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WEAKLY NONLINEAR MULTIPHASE GEOMETRIC OPTICS FOR HYPERBOLIC QUASILINEAR BOUNDARY VALUE PROBLEMS: CONSTRUCTION OF A LEADING PROFILE

CORENTIN KILQUE

Abstract. We investigate in this paper the existence of the leading profile of a WKB expansion for quasilinear initial boundary value problems with a highly oscillating forcing boundary term. The framework is weakly nonlinear, as the boundary term is of order $O(\varepsilon)$ where the frequencies are of order $O(1/\varepsilon)$. We consider here multiple phases on the boundary, generating a countable infinite number of phases inside the domain, and we therefore use an almost periodic functional framework. The major difficulties of this work are the lack of symmetry in the leading profile equation and the occurrence of infinitely many resonances (opposite to the simple phase case studied earlier) The leading profile is constructed as the solution of a quasilinear problem, which is solved using a priori estimates without loss of derivatives. The assumptions of this work are illustrated with the example of isentropic Euler equations in space dimension two.

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1. Introduction

We consider in this paper hyperbolic quasilinear initial boundary value problems with a highly oscillating forcing boundary term. We are interested in constructing, in the high frequency asymptotic, an approximate solution to this problem in the form of a WKB expansion. This is the type of question studied in hyperbolic geometric optics. The general idea is to consider an hyperbolic system, of which the source term, the initial term or the boundary term (in the case of boundary value problems) is highly oscillatory, namely with frequencies of order $1/\varepsilon$ and to look for an approximate solution to the system in the form of an asymptotic expansion. First the equation satisfied by the terms of the asymptotic expansion needs to be formally derived, and then to be solved in a suitable functional space. Once this formal series has been constructed, one may prove that the truncated sums actually approaches the exact solution in the high frequencies asymptotic. The present paper addresses the first part of this framework, and more precisely we prove existence and uniqueness for the leading profile of the asymptotic expansion.

The study of hyperbolic geometric optics goes back to [Lax57] for the study of the linear Cauchy problem. When the system is nonlinear, the multiplicity of phases in the source term, initial term or boundary term is important since nonlinear interactions between phases may occur. In the case of only one phase, the construction of an asymptotic expansion was first performed by [CB64]. For the justification of this asymptotic expansion we can refer to [JR92] in the semi-linear case and [Guè93] in the quasi-linear case. The first study of the multiphase case for the Cauchy problem goes back to [HMR86]. The question has then been largely resolved by J.L. Joly, G. Métivier and J. Rauch, see in particular [JMR93], [JMR94] and [JMR95]. The natural question is to obtain results for boundary value problems, similar to the ones for the Cauchy problems. In [Chi91], the author deals with a semi-linear boundary value problem for a system of two equations, the general case of multiple equations being treated for example in [Wil96] and [Wil00]. The quasi-linear case but with only one phase on the boundary is treated notably in [Wil02], [CGW11], [CW13] (the latter taking interest into the justification, which is not addressed in this paper) and [Her15]. This work is an extension to the multiphase case: we deal with the same quasi-linear boundary value problem, but with multiple phases on the boundary.

Because of the multiple frequencies on the boundary, the nonlinearity of the problem generates a countable infinite number of phases inside the domain, forcing us to consider an almost-periodic framework, the group of frequencies being, in general, not finitely generated. This almost-periodic functional framework has been previously used to construct approximate solutions to systems with multiple phases, for semi-linear systems in the context of Wiener algebras by [JMR94] for the Cauchy problem and [Wil96] for the boundary value problem. For quasi-linear systems, Bohr-Besicovich spaces are generally used, notably by [JMR95] for the Cauchy problem. In this work we attempt to achieve the next step, namely to obtain a similar result as the one of [JMR95], for quasi-linear boundary value problems. We adapt the functional framework of [JMR95] to the context of boundary value problems, by considering functions that are quasi-periodic with respect to the tangential fast variables and almost-periodic with respect to the normal fast variable. Concerning the regularity, we choose a Sobolev control for the (slow and fast) tangential variables, and a uniform control for the normal variables. The leading profile of the WKB expansion is then obtained as the solution of a quasilinear problem which takes into account the potentially infinite number of resonances between the phases. We solve this quasilinear problem in a classical way by proving estimates without loss of regularity. The example of gas dynamics is used all along the paper to illustrate the general assumptions that will be made during the analysis. The main difference between this paper and [JMR95] is the absence of symmetry in the problem. Indeed, starting with an evolution problem in time, we modify it to obtain a propagation problem in the normal variable x_d, with respect to which the system is not hyperbolic. In [JMR95], these symmetries are used for the a priori estimates to handle the resonance terms that appear in the equations. Even though it is relatively easy in our problem to create symmetries for the self-interaction terms, it is more delicate for the resonance

terms, which, unlike the case of [CGW11], are in infinite number. The last assumption of the paper is made to deal with this issue, and in essence controls the lack of symmetry of resonance terms. The notions associated to it appear in [Rau12, Chapter 11].

The proof of existence of a leading profile is divided in three parts: formal derivation of the equation satisfied by the leading profile, reduction and decoupling of these equations, and finally energy estimates on these equations. Formal derivation of the WKB cascade is quite classical in geometric optics, and consists on formally replacing the series in the exact system (system (2.1) in the following). As usual, this cascade is decoupled using projectors and an operator on the space of profiles. The said operator remains formal in this paper, and the projectors require a small divisors assumption to be rigorously defined. The second part of the proof takes interest into decoupling and reducing the system to a system for the oscillating resonant modes, a system for each oscillating non-resonant mode, and a system for the evanescent part. In order to do that, extending the system into modes, we begin by showing that the mean value satisfies a decoupled system with zero source term and boundary term, and is therefore zero. Next we prove that the outgoing modes are also zero, deriving energy estimates for them. For this purpose we use a suitable scalar product on the space of profiles, that requires a compact support in the normal direction. Therefore a finite speed propagation is proven beforehand. Once there are only incoming and evanescent modes, it is easy to determine a boundary condition for each mode from the original boundary condition, and therefore decouple the system. The equation satisfied by the evanescent part gives a formula for it using the double trace on the boundary, so its construction is quite straightforward. However we need to check that the constructed solution is actually in the space of evanescent profiles. To construct the oscillating parts we show a priori estimates without loss of regularity for the linearized oscillating systems, which allow us to prove the well-posedness of these linearized systems, and then by an iterative scheme the existence of solutions to the original systems. To derive the energy estimates we use an alternative scalar product that takes advantage of propagation in the normal direction. Several terms need to be handled. The transport and Burgers ones are quite classical to treat, and the assumption on the set of resonances is used to address the resonant ones. Since it is quite usual, we do not give the details of the construction of the solution to the linearized systems and the iterative schemes. Reassembling the constructed profiles we finally get the leading profile solution to the initial system.

The paper is organized as follows. After the first section devoted to this introduction, the second one introduces the problem and states the usual assumptions on the system. First the problem studied in this work is precisely described, and the example of Euler equations that will be used throughout the analysis is introduced, and then characteristic frequencies and the strict hyperbolicity assumption are looked at. Finally interest is made on properties and assumption about the boundary condition. Assumptions of this section ensure that the initial boundary problem is well posed locally in time for the exact solution. However, due to the high frequencies in the forcing term, we do not know if the lifespan of the exact solution is uniform with respect to the small wavelength. Third section is devoted to the functional framework of the paper. After a motivation of this framework with a formal study of the frequencies created inside the domain and some assumptions on it, we describe the spaces of profiles which will be used, and introduce scalar products on these functional spaces. After this rather long introduction of the problem and assumptions, the ansatz of the expansion and the main result are stated in section 4. The proof is then divided in two sections. The fifth one is a formal derivation of the equations satisfied by the leading profile. First the cascade of equations for the profiles is obtained by a formal WKB study. It gives rise to a certain fast problem, that is resolved in a second part, which allows to finally write the decoupled equations for the leading profile. The last section of this paper is the core of the proof. First some coefficients associated to resonances are introduced and the last assumption of this work is made about these coefficients to deal with the lack of symmetry in resonance terms. We proceed by making rigorous the results of the fifth section analysis which will be used after. Next part is devoted to reducing and decoupling the equations into an equation for the evanescent part, and equations for the resonant oscillating part and each non-resonant oscillating part. A lot of the techniques used in this part are used in

the next two ones, that achieves the main step of the proof, namely proving energy estimates on the linearized equations for the oscillating resonant and non-resonant parts. These estimates are used in the following part to construct an oscillating solution, using an iterative scheme. It is also proven that the constructed evanescent part belongs to the space of evanescent profiles. Finally a conclusion and some perspectives are drawn. In the appendix is presented three technical proofs that have been postponed, and are about notions of the second section.

In all the paper the letter C denotes a positive constant that may vary during the analysis, possibly without any mention being made, and for every matrix M, the notation ${}^tM$ refers to its real transpose.

2. Notations and assumptions

2.1. **Presentation of the problem.** Given a time $T > 0$ and an integer $d \geqslant 2$, let Ω_T be the domain $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$ its boundary. We denote as $t \in (-\infty, T]$ the time variable, $x = (y, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ the space variable, with $y \in \mathbb{R}^{d-1}$ the tangential variable and $x_d \in \mathbb{R}_+$ the normal variable, and at last $z = (t, x) = (t, y, x_d)$. We also denote by $z' = (t, y) \in \omega_T$ the variable of the boundary $\{x_d = 0\}$. For $i = 1, \dots, d$, we denote by ∂_i the operator of partial derivative with respect to x_i. Finally we denote as $\alpha \in \mathbb{R}^{d+1}$ and $\zeta \in \mathbb{R}^d$ the dual variables of $z \in \Omega_T$ and $z' \in \omega_T$. We consider the following problem

$$\begin{cases} L(u^\varepsilon, \partial_z)\, u^\varepsilon := \partial_t u^\varepsilon + \displaystyle\sum_{i=1}^{d} A_i(u^\varepsilon)\, \partial_i u^\varepsilon = 0 & \text{in } \Omega_T, \\ B\, u^\varepsilon_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon_{|t\leqslant 0} = 0, \end{cases} \tag{2.1}$$

where the unknown u^ε is a function from Ω_T to an open set $\mathcal{O}$ of $\mathbb{R}^N$ containing zero, with $N \geqslant 1$, the matrices A_j are regular functions of $\mathcal{O}$ with values in $\mathcal{M}_N(\mathbb{R})$ and the matrix B belongs to $\mathcal{M}_{M\times N}(\mathbb{R})$ and is of maximal rank. The integer M is made precise in Assumption 3 below. To simplify the notations and clarify the proofs we consider here linear boundary conditions, but it would be possible to deal with non-linear ones. Furthermore we assume the boundary to be noncharacteristic, that is the following assumption is made.

Assumption 1 (Noncharacteristic boundary)**.** *For all u in $\mathcal{O}$, the matrix $A_d(u)$ is invertible.*

The dependence on $\varepsilon > 0$ of system (2.1) comes from the source term $\varepsilon\, g^\varepsilon$ on the boundary ω_T, where the quasi-periodic function g^ε is defined, for z' in ω_T, as

$$g^\varepsilon(z') = G\left(z', \frac{z'\cdot\zeta_1}{\varepsilon}, \dots, \frac{z'\cdot\zeta_m}{\varepsilon}\right), \tag{2.2}$$

where G is a function of the Sobolev space $H^\infty(\mathbb{R}^d \times \mathbb{T}^m)$, with $m \geqslant 2$, that vanishes for negative times t and of zero mean with respect to θ in $\mathbb{T}^m$, and where $\zeta_1, \dots, \zeta_m$ are frequencies of $\mathbb{R}^d \setminus \{0\}$. Here the notation $\mathbb{T}$ stands for the torus $\mathbb{R}/2\pi\mathbb{Z}$. We denote by $\boldsymbol{\zeta}$ the m-tuple $\boldsymbol{\zeta} := (\zeta_1, \dots, \zeta_m)$. The function G being periodic and of zero mean with respect to θ, we may write

$$G(z', \theta) = \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} G_{\mathbf{n}}(z')\, e^{i\mathbf{n}\cdot\theta}, \tag{2.3}$$

where $G_{\mathbf{n}}$ is in $H^\infty(\mathbb{R}^d)$ and is zero for negative times t, for all $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$. The framework of weakly non-linear geometric optics is chosen here, namely we expect the leading profile in the asymptotic expansion to be of order ε, which explains the ε factor in front of g^ε in the boundary condition. Note that without loss of generality, we can assume that $\zeta_1, \dots, \zeta_m$ are linearly independent over $\mathbb{Q}$.

Condition $u^\varepsilon_{|t\leqslant 0} = 0$ in (2.1) expresses the nullity of the initial conditions. The time of existence $T > 0$ is not fixed at first and is likely to become sufficiently small to ensure existence of a leading profile.

The study of [Wil02], [CGW11] and [Her15] is here extended to several phases on the boundary. No assumption on the group of boundary frequencies generated by the frequencies $(\zeta_1, \dots, \zeta_m)$ is made, apart from it being finitely generated. In particular it may not be discrete.

We want to approximate the exact solution to (2.1), in the limit where ε goes to 0, by an approximate solution that behaves as ε in range, and $1/\varepsilon$ in frequency. This is the weakly nonlinear geometric optics framework, see [Rau12] and [Mét09]. Recall that in this paper we do not prove stability, i.e. that the approximate solution converges in some sense to the exact one, since we do not know if the latter exists on a time interval independent of ε. To obtain this kind of result, we first have to make several suitable assumptions about the original problem. The rest of this section is devoted to these assumptions, and focuses on the characteristic frequencies associated with the system. Let us first detail the example that inspires the general framework developed in this paper.

Example 2.1. The isentropic compressible Euler equations in two dimensions provide a system of the form of (2.1). Under regularity assumptions on the solution, the associated boundary value problem reads

$$\begin{cases} \partial_t V^\varepsilon + A_1(V^\varepsilon)\,\partial_1 V^\varepsilon + A_2(V^\varepsilon)\,\partial_2 V^\varepsilon = 0 & \text{in } \Omega_T, \\ B\,V^\varepsilon_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ V^\varepsilon_{|t\leqslant 0} = 0, \end{cases} \tag{2.4}$$

with $V^\varepsilon = (v^\varepsilon, \mathbf{u}^\varepsilon) \in \mathbb{R}^3$, where $v^\varepsilon \in \mathbb{R}_+^*$ represents the fluid volume, and $\mathbf{u}^\varepsilon \in \mathbb{R}^2$ its velocity, and where the functions A_1 and A_2 are defined on $\mathbb{R}_+^* \times \mathbb{R}^2$ as

$$A_1(V) := \begin{pmatrix} \mathbf{u}_1 & -v & 0 \\ -c(v)^2/v & \mathbf{u}_1 & 0 \\ 0 & 0 & \mathbf{u}_1 \end{pmatrix}, \qquad A_2(V) := \begin{pmatrix} \mathbf{u}_2 & 0 & -v \\ 0 & \mathbf{u}_2 & 0 \\ -c(v)^2/v & 0 & \mathbf{u}_2 \end{pmatrix}, \tag{2.5}$$

with $c(v) > 0$ representing the sound velocity in the fluid, which depends on its volume v. The noncharacteristic boundary Assumption 1 for system (2.4) is now discussed. In this article, we consider geometric optics expansions for system (2.1) constructed as perturbations around the equilibrium 0, performing a change of variables if necessary. For the Euler system the natural coefficients A_1, A_2 are rather used, and a perturbation around the equilibrium $V_0 = (v_0, 0, u_0)$ is considered, where $v_0 > 0$ is a fixed volume, and $(0, u_0)$ is an incoming *subsonic* velocity, that is such that $0 < u_0 < c_0$, where we denote $c_0 := c(v_0)$.

The Assumption 1 concerns in this case the invertibility of the matrix $A_2(V)$ for $V = (v, u_1, u_2) \in \mathbb{R}_+^* \times \mathbb{R}^2$ in the neighborhood V_0. The determinant of the matrix $A_2(V)$ is given by $\det A_2(V) = u_2\,(u_2^2 - c(v)^2)$, which is nonzero if the velocity u_2 satisfies $0 < u_2 < c(v)$. The equilibrium V_0 verifying this condition, every small enough neighborhood $\mathcal{O}$ of V_0 suits to satisfy Assumption 1.

The rest of the section is dedicated to the characteristic frequencies related to the problem and the associated assumptions.

2.2. **Strict hyperbolicity.** The following definition introduces the notion of characteristic frequency.

Definition 2.2. *For $\alpha = (\tau, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$, the symbol $L(0, \alpha)$ associated with $L(0, \partial_z)$ is defined as*

$$L(0, \alpha) := \tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0).$$

Then we define its characteristic polynomial as $p(\tau,\eta,\xi) := \det L\big(0,(\tau,\eta,\xi)\big)$. *We say that* $\alpha \in \mathbb{R}^{1+d}$ *is a* characteristic frequency *if it is a root of the polynomial* p, *and we denote by* $\mathcal{C}$ *the set of characteristic frequencies.*

The following assumption, called *strict hyperbolicity* (see [BGS07, Definition 1.2]), is made. Assumptions 1 of non-characteristic boundary and of hyperbolicity (whether strict or with constant multiplicity) are very usual, see e.g. [Wil96, CGW11, JMR95], and related to the structure of the problem. Assumption of hyperbolicity of constant multiplicity, which is more general than Assumption 2 of strict hyperbolicity, is sometimes preferred like in [CGW11, JMR95]. We chose here to work with the latter for technical reasons.

Assumption 2 (Strict hyperbolicity)**.** *There exist real functions* $\tau_1 < \cdots < \tau_N$, *analytic with respect to* (η,ξ) *in* $\mathbb{R}^d \setminus \{0\}$, *such that for all* $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$ *and for all* $\tau \in \mathbb{R}$, *the following factorisation is verified*

$$p(\tau,\eta,\xi) = \det\Big(\tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0)\Big) = \prod_{k=1}^{N} \big(\tau - \tau_k(\eta,\xi)\big),$$

where the eigenvalues $-\tau_k(\eta,\xi)$ *of the matrix* $A(\eta,\xi) := \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0)$ *are therefore simple. Consequently, for all* $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$, *the following decompositions of* $\mathbb{C}^N$ *into dimension 1 eigenspaces hold*

$$\mathbb{C}^N = \ker L\big(0,\tau_1(\eta,\xi),\eta,\xi\big) \oplus \cdots \oplus \ker L\big(0,\tau_N(\eta,\xi),\eta,\xi\big), \tag{2.6}$$

$$\mathbb{C}^N = A_d(0)^{-1} \ker L\big(0,\tau_1(\eta,\xi),\eta,\xi\big) \oplus \cdots \oplus A_d(0)^{-1} \ker L\big(0,\tau_N(\eta,\xi),\eta,\xi\big). \tag{2.7}$$

For $k = 1,\dots,N$ *and for* (η,ξ) *in* $\mathbb{R}^d \setminus \{0\}$, *we define the projectors* $\pi_k(\eta,\xi)$ *and* $\widetilde{\pi}_k(\eta,\xi)$, *respectively associated with decompositions* (2.6) *and* (2.7).

For $k = 1,\dots,N$ *and* $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$, *we also denote as* $E_k(\eta,\xi)$ *a unitary eigenvector generating the eigenspace* $\ker L\big(0,\tau_k(\eta,\xi),\eta,\xi\big)$, *so that*

$$E_1(\eta,\xi),\dots,E_N(\eta,\xi) \tag{2.8}$$

is a real normal basis of $\mathbb{C}^N$ *adapted to decomposition* (2.6). *Observe that the family*

$$A_d(0)^{-1}E_1(\eta,\xi),\dots,A_d(0)^{-1}E_N(\eta,\xi) \tag{2.9}$$

is therefore a real normal basis of $\mathbb{C}^N$ *adapted to decomposition* (2.7).

Remark 2.3. i) We will be led further on to consider the modified operator

$$\widetilde{L}(0,\partial_z) := A_d(0)^{-1}\, L(0,\partial_z).$$

This justifies the introduction of the modified symbol $\widetilde{L}(0,\alpha) := A_d(0)^{-1}\, L(0,\alpha)$, and thus of the projector $\widetilde{\pi}_k(\eta,\xi)$, the symbols $L(0,\alpha)$ and $\widetilde{L}(0,\alpha)$ having different ranges.

ii) Since the matrix $A(\eta,\xi) = \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0)$ is real for $(\eta,\xi) \in \mathbb{R}^d$ and the eigenvalues $\tau_1,\dots,\tau_N$ are real, decompositions (2.6) and (2.7) also hold in $\mathbb{R}^N$, but we are interested in the ones of $\mathbb{C}^N$ since some functions that will be studied are complex valued.

iii) If $\alpha = (\tau,\eta,\xi)$ is a characteristic frequency, then by definition and according to Assumption 2, the triplet (τ,η,ξ) satisfies

$$\prod_{k=1}^{N} \big(\tau - \tau_k(\eta,\xi)\big) = 0.$$

There exists therefore an integer k between 1 and N such that $\tau = \tau_k(\eta,\xi)$. In other words, the characteristic manifold $\mathcal{C}$ is the union of the N hypersurfaces given by $\{\tau = \tau_k(\eta,\xi)\}$, $k = 1,\dots,N$.

Remark 2.4. One can verify that in Assumption 2, the functions τ_k for $k = 1,\dots,N$ are positively homogeneous of degree 1 in $\mathbb{R}^d \setminus \{0\}$. The projectors π_k and $\widetilde{\pi}_k$ for $k = 1,\dots,N$ are therefore positively homogeneous of degree 0 in $\mathbb{R}^d \setminus \{0\}$.

Example 2.5. Returning to Example 2.1, for system (2.4) linearized around $V_0 = (v_0, 0, u_0)$, the characteristic polynomial p reads

$$p(\tau,\eta,\xi) = \det \begin{pmatrix} \tau + \xi\, u_0 & -v_0\, \eta & -v_0\, \xi \\ -c_0^2\, \eta / v_0 & \tau + \xi\, u_0 & 0 \\ -c_0^2\, \xi / v_0 & 0 & \tau + \xi\, u_0 \end{pmatrix} = (\tau + \xi\, u_0)\, \big((\tau + \xi\, u_0)^2 - c_0^2\, (\eta^2 + \xi^2) \big).$$

Thus the eigenvalues of the matrix $A(\eta,\xi) = \eta\, A_1(V_0) + \xi\, A_2(V_0)$ are the additive inverse of the roots with respect to τ of the polynomial p, given by

$$\tau_1(\eta,\xi) := -u_0\, \xi - c_0 \sqrt{\eta^2 + \xi^2}, \quad \tau_2(\eta,\xi) := -u_0\, \xi, \quad \tau_3(\eta,\xi) := -u_0\, \xi + c_0 \sqrt{\eta^2 + \xi^2}. \tag{2.10}$$

The functions τ_1, τ_2 and τ_3 are analytic and distinct in $\mathbb{R}^2 \setminus \{0\}$. System (2.4) is therefore strictly hyperbolic, which means that it satisfies Assumption 2 of strict hyperbolicity. We have represented in Figure 1 the characteristic frequencies $\alpha = (\tau,\eta,\xi) \in \mathbb{R}^3$ for system (2.4).

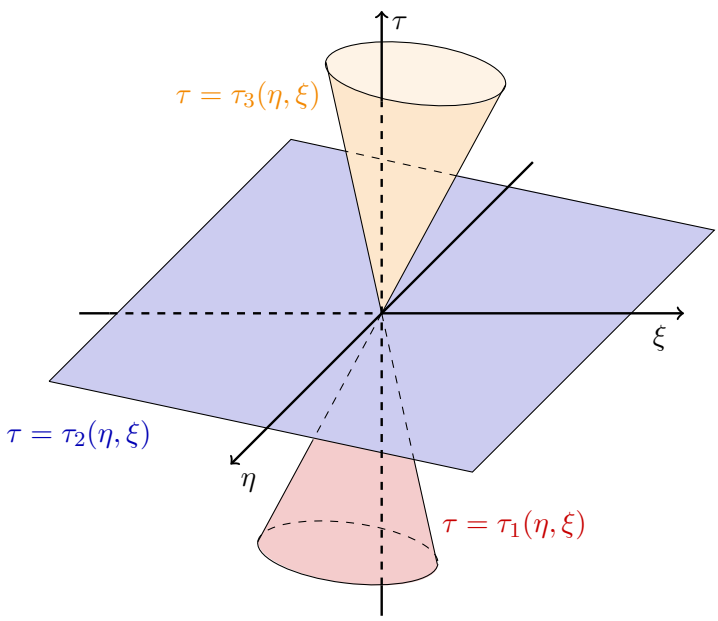

FIGURE 1. Characteristic frequencies for the isentropic compressible Euler system (2.4)

We now define projectors derived from the $\mathbb{C}^N$ decomposition (2.6), that we extend to non-characteristic frequencies. We also determine some equalities between the kernel and range of the projectors π_α and $\widetilde{\pi}_\alpha$ and of the matrices $L(0,\alpha)$ and $\widetilde{L}(0,\alpha)$. The proof is based on the one of [CG10, Lemma 3.2].

Definition 2.6. *Let $\alpha = (\tau,\eta,\xi) \in \mathbb{R}^{1+d} \setminus \{0\}$ be a characteristic frequency and k the integer between 1 and N such that $\tau = \tau_k(\eta,\xi)$. We denote by $\pi_\alpha := \pi_k(\eta,\xi)$ (resp. $\widetilde{\pi}_\alpha := \widetilde{\pi}_k(\eta,\xi)$) the projection from $\mathbb{C}^N$ onto the eigenspace $\ker L\big(0,\tau_k(\eta,\xi),\eta,\xi\big)$ (resp. the subspace $A_d(0)^{-1} \ker L\big(0,\tau_k(\eta,\xi),\eta,\xi\big)$) according to decomposition (2.6) (resp. (2.7)). If the frequency $\alpha \in \mathbb{R}^{1+d} \setminus \{0\}$ is not characteristic, we denote $\pi_\alpha = \widetilde{\pi}_\alpha := 0$ and if $\alpha = 0$ we denote $\pi_0 := \widetilde{\pi}_0 := I$. For all α in $\mathbb{R}^{d+1}$, we can verify that π_α satisfies*

$$\ker L(0,\alpha) = \ker \widetilde{L}(0,\alpha) = \operatorname{Im} \pi_\alpha, \tag{2.11}$$

and

$$\operatorname{Im} L(0,\alpha) = \ker \pi_\alpha, \tag{2.12}$$

and that the projector $\widetilde{\pi}_\alpha$ satisfies

$$\operatorname{Im} \widetilde{L}(0,\alpha) = \ker \widetilde{\pi}_\alpha, \tag{2.13}$$

recalling that $\widetilde{L}(0,\alpha)$ refers to the modified symbol $\widetilde{L}(0,\alpha) := A_d(0)^{-1}\, L(0,\alpha)$.

For all $\alpha \in \mathbb{R}^{1+d} \setminus \{0\}$, we denote by Q_α the partial inverse of the matrix $L(0,\alpha)$, namely the unique matrix Q_α such that $Q_\alpha\, L(0,\alpha) = L(0,\alpha)\, Q_\alpha = I - \pi_\alpha$. If $\alpha = 0$, we define $Q_\alpha := I$.

Proof. Consider $\alpha = (\tau,\eta,\xi)$ in $\mathbb{R}^{d+1}$. Equation (2.11) is satisfied by definition of π_α, and equality of the kernels $\ker L(0,\alpha)$ and $\ker \widetilde{L}(0,\alpha)$, the matrix $A_d(0)$ being invertible. Regarding equation (2.12), we first note that by the rank-nullity theorem and by definition of π_α, the subspaces $\operatorname{Im} L(0,\alpha)$ and $\ker \pi_\alpha$ have the same dimension. We denote by k_0 the integer between 1 and N such that $\tau = \tau_{k_0}(\eta,\xi)$. We consider then an element $L(0,\alpha)\, X$ of $\operatorname{Im} L(0,\alpha)$, with X in $\mathbb{C}^N$ that we decompose according to (2.6), as $X = \sum_{k=1}^N \pi_k(\eta,\xi)\, X$. For $k = 1,\dots,N$, the projector $\pi_k(\eta,\xi)$ admitting the eigenspace of the matrix $A(\eta,\xi)$ associated with the eigenvalue $-\tau_k(\eta,\xi)$ as range, we have

$$\begin{aligned} L(0,\alpha)\, X &= \sum_{k=1}^N \Big(\tau_{k_0}(\eta,\xi)\, I + A(\eta,\xi)\Big)\pi_k(\eta,\xi)\, X \\ &= \sum_{k\neq k_0} \big(\tau_{k_0}(\eta,\xi) - \tau_k(\eta,\xi)\big)\, \pi_k(\eta,\xi)\, X. \end{aligned}$$

Equation (2.11) being satisfied, we deduce that $L(0,\alpha)\, X$ belongs to

$$\bigoplus_{k\neq k_0} \ker L\big(0,(\tau_k(\eta,\xi),\eta,\xi)\big),$$

which, by definition of the projectors π_k, is equal to the kernel of $\pi_{k_0}(\eta,\xi) = \pi_\alpha$. With the equality of dimensions, equation (2.12) is therefore verified. The proof of equation (2.13) is similar: we consider X in $\mathbb{C}^N$ that we decompose as $X = \sum_{k=1}^N \pi_k(\eta,\xi)\, X$, and then we write

$$\begin{aligned} \widetilde{L}(0,\alpha)\, X &= A_d(0)^{-1} \sum_{k=1}^N \Big(\tau_{k_0}(\eta,\xi)\, I + A(\eta,\xi)\Big)\pi_k(\eta,\xi)\, X \\ &= A_d(0)^{-1} \sum_{k\neq k_0} \big(\tau_{k_0}(\eta,\xi) - \tau_k(\eta,\xi)\big)\, \pi_k(\eta,\xi)\, X, \end{aligned}$$

so that $\widetilde{L}(0,\alpha)\, X$ belongs to

$$\bigoplus_{k\neq k_0} A_d(0)^{-1} \ker L\big(0,(\tau_k(\eta,\xi),\eta,\xi)\big) = \ker \widetilde{\pi}_\alpha.$$

Once again by equality of dimensions it leads to equation (2.13). □

Remark 2.7. i) For every $k = 1,\dots,N$, the projectors $\pi_k(\eta,\xi)$ and $\widetilde{\pi}_k(\eta,\xi)$ are positively homogeneous of degree 0 in $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$. Furthermore, by strict hyperbolicity, the basis $E_1(\eta,\xi),\dots,E_N(\eta,\xi)$ and $A_d(0)^{-1}E_1(\eta,\xi),\dots,A_d(0)^{-1}E_N(\eta,\xi)$ are analytic with respect to $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$, and the maps $(\eta,\xi) \mapsto \pi_k(\eta,\xi)$ and $(\eta,\xi) \mapsto \widetilde{\pi}_k(\eta,\xi)$ are therefore analytic in $\mathbb{R}^d \setminus \{0\}$. Thus, by compactness of the sphere $\mathbb{S}^{d-1}$, for all $k = 1,\dots,N$, the projectors $\pi_k(\eta,\xi)$ and $\widetilde{\pi}_k(\eta,\xi)$ are uniformly bounded with respect to $(\eta,\xi) \in \mathbb{R}^d \setminus \{0\}$. The projectors π_α and $\widetilde{\pi}_\alpha$ are therefore bounded with respect to α in $\mathbb{R}^{1+d}$.

ii) Unlike the projectors π_k and $\widetilde{\pi}_k$, $k = 1,\dots,N$, the projectors π_α and $\widetilde{\pi}_\alpha$ are homogeneous of degree 0 with respect to α in $\mathbb{R}^{d+1}$, and not only *positively* homogeneous. Indeed, the claim is obvious if α is zero or noncharacteristic, and if α is a nonzero characteristic frequency, and λ a nonzero real number, then, since $\ker L(0,\lambda\,\alpha) = \ker L(0,\alpha)$ and $\operatorname{Im} L(0,\lambda\,\alpha) = \operatorname{Im} L(0,\alpha)$, we have $\pi_{\lambda\,\alpha} = \pi_\alpha$. The proof is the same for $\widetilde{\pi}_\alpha$.

2.3. **The uniform Kreiss-Lopatinskii condition and some preliminary results.** We define the following space of frequencies

$$\begin{aligned}\Xi &:= \{\zeta = (\sigma = \tau - i\gamma, \eta) \in (\mathbb{C} \times \mathbb{R}^{d-1})\backslash\{0\} \mid \gamma \geqslant 0\},\\ \Sigma &:= \left\{\zeta \in \Xi \mid \tau^2 + \gamma^2 + |\eta|^2 = 1\right\},\\ \Xi_0 &:= \{\zeta \in \Xi \mid \gamma = 0\},\\ \Sigma_0 &:= \Xi_0 \cap \Sigma.\end{aligned}$$

We also define the matrix that we get when applying the Laplace-Fourier transform to the operator $L(0, \partial_z)$. For all $\zeta = (\sigma, \eta) \in \Xi$, let

$$\mathcal{A}(\zeta) := -i\, A_d(0)^{-1}\Big(\sigma I + \sum_{i=1}^{d-1} \eta_j\, A_j(0)\Big).$$

The noncharacteristic boundary Assumption 1 is used here to define the matrix $\mathcal{A}(\zeta)$. We note that if $\zeta = (\tau, \eta) \in \Xi_0$, and if $i\xi$ is an imaginary eigenvalue of $\mathcal{A}(\zeta)$, then the frequency (τ, η, ξ) is a real characteristic frequency, and vice versa.

Hersh lemma [Her63, Lemma 1] ensures that for ζ in $\Xi\backslash\Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no eigenvalue of zero real part, and that the stable subspace associated with the eigenvalues of negative real part, denoted by $E_-(\zeta)$, is of constant dimension, denoted p. Furthermore, the integer p is obtained as the number of positive eigenvalues of the matrix $A_d(0)$. We denote by $E_+(\zeta)$ the unstable subspace $\mathcal{A}(\zeta)$ associated with eigenvalues of positive real part, that is of dimension $N - p$.

In [Kre70] (see also [CP82, Theorem 3.5] and [BGS07, Lemma 4.5]) it is shown that the stable and unstable subspaces $E_\pm$ extend continuously to the whole space Ξ in the strictly hyperbolic case (Assumption 2). We still denote by $E_\pm$ the extensions to Ξ. The main assumption of this work may now be stated, which, along with Assumptions 1 and 2, ensures that system (2.1) is well posed locally in time. Indeed the three assumptions 1, 2 and 3 are stable under small perturbations around the equilibrium. Just like Assumptions 1 and 2, the following assumption is very structural to the problem.

Assumption 3 (Uniform Kreiss-Lopatinskii condition)**.** *For all $\zeta \in \Xi$, we have*

$$\ker B \cap E_-(\zeta) = \{0\}.$$

In particular, it forces the rank of the matrix B to be equal to the dimension of $E_-(\zeta)$, namely $M = p$.

Remark 2.8. Historically, the first given definition of the uniform Kreiss-Lopatinskii condition did not involve the extension of E_- to Ξ_0. The original definition states that, for all $\zeta \in \Xi \setminus \Xi_0$,

$$\ker B \cap E_-(\zeta) = \{0\},$$

and that the linear map $\big(B_{|E_-(\zeta)}\big)^{-1}$ is uniformly bounded with respect to $\zeta \in \Xi \setminus \Xi_0$, see for instance [Sar65]. Indeed, the space $E_-(\zeta)$ being homogeneous of degree zero and continuous with respect to $\zeta \in \Xi$, and by compactness of the unitary sphere Σ, we note that Assumption 3 implies that the linear map $\big(B_{|E_-(\zeta)}\big)^{-1}$ is uniformly bounded with respect to $\zeta \in \Xi$.

