deterministic elliptic problem,

$$\begin{cases} -a(l(u))\Delta u = f(u)+h & \text{in } \mathcal{O},\\ u=0, & \text{on } \partial\mathcal{O}.\end{cases} \tag{3.1}$$

Since we are dealing with stationary solutions, the assumption imposed on function h does not depend on time, i.e., $h\in V^*$. The solutions to (3.1) are the so called steady-state solutions or equilibria and the formal definition is the following.

Definition 3.1. *A stationary or steady-state solution to problem (3.1) (also called equilibrium) is a function $u^*\in V\cap L^p(\mathcal{O})$ which fulfills*

$$a(l(u^*))((u^*,v)) = (f(u^*),v)+ < h,v >, \qquad \forall v\in V\cap L^p(\mathcal{O}),$$

or, in other words, is a solution of the elliptic equation,

$$a(l(u^*))\Delta u^* = f(u^*) + h, \quad \text{in} \quad V^* + L^q(\mathcal{O}). \tag{3.2}$$

Observe that a steady-state solution u^* to problem (3.1) can only be solution to the stochastic problem (1.1) (with $h(t) = h \in V^*$) if $g(t, u^*) = 0$ for all $t \in [\tau, +\infty)$, which is a very particular situation. Thus, our main interest is to study how the solutions to stochastic problem (1.1) behave around the equilibria of the deterministic problem (3.1). In this way, to establish some sufficient conditions ensuring the exponential decay of solutions to (1.1) towards some solutions of (3.1), we assume the existence of stationary solutions to (3.1) (see, for instance, [18, Theorem 3.8]). Notice that, when function f is more general, namely, which satisfies the conditions (1.3)-(1.4), it is not easy to argue. Therefore, in order to prove the existence of at least one nontrivial stationary solution to problem (3.1), the authors in [18] studied one particular, but very interesting case when $f : [0, 1] \to \mathbb{R}$ is given by $f(s) = s - s^3$, for $s \in [0, 1]$, the arguments were based on a fixed point theorem. Whereas, considering again the general form function f and under new suitable assumptions, the authors in [18] showed that any stationary solution is positive provided its existence is guaranteed [18, Chapter 3.2].

In the sequel, our goal is to establish sufficient conditions to prove exponential decay of variational solutions in mean square.

Definition 3.2. *A solution u to (1.1) is said to converge to (or to decay to) $u^* \in V \cap L^p(\mathcal{O})$ exponentially in mean square, if there exist $\alpha > 0$ and $M = M(u_0) > 0$ such that*

$$\mathbb{E}|u(t) - u^*|^2 \leq M e^{-\alpha(t-\tau)}, \qquad \forall t \geq \tau.$$

Definition 3.3. *A solution u to equation (1.1) is said to converge exponentially to $u^* \in V \cap L^p(\mathcal{O})$ almost surely, if there exists $\gamma > 0$ such that*

$$\limsup_{t\to+\infty} \frac{1}{t} \log |u(t) - u^*| \leq -\gamma, \qquad \textit{almost surely.}$$

In order to prove the exponential stability results, the following condition as in [6] is considered. Assume there exists a steady-state solution u^* of (3.1) such that g satisfies

g4) $|g(t, u)|^2 \leq \beta(t) + (\xi + \delta(t))|u - u^*|^2$, for all $u \in H$, where ξ is a positive constant, $\beta(t)$, $\delta(t)$ are nonnegative integrable functions, such that there exist real numbers $\theta > \alpha$, $M_\beta \geq 1$ and $M_\delta \geq 1$ with

$$\beta(t) \leq M_\beta e^{-\theta t} \quad \text{and} \quad \delta(t) \leq M_\delta e^{-\theta t}, \qquad \forall t \geq 0.$$

We will present in the next theorem that, any variational solution to (1.1) converges exponentially to u^* in mean square, showing that u^* is the only relevant stationary solution for the stochastic system. No matter how many steady-state solutions (3.1) may have, this u^* is attracting in mean square any other solution of the stochastic problem.

Theorem 3.4. *Assume (1.2)-(1.4) and g4) hold with*

$$(2\eta + \xi)\lambda_1^{-1} + 2L_a|l|\|u^*\|\lambda_1^{-1/2} < m, \tag{3.3}$$

where $a(\cdot)$ is supposed to be globally Lipschitz, the Lipschitz constant is still denoted the same by L_a. Then:

(i) Any variational solution $u(\cdot)$ of problem (1.1) converges to the stationary solution u^* of (3.1) exponentially in the mean square. That is, there exist $\alpha > 0$ and $M = M(u_0)$ such that,

$$\mathbb{E}|u(t) - u^*|^2 \leq M e^{-\alpha(t-\tau)}, \qquad t \geq \tau;$$

(ii) Any variational solution $u(t)$ of problem (1.1) converges to the stationary solution u^* of (3.1) almost surely exponentially.

Proof. (i) Since $(2\eta+\xi)\lambda_1^{-1}+2L_a|l|\|u^*\|\lambda_1^{-1/2}<m$, we can choose $0<\alpha<\theta$ such that,

$$(\alpha+2\eta+\xi)\lambda_1^{-1}+2L_a|l|\|u^*\|\lambda_1^{-1/2}-2m<0.$$

By applying the Itô formula to $e^{\alpha t}|u(t)-u^*|^2$ and taking expectation, we obtain

$$\begin{aligned} e^{\alpha t}\mathbb{E}|u(t)-u^*|^2 &= e^{\alpha\tau}\mathbb{E}|u_0-u^*|^2+\alpha\mathbb{E}\int_\tau^t e^{\alpha s}|u(s)-u^*|^2ds \\ &\quad +2\mathbb{E}\int_\tau^t e^{\alpha s}<a(l(u))\Delta u(s),u(s)-u^*>ds+2\mathbb{E}\int_\tau^t e^{\alpha s}(f(u(s)),u(s)-u^*)ds \\ &\quad +2\mathbb{E}\int_\tau^t e^{\alpha s}<h,u(s)-u^*>ds+\mathbb{E}\int_\tau^t e^{\alpha s}|g(s,u(s))|^2ds. \end{aligned}$$

As u^* is the stationary solution to problem (3.1), we have

$$-\mathbb{E}\int_\tau^t e^{\alpha s}<a(l(u^*))\Delta u^*,u(s)-u^*>ds=\mathbb{E}\int_\tau^t e^{\alpha s}(f(u^*),u(s)-u^*)ds+\mathbb{E}\int_\tau^t e^{\alpha s}<h,u(s)-u^*>ds.$$

It follows from the two above equalities that,

$$\begin{aligned} e^{\alpha t}\mathbb{E}|u(t)-u^*|^2 &= e^{\alpha\tau}\mathbb{E}|u_0-u^*|^2+\alpha\mathbb{E}\int_\tau^t e^{\alpha s}|u(s)-u^*|^2ds \\ &\quad +2\mathbb{E}\int_\tau^t e^{\alpha s}<a(l(u(s)))\Delta u(s)-a(l(u^*))\Delta u^*,u(s)-u^*>ds \\ &\quad +2\mathbb{E}\int_\tau^t e^{\alpha s}(f(u(s))-f(u^*),u(s)-u^*)ds+\mathbb{E}\int_\tau^t e^{\alpha s}|g(s,u(s))|^2ds. \end{aligned}$$

By means of assumptions (1.2), (1.4) and g4), together with the fact that a is Lipschitz and the Poincaré inequality, we derive

$$\begin{aligned} e^{\alpha t}\mathbb{E}|u(t)-u^*|^2 &\le e^{\alpha\tau}\mathbb{E}|u_0-u^*|^2+\mathbb{E}\int_\tau^t e^{\alpha s}\left(\beta(s)+\delta(s)|u(s)-u^*|^2\right)ds \\ &\quad +\Big((\alpha+2\eta+\xi)\,\lambda_1^{-1}+2L_a|l|\|u^*\|\lambda_1^{-1/2}-2m\Big)\,\mathbb{E}\int_\tau^t e^{\alpha s}\|u(s)-u^*\|^2ds. \end{aligned} \tag{3.4}$$

Thanks to the fact that $\Big((\alpha+2\eta+\xi)\lambda_1^{-1}+2L_a|l|\|u^*\|\lambda_1^{-1/2}-2m\Big)<0$, the last term of (3.4) is negative, we obtain

$$e^{\alpha t}\mathbb{E}|u(t)-u^*|^2\le e^{\alpha\tau}\mathbb{E}|u_0-u^*|^2+\int_\tau^t e^{\alpha s}\beta(s)ds+\int_\tau^t \delta(s)e^{\alpha s}\mathbb{E}|u(s)-u^*|^2ds.$$

Since $\theta>\alpha$, applying the Gronwall lemma to the above inequality, the result (i) is proved.

(ii) We now move to the second assertion, let N be a natural number, by applying the Itô formula to $|u(t)-u^*|^2$ and using fact that u^* is a steady-state solution, it follows that

$$\begin{aligned} |u(t)-u^*|^2 &= |u(N)-u^*|^2+2\int_N^t<a(l(u(s)))\Delta u(s)-a(l(u^*))\Delta u^*,u(s)-u^*>ds \\ &\quad +2\int_N^t(f(u(s))-f(u^*),u(s)-u^*)ds \\ &\quad +2\int_N^t(g(s,u(s)),u(s)-u^*)dW(s)+\int_N^t|g(s,u(s))|^2ds. \end{aligned}$$

Therefore, by (1.2)-(1.3), we have

$$\begin{aligned}
&|u(t)-u^*|^2 + 2m\int_N^t \|u(s)-u^*\|^2 ds \\
&\le 2\int_N^t | < (a(l(u(s))) - a(l(u^*)))\Delta u^*, u(s)-u^* > | ds \\
&\quad + |u(N)-u^*|^2 + 2\eta \int_N^t |u(s)-u^*|^2 ds \\
&\quad + 2\left|\int_N^t (g(s,u(s)), u(s)-u^*) dW(s)\right| + \int_N^t |g(s,u(s))|^2 ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{N\le t\le N+1} |u(t)-u^*|^2\right] + 2m\mathbb{E}\int_N^{N+1} \|u(s)-u^*\|^2 ds \\
&\le 4\mathbb{E}\left[\int_N^{N+1} | < (a(l(u(s))) - a(l(u^*)))\Delta u^*, u(s)-u^* > | ds\right] \\
&\quad + 2\mathbb{E}|u(N)-u^*|^2 + 4\eta\mathbb{E}\int_N^{N+1} |u(s)-u^*|^2 ds \\
&\quad + 4\mathbb{E}\left[\sup_{N\le t\le N+1}\left|\int_N^t (g(s,u(s)), u(s)-u^*) dW(s)\right|\right] + 2\mathbb{E}\left[\int_N^{N+1} |g(s,u(s))|^2 ds\right].
\end{aligned} \tag{3.5}$$

With the help of the Burkholder-Davis-Gundy and Young inequalities, we have

$$\begin{aligned}
&4\mathbb{E}\left[\sup_{N\le t\le N+1}\left|\int_N^t (g(s,u(s)), u(s)-u^*) dW(s)\right|\right] \\
&\le 4C_2\mathbb{E}\left[\int_N^{N+1} |g(s,u(s))|^2 |u(s)-u^*|^2 ds\right]^{\frac{1}{2}} \\
&\le 4C_2\mathbb{E}\left[\sup_{N\le t\le N+1} |u(t)-u^*|^2 \int_N^{N+1} |g(s,u(s))|^2 ds\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{N\le t\le N+1} |u(s)-u^*|^2\right] + 8C_2^2\mathbb{E}\left[\int_N^{N+1} |g(s,u(s))|^2 ds\right].
\end{aligned} \tag{3.6}$$

Proceeding now as in the proof of the previous theorem and substituting (3.6) into (3.5), it yields

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{N\le t\le N+1} |u(t)-u^*|^2\right] \\
&\le 2\mathbb{E}|u(N)-u^*|^2 + \left(-2m + 4L_a|l|\|u^*\|\lambda_1^{-1/2} + 4\eta\lambda_1^{-1}\right)\mathbb{E}\int_N^{N+1} \|u(s)-u^*\|^2 ds \\
&\quad + (8C_2^2+2)\mathbb{E}\int_N^{N+1} \left(\beta(s) + (\xi+\delta(s))|u(s)-u^*|^2\right) ds \\
&\le 2\mathbb{E}|u(N)-u^*|^2 + (8C_2^2+2)\int_N^{N+1} \left(\beta(s) + (\xi+\delta(s))\mathbb{E}|u(s)-u^*|^2\right) ds
\end{aligned}$$

□

The last step of above inequality is true thanks to assumption $(2\eta+\xi)\lambda_1^{-1} + 2L_a|l|\|u^*\|\lambda_1^{-1/2} < m$. Moreover, it follows from condition g4) that $\beta(t) \le M_\beta e^{-\theta t}$ and $\delta(t) \le M_\delta e^{-\theta t}$, $0 < \alpha < \theta$, $M_\beta \ge 1$ and

$M_\delta \geq 1$. Thus, taking into account the exponential decay in mean square stated in Theorem 3.4, there exists $M := M(\tau, u_0) > 0$, such that

$$\mathbb{E}\left[\sup_{N \leq t \leq N+1} |u(t) - u^*|^2\right] \leq M e^{-\alpha N}.$$

The proof is completed by using the Borel-Cantelli lemma (see [8] for a detailed explanation).

Remark 3.5. Notice that it is enough to assume that $(2\eta+\xi)\lambda_1^{-1} + 2L_a|l|\|u^*\|\lambda_1^{-1/2} < 2m$ in Theorem 3.4 instead of $(2\eta+\xi)\lambda_1^{-1} + 2L_a|l|\|u^*\|\lambda_1^{-1/2} < m$. However, in the next theorem it will be necessary to impose the latter, so we prefer to impose this one in both theorems.

We conclude this section with a result on the exponential stability of the steady-state solution in mean square, when this becomes also a solution of the stochastic equation.

THEOREM 3.6. *Assume (1.2)-(1.4) hold with*

$$2L_a|l|\|u^*\|\lambda_1^{-1/2} + 2\eta\lambda_1^{-1} + L_g\lambda_1^{-1} < 2m. \tag{3.7}$$

where $a(\cdot)$ is supposed to be globally Lipschitz, the Lipschitz constant is still denoted the same by L_a. Additionally, assume the nonlinear stochastic term g fulfills g3), and $g(t, u^*) = 0$ for all $t \geq \tau$. Then the solution to problem (1.1) converges to the stationary solution of (3.1) u^* exponentially in the mean square. Namely, there exists a real number $\gamma > 0$, such that

$$\mathbb{E}|u(t) - u^*|^2 \leq \mathbb{E}|u_0 - u^*|^2 e^{-\gamma(t-\tau)}, \qquad \forall t \geq \tau.$$

Proof. Since u^* is the stationary solution of (3.1), combined with (1.1), we derive

$$\begin{aligned} u(t) - u^* = u_0 - u^* &+ \int_\tau^t (a(l(u(s)))\Delta u(s) - a(l(u^*))\,\Delta u^*)ds \\ &+ \int_\tau^t (f(u(s)) - f(u^*))ds + \int_\tau^t (g(s, u(s)) - g(s, u^*))dW(s). \end{aligned}$$

Thanks to (3.7), we can choose a sufficiently small $\gamma > 0$, such that

$$\gamma\lambda_1^{-1} + 2L_a|l|\|u^*\|\lambda_1^{-1/2} + 2\eta\lambda_1^{-1} + L_g\lambda_1^{-1} - 2m < 0.$$

Applying now the Itô formula to $e^{\gamma t}|u(t) - u^*|^2$, taking expectation and using the same arguments as in Theorem 3.4, we obtain

$$\begin{aligned} e^{\gamma t}\mathbb{E}|u(t) - u^*|^2 &= e^{\gamma\tau}\mathbb{E}|u_0 - u^*|^2 + \gamma\mathbb{E}\int_\tau^t |u(s) - u^*|^2 ds \\ &\quad + 2\mathbb{E}\int_\tau^t e^{\gamma s} < a(l(u(s)))\Delta u(s) - a(l(u^*))\Delta u^*, u(s) - u^* > ds \\ &\quad + 2\mathbb{E}\int_\tau^t e^{\gamma s}(f(u(s)) - f(u^*), u(s) - u^*)ds + \mathbb{E}\int_\tau^t e^{\gamma s}|g(s, u(s)) - g(s, u^*)|^2 ds \\ &\leq e^{\gamma\tau}\mathbb{E}|u_0 - u^*|^2 + \gamma\lambda_1^{-1}\mathbb{E}\int_\tau^t e^{\gamma s}\|u(s) - u^*\|^2 ds \\ &\quad + \left(-2m + 2L_a|l|\|u^*\|\lambda_1^{-1/2} + 2\eta\lambda_1^{-1} + L_g\lambda_1^{-1}\right)\mathbb{E}\int_\tau^t e^{\gamma s}\|u(s) - u^*\|^2 ds. \end{aligned}$$

Due to the choice of γ, the result follows immediately. ∎

4. Attractors of nonlocal stochastic PDEs driven by colored noise.

Our aim now is to study the existence of attractors for the solution of problem (1.1). However, as it is well known, the theory of random dynamical systems has only been applied successfully to problems modeled by partial differential equations when the noise possesses a particular form: additive or multiplicative noise. These two cases have already been analyzed in [33]. Recently, B. X. Wang and his collaborators (see [17, 15, 22]) have been using an idea to approximate the nonlinear noise by a stochastic process (called colored noise), which basically is a Wong-Zakai approximation of the derivative of the Wiener process, providing a rigorous approximation of the cases with additive and multiplicative noise (as we explained in the Introduction). This is why, in this section, we study the long time behavior of the following non-autonomous nonlocal partial differential equations driven by colored noise,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t) + g(t,u)\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty), \\ u = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u(x,\tau) = u_\tau(x), & \text{in } \mathcal{O}, \end{cases} \tag{4.1}$$

where $\zeta_\delta(\theta_t\omega)$ is the colored noise with correlation time $\delta > 0$, functions a, f, h and g fulfill the same assumptions as in Section 2.

4.1. Cocycles for nonlocal PDEs.

To describe the global long time behavior of problem (4.1), it is necessary to establish the existence of a continuous non-autonomous cocycle for (4.1). Let us first recall some notions, definitions and lemmas which furnish the essential tools used throughout this section ([15, 17, 29, 31]).

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a standard probability space, where $\Omega = C_0(\mathbb{R},\mathbb{R}) := \{\omega\in C(\mathbb{R},\mathbb{R}) : \omega(0)=0\}$ with the open compact topology, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure on $(\Omega,\mathcal{F})$. In what follows, we will consider the Wiener shift $\{\theta_t\}_{t\in\mathbb{R}}$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ by

$$\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t), \qquad \text{for all } \omega\in\Omega, \ t\in\mathbb{R}.$$

It is known that $\mathbb{P}$ is an ergodic invariant measure for $\{\theta_t\}_{t\in\mathbb{R}}$, and the quadruple $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ forms a metric dynamical system (see [1]).

In the sequel, we use (X,d) to denote a complete separable metric space. If A and B are two nonempty subsets of X, then we use $dist_X(A,B) := \sup_{a\in A}\inf_{b\in B} d(a,b)$ to denote their Hausdorff semidistance.

Definition 4.1. *([28, Definition 2.6]) Let $D : \mathbb{R}\times\Omega \to 2^X$ be a set-valued mapping with closed nonempty images. We say D is measurable with respect to $\mathcal{F}$ in Ω, if the mapping $\omega\in\Omega \to d(x, D(\tau,\omega))$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable for every fixed $x\in X$ and $\tau\in\mathbb{R}$.*

Definition 4.2. *([28, Definition 2.7]) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $B = \{B(\tau,\omega) : \tau\in\mathbb{R}, \omega\in\Omega\}\in\mathcal{D}$. Then B is called a $\mathcal{D}$-pullback absorbing set for Φ, if for all $\tau\in\mathbb{R}$, $\omega\in\Omega$ and for every $B\in\mathcal{D}$, there exists $T = T(B,\tau,\omega) > 0$ such that*

$$\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)) \subset B(\tau,\omega) \qquad \text{for all } t\geq T.$$

Definition 4.3. *([28, Definition 2.8]) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$, the sequence*

$$\{\Phi(t_n,\tau-t_n,\theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \ \text{has a convergent subsequence in } X,$$

whenever $t_n\to\infty$ and $x_n\in D(\tau-t_n,\theta_{-t_n}\omega)$ with $\{D(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$.

Definition 4.4. *([28, Definition 2.9]) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (i)-(iii) are fulfilled:*

(i) $\mathcal{A}$ is measurable in the sense of Definition 4.1, and $\mathcal{A}(\tau,\omega)$ is compact for all $\tau\in\mathbb{R}$ and $\omega\in\Omega$.

(ii) $\mathcal{A}$ is invariant, that is, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = A(\tau+t,\theta_t\omega), \quad \forall t\geq 0.$$

(*iii*) *$\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, for every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for every $\tau \in \mathbb{R}$, $\omega \in \Omega$,*

$$\lim_{t\to\infty} d(\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0.$$

We have introduced all required definitions of stochastic dynamical systems, which later on will allow us to define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ for equation (4.1), such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau; \tau, \theta_{-\tau}\omega, u_\tau), \tag{4.2}$$

where $u(\cdot; \tau, \omega, u_\tau)$ denotes the solution to (4.1) which will be proved to exist in Section 4.3. Thus, Φ will be a continuous cocycle on H over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Moreover, let $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a tempered family of bounded nonempty subsets of H, that is, for every $\gamma > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t\to-\infty} e^{\gamma t}|D(\tau + t, \theta_t\omega)| = 0, \tag{4.3}$$

where $|D| = \sup_{u\in D} |u|$. Throughout this section, we will use $\mathcal{D}$ to denote the collection of all tempered families of bounded nonempty subsets of H, i.e.,

$$\mathcal{D} = \{D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satifies } (4.3)\}. \tag{4.4}$$

Remark 4.5. Although the cocycle generated by (4.1) depends on the parameter δ, we will omit this dependence in this section since it will be fixed from the beginning. Hence, we will use Φ instead of using the notation Φ_δ.

4.2. Properties of white and colored noises. We recall some known results for the Wiener process $W(t,\omega) = \omega(t)$ in [1] and the colored noise $\zeta_\delta(\theta_t\omega)$ in [17, 15], since they play important roles in the proof of the main theorems.

Lemma 4.6. *Let the correlation time $\delta \in (0, 1]$. There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset (still denoted by) Ω of full measure, such that for all $\omega \in \Omega$,*

(*i*)

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0; \tag{4.5}$$

(*ii*) *The mapping*

$$(t,\omega) \to \zeta_\delta(\theta_t\omega) = -\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}\theta_t\omega(s)ds \tag{4.6}$$

is a stationary solution (also called an Ornstein-Uhlenbeck process or a colored noise) of the one-dimensional stochastic differential equation $d\zeta_\delta + \frac{1}{\delta}\zeta_\delta dt = \frac{1}{\delta}dW$ with continuous trajectories, satisfying

$$\lim_{t\to\pm\infty} \frac{\zeta_\delta(\theta_t\omega)}{t} = 0 \quad \textit{for all} \ \ 0 < \delta \le 1, \tag{4.7}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \zeta_\delta(\theta_s\omega)ds = \mathbb{E}\zeta_\delta = 0, \quad \textit{uniformly for} \ \ 0 < \delta \le 1; \tag{4.8}$$

(*iii*) *For arbitrary $T > 0$, $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$, such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,*

$$\left|\int_0^t \zeta_\delta(\theta_s\omega)ds - \omega(t)\right| < \varepsilon. \tag{4.9}$$

Remark 4.7. Notice that, from (4.9), we can derive that there exist $\delta_0 = \delta_0(\tau, \omega, T)$ and $\tilde{c} = \tilde{c}(\tau, \omega, T) > 0$ such that, for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau + T]$,

$$\left|\int_0^t \zeta_\delta(\theta_s\omega)ds\right| \le \tilde{c}. \tag{4.10}$$

4.3. Well-posedness of problem (4.1). We are now in a position to show the existence and uniqueness of solution to equation (4.1) in the following sense.

DEFINITION 4.8. *A weak solution to problem* (4.1) *is a mapping* $u(\cdot;\tau,\omega,u_\tau):[\tau,T)\to H$, *for all* $T>\tau$ *with* $u(\tau)=u_\tau$, *satisfying for any* $\tau\in\mathbb{R}$, $\omega\in\Omega$,

$$u(\cdot;\tau,\omega,u_\tau)\in C(\tau,T;H)\cap L^2(\tau,T;V)\cap L^p(\tau,T;L^p(\mathcal{O})).$$

Moreover, for every $t>\tau$ *and* $v\in V+L^p(\mathcal{O})$,

$$\begin{aligned}(u,v)&=(u_\tau,v)+\int_\tau^t a(l(u))((u,v))ds+\int_\tau^t (f(u),v)ds\\&\quad+\int_\tau^t <h,v>ds+\int_\tau^t (g(s,u(s))\zeta_\delta(\theta_s\omega),v)ds.\end{aligned}$$

Note that, if we denote by A *the operator* $-\Delta$ *with homogeneous boundary condition, then the above equality can be written as*

$$\frac{du}{dt}+a(l(u))Au=f(u)+h(t)+g(t,u)\zeta_\delta(\theta_t\omega),\qquad in\quad V^*+L^q(\mathcal{O}).$$

THEOREM 4.9. *Assume that function* a *is locally Lipschitz and satisfies* (1.2), $f\in C(\mathbb{R})$ *fulfills* (1.3)-(1.4), $h\in L^2_{loc}(\mathbb{R}^+;V^*)$ *and* $l\in L^2(\mathcal{O})$. *Additionally, function* g *satisfies g1)-g3). Then, for each initial datum* $u_0\in H$, *there exists a unique weak solution to problem* (4.1) *in the sense of Definition 4.8. Moreover, this solution behaves continuously in* H *with respect to the initial values.*

Proof. Since equation (4.1) can be viewed as a deterministic problem parametrized by ω (cf. [22]), for every $T>\tau$ and $\omega\in\Omega$, we can prove (4.1) has a unique solution,

$$u(\cdot;\tau,\omega,u_\tau)\in C(\tau,T;H)\cap L^2(\tau,T;V)\cap L^p(\tau,T;L^p(\mathcal{O})),$$

by applying the Galerkin method and energy estimations [18, Chapter 3, Theorem 3.3]. □

In this subsection, we first derive uniform estimations on the solution of (4.1) and then prove $\mathcal{D}$-pullback asymptotic compactness by using the idea introduced by Ball in [2]. To this end, we need the following assumptions:

$h1)$ Suppose that

$$\int_{-\infty}^{\tau} e^{m\lambda_1 s}\|h(s)\|_*^2ds<\infty,\qquad\forall\tau\in\mathbb{R}.$$

For the existence of tempered random attractors, we need the assumption below:

$h2)$ For every $\gamma>0$, it holds

$$\lim_{t\to-\infty}e^{\gamma t}\int_{-\infty}^0 e^{m\lambda_1 s}\|h(s+t)\|_*^2ds=0.$$

It is worth stressing that h1) and h2) do not require $h(t)$ is bounded in V^* as $t\to\pm\infty$.

LEMMA 4.10. *Assume conditions of Theorem 4.9 and h1) hold. Then, for every* $\delta\in(0,1]$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T=T(\tau,\omega,\delta,D)>0$ *such that for all* $t\geq T$ *and* $\sigma\geq\tau-t$, *the solution of problem* (4.1) *satisfies,*

$$\begin{aligned}&|u(\sigma;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\leq e^{-m\lambda_1(\sigma-\tau)}\\&\qquad+\int_{-\infty}^{\sigma-\tau}e^{m\lambda_1(s-\sigma+t)}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_s\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds,\end{aligned}$$

$$\int_{\tau-t}^{\tau} e^{m\lambda_1(s-\tau)}\|u(s;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds \\ \le \frac{2}{m}+\frac{2}{m}\int_{-\infty}^{0} e^{m\lambda_1 s}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_s\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds,$$

and

$$\int_{\tau-t}^{\tau} e^{m\lambda_1(s-\tau)}|u(s;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|_p^p ds \\ \le \frac{1}{\alpha_2}+\frac{1}{\alpha_2}\int_{-\infty}^{0} e^{m\lambda_1 s}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_s\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds,$$

where $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$, and c is a constant which depends on α_2, p and L_g but not on δ.