It has already been discussed that for $\zeta \in \Xi\backslash\Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no imaginary eigenvalue. We now commit to describe more precisely the matrix $\mathcal{A}(\zeta)$ for ζ in Ξ_0 as well as the continuous extension to Ξ_0 of the spaces $E_\pm(\zeta)$. The following result, proved by Kreiss [Kre70] for the strictly hyperbolic case that is of interest here, Métivier [Mét00] for the constantly hyperbolic case, and extended by Métivier and Zumbrun [MZ05] to an even more general framework, gives a very useful decomposition of the matrix $\mathcal{A}(\zeta)$ when ζ belongs to Ξ_0.

Proposition 2.9 (Block structure)**.** *When Assumption 2 is satisfied, for all $\underline{\zeta} \in \Xi$, there exist a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Ξ, an integer $L \geqslant 1$, a partition $N = \rho_1 + \cdots + \rho_L$ and an invertible matrix T analytic in $\overline{\mathcal{V}}$ such that for all $\zeta \in \mathcal{V}$, we have*

$$T(\zeta)\,\mathcal{A}(\zeta)\,T(\zeta)^{-1} = \operatorname{diag}\big(\mathcal{A}_1(\zeta), \ldots, \mathcal{A}_L(\zeta)\big),$$

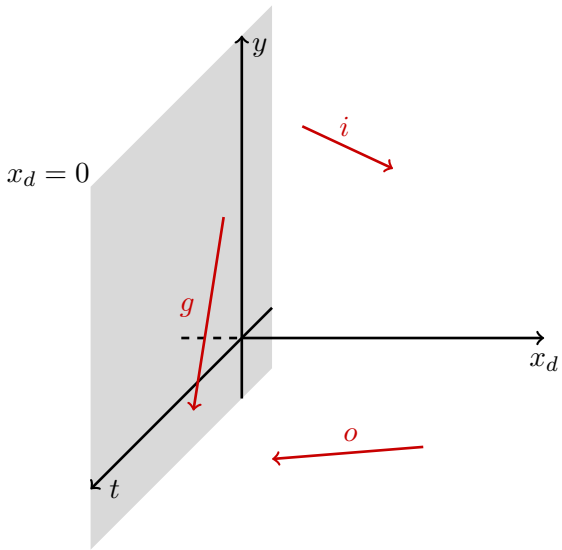

FIGURE 2. Incoming (i), outgoing (o) and glancing (g) vector field.

where for all j the matrix $\mathcal{A}_j(\zeta)$ is of size ρ_j and satisfies one of the following properties:

i) the real part of the matrix $\mathcal{A}_j(\zeta)$, defined by $(\mathcal{A}_j(\zeta) + \mathcal{A}_j(\zeta)^*)/2$, is positive-definite,
ii) the real part of the matrix $\mathcal{A}_j(\zeta)$ is negative-definite,
iii) $\rho_j = 1$, $\mathcal{A}_j(\zeta)$ is imaginary when γ is zero and $\partial_\gamma \mathcal{A}_j(\underline{\zeta}) \in \mathbb{R}^*$,
iv) $\rho_j > 1$, the coefficients of $\mathcal{A}_j(\zeta)$ are imaginary when γ is zero, there exists $\xi_j \in \mathbb{R}$ such that

$$\mathcal{A}_j(\underline{\zeta}) = \begin{pmatrix} i\,\xi_j & i & 0 \\ & \ddots & i \\ 0 & & i\,\xi_j \end{pmatrix},$$

and the bottom left coefficient of $\partial_\gamma \mathcal{A}_j(\underline{\zeta})$ is real and non zero.

This result, commonly referred to as "block structure" [BGS07, Section 5.1.2], is fundamental for the proof of Proposition 2.21 below. In the aim of describing the subspaces $E_\pm(\zeta)$ for $\zeta \in \Xi_0$, the vector fields associated with each real characteristic phase are now defined.

Definition 2.10. *Let $\alpha = (\tau, \eta, \xi) \in \mathbb{R}^{d+1} \setminus \{0\}$ be a characteristic frequency, and k the integer between 1 and N such that $\tau = \tau_k(\eta, \xi)$. The group velocity $\mathbf{v}_\alpha$ associated with α is defined as*

$$\mathbf{v}_\alpha := \nabla_{\eta,\xi}\, \tau_k(\eta, \xi).$$

We shall say that α is glancing (resp. incoming, outgoing) if $\partial_\xi \tau_k(\eta, \xi)$ is zero (resp. negative, positive). Then the vector field X_α associated with α is defined as

$$X_\alpha := \partial_t - \mathbf{v}_\alpha \cdot \nabla_x = \partial_t - \nabla_\eta \tau_k(\eta, \xi) \cdot \nabla_y - \partial_\xi \tau_k(\eta, \xi)\, \partial_{x_d}. \tag{2.14}$$

The vector field X_α is represented in Figure 2 in the glancing, incoming and outgoing case.

Example 2.11. We start by giving an example of a boundary condition for Example 2.1 satisfying the uniform Kreiss-Lopatinskii condition. For this purpose we look for a matrix B in $\mathcal{M}_{2,3}(\mathbb{R})$ of maximal rank, that generates strictly dissipative boundary conditions (see [BGS07, Definition 9.2]), namely such that its kernel, which is of dimension 1, is generated by a nonzero vector E satisfying

$${}^tE\, S(V)\, A_2(V)\, E < 0$$

for all V in the neighborhood of V_0, where the matrix $S(V)$ refers to a Friedrichs symmetrizer of the system. Such strictly dissipative boundary conditions satisfy in particular the uniform

Kreiss-Lopatinskii condition, see [BGS07, Proposition 4.4]. In our example, the following symmetrizer may be considered

$$S(V) = \operatorname{diag}\big(c(v)^2, v^2, v^2\big).$$

Recall that, in the notations of the example, a symmetrizer $S(V)$ is a positive definite matrix such that the matrices $S(V)\,A_1(V)$ and $S(V)\,A_2(V)$ are symmetric for all V in a neighborhood of V_0. It is then determined that a suitable vector E is given by $E = (v_0, 0, u_0)$, since in that case we have

$${}^tE\,S(V_0)\,A_2(V_0)\,E = u_0\,v_0^2\,(u_0^2 - c_0^2)$$

the right-hand side quantity being negative by assumption on V_0, so it stays negative in a neighborhood of V_0. Thus a matrix B of maximal rank whose kernel is generated by E is for example given by

$$B := \begin{pmatrix} 0 & v_0 & 0 \\ -u_0 & 0 & v_0 \end{pmatrix},$$

which gives an example of a boundary condition satisfying the uniform Kreiss-Lopatinskii condition for Example 2.1 of compressible isentropic Euler equations in dimension 2.

Interest is now made on the eigenvalues of the matrix $\mathcal{A}(\tau,\eta)$ for the system of Example 2.1. Their expressions, for $(\tau,\eta) \in \mathbb{R}^2 \setminus \{0\}$, depend on the sign of $\tau^2 - \eta^2\,(c_0^2 - u_0^2)$, as represented in Figure 3.

If $|\tau| > \sqrt{c_0^2 - u_0^2}\,|\eta|$, i.e. if $\zeta = (\tau,\eta)$ is in the so-called *hyperbolic* region $\mathcal{H}$ ([Ben14, Definition 2.1]), then the matrix $\mathcal{A}(\zeta)$ admits three simple imaginary eigenvalues given by

$$i\,\xi_1(\tau,\eta) := i\,\frac{\tau\,u_0 + \operatorname{sign}(\tau)\,c_0\,\sqrt{\tau^2 - \eta^2\,(c_0^2 - u_0^2)}}{c_0^2 - u_0^2}, \tag{2.15a}$$

$$i\,\xi_2(\tau,\eta) := i\,\frac{\tau\,u_0 - \operatorname{sign}(\tau)\,c_0\,\sqrt{\tau^2 - \eta^2\,(c_0^2 - u_0^2)}}{c_0^2 - u_0^2}, \tag{2.15b}$$

$$i\,\xi_3(\tau,\eta) := i\,\frac{-\tau}{u_0}, \tag{2.15c}$$

where $\operatorname{sign}(x) := x/|x|$ for $x \neq 0$. The number $\xi_1(\tau,\eta)$ being real, the frequency $\alpha_1(\tau,\eta) := \big(\tau,\eta,\xi_1(\tau,\eta)\big)$ is a real characteristic frequency. It is then determined that we have $\tau = \tau_3\big(\eta,\xi_1(\tau,\eta)\big)$ if $\tau > 0$ and $\tau = \tau_1\big(\eta,\xi_1(\tau,\eta)\big)$ if $\tau < 0$. A calculation gives, if $\tau > 0$,

$$\partial_\xi \tau_3\big(\eta,\xi_1(\tau,\eta)\big) = \sqrt{\frac{\tau^2 - \eta^2(c_0^2 - u_0^2)}{\eta^2 + \xi_1^2(\tau,\eta)}},$$

and, if $\tau < 0$,

$$\partial_\xi \tau_1\big(\eta,\xi_1(\tau,\eta)\big) = \sqrt{\frac{\tau^2 - \eta^2(c_0^2 - u_0^2)}{\eta^2 + \xi_1^2(\tau,\eta)}}.$$

Thus the frequency $\alpha_1(\tau,\eta) = \big(\tau,\eta,\xi_1(\tau,\eta)\big)$ is always outgoing. Likewise, it is determined that the real characteristic frequency $\alpha_2(\tau,\eta) := \big(\tau,\eta,\xi_2(\tau,\eta)\big)$ is always incoming, and the frequency $\alpha_3(\tau,\eta) := \big(\tau,\eta,\xi_3(\tau,\eta)\big)$ is incoming as well.

If ζ is located in the so-called *glancing* region $\mathcal{G}$, i.e. if $|\tau| = \sqrt{c_0^2 - u_0^2}\,|\eta|$, then the matrix $\mathcal{A}(\zeta)$ admits one imaginary simple eigenvalue $i\,\xi_3(\zeta)$ which is still given by formula (2.15c), and a double imaginary eigenvalue given by

$$i\,\xi_1(\zeta) = i\,\xi_2(\zeta) = i\,\frac{\tau\,u_0}{c_0^2 - u_0^2}.$$

In this case we still have $\tau = \tau_3\big(\eta,\xi_1(\tau,\eta)\big)$ if $\tau > 0$ and $\tau = \tau_1\big(\eta,\xi_1(\tau,\eta)\big)$ if $\tau < 0$, and regarding the characteristic frequency $\alpha_3(\tau,\eta) = \big(\tau,\eta,\xi_3(\tau,\eta)\big)$, we still have $\tau = \tau_2\big(\eta,\xi_3(\tau,\eta)\big)$. Thus it is determined that

$$\partial_\xi \tau_1\big(\eta,\xi_1(\tau,\eta)\big) = \partial_\xi \tau_3\big(\eta,\xi_1(\tau,\eta)\big) = 0,$$

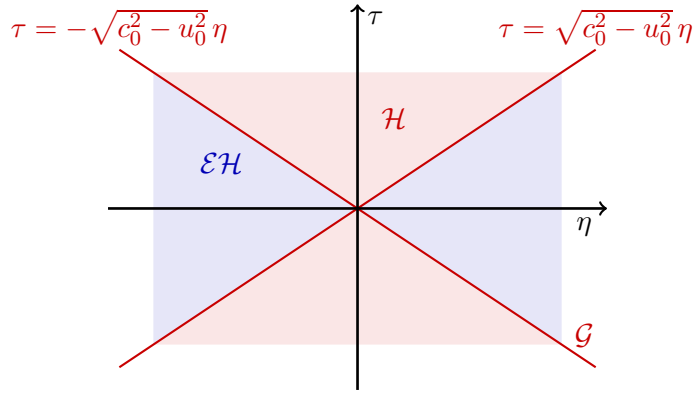

FIGURE 3. Areas of Ξ_0 for the isentropic compressible Euler equations

and therefore, regardless of the sign of τ, the frequency $\alpha_1(\tau,\eta)$ is glancing. As for it, the frequency $\alpha_3(\tau,\eta)$ is always incoming.

Finally if $|\tau| < \sqrt{c_0^2 - u_0^2}\,|\eta|$ and so if ζ is in the so-called *mixed* region $\mathcal{EH}$, then the matrix $\mathcal{A}(\zeta)$ has one simple imaginary eigenvalue $i\,\xi_3(\zeta)$ given by formula (2.15c), and two simple eigenvalues of nonzero real part (symmetric with respect to the imaginary axis), that are still denoted by $i\,\xi_1$ et $i\,\xi_2$ and which are given by

$$i\,\xi_1(\tau,\eta) := i\,\frac{\tau\,u_0 + i\,c_0\,\operatorname{sign}(\tau)\,\sqrt{\eta^2\,(c_0^2 - u_0^2) - \tau^2}}{c_0^2 - u_0^2},$$
$$i\,\xi_2(\tau,\eta) := i\,\frac{\tau\,u_0 - i\,c_0\,\operatorname{sign}(\tau)\,\sqrt{\eta^2\,(c_0^2 - u_0^2) - \tau^2}}{c_0^2 - u_0^2}.$$

The real characteristic frequency $\alpha_3(\tau,\eta) = \big(\tau,\eta,\xi_3(\tau,\eta)\big)$ is once again incoming.

In the following, in order for the matrix factor of the partial derivative with respect to the normal variable x_d in the equations to be the identity matrix, the modified operator $\widetilde{L}(u,\partial_z) := A_d(u)^{-1}\,L(u,\partial_z)$ shall be considered. For $j = 1,\dots,d-1$, we denote $\widetilde{A}_j := A_d^{-1}A_j$ and $\widetilde{A}_0 := A_d^{-1}$. The following lemma, which is a result of [Lax57] adapted by [CGW11, Lemma 2.11], shows that, under suitable assumptions, the operator $\widetilde{\pi}_\alpha\,\widetilde{L}(0,\partial_z)\,\pi_\alpha$ is given by a constant coefficient scalar transport operator, and therefore that the operator $\widetilde{\pi}_\alpha\,\widetilde{L}(0,\partial_z)$ acts on polarized profiles (i.e. profiles U such that $\pi_\alpha\,U = U$) as a much simpler operator.

Lemma 2.12 ([CGW11, Lax Lemma])**.** *Let $\alpha = (\tau,\eta,\xi) \in \mathbb{R}^{1+d}\setminus\{0\}$ be a real* non glancing *characteristic frequency and k the integer between 1 and N such that $\tau = \tau_k(\eta,\xi)$. Then we have*

$$\widetilde{\pi}_\alpha\,\widetilde{L}(0,\partial_z)\,\pi_\alpha = \frac{-1}{\partial_\xi\tau_k(\eta,\xi)}\,X_\alpha\,\widetilde{\pi}_\alpha\,\pi_\alpha,$$

where X_α is the vector field associated with α defined by (2.14). Then we denote

$$\widetilde{X}_\alpha := \frac{-1}{\partial_\xi\tau_k(\eta,\xi)}\,X_\alpha = \partial_{x_d} - \frac{1}{\partial_\xi\tau_k(\eta,\xi)}\,\partial_t + \frac{1}{\partial_\xi\tau_k(\eta,\xi)}\,\nabla_\eta\tau_k(\eta,\xi)\cdot\nabla_y.$$

For the sake of completeness, the proof of [CGW11] is recalled here.

Proof. According to identity (2.11), we have

$$\widetilde{L}\big(0,(\tau_k(\eta,\xi),\eta,\xi)\big)\,\pi_k(\eta,\xi) = \Big(\tau_k(\eta,\xi)\,\widetilde{A}_0(0) + \sum_{i=1}^{d-1}\eta_i\,\widetilde{A}_i(0) + \xi\,I\Big)\,\pi_k(\eta,\xi) = 0. \tag{2.16}$$

The Dunford formula and the implicit function theorem ensure that in the strictly hyperbolic case, the projectors π_k as well as the real functions τ_k are differentiable with respect to (η,ξ)

in $\mathbb{R}^d \setminus \{0\}$ (they even depend analytically of (η, ξ)). Thus identity (2.16) is differentiated with respect to ξ in a neighborhood of a frequency (η, ξ) in $\mathbb{R}^d \setminus \{0\}$ to obtain

$$\Big(\partial_\xi \tau_k(\eta,\xi)\, \widetilde{A}_0(0) + I\Big)\, \pi_k(\eta,\xi) + \Big(\tau_k(\eta,\xi)\, \widetilde{A}_0(0) + \sum_{i=1}^{d-1} \eta_i\, \widetilde{A}_i(0) + \xi\, I\Big)\, \partial_\xi \pi_k(\eta,\xi) = 0,$$

and therefore, multiplying by $\widetilde{\pi}_k(\eta,\xi)$ on the left, according to identity (2.13), we get

$$\widetilde{\pi}_k(\eta,\xi)\, \widetilde{A}_0(0)\, \pi_k(\eta,\xi) = -\frac{1}{\partial_\xi \tau_k(\eta,\xi)} \widetilde{\pi}_k(\eta,\xi)\, \pi_k(\eta,\xi). \tag{2.17}$$

Likewise, for $i = 1, \dots, d-1$, equality (2.16) is differentiated with respect to η_i and next multiplied by $\widetilde{\pi}_k(\eta,\xi)$ to obtain

$$\widetilde{\pi}_k(\eta,\xi) \Big(\partial_{\eta_j} \tau_k(\eta,\xi)\, \widetilde{A}_0(0) + \widetilde{A}_i(0)\Big)\, \pi_k(\eta,\xi) = 0.$$

With (2.17), we thus get

$$\widetilde{\pi}_k(\eta,\xi)\, \widetilde{A}_i(0)\, \pi_k(\eta,\xi) = \frac{\partial_{\eta_i} \tau_k(\eta,\xi)}{\partial_\xi \tau_k(\eta,\xi)} \widetilde{\pi}_k(\eta,\xi)\, \pi_k(\eta,\xi),$$

which concludes the proof of the lemma. □

The following results use the classical Lax Lemma, whose proof is similar to the one of Lemma 2.12. The result is recalled here.

Lemma 2.13 ([Lax57])**.** *Let $\alpha = (\tau, \eta, \xi) \in \mathbb{R}^{1+d} \setminus \{0\}$ be a real characteristic frequency and k the integer between 1 and N such that $\tau = \tau_k(\eta,\xi)$. Then we have*

$$\pi_\alpha\, L(0, \partial_z)\, \pi_\alpha = X_\alpha\, \pi_\alpha,$$

where X_α is the vector field associated with α defined by (2.14).

The first lemma below, quite standard, states that the group velocities $\mathbf{v}_\alpha$ are bounded. The result presented here is not optimal, considering the constant C can be taken equal to 1, but it is sufficient for our analysis, and its proof is simpler.

Lemma 2.14. *There exists a positive constant C such that, for $k = 1 \dots, N$ and (η, ξ) in $\mathbb{R}^d \setminus \{0\}$, we have*

$$|\nabla_{\eta,\xi} \tau_k(\eta,\xi)| \leqslant C \sup_{(\eta,\xi)\in\mathbb{S}^{d-1}} \rho\big(A(\eta,\xi)\big),$$

where we recall that $A(\eta,\xi)$ has been defined for (η,ξ) in $\mathbb{R}^d \setminus \{0\}$ in Assumption 2, and where $\rho\big(A(\eta,\xi)\big)$ refers to the spectral radius of the matrix $A(\eta,\xi)$. Then we denote by $\mathcal{V}^*$ the finite quantity

$$\mathcal{V}^* := C \sup_{(\eta,\xi)\in\mathbb{S}^{d-1}} \rho\big(A(\eta,\xi)\big),$$

which bounds the group velocities.

Proof. First note that the quantity $\mathcal{V}^*$ is actually finite. Indeed, according to Assumption 2, we have

$$\rho\big(A(\eta',\xi')\big) = \max_{k=1,\dots,N} \big|\tau_k(\eta',\xi')\big|,$$

and the real functions $\tau_1, \dots, \tau_N$ are analytic in $\mathbb{R}^d \setminus \{0\}$, and thus bounded on $\mathbb{S}^{d-1}$.

Now let (η,ξ) be in $\mathbb{R}^d \setminus \{0\}$ and k between 1 and N. According to Lax Lemma 2.13, we have, for (η',ξ') in $\mathbb{S}^{d-1}$,

$$\pi_k(\eta,\xi)\, A(\eta',\xi')\, \pi_k(\eta,\xi) = -d\tau_k(\eta,\xi) \cdot (\eta',\xi')\, \pi_k(\eta,\xi).$$

But since the following equality holds,

$$|\nabla_{\eta,\xi} \tau_k(\eta,\xi)| = \sup_{(\eta',\xi')\in\mathbb{S}^{d-1}} \big|d\tau_k(\eta,\xi) \cdot (\eta',\xi')\big|,$$

we obtain

$$|\nabla_{\eta,\xi}\tau_k(\eta,\xi)| \leqslant \|\pi_k(\eta,\xi)\| \sup_{(\eta',\xi')\in\mathbb{S}^{d-1}} \big\|A(\eta',\xi')\big\|. \tag{2.18}$$

On an other hand, because of Remark 2.7, there exists a positive constant C such that

$$\|\pi_k(\eta,\xi)\| \leqslant C, \quad k=1,\dots,N, \tag{2.19}$$

uniformly with respect to (η,ξ) in $\mathbb{R}^d\setminus\{0\}$. Finally, Assumption 2 claims that the matrix $A(\eta',\xi')$ is diagonalizable and well-conditioned, so there exists a positive constant C such that

$$\sup_{(\eta',\xi')\in\mathbb{S}^{d-1}} \big\|A(\eta',\xi')\big\| \leqslant C \sup_{(\eta',\xi')\in\mathbb{S}^{d-1}} \rho\big(A(\eta',\xi')\big). \tag{2.20}$$

Equations (2.18), (2.19), and (2.20) then lead to the result. □

The second result quantitatively links the vector $\widetilde{\pi}_\alpha\, E_k(\eta,\xi)$ to the vector $E_k(\eta,\xi)$, which will be useful in the following to get a control from below of the first vector. It is mentioned, for its second part, in [CG10].

Lemma 2.15. *Let $\alpha=(\tau,\eta,\xi)\in\mathbb{R}^{1+d}\setminus\{0\}$ be a real characteristic frequency and k the integer between 1 and N such that $\tau=\tau_k(\eta,\xi)$. Then we have*

$$\widetilde{\pi}_\alpha\, E_k(\eta,\xi) = -\partial_\xi\tau_k(\eta,\xi)\, A_d(0)^{-1}E_k(\eta,\xi).$$

In particular, if the frequency α is not glancing, the projector $\widetilde{\pi}_\alpha$ induces an isomorphism from $\operatorname{Im}\pi_\alpha$ to $\operatorname{Im}\widetilde{\pi}_\alpha$.

Proof. First the vector $E_k(\eta,\xi)$ is decomposed in basis (2.9) adapted to decomposition (2.7):

$$E_k(\eta,\xi) = \sum_{j=1}^N \lambda_j\, A_d(0)^{-1}E_j(\eta,\xi), \tag{2.21}$$

so that we have $\widetilde{\pi}_\alpha\, E_k(\eta,\xi) = \lambda_k\, A_d(0)^{-1}E_k(\eta,\xi)$. Thus the aim is to determine the coefficient λ_k. Given that $\pi_\alpha\, E_k(\eta,\xi) = E_k(\eta,\xi)$, and according to decomposition (2.21), we have

$$A_d(0)\,\pi_\alpha\, E_k(\eta,\xi) = \sum_{j=1}^N \lambda_j\, E_j(\eta,\xi),$$

thus

$$\pi_\alpha\, A_d(0)\,\pi_\alpha\, E_k(\eta,\xi) = \lambda_k\, E_k(\eta,\xi).$$

And we conclude using Lax Lemma 2.13 which claims that $\pi_\alpha\, A_d(0)\,\pi_\alpha = -\partial_\xi\tau_k(\eta,\xi)\,\pi_\alpha$.

To show that the projector $\widetilde{\pi}_\alpha$ induces an isomorphism from $\operatorname{Im}\pi_\alpha$ to $\operatorname{Im}\widetilde{\pi}_\alpha$, the two spaces $\operatorname{Im}\pi_\alpha$ and $\operatorname{Im}\widetilde{\pi}_\alpha$ having the same dimension, it is sufficient to prove that the intersection

$$\ker\widetilde{\pi}_\alpha \cap \operatorname{Im}\pi_\alpha$$

is trivial. So we consider a vector X of $\mathbb{C}^N$ belonging to this intersection. Because X belongs to $\operatorname{Im}\pi_\alpha$, by definition of the vector $E_k(\eta,\xi)$, it writes

$$X = \lambda\, E_k(\eta,\xi),$$

where k is the integer between 1 and N such that $\alpha=\big(\tau_k(\eta,\xi),\eta,\xi\big)$ and $\lambda\in\mathbb{R}$. According to the previous result, we have

$$\widetilde{\pi}_\alpha\, X = -\partial_\xi\tau_k(\eta,\xi)\, A_d(0)^{-1}X.$$

But we also have $\widetilde{\pi}_\alpha\, X = 0$ by assumption and $\partial_\xi\tau_k(\eta,\xi)\neq 0$, the frequency α being non-glancing. We therefore obtain $X=0$, which is the sought result. □

We are now in position to describe the decomposition of the stable subspace $E_-(\zeta)$ for $\zeta\in\Xi_0$, which uses the strict hyperbolicity Assumption 2.

Proposition 2.16 ([Wil96], Proposition 3.4)**.** *Consider $\zeta = (\tau, \eta) \in \Xi_0$. We denote by $i\,\xi_j(\zeta)$ for $j = 1, \dots, \mathcal{M}(\zeta)$ the distinct complex eigenvalues of the matrix $\mathcal{A}(\zeta)$, and if $\xi_j(\zeta)$ is real, we shall denote by $\alpha_j(\zeta) := (\tau, \eta, \xi_j(\tau, \eta))$ the associated real characteristic frequency. If $\xi_j(\zeta)$ is real, we also denote by k_j the integer between 1 and N such that $\tau = \tau_{k_j}(\eta, \xi_j(\zeta))$. Then the set $\{1, 2, \dots, \mathcal{M}(\zeta)\}$ decomposes as the disjoint union*

$$\{1, 2, \dots, \mathcal{M}(\zeta)\} = \mathcal{G}(\zeta) \cup \mathcal{I}(\zeta) \cup \mathcal{P}(\zeta) \cup \mathcal{O}(\zeta) \cup \mathcal{N}(\zeta), \tag{2.22}$$

where the sets $\mathcal{G}(\zeta)$, $\mathcal{I}(\zeta)$, $\mathcal{P}(\zeta)$, $\mathcal{O}(\zeta)$ and $\mathcal{N}(\zeta)$ correspond to indexes j such that respectively $\alpha_j(\zeta)$ is glancing, $\alpha_j(\zeta)$ is incoming, $\mathrm{Im}(\xi_j(\zeta))$ is positive, $\alpha_j(\zeta)$ is outgoing and $\mathrm{Im}(\xi_j(\zeta))$ is negative.

Then the following decomposition of $E_-(\zeta)$ holds

$$E_-(\zeta) = \bigoplus_{j \in \mathcal{G}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{I}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta), \tag{2.23}$$

where for each index j, the subspace $E_-^j(\zeta)$ is precisely described as follows.

i) If $j \in \mathcal{P}(\zeta)$, the space $E_-^j(\zeta)$ is the generalized eigenspace $\mathcal{A}(\zeta)$ associated with the eigenvalue $i\,\xi_j(\zeta)$.

ii) If $j \in \mathcal{R}(\zeta)$, we have $E_-^j(\zeta) = \ker L\big(0, \alpha_j(\zeta)\big)$, which is of dimension 1.

iii) If $j \in \mathcal{G}(\zeta)$, we denote by n_j the algebraic multiplicity of the imaginary eigenvalue $i\xi_j(\zeta)$. For small positive γ, the multiple eigenvalue $i\,\xi_j(\tau, \eta)$ splits into n_j simple eigenvalues, denoted by $i\,\xi_j^k(\tau - i\gamma, \eta)$, $k = 1, \dots, n_j$, all of nonzero real part. We denote by μ_j the number (independent of $\gamma > 0$) of the eigenvalues $i\,\xi_j^k(\tau - i\gamma, \eta)$ of negative real part. Then $E_-^j(\zeta)$ is of dimension μ_j and is generated by the vectors w satisfying $[\mathcal{A}(\zeta) - i\xi_j(\zeta)]^{\mu_j} w = 0$. Furthermore, if n_j is even, $\mu_j = n_j/2$ and if n_j is odd, μ_j is equal to $(n_j - 1)/2$ or $(n_j + 1)/2$.

Likewise, the unstable subspace $E_+(\zeta)$ decomposes as

$$E_+(\zeta) = \bigoplus_{j \in \mathcal{G}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{O}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{N}(\zeta)} E_+^j(\zeta), \tag{2.24}$$

with similar description of the subspaces $E_+^j(\zeta)$. In particular, if the set $\mathcal{G}(\zeta)$ is empty, then

$$\mathbb{C}^N = E_-(\zeta) \oplus E_+(\zeta).$$

Remark 2.17. The notation $\xi_j(\zeta)$ should not be taken for a function ξ_j depending on $\zeta \in \Xi_0$. Indeed for example the set $\mathcal{M}(\zeta)$ depends on ζ. However, note that the matrix $\mathcal{A}(\zeta)$ is homogeneous of degree 1 with respect to ζ in Ξ_0. Thus the number $\mathcal{M}(\zeta)$ as well as the cardinality of the sets $\mathcal{G}(\zeta)$, $\mathcal{I}(\zeta) \cup \mathcal{O}(\zeta)$ and $\mathcal{P}(\zeta) \cup \mathcal{N}(\zeta)$ depend only on the direction of ζ in Ξ_0. We therefore assume that, $\zeta \in \Xi_0$ being fixed, for $\lambda \in \mathbb{R}^*$, the indexes $1, \dots, \mathcal{M}(\lambda\,\zeta)$ are arranged in a way that, for $j = 1, \dots, \mathcal{M}(\zeta)$, we have

$$\xi_j(\lambda\,\zeta) = \lambda\,\xi_j(\zeta), \quad \text{so that} \quad \alpha_j(\lambda\,\zeta) = \lambda\,\alpha_j(\zeta).$$

With this ordering, we note that if for $\zeta \in \Xi_0$, the frequency $\alpha_j(\zeta)$ is glancing, incoming or outgoing (resp. $\xi_j(\zeta)$ is of nonzero imaginary part), i.e. if $j \in \mathcal{G}(\zeta) \cup \mathcal{O}(\zeta) \cup \mathcal{I}(\zeta)$ (resp. $j \in \mathcal{P}(\zeta) \cup \mathcal{N}(\zeta)$), then for $\lambda \in \mathbb{R}^*$, the frequency $\alpha_j(\lambda\,\zeta) = \lambda\,\alpha_j(\zeta)$ is still glancing, incoming or outgoing (resp. $\xi_j(\lambda\,\zeta)$ is still of nonzero imaginary part), that is to say $j \in \mathcal{G}(\lambda\,\zeta) \cup \mathcal{O}(\lambda\,\zeta) \cup \mathcal{I}(\lambda\,\zeta)$ (resp. $j \in \mathcal{P}(\lambda\,\zeta) \cup \mathcal{N}(\lambda\,\zeta)$). More precisely, if $j \in \mathcal{P}(\zeta)$ (resp. $\mathcal{N}(\zeta)$), then $j \in \mathcal{N}(-\zeta)$ (resp. $\mathcal{P}(-\zeta)$).

Definition 2.18. *Consider $\zeta \in \Xi_0$. We say that ζ is a glancing point and we denote $\zeta \in \mathcal{G}$ if, with notations of Proposition 2.16, there exists an index j between 1 and $\mathcal{M}(\zeta)$ such that $j \in \mathcal{G}(\zeta)$, in other words, if ζ is such that there exists a real nonzero number ξ such that the frequency (ζ, ξ) is characteristic and glancing.*

An assumption is now made, that helps to prove that the projectors associated with decomposition (2.23) are bounded uniformly with respect to ζ in Ξ_0. This assumption has already been made in [Sar65, Wil96], and seems essential, see [Wil00].

Assumption 4. *For all $k = 1, \dots, N$, and for all $(\underline{\eta}, \underline{\xi}) \in \mathbb{R}^d \setminus \{0\}$, we have*

$$\frac{\partial \tau_k}{\partial \xi}(\underline{\eta}, \underline{\xi}) = 0 \quad \Rightarrow \quad \frac{\partial^2 \tau_k}{\partial \xi^2}(\underline{\eta}, \underline{\xi}) \neq 0.$$

Remark 2.19. We will see during the proof of Proposition 2.21 in appendix A that assumption 4 implies that for all ζ in Ξ_0, for all index j in $\mathcal{G}(\zeta)$, we have $n_j = 2$, using the notations of Proposition 2.16. We deduce that $\mu_j = 1$ and that the component $E_-^j(\zeta)$ of the stable subspace $E_-(\zeta)$ is of dimension 1 and given by $\ker L\big(0, \alpha_j(\zeta)\big)$.

Definition 2.20. *For $\zeta \in \Xi_0$ and, using the notations of Proposition 2.16, for an index j in $\mathcal{G}(\zeta) \cup \mathcal{I}(\zeta)$, we denote by $\Pi_-^j(\zeta)$ the projection from $E_-(\zeta)$ on the component $E_-^j(\zeta)$ according to decomposition (2.23).*

We also denote by $\Pi_-^e(\zeta)$ the projection from $E_-(\zeta)$ on the elliptic stable component $E_-^e(\zeta) := \oplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta)$ according to decomposition (2.23).

Finally, if ζ is not glancing, that is if the set $\mathcal{G}(\zeta)$ is empty, then according to Proposition 2.16 we have the following decomposition of $\mathbb{C}^N$

$$\mathbb{C}^N = \bigoplus_{j \in \mathcal{O}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{N}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{I}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta). \tag{2.25}$$

In that case we denote by $\Pi_{\mathbb{C}^N}^e(\zeta)$ the projection from $\mathbb{C}^N$ on the stable elliptic component $E_-^e(\zeta)$ according to this decomposition, and by $\Pi_{\mathbb{C}^N}^{e,+}(\zeta)$ the projection from $\mathbb{C}^N$ on the unstable elliptic component $E_+^e(\zeta) := \oplus_{j \in \mathcal{N}(\zeta)} E_+^j(\zeta)$ according to the same decomposition.

The following proposition will be a key result in our analysis. It uses in a crucial way Assumption 4, as well as the strict hyperbolicity Assumption 2.

Proposition 2.21 ([Wil96])**.** *Under Assumptions 2 and 4, for $\zeta \in \Xi_0$, the projectors $\Pi_-^j(\zeta)$ for j in $\mathcal{G}(\zeta) \cup \mathcal{I}(\zeta)$, and the projectors $\Pi_-^e(\zeta)$ are uniformly bounded with respect to ζ in Ξ_0.*

The proof of this result, omitted in [Wil96] and which requires some work, is postponed until Appendix A.

Thanks to Assumption 4 we are also able to prove the following result, which continues Lemma 2.15, and establishes a control from below over the normal component of the group velocity, and therefore over the vector $\widetilde{\pi}_\alpha E_k(\eta, \xi)$ for all $\alpha = \big(\tau_k(\eta, \xi), \eta, \xi\big)$, involving the distance from $\big(\tau_k(\eta, \xi), \eta\big)$ to the glancing set $\mathcal{G}$. Its proof uses notations and results from the one of Proposition 2.21, and is therefore also skipped until Appendix A.