Proof. Multiplying by $u(\cdot)$ on both sides of (4.1) in H, we derive

$$\frac{d}{dt}|u|^2+2a(l(u))\|u\|^2=2(f(u),u)+2<h(t),u>+2\zeta_\delta(\theta_t\omega)(g(t,u),u). \tag{4.11}$$

It follows from (1.4) that

$$2(f(u),u)\le 2\int_{\mathcal{O}}(\kappa-\alpha_2|u|^p)\,dx\le 2\kappa|\mathcal{O}|-2\alpha_2|u|_p^p. \tag{4.12}$$

By the Young inequality, we have

$$2<h(t),u>\le\frac{2}{m}\|h(t)\|_*^2+\frac{m}{2}\|u\|^2. \tag{4.13}$$

Conditions g2)-g3) and the Young inequality yield that,

$$\begin{aligned} 2|\zeta_\delta(\theta_t\omega)(g(t,u),u)| &\le 2L_g^{1/2}|\zeta_\delta(\theta_t\omega)||u|^2 \\ &= 2L_g^{1/2}\int_{\mathcal{O}}|\zeta_\delta(\theta_t\omega)||u|^2dx \\ &\le \alpha_2\int_{\mathcal{O}}|u|^pdx+c|\mathcal{O}||\zeta_\delta(\theta_t\omega)|^{p/(p-2)}, \end{aligned} \tag{4.14}$$

where c is a constant depending on α_2, p and L_g.

Substituting (4.12)-(4.14) into (4.11), together with (1.2) and the Poincaré inequality, we have

$$\frac{d}{dt}|u|^2+m\lambda_1|u|^2+\frac{m}{2}\|u\|^2+\alpha_2|u|_p^p\le\frac{2}{m}\|h(t)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_t\omega)|^{p/(p-2)}\right)|\mathcal{O}|.$$

By straightforward computations with $u(\sigma;\tau-t,\theta_{-(\tau-t)}\omega,u_{\tau-t})$ and replacing ω by $\theta_{-t}\omega$, we obtain,

(4.15)
$$\begin{aligned}
|u(\sigma;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2 &+ \frac{m}{2}\int_{\tau-t}^{\sigma} e^{m\lambda_1(s-\sigma)}\|u(s;\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 ds\\
&+\alpha_2\int_{\tau-t}^{\sigma} e^{m\lambda_1(s-\sigma)}|u(s;\tau-t,\theta_{-\tau}\omega,u_{\tau-t}|_p^p ds\\
&\le e^{-m\lambda_1(\sigma-\tau+t)}|u_{\tau-t}|^2\\
&+\int_{\tau-t}^{\sigma} e^{m\lambda_1(s-\sigma)}\left(\frac{2}{m}\|h(s)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_s\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds\\
&\le e^{-m\lambda_1(\sigma-\tau+t)}|u_{\tau-t}|^2\\
&+\int_{-t}^{\sigma-\tau} e^{m\lambda_1(s-\sigma+\tau)}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_{s+\tau}\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds.
\end{aligned}$$

On the one hand, it follows from h1) that,

$$\int_{-\infty}^{\sigma-\tau} e^{m\lambda_1(s-\sigma+\tau)}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_{s+\tau}\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds<\infty. \tag{4.16}$$

On the other hand, as $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)\in\mathcal{D}$, we deduce that

$$e^{-m\lambda_1 t}|u_{\tau-t}|^2\le e^{-m\lambda_1 t}|D(\tau-t,\theta_{-t}\omega)|^2\to 0,\quad \text{as}\ \ t\to\infty.$$

Thus, there exists $T=T(\tau,\omega,D)>0$, such that for all $t\ge T$,

$$e^{-m\lambda_1(\sigma-\tau+t)}|u_{\tau-t}|^2\le 1,$$

which, along with (4.15) and (4.16), completes the proof. ∎

Corollary 4.11. *Assume the conditions of Theorem 4.9 and h2) hold. Then the continuous cocycle Φ associated with problem (4.1) possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ in H. Namely, for any given $\delta\in(0,1]$, every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we denote*

$$K(\tau,\omega)=\{u\in H:|u|^2\le R(\tau,\omega)\},$$

where

$$R(\tau,\omega)=1+\int_{-\infty}^{0} e^{m\lambda_1 s}\left(\frac{2}{m}\|h(s+\tau)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_{s+\tau}\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds.$$

Proof. Since for every $\tau\in\mathbb{R}$, $R(\tau,\cdot):\Omega\to\mathbb{R}$ is $(\mathcal{F},\mathcal{B})$-measurable, we know that $K(\tau,\cdot):\Omega\to 2^H$ is a measurable set-valued mapping. Also, it follows from Lemma 4.10 that for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D\in\mathcal{D}$, there exists $T=T(\tau,\omega,D)>0$, such that for all $t\ge T$,

$$\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))=u(\tau;\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-t}\omega))\subset K(\tau,\omega).$$

Therefore, to finish this proof, it only remains to show K belongs to $\mathcal{D}$. Let γ be an arbitrary positive number, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we have that

$$\begin{aligned}
&\lim_{t\to-\infty} e^{\gamma t}|K(\tau+t,\theta_t\omega)|=\lim_{t\to-\infty} e^{\gamma t}R(\tau+t,\theta_t\omega)\\
&=\lim_{t\to-\infty} e^{\gamma t}\left(1+\int_{-\infty}^{0} e^{m\lambda_1 s}\left(\frac{2}{m}\|h(s+\tau+t)\|_*^2+\left(2\kappa+c|\zeta_\delta(\theta_{s+\tau+t}\omega)|^{p/(p-2)}\right)|\mathcal{O}|\right)ds\right)=0,
\end{aligned}$$

thanks to h2). The desired result is proved. ∎

Next, let us discuss the asymptotic compactness of the continuous cocycle Φ related to problem (4.1). Indeed, we prove that the sequence of solutions of (4.1) is compact in H.

LEMMA 4.12. *Under assumptions of Lemma 4.10, the continuous cocycle Φ associated with problem (4.1) is $\mathcal{D}$-pullback asymptotically compact in H. That is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and $t_n \to \infty$, the initial data $u_{\tau,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{\tau,n}) = u(\tau;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})\}$ (solutions to problem (4.1)) has a convergence subsequence in H.*

Proof. Let $\{u_{\tau,n}\}_{n=1}^{\infty}$ be a sequence in $D(\tau - t_n, \theta_{-t_n}\omega)$, Lemma 4.10 implies that there exists $T := T(\tau,\omega,D) > 0$, such that for all $t_n > T$, we have

$$\{u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}) \ \text{ is bounded in } \ L^{\infty}(\tau - T, \tau; H) \cap L^2(\tau - T, \tau; V) \cap L^p(\tau - T, \tau; L^p(\mathcal{O})). \tag{4.17}$$

On the one hand, making use of (1.5) and (4.17), we obtain

$$\{f(u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}))\} \ \text{ is bounded in } \ L^q(\tau - T, \tau; L^q(\mathcal{O})). \tag{4.18}$$

In addition, it follows from conditions g2)-g3) that

$$\{g(\cdot, u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}))\} \ \text{ is bounded in } \ L^2(\tau - T, \tau; H). \tag{4.19}$$

On the other hand, by (1.2) and (4.17), we have

$$\begin{aligned}
&\int_{\tau-T}^{\tau} |a(l(u(s;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})))|^2 \| - \Delta u(s;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})\|_*^2 ds \\
&\le \widetilde{m}^2 C \int_{\tau-T}^{\tau} \|u(s;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})\|^2 ds,
\end{aligned}$$

which implies that

$$a(l(u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})))\Delta u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}) \ \text{ is bounded in } \ L^2(\tau - T, \tau; V^*). \tag{4.20}$$

Consequently, it follows from (4.18)-(4.20) that

$$\left\{\frac{d}{dt} u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})\right\} \in L^2(\tau - T, \tau; V^*) + L^q(\tau - T, \tau; L^q(\mathcal{O})) + L^2(\tau - T, \tau; H). \tag{4.21}$$

Since the embedding $V \hookrightarrow H$ is compact, by (4.17), (4.21) and Aubin-Lions compactness Lemma, we infer that there exists $u \in L^2(\tau - T, \tau; H)$ such that, up to a subsequence,

$$u(\cdot;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}) \to u \ \text{ strongly in } \ L^2(\tau - T, \tau; H). \tag{4.22}$$

Therefore, by choosing a further subsequence (still denoted the same), we obtain,

$$u(\tau - s;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}) \to u(\tau - s) \ \text{ strongly in } \ H \quad \text{for almost all} \quad s \in (0, T). \tag{4.23}$$

Since $0 < s < T$, by (4.23), there exists a constant $0 < T' < T$, such that, the convergence (4.22) is true for $s \in (\tau - T, \tau - T')$. Then by the continuity of solution with initial data in H, we obtain from (4.23) that

$$\begin{aligned}
u(\tau;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n}) &= u(\tau;\tau - s, \theta_{-\tau}\omega, u(\tau - s;\tau - t_n, \theta_{-\tau}\omega, u_{\tau,n})) \\
&\to u(\tau, \tau - s, \theta_{-\tau}\omega, u(\tau - s)),
\end{aligned}$$

which implies the continuous cocycle Φ associated with (4.1) is $\mathcal{D}$-pullback asymptotically compact in H. The proof is finished. ∎

As an immediate consequence of Lemma 4.12, we obtain the following $\mathcal{D}$-pullback asymptotic compactness of the continuous cocycle Φ associated with (4.1).

THEOREM 4.13. *Assume function a is locally Lipschitz and satisfies (1.2), $f \in C(\mathbb{R})$ fulfills (1.3)-(1.4), $h \in L^2_{loc}(\mathbb{R}^+; V^*)$ satisfies h1)-h2), and $l \in L^2(\mathcal{O})$. In addition, function g satisfies g1)-g3). Then the continuous cocycle Φ associated to problem (4.1) has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in H.*

Proof. The result follows from Definition 4.4 immediately combining Corollary 4.11 and Lemma 4.12, for more details, see [28, Proposition 2.10]. □

Remark 4.14. The results in this Section hold true if we impose a different set of assumptions on function g. Namely, assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that for all $t, s \in \mathbb{R}$,

$$|g(t,s)| \le d_1|s|^{r_1-1} + \psi_1(t), \tag{4.24}$$

$$\left|\frac{\partial g}{\partial s}(t,s)\right| \le d_2|s|^{r_1-2} + \psi_2(t), \tag{4.25}$$

where $2 \le r_1 < q_1$, d_1 and d_2 are nonnegative constants, $\psi_1 \in L^{p_1}_{loc}(\mathbb{R}; L^{p_1}(\mathcal{O}))$ and $\psi_2 \in L^{\infty}_{loc}(\mathbb{R}; L^{\infty}(\mathcal{O}))$ (p_1 is the conjugated number with q_1). Then, Theorem 4.13 holds true assuming that function g satisfies (4.24)-(4.25) instead of g1)-g3) (see [22] for a similar situation).

5. Convergence of random attractors for stochastic nonlocal PDEs with additive noise. As we mentioned before, since it is not known how to apply the theory of random dynamical systems to study the long time behavior of problem (1.1), we have applied an approximation of this problem in Section 4 by using colored noise and proved that the approximate problem possesses a random attractor. In the next two sections, we will consider two particular cases of equation (1.1) which have been analyzed already within the framework of random dynamical systems (see [33]). When the stochastic forcing term $g(t, u(t))$ in (1.1) is linear (such as $g(t, u) = \sigma u$, multiplicative noise) or independent on u (such as, $g(t, u) = \phi$, additive noise), the existence of random attractors to problem (1.1) can be constructed via performing a conjugation which transforms the stochastic equation into a random one. Therefore, a sensible question is: if we study long time behavior of problem (4.1) with additive colored noise or multiplicative colored noise, what is the relationship between problem (1.1) and problem (4.1) with additive/multiplicative noise when the parameter δ goes to zero? We will answer this question in the remaining parts of this paper.

To simplify the presentation, in the following lines we assume $h(t) = 0$, which means we will study the dynamics of the stochastic autonomous PDEs. Actually, the ideas to work on the stochastic non-autonomous PDEs are the same (as have been done in the previous sections). In [33, Section 4], the authors investigated the existence of random attractors of the following stochastic nonlocal PDEs driven by a white noise,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \phi\frac{dW(t)}{dt}, & \text{in } \mathcal{O} \times (\tau, \infty), \\ u = 0, & \text{on } \partial\mathcal{O} \times (\tau, \infty), \\ u(x, \tau) = u_0, & \text{in } \mathcal{O}, \end{cases} \tag{5.1}$$

where $\phi \in V \cap H^2(\mathcal{O})$, functions a and f satisfy conditions (1.2)-(1.4) with $p = 2$ and $\beta = C_f$, respectively. The main idea is to apply a conjugation given by a transformation involving an Ornstein-Uhlenbeck process: $v(t) = u(t) - \phi z^*(\theta_t\omega)$, which takes (5.1) into

$$\begin{aligned} \frac{\partial v}{\partial t} &= a(l(v) + z^*(\theta_t\omega)l(\phi))\Delta v(t) + f(v + \phi z^*(\theta_t\omega)) \\ &\quad + \phi z^*(\theta_t\omega) + a(l(v) + z^*(\theta_t\omega)l(\phi))z^*(\theta_t\omega)\Delta\phi. \end{aligned} \tag{5.2}$$

Motivated by [15], we now study the same problem but driven by a colored noise,

$$\begin{cases} \frac{\partial u_\delta}{\partial t} - a(l(u_\delta))\Delta u_\delta = f(u_\delta) + \phi\zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty), \\ u_\delta = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u_\delta(x,\tau) = u_{0,\delta}, & \text{in } \mathcal{O}. \end{cases} \tag{5.3}$$

We now transform (5.3) via the solution of the following random equation driven by colored noise,

$$\frac{dy_\delta}{dt} = -\eta y_\delta + \zeta_\delta(\theta_t\omega). \tag{5.4}$$

For almost all $\omega\in\Omega$, one special solution of (5.4) can be represented by

$$Y_\delta(t,\omega) = e^{-\eta t}\int_{-\infty}^{t} e^{\eta s}\zeta_\delta(\theta_s\omega)ds,$$

which, in fact, can be rewritten as $Y_\delta(t,\omega) = y_\delta(\theta_t\omega)$, where $y_\delta : \Omega\to\mathbb{R}$ is a well-defined random variable given by $y_\delta(\omega) := \int_{-\infty}^{0} e^{\eta s}\zeta_\delta(\theta_s\omega)ds$. Let us recall the properties of y_δ for later purpose.