Lemma 2.22. *There exists a positive constant $C > 0$ such that, if the real frequency $\alpha = (\tau, \eta, \xi)$ in $\mathbb{R}^{1+d} \setminus \{0\}$ is characteristic, and if k between 1 and N is such that $\tau = \tau_k(\eta, \xi)$, then we have*

$$|\partial_\xi \tau_k(\eta, \xi)| \geqslant C \frac{\operatorname{dist}\big((\tau, \eta), \mathcal{G}\big)^{1/2}}{|(\tau, \eta)|^{1/2}}.$$

Using Lemma 2.15, we therefore obtain the following estimate

$$|\widetilde{\pi}_\alpha E_k(\eta, \xi)| \geqslant C \frac{\operatorname{dist}\big((\tau, \eta), \mathcal{G}\big)^{1/2}}{|(\tau, \eta)|^{1/2}}. \tag{2.26}$$

3. Functional framework

3.1. **Set of frequencies inside the domain.** To define the functional framework that will be used, we need first to determine a priori which frequencies may appear in the solution to (2.1). For a detailed discussion of this analysis, reference is made to [Rau12, Chapters 9 and 10] and [MA88]. The presence on the boundary of the frequencies $\zeta_1, \dots, \zeta_m$ creates, by nonlinear interaction, the following group of frequencies on the boundary

$$\mathcal{F}_b := \zeta_1 \mathbb{Z} + \cdots + \zeta_m \mathbb{Z} \subset \mathbb{R}^d. \tag{3.1}$$

The assumption is now made that this group does not contain any glancing point, which have been introduced in Definition 2.18. This assumption is often made, and allows to avoid complications created by the glancing modes, see e.g. [CG10, CGW11].

Assumption 5. *We have*

$$\big(\mathcal{F}_b \setminus \{0\}\big) \cap \mathcal{G} = \emptyset.$$

In other words, with the notations of Proposition 2.16, for all $\zeta \in \mathcal{F}_b \setminus \{0\}$, the set $\mathcal{G}(\zeta)$ of indexes j between 1 and $\mathcal{M}(\zeta)$ such that the characteristic frequency $\big(\zeta, \xi_j(\zeta)\big)$ is glancing, is an empty set.

However, attention must be paid on the fact that despite Assumption 5, the set $\mathcal{F}_b \setminus \{0\}$ may contain frequencies arbitrary close to the set of glancing frequencies $\mathcal{G}$, namely frequencies ζ admitting a lifting inside the domain $\alpha = \big(\tau_k(\eta, \xi), \eta, \xi\big)$ of which the normal component of the group velocity given by $-\partial_\xi \tau_k(\eta, \xi)$ is arbitrary close to zero. This phenomenon is well illustrated in Example 3.1 of compressible isentropic Euler equations below. In the following, we will need a control on the projectors $\Pi^{\mathrm{e}}_{\mathbb{C}^N}(\zeta)$ for ζ in the group $\mathcal{F}_b \setminus \{0\}$, defined for ζ non glancing. Indeed the norm of this projector increases when ζ gets close to the glancing set $\mathcal{G}$. This is why a small divisor assumption is now made, that gives a control over the distance between ζ in $\mathcal{F}_b$ and $\mathcal{G}$ for large ζ, notably leading to Proposition 6.16 below.

Assumption 6. *There exists a real number a_1 and a positive constant c such that for all ζ in $\mathcal{F}_b \setminus \{0\}$, we have*

$$\mathrm{dist}\,\big(\zeta, \mathcal{G}\big) \geqslant c\,|\zeta|^{-a_1}.$$

Note that Assumption 5 is a consequence of Assumption 6, so could be omitted. However we have chosen to keep both assumptions because they play two different roles in the proofs. Small divisors Assumption 6 is quite unusual, and plays a technical role in the proofs.

The operator $L(0, \partial_z)$ being hyperbolic, the frequencies on the boundary $\zeta \in \mathcal{F}_b \setminus \{0\}$ are then lifted inside the domain into frequencies (ζ, ξ). We will see that the polarization conditions for the leading profile cancel the modes associated with noncharacteristic frequencies. Therefore, since we are interested in bounded solutions, at this point only the incoming and evanescent characteristic frequencies lifted from frequencies on the boundary are created. Assumption 5 is used here to exclude the possibility of creating glancing frequencies $\big(\zeta, \xi_j(\zeta)\big)$, that is with $j \in \mathcal{G}(\zeta)$. Thus, at this stage, the set of frequencies $\{0\} \cup \mathcal{F}^{\mathrm{in}} \cup \mathcal{F}^{\mathrm{ev}}$ has been obtained for the leading profile, where the sets $\mathcal{F}^{\mathrm{in}}$ and $\mathcal{F}^{\mathrm{ev}}$ are given by

(3.2)

$$\mathcal{F}^{\mathrm{in}} := \Big\{\big(\zeta, \xi_j(\zeta)\big),\ \zeta \in \mathcal{F}_b \setminus \{0\}\,,\ j \in \mathcal{I}(\zeta)\Big\}, \quad \mathcal{F}^{\mathrm{ev}} := \Big\{\big(\zeta, \xi_j(\zeta)\big),\ \zeta \in \mathcal{F}_b \setminus \{0\}\,,\ j \in \mathcal{P}(\zeta)\Big\}.$$

Apart from exceptional cases, the set $\mathcal{F}^{\mathrm{in}}$ is not finitely generated, which imposes an almost-periodic framework for the normal fast variable.

Interest is now made on resonances that may occur inside the domain. By nonlinear interaction, two frequencies $\alpha_{j_p}(\zeta_p) = \big(\zeta_p, \xi_{j_p}(\zeta_p)\big)$ and $\alpha_{j_q}(\zeta_q) = \big(\zeta_q, \xi_{j_q}(\zeta_q)\big)$ of $\mathcal{F}^{\mathrm{in}}$ may resonate to create a characteristic frequency $\alpha_{j_r}(\zeta_r) = \big(\zeta_r, \xi_{j_r}(\zeta_r)\big)$ in the following way:

$$n_p\,\alpha_{j_p}(\zeta_p) + n_q\,\alpha_{j_q}(\zeta_q) = n_r\,\alpha_{j_r}(\zeta_r), \quad n_p, n_q, n_r \in \mathbb{Z} \setminus \{0\}\,.$$

If the index j_r belongs to the set $\mathcal{O}(\zeta_r)$, that is to say if $\alpha_{j_r}(\zeta_r)$ is an outgoing real characteristic frequency, a new frequency inside the domain is thus created, which does not already belong to the initial set $\mathcal{F}^{\mathrm{in}}$ defined above. The simplifying assumption that it does not occur is made, so there is no outgoing characteristic frequency created through a resonant triplet. More precisely, we assume that the outgoing and the incoming frequencies do not resonate one with the other. The set of outgoing frequencies $\mathcal{F}^{\mathrm{out}}$ is defined as

$$\mathcal{F}^{\mathrm{out}} := \Big\{\big(\zeta, \xi_j(\zeta)\big),\ \zeta \in \mathcal{F}_b \setminus \{0\}\,,\ j \in \mathcal{O}(\zeta)\Big\}. \tag{3.3}$$

Assumption 7. *i) There does not exist a couple (α_p, α_q) of incoming characteristic frequencies $\mathcal{F}^{\mathrm{in}}$ and a couple of integers (n_p, n_q) such that the frequency*

$$n_p\,\alpha_p + n_q\,\alpha_q$$

is real, characteristic and outgoing.

ii) There does not exist a couple (α_p, α_q) of outgoing characteristic frequencies $\mathcal{F}^{\text{out}}$ and a couple of integers (n_p, n_q) such that the frequency

$$n_p\,\alpha_p + n_q\,\alpha_q$$

is real, characteristic and incoming.

This is a strong assumption, that, up to our knowledge, cannot be found in the literature. It implies that there is no outgoing mode in the leading profile of the expansion. For example in [CGW11], there is no such assumption, but since there is only one phase on the boundary, there are a finite number of resonances, conversely to our case. When there are outgoing modes, they are coupled with incoming ones through the trace on the boundary. This is an issue since it seems that the suitable functional framework for outgoing modes, tailored for evolution in time, is different from the one for incoming modes, adapted to propagation in normal direction. It also complicates the iterative process used to construct a solution. We have chosen here to focus on the construction of a functional framework for incoming modes that allows to solve the problem.

Note that if three real characteristic frequencies $\alpha_p, \alpha_q, \alpha_r$ resonate as

$$n_p\,\alpha_p + n_q\,\alpha_q = n_r\,\alpha_r, \quad n_p, n_q, n_r \in \mathbb{Z},$$

then according to the previous assumption, the frequencies $\alpha_p, \alpha_q, \alpha_r$ are either all incoming or all outgoing. On an other hand, despite Assumption 7, there may exist a countable infinite number of resonances between incoming frequencies, as it is the case in Example 3.1 of compressible isentropic Euler equations in dimension 2.

At this stage, for a new frequency to be created from $\mathcal{F}^{\text{in}}$, there must exist a resonance between two frequencies of $\mathcal{F}^{\text{in}}$, that creates a real characteristic frequency which does not already belong to $\mathcal{F}^{\text{in}}$. The frequencies in $\mathcal{F}^{\text{in}}$ are incoming, and according to Assumption 7 above, a resonance between two incoming frequencies may only produce an incoming frequency, which already belongs to $\mathcal{F}^{\text{in}}$. There is therefore no new frequency created, and the final set of frequencies inside the domain created by nonlinear interaction on the boundary and lifting inside the domain is given by

$$\mathcal{F} := \{0\} \cup \Big\{\alpha_j(\zeta),\ \zeta \in \mathcal{F}_b \setminus \{0\}\,,\ j \in \mathcal{I}(\zeta) \cup \mathcal{P}(\zeta)\Big\}. \tag{3.4}$$

We expect for the leading profile of the solution to (2.1) to feature all frequencies in $\mathcal{F}$ created by lifting. It leads to consider, to maintain generality, all frequencies in $\mathcal{F}$. Yet it seems unlikely that the group generated by $\mathcal{F}$ may be finitely generated, which a priori excludes an asymptotic expansion of the solution u^ε in the form of quasi-periodic functions. Following [JMR95] and [CGW11] after, a quasi-periodic framework is nevertheless considered for the tangential fast variables (the group of frequencies on the boundary being finitely generated), but an almost-periodic framework for the normal fast variable is considered. The next subsection is devoted to that question and describes the functional framework used in this analysis. This part is ended by verifying the different assumptions and assertions made in this subsection for Example 2.1 of compressible isentropic Euler equations in dimension 2.

Example 3.1. The notations of Example 2.1 and those after are used. Assumptions 5,6 and 7 concern the group of frequencies on the boundary $\mathcal{F}_b$, thus adequate frequencies on the boundary must be considered for Example 2.1. To simplify the calculations, we take two frequencies ζ^1 and ζ^δ given by $\zeta^1 := (c_0\,\eta_0, \eta_0)$ and $\zeta^\delta := (c_0\,\delta\,\eta_0, \eta_0)$, with $\eta_0 > 0$ and δ an irrational number strictly larger than 1, so that ζ^1 and ζ^δ are both in the hyperbolic region $\mathcal{H}$. Recall that $c_0 = c(v_0) > 0$ refers to the sound velocity and that the equilibrium $V_0 = (v_0, 0, u_0)$ satisfies $0 < u_0 < c_0$. The boundary frequencies lattice $\mathcal{F}_b$ is therefore given in this example by

$$\mathcal{F}_b = \big\{\big(c_0\,\eta_0\,(p + \delta\,q), \eta_0\,(p+q)\big) \mid p, q \in \mathbb{Z}\big\} \subset \mathbb{R}^2.$$

We denote by $\zeta_{p,q} := (\tau_{p,q}, \eta_{p,q}) := \big(c_0\,\eta_0\,(p + \delta q), \eta_0\,(p+q)\big)$ the frequency of $\mathcal{F}_b$ given by $p\,\zeta^1 + q\,\zeta^\delta$, for p, q in $\mathbb{Z}$.

	$p/q > K_+$	$K_+ > p/q > -\delta$	$p/q < -\delta$
$q > 0$	$\zeta_{p,q}$ is in the hyperbolic region $\mathcal{H}$ with $\tau_{p,q} > 0$	$\zeta_{p,q}$ is in the mixed region $\mathcal{EH}$	$\zeta_{p,q}$ is in the hyperbolic region $\mathcal{H}$ with $\tau_{p,q} < 0$
$q < 0$	$\zeta_{p,q}$ is in the hyperbolic region $\mathcal{H}$ with $\tau_{p,q} < 0$	$\zeta_{p,q}$ is in the mixed region $\mathcal{EH}$	$\zeta_{p,q}$ is in the hyperbolic region $\mathcal{H}$ with $\tau_{p,q} > 0$

FIGURE 4. Position of $\zeta_{p,q}$ depending on p and q.

A nonzero frequency $\zeta_{p,q}$ is glancing if and only if $|\tau_{p,q}| = \sqrt{c_0^2 - u_0^2}\,|\eta_{p,q}|$, that is to say if and only if [1]

$$\frac{p}{q} \in \left\{ \frac{\sqrt{1-M^2}-\delta}{1-\sqrt{1-M^2}}, \frac{-\sqrt{1-M^2}-\delta}{1+\sqrt{1-M^2}} \right\}. \tag{3.5}$$

We have denoted by M the Mach number given by $M := u_0/c_0$, belonging to $(0,1)$. If the two real numbers

$$K_- := \frac{\sqrt{1-M^2}-\delta}{1-\sqrt{1-M^2}} \quad \text{et} \quad K_+ := \frac{-\sqrt{1-M^2}-\delta}{1+\sqrt{1-M^2}} \tag{3.6}$$

are irrational, then there does not exist a relation of the form (3.5), and there is therefore no glancing frequency in the group $\mathcal{F}_b \setminus \{0\}$. Thus we make the assumption that K_- and K_+ are irrational, so that the assumption 5 is verified. We may for example take $M = \sqrt{3}/2$ and $\delta > 1$ irrational. We summarize now the different areas where the frequencies $\zeta_{p,q}$ may be, depending on p and q. First note that we have $0 > K_+ > K_- > -\delta$. Recall that the frequency $\zeta_{p,q}$ is in the hyperbolic region (resp. mixed region) if and only if $|\tau_{p,q}| > \sqrt{c_0^2 - u_0^2}\,|\eta_{p,q}|$ (resp. $|\tau_{p,q}| < \sqrt{c_0^2 - u_0^2}\,|\eta_{p,q}|$). We thus infer the classification given in Figure 4. The calculation steps are not detailed, but one case is treated in more details below.

The remark concerning the group velocities that follows Assumption 5 is now illustrated. For that purpose a sequence of frequencies on the boundary that draw near the glancing set is considered, see Figure 5. The real number K_+ given (3.6) being irrational, there exist two sequences $(p_k)_k$ and $(q_k)_k$ of integers such that for $k \geqslant 0$, p_k and q_k are coprime, $q_k > 0$, and such that

$$\frac{p_k}{q_k} \xrightarrow[k\to+\infty]{>} \frac{-\sqrt{1-M^2}-\delta}{1+\sqrt{1-M^2}} = K_+.$$

Note that since $K_+ > K_- > -\delta$, for all $k \geqslant 0$, we have $p_k/q_k > K_- > -\delta$, so on one hand we have $p_k + \delta q_k > 0$, and on the other hand, independently of the sign of $p_k + q_k$, we have $p_k + \delta q_k > \sqrt{1-M^2}\,|p_k + q_k|$. The frequencies ζ_{p_k,q_k} are therefore in the hyperbolic region $\mathcal{H}$ with $\tau > 0$, and draw near the glancing region $\mathcal{G}$, see Figure 5. Since $\tau_{p_k,q_k} > 0$, according to Example 2.11, the last component of the group velocity associated with the frequency on the inside $\alpha_1(\tau_{p_k,q_k}, \eta_{p_k,q_k})$ is given by

$$\partial_\xi \tau_3\big(\eta_{p_k,q_k}, \xi_1(\tau_{p_k,q_k}, \eta_{p_k,q_k})\big) \leqslant \sqrt{\frac{\tau_{p_k,q_k}^2 - \eta_{p_k,q_k}^2\,(c_0^2 - u_0^2)}{\eta_{p_k,q_k}^2}} \tag{3.7}$$

$$= c_0 \sqrt{q_k\left(1+\sqrt{1-M^2}\right)\frac{p_k + \delta q_k - (p_k+q_k)(1-M^2)}{(p_k+q_k)^2}} \sqrt{\frac{p_k}{q_k} - \frac{-\sqrt{1-M^2}-\delta}{1+\sqrt{1-M^2}}}. \tag{3.8}$$

Note that the quantity under the first square root sign of (3.8) is non negative, since for all $k \geqslant 0$, $q_k > 0$ and $p_k/q_k > K_-$, so that $p_k + \delta q_k - (p_k+q_k)(1-M^2) > 0$. This quantity being bounded, the quantity (3.8) converges towards zero by construction of the integers (p_k, q_k). We see that the normal group velocity (3.7) of the hyperbolic frequencies ζ_{p_k,q_k} converges towards

[1] If $q = 0$, then according to the relation $|\tau_{p,q}| = \sqrt{c_0^2 - u_0^2}\,|\eta_{p,q}|$, we have $p^2 = (1 - u_0^2/c_0^2)\,p^2$ so p is also zero, i.e. the frequency $\zeta_{p,q}$ is zero, which is excluded by assumption.

zero as k goes to infinity. It shows that the normal group velocity of the hyperbolic frequencies, although nonzero, may be arbitrary close to zero.

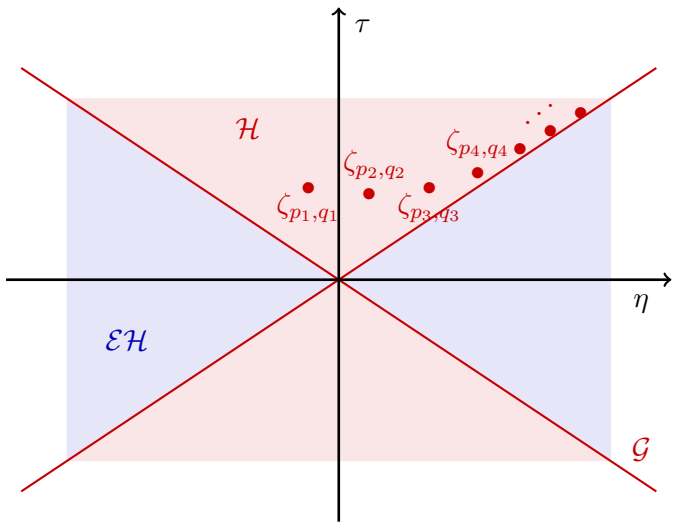

FIGURE 5. Sequence of frequencies that draw near the glancing region, of which the normal group velocity goes to zero.

Interest is now made on Assumption 7 and into the resonances between real characteristic frequencies. We recall the notations of Example 2.11, and we first determine that, in the hyperbolic region, the eigenvalues $i\,\xi_1(\zeta)$, $i\,\xi_2(\zeta)$ and $i\,\xi_3(\zeta)$, defined by (2.15), are given, for $\zeta = \zeta_{p,q}$ in $\mathcal{F}_b \setminus \{0\}$, by

$$i\,\xi_1(\zeta_{p,q}) = i\,\eta_0\,\frac{M\,(p+\delta q) + \operatorname{sign}(p+\delta q)\,\sqrt{(p+q)^2\,M^2 + 2pq(\delta-1) + q^2(\delta^2-1)}}{1-M^2},$$

$$i\,\xi_2(\zeta_{p,q}) = i\,\eta_0\,\frac{M\,(p+\delta q) - \operatorname{sign}(p+\delta q)\,\sqrt{(p+q)^2\,M^2 + 2pq(\delta-1) + q^2(\delta^2-1)}}{1-M^2},$$

$$i\,\xi_3(\zeta_{p,q}) = -i\,\eta_0\,\frac{p+\delta q}{M}.$$

The case of the glancing region is excluded by assumption, and the one of the mixed region is included in the following, considering there is in this case only one imaginary eigenvalue, which is the linear eigenvalue $i\,\xi_3(\zeta)$. We first observe that, the eigenvalue $i\,\xi_3(\zeta_{p,q})$ begin linear, it generates resonances of the form

$$\alpha_3(\zeta_{p,q}) + \alpha_3(\zeta_{r,s}) = \alpha_3(\zeta_{p+r,q+s}), \quad \forall p,q,r,s \in \mathbb{Z}.$$

The frequency $\alpha_3(\zeta_{p,q})$ being always incoming, there are therefore already an infinite number of resonances between incoming frequencies. From now on the notation $\alpha_{1,2}$ refers to one of the characteristic frequency α_1 or α_2. Since by linearity of α_3, the resonance between two frequencies α_3 and a frequency $\alpha_{1,2}$ is impossible, the two following cases of resonance are still to be investigated:

$$\alpha_{1,2}(\zeta_{p,q}) + \alpha_{1,2}(\zeta_{r,s}) = \alpha_3(\zeta_{p+r,q+s}) \quad \text{et} \quad \alpha_{1,2}(\zeta_{p,q}) + \alpha_{1,2}(\zeta_{r,s}) = \alpha_{1,2}(\zeta_{p+r,q+s}).$$

In the first case, it is equivalent to the relation

$$\begin{aligned}\big[(p+q)^2-(r+s)^2\big]\,M^8 + 2\big[(p+q)^2-(r+s)^2\big]\big(2(rs-pq)(\delta-1)+(s^2-q^2)(\delta^2-1)\big)\,M^6 \\ + C_4(p,q,r,s,\delta)\,M^4 + C_2(p,q,r,s,\delta)\,M^2 + (p+r+\delta q+\delta s)^2 = 0\end{aligned} \tag{3.9}$$

where coefficients $C_4(p,q,r,s,\delta)$ and $C_2(p,q,r,s,\delta)$ are polynomial in their variables. Two cases may now occur, depending on whether the coefficient in front of M^8 in equation (3.9) is zero or not.

i) Either we have $(p+q)^2-(r+s)^2 \neq 0$, in which case equation (3.9) is a polynomial equation of degree 4 in $\mathbb{Q}[\delta]$ satisfied by M^2.
ii) Or we have $(p+q)^2=(r+s)^2$, and in this case (3.9) is a polynomial equation of degree at most 2 satisfied by M^2 in $\mathbb{Q}[\delta]$. If once again the coefficients $C_4(p,q,r,s,\delta)$ and $C_2(p,q,r,s,\delta)$ in front of M^4 and M^2 are zero, then we get $p+r+\delta q+\delta s=0$. Therefore we have $r=-p$ and $s=-q$, that is to say $\zeta_{r,s}=-\zeta_{p,q}$, so the studied resonance is actually self-interaction of $\zeta_{p,q}$ with itself to generate the zero frequency. Thus, if $(p+q)^2=(r+s)^2$, the only cases of a real resonance are those where M^2 is a root of a polynomial of degree 1 or 2 in $\mathbb{Q}[\delta]$.

It has therefore been determined that for a resonance of the first type to occur (not of self-interaction type), then M^2 needs to be a root of a polynomial of degree at most 4 in $\mathbb{Q}[\delta]$.

For the second type of resonance, such a relation is verified if and if only if the following relation holds

$$(ps-qr)^2(1-M^2)(\delta-1)^2=0, \tag{3.10}$$

that is to say, since we have $0<M<1$ and $\delta>1$, if and only if the two frequencies $\zeta_{p,q}$ and $\zeta_{r,s}$ are collinear. Then one may write $\zeta_{p,q}=\lambda\,\zeta_{t,w}$ and $\zeta_{r,s}=\mu\,\zeta_{t,w}$, with $\zeta_{t,w}$ a hyperbolic frequency, and λ,μ in $\mathbb{Z}^*$. Next verify that, since we have $\operatorname{sign}(\lambda\,(t+\delta w))=\operatorname{sign}(\lambda)\operatorname{sign}(t+\delta w)$, the two following relations hold $\xi_1(\lambda\,\zeta_{t,w})=\lambda\,\xi_1(\zeta_{t,w})$ and $\xi_2(\lambda\,\zeta_{t,w})=\lambda\,\xi_2(\zeta_{t,w})$. The same holds for the two frequencies $\mu\,\zeta_{t,w}$ and $(\lambda+\mu)\,\zeta_{t,w}$. The only two resonances that may occur are therefore

$$\lambda\,\alpha_1(\zeta_{t,w})+\mu\,\alpha_1(\zeta_{t,w})=(\lambda+\mu)\,\alpha_1(\zeta_{t,w}) \quad \text{et} \quad \lambda\,\alpha_2(\zeta_{t,w})+\mu\,\alpha_2(\zeta_{t,w})=(\lambda+\mu)\,\alpha_2(\zeta_{t,w}),$$

which both are actually self-interaction of frequencies $\alpha_1(\zeta_{t,w})$ and $\alpha_2(\zeta_{t,w})$ with themselves: the evolution of the harmonics λ and μ are coupled with the one of $\lambda+\mu$. In particular, the three frequencies implied in this resonance are either all incoming or all outgoing. Thus, if M^2 is not a root of a polynomial of degree at most 4 in $\mathbb{Q}[\delta]$, Assumption 7 is verified for the compressible isentropic Euler equations in dimension 2, with the group of frequencies on the boundary that has been considered.

We finally dig into the small divisors Assumption 6. One can check that[2], depending on the sign of p,q, the distance between $\zeta_{p,q}$ and the glancing set $\mathcal{G}$ is given by

$$\operatorname{dist}(\zeta_{p,q},\mathcal{G})=C\left|(p+\delta q)\pm\sqrt{1-M^2}(p+q)\right|=Cq\big(1\pm\sqrt{1-M^2}\big)\left|\frac{p}{q}-K_\pm\right|.$$

If p and q are not of the same size scale, then the same holds for $p+\delta q$ and $p+q$, so the previous distance can be lower bounded by a positive constant. We thus may in the following assume that

$$C_1|p|\leqslant|q|\leqslant C_2|p|. \tag{3.11}$$

According to Roth theorem, see [Sch91, Theorem 2A], if the real numbers K_+ and K_- given by (3.6) are algebraic numbers (and irrational, which has been previously assumed), then they satisfy

$$\left|K_\pm-\frac{p}{q}\right|\geqslant C\,|q|^{-(2+\varepsilon)},$$

for all $q\in\mathbb{N}^*$, $p\in\mathbb{Z}$ and $\varepsilon>0$. So for all $\zeta_{p,q}$ in $\mathcal{F}_b\setminus\{0\}$, we get, using (3.11),

$$\operatorname{dist}(\zeta_{p,q},\mathcal{G})\geqslant C|q|^{-3/2}\geqslant C|\zeta_{p,q}|^{-3/2},$$

and Assumption 6 is therefore verified.

In conclusion, for the compressible isentropic Euler equations in dimension 2 to satisfies Assumptions 5, 6 and 7, it is therefore sufficient that the Mach number M and the parameter $\delta>1$ are such that K_+ and K_- are irrational algebraic numbers and that M^2 is not a polynomial solution of degree at most 4 in $\mathbb{Q}[\delta]$. The set of solutions to such equations being countable, one

[2]Using the fact that the glancing set is constituted here of two lines, an elementary geometrical argument allows to reduce to the distance with respect to η only, where the constant C is given by

$$C:=\frac{\eta_0\,\sin\arctan(\sqrt{c_0^2-u_0^2})}{\sqrt{1-M^2}}.$$

may convince himself that the set of real numbers M satisfying these properties is not empty. One may for example choose $M := \sqrt{3}/2$, which gives $K_- = 1 - 2\delta$ and $K_+ = -(1+2\delta)/3$, and also choose $\delta = \sqrt[7]{2} > 1$. In this way the real numbers K_+ and K_- are actually irrational algebraic numbers. On an other hand, a relation of the form (3.9) cannot be satisfied, except for trivial cases on p, q, r, s, because one can check that it forms an algebraic equation of degree 4 in δ, which is not an algebraic number of degree 4.

3.2. **Spaces of profiles.** According to the analysis of which frequencies may appear in the solution to (2.1), we must define for the leading profile a functional framework that allows to consider superposition of waves of the form

$$a(z)\, e^{i\,\alpha\cdot z/\varepsilon},$$

for α a characteristic frequency in $\mathcal{F}$. Since we choose a quasi-periodic framework for the fast tangential variables, we write such a wave as

$$a(z)\, e^{i\,n_1\,\zeta_1\cdot z'/\varepsilon} \cdots e^{i\,n_m\,\zeta_m\cdot z'/\varepsilon}\, e^{i\,\xi\,x_d/\varepsilon},$$

where $\alpha = (\zeta, \xi)$ with $\zeta = n_1\,\zeta_1 + \cdots + n_m\,\zeta_m \in \mathcal{F}_b$ and $\xi \in \mathbb{C}$. Next we denote by $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$ the fast tangential variables which substitute to $(z'\cdot\zeta_1/\varepsilon, \ldots, z'\cdot\zeta_m/\varepsilon)$ and $\psi_d \in \mathbb{R}_+$ the fast normal variable substituting to x_d/ε.

For each integer $s \geqslant 0$ and for $T > 0$, we denote by $H^s_+(\omega_T \times \mathbb{T}^m)$ the space of functions of $(t, y, \theta) \in \omega_T \times \mathbb{T}^m$, zero for negative times t, of which all derivatives of order less or equal to s belong to $L^2(\omega_T \times \mathbb{T}^m)$.

Now we describe the general space that will contain the oscillating and evanescent profiles spaces. We choose only a uniform control with respect to the fast and slow normal variables since it would be difficult to control derivatives of the leading profile with respect to these two variables.

Definition 3.2. *For an integer $s \geqslant 0$ and for $T > 0$, we define the space $\mathcal{E}_{s,T}$ as the set of functions U of $(z', x_d, \theta, \psi_d) \in \omega_T \times \mathbb{R}_+ \times \mathbb{T}^m \times \mathbb{R}_+$, bounded continuous with respect to (x_d, ψ_d) in $\mathbb{R}_+ \times \mathbb{R}_+$ with values in $H^s_+(\omega_T \times \mathbb{T}^m)$, equipped with the obvious norm*

$$\|U\|_{\mathcal{E}_{s,T}} := \sup_{x_d>0, \psi_d>0} \|U(\,.\,, x_d,\,.\,, \psi_d)\|_{H^s_+(\omega_T\times\mathbb{T}^m)}\,.$$

We may now introduce the space of oscillating profiles, corresponding to real frequencies $\alpha_j(\zeta)$. Following [JMR95], we choose a quasi-periodic framework with respect to θ and an almost-periodic one with respect to ψ_d, namely we consider the closure of the space of trigonometric polynomials with respect to ψ_d in the space $\mathcal{E}_{s,T}$ of quasi-periodic functions with respect to θ. See [Cor09, Chapters 3, 4] for more details about almost-periodic functions with values in a Banach space.

Definition 3.3. *We call a trigonometric polynomial with respect to ψ_d every function U of $\mathcal{E}_{s,T}$ that writes as a finite sum in real numbers ξ,*

$$U(z, \theta, \psi_d) = \sum_{\xi} U_\xi(z, \theta)\, e^{i\,\psi_d\,\xi},$$

with U_ξ in $\mathcal{C}_b(\mathbb{R}^+_{x_d}, H^s_+(\omega_T \times \mathbb{T}^m))$ for all ξ.

The space of oscillating profiles $\mathcal{P}^{\mathrm{osc}}_{s,T}$ is then defined as the closure in $\mathcal{E}_{s,T}$ of the set of trigonometric polynomials with respect to ψ_d. This space is equipped with the norm of $\mathcal{E}_{s,T}$.

Concerning evanescent profiles, corresponding to frequencies $\alpha_j(\zeta)$ with $j \in \mathcal{P}(\zeta)$, we consider quasi-periodic functions with respect to θ. The factors $e^{i\,\xi\,\psi_d}$ with $\operatorname{Im}\xi > 0$ are expressed through a convergence to zero as ψ_d goes to infinity.

Definition 3.4. *For $s \geqslant 0$ and $T > 0$, the space $\mathcal{P}^{\mathrm{ev}}_{s,T}$ of evanescent profiles is defined as the set of functions U of $\mathcal{E}_{s,T}$, converging to zero in $H^s(\omega_T \times \mathbb{T}^m)$ as ψ_d goes to infinity (for every fixed $x_d \geqslant 0$). The space $\mathcal{P}^{\mathrm{ev}}_{s,T}$ is equipped with the norm of $\mathcal{E}_{s,T}$.*

We may now describe the space of profiles, constructed as the sum of an oscillating and an evanescent part.

Definition 3.5. *For $T > 0$ and $s \geqslant 0$, we define the space of profiles $\mathcal{P}_{s,T}$ of regularity of order s as*

$$\mathcal{P}_{s,T} := \mathcal{P}^{\mathrm{osc}}_{s,T} \oplus \mathcal{P}^{\mathrm{ev}}_{s,T},$$

equipped with the obvious norm. If U belongs to $\mathcal{P}_{s,T}$, we denote by $U^{\mathrm{osc}} \in \mathcal{P}^{\mathrm{osc}}_{s,T}$ and $U^{\mathrm{ev}} \in \mathcal{P}^{\mathrm{ev}}_{s,T}$ the profiles such that $U = U^{\mathrm{osc}} + U^{\mathrm{ev}}$.

The proof of the fact that the spaces $\mathcal{P}^{\mathrm{osc}}_{s,T}$ and $\mathcal{P}^{\mathrm{osc}}_{s,T}$ are indeed in a direct sum is presented later, after the introduction of a scalar product used in the proof.

One can find in [JMR95] a partial proof of the following result, that we recall here for the sake of clarity.

Lemma 3.6 ([JMR95, Lemma 6.1.2])**.** *For all $T > 0$ and for $s > (d+m)/2$, the spaces $\mathcal{E}_{s,T}$, $\mathcal{P}^{\mathrm{osc}}_{s,T}$, $\mathcal{P}^{\mathrm{ev}}_{s,T}$ and $\mathcal{P}_{s,T}$ are all normed algebras. Furthermore, if U and V decomposes in $\mathcal{P}_{s,T} = \mathcal{P}^{\mathrm{osc}}_{s,T} \oplus \mathcal{P}^{\mathrm{ev}}_{s,T}$ as $U = U^{\mathrm{osc}} + U^{\mathrm{ev}}$ and $V = V^{\mathrm{osc}} + V^{\mathrm{ev}}$, then the oscillating part of the profile UV is given by $U^{\mathrm{osc}}V^{\mathrm{osc}}$ and and its evanescent part by $U^{\mathrm{osc}}V^{\mathrm{ev}} + U^{\mathrm{ev}}V^{\mathrm{osc}} + U^{\mathrm{ev}}V^{\mathrm{ev}}$.*

Moreover, for $T > 0$ and $s \geqslant 0$, the spaces $\mathcal{E}_{s,T}$ and $\mathcal{P}^{\mathrm{osc}}_{s,T}$ are Banach spaces.

Proof. The algebra properties for $\mathcal{E}_{s,T}$ and $\mathcal{P}^{\mathrm{ev}}_{s,T}$ arise immediately from the one of $H^s_+(\omega_T \times \mathbb{T}^m)$. The same holds for $\mathcal{P}^{\mathrm{osc}}_{s,T}$ since the set of trigonometric polynomials is stable under multiplication. Finally, this algebra property for $H^s_+(\omega_T \times \mathbb{T}^m)$ shows that if U belongs to $\mathcal{P}^{\mathrm{osc}}_{s,T}$ and V to $\mathcal{P}^{\mathrm{ev}}_{s,T}$, then the product UV belongs to $\mathcal{P}^{\mathrm{ev}}_{s,T}$, so the space $\mathcal{P}_{s,T}$ is also an algebra.