Lemma 5.1. *([17, Lemma 3.2]) Let y_δ be the random variable defined above. Then the mapping*

$$(t,\omega)\to y_\delta(\theta_t\omega) = e^{-\eta t}\int_{-\infty}^{t} e^{\eta s}\zeta_\delta(\theta_s\omega)ds \tag{5.5}$$

is a stationary solution of (5.4) with continuous trajectories. In addition, $\mathbb{E}(y_\delta) = 0$ and for almost all $\omega\in\Omega$,

$$\lim_{\delta\to 0} y_\delta(\theta_t\omega) = z^*(\theta_t\omega) \quad \textit{uniformly on } [\tau,\tau+T] \textit{ with } \tau\in\mathbb{R},\ T>0; \tag{5.6}$$

$$\lim_{t\to\pm\infty}\frac{|y_\delta(\theta_t\omega)|}{|t|} = 0 \quad \textit{uniformly for } 0<\delta<\tilde{\eta}; \tag{5.7}$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t y_\delta(\theta_r\omega)dr = 0 \quad \textit{uniformly for } 0<\delta<\tilde{\eta}; \tag{5.8}$$

$$\lim_{\delta\to 0}\mathbb{E}(|y_\delta(\omega)|) = \mathbb{E}(|z^*(\omega)|), \tag{5.9}$$

where $\tilde{\eta} = \min\{1, \frac{1}{2\eta}\}$, $z^*(\omega)$ is the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation (see [33, Section 2]) given by

$$z^*(\omega) = -\eta\int_{-\infty}^{0} e^{\eta s}\omega(s)\,ds.$$

Remark 5.2. In this manuscript, in order to simplify the computations, we take $\eta = 1$ in equation (5.4), then the results of Lemma 5.1 are true for $\eta = 1$.

Now, define a new variable

$$v_\delta(t) = u_\delta(t) - \phi y_\delta(\theta_t\omega), \tag{5.10}$$

where we denote by $u_\delta(\cdot) = u_\delta(\cdot;\tau,\omega,u_{0,\delta})$ the solution of equation (5.3). It follows from (5.3) and (5.10) that

$$\begin{aligned} \frac{\partial v_\delta}{\partial t} &= a(l(v_\delta) + y_\delta(\theta_t\omega)l(\phi))\Delta v_\delta + f(v_\delta + \phi y_\delta(\theta_t\omega)) \\ &\quad + \phi y_\delta(\theta_t\omega) + a(l(v_\delta) + y_\delta(\theta_t\omega)l(\phi))y_\delta(\theta_t\omega)\Delta\phi, \end{aligned} \tag{5.11}$$

with initial value $v_\delta(\tau) = u_\delta(\tau) - \phi y_\delta(\theta_\tau\omega) := v_{0,\delta}$. In a similar way as [33, Theorem 7], we are able to prove that, problem (5.11) with initial value $v_{0,\delta} \in H$ and Dirichlet boundary condition possesses a unique weak solution,

$$v_\delta(\cdot;\tau,\omega,v_{0,\delta}) \in C(\tau,T;H) \cap L^2(\tau,T;V),$$

for every $T > \tau$. In addition, this solution is continuous with respect to the initial value $v_{0,\delta}$ in H. Furthermore, this weak solution is a strong solution, namely, for the initial value $v_{0,\delta} \in V \cap H^2(\mathcal{O})$,

$$v_\delta(\cdot;\tau,\omega,v_{0,\delta}) \in C(\tau,T;V) \cap L^2(\tau,T;V \cap H^2(\mathcal{O})).$$

Let us define a mapping $\Xi_\delta : \mathbb{R}^+ \times \Omega \times H \to H$ such that

$$\Xi_\delta(t,\omega,u_{0,\delta}) = v_\delta(t;0,\omega,v_{0,\delta}), \qquad \forall v_{0,\delta} \in H, \quad \forall \omega \in \Omega. \tag{5.12}$$

Thanks to the conjugation, there is a mapping $\Psi_\delta : \mathbb{R}^+ \times \Omega \times H \to H$ satisfying

$$\begin{aligned}\Psi_\delta(t,\omega,u_{0,\delta}) &= u_\delta(t;0,\omega,u_{0,\delta}) \\ &= v_\delta(t;0,\omega,u_{0,\delta} - \phi y_\delta(\omega)) + \phi y_\delta(\theta_t\omega), \qquad \forall u_{0,\delta} \in H, \quad \forall \omega \in \Omega.\end{aligned} \tag{5.13}$$

Theorem 5.3. *([33, Theorem 9]) Suppose that a is locally Lipschitz and fulfills (1.2), $f \in C(\mathbb{R})$ satisfies (1.3) and (1.5) with $p = 2$ and $\beta = C_f$, $\phi \in V \cap H^2(\mathcal{O})$ and $l \in L^2(\mathcal{O})$. Also, let $m\lambda_1 > 4C_f$. Then, there exists a random $\mathcal{D}_F$-attractor $\mathcal{A}(\omega)$ (where $\mathcal{D}_F$ is the universe of fixed bounded sets) for the dynamical system $\Psi(t,\omega,u_0)$. In addition, the $\mathcal{D}_F$-pullback absorbing set $B_0 = \{B_0(\omega) : \omega \in \Omega\} \in \mathcal{D}$ in H is given by*

$$B_0(\omega) = \{u \in H : |u|^2 \le \lambda_1^{-1} R_0(\omega)\}, \quad \text{for almost all} \quad \omega \in \Omega,$$

with

$$\begin{aligned}R_0(\omega) &= 2\|\phi\|^2|z^*(\omega)|^2 + \frac{8C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{4\lambda_1 C_f^2|\mathcal{O}|}{(m\lambda_1 - 4C_f)^2} \\ &\quad + \frac{4 + 2\lambda_1 C_f m + m\lambda_1 - 4C_f + 2C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} \\ &\quad + \left(4m^{-1} + 2\lambda_1 C_f\right)\int_{-\infty}^0 e^{(m\lambda_1 - 4C_f)t}\left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 dt \\ &\quad + 2\int_{-1}^0 e^{(m\lambda_1 - 4C_f)t}\left(\lambda_1 C_f|\mathcal{O}| + \left(C_f\lambda_1 + \lambda_1 C_f^{-1}\right)|z^*(\theta_t\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2\right)dt.\end{aligned}$$

Theorem 5.4. *Assume the conditions in Theorem 5.3 are true. Then, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, (5.3) has a random $\mathcal{D}_F$-attractor $\mathcal{A}_\delta(\omega)$ associated to the dynamical system $\Psi_\delta(t,\omega,u_{0,\delta})$. In addition, the $\mathcal{D}_F$-pullback absorbing set $B_\delta := \{B_\delta(\omega) : \omega \in \Omega\} \in \mathcal{D}$ in H is given by*

$$B_\delta(\omega) = \{u \in H : |u|^2 \le \lambda_1^{-1} R_\delta(\omega)\},$$

with

$$\begin{aligned}R_\delta(\omega) &= 2\|\phi\|^2|y_\delta(\omega)|^2 + \frac{8C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{4\lambda_1 C_f^2|\mathcal{O}|}{(m\lambda_1 - 4C_f)^2} \\ &\quad + \frac{4 + 2\lambda_1 C_f m + m\lambda_1 - 4C_f + 2C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} \\ &\quad + \left(4m^{-1} + 2\lambda_1 C_f\right)\int_{-\infty}^0 e^{(m\lambda_1 - 4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 dt \\ &\quad + 2\int_{-1}^0 e^{(m\lambda_1 - 4C_f)t}\left(\lambda_1 C_f|\mathcal{O}| + (C_f\lambda_1 + \lambda_1 C_f^{-1})|y_\delta(\theta_t\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2\right)dt.\end{aligned}$$

Proof. The idea to prove the existence of random $\mathcal{D}_F$-attractor to (5.3) is the same as [33, Theorem 9]. Namely, looking for a random compact absorbing set $B_\delta(\omega)$ (which will be given by the ball of center 0 and radius $R_\delta(\omega)$ in V) absorbing every bounded deterministic set $D \subset H$, together with the compact embedding $V \hookrightarrow H$, we achieve the goal. Firstly, multiplying (5.11) by $v_\delta(t) := v_\delta(t;\tau,\omega,v_{0,\delta})$ in H, by (1.2), we obtain

$$\frac{d}{dt}|v_\delta(t)|^2 + 2m\|v_\delta(t)\|^2 \le 2(f(v_\delta(t)+\phi y_\delta(\theta_t\omega)),v_\delta(t)) + 2y_\delta(\theta_t\omega)(\phi,v_\delta(t)) + 2\widetilde{m}\|\phi\|\|v_\delta(t)\|,$$

with the help of (1.5), the Young and Poincaré inequalities, we have

$$\begin{aligned}\frac{d}{dt}|v_\delta(t)|^2 + m\|v_\delta(t)\|^2 &\le (-m\lambda_1 + 2C_f(\mu_1+1)+\mu_2)|v_\delta(t)|^2 + \frac{C_f|\mathcal{O}|}{\mu_1}\\ &\quad + \left(\frac{C_f}{\mu_1\lambda_1}+\frac{1}{\mu_2\lambda_1}\right)|y_\delta(\theta_t\omega)|^2\|\phi\|^2 + \frac{\widetilde{m}^2}{\mu_3}\|\phi\|^2 + \mu_3\|v_\delta(t)\|^2.\end{aligned} \tag{5.14}$$

Letting $\mu_1 = \frac{1}{2}$, $\mu_2 = C_f$ and $\mu_3 = \frac{m}{2}$ in (5.14), we derive

$$\begin{aligned}\frac{d}{dt}|v_\delta(t)|^2 &\le -(m\lambda_1 - 4C_f)|v_\delta(t)|^2 + 2C_f|\mathcal{O}|\\ &\quad + \left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 - \frac{m}{2}\|v_\delta(t)\|^2.\end{aligned} \tag{5.15}$$

Neglecting the last term of (5.15) and integrating in $[t_0,-1]$ with $t_0 \le -1$, we have

$$\begin{aligned}|v_\delta(-1)|^2 &\le e^{-(m\lambda_1-4C_f)(-1-t_0)}\Bigg[\int_{t_0}^{-1}\left(2C_f|\mathcal{O}| + \left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2\right)\\ &\qquad \times e^{(m\lambda_1-4C_f)(t-t_0)}dt + |v_\delta(t_0)|^2\Bigg]\\ &\le e^{-(m\lambda_1-4C_f)(-1-t_0)}|v_\delta(t_0)|^2\\ &\quad + \int_{t_0}^{-1} e^{-(m\lambda_1-4C_f)(-t-1)}\left(2C_f|\mathcal{O}| + \left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2\right)dt\\ &\le e^{(m\lambda_1-4C_f)}\Bigg[e^{(m\lambda_1-4C_f)t_0}|v_\delta(t_0)|^2\\ &\quad + \int_{t_0}^{-1} e^{(m\lambda_1-4C_f)t}\left(2C_f|\mathcal{O}| + \left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2\right)dt\Bigg].\end{aligned}$$

Consequently, for a given $B(0,\rho_\delta) \subset H$, there exists $T(\omega,\rho_\delta) \le -1$, such that for all $t_0 \le T(\omega,\rho_\delta)$ and for all $u_0 \in B(0,\rho_\delta)$,

$$|v_\delta(-1;t_0,\omega,u_\delta(t_0) - \phi y_\delta(\theta_{t_0}\omega))|^2 \le r_{3,\delta}^2(\omega),$$

with

$$r_{3,\delta}^2(\omega) = 1 + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + \int_{-\infty}^{-1} e^{(m\lambda_1-4C_f)(t+1)}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 dt,$$

which is well defined. Indeed, it is enough to choose $T(\omega,\rho_\delta)$ such that, for any $t_0 \le T(\omega,\rho_\delta)$, we have

$$\begin{aligned}e^{(m\lambda_1-4C_f)(t_0+1)}|v_\delta(t_0)|^2 &= e^{(m\lambda_1-4C_f)(t_0+1)}|u_\delta(t_0) - \phi y_\delta(\theta_{t_0}\omega)|^2\\ &\le 2e^{(m\lambda_1-4C_f)(t_0+1)}\left(\rho_\delta^2 + |\phi|^2|y_\delta(\theta_{t_0}\omega)|^2\right)\\ &\le 1.\end{aligned}$$

From (5.15), for $t \in [-1, 0]$, we have

$$|v_\delta(t)|^2 \leq e^{-(m\lambda_1 - 4C_f)(t+1)} \left[\int_{-1}^{t} \left(2C_f|\mathcal{O}| + \left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 - \frac{m}{2}\|v_\delta(s)\|^2 \right) e^{(m\lambda_1 - 4C_f)(s+1)} ds + |v_\delta(-1)|^2 \right].$$

Therefore,

$$|v_\delta(t)|^2 \leq e^{-(m\lambda_1 - 4C_f)(t+1)}|v_\delta(-1)|^2 + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + \int_{-1}^{t} e^{-(m\lambda_1 - 4C_f)(t-s)} \left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 ds,$$

and

$$\int_{-1}^{0} e^{(m\lambda_1 - 4C_f)s}\|v_\delta(s)\|^2 ds \leq \frac{2}{m} e^{-(m\lambda_1 - 4C_f)}|v_\delta(-1)|^2 + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{2}{m}\int_{-1}^{0} e^{(m\lambda_1 - 4C_f)s} \left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 ds.$$