As for them, the completeness properties are obvious. □

3.3. **Scalar products on the space of oscillating profiles.** We now define three scalar products that will be useful in the following, notably to obtain a priori estimates. This part is adapted from [JMR95] to the framework of boundary value problems. We shall use a scalar product with the time variable t fixed (as in [JMR95], which is a priori adapted to the Cauchy problem) as well as a scalar product with the space variable x_d fixed, more adapted to the initial boundary problem.

For U, V two functions of $\omega_T \times \mathbb{R}_+ \times \mathbb{T}^m \times \mathbb{R}_+$, we denote, when the formulas are licit, for $x_d > 0$,

$$\langle U \,|\, V \rangle_{\mathrm{in}}(x_d) := \lim_{R \to +\infty} \frac{1}{R} \int_0^R \langle U \,|\, V \rangle_{L^2(\omega_T \times \mathbb{T}^m)}(x_d, \psi_d)\, d\psi_d, \tag{3.12}$$

for $0 < t < T$,

$$\langle U \,|\, V \rangle_{\mathrm{out}}(t) := \lim_{R \to +\infty} \frac{1}{R} \int_0^R \langle U \,|\, V \rangle_{L^2(\mathbb{R}^{d-1} \times \mathbb{R}_+ \times \mathbb{T}^m)}(t, \psi_d)\, d\psi_d. \tag{3.13}$$

and, if $\mathcal{K}$ is a domain of Ω_T bounded in the x_d direction,

$$\langle U \,|\, V \rangle_{\mathcal{K}} := \lim_{R \to +\infty} \frac{1}{R} \int_0^R \langle U \,|\, V \rangle_{L^2(\mathcal{K} \times \mathbb{T}^m)}(\psi_d)\, d\psi_d. \tag{3.14}$$

The first scalar product is suited to the study of incoming modes when the second one is for the outgoing modes, and the last one will be used to prove the finite speed propagation of the leading profile oscillating part.

If U and V are trigonometric polynomials of $\mathcal{P}^{\mathrm{osc}}_{s,T}$ of the form[3]

$$U(z, \theta, \psi_d) = \sum_{\mathbf{n} \in \mathbb{Z}^m} \sum_{\xi \in \mathbb{R}} U_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d}, \quad V(z, \theta, \psi_d) = \sum_{\mathbf{n} \in \mathbb{Z}^m} \sum_{\xi \in \mathbb{R}} V_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d},$$

[3]The sums in ξ are necessarily countable.

then [Cor09, Theorem 3.4, Remark 4.17] ensure that the scalar products $\langle U \,|\, V\rangle_{\rm in}(x_d)$ and $\langle U \,|\, V\rangle_{\mathcal{K}}$ are well-defined and satisfies, for $x_d \geqslant 0$,

$$\langle U \,|\, V\rangle_{\rm in}(x_d) = (2\pi)^m \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} \langle U_{\mathbf{n},\xi} \,|\, V_{\mathbf{n},\xi}\rangle_{L^2(\omega_T)}(x_d), \tag{3.15}$$

and

$$\langle U \,|\, V\rangle_{\mathcal{K}} = (2\pi)^m \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} \langle U_{\mathbf{n},\xi} \,|\, V_{\mathbf{n},\xi}\rangle_{L^2(\mathcal{K})}. \tag{3.16}$$

Indeed, for each function U of $\mathcal{P}^{\rm osc}_{s,T}$, its trace with respect to ψ_d belongs to $L^2(\mathcal{K}\times\mathbb{T}^m)$ for all $\psi_d \geqslant 0$, and its trace with respect to x_d, ψ_d belongs to $L^2(\omega_T\times\mathbb{T}^m)$ for all $x_d, \psi_d \geqslant 0$. If U and V are moreover of compact support with respect to x_d and if $s \geqslant 1$, then the traces of U and V with respect to t, ψ_d belong to $L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+\times\mathbb{T}^m)$, and the same results from [Cor09] ensure that the scalar product $\langle U \,|\, V\rangle_{\rm out}(t)$ is well-defined and satisfy, for $t > 0$,

$$\langle U \,|\, V\rangle_{\rm out}(t) = (2\pi)^m \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} \langle U_{\mathbf{n},\xi} \,|\, V_{\mathbf{n},\xi}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t). \tag{3.17}$$

In short, scalar products (3.12) and (3.14) (resp. (3.13)) are well-defined on the space of profiles $\mathcal{P}^{\rm osc}_{s,T}$, $s \geqslant 0$ (resp. for profiles of $\mathcal{P}^{\rm osc}_{s,T}$ with compact support with respect to x_d with $s \geqslant 1$), and formulas (3.15), (3.16) and (3.17) are satisfied in this case.

The incoming scalar product (3.12) is used to prove the following result.

Lemma 3.7. *For all $T > 0$, the spaces $\mathcal{P}^{\rm osc}_{0,T}$ and $\mathcal{P}^{\rm ev}_{0,T}$ are in direct sum.*

Proof. Consider a profile U in $\mathcal{P}^{\rm osc}_{0,T} \cap \mathcal{P}^{\rm ev}_{0,T}$ that writes, because it is an oscillating profile,

$$U(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} U_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d},$$

the sum in ξ being countable. The profile U also being evanescent, for all $x_d \geqslant 0$, the function $\psi_d \mapsto U(.,x_d,.,\psi_d)$ converges to zero in $L^2(\omega_T\times\mathbb{T}^m)$ as ψ_d goes to infinity. Thus, for all $x_d \geqslant 0$, we have

$$\langle U \,|\, U\rangle_{\rm in}(x_d) = 0,$$

since the mean value (in terms of (3.12)) of a continuous function on $\mathbb{R}_+$ converging to zero at infinity is zero. But, since U is an oscillating profile, we have

$$\langle U \,|\, U\rangle_{\rm in}(x_d) = (2\pi)^m \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} \|U_{\mathbf{n},\xi}\|^2_{L^2(\omega_T)}(x_d),$$

so, for all $\mathbf{n}, \xi$, the function $U_{\mathbf{n},\xi}(x_d)$ is zero in $L^2(\omega_T\times\mathbb{T}^m)$, and the profile U is therefore zero as well. □

4. Ansatz and main result

We seek to construct an approximate solution to (2.1) under the form of a formal series $u^{\varepsilon,\rm app}(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)$, where $u^{\varepsilon,\rm app}$ is given by

$$u^{\varepsilon,\rm app}(z,\theta,\psi_d) := \sum_{k\geqslant 1} \varepsilon^k\, U_k(z,\theta,\psi_d), \tag{4.1}$$

with at least U_1 in $\mathcal{P}_{s,T}$ for some $s \geqslant 0$. As for them, correctors $(U_k)_{k\geqslant 2}$ a priori exhibit frequencies that may not be characteristic. The convergence of the associated series then relies on a small divisor assumption which is different from the previously made small divisor assumption 6. Thus we only consider correctors as formal trigonometric series involving all frequencies in the group $\langle\mathcal{F}\rangle$ generated by the set $\mathcal{F}$.

We are now in place to state the main result of this work. It is placed under Assumptions 1 to 7, as well as Assumption 8 that will be made further on. We denote by h an integer larger or equal to $(3+a_1)/2$ where a_1 is the real number introduced in the small divisors Assumption 6. Then we denote $s_0 := h + (d+m)/2$.

Theorem 4.1. *Let s be an integer such that $s \geqslant s_0$. Under previously listed assumptions, there exists a time $T > 0$ such that system (5.11) described below and that governs the evolution of the leading profile in the asymptotic expansion (4.1) admits a unique solution U_1 in $\mathcal{P}_{s,T}$.*

We recall that we have considered a forcing term G in $H^\infty(\mathbb{R}^d \times \mathbb{T}^m)$, zero for negative times t and of zero mean with respect to θ in $\mathbb{T}^m$, but the infinite regularity assumption is made only for simplicity, and the estimates, and thus the existence time T, only depend on the $H^s(\mathbb{R}^d \times \mathbb{T}^m)$ norm of G. More precisely, the existence time T depends on the operator $L(0, \partial_z)$, the frequencies on the boundary $\zeta_1, \dots, \zeta_m$, the order of regularity s, and the $H^s(\omega_T \times \mathbb{T}^m)$ norm of the forcing term G.

The formal WKB study shows that the function

$$z \mapsto \varepsilon\, U_1(z, z' \cdot \zeta_1/\varepsilon, \dots, z' \cdot \zeta_m/\varepsilon, x_d/\varepsilon)$$

is formally an approximate solution to system (2.1).

Last two sections of the article are devoted to the proof of Theorem 4.1. In section 5 we start by formally deriving the cascade of equations that must be verified by the amplitudes $(U_k)_{k\geqslant 1}$. By resolving, first formally and then rigorously for a part of it (in section 6), a fast problem, this cascade is triangularized and a system of equations for the leading profile is extracted from it. Next, in section 6, after a discussion about the different types of resonances that appear, the system is reduced to two decoupled systems for the oscillating and evanescent parts, the one on the oscillating part involving only the incoming phases. The oscillating system is even decoupled in a system for all resonant modes, and a system for each non-resonant mode. Then a priori estimates are proved for the linearized system for the oscillating parts, leading to the construction of solutions to these linearized systems. Such estimates prove in particular that the solution to (5.11) is unique. An iterative scheme is then used to construct solutions to the nonlinear systems, and the evanescent part is finally determined.

Unlike in [JMR95, Part 6], from which the following is mainly inspired, there is no symmetry in the system, since it is not hyperbolic as a propagation system in the variable x_d. This lack of symmetry is a genuine obstacle to deal with the resonance terms (that are in infinite number) in the a priori estimates. Assumption 8, concerning all resonances with the possible exception of a finite number of them, allows to work around the problem and to obtain estimates for the associated terms. Assumption 8 will be carefully verified for the Euler system.

5. Formal WKB study

5.1. **Cascade of equations for the profiles.** We seek to formally determine the equations the sequence of profiles $(U_n)_{n\geqslant 1}$ must satisfy for the formal series $z \mapsto u^{\varepsilon,\mathrm{app}}(z, z' \cdot \zeta_1/\varepsilon, \dots, z' \cdot \zeta_m/\varepsilon, x_d/\varepsilon)$ given by (4.1) to be solution to system (2.1). In the following we wish for the coefficient (a priori dependent on $u^{\varepsilon,\mathrm{app}}$) in factor of the partial derivative with respect to x_d to be the identity matrix, so that its differential is zero. The analogous property for the partial derivative in time is crucial in [JMR95] from which we mainly draw our analysis. This choice is justified here by the particular role of the variable x_d in a priori estimates for the principal profile. This is why we are interested from now on in the following equivalent system

$$\begin{cases} \widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)\, u^{\varepsilon,\mathrm{app}} = 0 & \text{in } \Omega_T, \\ B\, u^{\varepsilon,\mathrm{app}}_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ u^{\varepsilon,\mathrm{app}}_{|t\leqslant 0} = 0 \end{cases} \tag{5.1}$$

where we have denoted

$$\widetilde{L}(u, \partial_z) := A_d(u)^{-1}\, L(u, \partial_z) = \widetilde{A}_0(u)\, \partial_t + \sum_{i=1}^{d-1} \widetilde{A}_i(u)\, \partial_i + \partial_d,$$

with $\widetilde{A}_i(u) := A_d(u)^{-1}\, A_i(u)$ for $i = 0, \dots, d-1$, and $A_0(u) := I$. In the following we may use the notation $\partial_0 := \partial_t$.

5.1.1. *WKB Cascade inside the domain.* We are now able to write the equations verified by the profiles U_k, $k \geqslant 1$, by formally replacing $u^{\varepsilon,\mathrm{app}}$ by its formal expansion

$$u^{\varepsilon,\mathrm{app}} = \sum_{k\geqslant 1} \varepsilon^k\, U_k$$

in system (5.1). First, we note that the following Taylor expansion is verified, for $i = 0, \dots, d-1$:

$$\widetilde{A}_i(u^{\varepsilon,\mathrm{app}}) = \widetilde{A}_i(0) + \varepsilon\, d\widetilde{A}_i(0)\cdot U_1 + \sum_{k\geqslant 2} \varepsilon^k \Big[d\widetilde{A}_i(0)\cdot U_k + G^i_{k-1}\Big],$$

where, for $k \geqslant 2$, G^i_{k-1} only depends on $U_1, \dots, U_{k-1}$. The operator $\widetilde{L}\big(u^{\varepsilon,\mathrm{app}}, \partial_z\big)$ thus writes

$$\widetilde{L}\big(u^{\varepsilon,\mathrm{app}}, \partial_z\big) = \widetilde{L}(0, \partial_z) + \sum_{k\geqslant 1} \varepsilon^k \widetilde{L}_k(U_k, \partial_{z'}), \tag{5.2}$$

with

$$\widetilde{L}_1\big(U_1, \partial_{z'}\big) := \sum_{i=0}^{d-1} d\widetilde{A}_i(0)\cdot U_1\, \partial_i, \qquad \widetilde{L}_k\big(U_k, \partial_{z'}\big) := \sum_{i=0}^{d-1} \Big(d\widetilde{A}_i(0)\cdot U_k + G^i_{k-1}\Big)\, \partial_i, \quad \forall k \geqslant 2.$$

For $k \geqslant 2$, despite the fact that only the U_k dependency is indicated in the operators $\widetilde{L}_k(U_k, \partial_{z'})$, these ones also depend on the profiles $U_1, \dots, U_{k-1}$, via the functions G^i_{k-1}. The operator $\widetilde{L}_1(U_1, \partial_{z'})$ depends however only on U_1.

We see here the benefit of considering the modified operator $\widetilde{L}(u, \partial_z)$: there is an x_d derivative only in the leading operator $\widetilde{L}(0, \partial_z)$, and not in the other operators $\widetilde{L}_k(U_k, \partial_{z'})$. Furthermore, we verify that

$$\begin{aligned} &\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)\Big[u^{\varepsilon,\mathrm{app}}(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)\Big] \\ &\quad = \Big[\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)\, u^\varepsilon + \frac{1}{\varepsilon}\sum_{j=1}^m \widetilde{L}(u^{\varepsilon,\mathrm{app}}, \zeta_j)\, \partial_{\theta_j} u^{\varepsilon,\mathrm{app}} + \frac{1}{\varepsilon}\partial_{\psi_d} u^{\varepsilon,\mathrm{app}}\Big] \\ &\qquad (z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon). \end{aligned} \tag{5.3}$$

where, for $j = 1, \dots, m$, the symbol $\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \zeta_j)$ is defined by $\sum_{i=0}^{d-1} \zeta_j^i\, \widetilde{A}_i(u^{\varepsilon,\mathrm{app}})$ with $\zeta_j = (\zeta_j^0, \dots, \zeta_j^{d-1})$. Expansion (5.2) of the operator $\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)$ leads to the analogous expansions of the operators $\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \zeta_j)$ for $j = 1, \dots, m$:

$$\widetilde{L}\big(u^{\varepsilon,\mathrm{app}}, \zeta_j\big) = \widetilde{L}(0, \zeta_j) + \sum_{k\geqslant 1} \varepsilon^k \widetilde{L}_k(U_k, \zeta_j) \tag{5.4}$$

where

$$\widetilde{L}_1\big(U_1, \zeta_j\big) := \sum_{i=0}^{d-1} \zeta_j^i\, d\widetilde{A}_i(0)\cdot U_1, \qquad \widetilde{L}_k\big(U_k, \zeta_j\big) := \sum_{i=0}^{d-1} \zeta_j^i \Big(d\widetilde{A}_i(0)\cdot U_k + G^i_{k-1}\Big), \quad \forall k \geqslant 2.$$

Thus, according to expansions (5.2), (5.3) and (5.4), the following asymptotic expansion holds

$$\begin{aligned} &\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)\Big[u^{\varepsilon,\mathrm{app}}(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)\Big] \\ &\quad = \frac{1}{\varepsilon}\Big\{\sum_{j=1}^m \widetilde{L}(0, \zeta_j)\, \partial_{\theta_j} u^{\varepsilon,\mathrm{app}} + \partial_{\psi_d} u^{\varepsilon,\mathrm{app}}\Big\} + \widetilde{L}(0, \partial_z)\, u^\varepsilon + \sum_{j=1}^m \widetilde{L}_1(U_1, \zeta_j)\, \partial_{\theta_j} u^{\varepsilon,\mathrm{app}} \\ &\qquad + \sum_{k\geqslant 1} \varepsilon^k \Big\{\widetilde{L}_k(U_k, \partial_{z'})\, u^{\varepsilon,\mathrm{app}} + \sum_{j=1}^m \widetilde{L}_{k+1}(U_{k+1}, \zeta_j)\, \partial_{\theta_j} u^{\varepsilon,\mathrm{app}}\Big\}, \end{aligned}$$

where the right hand side is evaluated in $(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)$. The operator $L(u^{\varepsilon,\mathrm{app}}, \partial_z)$ applied to $u^{\varepsilon,\mathrm{app}}(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)$ is therefore given by the formal series

(5.5)
$$\widetilde{L}(u^{\varepsilon,\mathrm{app}}, \partial_z)\Big[u^{\varepsilon,\mathrm{app}}(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon)\Big] = \sum_{k\geqslant 0} \varepsilon^k\, W_k(z, z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon, x_d/\varepsilon),$$

where, if the variables θ and ψ_d are substituted to $(z'\cdot\zeta_1/\varepsilon, \dots, z'\cdot\zeta_m/\varepsilon)$ and x_d/ε, the amplitudes $(W_k)_{k\geqslant 0}$ of the formal series (5.5) are given by

$$W_0 := \Big\{\sum_{j=1}^m \widetilde{L}(0, \zeta_j)\,\partial_{\theta_j} + \partial_{\psi_d}\Big\} U_1, \tag{5.6a}$$

$$W_1 := \Big\{\sum_{j=1}^m \widetilde{L}(0, \zeta_j)\,\partial_{\theta_j} + \partial_{\psi_d}\Big\} U_2 + \Big\{\widetilde{L}(0, \partial_z) + \sum_{j=1}^m \widetilde{L}_1(U_1, \zeta_j)\,\partial_{\theta_j}\Big\} U_1, \tag{5.6b}$$

and for $k \geqslant 2$,

$$\begin{aligned} W_k := \Big\{\sum_{j=1}^m \widetilde{L}(0, \zeta_j)\,\partial_{\theta_j} + \partial_{\psi_d}\Big\} U_{k+1} + \Big\{\widetilde{L}(0, \partial_z) + \sum_{j=1}^m \widetilde{L}_1(U_1, \zeta_j)\,\partial_{\theta_j}\Big\} U_k \\ + \sum_{l=1}^{k-1} \Big\{\widetilde{L}_{k-l}(U_{k-l}, \partial_{z'}) + \sum_{j=1}^m \widetilde{L}_{k-l+1}(U_{k-l+1}, \zeta_j)\,\partial_{\theta_j}\Big\} U_l. \end{aligned} \tag{5.6c}$$

Formulas (5.6a) and (5.6b) correspond to the analogous ones in [CGW11, (1.33), (1.46)] in the case $m = 1$.

Thus, for the formal series (4.1) to be solution to (2.1), the formal series (5.5) must be zero, or equivalently

$$W_k = 0, \qquad \forall k \geqslant 0.$$

We note that each equation $W_k = 0$ involves the fast operator

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d}) := \sum_{j=1}^m \widetilde{L}(0, \zeta_j)\,\partial_{\theta_j} + \partial_{\psi_d},$$

which is linear and has constant coefficients, as customary in weakly nonlinear geometric optics, see for example [Rau12, Section 9.4]. The subject of the following part is to study this operator in order to rewrite equations (5.6) in an equivalent manner. Before that the WKB cascades on the boundary and at initial time are determined.

5.1.2. *WKB cascade on the boundary.* Since we want the formal series (4.1) to satisfy the boundary condition

$$B\, u^{\varepsilon,\mathrm{app}}_{|x_d=0} = \varepsilon\, g^\varepsilon,$$

the profiles $(U_k)_{k\geqslant 1}$ must verify, using variables (z', θ), the following boundary conditions

$$\begin{aligned} (B\,U_1)_{|x_d=0,\psi_d=0} &= G \\ (B\,U_k)_{|x_d=0,\psi_d=0} &= 0, \quad k \geqslant 2. \end{aligned}$$

5.1.3. *Initial conditions.* In a similar manner, the profiles $(U_k)_{k\geqslant 1}$ must satisfy the following initial conditions

$$(U_k)_{|t\leqslant 0} = 0, \quad k \geqslant 1.$$

5.2. **Resolution of the fast problem $\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = H$.** In this part we seek to resolve in the formal trigonometric series framework the equation

$$\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = H,$$

and more precisely, to formally determine the kernel and range of the operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$. We follow, in a formal manner, the analysis of [Les07, Part 3]. Thus we consider U writing

$$U(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d} + \sum_{\mathbf{n}\in\mathbb{Z}^m} U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\, e^{i\,\mathbf{n}\cdot\theta},$$

and H writing

$$H(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} H^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d} + \sum_{\mathbf{n}\in\mathbb{Z}^m} H^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\, e^{i\,\mathbf{n}\cdot\theta},$$

where, for all $\mathbf{n}$ in $\mathbb{Z}^m$, the sum in ξ is countable. Then, by definition of the fast operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$, we get

$$\begin{aligned}\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U(z,\theta,\psi_d) = &\sum_{\mathbf{n}\in\mathbb{Z}^m} \sum_{\xi\in\mathbb{R}} i\,\widetilde{L}\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)\, U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d}\\ &+ \sum_{\mathbf{n}\in\mathbb{Z}^m} \big\{ i\,\widetilde{L}\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},0)\big) + \partial_{\psi_d}\big\} U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\, e^{i\,\mathbf{n}\cdot\theta},\end{aligned}$$

where we recall that $\boldsymbol{\zeta}$ refers to the m-tuple of elements of $\mathbb{R}^d$ given by $\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_m)$. Therefore, the profile U is a solution to $\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U = H$ if and only if, for all $\mathbf{n}$ in $\mathbb{Z}^m$ and for all ξ in $\mathbb{R}$, we have

$$i\,\widetilde{L}\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)\, U^{\mathrm{osc}}_{\mathbf{n},\xi}(z) = H^{\mathrm{osc}}_{\mathbf{n},\xi}(z), \tag{5.7a}$$

and

$$\Big(i\,\widetilde{L}\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},0)\big) + \partial_{\psi_d}\Big)\, U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d) = H^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d). \tag{5.7b}$$

For $\mathbf{n}$ in $\mathbb{Z}^m$ and for ξ in $\mathbb{R}$, equation (5.7a) admits a solution if and only if $H^{\mathrm{osc}}_{\mathbf{n},\xi}$ belongs to the range of the matrix $\widetilde{L}(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi))$, that is to say, according to Definition 2.6, the kernel $\ker \widetilde{\pi}_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}$. According to Definition 2.6 of the partial inverse Q_α, every solution is therefore of the form

$$U^{\mathrm{osc}}_{\mathbf{n},\xi} = X_{\mathbf{n},\xi} - i\, Q_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, A_d(0)\, H^{\mathrm{osc}}_{\mathbf{n},\xi},$$

with $X_{\mathbf{n},\xi}$ an element of $\operatorname{Im} \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}$, and thus satisfies

$$U^{\mathrm{osc}}_{\mathbf{n},\xi} = \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U^{\mathrm{osc}}_{\mathbf{n},\xi} - i\, Q_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, A_d(0)\, H^{\mathrm{osc}}_{\mathbf{n},\xi}.$$

As for it, the differential equation (5.7b) admits a formal solution for every $\mathbf{n}$ in $\mathbb{Z}^m$. For $\mathbf{n} = 0$, the solution is formally given by

$$U^{\mathrm{ev}}_{\mathbf{0}}(z,\psi_d) = -\int_{\psi_d}^{+\infty} H^{\mathrm{ev}}_{\mathbf{0}}(z,s)\, ds,$$

and, for $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$, according to Duhamel's principle, by

$$\begin{aligned}U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d) =& e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(z,0) + \int_0^{\psi_d} e^{(\psi_d - s)\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, H^{\mathrm{ev}}_{\mathbf{n}}(z,s)\, ds\\ &- \int_{\psi_d}^{+\infty} e^{(\psi_d - s)\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \big(I - \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\, H^{\mathrm{ev}}_{\mathbf{n}}(z,s)\, ds,\end{aligned} \tag{5.8}$$

noting that $i\,\widetilde{L}\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},0)\big) = -\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})$. Indeed, according to Assumption 5, the frequency $\mathbf{n}\cdot\boldsymbol{\zeta}$ is not glancing and the projector $\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})$ is thus well defined. The Duhamel's principle then applies separately to $\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbb{C}^N}$ and $(I - \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta}))\, U^{\mathrm{ev}}_{\mathbb{C}^N}$. The first term of the right-hand side of (5.8) is therefore well-defined, and the integral of the second one converges since, according to Proposition 6.16 proved in appendix, the matrix $e^{t\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})$ is bounded by a decaying exponential for $t \geqslant 0$. However we do not know if the integral of the third term converges.

Indeed, $H^{\mathrm{ev}}_{\mathbf{n}}$ is only converging to zero at infinity, and the matrix $e^{t\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,(I-\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta}))$ is simply bounded for $t\leqslant 0$, according to Proposition 6.16. The issue is essentially the same for the integral defining $U^{\mathrm{ev}}_{\mathbf{0}}$. The following result is deduced from this analysis.

Lemma 5.1 ([Les07, Theorem 2.14])**.** *The equation $\mathcal{L}(\partial_\theta,\partial_{\psi_d})\,U=H$ admits a solution in the framework of formal trigonometric series if and only if $\widetilde{\mathbf{E}^i}\,H=0$, and every solution is of the form*

$$U=\mathbf{E}\,U+\mathbf{Q}\,H,$$

where projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$ and operator $\mathbf{Q}$ are formally defined further on. In particular we have

$$\ker\mathcal{L}(\partial_\theta,\partial_{\psi_d})=\operatorname{Im}\mathbf{E}\quad\text{and}\quad\operatorname{Im}\mathcal{L}(\partial_\theta,\partial_{\psi_d})=\ker\widetilde{\mathbf{E}^i}.$$

If U is given by

$$U(z,\theta,\psi_d)=\sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}}U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d}+\sum_{\mathbf{n}\in\mathbb{Z}^m}U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\,e^{i\,\mathbf{n}\cdot\theta},$$

then $\widetilde{\mathbf{E}^i}\,U$ is defined as

$$\widetilde{\mathbf{E}^i}\,U(z,\theta,\psi_d):=\sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}}\widetilde{\pi}_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d},\tag{5.9}$$

$\mathbf{E}\,U$ as

$$\begin{aligned}\mathbf{E}\,U(z,\theta,\psi_d):=&\sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}}\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\;U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d}\\&+\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}e^{\psi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{\mathrm{ev}}_{\mathbf{n}}(z,0)\,e^{i\,\mathbf{n}\cdot\theta},\end{aligned}\tag{5.10}$$

and $\mathbf{Q}\,U$ as

$$\begin{aligned}\mathbf{Q}\,U(z,\theta,\psi_d):=&-\sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}}i\,Q_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,A_d(0)\,U^{\mathrm{osc}}_{\mathbf{n},\xi}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d}-\int_{\psi_d}^{+\infty}U^{\mathrm{ev}}_{\mathbf{0}}(z,s)\,ds\\&+\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\Bigg(\int_0^{\psi_d}e^{(\psi_d-s)\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{\mathrm{ev}}_{\mathbf{n}}(z,s)\,ds\\&-\int_{\psi_d}^{+\infty}e^{(\psi_d-s)\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\big(I-\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,U^{\mathrm{ev}}_{\mathbf{n}}(z,s)\,ds\Bigg)\,e^{i\,\mathbf{n}\cdot\theta}.\end{aligned}$$

Note that, for now, the operator $\mathbf{Q}$ and the projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$ are only formally defined. The projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$, that are the only one involved in the leading profile equations, can be defined in the space $\mathcal{P}_{s,T}$, and this result will constitute a part of the following section. However, the operator $\mathbf{Q}$ cannot be rigorously defined in the functional framework used here (the issues are the absence of some small divisor control - in the same manner as in [JMR95], as well as a lack of exponential decay for the evanescent profiles).

5.3. **System of equations satisfied by the leading profile.** According to expressions (5.6a) and (5.6b) of the amplitudes W_0 and W_1, and using the previous Lemma 5.1, we get the following system of equations for the leading profile U_1, simply denoted from now on by U:

$$\mathbf{E}\,U=U\tag{5.11a}$$

$$\widetilde{\mathbf{E}^i}\Big[\widetilde{L}(0,\partial_z)\,U+\sum_{j=1}^{m}\widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U\Big]=0\tag{5.11b}$$

$$B\,U_{|x_d=0,\psi_d=0}=G\tag{5.11c}$$

$$U_{|t\leqslant 0}=0.\tag{5.11d}$$

We note that the leading profile U_1 is polarized, in the sense that it satisfies equation (5.11a), so according to Formula (5.10) defining projector $\mathbf{E}$, only the characteristic frequencies occur in its Fourier expansion. We shall see in the next section that the oscillating part U^{osc} of profile U satisfies the problem

$$\mathbf{E}\, U^{\mathrm{osc}} = U^{\mathrm{osc}} \tag{5.12a}$$

$$\widetilde{\mathbf{E}^i}\Big[\widetilde{L}(0,\partial_z)\, U^{\mathrm{osc}} + \sum_{j=1}^{m} \widetilde{L}_1(U^{\mathrm{osc}},\zeta_j)\, \partial_{\theta_j} U^{\mathrm{osc}}\Big] = 0 \tag{5.12b}$$

$$B\left(U^{\mathrm{osc}} + U^{\mathrm{ev}}\right)_{|x_d=0,\psi_d=0} = G \tag{5.12c}$$

$$U^{\mathrm{osc}}_{|t\leqslant 0} = 0. \tag{5.12d}$$

The question is to know whether or not boundary condition (5.12c) determines on its own the trace $U^{\mathrm{osc}}_{|x_d=0,\psi_d=0}$. As already explained in [CGW11] and [CW17], the answer depends on the existence of a resonance between two incoming frequencies that generates an outgoing frequency. Such a resonance pattern is excluded by Assumption 7. In this case the boundary condition (5.12c) also determines the trace $U^{\mathrm{ev}}_{|x_d=0,\psi_d=0}$, which, according to the polarization type condition, immediately leads to the construction of the evanescent part of U.

6. Construction of the leading profile

Now that the system that must be verified by the leading profile U has been formally determined, we are in position to construct a solution to it. We begin in a first part by a discussion about the different types of resonances that may appear in the system. In particular, the technical assumption is made that the resonances for which the lack of symmetry is not controlled are in finite number. This assumption is made to deal with the lack of symmetry in the resonances terms compared to the case of [JMR95]. In a second part, the projectors appearing in system (5.11) as well as some results of section 5 are made rigorous, using the small divisors Assumption 6. Then we proceed to the three main steps of the proof, namely the decoupling of system (5.11), derivation of energy estimates for the linearized systems and construction of the solution. First step is achieved in section 6.3 and consists in reducing the system (5.11) to a system for the evanescent part, a system for the oscillating incoming resonant part and a system for each oscillating incoming non-resonant part. Writing equation (5.11b) in extension with modes, we isolate the system satisfied by the mean value to show that it is zero. Then it is proven, using an energy estimate on the oscillating outgoing part, that every outgoing mode is zero. To obtain this estimate, we use a scalar product defined only for profiles with bounded x_d-support. Therefore we need to show beforehand that a solution to (5.11) propagates in the normal direction with finite velocity. The proof of this result is postponed after the introduction of the techniques used in it, in section 6.4. Finally, using the fact that incoming modes are zero, the boundary condition (5.11c) can be decoupled for each evanescent and incoming oscillating mode, which will conclude the decoupling of the system. The derivation of a priori estimates without loss of derivatives is performed in sections 6.4 and 6.5, one for the resonant part and one for each non-resonant part, namely Burgers type equations. Each non-resonant mode must be treated separately to avoid a factor unbounded with respect to the frequency, but part 6.5 presents no additional difficulty since it reuses techniques displayed in the previous one. The derivation of a priori estimates for the linearized oscillating resonant system is presented in section 6.4, beginning by the L^2 estimate. It is obtained taking the incoming modes suited scalar product between the linearized propagation equation and a modified profile. Four terms need to be addressed, the transport and Burgers type (corresponding to self-interaction) ones are treated classically (with an integration by part and using symmetry in the self-interaction terms), while the two resonant ones are handled using the technical assumption on resonance terms. More precisely this assumption asserts that these resonance terms (except for a finite number of them, that are treated separately) are such that the lack of symmetry in it is controlled in a way that the techniques used for the Burgers type terms can be adapted. The same method for energy

estimates is used with a different scalar product to prove the finite speed propagation in this section. Then the estimates for derivatives are obtained classically using commutator estimates. Section 6.6 is devoted to construction of the solution. The oscillating part is constructed using an usual procedure which is not detailed, and consists in proving existence of a linearized solution with a finite difference scheme using the a priori estimates previously derived, and then existence of the sought solution using an iterative scheme. Uniqueness is deduced from the a priori estimates. As for the evanescent part, its expression is prescribed by a polarization type condition, and we prove that the constructed profile belongs to the space of evanescent profiles. It will achieve the proof of Theorem 4.1, and finally section 6.7 draws a conclusion and some perspectives.

6.1. **Resonance coefficient and additional assumption.** The sets defined below permit to gather the characteristic frequencies according to collinearity.

Definition 6.1. *We consider the subset of $\mathbb{Z}^m \setminus \{0\}$, denoted by $\mathcal{B}_{\mathbb{Z}^m}$, constituted of all m-tuples of coprime integers of which the first nonzero term is positive:*

$$\mathcal{B}_{\mathbb{Z}^m} := \left\{ (n_1, \dots, n_m) \in \mathbb{Z}^m \setminus \{0\} \,\middle|\, \begin{array}{l} n_1 \wedge \cdots \wedge n_m = 1, \\ \exists k \in \{0, \dots, m-1\}\,,\ n_1, \dots, n_k = 0, n_{k+1} > 0 \end{array} \right\}.$$

One can verify that for all $\mathbf{n}$ of $\mathbb{Z}^m \setminus \{0\}$, there exists a unique element $\mathbf{n}_0$ of $\mathcal{B}_{\mathbb{Z}^M}$ and a unique nonzero integer λ such that $\mathbf{n} = \lambda\, \mathbf{n}_0$.

Then we introduce the following notation for real characteristic frequencies lifted from frequencies on the boundary.

Definition 6.2. *For $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$, we denote by $\mathcal{C}(\mathbf{n})$ the finite set of real numbers ξ such that the frequency $(\mathbf{n} \cdot \boldsymbol{\zeta}, \xi)$ is real and characteristic, namely*

$$\mathcal{C}(\mathbf{n}) := \{\xi \in \mathbb{R} \mid (\mathbf{n} \cdot \boldsymbol{\zeta}, \xi) \in \mathcal{C}\}\,.$$

We also denote by $\mathcal{C}_{\text{in}}(\mathbf{n})$ (resp. $\mathcal{C}_{\text{out}}(\mathbf{n})$) the set of real numbers ξ such that the frequency $(\mathbf{n} \cdot \boldsymbol{\zeta}, \xi)$ is real, characteristic and incoming (resp. outgoing), namely

$$\mathcal{C}_{\text{in}}(\mathbf{n}) = \{\xi_j(\mathbf{n} \cdot \boldsymbol{\zeta}) \mid j \in \mathcal{I}(\mathbf{n} \cdot \boldsymbol{\zeta})\}\,, \quad \mathcal{C}_{\text{out}}(\mathbf{n}) = \{\xi_j(\mathbf{n} \cdot \boldsymbol{\zeta}) \mid j \in \mathcal{O}(\mathbf{n} \cdot \boldsymbol{\zeta})\}\,,$$

with notations of Proposition 2.16.

We recall that according to Assumption 5, there is no glancing frequency in $\mathcal{F}$, so the disjoint union

$$\mathcal{C}(\mathbf{n}) = \mathcal{C}_{\text{in}}(\mathbf{n}) \cup \mathcal{C}_{\text{out}}(\mathbf{n})$$

is satisfied for all $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$. All *real* characteristic frequencies have been considered here, but there may also exist non-real characteristic frequencies lifted from $\mathbf{n} \cdot \boldsymbol{\zeta}$.

Remark 6.3. One can check that, according to Remark 2.17, the sets $\mathcal{C}$, $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ are homogeneous of degree 1. Thus, if $\mathbf{n}$ belongs to $\mathbb{Z}^m \setminus \{0\}$ and ξ to $\mathcal{C}(\mathbf{n})$, and if $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$ and λ in $\mathbb{Z}^*$ are such that $\mathbf{n} = \lambda\, \mathbf{n}_0$, then there exists ξ_0 in $\mathcal{C}(\mathbf{n}_0)$ such that $\xi = \lambda\, \xi_0$.

We now introduce some notations for the resonances.

Definition 6.4. *For $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$ and ξ in $\mathcal{C}(\mathbf{n})$, we denote by $E(\mathbf{n}, \xi)$ the vector of the basis $E_1, \dots, E_N$ of $\mathbb{C}^N$ given by (2.8) that generates the line $\ker L\big(0, (\mathbf{n} \cdot \boldsymbol{\zeta}, \xi)\big)$.*

Remark 6.5. Note that for $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$, ξ in $\mathcal{C}(\mathbf{n})$ and λ in $\mathbb{Z}^*$, since the linear subspaces $\ker L\big(0, (\mathbf{n} \cdot \boldsymbol{\zeta}, \xi)\big)$ and $\ker L\big(0, (\lambda \mathbf{n} \cdot \boldsymbol{\zeta}, \lambda \xi)\big)$ are equal, we infer $E(\mathbf{n}, \xi) = E(\lambda \mathbf{n}, \lambda \xi)$, so the vector $E(\mathbf{n}, \xi)$ is homogeneous of degree 0.

The following definition is based on [Rau12, Chapter 11].

Definition 6.6. *Let $\mathbf{n}_p$, $\mathbf{n}_q$ be two elements of $\mathbb{Z}^m \setminus \{0\}$, and let (ξ_p, ξ_q) in $\mathcal{C}(\mathbf{n}_p) \times \mathcal{C}(\mathbf{n}_q)$ be such that the frequency*

$$(\mathbf{n}_p \cdot \boldsymbol{\zeta}, \xi_p) + (\mathbf{n}_q \cdot \boldsymbol{\zeta}, \xi_q) =: (\mathbf{n}_r \cdot \boldsymbol{\zeta}, \xi_r)$$

is real and characteristic (i.e. such that there is a resonance). Then the resonance coefficient $\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)$ *is defined by the equation*

$$\widetilde{\pi}_{(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r)}\,\widetilde{L}_1\big(E(\mathbf{n}_p,\xi_p),\mathbf{n}_q\cdot\boldsymbol{\zeta}\big)\,E(\mathbf{n}_q,\xi_q)=\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)\,\widetilde{\pi}_{(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r)}\,E(\mathbf{n}_r,\xi_r).$$

This coefficient exists by definition of the projectors $\widetilde{\pi}_k$*, for* $k=1,\dots,N$ *and according to Lemma 2.15.*

Remark 6.7. i) Since all quantities involved in the definition of Γ are homogeneous of degree 0 or 1, the coefficient Γ is homogeneous of degree 1, i.e. for all $\mathbf{n}_p$, $\mathbf{n}_q$ in $\mathbb{Z}^m\setminus\{0\}$, (ξ_p,ξ_q) in $\mathcal{C}(\mathbf{n}_p)\times\mathcal{C}(\mathbf{n}_q)$ such that the frequency $(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)$ is real characteristic (that is such that there is a resonance) and for all λ in $\mathbb{Z}^*$, we have

$$\Gamma\big((\lambda\mathbf{n}_p,\lambda\xi_p),(\lambda\mathbf{n}_q,\lambda\xi_q)\big)=\lambda\,\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big). \tag{6.1}$$

By definition and for the same reason, we also have, for $\mathbf{n}_0$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ_0 in $\mathcal{C}(\mathbf{n}_0)$, and for λ_1,λ_2 in $\mathbb{Z}^*$,

$$\Gamma\big(\lambda_1(\mathbf{n}_0,\xi_0),\lambda_2(\mathbf{n}_0,\xi_0)\big)=\lambda_2\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big). \tag{6.2}$$

ii) Since according to Remark 2.7 the projectors $\widetilde{\pi}_\alpha$ are bounded and the vectors $E(\mathbf{n},\xi)$ are of norm 1, for all $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and all ξ in $\mathcal{C}(\mathbf{n})$, we have

$$\big|\Gamma\big((\mathbf{n},\xi),(\mathbf{n},\xi)\big)\big|\leqslant C\frac{|\mathbf{n}|}{\big|\widetilde{\pi}_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,E(\mathbf{n},\xi)\big|}.$$

Therefore, according to the lower bound (2.26) of Lemma 2.22 and the small divisors Assumption 6, for all $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and all ξ in $\mathcal{C}(\mathbf{n})$, we have

$$\big|\Gamma\big((\mathbf{n},\xi),(\mathbf{n},\xi)\big)\big|\leqslant C|\mathbf{n}|^h, \tag{6.3}$$

where h is an integer larger than $(3+a_1)/2$ with notation of Assumption 6.

iii) The quantity $\widetilde{\pi}_{(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r)}\,\widetilde{L}_1\big(E(\mathbf{n}_p,\xi_p),\mathbf{n}_q\cdot\boldsymbol{\zeta}\big)\,E(\mathbf{n}_q,\xi_q)$ being homogeneous of degree 1 with respect to $\mathbf{n}_q$, the resonance coefficient $\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)$ formally corresponds to a partial derivative with respect to the fast tangential variables, applied to the profile associated with the frequency $(\mathbf{n}_q\cdot\zeta,\xi_q)$.

The coefficients Γ defined above shall appear in the computations to obtain a priori estimates for system (5.11). In particular, when these coefficients present some symmetry property, the associated resonance is easy to control in the a priori estimates. Thus we discriminate the resonances satisfying this symmetry property from the others.

Definition 6.8. *Fix a constant* $C_0>0$*. Let* $\mathbf{n}_r$ *be in* $\mathcal{B}_{\mathbb{Z}^m}$*, and* ξ_r *in* $\mathcal{C}(\mathbf{n}_r)$*. We consider the set of 7-tuples* $(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)$ *with* $\lambda_p,\lambda_q,\lambda_r$ *in* $\mathbb{Z}^*$*,* $\mathbf{n}_p,\mathbf{n}_q$ *in* $\mathcal{B}_{\mathbb{Z}^m}$*,* (ξ_p,ξ_q) *in* $\mathcal{C}(\mathbf{n}_p)\times\mathcal{C}(\mathbf{n}_q)$*,* $(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)$ *and* $(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)$ *non collinear and* $\lambda_p,\lambda_q,\lambda_r$ *coprime numbers, that resonate to give the resonance* $\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi)$ *in the following way*

$$\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)=\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r).$$

This set is written as the disjoint union

$$\mathcal{R}_1(\mathbf{n}_r,\xi_r)\sqcup\mathcal{R}_2(\mathbf{n}_r,\xi_r),$$

where the sets $\mathcal{R}_1(\mathbf{n}_r,\xi_r)$ *and* $\mathcal{R}_2(\mathbf{n}_r,\xi_r)$ *are defined as follows.*

i) The set $\mathcal{R}_1(\mathbf{n}_r,\xi_r)$ *is constituted of 7-tuples* $(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)$ *satisfying*

$$\big|\Gamma\big((\lambda_p\,\mathbf{n}_p,\lambda_p\,\xi_p),(\lambda_q\,\mathbf{n}_q,\lambda_q\,\xi_q)\big)+\Gamma\big((\lambda_p\,\mathbf{n}_p,\lambda_p\,\xi_p),(-\lambda_r\,\mathbf{n}_r,-\lambda_r\,\xi_r)\big)\big|\leqslant C_0\big|(\lambda_p\,\mathbf{n}_p,\lambda_p\,\xi_p)\big| \tag{6.4}$$

where $C_0>0$ *is the constant which have been fixed in the beginning and which does not depend on* λ_p*,* λ_q*,* λ_r*,* $\mathbf{n}_p$*,* $\mathbf{n}_q$*,* $\mathbf{n}_r$*,* ξ_p*,* ξ_q*,* ξ_r*. These resonances are said to be of type 1.*

ii) The 7-tuples $(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)$ *which do not satisfy the previous property constitute the set of type 2 resonances, denoted by* $\mathcal{R}_2(\mathbf{n}_r,\xi_r)$*.*

Remark 6.9. i) Note that the sets $\mathcal{R}_1(\mathbf{n},\xi)$ and $\mathcal{R}_2(\mathbf{n},\xi)$ depend on the constant $C_0>0$ fixed at the beginning, although this dependence is not indicated.

ii) According to Assumption 7, note that if the frequency $(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)$ is outgoing (resp. incoming), then both sets $\mathcal{R}_1(\mathbf{n},\xi)$ and $\mathcal{R}_2(\mathbf{n},\xi)$ are constituted only of 7-tuples corresponding to outgoing (resp. incoming) frequencies.

iii) Note that since the coefficients Γ are not symmetrical, the type of a resonance

$$\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)=\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r),$$

depends on the way it is written. However, since condition (6.4) is symmetrical in (q,r), the resonance

$$\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)=\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r),$$

is of type 1 if and only if the resonance

$$\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)-\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r)=-\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q),$$

is of type 1.

iv) Also note that if a resonance of the form

$$\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)=\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r),$$

holds, then for k in $\mathbb{Z}^*$, the following resonance relation is also satisfied

$$k\,\lambda_p\,(\mathbf{n}_p\cdot\boldsymbol{\zeta},\xi_p)+k\,\lambda_q\,(\mathbf{n}_q\cdot\boldsymbol{\zeta},\xi_q)=k\,\lambda_r\,(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r).$$

This explains the choice made in Definition 6.8 to consider only 3-tuples $(\lambda_p,\lambda_q,\lambda_r)$ of coprime integers and m-tuples $\mathbf{n}_p$, $\mathbf{n}_q$ and $\mathbf{n}_r$ of $\mathcal{B}_{\mathbb{Z}^m}$.

The previous definition leads to the last assumption of this work. It is made for technical reasons, but, up to our knowledge, it is not a necessary assumption.

Assumption 8. *There exists a constant $C_0>0$ such that the sets $\mathcal{R}_1(\mathbf{n},\xi)$ and $\mathcal{R}_2(\mathbf{n},\xi)$, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ in $\mathcal{C}(\mathbf{n})$, defined in Definition 6.8 satisfy the two following properties.*

a) The sets of incoming resonances of type 2 and outgoing resonances of types 1 and 2

$$\bigcup_{\substack{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi\in\mathcal{C}_{\mathrm{in}}(\mathbf{n})}}\mathcal{R}_2(\mathbf{n},\xi),\qquad \bigcup_{\substack{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi\in\mathcal{C}_{\mathrm{out}}(\mathbf{n})}}\big(\mathcal{R}_1(\mathbf{n},\xi)\cup\mathcal{R}_2(\mathbf{n},\xi)\big),$$

are finite sets.

b) For all incoming frequency $(\mathbf{n},\xi)$ of $\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\mathrm{in}}(\mathbf{n})$ such that the set $\mathcal{R}_1(\mathbf{n},\xi)$ is nonempty, the following lower bound holds

$$\big|\widetilde{\pi}_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}E(\mathbf{n},\xi)\big|\geqslant\frac{1}{C_0}. \tag{6.5}$$

Remark 6.10. i) The self-interaction between two collinear frequencies always constitute a resonance, but these terms should not be an issue in the analysis, since they induce terms of Burgers type, which are commonly treated in the estimates. However the resonances of type 2 are difficult to control, that is why a finiteness assumption is made on this set, whereas property (6.4) satisfied by resonances of type 1 allows to treat an infinite number of them (to the prize of a uniform control). Such an infinity of resonances appears irremediably in Example 2.1 of compressible isentropic Euler equations in dimension 2. It constitute one of the main additional difficulty addressed here in comparison to the monophase case of [CGW11].

ii) We already know that for $(\mathbf{n},\xi)$ in $\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}(\mathbf{n})$, the vector $\widetilde{\pi}_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}E(\mathbf{n},\xi)$ is bounded, according to Remark 2.7, and that it is allowed to go to zero but in a controlled way, according to estimate (2.26) and Assumption 6. In the case of an infinite number of resonances, namely for incoming resonances of type 1, we also need to make sure that these vectors do not go to zero, for a technical reason explained below. This is why we assume the uniform lower bound (6.5) of Assumption 8. This assumption excludes the possibility of the existence of a sequence of frequencies $(\mathbf{n},\xi)$ such that $\mathcal{R}_1(\mathbf{n},\xi)$ is nonempty, converging to the glancing set $\mathcal{G}$.

According to Assumption 7, it has already been established that the sets of incoming and outgoing frequencies $\mathcal{F}^{\rm in}$ and $\mathcal{F}^{\rm out}$ defined in (3.2) and (3.3) do not resonate with each other. This decomposition of the frequencies set in sets that do not resonate with each other is now to be refined, which will allow to decouple the studied system according to these sets.

Definition 6.11. *In this definition we confuse the frequency $(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)$ with the couple $(\mathbf{n},\xi)$. Let $C_0>0$ be the constant fixed in Assumption 8. We denote by $\mathcal{F}^{\rm out}_{\rm res}$ the set of outgoing frequencies $(\mathbf{n},\xi)$ of $\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm out}(\mathbf{n})$ involved in resonances of type 1 or 2, namely such that $\mathcal{R}_1(\mathbf{n},\xi)\cup\mathcal{R}_2(\mathbf{n},\xi)$ is nonempty. Then the following disjoint union holds*

$$\{(\mathbf{n},\xi)\in\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm out}(\mathbf{n})\}=\mathcal{F}^{\rm out}_{\rm res}\sqcup\bigsqcup_{(\mathbf{n},\xi)\in(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm out}(\mathbf{n}))\setminus\mathcal{F}^{\rm out}_{\rm res}}\{(\mathbf{n},\xi)\}\,,\tag{6.6}$$

where the set involved in the disjoint union do not resonate with each other. The set $\mathcal{F}^{\rm in}_{\rm res}$ is defined in a similar way for incoming frequencies, so that the following decomposition holds

$$\{(\mathbf{n},\xi)\in\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n})\}=\mathcal{F}^{\rm in}_{\rm res}\sqcup\bigsqcup_{(\mathbf{n},\xi)\in(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}))\setminus\mathcal{F}^{\rm in}_{\rm res}}\{(\mathbf{n},\xi)\}\,,\tag{6.7}$$

where the set involved in the disjoint union do not resonate with each other.

Remark 6.12. In Assumption 8, the bound (6.5) a priori applies to couples $(\mathbf{n},\xi)$ such that the set $\mathcal{R}_1(\mathbf{n},\xi)$ is nonempty. But since according to Assumption 8 there is only a finite number of type 2 resonances, we can assume without loss of generality that this bound also applies to couples $(\mathbf{n},\xi)$ such that $\mathcal{R}_1(\mathbf{n},\xi)$ is empty but $\mathcal{R}_2(\mathbf{n},\xi)$ is not, namely to all elements of $\mathcal{F}^{\rm in}_{\rm res}$. Therefore for all $(\mathbf{n},\xi)$ in $\mathcal{F}^{\rm in}_{\rm res}$, the following bound holds

$$\big|\Gamma\big((\mathbf{n},\xi),(\mathbf{n},\xi)\big)\big|\leqslant C|\mathbf{n}|.\tag{6.8}$$

Note that the previous estimate differs from (6.3) by a linear control and not an algebraic control of degree h.

Finally the projectors analogous to $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$, selecting only the incoming resonant frequencies, are defined, and we verify after that Assumption 8 for the Euler equations example considered in this paper.

Definition 6.13. *For all formal trigonometric series U writing*

$$U(z,\theta,\psi_d)=\sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}}U^{\rm osc}_{\mathbf{n},\xi}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d}+\sum_{\mathbf{n}\in\mathbb{Z}^m}U^{\rm ev}_{\mathbf{n}}(z,\psi_d)\,e^{i\,\mathbf{n}\cdot\theta},$$

the series $\widetilde{\mathbf{E}^i}^{\rm in}_{\rm res}U$ is defined as

$$\widetilde{\mathbf{E}^i}^{\rm in}_{\rm res}U(z,\theta,\psi_d):=\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{\pi}_{(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta},\lambda\xi_0)}\,U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d},\tag{6.9}$$

and $\mathbf{E}^{\rm in}_{\rm res}U$ as

$$\mathbf{E}^{\rm in}_{\rm res}U(z,\theta,\psi_d):=\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}}\sum_{\lambda\in\mathbb{Z}^*}\pi_{(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta},\lambda\xi_0)}\,U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}.\tag{6.10}$$

Example 6.14. We return to Example 2.1 of compressible isentropic Euler equations in dimension 2, for which we check the last assumption of this work, namely Assumption 8 about the control of resonances. Recall the notations and results of Example 2.1 and those after, and notably the analysis of the resonances made in Example 3.1, and consider a Mach number M satisfying the previously made assumptions. Is has been shown that if M^2 is not an algebraic number of degree less than 4 in $\mathbb{Q}[\delta]$, then the only resonances (except for the self-interactions) occurring are those involving the linear frequency $\alpha_3(\zeta)$, which are in infinite number, even with collinearity, and between incoming frequencies. It will be shown that there exists a constant $C_0>0$ such that these resonances satisfy property (6.4) and such that all frequencies $\alpha_3(\zeta_{p,q})$ satisfy the lower bound (6.5), proving that system (2.4) verifies Assumption 8. First we look for the coefficients

Γ for this type of resonances. Consider (p,q) and (r,s) in $\mathbb{Z}^2\setminus\{0\}$. The aim is to determine the coefficient

$$\Gamma\big((p,q),\xi_3(\zeta_{p,q}),(r,s),\xi_3(\zeta_{r,s})\big)$$

relative to the resonance

$$\alpha_3(\zeta_{p,q})+\alpha_3(\zeta_{r,s})=\alpha_3(\zeta_{p+r,q+s}),$$

denoted more briefly by $\Gamma\big((p,q),(r,s)\big)$. Since, for ζ in $\mathbb{R}^2\setminus\{0\}$, the real characteristic frequency $\alpha_3(\zeta)=(\tau,\eta,\xi_3(\zeta))$ satisfies $\tau=\tau_2\big(\eta,\xi_3(\zeta)\big)$, we obtain

$$E\big((p,q),\xi_3(\zeta_{p,q})\big)=E_2\big(\eta_{p,q},\xi_3(\zeta_{p,q})\big),\quad E\big((r,s),\xi_3(\zeta_{r,s})\big)=E_2\big(\eta_{r,s},\xi_3(\zeta_{r,s})\big),$$
$$E\big((p+r,q+s),\xi_3(\zeta_{p+r,q+s})\big)=E_2\big(\eta_{p+r,q+s},\xi_3(\zeta_{p+r,q+s})\big),$$

where $E_2(\eta,\xi)$ is the vector of basis (2.8) of $\mathbb{C}^N$ given in this example by

$$E_2(\eta,\xi):=\frac{1}{\sqrt{\eta^2+\xi^2}}\begin{pmatrix}0\\ \xi\\ -\eta\end{pmatrix},\quad (\eta,\xi)\in\mathbb{R}^2\setminus\{0\}\,.$$

Thus the associated vector of basis (2.9) is given by

$$A_2(V_0)^{-1}\,E_2(\eta,\xi)=\frac{1}{(u_0^2-c_0^2)\sqrt{\eta^2+\xi^2}}\begin{pmatrix}-\eta\,v_0\\ \xi\,\frac{u_0^2-c_0^2}{u_0}\\ -\eta\,u_0\end{pmatrix},\quad (\eta,\xi)\in\mathbb{R}^2\setminus\{0\}\,.$$

Also computing the vectors $E_1(\eta,\xi)$ and $E_3(\eta,\xi)$ for $(\eta,\xi)\in\mathbb{R}^2\setminus\{0\}$, it is determined that the projector $\widetilde{\pi}_{((p+r,q+s)\cdot\boldsymbol{\zeta},\xi_3(\zeta_{p+r,q+s}))}$ occurring in the coefficient $\Gamma\big((p,q),(r,s)\big)$ is given in this example by $\widetilde{\pi}_2(\eta_{p+r,q+s},\xi_3(\zeta_{p+r,q+s}))$ where, for (η,ξ) in $\mathbb{R}^2\setminus\{0\}$,

$$\widetilde{\pi}_2(\eta,\xi)=\frac{1}{u_0(u_0^2-c_0^2)(\eta^2+\xi^2)}\begin{pmatrix}-u_0\,c_0^2\,\eta^2 & -u_0^2\,v_0\,\eta\,\xi & u_0^2\,v_0\,\eta^2\\ \eta\,\xi\,c_0^2\,(u_0^2-c_0^2)/v_0 & \xi^2\,u_0\,(u_0^2-c_0^2) & -\eta\,\xi\,u_0\,(u_0^2-c_0^2)\\ -\eta^2\,c_0^2\,u_0^2/v_0 & -\eta\,\xi\,u_0^3 & u_0^3\,\eta^2\end{pmatrix}.$$

Thus the vector $\widetilde{\pi}_2(\eta,\xi)\,E_2(\eta,\xi)$ is given by $u_0\,A_2(V_0)^{-1}\,E_2(\eta,\xi)$, for (η,ξ) in $\mathbb{R}^2\setminus\{0\}$. It ensures in particular that Assumption 8 is verified, since the following uniform lower bound holds

$$\begin{aligned}\big|\widetilde{\pi}_{\alpha_3(\zeta_{p,q})}E\big((p,q),\xi_3(\zeta_{p,q})\big)\big|&=\big|\widetilde{\pi}_2\big(\eta_{p,q},\xi_3(\zeta_{p,q})\big)\,E_2\big(\eta_{p,q},\xi_3(\zeta_{p,q})\big)\big|\\ &=\big|u_0\,A_2(V_0)^{-1}\,E_2\big(\eta_{p,q},\xi_3(\zeta_{p,q})\big)\big|\geqslant C\,\big|E_2\big((p,q),\xi_3(\zeta_{p,q})\big)\big|=C.\end{aligned}$$

Returning to the determination of coefficients $\Gamma\big((p,q),(r,s)\big)$, by computing differentials $d\widetilde{A}_i(V_0)$, we finally get

$$\begin{aligned}&\widetilde{L}_1\big(E_2\big(\eta_{p,q},\xi_3(\zeta_{p,q})\big),\zeta_{r,s}\big)\,E_2\big(\eta_{r,s},\xi_3(\zeta_{r,s})\big)\\ &\qquad=\frac{\tau_{p,q}\,\eta_{r,s}-\tau_{r,s}\,\eta_{p,q}}{\sqrt{\eta_{p,q}^2+\tau_{p,q}^2/u_0^2}\sqrt{\eta_{r,s}^2+\tau_{r,s}^2/u_0^2}}\begin{pmatrix}v_0\,\eta_{r,s}/[u_0\,(u_0^2-c_0^2)]\\ \tau_{r,s}/u_0^3\\ \eta_{r,s}/(u_0^2-c_0^2)\end{pmatrix}.\end{aligned}$$

The formula $\xi_3(\zeta) = -\tau/u_0$, for $\zeta = (\tau, \eta)$ in $\mathbb{R}^2 \setminus \{0\}$ has been used here. Then we have

$$\widetilde{\pi}_2(\eta_{p+r,q+s}, \xi_3(\zeta_{p+r,q+s}))\, \widetilde{L}_1\big(E_2\big(\eta_{p,q}, \xi_3(\zeta_{p,q})\big), \zeta_{r,s}\big)\, E_2\big(\eta_{r,s}, \xi_3(\zeta_{r,s})\big)$$

$$= \frac{(\tau_{p,q}\,\eta_{r,s} - \tau_{r,s}\,\eta_{p,q})(\tau_{p+r,q+s}\,\tau_{r,s} + u^2\,\eta_{p+r,q+s}\,\eta_{r,s})}{u_0^2\,(u_0^2 - c_0^2)\sqrt{\eta_{p,q}^2 + \tau_{p,q}^2/u_0^2}\sqrt{\eta_{r,s}^2 + \tau_{r,s}^2/u_0^2}\big(\eta_{p+r,q+s}^2 + \tau_{p+r,q+s}^2/u_0^2\big)} \begin{pmatrix} \eta_{p+r,q+s}\, v_0/u_0 \\ \tau_{p+r,q+s}\,(u_0^2 - c_0^2)/u_0^3 \\ \eta_{p+r,q+s} \end{pmatrix}.$$

We deduce from the relation $\widetilde{\pi}_2(\eta, \xi)\, E_2(\eta, \xi) = u_0\, A_2(V_0)^{-1}\, E_2(\eta, \xi)$ the following formula for the pursued coefficient $\Gamma\big((p,q),(r,s)\big)$:

$$\Gamma\big((p,q),(r,s)\big) = -\frac{(\tau_{p,q}\,\eta_{r,s} - \tau_{r,s}\,\eta_{p,q})(\tau_{p+r,q+s}\,\tau_{r,s} + u_0^2\,\eta_{p+r,q+s}\,\eta_{r,s})}{u_0^4\,\sqrt{\eta_{p,q}^2 + \tau_{p,q}^2/u_0^2}\sqrt{\eta_{r,s}^2 + \tau_{r,s}^2/u_0^2}\sqrt{\eta_{p+r,q+s}^2 + \tau_{p+r,q+s}^2/u_0^2}}. \tag{6.11}$$

We now check estimate (6.4) with these coefficients. Let (p,q), (r,s) and (t,w) be in $\mathbb{Z}^2 \setminus \{0\}$ such that $(p,q) + (r,s) + (t,w) = (0,0)$. One can verify, using the formulas $\tau_{p,q} = c_0\,(p + \delta q)\,\eta_0$, $\eta_{p,q} = (p+q)\,\eta_0$, $p + r = -t$ and $q + s = -w$, that we get

$$\Gamma\big((p,q),(r,s)\big)$$
$$= \frac{(1-\delta)(ps - qr)\big((t+\delta w)\,(r+\delta s) + M^2\,(t+w)\,(r+s)\big)}{\sqrt{(p+q)^2 + (p+\delta q)^2/M^2}\sqrt{(r+s)^2 + (r+\delta s)^2/M^2}\sqrt{(t+w)^2 + (t+\delta w)^2/M^2}}\,\frac{\eta_0\,c_0^3}{u_0^4}.$$

Since $ps - qr = -(pw - qt)$, we finally have

$$\Gamma\big((p,q),(r,s)\big) + \Gamma\big((p,q),(t,w)\big) = 0,$$

so estimate (6.4) is in particular trivially satisfied and therefore so is Assumption 8. Note that in this example, the set $\mathcal{F}_{\text{res}}^{\text{out}}$ is empty and the set $\mathcal{F}_{\text{res}}^{\text{in}}$ is given by

$$\mathcal{F}_{\text{res}}^{\text{in}} = \{\alpha_3(\zeta) \,|\, \zeta \in \mathcal{F}_b\}.$$

It concludes the analysis in this paper of the example of compressible isentropic Euler equations in dimension 2, which, with the chosen parameters, satisfies all assumptions of this work.

Remark 6.15. Note that the coefficient Γ determined above corresponds to the one in [Rau12, (11.5.9)].

In the following subsection, some rigorous results on projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$ will be proved, using the small divisors Assumption 6.

6.2. **Rigorous definition of projectors $\mathbf{E}$, $\widetilde{\mathbf{E}^i}$, $\mathbf{E}_{\text{res}}^{\text{in}}$ and $\widetilde{\mathbf{E}^i}_{\text{res}}^{\text{in}}$.** This part follows [JMR95, Section 6.2]. Before considering the projectors $\mathbf{E}$, $\widetilde{\mathbf{E}^i}$, $\mathbf{E}_{\text{res}}^{\text{in}}$ and $\widetilde{\mathbf{E}^i}_{\text{res}}^{\text{in}}$, we state the following controls over spectral projectors, that will be used to rigorously define the projectors $\mathbf{E}$ and $\mathbf{E}_{\text{res}}^{\text{in}}$. The proof of these controls uses notations and results from the one of Proposition 2.21, and is therefore postponed after it, in Appendix A.

Proposition 6.16. *Under Assumption 6, there exists a constant $c_1 > 0$ and a real number b_1 such that, for all ζ in $\mathcal{F}_b \setminus \{0\}$, the following estimates hold*

$$\left|e^{t\mathcal{A}(\zeta)}\,\Pi_-^e(\zeta)\right| \leqslant c_1\, e^{-c_1\, t\,|\zeta|^{-b_1}} \leqslant c_1, \qquad \forall t \geqslant 0, \tag{6.12a}$$

$$\left|e^{t\mathcal{A}(\zeta)}\,\Pi_{\mathbb{C}^N}^e(\zeta)\right| \leqslant c_1\,|\zeta|^{b_1}\, e^{-c_1\, t\,|\zeta|^{-b_1}}, \qquad \forall t \geqslant 0, \tag{6.12b}$$

$$\left|e^{t\mathcal{A}(\zeta)}\big(I - \Pi_{\mathbb{C}^N}^e(\zeta)\big)\right| \leqslant c_1\,|\zeta|^{b_1}, \qquad \forall t \leqslant 0. \tag{6.12c}$$

We are now in position to rigorously define projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$. The result concerning the oscillating part comes from [JMR95, Proposition 6.2.1] and [CGW11, Proposition 2.2], but the proof is recalled here. The result concerning the evanescent part is simpler, and reduces to prove that some series converges.

Lemma 6.17 ([JMR95, Proposition 6.2.1])**.** *For all $T > 0$ and $s \geqslant 0$, the projectors $\widetilde{\mathbf{E}^i}$ and $\mathbf{E}$ defined by (5.9) and (5.10) on the space of trigonometric polynomials each admit a unique extension from the space $\mathcal{P}^{\rm osc}_{s,T}$ to itself. Moreover, for $T_0 > 0$, their norm is uniformly bounded with respect to T in $]0, T_0]$.*

On an other hand, for $T > 0$ and $s \geqslant 0$, the projector $\mathbf{E}$ is well-defined from the space $\mathcal{P}^{\rm ev}_{s+\lceil b_1\rceil,T}$ to the space $\mathcal{P}^{\rm ev}_{s,T}$. Furthermore it is uniformly bounded with respect to s and T. Recall that b_1 refers to the real number of Proposition 6.16.

Finally, for $T > 0$ and $s \geqslant 0$, the projectors $\widetilde{\mathbf{E}^i}^{\rm in}_{\rm res}$ and $\mathbf{E}^{\rm in}_{\rm res}$ defined as (6.9) and (6.10) on the space of trigonometric polynomials each admit a unique extension from the space $\mathcal{P}^{\rm osc}_{s,T}$ to itself. Moreover, for $T_0 > 0$, their norm is uniformly bounded with respect to T in $]0, T_0]$.

Proof. First the oscillating case is investigated. We consider $U^{\rm osc}$ a trigonometric polynomial writing

$$U^{\rm osc}(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}} U^{\rm osc}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d},$$

where each sum in ξ is finite, and we denote, for $\mathbf{n}\in\mathbb{Z}^m$,

$$U^{\rm osc}_{\mathbf{n}}(z,\psi_d) := \sum_{\xi} U^{\rm osc}_{\mathbf{n},\xi}(z)\, e^{i\,\xi\,\psi_d}. \tag{6.13}$$

According to formula (5.10) for the projector $\mathbf{E}$ and since the projectors π_α are zero for every noncharacteristic frequency α, we obtain, for $\mathbf{n}\in\mathbb{Z}^m$,

$$\mathbf{E}\, U^{\rm osc}_{\mathbf{n}}(z,\psi_d) = \sum_{\xi\in\mathcal{C}(\mathbf{n})} \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U^{\rm osc}_{\mathbf{n},\xi}(z)\, e^{i\,\xi\,\psi_d},$$

so that, according to Remark 2.7 ensuring that the projectors π_α are uniformly bounded,

$$|\mathbf{E}\, U^{\rm osc}_{\mathbf{n}}(z,\psi_d)|^2 \leqslant C\, N \sum_{\xi\in\mathcal{C}(\mathbf{n})} \left|U^{\rm osc}_{\mathbf{n},\xi}(z)\right|^2,$$

using the fact that the cardinality of $\mathcal{C}(\mathbf{n})$ is at most N. On an other hand, according to (6.13), for $\mathbf{n}$ in $\mathbb{Z}^m$ and ξ in $\mathbb{R}$, we have

$$U^{\rm osc}_{\mathbf{n},\xi}(z) = \lim_{R\to+\infty}\frac{1}{R}\int_0^R U^{\rm osc}_{\mathbf{n}}(z,\psi_d)\, e^{-i\,\xi\,\psi_d}\, d\psi_d,$$

so that, using Cauchy-Schwarz inequality,

$$|\mathbf{E}\, U^{\rm osc}_{\mathbf{n}}(z,\psi_d)|^2 \leqslant C\, N^2 \lim_{R\to+\infty}\frac{1}{R}\int_0^R |U^{\rm osc}_{\mathbf{n}}(z,\psi_d)|^2\, d\psi_d.$$

Then Fatou's lemma is applied to the sum with respect to $\mathbf{n}$ in $\mathbb{Z}^m$ and the integration with respect to z' in ω_T to get

$$\sum_{\mathbf{n}\in\mathbb{Z}^m} \|\mathbf{E}\, U^{\rm osc}_{\mathbf{n}}(.,x_d,\psi_d)\|^2_{L^2(\omega_T)} \leqslant C\, N^2 \liminf_{R\to+\infty}\frac{1}{R}\int_0^R \sum_{\mathbf{n}\in\mathbb{Z}^m} \|U^{\rm osc}_{\mathbf{n}}(.,x_d,\psi_d)\|^2_{L^2(\omega_T)}\, d\psi_d,$$

thus

$$\|\mathbf{E}\, U(.,x_d,.,\psi_d)\|^2_{L^2(\omega_T\times\mathbb{T}^m)} \leqslant C\, N^2 \sup_{\psi_d>0} \|U^{\rm osc}(.,x_d,.,\psi_d)\|^2_{L^2(\omega_T\times\mathbb{T}^m)},$$

that is to say

$$\|\mathbf{E}\, U\|_{\mathcal{E}_{0,T}} \leqslant \sqrt{C}\, N\, \|U\|_{\mathcal{E}_{0,T}}.$$

The projector $\mathbf{E}$ thus admits a uniformly bounded unique extension from $\mathcal{P}^{\rm osc}_{0,T}$ to $\mathcal{P}^{\rm osc}_{0,T}$. The result for the space $\mathcal{P}^{\rm osc}_{s,T}$ for $s\geqslant 1$ is obtained by observing that the projector $\mathbf{E}$ commutes with

the partial derivatives with respect to z' and θ. The same argument applies to the projector $\widetilde{\mathbf{E}^i}$, which concludes the proof relative to the oscillating part.