Thus, we conclude for a given $B(0, \rho_\delta) \subset H$, there exists $T(\omega, \rho_\delta) \leq -1$, such that for all $t_0 \leq T(\omega, \rho_\delta)$ and for all $u_0 \in B(0, \rho_\delta)$,

$$|v_\delta(t)|^2 \leq e^{-(m\lambda_1 - 4C_f)(t+1)} r_{3,\delta}^2(\omega) + \frac{2C_f|\mathcal{O}|}{m\lambda_1 - 4C_f} + \int_{-1}^{t} e^{-(m\lambda_1 - 4C_f)(t-s)} \left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 ds,$$

and

$$\int_{-1}^{0} e^{(m\lambda_1 - 4C_f)s}\|v_\delta(s)\|^2 ds \leq \frac{2}{m} e^{-(m\lambda_1 - 4C_f)} r_{3,\delta}^2(\omega) + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{2}{m}\int_{-1}^{0} e^{(m\lambda_1 - 4C_f)s} \left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 ds. \tag{5.16}$$

To obtain a bounded absorbing set in V, multiplying (5.11) by $-\Delta v_\delta(t)$, making use of (1.2), (1.5), the Poincaré and Young inequalities, we have

$$\frac{d}{dt}\|v_\delta(t)\|^2 \leq -(m\lambda_1 - 4C_f)\|v_\delta(t)\|^2 + \lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v_\delta(t)|^2 + \left(C_f\lambda_1 + \frac{\lambda_1}{C_f} \right) |y_\delta(\theta_t\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2.$$

Integrating the above inequality between s and 0, where $s \in [-1, 0]$, we have

$$\|v_\delta(0)\|^2 \leq e^{(m\lambda_1 - 4C_f)s}\|v_\delta(s)\|^2 + \int_{s}^{0} e^{(m\lambda_1 - 4C_f)t} \Big(\lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v_\delta(t)|^2 + (C_f\lambda_1 + \lambda_1 C_f^{-1})|y_\delta(\theta_t\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2 \Big) dt.$$

Integrating again the above inequality in $[-1,0]$, together with the above inequality, it follows

$$\|v_\delta(0)\|^2 \le \frac{2}{m}e^{-(m\lambda_1-4C_f)}r^2_{3,\delta}(\omega) + \frac{4C_f|\mathcal{O}|}{m(m\lambda_1-4C_f)} + \frac{2}{m}\int_{-1}^0 e^{(m\lambda_1-4C_f)s}$$
$$\times\left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 ds + \int_{-1}^0 e^{(m\lambda_1-4C_f)t}$$
$$\times\left(\lambda_1 C_f|\mathcal{O}| + \lambda_1 C_f|v(t)|^2 + (C_f\lambda_1 + \lambda_1 C_f^{-1})|y_\delta(\theta_t\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2\right)dt.$$

Consequently, there exists $r_{4,\delta}(\omega)$ satisfying, for a given $\rho_\delta>0$, there exists $T(\omega,\rho_\delta)\le -1$, such that for all $t_0\le T(\omega,\rho_\delta)$ and $|u_{0,\delta}|\le\rho_\delta$,

$$\|u_\delta(0;t_0,\omega,u_{0,\delta})\|^2 = \|v_\delta(0;t_0,\omega,u_{0,\delta}-\phi y_\delta(\theta_{t_0}\omega)) + \phi y_\delta(\omega)\|^2 \le r^2_{4,\delta}(\omega),$$

where

$$r^2_{4,\delta}(\omega) = 2\|\phi\|^2|y_\delta(\omega)|^2 + \left(4m^{-1}+2\lambda_1 C_f\right)r^2_{3,\delta}(\omega) + \frac{8C_f|\mathcal{O}|}{m(m\lambda_1-4C_f)} + \frac{4\lambda_1 C_f^2|\mathcal{O}|}{(m\lambda_1-4C_f)^2}$$
$$+\left(4m^{-1}+2\lambda_1 C_f\right)\int_{-\infty}^0 e^{(m\lambda_1-4C_f)s}\left(\frac{|y_\delta(\theta_s\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_s\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m}\right)\|\phi\|^2 ds$$
$$+2\int_{-1}^0 e^{(m\lambda_1-4C_f)s}\left(\lambda_1 C_f|\mathcal{O}| + (C_f\lambda_1+\lambda_1 C_f^{-1})|y_\delta(\theta_s\omega)|^2|\phi|^2 + \frac{\widetilde{m}^2}{m}|\Delta\phi|^2\right)ds.$$

Thus, we conclude from [33, Theorem 1] that, there exists a unique random attractor $\mathcal{A}_\delta(\omega)$ to equation (5.3) with respect to deterministic bounded sets. ∎

THEOREM 5.5. *Let conditions of Theorem 5.3 hold. Then, for almost all $\omega\in\Omega$, we have*

$$\lim_{\delta\to 0} R_\delta(\omega) = R_0(\omega),$$

where $R_0(\omega)$ and $R_\delta(\omega)$ are given in theorems 5.3 and 5.4, respectively.

Proof. From (5.6), we obtain

$$\lim_{\delta\to 0} y_\delta(\omega) = z^*(\omega). \tag{5.17}$$

On the one hand, (5.7) implies that there exist $r<0$ and $\delta_0>0$, such that for all $0<\delta<\delta_0$,

$$|y_\delta(\theta_t\omega)| \le |t|, \qquad \forall t\le r. \tag{5.18}$$

Notice that,

$$\int_{-\infty}^0 e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)||\phi||^2 dt$$
$$= \int_{-\infty}^r e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)||\phi||^2 dt$$
$$+\int_r^0 e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)||\phi||^2 dt.$$

Therefore, for all $0<\delta<\delta_0$, it follows from (5.18) that

$$\int_{-\infty}^r e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)||\phi||^2 dt$$
$$\le \int_{-\infty}^r e^{(m\lambda_1-4C_f)t}\left(\frac{|t|^2}{\lambda_1 C_f} + \frac{2C_f|t|^2}{\lambda_1}\right)\|\phi\|^2 dt < \infty.$$

By means of the above inequality, (5.6) and dominated convergence theorem, we have

$$\begin{aligned}\lim_{\delta\to 0}\int_{-\infty}^{r} e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f}+\frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)\|\phi\|^2dt \\ =\int_{-\infty}^{r} e^{(m\lambda_1-4C_f)t}\left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f}+\frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1}\right)\|\phi\|^2dt.\end{aligned}\tag{5.19}$$

On the other hand, by (5.6), the continuity of $y_\delta(\theta_t\omega)$ and the dominated convergence theorem, it follows

$$\begin{aligned}\lim_{\delta\to 0}\int_{r}^{0} e^{(m\lambda_1-4C_f)t}\left(\frac{|y_\delta(\theta_t\omega)|^2}{\lambda_1 C_f}+\frac{2C_f|y_\delta(\theta_t\omega)|^2}{\lambda_1}\right)\|\phi\|^2dt \\ =\int_{r}^{0} e^{(m\lambda_1-4C_f)t}\left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f}+\frac{2C_f|z^*(\theta_t\omega)|^2}{\lambda_1}\right)\|\phi\|^2dt.\end{aligned}\tag{5.20}$$

By similar arguments to (5.20), it is easy to check

$$\begin{aligned}\lim_{\delta\to 0}\int_{-1}^{0} e^{(m\lambda_1-4C_f)t}\left(C_f\lambda_1+\lambda_1 C_f^{-1}\right)|y_\delta(\theta_t\omega)|^2\|\phi\|^2dt \\ =\int_{-1}^{0} e^{(m\lambda_1-4C_f)t}\left(C_f\lambda_1+\lambda_1 C_f^{-1}\right)|z^*(\theta_t\omega)|\|\phi\|^2dt.\end{aligned}\tag{5.21}$$

The conclusion of this theorem follows from (5.19)-(5.21). The proof is complete. ∎

Lemma 5.6. *Under assumptions of Theorem 5.3, let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence satisfying $\delta_n\to 0$ as $n\to+\infty$. Let u_{δ_n} and u be the solutions of (5.3) and (5.1) with initial values u_{0,δ_n} and u_0, respectively. If $u_{0,\delta_n}\to u_0$ strongly in H as $n\to+\infty$, then for almost all $\omega\in\Omega$ and $t\geq\tau$,*

$$u_{\delta_n}(t;\tau,\omega,u_{0,\delta_n})\to u(t;\tau,\omega,u_0)\qquad\text{strongly in}\quad H\quad\text{as}\quad n\to+\infty.$$

Proof. The proof is similar to [16, Lemma 4.4] and we omit the details here. ∎

Lemma 5.7. *Assume conditions of Theorem 5.3 hold, let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence so that $\delta_n\to 0$ as $n\to+\infty$. Let v_{δ_n} and v be the solutions of problems (5.11) and (5.2) with initial data v_{0,δ_n} and v_0, respectively. If $v_{0,\delta_n}\to v_0$ weakly in H as $n\to+\infty$, then for almost all $\omega\in\Omega$,*

$$v_{\delta_n}(r;\tau,\omega,v_{0,\delta_n})\to v(r;\tau,\omega,v_0)\quad\text{weakly in}\quad H,\quad\forall r\geq\tau,\tag{5.22}$$

and

$$v_{\delta_n}(\cdot;\tau,\omega,v_{0,\delta_n})\to v(\cdot;\tau,\omega,v_0)\quad\text{weakly in}\quad L^2(\tau,\tau+T;V),\quad\forall T>0.\tag{5.23}$$

Proof. The results follow analogously to the proof of existence of solutions to problem (5.11) [15, Lemma 3.5]. We therefore omit the details. ∎

Lemma 5.8. *Suppose conditions of Theorem 5.3 hold, let $\omega\in\Omega$ be fixed. If $\delta_n\to 0$ as $n\to+\infty$ and $u_{\delta_n}\in\mathcal{A}_{\delta_n}(\omega)$, then the sequence $\{u_{\delta_n}\}_{n=1}^{\infty}$ has a convergent subsequence in H.*

Proof. Since $\delta_n\to 0$ as $n\to+\infty$, by Theorem 5.5, we obtain for almost all $\omega\in\Omega$, there exists $N=N(\omega)$, such that for all $n\geq N$

$$R_{\delta_n}(\omega)\leq 2R_0(\omega).\tag{5.24}$$

Thanks to $u_n:=u_{\delta_n}(t;\tau,\omega,u_{0,\delta_n})\in\mathcal{A}_{\delta_n}(\omega)$ and $\mathcal{A}_{\delta_n}(\omega)\subset R_{\delta_n}(\omega)$, hence for all $n\geq N$, we have

$$|u_n|^2\leq 2\lambda_1^{-1}R_0(\omega).\tag{5.25}$$

In fact, (5.25) implies u_n is bounded in H, thus, up to a subsequence (relabeled the same), we have

$$u_n \to \tilde{u} \qquad \text{weakly in} \quad H. \tag{5.26}$$

In what follows, we prove that the weak convergence in (5.26) is actually a strong one. On the one hand, since $u_n \in \mathcal{A}_{\delta_n}(\omega)$, making use of the invariance of $\mathcal{A}_{\delta_n}$, for every $k \geq 1$, there exists $u_{n,k}(\omega) \in \mathcal{A}_{\delta_n}(\theta_{-k}\omega)$ such that

$$u_n = \Psi_{\delta_n}(k, \theta_{-k}\omega, u_{n,k}) = u_{\delta_n}(0; -k, \omega, u_{n,k}). \tag{5.27}$$

Since $u_{n,k} \in \mathcal{A}_{\delta_n}(\theta_{-k}\omega)$ and $\mathcal{A}_{\delta_n}(\theta_{-k}\omega) \subset B_{\delta_n}(\theta_{-k}\omega)$, by (5.24), we infer that for each $k \geq 1$ and $n \geq N := N(\theta_{-k}\omega)$,

$$|u_{n,k}|^2 \leq 2\lambda_1^{-1} R_0(\theta_{-k}\omega). \tag{5.28}$$

On the other hand, by (5.10), we have

$$v_{\delta_n}(0; -k, \omega, v_{n,k}) = u_{\delta_n}(0; -k, \omega, u_{n,k}) - \phi y_{\delta_n}(\omega), \tag{5.29}$$

where $v_{n,k} = u_{n,k} - \phi y_{\delta_n}(\theta_{-k}\omega)$. Therefore, (5.27) and (5.29) imply

$$u_n = v_{\delta_n}(0; -k, \omega, v_{n,k}) + \phi y_{\delta_n}(\omega). \tag{5.30}$$

By (5.28), we have

$$|v_{n,k}|^2 \leq 2|u_{n,k}|^2 + 2|\phi|^2 |y_{\delta_n}(\omega)|^2 \leq 4\lambda_1^{-1} R_0(\theta_{-k}\omega) + 2|\phi|^2 |y_{\delta_n}(\omega)|^2. \tag{5.31}$$

It follows from (5.6) and (5.31) that there exists $N_1 := N_1(\omega, k)$ such that for every $k \geq 1$ and $n \geq N_1$,

$$|v_{n,k}|^2 \leq 4\lambda_1^{-1} R_0(\theta_{-k}\omega) + 4|\phi|^2 (1 + |z^*(\omega)|^2). \tag{5.32}$$

Notice that (5.6), (5.28) and (5.30) imply, as $n \to +\infty$,

$$v_{\delta_n}(0; -k, \omega, v_{n,k}) \to \tilde{v} \qquad \text{weakly in} \quad H \qquad \text{with} \quad \tilde{v} = \tilde{u} - \phi z^*(\omega). \tag{5.33}$$