Concerning the evanescent part of $\mathbf{E}$, it must be proved that if U^{ev} writing

$$U^{\mathrm{ev}}(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m} U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\, e^{i\,\mathbf{n}\cdot\theta}$$

belongs to $\mathcal{P}^{\mathrm{ev}}_{s+\lceil b_1\rceil,T}$, then $\mathbf{E}\,U^{\mathrm{ev}}$ is in $\mathcal{P}^{\mathrm{ev}}_{s,T}$. By definition of the norm of $\mathcal{E}_{s,T}$ and according to the Parseval's identity, we obtain

$$\begin{aligned}\|\mathbf{E}\,U^{\mathrm{ev}}\|^2_{\mathcal{E}_{s,T}} &= \sup_{x_d>0,\psi_d>0} \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l} \left\| e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\right\|^2_{H^l_+(\omega_T)} \\ &\leqslant \sup_{x_d>0,\psi_d>0} \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l}\, c_1^2\, |\mathbf{n}\cdot\boldsymbol{\zeta}|^{2\,b_1}\, \|U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\|^2_{H^l_+(\omega_T)}\,,\end{aligned}$$

according to estimate (6.12b) of Proposition 6.16 and recalling the notations of this result. It leads to the following estimate

$$\|\mathbf{E}\,U^{\mathrm{ev}}\|_{\mathcal{E}_{s,T}} \leqslant C\, \|U^{\mathrm{ev}}\|_{\mathcal{E}_{s+\lceil b_1\rceil,T}}\,.$$

We investigate now the convergence towards zero in $H^s(\omega_T\times\mathbb{T}^m)$ of the profile $\mathbf{E}\,U^{\mathrm{ev}}(x_d,\psi_d)$, for every fixed $x_d\geqslant 0$. Consider $\varepsilon>0$. Using the convergence of the following sum,

$$\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l}\, |\mathbf{n}\cdot\boldsymbol{\zeta}|^{2\,b_1}\, \|U^{\mathrm{ev}}_{\mathbf{n}}(.,x_d,0)\|^2_{H^l_+(\omega_T)}\,,$$

which is bounded by $C\,\|U^{\mathrm{ev}}\|^2_{\mathcal{E}_{s+\lceil b_1\rceil,T}}$, there exists $M>0$ such that

$$\sum_{|\mathbf{n}|>M} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l}\, |\mathbf{n}\cdot\boldsymbol{\zeta}|^{2\,b_1}\, \|U^{\mathrm{ev}}_{\mathbf{n}}(.,x_d,0)\|^2_{H^l_+(\omega_T)} \leqslant \varepsilon.$$

Then we have

$$\begin{aligned}\|\mathbf{E}\,U^{\mathrm{ev}}(x_d,\psi_d)\|^2_{H^s(\omega_T\times\mathbb{T}^m)} &= \sum_{0<|\mathbf{n}|\leqslant M} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l} \left\| e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\right\|^2_{H^l_+(\omega_T)} \\ &\quad + \sum_{|\mathbf{n}|>M} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l} \left\| e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\right\|^2_{H^l_+(\omega_T)}.\end{aligned}$$

According to estimate (6.12b) of Proposition 6.16 and by construction of M, the second sum of the right-hand side is less or equal to $c_1^2\,\varepsilon$. For the first one, according to the same estimate (6.12b), we have

$$\begin{aligned}&\sum_{0<|\mathbf{n}|\leqslant M} \sum_{l=0}^{s} (1+|\mathbf{n}|^2)^{s-l} \left\| e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\right\|^2_{H^l_+(\omega_T)} \\ &\qquad \leqslant \sum_{0<|\mathbf{n}|\leqslant M} \sum_{l=0}^{s} c_1^2\, (1+|\mathbf{n}|^2)^{s-l}\, |\mathbf{n}\cdot\boldsymbol{\zeta}|^{2\,b_1}\, e^{-2\,c_1\,\psi_d\,|\mathbf{n}\cdot\boldsymbol{\zeta}|^{-b_1}}\, \|U^{\mathrm{ev}}_{\mathbf{n}}(.,0)\|^2_{H^l_+(\omega_T)}\,.\end{aligned}$$

The right-hand side is a finite sum of functions of ψ_d converging to zero at infinity, so there exists $B>0$ such that for $|\psi_d|>B$, the right-hand side is less or equal to ε. We get finally

$$\|\mathbf{E}\,U^{\mathrm{ev}}(x_d,\psi_d)\|^2_{H^s(\omega_T\times\mathbb{T}^m)} \leqslant (1+c_1^2)\,\varepsilon,$$

for all ψ_d such that $|\psi_d|>B$, and the aimed convergence follows. The profile $\mathbf{E}\,U^{\mathrm{ev}}$ thus satisfies the condition of Definition 3.4 of the evanescent profiles of $\mathcal{P}^{\mathrm{ev}}_{s,T}$, concluding the proof.

Concerning the projectors $\mathbf{E}^{\mathrm{in}}_{\mathrm{res}}$ and $\widetilde{\mathbf{E}^i}^{\mathrm{in}}_{\mathrm{res}}$, the proof is analogous to the one for the oscillating part of $\mathbf{E}$ and $\widetilde{\mathbf{E}}$. □

Now that the projectors $\mathbf{E}$, $\widetilde{\mathbf{E}^i}$, $\mathbf{E}^{\mathrm{in}}_{\mathrm{res}}$ and $\widetilde{\mathbf{E}^i}^{\mathrm{in}}_{\mathrm{res}}$ are well-defined, it can be proved that the kernel in $\mathcal{P}_{s,T}$ of the fast operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$ is actually given by the range of the projector $\mathbf{E}$.

Definition 6.18. *For $s \geqslant 0$ and $T > 0$, we denote by $\mathcal{N}_{s,T}$ the range in $\mathcal{P}_{s,T}$ of $\mathcal{P}_{s+\lceil b_1 \rceil,T}$ projected by $\mathbf{E}$. We also denote $\mathcal{N}^{\mathrm{osc}}_{s,T} := \mathcal{N}_{s,T} \cap \mathcal{P}^{\mathrm{osc}}_{s,T}$ and $\mathcal{N}^{\mathrm{ev}}_{s,T} := \mathcal{N}_{s,T} \cap \mathcal{P}^{\mathrm{ev}}_{s,T}$.*

Lemma 6.19 ([JMR95, Lemma 6.2.3.])**.** *The space $\mathcal{N}_{s,T}$ is the kernel in $\mathcal{P}_{s,T}$ of the operator $\mathcal{L}(\partial_\theta, \partial_{\psi_d})$.*

Proof. Since $\mathcal{N}_{s,T}$ is equal to the kernel $\ker I - \mathbf{E}$ in $\mathcal{P}_{s,T}$, it must be shown that the equality of kernels $\ker I - \mathbf{E} = \ker \mathcal{L}(\partial_\theta, \partial_{\psi_d})$ holds in $\mathcal{P}_{s,T}$. Let $U = U^{\mathrm{osc}} + U^{\mathrm{ev}}$ be in $\ker \mathcal{L}(\partial_\theta, \partial_{\psi_d})$, and write

$$U(z, \theta, \psi_d) = U^*(z, \psi_d) + \sum_{\mathbf{n} \in \mathbb{Z}^m \setminus \{0\}} U_{\mathbf{n}}(z, \psi_d)\, e^{i\,\mathbf{n}\cdot\theta},$$

where $U_{\mathbf{n}}$ decomposes in $\mathcal{P}_{s,T} = \mathcal{P}^{\mathrm{osc}}_{s,T} \oplus \mathcal{P}^{\mathrm{ev}}_{s,T}$ as $U_{\mathbf{n}} = U^{\mathrm{osc}}_{\mathbf{n}} + U^{\mathrm{ev}}_{\mathbf{n}}$ for all $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$. Then one gets the following differential equations,

$$\partial_{\psi_d} U^* = 0, \qquad \text{and} \qquad \big(- \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta}) + \partial_{\psi_d} \big)\, U_{\mathbf{n}} = 0, \quad \forall \mathbf{n} \in \mathbb{Z}^m \setminus \{0\}\,.$$

Thus, on one hand, it follows $U^*(z, \psi_d) = U^*(z)$, which therefore belongs to $\mathcal{P}^{\mathrm{osc}}_{s,T}$. On the other hand, for every $\mathbf{n}$ in $\mathbb{Z}^m$, the amplitude $U_{\mathbf{n}}$ admits the following expansion, according to decomposition (2.25) of $\mathbb{C}^N$ into subspaces stable under the action of $\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})$,

$$U_{\mathbf{n}} = \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}} + \Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}} + \sum_{\xi \in \mathcal{C}(\mathbf{n})} \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}},$$

where, recalling the notations of Definition 2.20, $\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}$ (resp. $\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}$) belongs to the stable (resp. unstable) elliptic component $E^e_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ (resp. $E^e_+(\mathbf{n}\cdot\boldsymbol{\zeta})$), and for $\xi = \xi_j(\mathbf{n}\cdot\boldsymbol{\zeta}) \in \mathcal{C}(\mathbf{n})$, $\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}$ belongs to the subspace $\ker L\big(0, \alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)$ (recall that according to proposition 2.16, for j in $\mathcal{I}(\mathbf{n}\cdot\boldsymbol{\zeta})$, we have $E^j_-(\mathbf{n}\cdot\boldsymbol{\zeta}) = \ker L\big(0, \alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)$, and a similar result holds for j in $\mathcal{O}(\mathbf{n}\cdot\boldsymbol{\zeta})$). Using that $\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta}) = -i\widetilde{L}(0, \mathbf{n}\cdot\boldsymbol{\zeta})$ and the property of the projectors $\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}$, we get the following differential equations for each part,

$$\partial_{\psi_d} \big(\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}\big) = \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})\, \big(\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}\big), \tag{6.14a}$$

$$\partial_{\psi_d} \big(\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}\big) = \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})\, \big(\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}\big), \tag{6.14b}$$

$$\partial_{\psi_d} \big(\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}\big) = i\,\xi\, \big(\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}\big), \qquad \forall \xi \in \mathcal{C}(\mathbf{n}). \tag{6.14c}$$

Resolving equation (6.14a), one get

$$\Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, \psi_d) = e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, 0),$$

which goes to zero as ψ_d goes to infinity, so belongs to $\mathcal{P}^{\mathrm{ev}}_{s,T}$. In the same way, we get, with (6.14b),

$$\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, \psi_d) = e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, 0).$$

But since $\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})$ is the projector on the unstable elliptic component, if $\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, 0)$ is nonzero then $\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}$ is unbounded, so we have

$$\Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}} = 0.$$

Finally, (6.14c) gives, for ξ in $\mathcal{C}(\mathbf{n})$,

$$\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}(z, \psi_d) = e^{i\,\xi\,\psi_d}\, \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}(z, 0),$$

which belongs to $\mathcal{P}^{\mathrm{osc}}_{s,T}$. To summarize, $U_{\mathbf{n}} = U^{\mathrm{osc}}_{\mathbf{n}} + U^{\mathrm{ev}}_{\mathbf{n}}$ is given by

$$U^{\mathrm{osc}}_{\mathbf{n}}(z, \psi_d) = \sum_{\xi \in \mathcal{C}(\mathbf{n})} \pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U_{\mathbf{n}}(z, 0)\, e^{i\,\xi\,\psi_d}, \qquad U^{\mathrm{ev}}_{\mathbf{n}}(z, \psi_d) = e^{\psi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^{e,+}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U_{\mathbf{n}}(z, 0),$$

so the equality $\mathbf{E}\, U = U$ clearly holds.

Conversely, if $U^{\mathrm{ev}} = \mathbf{E}\, U^{\mathrm{ev}}$, it immediately leads to $\mathcal{L}(\partial_\theta, \partial_{\psi_d})\, U^{\mathrm{ev}} = 0$. On an other hand, supposing $U^{\mathrm{osc}} = \mathbf{E}\, U^{\mathrm{osc}}$, we consider a sequence $(U^{\mathrm{osc}}_\nu)_\nu$ of trigonometric polynomials converging

in $\mathcal{E}_{s+\lceil b_1\rceil,T}$ towards $U^{\rm osc}$. By continuity of the projector $\mathbf{E}$, the sequence of trigonometric polynomials $(\mathbf{E}\,U_\nu^{\rm osc})_\nu$ converges in $\mathcal{E}_{s,T}$ towards $U^{\rm osc}$. But one can check immediately that these trigonometric polynomials satisfy $\mathcal{L}(\partial_\theta,\partial_{\psi_d})\,\mathbf{E}\,U_\nu^{\rm osc}=0$, then passing to the limit yields to $\mathcal{L}(\partial_\theta,\partial_{\psi_d})\,U^{\rm osc}=0$. □

Remark 6.20. In the proof above, it has been proven in particular that if $U=U^{\rm osc}+U^{\rm ev}$ belongs to $\mathcal{N}_{s,T}^{\rm osc}\oplus\mathcal{N}_{s,T}^{\rm ev}$, then the profile $U^{\rm osc}$ writes

$$U^{\rm osc}(z,\theta,\psi_d)=U^*(z)+\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\sum_{\xi\in\mathcal{C}(\mathbf{n})}U_{\mathbf{n},\xi}^{\rm osc}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d},\tag{6.15}$$

with, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ in $\mathcal{C}(\mathbf{n})$, $\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U_{\mathbf{n},\xi}^{\rm osc}=U_{\mathbf{n},\xi}^{\rm osc}$, and the profile $U^{\rm ev}$ writes

$$U^{\rm ev}(z,\theta,\psi_d)=\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}e^{\psi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi_{\mathbb{C}^N}^e(\mathbf{n}\cdot\boldsymbol{\zeta})\,U_{\mathbf{n}}^{\rm ev}(z,0)\,e^{i\,\mathbf{n}\cdot\theta}.\tag{6.16}$$

The previous remark leads to the following result, which links the norm $\mathcal{C}_b(\mathbb{R}_{\psi_d}^+,L^2(\omega_T\times\mathbb{T}^m))$ of a profile of $\mathcal{N}_{0,T}^{\rm osc}$ and its incoming scalar product (3.12) with itself. This result will be used in the following to deduce from a priori estimates on the scalar product a priori estimates on the norm $\mathcal{P}_{s,T}^{\rm osc}$. It is analogous to [JMR95, Lemma 6.2.4], in a weaker form (because of a lack of symmetry in our context).

Lemma 6.21. *There exists a constant $C>0$ such that for every profile $U^{\rm osc}$ of $\mathcal{N}_{0,T}^{\rm osc}$, we have, for $x_d\geqslant 0$,*

$$C\,\|U^{\rm osc}\|^2_{\mathcal{C}_b(\mathbb{R}_{\psi_d}^+,L^2(\omega_T\times\mathbb{T}^m))}(x_d)\leqslant\langle U^{\rm osc}\,|\,U^{\rm osc}\rangle_{\rm in}(x_d)\leqslant\|U^{\rm osc}\|^2_{\mathcal{C}_b(\mathbb{R}_{\psi_d}^+,L^2(\omega_T\times\mathbb{T}^m))}(x_d).$$

Proof. The second inequality is obvious by definition of the scalar product $\langle.\,|\,.\rangle_{\rm in}$, since we have

$$\langle U^{\rm osc}\,|\,U^{\rm osc}\rangle_{\rm in}(x_d)=\lim_{R\to+\infty}\frac{1}{R}\int_0^R\|U^{\rm osc}\|^2_{L^2(\omega_T\times\mathbb{T}^m)}(x_d,\psi_d)\,d\psi_d\leqslant\sup_{\psi_d\geqslant 0}\|U^{\rm osc}\|^2_{L^2(\omega_T\times\mathbb{T}^m)}(x_d,\psi_d).$$

On the other hand, according to Remark 6.20, if $U^{\rm osc}$ belongs to $\mathcal{N}_{0,T}^{\rm osc}$, then the profile writes

$$U^{\rm osc}(z,\theta,\psi_d)=U^*(z)+\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\sum_{\xi\in\mathcal{C}(\mathbf{n})}U_{\mathbf{n},\xi}^{\rm osc}(z)\,e^{i\,\mathbf{n}\cdot\theta}\,e^{i\,\xi\,\psi_d},$$

where, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ in $\mathcal{C}(\mathbf{n})$, each amplitude satisfies $U_{\mathbf{n},\xi}^{\rm osc}=\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U_{\mathbf{n},\xi}^{\rm osc}$. The Parseval's identity then gives

$$\|U^{\rm osc}\|^2_{L^2(\omega_T\times\mathbb{T}^m)}(x_d,\psi_d)=\|U^*\|^2_{L^2(\omega_T)}+\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\left\|\sum_{\xi\in\mathcal{C}(\mathbf{n})}U_{\mathbf{n},\xi}^{\rm osc}(z)\,e^{i\,\xi\,\psi_d}\right\|^2_{L^2(\omega_T)}.$$

Therefore, since for all $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$, the set $\mathcal{C}(\mathbf{n})$ is of cardinality at most N, we have

$$\begin{aligned}\|U^{\rm osc}\|^2_{L^2(\omega_T\times\mathbb{T}^m)}(x_d,\psi_d)&\leqslant\|U^*\|^2_{L^2(\omega_T)}+N\sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\sum_{\xi\in\mathcal{C}(\mathbf{n})}\left\|U_{\mathbf{n},\xi}^{\rm osc}\right\|^2_{L^2(\omega_T)}(x_d)\\&\leqslant C\,\langle U^{\rm osc}\,|\,U^{\rm osc}\rangle_{\rm in}(x_d),\end{aligned}$$

according to formula (3.15). The first inequality of Lemma 6.21 follows finally by passing to the supremum in $\psi_d\geqslant 0$. □

6.3. **Reducing the system.** It is shown in this part that in every solution to system (5.11) there occur only incoming modes (in particular every solution is of zero mean), and every solution is supported in a finite interval in x_d. We also show that system (5.11) decouples according to the oscillating and the evanescent part, and even, for the oscillating part, according to the set $\mathcal{F}_{\rm res}^{\rm in}$ of resonant modes and each non resonant mode. More precisely, the following result is proved. Recall that s_0 is given by $s_0=h+(d+m)/2$ where h is an integer greater or equal to $(3+a_1)/2$ occurring in estimate (6.3), with a_1 the real number of Assumption 6.

Proposition 6.22. *Consider $T > 0$, and $s > s_0$. Every solution U in $\mathcal{P}_{s,T}$ of system (5.11) is such that its oscillating part $U^{\rm osc}$ features only incoming modes. Furthermore, system (5.11) on $U = U^{\rm osc} + U^{\rm ev}$ in $\mathcal{P}_{s,T}$ is equivalent to the following decoupled systems, the first one involving the resonant incoming modes,*

$$\mathbf{E}^{\rm in}_{\rm res}\, U^{\rm osc}_{\rm res} = U^{\rm osc}_{\rm res} \tag{6.17a}$$

$$\widetilde{\mathbf{E}^{i}}^{\rm in}_{\rm res}\Big[\widetilde{L}(0,\partial_z)\,\beta_T U^{\rm osc}_{\rm res} + \sum_{j=1}^{m}\widetilde{L}_1(\beta_T U^{\rm osc}_{\rm res},\zeta_j)\,\partial_{\theta_j}\beta_T U^{\rm osc}_{\rm res}\Big] = 0 \tag{6.17b}$$

$$\big(U^{\rm osc}_{\rm res}\big)_{|x_d=0,\psi_d=0} = H^{\rm osc}_{\rm res} \tag{6.17c}$$

$$\big(U^{\rm osc}_{\rm res}\big)_{|t\leqslant 0} = 0, \tag{6.17d}$$

then the system verified by each non resonant incoming mode, for $(\mathbf{n}_0,\xi_0)$ in $\big(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0)\big)\setminus\mathcal{F}^{\rm in}_{\rm res}$,

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}S_{\mathbf{n}_0,\xi_0} + \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)S_{\mathbf{n}_0,\xi_0}\partial_\Theta S_{\mathbf{n}_0,\xi_0} = 0 \tag{6.18a}$$

$$\big(S_{\mathbf{n}_0,\xi_0}\big)_{|x_d=0} = h_{\mathbf{n}_0,\xi_0} \tag{6.18b}$$

$$\big(S_{\mathbf{n}_0,\xi_0}\big)_{|t\leqslant 0} = 0, \tag{6.18c}$$

and finally the system for the evanescent part $U^{\rm ev}$,

$$\mathbf{E}\,U^{\rm ev} = U^{\rm ev} \tag{6.19a}$$

$$U^{\rm ev}_{|x_d=0,\psi_d=0} = H^{\rm ev}, \tag{6.19b}$$

where, if the solution $U^{\rm osc}$ (occurring only incoming modes and being polarized) writes

$$U^{\rm osc}(z,\theta,\psi_d) = \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}_{\rm in}(\mathbf{n}_0)}}\ \sum_{\lambda\in\mathbb{Z}^*}\sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,E(\mathbf{n}_0,\xi_0),$$

where $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$ are scalar functions, then the resonant part $U^{\rm osc}_{\rm res}$ is given by

$$U^{\rm osc}_{\rm res}(z,\theta,\psi_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}}\ \sum_{\lambda\in\mathbb{Z}^*}\sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,E(\mathbf{n}_0,\xi_0),$$

and the scalar component $S_{\mathbf{n}_0,\xi_0}:\Omega_T\times\mathbb{T}\to\mathbb{C}$ for each non resonant direction $(\mathbf{n}_0,\xi_0)$ in $\big(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0)\big)\setminus\mathcal{F}^{\rm in}_{\rm res}$, is given by

$$S_{\mathbf{n}_0,\xi_0}(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*}\sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\lambda\,\Theta},$$

where the function β_T of x_d, of class $\mathcal{C}^\infty$, equals 1 on $[0,\mathcal{V}^*T]$ and 0 on $[2\mathcal{V}^*T,+\infty)$ (where $\mathcal{V}^*$ has been defined in Lemma 2.14), and where $H^{\rm osc}_{\rm res}$, $h_{\mathbf{n}_0,\xi_0}$ for $(\mathbf{n}_0,\xi_0)$ in $\big(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0)\big)\setminus\mathcal{F}^{\rm in}_{\rm res}$, and $H^{\rm ev}$ are defined from G by the formulas

$$H^{\rm osc}_{\rm res}(z',\theta) := \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}}\ \sum_{\lambda\in\mathbb{Z}^*}\Pi_-^{j(\lambda\mathbf{n}_0,\lambda\xi_0)}(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\lambda\mathbf{n}_0}(z')\,e^{i\lambda\mathbf{n}_0\cdot\theta}, \tag{6.20a}$$

$$h_{\mathbf{n}_0,\xi_0}(z',\Theta) := \sum_{\lambda\in\mathbb{Z}^*}\Big\langle\Pi_-^{j(\lambda\mathbf{n}_0,\lambda\xi_0)}(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\lambda\mathbf{n}_0}(z')\,\Big|\,E(\mathbf{n}_0,\xi_0)\Big\rangle_{\mathbb{C}^N}e^{i\lambda\Theta}, \tag{6.20b}$$

$$H^{\rm ev}(z',\theta) := \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}}\Pi_-^{e}(\mathbf{n}\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\mathbf{n}}(z')\,e^{i\mathbf{n}\cdot\theta}, \tag{6.20c}$$

where, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ in $\mathcal{C}_{\rm in}(\mathbf{n})$, $j(\mathbf{n},\xi)$ is the index such that $\xi=\xi_{j(\mathbf{n},\xi)}(\mathbf{n}\cdot\boldsymbol{\zeta})$. Recall that amplitudes $G_{\mathbf{n}}$ of the function G have been defined by (2.3), and that projectors $\mathbf{E}^{\rm in}_{\rm res}$ and $\widetilde{\mathbf{E}^{i}}^{\rm in}_{\rm res}$ have been introduced in Definition 6.13. Note that in these notations, solution U decomposes as

$$U(z,\theta,\psi_d) = U^{\rm osc}_{\rm res}(z,\theta,\psi_d) + \sum_{\substack{(\mathbf{n}_0,\xi_0)\in\\ (\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0))\setminus\mathcal{F}^{\rm in}_{\rm res}}} S_{\mathbf{n}_0,\xi_0}(z,\mathbf{n}_0\cdot\theta+\xi_0\,\psi_d)\,E(\mathbf{n}_0,\xi_0) + U^{\rm ev}(z,\theta,\psi_d).$$

To prove Theorem 4.1, it is therefore equivalent to prove that there exists solutions $U^{\rm osc}_{\rm res}$, $S_{\mathbf{n}_0,\xi_0}$ and $U^{\rm ev}$ to systems (6.17), (6.18) and (6.19).

In this part dedicated to the proof of Proposition 6.22, we consider a solution U to (5.11) sufficiently regular, and we start by showing that its mean value U^* is zero, by extracting from (5.11) a homogeneous linear hyperbolic system satisfied by it. To show that there is no outgoing mode, the scalar product (3.13) for outgoing modes is used, which is defined for profiles of compact support with respect to x_d. Thus we must prove before that the considered solution U is of compact support with respect to x_d. Then the outgoing modes are isolated in equation (5.11), deducing that they are zero. First the left term of equation (5.11b) is rewritten.

6.3.1. *Rewriting the evolution equation.* According to remark 6.20, since U satisfies the polarization condition (5.11a), and according to Remark 6.20, the profile writes $U = U^{\rm osc} + U^{\rm ev}$, where

$$U^{\rm osc}(z,\theta,\psi_d) = U^*(z) + \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \sum_{\xi\in\mathcal{C}(\mathbf{n})} U^{\rm osc}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\psi_d}, \tag{6.21a}$$

and

$$U^{\rm ev}(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} e^{\psi_d\,\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\rm ev}_{\mathbf{n}}(z,0)\, e^{i\mathbf{n}\cdot\theta}, \tag{6.21b}$$

with $\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\, U^{\rm osc}_{\mathbf{n},\xi} = U^{\rm osc}_{\mathbf{n},\xi}$ for all $\mathbf{n},\xi$. Then $U^{\rm osc}$ is rewritten to take advantage of collinearities, using notations of Part 6.1. Let $\mathbf{n}$ be in $\mathbb{Z}^m\setminus\{0\}$, and $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$, λ in $\mathbb{Z}^*$ such that $\mathbf{n} = \lambda\,\mathbf{n}_0$, and let also ξ be in $\mathcal{C}(\mathbf{n})$. Since the set $\mathcal{C}$ is homogeneous of degree 1, we have $\mathcal{C}(\mathbf{n}) = \lambda\,\mathcal{C}(\mathbf{n}_0)$, so there exists ξ_0 in $\mathcal{C}(\mathbf{n}_0)$ such that $\xi = \lambda\,\xi_0$. By polarization of the profile U, the amplitude $U^{\rm osc}_{\mathbf{n},\xi}$ belongs to the kernel of $L\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)$, which is given according to Definition 6.4 by $\operatorname{Vect} E(\mathbf{n},\xi) = \operatorname{Vect} E(\mathbf{n}_0,\xi_0)$. One may thus write

$$U^{\rm osc}_{\mathbf{n},\xi}(z) = \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, E(\mathbf{n}_0,\xi_0),$$

where $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$ is a scalar function defined on Ω_T. Since the profile $U^{\rm osc}$ is assumed to be real, coefficients $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$ satisfy $\sigma_{-\lambda,\mathbf{n}_0,\xi_0} = \overline{\sigma_{\lambda,\mathbf{n}_0,\xi_0}}$ for all λ, $\mathbf{n}_0$ and ξ_0. In this notation, the profile $U^{\rm osc}$ writes

$$U^{\rm osc}(z,\theta,\psi_d) = U^*(z) + \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, E(\mathbf{n}_0,\xi_0).$$

Note that according to identity (3.17) and since the vectors $E(\mathbf{n}_0,\xi_0)$ are of norm 1, when it is well-defined, the scalar product $\langle U^{\rm osc}\,|\,U^{\rm osc}\rangle_{\rm out}(t)$ is given in these notations by

$$\langle U^{\rm osc}\,|\,U^{\rm osc}\rangle_{\rm out}(t) = (2\pi)^m\, \|U^*\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) + (2\pi)^m \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t).$$

Recall that the scalar product $\langle .\,|\,.\rangle_{\rm out}$ is defined only for profiles with compact support in the normal direction, but we will prove that every solution of (5.11) is indeed compactly supported in the x_d-direction before using this scalar product.

Since the projector $\widetilde{\mathbf{E}^i}$ occurring in equation (5.11b) only acts on oscillating profiles, the oscillating part of the term $\widetilde{L}(0,\partial_z)\,U + \sum_{j=1}^m \widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U$ must be determined. On one hand, the oscillating part of $\widetilde{L}(0,\partial_z)\,U$ is given by

$$\widetilde{L}(0,\partial_z)\,U^{\rm osc} = \widetilde{L}(0,\partial_z)\,U^* + \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{L}(0,\partial_z)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, E(\mathbf{n}_0,\xi_0).$$

These two terms correspond to terms (6.23a) and (6.23b) of equation (6.23) below.

On the other hand, according to Lemma 3.6 concerning the algebra properties of the space of profiles $\mathcal{P}_{s,T}$, the oscillating part of the quadratic term $\sum_{j=1}^m \widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U$ is given by

$$\sum_{j=1}^m \widetilde{L}_1(U^{\rm osc},\zeta_j)\,\partial_{\theta_j}U^{\rm osc} = \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{L}_1(U^*,i\,\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_0,\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}$$
$$+\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathcal{B}_{\mathbb{Z}^m}}\sum_{\substack{\xi_1\in\mathcal{C}(\mathbf{n}_1)\\ \xi_2\in\mathcal{C}(\mathbf{n}_2)}}\sum_{\lambda_1,\lambda_2\in\mathbb{Z}^*}\widetilde{L}_1(E(\mathbf{n}_1,\xi_1),i\,\lambda_2\mathbf{n}_2\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_2,\xi_2)$$
$$\sigma_{\lambda_1,\mathbf{n}_1,\xi_1}\,\sigma_{\lambda_2,\mathbf{n}_2,\xi_2}\,e^{i\,(\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\theta}\,e^{i\,(\lambda_1\xi_1+\lambda_2\xi_2)\,\psi_d}. \tag{6.22}$$

The first term of the right hand side of equation (6.22) corresponds to term (6.23c) of equation (6.23) below. In the second term of the right hand side of equation (6.22), since the projectors $\widetilde{\pi}_\alpha$ appear in the projector $\widetilde{\mathbf{E}^i}$, only the frequencies $\big((\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\boldsymbol{\zeta},\lambda_1\xi_1+\lambda_2\xi_2\big)$ that are characteristic will be preserved.

i) If $\mathbf{n}_1=\mathbf{n}_2$, $\xi_1=\xi_2$ and $\lambda_1=-\lambda_2$, the created frequency is zero, so it is characteristic. This non oscillating term corresponds to term (6.23d) of equation (6.23) below.
ii) If $\mathbf{n}_1=\mathbf{n}_2$, $\xi_1=\xi_2$ and $\lambda_1+\lambda_2\neq 0$, then the nonzero frequency obtained is given by $(\lambda_1+\lambda_2)\,(\mathbf{n}_1\cdot\boldsymbol{\zeta},\xi_1)$ which is characteristic. This is called *self-interaction* of the frequency $(\mathbf{n}_1\cdot\boldsymbol{\zeta},\xi_1)$ with itself, and constitutes term (6.23e) of equation (6.23).
iii) Finally, in the remaining cases, if the nonzero frequency obtained $\lambda_1\,(\mathbf{n}_1\cdot\boldsymbol{\zeta},\xi_1)+\lambda_2\,(\mathbf{n}_2\cdot\boldsymbol{\zeta},\xi_2)$ is characteristic, then it corresponds to a resonance in the sense of Definition 6.8. Namely there exist λ_0 in $\mathbb{Z}^*$, $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$ and ξ_0 in $\mathcal{C}(\mathbf{n}_0)$ such that
$$\lambda_1\,(\mathbf{n}_1\cdot\boldsymbol{\zeta},\xi_1)+\lambda_2\,(\mathbf{n}_2\cdot\boldsymbol{\zeta},\xi_2)=\lambda_0\,(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0),$$
thus there exists ℓ in $\mathbb{Z}^*$ such that $(\lambda_1,\lambda_2,\lambda_0)=\ell\,(\lambda_p,\lambda_q,\lambda_r)$ where the 7-tuple $(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)$ belongs to one of the sets $\mathcal{R}_1(\mathbf{n}_0,\xi_0)$ or $\mathcal{R}_2(\mathbf{n}_0,\xi_0)$. These resonances constitute terms (6.23f) and (6.23g) of equation (6.23).

According to the expression of the projector $\widetilde{\mathbf{E}^i}$ and since $\widetilde{\pi}_0=\mathrm{Id}$, the term $\widetilde{\mathbf{E}^i}\big[\widetilde{L}(0,\partial_z)\,U+\sum_{j=1}^m\widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U\big]$ is thus given by

$$\widetilde{\mathbf{E}^i}\Big[\widetilde{L}(0,\partial_z)\,U+\sum_{j=1}^m\widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U\Big]=\widetilde{L}(0,\partial_z)\,U^* \tag{6.23a}$$
$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\widetilde{L}(0,\partial_z)\,E(\lambda\mathbf{n}_0,\lambda\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}, \tag{6.23b}$$

constituting the transport terms of the mean value and the oscillating part, then the terms of resonances with the mean value as well as the resonances creating a zero frequency

$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\widetilde{L}_1(U^*,i\,\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\lambda\mathbf{n}_0,\lambda\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d} \tag{6.23c}$$
$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{L}_1(E(-\lambda\mathbf{n}_0,-\lambda\xi_0),i\,\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\lambda\mathbf{n}_0,\lambda\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\sigma_{-\lambda,\mathbf{n}_0,\xi_0}, \tag{6.23d}$$

and finally the self-interaction term

$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}}\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\widetilde{L}_1(E(\lambda_1\mathbf{n}_0,\lambda_1\xi_0),i\,\lambda_2\,\mathbf{n}_0\cdot\boldsymbol{\zeta})$$
$$E(\lambda_2\mathbf{n}_0,\lambda_2\xi_0)\,\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}, \tag{6.23e}$$

and the resonances of types 1 and 2 terms

$$+ \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \widetilde{L}_1(E(\ell\lambda_p\mathbf{n}_p,\ell\lambda_p\xi_p), i\ell\lambda_q\mathbf{n}_q\cdot\boldsymbol{\zeta}) \tag{6.23f}$$
$$E(\ell\lambda_q\mathbf{n}_q,\ell\lambda_q\xi_q)\,\sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\, e^{i\,\ell\,\lambda_r\,\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}$$

$$+ \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \widetilde{L}_1(E(\ell\lambda_p\mathbf{n}_p,\ell\lambda_p\xi_p), i\ell\lambda_q\mathbf{n}_q\cdot\boldsymbol{\zeta}) \tag{6.23g}$$
$$E(\ell\lambda_q\mathbf{n}_q,\ell\lambda_q\xi_q)\,\sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\, e^{i\,\ell\,\lambda_r\,\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}.$$

The homogeneity of degree zero of the projectors $\widetilde{\pi}_\alpha$ has been used here. In the following, Definition 6.6 of coefficients Γ will be used to rewrite the different terms of (6.23). In equation (6.23), the vectors $E(\mathbf{n},\xi)$ being homogeneous of degree 0, coefficients $\lambda,\lambda_1,\lambda_2,\lambda_p,\lambda_q$ and ℓ may or may not appear. They are indicated here because they will be useful in a computation below. They may however be removed without any mention being made.