Next, using energy estimations, we evaluate the limit of norm $|v_{\delta_n}(0; -k, \omega, v_{n,k})|$ for each k as $n \to +\infty$. By (5.32) we know that for each $k \geq 1$, the sequence $\{v_{n,k}\}_{n=1}^{\infty}$ is bounded in H, hence by a diagonal process, we can find a subsequence (relabeled the same) such that for each $k \geq 1$, there exists $\bar{v}_k \in H$ such that

$$v_{n,k} \to \bar{v}_k \quad \text{weakly in} \quad H \qquad \text{as} \quad n \to +\infty. \tag{5.34}$$

Lemma 5.7 and (5.34) conclude, as $n \to +\infty$,

$$v_{\delta_n}(0; -k, \omega, v_{n,k}) \to v(0; -k, \omega, \bar{v}_k) \quad \text{weakly in} \quad H, \tag{5.35}$$

and

$$v_{\delta_n}(\cdot; -k, \omega, v_{n,k}) \to v(\cdot; -k, \omega, \bar{v}_k) \quad \text{weakly in} \quad L^2(\tau, \tau + T; V). \tag{5.36}$$

By the uniqueness of limit, from (5.33) and (5.36), we obtain

$$v(0; -k, \omega, \bar{v}_k) = \tilde{v}. \tag{5.37}$$

By energy equality and (5.11), we have

$$\begin{aligned} \frac{d}{dt}|v_{\delta_n}(t)|^2 + 2a(l(v_{\delta_n}) + y_{\delta_n}(\theta_t\omega) l(\phi)) \|v_{\delta_n}(t)\|^2 &= 2(f(v_{\delta_n} + \phi y_{\delta_n}(\theta_t\omega)), v_{\delta_n}(t)) \\ &\quad + 2y_{\delta_n}(\theta_t\omega)(\phi, v_{\delta_n}(t)) - 2a(l(v_{\delta_n}) + y_{\delta_n}(\theta_t\omega) l(\phi))((\phi, v_{\delta_n})), \end{aligned}$$

i.e.,

$$\frac{d}{dt}|v_{\delta_n}(t)|^2 + m\lambda_1|v_{\delta_n}(t)|^2 + \Theta(v_{\delta_n}(t)) = 2(f(v_{\delta_n} + \phi y_{\delta_n}(\theta_t\omega)), v_{\delta_n}(t)) + 2y_{\delta_n}(\theta_t\omega)(\phi, v_{\delta_n}(t)) - 2a(l(v_{\delta_n}) + y_{\delta_n}(\theta_t\omega)l(\phi))((\phi, v_{\delta_n})), \tag{5.38}$$

where $\Theta(v_{\delta_n}(t)) = 2a(l(v_{\delta_n})+y_{\delta_n}(\theta_t\omega)l(\phi))\|v_{\delta_n}(t)\|^2 - m\lambda_1|v_{\delta_n}(t)|^2$, which is a functional in V. Multiplying (5.38) by $e^{m\lambda_1 t}$ and integrating it from $-k$ to 0, we obtain

$$\begin{aligned}|v_{\delta_n}(0;-k,\omega,v_{n,k})|^2 = e^{-m\lambda_1 k}|v_{n,k}|^2 - \int_{-k}^0 e^{m\lambda_1 t}\Theta(v_{\delta_n}(t;-k,\omega,v_{n,k}))dt \\ + 2\int_{-k}^0 e^{m\lambda_1 t}(f(v_{\delta_n}(t;-k,\omega,v_{n,k}) + \phi y_{\delta_n}(\theta_t\omega)), v_{\delta_n}(t;-k,\omega,v_{n,k}))dt \\ + 2\int_{-k}^0 e^{m\lambda_1 t} y_{\delta_n}(\theta_t\omega)(\phi, v_{\delta_n}(t;-k,\omega,v_{n,k}))dt \\ - 2\int_{-k}^0 e^{m\lambda_1 t} a(l(v_{\delta_n}(t;-k,\omega,v_{n,k})) + y_{\delta_n}(\theta_t\omega)l(\phi))((\phi, v_{\delta_n}(t;-k,\omega,v_{n,k})))dt.\end{aligned}$$

Similarly, by (5.2),(5.33) and (5.37), we have

$$\begin{aligned}|\tilde{v}|^2 := |\tilde{v}(0;-k,\omega,\bar{v}_k)|^2 = e^{-m\lambda_1 k}|\bar{v}_k|^2 - \int_{-k}^0 e^{m\lambda_1 t}\Theta(v(t;-k,\omega,\bar{v}_k))dt \\ + 2\int_{-k}^0 e^{m\lambda_1 t}(f(v(t;-k,\omega,\bar{v}_k) + \phi z^*(\theta_t\omega)), v(t;-k,\omega,\bar{v}_k))dt \\ + 2\int_{-k}^0 e^{m\lambda_1 t} z^*(\theta_t\omega)(\phi, v(t;-k,\omega,\bar{v}_k))dt \\ - 2\int_{-k}^0 e^{m\lambda_1 t} a(l(v(t;-k,\omega,\bar{v}_k)) + z^*(\theta_t\omega)l(\phi))((\phi, v(t;-k,\omega,\bar{v}_k)))dt.\end{aligned} \tag{5.39}$$

It is obvious that

$$\begin{aligned}&\limsup_{n\to\infty}|v_{\delta_n}(0;-k,\omega,v_{n,k})|^2 \\ &\le e^{-m\lambda_1 k}\left(4\lambda_1^{-1}R_0(\theta_{-k}\omega) + 4|\phi|^2\left(1+|z^*(\omega)|^2\right)\right) + |\tilde{v}|^2 - e^{-m\lambda_1 k}|\bar{v}_k|^2 \\ &\le e^{-m\lambda_1 k}\left(4\lambda_1^{-1}R_0(\theta_{-k}\omega) + 4|\phi|^2\left(1+|z^*(\omega)|^2\right)\right) + |v(0;-k,\omega,\bar{v}_k)|^2.\end{aligned} \tag{5.40}$$

Notice that, from (5.37) we know for $n \to +\infty$,

$$v(0;-k,\omega,\bar{v}_k) = \tilde{v} = u(0;-k,\omega,\bar{u}_k) - \phi z^*(\omega) := \tilde{u} - \phi z^*(\omega). \tag{5.41}$$

By (5.30), we find

$$v_{\delta_n}(0;-k,\omega,v_{n,k}) = u_n - \phi y_{\delta_n}(\omega). \tag{5.42}$$

It follows from (5.40)-(5.42) that

$$\limsup_{n\to\infty}|u_n - \phi y_n(\omega)| \le e^{-m\lambda_1 k}\left(4\lambda_1^{-1}R_0(\theta_{-k}\omega) + 4|\phi|^2\left(1+|z^*(\omega)|^2\right)\right) + |\tilde{u} - \phi z^*(\omega)|^2. \tag{5.43}$$

Since R_0 and z^* are tempered, we have

$$\limsup_{k\to\infty} e^{-m\lambda_1 k}\left(4\lambda_1^{-1}R_0(\theta_{-k}\omega) + 4|\phi|^2\left(1+|z^*(\omega)|^2\right)\right) = 0.$$

Let $k \to +\infty$ in (5.43), we obtain

$$\limsup_{n\to\infty} |u_n - \phi y_n(\omega)| \le |\tilde{u} - \phi z^*(\omega)|. \tag{5.44}$$

(5.26) and (5.6) lead us to

$$u_n - \phi y_n(\omega) \to \tilde{u} - \phi z^*(\omega) \quad \text{weakly in} \quad H,$$

together with (5.44), we have

$$u_n - \phi y_n(\omega) \to \tilde{u} - \phi z^*(\omega) \quad \text{strongly in} \quad H. \tag{5.45}$$

Therefore, by (5.6), we conclude that

$$u_n \to \tilde{u} \quad \text{strongly in} \quad H,$$

as desired. This completes the proof. ∎

We are now ready to establish the upper semicontinuity of random attractors as $\delta \to 0$.

Theorem 5.9. *Suppose that a is locally Lipschitz and fulfills (1.2), $f \in C(\mathbb{R})$ satisfies (1.3) and (1.5) with $p = 2$ and $\beta = C_f$, respectively, $\phi \in V \cap H^2(\mathcal{O})$ and $l \in L^2(\mathcal{O})$. Also, let $m\lambda_1 > 4C_f$. Then for almost all $\omega \in \Omega$,*

$$\lim_{\delta\to 0} dist_H(\mathcal{A}_\delta(\omega), \mathcal{A}(\omega)) = 0.$$

Proof. For every fixed $\omega \in \Omega$, define

$$\begin{aligned}\bar{B}(\omega) = \Bigg\{ u \in H : |u|^2 \le \lambda_1^{-1} \Bigg(& 2\|\phi\|^2 |z^*(\omega)|^2 + \frac{8C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} + \frac{4\lambda_1 C_f^2 |\mathcal{O}|}{(m\lambda_1 - 4C_f)^2} \\ & + \frac{4 + 2\lambda_1 C_f m + m\lambda_1 - 4C_f + 2C_f|\mathcal{O}|}{m(m\lambda_1 - 4C_f)} \\ & + \left(4m^{-1} + 2\lambda_1 C_f\right) \int_{-\infty}^0 e^{(m\lambda_1 - 4C_f)t} \left(\frac{|z^*(\theta_t\omega)|^2}{\lambda_1 C_f} + \frac{2C_f |z^*(\theta_t\omega)|^2}{\lambda_1} + \frac{2\widetilde{m}^2}{m} \right) \|\phi\|^2 dt \\ & + 2\int_{-1}^0 e^{(m\lambda_1 - 4C_f)t} \left(\lambda_1 C_f |\mathcal{O}| + \left(C_f \lambda_1 + \lambda_1 C_f^{-1} \right) |z^*(\theta_t\omega)|^2 |\phi|^2 + \frac{\widetilde{m}^2}{m} |\Delta\phi|^2 \right) dt \Bigg) \Bigg\}.\end{aligned}$$

By Theorem 5.3, we know $\bar{B} := \{\bar{B}(\omega) : \omega \in \Omega\}$ is also a $\mathcal{D}_F$-(pullback) random absorbing set for Ψ. Let B_δ be the $\mathcal{D}_F$-(pullback) random absorbing set of Ψ_δ given by Theorem 5.4, it follows from Theorem 5.5 that

$$\lim_{\delta\to 0} |B_\delta(\omega)| = |\bar{B}(\omega)| \quad \text{for almost all} \quad \omega \in \Omega.$$

Which, together with Lemmas 5.6 and 5.8, completes the proof by applying [27, Theorem 3.1]. ∎

Remark 5.10. Notice that, if for every $\omega \in \Omega$, the set $\bigcup_{\delta\in(0,1]} \mathcal{A}_\delta(\omega)$ is precompact in H, the results of Lemma 5.8 hold true automatically [27]. Indeed, in our case, we define the absorbing set $B_\delta(\omega) = \{u \in H : |u| \le \lambda_1^{-1} R_\delta(\omega)\}$ (Theorem 5.4) for every $\delta \in (0, 1]$, it is clear that the upper bound of $B_\delta(\omega)$ is uniform with respect to δ. In fact, using the similar arguments as Theorem 5.5, with the help of the properties of $y_\delta(\theta_t\omega)$ (cf. (5.6)-(5.8)), it is enough to show that $|B_\delta(\omega)| \le C(\omega)$, where $C(\omega)$ is a positive constant which does not depend on δ. Therefore, we can replace the complicated proof of Lemma 5.6 by this conclusion to prove the upper semicontinuity of random attractors (cf. Theorem 5.9).

6. Convergence of random attractors for stochastic nonlocal PDEs with multiplicative noise. We conclude our paper with studying the following stochastic nonlocal partial differential equations driven by colored noise,

$$\begin{cases} \frac{\partial u_\delta}{\partial t} - a(l(u_\delta))\Delta u_\delta = f(u_\delta) + \sigma u \zeta_\delta(\theta_t\omega), & \text{in } \mathcal{O}\times(\tau,\infty), \\ u_\delta = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u_\delta(x,\tau) = u_{0,\delta}, & \text{in } \mathcal{O}, \end{cases} \tag{6.1}$$

which is an approximation of the following one studied in [33],

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + \sigma u \circ \frac{dW}{dt}, & \text{in } \mathcal{O}\times(\tau,\infty), \\ u = 0, & \text{on } \partial\mathcal{O}\times(\tau,\infty), \\ u(x,\tau) = u_0, & \text{in } \mathcal{O}, \end{cases} \tag{6.2}$$

where $\circ$ denotes the Stratonovich sense in stochastic term. On account of the change of variable $v(t) = e^{-\sigma z^*(\theta_t\omega)}u(t)$, (6.2) can be written as,

$$\frac{dv}{dt} - a\left(l(v)e^{\sigma z^*(\theta_t\omega)}\right)\Delta v = e^{-\sigma z^*(\theta_t\omega)} f(ve^{\sigma z^*(\theta_t\omega)}) + v\sigma z^*(\theta_t\omega). \tag{6.3}$$

Analogously, to study the pathwise dynamics of problem (6.1), we need to transform the stochastic equations into random ones parameterized by $\omega\in\Omega$. Let

$$v_\delta(t) = u_\delta(t)e^{-\sigma y_\delta(\theta_t\omega)}. \tag{6.4}$$

Then, (6.1) and (6.4) imply that

$$\frac{dv_\delta}{dt} - a\left(l(v_\delta)e^{\sigma y_\delta(\theta_t\omega)}\right)\Delta v_\delta = e^{-\sigma y_\delta(\theta_t\omega)} f(v_\delta e^{\sigma y_\delta(\theta_t\omega)}) + v_\delta(t)\sigma y_\delta(\theta_t\omega), \tag{6.5}$$

with initial value $v_{0,\delta} := v_\delta(\tau) = u_0 e^{-\sigma y_\delta(\theta_\tau\omega)}$.

Proposition 6.1. *Suppose assumptions (1.2)-(1.5) are true with $p = 2$ and $\beta = C_f$, respectively. Then, for almost all $\omega\in\Omega$, function $a(\omega,\cdot) = a\left(l(\cdot)e^{\sigma y_\delta(\theta_t\omega)}\right) \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz and satisfies (1.2). Furthermore, there exists a constant $C_{F,\delta}$ depending on ω, σ, C_f and η, such that,*

$$|F(\omega,s)| \le C_{F,\delta}(1+|s|) \quad \text{and} \quad (F(\omega,s)-F(\omega,r))(s-r) \le \eta|s-r|^2, \qquad \forall s,r\in\mathbb{R},$$

where $F(\omega,s) = e^{-\sigma y_\delta(\omega)} f(e^{\sigma y_\delta(\omega)}s) + \sigma y_\delta(\omega)s$.