6.3.2. *The mean value is zero.* We prove now that the mean value U^* is zero, by extracting the system verified by it. According to equation (6.23), the mean value of the term $\widetilde{\mathbf{E}^i}\left(\widetilde{L}(0,\partial_z)\,U + \sum_{j=1}^m \widetilde{L}_1(U,\zeta_j)\,\partial_{\theta_j}U\right)$ is a priori given by

$$\widetilde{L}(0,\partial_z)\,U^* + \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{L}_1(E(\mathbf{n}_0,\xi_0), i\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_0,\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\sigma_{-\lambda,\mathbf{n}_0,\xi_0}. \tag{6.24}$$

The change of variables $\lambda=-\lambda$ then shows that term (6.24) is actually zero. Indeed one can compute

$$\begin{aligned}
&\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{L}_1(E(\mathbf{n}_0,\xi_0), i\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_0,\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\sigma_{-\lambda,\mathbf{n}_0,\xi_0}\\
&= \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{L}_1(E(\mathbf{n}_0,\xi_0), -i\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_0,\xi_0)\,\sigma_{-\lambda,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\\
&= -\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{L}_1(E(\mathbf{n}_0,\xi_0), i\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,E(\mathbf{n}_0,\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\sigma_{-\lambda,\mathbf{n}_0,\xi_0} = 0.
\end{aligned}$$

The second term of (6.24) being zero, the non oscillating terms of (6.23) are given by the term $\widetilde{L}(0,\partial_z)\,U^*$ only. Thus, using system (5.11), we see that the mean value U^* satisfies the decoupled system

$$\begin{cases}
\widetilde{L}(0,\partial_z)\,U^* = 0,\\[2ex]
B\,U^*_{|x_d=0} = 0,\\[2ex]
U^*_{|t\leqslant 0} = 0
\end{cases}$$

since G is of zero mean value. The mean value U^* therefore satisfies a boundary value problem verifying the uniform Kreiss-Lopatinskii condition with a strictly hyperbolic operator $\widetilde{L}(0,\partial_z)$. According to [Kre70], the problem is thus well-posed so U^* is zero on Ω_T.

At this point we should note that equation (6.23) can be decoupled between incoming and outgoing modes, thanks to Assumption 7 and the nullity of terms (6.23a), (6.23c) and (6.23d). The difficulty in decoupling the system now relies on decoupling the boundary condition, and we use the fact that there is no outgoing mode to do it. In its turn the nullity of outgoing modes relies on the nullity of the mean value.

6.3.3. *Finite speed propagation.* It can be proved that if U is a smooth enough solution to (5.11), then it is supported in a finite interval in x_d. More precisely the following result is verified.

Lemma 6.23. *Consider $T > 0$ and $s > s_0$, and let U in $\mathcal{P}_{s,T}$ be a solution to system (5.11). Then its oscillating part $U^{\rm osc}$ is zero outside the dihedron $\{(t, y, x_d) \in \Omega_T \,|\, 0 \leqslant x_d \leqslant \mathcal{V}^* t\}$ (see Figure 6).*

The proof of this lemma uses techniques developed below, so it is postponed, in order to focus on the derivation of a priori estimates. We use the fact that U travels at finite speed in the normal direction, according to Lemma 2.14.

According to this result, in system (5.11) and the associated linearized systems, the profile $U^{\rm osc}$ can be replaced by $\beta_T\, U^{\rm osc}$, where β_T is the function of $\mathcal{C}_0^\infty(\mathbb{R}^+_{x_d})$ introduced in Proposition 6.22, equating 1 on $[0, \mathcal{V}^* T]$ and 0 on $[2\mathcal{V}^* T, +\infty)$. In the following, the scalar product (3.13) suited for outgoing profiles can be used, since it is well-defined for profile of compact support with respect to x_d.

6.3.4. *There is no outgoing mode.* The aim is now to determine the equations satisfied by the outgoing modes. According to equation (6.23), since the mean value U^* is zero, the following equality holds

$$\sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}_{\rm out}(\mathbf{n}_0)}} \sum_{\lambda \in \mathbb{Z}^*} \widetilde{X}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, \sigma_{\lambda, \mathbf{n}_0, \xi_0}\, e^{i\,\lambda \mathbf{n}_0 \cdot \theta}\, e^{i\,\lambda \xi_0\, \psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0) \tag{6.25a}$$

$$+ \sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}_{\rm out}(\mathbf{n}_0)}} \sum_{\lambda \in \mathbb{Z}^*} \sum_{\substack{\lambda_1, \lambda_2 \in \mathbb{Z}^* \\ \lambda_1 + \lambda_2 = \lambda}} i\, \sigma_{\lambda_1, \mathbf{n}_0, \xi_0}\, \sigma_{\lambda_2, \mathbf{n}_0, \xi_0}\, \Gamma\big(\lambda_1(\mathbf{n}_0, \xi_0), \lambda_2(\mathbf{n}_0, \xi_0)\big) \tag{6.25b}$$

$$e^{i\,\lambda \mathbf{n}_0 \cdot \theta}\, e^{i\,\lambda \xi_0\, \psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0)$$

$$+ \sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}_{\rm out}(\mathbf{n}_0)}} \sum_{\substack{(\lambda_p, \lambda_q, \lambda_r, \mathbf{n}_p, \mathbf{n}_q, \xi_p, \xi_q) \\ \in \mathcal{R}_1(\mathbf{n}_0, \xi_0) \cup \mathcal{R}_2(\mathbf{n}_0, \xi_0)}} \sum_{\ell \in \mathbb{Z}^*} i\, \sigma_{\ell \lambda_p, \mathbf{n}_p, \xi_p}\, \sigma_{\ell \lambda_q, \mathbf{n}_q, \xi_q}\, \Gamma\big(\ell \lambda_p(\mathbf{n}_p, \xi_p), \ell \lambda_q(\mathbf{n}_q, \xi_q)\big) \tag{6.25c}$$

$$e^{i\,\ell\, \lambda_r \mathbf{n}_0 \cdot \theta}\, e^{i\,\ell\, \lambda_r \xi_0\, \psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0) = 0.$$

The Lax Lemma 2.12 has been used here to rewrite the term (6.23b) as the term (6.25a), using that, by definition, we have $E(\mathbf{n}_0, \xi_0) = \pi_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0)$, and Definition 6.6 of coefficients Γ has also been used to rewrite the terms (6.23e) and (6.23g) as (6.25b) and (6.25c). Note that according to Assumption 7, all modes $U^{\rm osc}_{\mathbf{n}, \xi}$ involved in equation (6.25) are outgoing modes, and also that the equations are now scalar up to a constant vector depending only on the directions.

Equation (6.25) is coupled to the initial condition

$$\big(U^{\rm osc}_{\mathbf{n}, \xi}\big)_{|t \leqslant 0} = 0, \quad \mathbf{n} \in \mathbb{Z}^m \setminus \{0\}\,, \xi \in \mathcal{C}_{\rm out}(\mathbf{n}). \tag{6.26}$$

We thus seek to solve the problem (6.25), (6.26). We will prove a priori estimates for this purpose, using the scalar product (3.13). The decomposition (6.6) of Definition 6.11 will be used, and the set $\mathcal{F}^{\rm out}_{\rm res}$ of outgoing frequencies involved in resonances will be treated separately, which is finite according to Assumption 8.

Non resonant modes. First the modes that are not involved in resonances are investigated, namely we consider $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$ and ξ_0 in $\mathcal{C}_{\rm out}(\mathbf{n}_0)$ such that $(\mathbf{n}_0, \xi_0)$ does not belong to $\mathcal{F}^{\rm out}_{\rm res}$. The sets $\mathcal{R}_1(\mathbf{n}_0, \xi_0)$ and $\mathcal{R}_2(\mathbf{n}_0, \xi_0)$ are therefore empty, so, according to equation (6.25), we obtain

$$\sum_{\lambda \in \mathbb{Z}^*} \widetilde{X}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, \sigma_{\lambda, \mathbf{n}_0, \xi_0}\, e^{i\,\lambda \mathbf{n}_0 \cdot \theta}\, e^{i\,\lambda \xi_0\, \psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0) \tag{6.27a}$$

$$+ \sum_{\lambda \in \mathbb{Z}^*} \sum_{\substack{\lambda_1, \lambda_2 \in \mathbb{Z}^* \\ \lambda_1 + \lambda_2 = \lambda}} i\, \sigma_{\lambda_1, \mathbf{n}_0, \xi_0}\, \sigma_{\lambda_2, \mathbf{n}_0, \xi_0}\, \lambda_2\, \Gamma\big((\mathbf{n}_0, \xi_0), (\mathbf{n}_0, \xi_0)\big) \tag{6.27b}$$

$$e^{i\,\lambda \mathbf{n}_0 \cdot \theta}\, e^{i\,\lambda \xi_0\, \psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0 \cdot \boldsymbol{\zeta}, \xi_0)}\, E(\mathbf{n}_0, \xi_0) = 0.$$

Here we have used identity (6.2) to get the term (6.27b). Note that if, for z in Ω_T and Θ in $\mathbb{T}$, we define

$$S_{\mathbf{n}_0,\xi_0}(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\lambda\Theta},$$

then one can check that the real valued function $S_{\mathbf{n}_0,\xi_0}$ satisfies the following scalar Burgers equation

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)} S_{\mathbf{n}_0,\xi_0} + \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) S_{\mathbf{n}_0,\xi_0}\partial_\Theta S_{\mathbf{n}_0,\xi_0} = 0,$$

that could be solved classically. Indeed, recall that $\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}$, defined in Lemma 2.12, is given by

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)} = \frac{-1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\partial_t + \frac{1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\nabla_\eta \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\cdot\nabla_y + \partial_{x_d}.$$

If, for $k=1,\dots,m$, we denote by η_k the last $d-1$ coordinates of ζ_k, then we have denoted by $\boldsymbol{\eta}$ the m-tuple $\boldsymbol{\eta} := (\eta_1,\dots,\eta_m)$, in a similar way than $\boldsymbol{\zeta}$. In this notation, for each $\mathbf{n}_0$, the frequency $\mathbf{n}_0\cdot\boldsymbol{\eta}$ is given by the $d-1$ last coordinates of $\mathbf{n}_0\cdot\boldsymbol{\zeta}$.

We choose however to explain on this easy example the techniques that shall be applied in the following to equations that go beyond the scope of the mere Burgers equations.

We take the scalar product $\langle . \,|\, . \rangle_{\text{out}}$ of equality (6.27) with the quantity

$$\sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \frac{\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)}{\big|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\big|^2},$$

to obtain

$$\sum_{\lambda\in\mathbb{Z}^*} \Big\langle \widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \sigma_{\lambda,\mathbf{n}_0,\xi_0} \,\Big|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\Big\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) \tag{6.28a}$$

$$+\sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\,\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) = 0. \tag{6.28b}$$

Note that the scalar product is well defined since U^{osc} is of compact support with respect to x_d. An integration by parts shows that the transport term (6.28a) satisfies

$$\begin{aligned} 2\,\mathrm{Re}\ (6.28a) = &\sum_{\lambda\in\mathbb{Z}^*} \frac{-1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\,\partial_t\,\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t)\\ &-\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}(t,0). \end{aligned}$$

We have denoted by $k(\mathbf{n}_0,\xi_0)$ the integer between 1 and N such that if $(\tau,\eta,\xi_0) := (\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)$, then $\tau = \tau_{k(\mathbf{n}_0,\xi_0)}(\eta,\xi_0)$. It leads to the following equality

$$\begin{aligned} 2\,\mathrm{Re}\ (6.28a) = &\frac{-1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\,\frac{d}{dt}\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t)\\ &-\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}(t,0). \end{aligned} \tag{6.29}$$

Then the Burgers term (6.28b) is studied, and more precisely the following sums, that is S given by

$$S := \sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda\,\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t,x_d).$$

and, for $j=1,2$,

$$S_j := \sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_j\,\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t,x_d),$$

the term (6.28b) being given by $\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\,S_2$. First, one can verify that $S = S_1 + S_2$. But, on one hand, we have immediately $S_1 = S_2$. On the other hand, the following equality holds:

$$S = \sum_{\lambda\in\mathbb{Z}^*}\ \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda\,\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\overline{\sigma_{\lambda,\mathbf{n}_0,\xi_0}}\,|\,\overline{\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t,x_d)$$

then, with the consecutive changes of variables $\lambda_2 = \lambda_1+\lambda_2$ and $\lambda_1 = -\lambda_1$,

$$\begin{aligned} S &= \sum_{\lambda\in\mathbb{Z}^*}\ \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_2-\lambda_1=\lambda}} i\,\lambda_2\,\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\overline{\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}}\,|\,\overline{\sigma_{\lambda,\mathbf{n}_0,\xi_0}}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t,x_d)\\ &= \sum_{\lambda\in\mathbb{Z}^*}\ \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_2+\lambda_1=\lambda}} i\,\lambda_2\,\langle \sigma_{-\lambda_1,\mathbf{n}_0,\xi_0}\,\overline{\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}}\,|\,\overline{\sigma_{\lambda,\mathbf{n}_0,\xi_0}}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t,x_d) \end{aligned}$$

finally, since we have $\sigma_{-\lambda_1,\mathbf{n}_0,\xi_0} = \overline{\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}}$ (the profile U^{osc} being real), one gets

$$\begin{aligned} S &= \sum_{\lambda\in\mathbb{Z}^*}\ \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_2+\lambda_1=\lambda}} i\,\lambda_2\,\langle \overline{\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}}\,\overline{\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}}\,|\,\overline{\sigma_{\lambda,\mathbf{n}_0,\xi_0}}\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t,x_d)\\ &= -\overline{S_2}. \end{aligned}$$

It follows from $S = 2S_2$ that $S_2 = 0$, so the term (6.28b) is zero. With equalities (6.28) and (6.29), we thus obtain

$$\frac{d}{dt}\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t) + \partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}\,(t,0) = 0,$$

and therefore, with the initial condition (6.26), for $t\geqslant 0$, we get

$$\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}\,(t) + \partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\int_0^t\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}\,(\rho,0)\,d\rho = 0.$$

Since the quantity $\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)$ is positive (the frequency $(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)$ being outgoing), we deduce that $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$ is zero for all λ in $\mathbb{Z}^*$, and $U^{\mathrm{osc}}_{\lambda\mathbf{n}_0,\lambda\xi_0}$ is therefore zero for $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$, ξ_0 in $\mathcal{C}_{\mathrm{out}}(\mathbf{n}_0)$ such that $\mathcal{R}_2(\mathbf{n}_0,\xi_0)$ and $\mathcal{R}_2(\mathbf{n}_0,\xi_0)$ are empty, and λ in $\mathbb{Z}^*$.

Resonant modes. Outgoing modes involved in resonances are now investigated, namely the couples $(\mathbf{n}_0,\xi_0)$ of the set $\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}$, that are coupled through equation (6.25), because of the resonances, and therefore must be treated all together. From equation (6.25) is deduced the equation for the resonant modes, involving, in addition to a transport and a Burgers terms, a resonant one. There holds

$$\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}}\ \sum_{\lambda\in\mathbb{Z}^*}\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0) \tag{6.30a}$$

$$+\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}}\ \sum_{\lambda\in\mathbb{Z}^*}\ \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0) \tag{6.30b}$$

$$+\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}}\ \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\\ \in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}}\ \sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)\,e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\,e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0) = 0. \tag{6.30c}$$

Identities (6.1) and (6.2) have been used here. Two profiles constructed from the resonant modes are defined here, that will be used in the following

$$U^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}(z,\theta,\psi_d) := \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, E(\mathbf{n}_0,\xi_0),$$

$$\widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}(z,\theta,\psi_d) := \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0).$$

Taking the scalar product $\langle\,.\,|\,.\,\rangle_{\mathrm{out}}$ of equality (6.30) with the profile $\widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}$, one gets

$$\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 \left\langle \widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \sigma_{\lambda,\mathbf{n}_0,\xi_0} \,\middle|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) \tag{6.31a}$$

$$+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 \tag{6.31b}$$

$$\left\langle \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,\middle|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t)$$

$$+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\\ \in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big) \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 \tag{6.31c}$$

$$\left\langle \sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\, \sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q} \,\middle|\, \sigma_{\ell\lambda_r,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) = 0.$$

The first two terms are treated in the same way as for the non resonant modes (see above), so we obtain

$$\begin{aligned} 2\,\mathrm{Re}\,(6.31a) &= -\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \frac{\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2}{\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)} \frac{d}{dt} \sum_{\lambda\in\mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) \\ &\quad - \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}(t,0) \\ &\leqslant -C\,\frac{d}{dt}\left\langle \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}} \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\right\rangle_{\mathrm{out}}(t), \end{aligned} \tag{6.32}$$

with $C>0$, using that $\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}$ is finite and that the group velocity $\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)$ is positive. On an other hand, with the same techniques than for the non resonant modes, one gets (6.31b) $=0$. Finally the resonance term (6.31c) is investigated. Since the sets $\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}$ and $\bigcup_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}}\big(\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)\big)$ are finite, the following bound holds

$$\left|\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)\right| \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 \leqslant C,$$

where the constant $C>0$ is independent of $\mathbf{n}_p,\mathbf{n}_q,\mathbf{n}_r,\xi_p,\xi_q$ and ξ_r. Thus a term of the form $\langle f*g\,|\,g\rangle$ is obtained in (6.31c), which is estimated using Cauchy-Schwarz and Young inequalities, and the injection of $L^2(\mathbb{T}^m)$ into $L^1(\mathbb{T}^m)$, which gives the following estimate on the term (6.31c):

$$|2\,\mathrm{Re}\,(6.31c)| \leqslant C\,\|U^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\|_{\mathcal{E}_{s,T}} \left\langle \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}} \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\right\rangle_{\mathrm{out}}(t). \tag{6.33}$$

It follows from equations (6.30), (6.32) and (6.33) the differential inequality

$$\frac{d}{dt}\left\langle \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}} \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\right\rangle_{\mathrm{out}}(t) \leqslant C\,\|U^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\|_{\mathcal{E}_{s,T}} \left\langle \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}} \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\right\rangle_{\mathrm{out}}(t)$$

The initial conditions (6.26) ensure that $\big(\widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\big)_{|t=0}=0$, so, for $t\geqslant 0$, we have

$$\left\langle \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}} \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathcal{R}^{\mathrm{out}}}\right\rangle_{\mathrm{out}}(t) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(t) = 0.$$

Thus, for all $(\mathbf{n}_0,\xi_0)$ in $\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}$ and λ in $\mathbb{Z}^*$, and for all $t\geqslant 0$, the function $\sigma_{\lambda,\mathbf{n}_0,\xi_0}(t,.)$ is zero, therefore the same holds for the outgoing amplitude $U^{\mathrm{osc}}_{\lambda\mathbf{n}_0,\lambda\xi_0}$, for all $(\mathbf{n}_0,\xi_0)$ in $\mathcal{F}^{\mathrm{out}}_{\mathrm{res}}$ and λ in $\mathbb{Z}^*$.

In conclusion, it has been proven that for every profile U regular enough solution to system (5.11), its mean value U^* as well as each of its outgoing modes $U^{\rm osc}_{\mathbf{n},\xi}$, $\mathbf{n} \in \mathbb{Z}^m \setminus \{0\}$, $\xi \in \mathcal{C}_{\rm out}(\mathbf{n})$, are zero.

6.3.5. *Decoupling the system.* Because of the algebra property of the space of profiles $\mathcal{P}_{s,T}$, and since the projectors $\mathbf{E}$ and $\widetilde{\mathbf{E}^i}$ preserve the decomposition $\mathcal{P}_{s,T} = \mathcal{P}^{\rm osc}_{s,T} \oplus \mathcal{P}^{\rm ev}_{s,T}$, equations (5.11a) and (5.11b) decoupled according to the oscillating and evanescent parts, and the same holds for equation (5.11d). The evanescent part therefore satisfies the equation

$$\mathbf{E}\, U^{\rm ev} = U^{\rm ev}$$

and the oscillating part the equations

$$\mathbf{E}\, U^{\rm osc} = U^{\rm osc}$$

$$\widetilde{\mathbf{E}^i}\Big[\widetilde{L}(0,\partial_z)\, U^{\rm osc} + \sum_{j=1}^m \widetilde{L}_1(U^{\rm osc},\zeta_j)\, \partial_{\theta_j} U^{\rm osc}\Big] = 0.$$

The second equation may be rewritten as, using notations for $U^{\rm osc}$ that have been already introduced,

$$\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}_{\rm in}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0) \tag{6.34a}$$

$$+ \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}_{\rm in}(\mathbf{n}_0)}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\, \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \tag{6.34b}$$

$$e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)$$

$$+ \sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}_{\rm in}(\mathbf{n}_0)}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\\ \in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\ell\, \sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\, \sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\, \Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big) \tag{6.34c}$$

$$e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0) = 0.$$

This equation decouples according to the set $\mathcal{F}^{\rm in}_{\rm res}$ of resonant modes, and each of the non resonant mode. For each mode $(\mathbf{n}_0,\xi_0)$ in $(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0))\setminus\mathcal{F}^{\rm in}_{\rm res}$, which is therefore such that the sets $\mathcal{R}_1(\mathbf{n}_0,\xi_0)$ and $\mathcal{R}_2(\mathbf{n}_0,\xi_0)$ are empty, we define

$$S_{\mathbf{n}_0,\xi_0}(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\lambda\,\Theta}.$$

Then, according to equations (6.34a) and (6.34b), this function satisfies the following scalar Burgers equation

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)} S_{\mathbf{n}_0,\xi_0} + \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\, S_{\mathbf{n}_0,\xi_0}\, \partial_\Theta S_{\mathbf{n}_0,\xi_0} = 0.$$

On an other hand the resonant modes of $\mathcal{F}^{\rm in}_{\rm res}$ satisfy the independent equation

$$\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)$$

$$+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\, \sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)$$

$$e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)$$

$$+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\\ \in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\ell\, \sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\, \sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\, \Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)$$

$$e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0) = 0,$$

that may be rewritten, with already introduced notations (see Definition 6.13), as

$$\widetilde{\mathbf{E}}^{i\,\mathrm{in}}_{\mathrm{res}}\Big[\widetilde{L}(0,\partial_z)\,U^{\mathrm{osc}}_{\mathrm{res}}+\sum_{j=1}^{m}\widetilde{L}_1(U^{\mathrm{osc}}_{\mathrm{res}},\zeta_j)\,\partial_{\theta_j}U^{\mathrm{osc}}_{\mathrm{res}}\Big]=0.$$

Note that by assumption on the set $\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}$, all modes involved in this equation are part of the set $\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}$. Furthermore it is clear that the polarization condition $\mathbf{E}\,U^{\mathrm{osc}}=U^{\mathrm{osc}}$ as well as the initial condition decouple in the same way. Therefore, to conclude the proof of Proposition 6.22, it must be shown that the boundary condition also decouples in this manner.

6.3.6. *Determination of the trace on the boundary.* It is possible to determine the traces on the boundary $(U^{\mathrm{osc}}_{\mathrm{res}})_{|x_d=0,\psi_d=0}$, $(S_{\mathbf{n}_0,\xi_0})_{x_d=0}$ and $U^{\mathrm{ev}}_{|x_d=0,\psi_d=0}$ from the boundary condition (5.11c) using the fact that there are only incoming modes, which will prove the intended decoupling of the system. According to polarization conditions (6.21a) and (6.21b), and since there are only incoming frequencies, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$, boundary condition (5.11c) writes

$$B\Big[\sum_{\xi\in\mathcal{C}_{\mathrm{in}}(\mathbf{n})}\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U^{\mathrm{osc}}_{\mathbf{n},\xi}(z',0)+\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{\mathrm{ev}}_{\mathbf{n}}(z',0,0)\Big]=G_{\mathbf{n}}(z'),\tag{6.35}$$

where the amplitudes $G_{\mathbf{n}}$ have been defined with the formula (2.3). For all ξ in $\mathcal{C}_{\mathrm{in}}(\mathbf{n})$, the term $\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U^{\mathrm{osc}}_{\mathbf{n},\xi}$ belongs to $\ker L\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)$ which is included in $E_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ according to Proposition 2.16, since the frequency $(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)$ is incoming. In the same way, according to the definition of the projector $\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})$, the term $\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{\mathrm{ev}}_{\mathbf{n}}(z,0)$ belongs to the space $E_-(\mathbf{n}\cdot\boldsymbol{\zeta})$. The vector on which acts the matrix B in (6.35) therefore belongs to $E_-(\mathbf{n}\cdot\boldsymbol{\zeta})$, and the matrix B restricted to this subspace is invertible according to the uniform Kreiss-Lopatinskii condition Assumption 3. It then follows by projecting on the spaces $E^{j}_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ and $E^{e}_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ the following boundary conditions

$$\pi_{(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)}\,U^{\mathrm{osc}}_{\mathbf{n},\xi}(z',0)=\Pi^{j(\mathbf{n},\xi)}_-(\mathbf{n}\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\mathbf{n}}(z'),\quad \xi\in\mathcal{C}_{\mathrm{in}}(\mathbf{n}),\tag{6.36a}$$

$$\Pi^{e}_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{\mathrm{ev}}_{\mathbf{n}}(z',0,0)=\Pi^{e}_-(\mathbf{n}\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\mathbf{n}}(z'),\tag{6.36b}$$

where, for ξ in $\mathcal{C}_{\mathrm{in}}(\mathbf{n})$, $j(\mathbf{n},\xi)$ is the index such that $\xi=\xi_{j(\mathbf{n},\xi)}(\mathbf{n}\cdot\boldsymbol{\zeta})$. Therefore, according to (6.36) and the polarization conditions (6.21a) and (6.21b), the profiles $H^{\mathrm{osc}}_{\mathrm{res}}$ and H^{ev} defined by (6.20) are such that $(U^{\mathrm{osc}}_{\mathrm{res}})_{|x_d=0,\psi_d=0}=H^{\mathrm{osc}}_{\mathrm{res}}$ and $U^{\mathrm{ev}}_{|x_d=0,\psi_d=0}=H^{\mathrm{ev}}$. On an other hand, since, according to (6.36a), we have, for all $(\mathbf{n}_0,\xi_0)$ in $(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\mathrm{in}}(\mathbf{n}_0))\setminus\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}$,

$$(\sigma_{\lambda,\mathbf{n}_0,\xi_0})_{|z_d=0}\,E(\mathbf{n}_0,\xi_0)=\Pi^{j(\lambda\mathbf{n}_0,\lambda\xi_0)}_-(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta})}\big)^{-1}G_{\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta}},$$

and since the vectors $E(\mathbf{n}_0,\xi_0)$ are of norm 1, the function $h_{\mathbf{n}_0,\xi_0}$ defined by formula (6.20b) satisfies $(S_{\mathbf{n}_0,\xi_0})_{x_d=0}=h_{\mathbf{n}_0,\xi_0}$. We finally check that the boundary terms $H^{\mathrm{osc}}_{\mathrm{res}}$, $h_{\mathbf{n}_0,\xi_0}$ and H^{ev} are controlled in $H^s(\omega_T)$ by G. On one hand, according to the uniform Kreiss-Lopatinskii condition 3, the inverse matrix $(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})})^{-1}$ is uniformly bounded, see Remark 2.8. On the other hand, according to Proposition 2.21, projectors $\Pi^{j(\mathbf{n},\xi)}_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ and $\Pi^{e}_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ are uniformly bounded with respect to $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$. According to formulas (2.3) and (6.20) and the Parseval's identity, the sought control in $L^2(\omega_T)$ is ensured. The control in $H^s(\omega_T\times\mathbb{T}^m)$ for all $s\geqslant 0$ follows, using that the quantities $\Pi^{j(\mathbf{n},\xi)}_-(\mathbf{n}\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\big)^{-1}$ and $\Pi^{e}_-(\mathbf{n}\cdot\boldsymbol{\zeta})\,\big(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\big)^{-1}$ do not depend on z' in ω_T. Therefore we obtain

$$\|H^{\mathrm{osc}}_{\mathrm{res}}\|^2_{H^s(\omega_T\times\mathbb{T}^m)}+\sum_{\substack{(\mathbf{n}_0,\xi_0)\in\\(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\mathrm{in}}(\mathbf{n}_0))\setminus\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}}}\|h_{\mathbf{n}_0,\xi_0}\|^2_{H^s(\omega_T\times\mathbb{T})}+\|H^{\mathrm{ev}}\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\leqslant C\,\|G\|^2_{H^s(\omega\times\mathbb{T}^m)}\,,\tag{6.37}$$

where the positive constant C does not depend on T or s. This completes the proof of Proposition 6.22.

6.4. **A priori estimate on the linearized system for the oscillating resonant part.** According to Proposition 6.22, the study may be narrowed down to the one of systems (6.17), (6.18) and (6.19). This part deals with the first one, and we will prove a priori estimates on the associated linearized system, which will be used to show the convergence of an iterative scheme. Recall that s_0 is given by $s_0 = h + (d+m)/2$ where h is an integer greater than $(3+a_1)/2$, occurring in estimate (6.3), whith a_1 the real number of Assumption 6.

Proposition 6.24. *Consider $s > s_0$ and let $U^{\rm osc}_{\rm res}$ be in $\mathcal{P}^{\rm osc}_{s,T}$, $V^{\rm osc}_{\rm res}$ in $\mathcal{N}^{\rm osc}_{s,T}$ both involving only incoming resonant modes, and $F^{\rm osc}_{\rm res}$ in $\mathcal{P}^{\rm osc}_{s,T}$, satisfying the system*

$$\mathbf{E}^{\rm in}_{\rm res}\, U^{\rm osc}_{\rm res} = U^{\rm osc}_{\rm res} \tag{6.38a}$$

$$\widetilde{\mathbf{E}^{i}}^{\rm in}_{\rm res}\Big[\widetilde{L}(0,\partial_z)\,\beta_T U^{\rm osc}_{\rm res} + \sum_{j=1}^{m} \widetilde{L}_1(\beta_T V^{\rm osc}_{\rm res},\zeta_j)\,\partial_{\theta_j}\beta_T U^{\rm osc}_{\rm res}\Big] = \widetilde{\mathbf{E}^{i}}\, F^{\rm osc}_{\rm res} \tag{6.38b}$$

$$\big(U^{\rm osc}_{\rm res}\big)_{|x_d=0,\psi_d=0} = H^{\rm osc}_{\rm res} \tag{6.38c}$$

$$\big(U^{\rm osc}_{\rm res}\big)_{|t\leqslant 0} = 0, \tag{6.38d}$$

where $H^{\rm osc}$ is defined by equation (6.20a). Then the profile $U^{\rm osc}_{\rm res}$ satisfies the a priori estimate

$$\|U^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}} \leqslant C_1\, e^{C(V)\,\mathcal{V}^* T}\, \|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)} + \mathcal{V}^* T\, e^{C(V)\mathcal{V}^* T}\, \|F^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}}\,, \tag{6.39}$$

where $C(V) := C_1\big(1 + \|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}}\big)$, with $C_1 > 0$ a positive constant depending only on the operator $L(0,\partial_z)$ and of s. Recall that the real number $\mathcal{V}^*$, which bounds the group velocities $\mathbf{v}_\alpha$, has been defined in Lemma 2.14.

Consider from now on an integer $s > s_0$.

6.4.1. *Rewriting the linearized oscillating system.* In system (6.38) which is the linearization of system (6.17) around $V^{\rm osc}_{\rm res}$ in $\mathcal{P}^{\rm osc}_{s,T}$, a source term $F^{\rm osc}_{\rm res}$ in $\mathcal{P}^{\rm osc}_{s,T}$ has been added, which will be useful to deduce from the L^2 estimate the higher order estimates, as well as in the iterative schemes used to construct solutions to the linearized system (6.38) and to system (6.17). To simplify the equations, the function β_T will be omitted in the following.

The analysis conducted in the previous subsection is now reproduced to rewrite the left term of equality (6.38b). Since the profile $U^{\rm osc}_{\rm res}$ satisfies the polarization condition (6.38a) and involves only incoming modes, according to Remark 6.20, it writes

$$U^{\rm osc}_{\rm res}(z,\theta,\psi_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\lambda\in\mathbb{Z}^*} U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z)\, e^{i\lambda\mathbf{n}_0\cdot\theta}\, e^{i\lambda\xi_0\,\psi_d}. \tag{6.40}$$

with $U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0} = \pi_{(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta},\lambda\xi_0)}\, U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}$ for all $\mathbf{n}_0,\xi_0,\lambda$. In the same way, since $V^{\rm osc}_{\rm res}$ is in $\mathcal{N}^{\rm osc}_{s,T}$ with only incoming resonant modes, we have

$$V^{\rm osc}_{\rm res}(z,\theta,\psi_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\lambda\in\mathbb{Z}^*} V^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z)\, e^{i\lambda\mathbf{n}_0\cdot\theta}\, e^{i\lambda\xi_0\,\psi_d}. \tag{6.41}$$

with $V^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0} = \pi_{(\lambda\mathbf{n}_0\cdot\boldsymbol{\zeta},\lambda\xi_0)}\, V^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}$ for all $\mathbf{n}_0,\xi_0,\lambda$. Once again, for $(\mathbf{n}_0,\xi_0)$ in $\mathcal{F}^{\rm in}_{\rm res}$ and λ in $\mathbb{Z}^*$, since the profiles U and V are polarized, we write

$$U^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z) = \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, E(\mathbf{n}_0,\xi_0),$$

$$V^{\rm osc}_{\lambda\mathbf{n}_0,\lambda\xi_0}(z) = \omega_{\lambda,\mathbf{n}_0,\xi_0}(z)\, E(\mathbf{n}_0,\xi_0).$$

Note that according to identity (3.15), the scalar product $\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d)$ is given in this notation by

$$\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d) = (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\lambda\in\mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\omega_T)}(x_d).$$

Since the projector $\widetilde{\mathbf{E}}^{i\,\mathrm{in}}_{\mathrm{res}}$ is applied to the source term $F^{\mathrm{osc}}_{\mathrm{res}}$, one can assume without loss of generality that the latter writes

$$F^{\mathrm{osc}}_{\mathrm{res}}(z,\theta,\psi_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} F_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\mathbf{n}\cdot\theta}\, e^{i\xi\,\psi_d}.$$

We then denote, for $(\mathbf{n}_0,\xi_0)$ in $\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}$ and λ in $\mathbb{Z}^*$, by $f_{\lambda,\mathbf{n}_0,\xi_0}$ the scalar function of Ω_T such that

$$\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, F_{\lambda,\mathbf{n}_0,\xi_0} = f_{\lambda,\mathbf{n}_0,\xi_0}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)$$

so that $\widetilde{\mathbf{E}}^{i\,\mathrm{in}}_{\mathrm{res}}\, F^{\mathrm{osc}}_{\mathrm{res}}$ writes

$$\widetilde{\mathbf{E}}^{i\,\mathrm{in}}_{\mathrm{res}}\, F^{\mathrm{osc}}_{\mathrm{res}}(z,\theta,\psi_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} f_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\mathbf{n}\cdot\theta}\, e^{i\xi\,\psi_d} \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0).$$

According to estimate (6.5) of Assumption 8, there exists a positive constant C such that for all $(\mathbf{n}_0,\xi_0)$ in $\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}$ and all λ in $\mathbb{Z}^*$, we have

$$\|f_{\lambda,\mathbf{n}_0,\xi_0}\|_{L^2(\Omega_T)} \leqslant C\, \|F_{\lambda,\mathbf{n}_0,\xi_0}\|_{L^2(\Omega_T)}\,. \tag{6.42}$$

In this notation, the resonant incoming modes satisfy the following coupled equation, connecting the source term $\widetilde{\mathbf{E}}^{i\,\mathrm{in}}_{\mathrm{res}}\, F^{\mathrm{osc}}_{\mathrm{res}}$

$$\sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} f_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0) \tag{6.43a}$$

with the sum of a transport term, corresponding to $\widetilde{L}(0,\partial_z)\, U^{\mathrm{osc}}_{\mathrm{res}}$,

$$= \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\, e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0), \tag{6.43b}$$

a self-interaction term,

$$\begin{aligned} &+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\omega_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \\ &\qquad e^{i\,\lambda\mathbf{n}_0\cdot\theta}\, e^{i\,\lambda\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0), \end{aligned} \tag{6.43c}$$

and resonance terms of type 1,

$$\begin{aligned} &+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\omega_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big) \\ &\qquad e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0), \end{aligned} \tag{6.43d}$$

and of type 2,

$$\begin{aligned} &+ \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\omega_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big) \\ &\qquad e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\, e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\, \widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0). \end{aligned} \tag{6.43e}$$

Note that in terms (6.43c), (6.43d) and (6.43e), factors $\lambda_2\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)$ and $\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)$ imply that something like a derivative with respect to θ is applied to U. To obtain estimates without loss of derivatives, one therefore needs the θ derivative to apply to the coefficient $V^{\mathrm{osc}}_{\mathrm{res}}$, which is the whole point of the following paragraph. Terms (6.43c) and (6.43e) are treated in the same way as the corresponding terms for the outgoing modes, whereas Assumption 8 is used to treat term (6.43d) of resonances of type 1.