In what follows, we will use $v_\delta(\cdot;\tau,\omega,v_{0,\delta})$ to denote the solution of equation (6.5). In a similar way as [33, Theorem 3], we deduce (6.5) has a unique weak solution in the sense of [33, Definition 7] which belongs to $L^2(\tau,T;V)\cap L^\infty(\tau,T;H)$ for every $T\ge\tau$. At this point, thanks to the transformation (6.4), there exists a unique weak solution $u_\delta(\cdot;\tau,\omega,u_{0,\delta}) \in L^2(\tau,T;V)\cap L^\infty(\tau,T;H)$ for every $T\ge\tau$. In addition, this solution behaves continuously in H with respect to the initial value.

Define a mapping $\Sigma_\delta : \mathbb{R}^+\times\Omega\times H \to H$, such that for every $t\in\mathbb{R}^+$,

$$\Sigma_\delta(t,\omega,v_{0,\delta}) = v_\delta(t;0,\omega,v_{0,\delta}), \qquad \forall v_{0,\delta}\in H, \quad \forall\omega\in\Omega.$$

Thanks to the conjugation [33, Lemma 1], there is a mapping $\Phi_\delta : \mathbb{R}^+\times\Omega\times H\to H$ such that for all $t\in\mathbb{R}^+$,

$$\Phi_\delta(t,\omega,u_{0,\delta}) = u_\delta(t;0,\omega,u_{0,\delta}) := v_\delta(t;0,\omega,e^{-\sigma y_\delta(\omega)}v_{0,\delta})e^{\sigma y_\delta(\theta_t\omega)}, \qquad \forall u_{0,\delta}\in H, \quad \forall\omega\in\Omega.$$

Theorem 6.2. *([33, Theorem 5]) Assume that function $a\in C(\mathbb{R};\mathbb{R}^+)$ fulfills (1.2), function f satisfies (1.3) and (1.5) with $p=2$ and $\beta=C_f$, respectively, $l\in L^2(\mathcal{O})$. Also, let $m\lambda_1 > 3C_f$. Then there exists a* *unique random attractor $\mathcal{A}(\omega)$ for the dynamical system $\Phi(t,\omega,u)$ associated to problem (6.2). Additionally, this $\mathcal{D}_F$-pullback absorbing set $B_0 := \{B_0(\omega) : \omega \in \Omega\}$ in H is given by*

$$B_0(\omega) = \{u \in H : |u|^2 \leq \lambda_1^{-1} R_0(\omega)\},$$

with

$$\begin{aligned} R_0(\omega) = \frac{1}{m} e^{\int_{-1}^{0} 2\sigma z^*(\theta_s\omega)ds + 2\sigma z^*(\omega)} \\ \times \left(1 + C_f|\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma z^*(\theta_s\omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma z^*(\theta_\tau\omega)d\tau} ds\right) \\ + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right) \int_{-1}^{0} e^{-2\sigma z^*(\theta_s\omega) + (m\lambda_1 - 3C_f)s + 2\sigma z^*(\omega) + \int_s^0 2\sigma z^*(\theta_r\omega)dr} ds. \end{aligned}$$

THEOREM 6.3. *Under assumptions of Theorem 6.2, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, equation (6.1) generates a random dynamical system $\Phi_\delta(t,\omega,u_{0,\delta})$, which possesses a unique random attractor $\mathcal{A}_\delta(\omega)$. Additionally, the $\mathcal{D}_F$-pullback absorbing set $B_\delta := \{B_\delta(\omega) : \omega \in \Omega\}$ in H is given by*

$$B_\delta(\omega) = \{u \in H : \ |u|^2 \leq \lambda_1^{-1} R_\delta(\omega)\},$$

with

$$\begin{aligned} R_\delta(\omega) = \frac{1}{m} e^{\int_{-1}^{0} 2\sigma y_\delta(\theta_s\omega)ds + 2\sigma y_\delta(\omega)} \\ \times \left(1 + C_f|\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma y_\delta(\theta_s\omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds\right) \\ + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right) \int_{-1}^{0} e^{-2\sigma y_\delta(\theta_s\omega) + (m\lambda_1 - 3C_f)s + 2\sigma y_\delta(\omega) + \int_s^0 2\sigma y_\delta(\theta_r\omega)dr} ds. \end{aligned}$$

Proof. The same method as [33, Theorem 5] will be used to prove this result. We first derive the boundedness of $v_\delta(\cdot) := v_\delta(\cdot; t_0, \omega, v_{0,\delta})$ in H for all $t \in [t_0, -1]$ with $t_0 \leq -1$, where $v_{0,\delta} = e^{-\sigma y_\delta(\theta_{t_0}\omega)} u_0$ and $u_0 \in D$ (a deterministic bounded set). Firstly, multiplying (6.5) by v_δ in H, thanks to (1.5) and the Young inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|v_\delta(t)|^2 + a(e^{\sigma y_\delta(\theta_t\omega)} l(v_\delta))\|v_\delta(t)\|^2 \\ \leq \frac{1}{2} e^{-2\sigma y_\delta(\theta_t\omega)} C_f|\mathcal{O}| + \left(\frac{3C_f}{2} + \sigma y_\delta(\theta_t\omega)\right)|v_\delta(t)|^2, \end{aligned}$$

thanks to the Poincaré inequality and (1.2), we have

$$\frac{d}{dt}|v_\delta(t)|^2 + m\|v_\delta(t)\|^2 \leq (-m\lambda_1 + 3C_f + 2\sigma y_\delta(\theta_t\omega))|v_\delta(t)|^2 + e^{-2\sigma y_\delta(\theta_t\omega)} C_f|\mathcal{O}|. \tag{6.6}$$

Integrating (6.6) between t_0 and -1, it follows

$$\begin{aligned} |v_\delta(-1)|^2 \leq e^{(m\lambda_1 - 3C_f)} \Big[e^{(m\lambda_1 - 3C_f)t_0 + \int_{t_0}^{-1} 2\sigma y_\delta(\theta_s\omega)ds} |v_\delta(t_0)|^2 \\ + C_f|\mathcal{O}| \int_{t_0}^{-1} e^{-2\sigma y_\delta(\theta_s\omega)} e^{(m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds \Big]. \end{aligned}$$

Consequently, for a given deterministic bounded set $D \subset H$, there exist a constant $\rho_\delta > 0$ and $T(\omega, \rho_\delta) \leq -1$, $\mathbb{P}$-a.e., such that, for any $u_{0,\delta} \in D \subset B(0, \rho_\delta)$, for all $t_0 \leq T(\omega, \rho_\delta)$, we have

$$\left| v_\delta\left(-1; t_0, \omega, e^{-\sigma y_\delta(\theta_{t_0}\omega)} u_{0,\delta}\right)\right|^2 \leq r_{1,\delta}^2(\omega),$$

with

$$r_{1,\delta}^2(\omega) = e^{(m\lambda_1 - 3C_f)} \left(1 + C_f|\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma y_\delta(\theta_s\omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds\right).$$

Secondly, we show $v \in L^\infty(-1, t; H) \cap L^2(-1, t; V)$ with $t \in [-1, 0]$ by energy estimations. Integrating (6.6) from -1 to t with $t \in [-1, 0]$, we obtain

$$\begin{aligned} |v_\delta(t)|^2 \le\ & e^{-(m\lambda_1 - 3C_f)(t+1) + \int_{-1}^t 2\sigma y_\delta(\theta_s\omega)ds} |v_\delta(-1)|^2 \\ & + C_f|\mathcal{O}| \int_{-1}^t e^{-2\sigma y_\delta(\theta_s\omega) + (3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds \\ & - m\int_{-1}^t e^{(3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma y_\delta(\theta_\tau\omega)d\tau} \|v_\delta(s)\|^2 ds. \end{aligned} \tag{6.7}$$

Therefore, by similar arguments, we conclude that for a given deterministic subset $D \subset B(0, \rho_\delta) \subset H$, there exists $T(\omega, \rho_\delta) \le -1$, $\mathbb{P}$-a.e., such that for all $t_0 \le T(\omega, \rho_\delta)$, for all $u_{0,\delta} \in D$, we have

$$\begin{aligned} |v_\delta(t)|^2 \le\ & e^{-(m\lambda_1 - 3C_f)(t+1) + \int_{-1}^t 2\sigma y_\delta(\theta_s\omega)ds} r_{1,\delta}^2(\omega) \\ & + C_f|\mathcal{O}| \int_{-1}^t e^{-2\sigma y_\delta(\theta_s\omega) + (3C_f - m\lambda_1)(t-s) + \int_s^t 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds, \end{aligned}$$

and

$$\begin{aligned} \int_{-1}^0 e^{(m\lambda_1 - 3C_f)s + \int_s^0 2\sigma y_\delta(\theta_\tau\omega)d\tau} \|v_\delta(s)\|^2 ds \le\ & \frac{1}{m} e^{-(m\lambda_1 - 3C_f) + \int_{-1}^0 2\sigma y_\delta(\theta_s\omega)ds} r_{1,\delta}^2(\omega) \\ & + \frac{C_f|\mathcal{O}|}{m} \int_{-1}^0 e^{-2\sigma y_\delta(\theta_s\omega) + (m\lambda_1 - 3C_f)s + \int_s^0 2\sigma y_\delta(\theta_\tau\omega)d\tau} ds. \end{aligned} \tag{6.8}$$

Thirdly, the boundedness of $v_\delta(\cdot)$ in V for all $t \in [-1, 0]$ and the compact embedding $V \hookrightarrow H$ ensure the existence of a compact absorbing ball in H. To obtain a bound in V, we first need to ensure the existence of strong solutions, by slightly improving the regularity of initial value, namely, $u_{0,\delta} \in V$, but assumptions imposed on functions a and f are the same, this result holds, for more details, see [32, Theorem 2.9]. Multiplying (6.5) by $-\Delta v_\delta(t)$, with the help of (1.3) and the Young inequality, we derive

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|v_\delta(t)\|^2 + a(e^{\sigma y_\delta(\theta_t\omega)} l(v_\delta))| - \Delta v_\delta(t)|^2 \\ & \le \frac{1}{m} e^{-2\sigma y_\delta(\theta_t\omega)} C_f^2|\mathcal{O}| + \frac{C_f^2}{m}|v_\delta(t)|^2 + \frac{m}{2}|\Delta v(t)|^2 + \sigma y_\delta(\theta_t\omega)\|v(t)\|^2. \end{aligned} \tag{6.9}$$

Using the Poincaré inequality, (6.9) can be bounded by

$$\begin{aligned} \frac{d}{dt}\|v_\delta(t)\|^2 & \le -m|\Delta v_\delta(t)|^2 + \frac{2}{m} C_f^2|\mathcal{O}| e^{-2\sigma y_\delta(\theta_t\omega)} + \frac{2C_f^2}{m}|v(t)|^2 + 2\sigma y_\delta(\theta_t\omega)\|v_\delta(t)\|^2 \\ & \le \left(-m\lambda_1 + \frac{2C_f^2}{m\lambda_1} + 2\sigma y_\delta(\theta_t\omega)\right)\|v_\delta(t)\|^2 + \frac{2}{m} C_f^2|\mathcal{O}| e^{-2\sigma y_\delta(\theta_t\omega)}. \end{aligned} \tag{6.10}$$

Integrating (6.10) between s and 0 with $s \in [-1, 0]$, we obtain

$$\begin{aligned} \|v_\delta(0)\|^2 \le\ & e^{(m\lambda_1 - 2C_f^2/m\lambda_1)s + \int_s^0 2\sigma y_\delta(\theta_\tau\omega)d\tau} \|v_\delta(s)\|^2 \\ & + \frac{2}{m} C_f^2|\mathcal{O}| \int_s^0 e^{-2\sigma y_\delta(\theta_\tau\omega) + (m\lambda_1 - 2C_f^2/m\lambda_1)\tau + \int_\tau^0 2\sigma y_\delta(\theta_t\omega)dt} d\tau. \end{aligned}$$

Integrating the above inequality again in $[-1,0]$, we have

$$\begin{aligned}\|v_\delta(0)\|^2 \le& \int_{-1}^{0} e^{(m\lambda_1-2C_f^2/m\lambda_1)s+\int_s^0 2\sigma y_\delta(\theta_\tau\omega)d\tau}\|v_\delta(s)\|^2 ds\\ &+\frac{2}{m}C_f^2|\mathcal{O}|\int_{-1}^{0} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-2C_f^2/m\lambda_1)s+\int_s^0 2\sigma y_\delta(\theta_r\omega)dr}ds.\end{aligned}$$

Thanks to assumption $3C_f < m\lambda_1$, it is easy to check $m\lambda_1 - 3C_f < m\lambda_1 - \frac{2C_f^2}{m\lambda_1}$, together with (6.8), we have

$$\begin{aligned}\|v_\delta(0)\|^2 \le& \frac{1}{m}e^{-(m\lambda_1-3C_f)+\int_{-1}^0 2\sigma y_\delta(\theta_s\omega)ds}r_{1,\delta}^2(\omega)\\ &+\left(\frac{1}{m}C_f|\mathcal{O}|+\frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma y_\delta(\theta_r\omega)dr}ds.\end{aligned}$$

Therefore, it is straightforward that

$$\begin{aligned}\|u_\delta(0)\|^2 =& \|v_\delta(0)e^{\sigma y_\delta(\omega)}\|^2\\ \le& \frac{1}{m}e^{-(m\lambda_1-3C_f)+2\sigma y_\delta(\omega)+\int_{-1}^0 2\sigma y_\delta(\theta_s\omega)ds}r_{1,\delta}^2(\omega)\\ &+\left(\frac{1}{m}C_f|\mathcal{O}|+\frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma y_\delta(\theta_s\omega)+2\sigma y_\delta(\omega)+(m\lambda_1-3C_f)s+\int_s^0 2\sigma y_\delta(\theta_r\omega)dr}ds.\end{aligned}$$

Consequently, there exists $r_{2,\delta}(\omega)$ such that for a given $\rho_\delta > 0$, there exists $\tilde{T}(\omega,\rho_\delta) \le -1$ satisfying, for all $t_0 \le \tilde{T}(\omega,\rho_\delta)$ and $u_{0,\delta} \in H$ with $|u_{0,\delta}| \le \rho_\delta$,

$$\|u_\delta(0;t_0,\omega,u_{0,\delta})\|^2 \le r_{2,\delta}(\omega),$$

where

$$\begin{aligned}r_{2,\delta}^2(\omega) =& \frac{1}{m}e^{\int_{-1}^0 2\sigma y_\delta(\theta_s\omega)ds+2\sigma y_\delta(\omega)ds}\\ &\times\left(1+C_f|\mathcal{O}|\int_{-\infty}^{-1} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau}ds\right)\\ &+\left(\frac{1}{m}C_f|\mathcal{O}|+\frac{2}{m}C_f^2|\mathcal{O}|\right)\int_{-1}^{0} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+2\sigma y_\delta(\omega)+\int_s^0 2\sigma y_\delta(\theta_r\omega)dr}ds.\end{aligned}$$

From (5.7), we know that for a given $\varepsilon = \frac{m\lambda_1-3C_f}{8|\sigma|}$, there exists $T_1(\varepsilon,\omega) < 0$, such that for all $t \le T_1$, we have

$$|y_\delta(\theta_t\omega)| \le -\frac{m\lambda_1-3C_f}{8|\sigma|}t. \tag{6.11}$$

Similarly, it follows from (5.8), for any $\varepsilon > 0$, there exists $T_2(\varepsilon,\omega) < 0$, such that for all $t \le T_2$,

$$\left|\int_0^t y_\delta(\theta_\tau\omega)d\tau\right| \le -\frac{m\lambda_1-3C_f}{8|\sigma|}t. \tag{6.12}$$

Therefore,

$$\begin{aligned}&\int_{-\infty}^{-1} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau}ds\\ &=\int_{-\infty}^{\min\{T_1,T_2\}} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau}ds\\ &\quad+\int_{\min\{T_1,T_2\}}^{-1} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1} 2\sigma y_\delta(\theta_\tau\omega)d\tau}ds = I_1+I_2.\end{aligned}$$

The continuity of $y_\delta(\omega)$ guarantees the boundedness of I_2. It remains to show I_1 is bounded, it follows from (6.11)-(6.12) that

$$\begin{aligned}
&\int_{-\infty}^{\min\{T_1,T_2\}} e^{-2\sigma y_\delta(\theta_s\omega)+(m\lambda_1-3C_f)s+\int_s^{-1}2\sigma y_\delta(\theta_\tau\omega)d\tau}ds \\
&\leq \int_{-\infty}^{\min\{T_1,T_2\}} e^{2|\sigma||y_\delta(\theta_s\omega)|+(m\lambda_1-3C_f)s+|\int_s^{-1}2\sigma y_\delta(\theta_\tau\omega)d\tau|}ds \\
&\leq \int_{-\infty}^{\min\{T_1,T_2\}} e^{(m\lambda_1-3C_f)(s+1/4)}ds < \infty.
\end{aligned}$$

Thus, we conclude from [33, Theorem 2] that there exists a unique random attractor $\mathcal{A}_\delta(\omega)$ to problem (6.1). □

THEOREM 6.4. *Suppose the conditions of Theorem 6.2 are true. Then, for almost all $\omega\in\Omega$,*

$$\lim_{\delta\to 0} R_\delta(\omega) = R_0(\omega),$$

where $R_0(\omega)$ and $R_\delta(\omega)$ are given in Theorems 6.2 and 6.3, respectively.

Proof. The proof of this theorem is based on the properties of $y_\delta(\theta_t\omega)$ (cf. (5.6)-(5.7)). Since the idea and technique to prove this result are the same as Theorems 5.5, we omit the details. □

LEMMA 6.5. *Assume the conditions of Theorem 6.2 are true, let $\{\delta_n\}_{n=1}^\infty$ be a sequence so that $\delta_n\to 0$ as $n\to+\infty$. Let v_{δ_n} and v be the solutions of problem (6.1) and (6.3) with initial data v_{0,δ_n} and v_0, respectively. If $v_{0,\delta_n}\to v_0$ weakly in H as $n\to+\infty$, then for almost all $\omega\in\Omega$,*

$$v_{\delta_n}(r;\tau,\omega,v_{0,\delta_n}) \to v(r;\tau,\omega,v_0) \quad \textit{weakly in} \quad H, \quad \forall r\geq\tau, \tag{6.13}$$

and

$$v_{\delta_n}(\cdot;\tau,\omega,v_{0,\delta_n}) \to v(\cdot;\tau,\omega,v_0) \quad \textit{strongly in} \quad L^2(\tau,\tau+T;H), \quad \forall T>0. \tag{6.14}$$

Proof. The proof is similar to [15, Lemma 3.5] and thus is omitted here. □

LEMMA 6.6. *Assume the conditions of Theorem 6.2 are true and a is locally Lipschitz. let $\{\delta_n\}_{n=1}^\infty$ be a sequence so that $\delta_n\to 0$ as $n\to+\infty$. Let v_{δ_n} and v be the solutions of problem (6.1) and (6.3) with initial data v_{0,δ_n} and v_0, respectively. If $v_{0,\delta_n}\to v_0$ in H as $n\to+\infty$, then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $t\geq\tau$,*

$$v_{\delta_n}(t;\tau,\omega,v_{0,\delta_n}) \to v(t;\tau,\omega,v_0) \quad \textit{in} \quad H, \quad \forall t\geq\tau, \tag{6.15}$$

Proof. The proof is similar to [16, Lemma 3.8] and thus is omitted here. □

Now, we prove the uniform compactness of the family of random attractors $\mathcal{A}_\delta(\omega)$.

LEMMA 6.7. *Assume the conditions of Lemma 6.6 hold, let $\omega\in\Omega$ is fixed. If $\delta_n\to 0$ as $n\to+\infty$ and $u_n\in\mathcal{A}_{\delta_n}(\omega)$, then the sequence $\{u_n\}_{n=1}^\infty$ has a convergent subsequence in H.*

Proof. Since $u_n\in\mathcal{A}_{\delta_n}(\omega)$, it follows from the invariance of $\mathcal{A}_{\delta_n}$, there exists $u_{n,-1}\in\mathcal{A}_{\delta_n}(\theta_{-1}\omega)$, such that

$$u_n = \Phi_\delta(1,\theta_{-1}\omega,u_{n,-1}) = u_{\delta_n}(0;-1,\omega,u_{n,-1}). \tag{6.16}$$

On the one hand, we deduce from Theorem 6.4 that there exists $N_1 = N_1(\omega) \geq 1$, such that for all $n \geq N_1$,

$$
\begin{aligned}
R_{\delta_n}(\theta_{-1}\omega) \leq & 1 + \frac{1}{m} e^{\int_{-1}^{0} 2\sigma y_{\delta_n}(\theta_{s-1}\omega)ds + 2\sigma y_{\delta_n}(\theta_{-1}\omega)} \\
& \times \left(1 + C_f|\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma y_{\delta_n}(\theta_{s-1}\omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma y_{\delta_n}(\theta_{\tau-1}\omega)d\tau} ds\right) \\
& + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right) \int_{-1}^{0} e^{-2\sigma y_{\delta_n}(\theta_{s-1}\omega) + (m\lambda_1 - 3C_f)s + 2\sigma y_{\delta_n}(\theta_{-1}\omega) + \int_s^0 2\sigma y_{\delta_n}(\theta_{r-1}\omega)dr} ds.
\end{aligned}
$$

Thanks to $u_{n,-1} \in \mathcal{A}_{\delta_n}(\theta_{-1}\omega) \subset B_{\delta_n}(\theta_{-1}\omega)$, by Theorem 6.3 and (6.16), we obtain for all $n \geq N_1$,

(6.17)
$$
\begin{aligned}
|u_{n,-1}|^2 \leq \lambda_1^{-1} \Bigg(& 1 + \frac{1}{m} e^{\int_{-1}^{0} 2\sigma y_{\delta_n}(\theta_{s-1}\omega)ds + 2\sigma y_{\delta_n}(\theta_{-1}\omega)} \\
& \times \left(1 + C_f|\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma y_{\delta_n}(\theta_{s-1}\omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma y_{\delta_n}(\theta_{\tau-1}\omega)d\tau} ds\right) \\
& + \left(\frac{1}{m}C_f|\mathcal{O}| + \frac{2}{m}C_f^2|\mathcal{O}|\right) \int_{-1}^{0} e^{-2\sigma y_{\delta_n}(\theta_{s-1}\omega) + (m\lambda_1 - 3C_f)s + 2\sigma y_{\delta_n}(\theta_{-1}\omega) + \int_s^0 2\sigma y_{\delta_n}(\theta_{r-1}\omega)dr} ds\Bigg).
\end{aligned}
$$

On the other hand, by (6.4), we have

$$v_{\delta_n}(s; -1, \omega, v_{n,-1}) = u_{\delta_n}(s; -1, \omega, u_{n,-1}) e^{-\sigma y_{\delta_n}(\theta_s\omega)},$$

and

$$v_{n,-1} = u_{n,-1} e^{-\sigma y_{\delta_n}(\theta_{-1}\omega)}. \tag{6.18}$$

By (5.6), we know

$$\lim_{\delta_n \to 0} e^{-\sigma y_{\delta_n}(\theta_{-1}\omega)} = e^{-\sigma z^*(\theta_{-1}\omega)},$$

which, along with (6.17)-(6.18) shows that the sequence $\{v_{n,-1}\}_{n=1}^{\infty}$ is bounded in H. Therefore, there exist a subsequence $\{v_{n,-1}\}$ (relabeled the same) and v_{-1} such that $v_{n,-1} \to v_{-1}$ weakly in H. Lemma 6.5 ensures the existence of $\bar{v} := \bar{v}(\cdot; -1, \omega, v_{-1}) \in L^2(-1, 0; H)$ such that, up to a subsequence,

$$v_{\delta_n}(\cdot; -1, \omega, v_{n,-1}) \to \bar{v} \text{ strongly in } L^2(-1, 0; H),$$

which implies, up to a further subsequence,

$$v_{\delta_n}(s; -1, \omega, v_{n,-1}) \to \bar{v}(s) \text{ strongly in } H, \qquad \text{a.e. } s \in (-1, 0). \tag{6.19}$$

By (5.6), (6.18)-(6.19), we obtain

$$u_{\delta_n}(s; -1, \omega, u_{n,-1}) \to e^{\sigma z^*(\theta_s\omega)} \bar{v}(s) \text{ strongly in } H, \qquad \text{a.e. } s \in (-1, 0). \tag{6.20}$$

Since $\delta_n \to 0$ as $n \to +\infty$, it follows from Lemma 6.6 and (6.20) that,

$$u_{\delta_n}(0; s, \omega, u_{\delta_n}(s; -1, \omega, u_{n,-1})) \to u(0; s, \omega, e^{\sigma z^*(\theta_s\omega)} \bar{v}(s)) \quad \text{strongly in } H, \tag{6.21}$$

where u is solution of (6.2). By cocycle property,

$$u_{\delta_n}(0; s, \omega, u_{\delta_n}(s; -1, \omega, u_{n,-1})) = u_{\delta_n}(0; -1, \omega, u_{n,-1}).$$

Therefore, by (6.21) we have

$$u_{\delta_n}(0; -1, \omega, u_{n,-1}) \to u(0; s, \omega, e^{\sigma z^*(\theta_s\omega)} \bar{v}(s)) \quad \text{strongly in } H,$$

together with (6.16), the proof is complete. □

We finally present the upper semicontinuity of random (pullback) attractors as $\delta \to 0$.

THEOREM 6.8. *Assume that function $a \in C(\mathbb{R}; \mathbb{R}^+)$ fulfills (1.2), function f satisfies (1.3) and (1.5) with $p = 2$ and $\beta = C_f$, respectively. Also, let $m\lambda_1 > 3C_f$ and $l \in L^2(\mathcal{O})$. Then, for almost all $\omega \in \Omega$,*

$$\lim_{\delta \to 0} dist_H(\mathcal{A}_\delta(\omega), \mathcal{A}(\omega)) = 0. \tag{6.22}$$

Proof. For every fixed $\omega \in \Omega$, let

$$\begin{aligned}\tilde{B}(\omega) = \Bigg\{ u \in H : |u|^2 \le \lambda_1^{-1} \Bigg(&\frac{1}{m} e^{\int_{-1}^{0} 2\sigma z^*(\theta_s \omega) ds + 2\sigma z^*(\omega)} \\ &\times \left(1 + C_f |\mathcal{O}| \int_{-\infty}^{-1} e^{-2\sigma z^*(\theta_s \omega) + (m\lambda_1 - 3C_f)s + \int_s^{-1} 2\sigma z^*(\theta_\tau \omega) d\tau} ds \right) \\ &+ \left(\frac{1}{m} C_f |\mathcal{O}| + \frac{2}{m} C_f^2 |\mathcal{O}| \right) \int_{-1}^{0} e^{-2\sigma z^*(\theta_s \omega) + (m\lambda_1 - 3C_f)s + 2\sigma z^*(\omega) + \int_s^0 2\sigma z^*(\theta_r \omega) dr} ds \Bigg) \Bigg\}.\end{aligned}$$

By Theorem 6.2 we see $\tilde{B} := \{\tilde{B}(\omega), \omega \in \Omega\}$ belongs to $\mathcal{D}$. Moreover, Theorem 6.4 implies

$$\lim_{\delta \to 0} |B_\delta(\omega)| = |\tilde{B}(\omega)|, \quad \text{for almost all} \quad \omega \in \Omega.$$

Combine above equality with Lemmas 6.5 and 6.7, we finish the proof of this theorem by [27, Theorem 3.1]. ∎

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