6.4.2. *L^2 estimate.* This subsection is devoted to the proof of the following lemma.

Lemma 6.25. *Consider $s > s_0$ and $T > 0$, and let $U^{\rm osc}_{\rm res}$ be in $\mathcal{P}^{\rm osc}_{1,T}$, $F^{\rm osc}_{\rm res}$ in $\mathcal{P}^{\rm osc}_{0,T}$ and $V^{\rm osc}_{\rm res}$ in $\mathcal{N}^{\rm osc}_{s,T}$, only involving resonant incoming modes, satisfying system (6.38). Then the following estimate holds for $x_d \geqslant 0$,*

$$\frac{d}{dx_d}\,\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d) \leqslant C\,\langle F^{\rm osc}_{\rm res}\,|\,F^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d) + C\left(1+\|V^{\rm osc}_{\rm res}\|_{\mathcal{E}_{s,T}}\right)\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d). \tag{6.44}$$

Proof. Consider the modified profile $\widetilde{U}^{\rm osc}_{\rm res}$ given by

$$\widetilde{U}^{\rm osc}_{\rm res}(z,\theta,\psi_d) := \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}}\ \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\frac{\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)}{|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)|^2}.$$

Note that despite the factor $|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)|^{-1}$ that could present a problem, the profile $\widetilde{U}^{\rm osc}_{\rm res}$ is well defined, since the set $\mathcal{F}^{\rm in}_{\rm res}$ satisfies the property (6.5) ensuring that these factors are uniformly lower bounded. They have been introduced to balance the factors $|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)|$ which will occur in the estimate. These factors may not be uniformly bounded with respect to $(\mathbf{n}_0,\xi_0)$ varying in the (potentially infinite) set of directions $(\mathbf{n}_0,\xi_0)$ of $\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0)$ such that $\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)$ is empty, justifying the choice to treat them separately below.

Taking the double of the real part of the scalar product (3.12) of equality (6.43) with the profile $\widetilde{U}^{\rm osc}_{\rm res}$, one gets an equality, with on one side the term

$$2\,{\rm Re}\left\langle \widetilde{\mathbf{E}^i}^{\rm in}_{\rm res}\,F^{\rm osc}_{\rm res}\,\middle|\,\widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d),$$

which is estimated in the following way:

$$\begin{aligned}\left|2\,{\rm Re}\left\langle \widetilde{\mathbf{E}^i}^{\rm in}_{\rm res}\,F^{\rm osc}_{\rm res}\,\middle|\,\widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d)\right| &= \left|2\,{\rm Re}\left\langle \widehat{F}^{\rm osc}_{\rm res}\,\middle|\,\widehat{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d)\right| \\ &\leqslant C\left\langle \widehat{F}^{\rm osc}_{\rm res}\,\middle|\,\widehat{F}^{\rm osc}_{\rm res}\right\rangle^{1/2}_{\rm in}(x_d)\left\langle \widehat{U}^{\rm osc}_{\rm res}\,\middle|\,\widehat{U}^{\rm osc}_{\rm res}\right\rangle^{1/2}_{\rm in}(x_d) \\ &\leqslant C\,\langle F^{\rm osc}_{\rm res}\,|\,F^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d) + C\,\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d),\end{aligned} \tag{6.45}$$

where it has been denoted

$$\widehat{F}^{\rm osc}_{(\mathbf{n}_0,\xi_0)}(z,\theta,\psi_d) := \sum_{\lambda\in\mathbb{Z}^*} f_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\frac{\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)}{\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|},$$

and

$$\widehat{U}^{\rm osc}_{(\mathbf{n}_0,\xi_0)}(z,\theta,\psi_d) := \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\frac{\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)}{\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|},$$

so that the profile $\widehat{U}^{\rm osc}_{\rm res}$ is such that its scalar product with itself equals the one of $U^{\rm osc}_{(\mathbf{n}_0,\xi_0)}$ with itself, and according to estimate (6.42), the scalar product of $\widehat{F}^{\rm osc}_{\rm res}$ with itself is bounded, up to a positive multiplicative constant, by the one of $F^{\rm osc}_{\rm res}$ with itself. Since the lower bound (6.5) is in general not verified by the non resonant modes, the analogue of estimate (6.45) seems false, explaining why these modes cannot be treated in the same way as the resonant modes in this subsection, which leads us to go back to scalar equations for the first ones.

Now the right hand side terms of the equality obtained by taking the double of the real part of the scalar product of equality (6.43) with $\widetilde{U}^{\rm osc}_{\rm res}$ are investigated. The analysis of the terms corresponding to terms (6.43b), (6.43c) and (6.43e) is analogous to the one made for the outgoing modes.

Concerning the transport term (6.43b), identity (3.15) and an integration by parts lead to

$$2\,\mathrm{Re}\left\langle (6.43b) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d) = \frac{d}{dx_d} \langle U^{\mathrm{osc}}_{\mathrm{res}} \,|\, U^{\mathrm{osc}}_{\mathrm{res}} \rangle_{\mathrm{in}} (x_d) + (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \frac{-1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1})}(T). \tag{6.46}$$

Note that since all modes are incoming here, the quantity $-\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)$ is positive for all $(\mathbf{n}_0,\xi_0)$, which will allow us to omit the second term on the right of the equality in the estimates below.

For the self-interaction term (6.43c) one can compute,

$$\left\langle (6.43c) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d)$$

$$= (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\lambda_2\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)} (x_d) \tag{6.47a}$$

$$= (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\lambda\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)} (x_d) \tag{6.47b}$$

$$- (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\lambda_1\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)} (x_d). \tag{6.47c}$$

But with already detailed computations, one gets (6.47b) $= -\overline{(6.47a)}$, so

$$2\,\mathrm{Re}\left\langle (6.43c) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d) = - (2\pi)^m \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\lambda_1\, \Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big) \langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)} (x_d).$$

Note that this term differs from (6.47a) because of the coefficient λ_1 instead of λ_2, which makes the derivatives with respect to θ apply on the coefficient $V^{\mathrm{osc}}_{\mathrm{res}}$ instead of on the unknown $U^{\mathrm{osc}}_{\mathrm{res}}$. Upper bound (6.8) therefore leads to

$$\left|2\,\mathrm{Re}\left\langle (6.43c) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d)\right| \leqslant C \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} |\lambda_1\mathbf{n}_0| \left|\langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\, \sigma_{\lambda_2,\mathbf{n}_0,\xi_0} \,|\, \sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)} (x_d)\right|.$$

The term on the right of the equality is of the form $\langle fg \,|\, g\rangle$, so we get

$$\left|2\,\mathrm{Re}\left\langle (6.43c) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d)\right| \leqslant C\, \|V^{\mathrm{osc}}_{\mathrm{res}}\|_{\mathcal{E}_{s,T}} \langle U^{\mathrm{osc}}_{\mathrm{res}} \,|\, U^{\mathrm{osc}}_{\mathrm{res}} \rangle_{\mathrm{in}} (x_d). \tag{6.48}$$

For term (6.43e) of type 2 resonances, we write

$$\left\langle (6.43e) \,\middle|\, \widetilde{U}^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d) = \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\mathrm{in}}_{\mathrm{res}}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_2(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\ell\, \Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big) \left\langle \omega_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\, \sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q} \,\middle|\, \sigma_{\ell\lambda_r,\mathbf{n}_r,\xi_r} \right\rangle_{L^2(\omega_T)} (x_d).$$

Then the following upper bound is derived, for all $(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\in\mathcal{R}_2(\mathbf{n}_0,\xi_0)$ with $\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}$, $\xi_0\in\mathcal{C}_{\mathrm{in}}(\mathbf{n}_0)$ (which constitutes a finite set, see Assumption 8),

$$\left|\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)\right| \leqslant C|\lambda_p|\,|\mathbf{n}_p,\xi_p|\,,$$

where the constant $C > 0$ is independent of $\mathbf{n}_p, \mathbf{n}_q, \mathbf{n}_0, \xi_p, \xi_q$ and ξ_0. Once again, with this bound, the derivative with respect to θ no longer apply on $U^{\rm osc}_{\rm res}$ but only on $V^{\rm osc}_{\rm res}$. The following estimate is thus deduced:

$$\left|2\,\mathrm{Re}\left\langle (6.43e) \,\middle|\, \widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d)\right| \leqslant C\,\|V^{\rm osc}_{\rm res}\|_{\mathcal{E}_{s,T}}\,\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d). \tag{6.49}$$

Finally term (6.43d) of type 1 resonances is investigated, which is treated following [Rau12, Chapter 11]. Once again the aim is to have a derivative applying totally on $V^{\rm osc}_{\rm res}$. First the set on which the sum (6.43d) is taken is parameterized in a different way. The set $\mathcal{R}_1$ of type 1 incoming resonant 6-tuples is defined as

$$\mathcal{R}_1 := \left\{ \begin{array}{c} (\ell\,\lambda_p\,\mathbf{n}_p, \ell\,\lambda_p\,\xi_p, \ell\,\lambda_q\,\mathbf{n}_q, \\ \ell\,\lambda_q\,\xi_q, -\ell\,\lambda_r\,\mathbf{n}_0, -\ell\,\lambda_r\,\xi_0) \end{array} \middle| \begin{array}{c} \ell\in\mathbb{Z}^*,\ \mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m},\ \xi_0\in\mathcal{C}_{\rm in}(\mathbf{n}_0), \\ (\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0) \end{array} \right\}.$$

Note that if $(\mathbf{n}_p, \xi_p, \mathbf{n}_q, \xi_q, \mathbf{n}_r, \xi_r)$ is in $\mathcal{R}_1$, then $\mathbf{n}_p + \mathbf{n}_q + \mathbf{n}_r = 0$, and $\xi_p + \xi_q + \xi_r = 0$. We also see that, according to remark 6.9, a 6-tuple $(\mathbf{n}_p, \xi_p, \mathbf{n}_q, \xi_q, \mathbf{n}_r, \xi_r)$ is in $\mathcal{R}_1$ if and only if the symmetrical 6-tuple $(\mathbf{n}_p, \xi_p, \mathbf{n}_r, \xi_r, \mathbf{n}_q, \xi_q)$ is in $\mathcal{R}_1$. According to identity (6.1), we have

$$\begin{aligned}
(6.43d) &= \sum_{(\mathbf{n}_0,\xi_0)\in\mathcal{F}^{\rm in}_{\rm res}} \sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0)}} \sum_{\ell\in\mathbb{Z}^*} i\,\omega_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\Gamma\big((\ell\lambda_p\mathbf{n}_p,\ell\lambda_p\xi_p),(\ell\lambda_q\mathbf{n}_q,\ell\lambda_q\xi_q)\big) \\
&\qquad e^{i\,\ell\lambda_r\mathbf{n}_0\cdot\theta}\,e^{i\,\ell\lambda_r\xi_0\,\psi_d}\,\widetilde{\pi}_{(\ell\lambda_r\mathbf{n}_0\cdot\boldsymbol{\zeta},\ell\lambda_r\xi_0)}\,E(\ell\lambda_r\mathbf{n}_0,\ell\lambda_r\xi_0) \\
&= \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} i\,\omega_{\mathbf{n}_p,\xi_p}\,\sigma_{\mathbf{n}_q,\xi_q}\,\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)e^{-i\,\mathbf{n}_r\cdot\theta}\,e^{-i\xi_r\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_r\cdot\boldsymbol{\zeta},\xi_r)}\,E(\mathbf{n}_r,\xi_r).
\end{aligned}$$

If $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$ and ξ in $\mathcal{C}(\mathbf{n})$ write as $(\mathbf{n},\xi) = \lambda\,(\mathbf{n}_0,\xi_0)$ with $\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}$, $\xi\in\mathcal{C}(\mathbf{n}_0)$ and $\lambda\in\mathbb{Z}^*$, we have denoted

$$\sigma_{\mathbf{n},\xi} := \sigma_{\lambda,\mathbf{n}_0,\xi_0}, \quad \text{and} \quad \omega_{\mathbf{n},\xi} := \omega_{\lambda,\mathbf{n}_0,\xi_0}.$$

Therefore we have

$$\begin{aligned}
\left\langle (6.43d) \,\middle|\, \widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d) &= \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} i\,\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)\,\big\langle \omega_{\mathbf{n}_p,\xi_p}\,\sigma_{\mathbf{n}_q,\xi_q}\,\big|\,\sigma_{-\mathbf{n}_r,-\xi_r}\big\rangle_{L^2(\omega_T)}(x_d) \\
&= \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} i\,\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big)\overline{\big\langle \omega_{-\mathbf{n}_p,-\xi_p}\,\sigma_{-\mathbf{n}_r,-\xi_r}\,\big|\,\sigma_{\mathbf{n}_q,\xi_q}\big\rangle_{L^2(\omega_T)}}(x_d) \\
&= \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} -i\,\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_r,\xi_r)\big)\overline{\big\langle \omega_{\mathbf{n}_p,\xi_p}\,\sigma_{\mathbf{n}_q,\xi_q}\,\big|\,\sigma_{-\mathbf{n}_r,-\xi_r}\big\rangle_{L^2(\omega_T)}}(x_d).
\end{aligned}$$

We have used here the fact that $\overline{\omega_{\mathbf{n}_p,\xi_p}} = \omega_{-\mathbf{n}_p,-\xi_p}$, the profile $V^{\rm osc}$ being real, a change of variables $(\mathbf{n}_p,\mathbf{n}_r,\mathbf{n}_q,\xi_p,\xi_r,\xi_q) = -(\mathbf{n}_p,\mathbf{n}_q,\mathbf{n}_r,\xi_p,\xi_q,\xi_r)$, the fact that $-\mathcal{R}_1 = \mathcal{R}_1$ and the identity (6.1). Thus we obtain

$$\begin{aligned}
&2\,\mathrm{Re}\left\langle (6.43d) \,\middle|\, \widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d) \\
&\quad= \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} i\Big\{\Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_q,\xi_q)\big) + \Gamma\big((\mathbf{n}_p,\xi_p),(\mathbf{n}_r,\xi_r)\big)\Big\}\,\big\langle \omega_{\mathbf{n}_p,\xi_p}\,\sigma_{\mathbf{n}_q,\xi_q}\,\big|\,\sigma_{-\mathbf{n}_r,-\xi_r}\big\rangle_{L^2(\omega_T)}(x_d).
\end{aligned}$$

Using the uniform estimate (6.4) given by Assumption 8, one obtains

$$\begin{aligned}
\left|2\,\mathrm{Re}\left\langle (6.43d) \,\middle|\, \widetilde{U}^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d)\right| &\leqslant C \sum_{\substack{(\mathbf{n}_p,\xi_p,\mathbf{n}_q,\xi_q,\\ \mathbf{n}_r,\xi_r)\in\mathcal{R}_1}} |(\mathbf{n}_p,\xi_p)|\,\left|\big\langle \omega_{\mathbf{n}_p,\xi_p}\,\sigma_{\mathbf{n}_q,\xi_q}\,\big|\,\sigma_{-\mathbf{n}_r,-\xi_r}\big\rangle_{L^2(\omega_T)}(x_d)\right| \\
&\leqslant C\,\left\|V^{\rm osc}_{\mathcal{R}^{\rm in}}\right\|_{\mathcal{E}_{s,T}}\,\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}(x_d).
\end{aligned} \tag{6.50}$$

Only the profiles $V^{\rm osc}_{\rm res}$ and $U^{\rm osc}_{\rm res}$ appear in the estimate since only frequencies of $\mathcal{F}^{\rm in}_{\rm res}$ occur in $\mathcal{R}_1$.

Equations (6.43) and (6.46) and estimates (6.45), (6.48), (6.49) and (6.50) finally lead to

$$\frac{d}{dx_d}\,\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d) \leqslant C\,\langle F^{\rm osc}_{\rm res}\,|\,F^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d) + C\left(1+\|V^{\rm osc}_{\rm res}\|_{\mathcal{E}_{s,T}}\right)\langle U^{\rm osc}_{\rm res}\,|\,U^{\rm osc}_{\rm res}\rangle_{\rm in}\,(x_d), \tag{6.51}$$

which is the expected differential inequality. □

6.4.3. *Proof of Lemma 6.23.* All requisite techniques to show Lemma 6.23 have now been developed, so the proof is given here. It follows [BGS07, Section 1.3.1]. Recall that at this stage, we have considered a solution U to (5.11) regular enough, and we have shown that its mean value is zero.

Proof (Lemma 6.23). It has been shown that if U is a solution to (5.11), then, with already introduced notations, we have

$$\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0) \tag{6.52a}$$

$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0) \tag{6.52b}$$

$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}}\sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)\,e^{i\,\ell\,\lambda_r\mathbf{n}_0\cdot\theta}\,e^{i\,\ell\,\lambda_r\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)=0. \tag{6.52c}$$

Let us point out that despite formula (6.52) looks like formula (6.34), the former only uses the fact that the mean value U^* is zero, and involves both incoming and outgoing modes.

For $0\leqslant t_0\leqslant T$ and $x_d^0\geqslant 0$, consider the domain $\mathcal{K}(t_0,x_d^0)$, bounded with respect to x_d, given by

$$\mathcal{K}(t_0,x_d^0) := \left\{(t,y,x_d)\in\Omega_T\,\middle|\,\mathcal{V}^*t\leqslant x_d\leqslant x_d^0+\mathcal{V}^*(t_0-t),\ 0\leqslant t\leqslant t_0\right\},$$

see Figure 6. Let us prove that $U^{\rm osc}$ is zero on the upper boundary of this domain, namely for $t=t_0$ and $\mathcal{V}^*t_0\leqslant x_d\leqslant x_d^0$, which suffices to prove that $U^{\rm osc}$ is zero outside $\{0\leqslant x_d\leqslant\mathcal{V}^*t\}$ for all t in $[0,T]$. Take the scalar product (3.14) of expression (6.52) with the modified profile

$$-\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\,\lambda\xi_0\,\psi_d}\,\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0),$$

to obtain, according to (3.16),

(6.53a)
$$\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\left\langle X_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\middle|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathcal{K}(t_0,x_d^0))}\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|^2$$

(6.53b)
$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\,\lambda_2\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\left\langle\sigma_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,\middle|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathcal{K}(t_0,x_d^0))}$$
$$\big(-\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|^2$$

(6.53c)
$$+\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\substack{(\lambda_p,\lambda_q,\lambda_r,\mathbf{n}_p,\mathbf{n}_q,\\ \xi_p,\xi_q)\in\mathcal{R}_1(\mathbf{n}_0,\xi_0)\cup\mathcal{R}_2(\mathbf{n}_0,\xi_0)}}\sum_{\ell\in\mathbb{Z}^*} i\,\ell\,\Gamma\big(\lambda_p(\mathbf{n}_p,\xi_p),\lambda_q(\mathbf{n}_q,\xi_q)\big)\big(-\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)$$
$$\left\langle\sigma_{\ell\lambda_p,\mathbf{n}_p,\xi_p}\,\sigma_{\ell\lambda_q,\mathbf{n}_q,\xi_q}\,\middle|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathcal{K}(t_0,x_d^0))}\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|^2=0.$$

Term (6.53a) is obtained by noting that, with the notations of Definition 2.10 and Lemma 2.12, we have $\big(-\partial_\xi\tau_k(\eta,\xi)\big)\,\widetilde{X}_\alpha=X_\alpha$. First term (6.53a) is investigated. According to Green's formula, for all $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$, ξ_0 in $\mathcal{C}(\mathbf{n}_0)$ and λ in $\mathbb{Z}^*$, we obtain

$$2\,\mathrm{Re}\left\langle X_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\middle|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathcal{K}(t_0,x_d^0))}=2\,\mathrm{Re}\int_{\partial\mathcal{K}(t_0,x_d^0)}\big(n_t+\vec{n}_x\cdot\mathbf{v}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\big)|\sigma_{\lambda,\mathbf{n}_0,\xi_0}|^2\,dS,$$

where the notation $\mathbf{v}_\alpha$ has been introduced in Definition 2.10, $\vec{n}:=(n_t,\vec{n}_x)$ is the outward normal vector associated with $\partial\mathcal{K}(t_0,x_d^0)$, and dS is the surface measure. The vector $\vec{n}$ is given (see Figure 6), for the upper boundary by $\vec{n}=(1,0,\dots,0)$, for the lower boundary by $\vec{n}=(-1,0,\dots,0)$, for the left boundary by $\vec{n}=(\mathcal{V}^*,0,\dots,0,-1)/\sqrt{1+(\mathcal{V}^*)^2}$ and for the right boundary by $\vec{n}=(\mathcal{V}^*,0,\dots,0,1)/\sqrt{1+(\mathcal{V}^*)^2}$. Thus we get

$$\begin{aligned}
&2\,\mathrm{Re}\left\langle X_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,\middle|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\right\rangle_{L^2(\mathcal{K}(t_0,x_d^0))}\\
&=2\,\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times[\mathcal{V}^*t_0,x_d^0])}(t_0)-2\,\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,x_d^0])}(0)\\
&\quad+\frac{2}{\sqrt{1+(\mathcal{V}^*)^2}}\int_{\{(t,y,\mathcal{V}^*t),0\leqslant t\leqslant t_0\}}\big(\mathcal{V}^*-\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)|\sigma_{\lambda,\mathbf{n}_0,\xi_0}|^2\,dS\\
&\quad+\frac{2}{\sqrt{1+(\mathcal{V}^*)^2}}\int_{\{(t,y,x_d^0+\mathcal{V}^*(t_0-t)),0\leqslant t\leqslant t_0\}}\big(\mathcal{V}^*+\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)|\sigma_{\lambda,\mathbf{n}_0,\xi_0}|^2\,dS.
\end{aligned}$$

Then note that on one hand we have $\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}_+)}(0)=0$ according to the initial condition (5.11d), and on the other hand, according to Lemma 2.14, the quantities $\big(\mathcal{V}^*-\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)$ and $\big(\mathcal{V}^*+\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)$ are non-negative. Therefore,

$$2\,\mathrm{Re}\,(6.53a)\geqslant 2\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times[\mathcal{V}^*t_0,x_d^0])}(t_0)\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|^2. \tag{6.54}$$

As for them, terms (6.53b) and (6.53c) are treated the same way as before. For the self-interaction term (6.53b), it is proved in the same manner than term (6.28b) that it satisfies

$$2\,\mathrm{Re}\,(6.53b)=0.$$

For the resonance term (6.53c), the same techniques as for terms (6.49) and (6.50) are used. According to Lemma 2.14, the group velocities $\big(-\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\big)$ can be uniformly bounded to obtain

$$|2\,\mathrm{Re}\,(6.53c)|\leqslant C\,\|U^{\mathrm{osc}}\|_{\mathcal{E}_{s,T}}\sum_{\substack{\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^m}\\ \xi_0\in\mathcal{C}(\mathbf{n}_0)}}\sum_{\lambda\in\mathbb{Z}^*}\|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathcal{K}(t_0,x_d^0))}\left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)\right|^2. \tag{6.55}$$

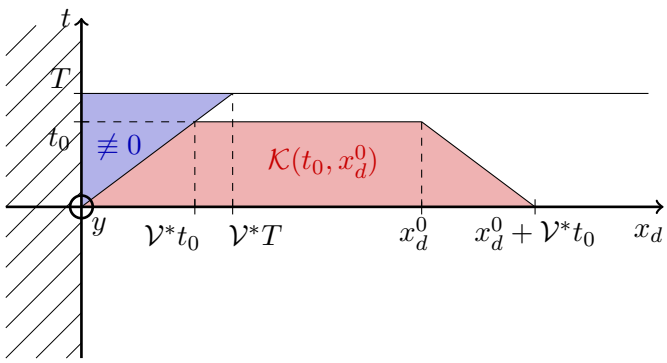

FIGURE 6. Propagation zone.

Noting that

$$\sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda \in \mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathcal{K}(t_0,x_d^0))} \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2$$
$$= \int_0^{t_0} \sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda \in \mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times[\mathcal{V}^* t, x_d^0 + \mathcal{V}^*(t_0 - t)])}(t) \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 dt,$$

using equations (6.53), (6.54) and (6.55) and according to the Grönwall's inequality, it follows

$$\sum_{\substack{\mathbf{n}_0 \in \mathcal{B}_{\mathbb{Z}^m} \\ \xi_0 \in \mathcal{C}(\mathbf{n}_0)}} \sum_{\lambda \in \mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times[\mathcal{V}^* t_0, x_d^0])}(t_0) \left|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\, E(\mathbf{n}_0,\xi_0)\right|^2 = 0.$$

Therefore, for all $\mathbf{n}_0$ in $\mathcal{B}_{\mathbb{Z}^m}$, ξ_0 in $\mathcal{C}(\mathbf{n}_0)$ and λ in $\mathbb{Z}^*$, the function $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$ is zero on $\{t = t_0, \mathcal{V}^* t_0 \leqslant x_d \leqslant x_d^0\}$, so the profile U^{osc} is also zero in this set, concluding the proof of the Lemma. □

6.4.4. *Estimating the derivatives.* Returning to the proof of the a priori estimate, Proposition 6.24 is proved here using estimate (6.44) of Lemma 6.25. Consider a multi-index α of $\mathbb{N}^{d+m}$ such that $|\alpha| \leqslant s$. Since the operator $\partial^\alpha_{z',\theta}$ commutes with the projectors $\mathbf{E}^{\mathrm{in}}_{\mathrm{res}}$ and $\widetilde{\mathbf{E}^i}^{\mathrm{in}}_{\mathrm{res}}$, the profile $\partial^\alpha_{z',\theta} U^{\mathrm{osc}}$ satisfies a system of the form (6.38), with $\partial^\alpha_{z',\theta} G$ as boundary term, and $\widetilde{\mathbf{E}^i}^{\mathrm{in}}_{\mathrm{res}}\big[\partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}} + F_\alpha\big]$ as source term, where F_α is the following commutator

$$F_\alpha := \Big[\sum_{j=1}^m \widetilde{L}_1(V^{\mathrm{osc}}_{\mathrm{res}}, \zeta_j)\, \partial_{\theta_j}, \partial^\alpha_{z',\theta}\Big] U^{\mathrm{osc}}_{\mathrm{res}}.$$

Thus, according to estimate (6.44) and applying the triangle inequality, we get

$$\frac{d}{dx_d} \left\langle \partial^\alpha_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \,\middle|\, \partial^\alpha_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}}(x_d) \leqslant C \left\langle \partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}} \,\middle|\, \partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}}(x_d) + C \left\langle F_\alpha \,\middle|\, F_\alpha \right\rangle_{\mathrm{in}}(x_d)$$
$$+ C\left(1 + \|V^{\mathrm{osc}}_{\mathrm{res}}\|_{\mathcal{E}_{s,T}}\right) \left\langle \partial^\alpha_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \,\middle|\, \partial^\alpha_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}}(x_d). \tag{6.56}$$

Note that according to Lemma 6.21 we have

$$\left\langle F_\alpha \,\middle|\, F_\alpha \right\rangle_{\mathrm{in}}(x_d) \leqslant \|F_\alpha\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d}, L^2(\omega_T \times \mathbb{T}^m))}(x_d),$$

and in the same way

$$\left\langle \partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}} \,\middle|\, \partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}}(x_d) \leqslant \left\|\partial^\alpha_{z',\theta} F^{\mathrm{osc}}_{\mathrm{res}}\right\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d}, L^2(\omega_T \times \mathbb{T}^m))}(x_d).$$

On an other hand, according to the algebra property of $H^s(\omega_T \times \mathbb{T}^m)$ (since $s > (d+m)/2+1$) and the commutator estimate [BGS07, Proposition C.13], we obtain

$$\begin{aligned}\|F_\alpha\|^2_{\mathcal{C}_b(\mathbb{R}_+,L^2(\omega_T\times\mathbb{T}^m))}(x_d) &\leqslant C\,\|V^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}_+,H^s(\omega_T\times\mathbb{T}^m))}(x_d)\,\|U^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}_+,H^s(\omega_T\times\mathbb{T}^m))}(x_d)\\ &\leqslant C\,\|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}}\,\|U^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}_+,H^s(\omega_T\times\mathbb{T}^m))}(x_d).\end{aligned}$$

Finally, by definition of the $H^s(\omega_T \times \mathbb{T}^m)$ norm and according to Lemma 6.21, we get

$$\begin{aligned}\|U^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}_+,H^s(\omega_T\times\mathbb{T}^m))}(x_d) &= \sum_{|\alpha|\leqslant s}\left\|\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\|^2_{\mathcal{C}_b(\mathbb{R}_+,L^2(\omega_T\times\mathbb{T}^m))}(x_d)\\ &\leqslant C\sum_{|\alpha|\leqslant s}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d).\end{aligned}\tag{6.57}$$

Therefore, by summing equations (6.56) for $|\alpha| \leqslant s$, one gets

$$\begin{aligned}\frac{d}{dx_d}\sum_{|\alpha|\leqslant s}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d) &\leqslant C\,\|F^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d},H^s(\omega_T\times\mathbb{T}^m))}(x_d)\\ &\quad + C\left(1+\|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}}\right)\sum_{|\alpha|\leqslant s}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d).\end{aligned}\tag{6.58}$$

Thus, according to Grönwall's inequality,

$$\begin{aligned}\sum_{|\alpha|\leqslant s}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(x_d) &\leqslant e^{C(V)\,x_d}\sum_{|\alpha|\leqslant s}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(0)\\ &\quad + \int_0^{x_d} e^{C(V)(x_d-x'_d)}\,\|F^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d},H^s(\omega_T\times\mathbb{T}^m))}(x'_d)\,dx'_d,\end{aligned}\tag{6.59}$$

where $C(V) = C(1+\|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}})$. The trace on the boundary $U^{\rm osc}_{\rm res}$ is therefore given by $\left(U^{\rm osc}_{\rm res}\right)_{|x_d=0,\psi_d=0} = H^{\rm osc}_{\rm res}$ where $H^{\rm osc}_{\rm res}$ is determined by equation (6.20a). Remark 2.8, Proposition 2.21 and Lemma 6.21 ensure that, for $|\alpha| \leqslant s$,

$$\begin{aligned}\left\langle\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\,\middle|\,\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}\right\rangle_{\rm in}(0) &\leqslant \left\|\partial^\alpha_{z',\theta}U^{\rm osc}_{\rm res}(0)\right\|^2_{L^2(\omega_T\times\mathbb{T}^m)} = \left\|\partial^\alpha_{z',\theta}H^{\rm osc}_{\rm res}\right\|^2_{L^2(\omega_T\times\mathbb{T}^m)}\\ &\leqslant C\left\|\partial^\alpha_{z',\theta}G\right\|^2_{L^2(\omega_T\times\mathbb{T}^m)}.\end{aligned}\tag{6.60}$$

It thus follows, with (6.57), (6.59) and (6.60),

$$\begin{aligned}\|U^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d},H^s(\omega_T\times\mathbb{T}^m))}(x_d) &\leqslant Ce^{C(V)\,x_d}\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\\ &\quad + \int_0^{x_d} e^{C(V)(x_d-x'_d)}\,\|F^{\rm osc}_{\rm res}\|^2_{\mathcal{C}_b(\mathbb{R}^+_{\psi_d},H^s(\omega_T\times\mathbb{T}^m))}(x'_d)\,dx'_d.\end{aligned}\tag{6.61}$$

Because of the function β_T in equation (6.38b), it is possible to bound x_d by $2\mathcal{V}^*T$ then to pass to the upper bound with respect to x_d in estimate (6.61) to obtain the required estimate (6.39), concluding the proof of Proposition 6.24.

6.5. **A priori estimate for the linearized Burgers equations.** We prove now a priori estimates for the linearized Burgers equations (6.18), corresponding to the non-resonant incoming modes. These a priori estimates will be used to prove the existence of solution to these Burgers equations. However the estimates will have to be summed with respect to $(\mathbf{n}_0, \xi_0)$, so we wish for constants independent of $(\mathbf{n}_0, \xi_0)$. This part is devoted to the proof of the following result.

Proposition 6.26. *Consider* $(\mathbf{n}_0,\xi_0)\in(\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0))\setminus\mathcal{F}^{\rm in}_{\rm res}$, $s>s_0$ *and let* $S_{\mathbf{n}_0,\xi_0}$, $W_{\mathbf{n}_0,\xi_0}$ *and* $F_{\mathbf{n}_0,\xi_0}$ *be in* $\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))$ *of zero mean, satisfying the scalar boundary value problem*

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}S_{\mathbf{n}_0,\xi_0}+\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)W_{\mathbf{n}_0,\xi_0}\partial_\Theta S_{\mathbf{n}_0,\xi_0}=F_{\mathbf{n}_0,\xi_0} \tag{6.62a}$$

$$\big(S_{\mathbf{n}_0,\xi_0}\big)_{|x_d=0}=h_{\mathbf{n}_0,\xi_0} \tag{6.62b}$$

$$\big(S_{\mathbf{n}_0,\xi_0}\big)_{|t\leqslant 0}=0, \tag{6.62c}$$

where $h_{\mathbf{n}_0,\xi_0}$ *is defined by equation* (6.20b). *Then the function* $S_{\mathbf{n}_0,\xi_0}$ *satisfies the a priori estimate*

$$\begin{aligned}\|S_{\mathbf{n}_0,\xi_0}\|^2_{\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))}&\\ \leqslant C_1\,e^{C(W)\,\mathcal{V}^*T}\,\|h_{\mathbf{n}_0,\xi_0}\|^2_{H^s(\omega_T\times\mathbb{T}^m)}&+\mathcal{V}^*T\,e^{C(W)\mathcal{V}^*T}\,\|F_{\mathbf{n}_0,\xi_0}\|^2_{\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))}\,,\end{aligned} \tag{6.63}$$

where $C(W):=C_1\big(1+\|W_{\mathbf{n}_0,\xi_0}\|^2_{\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\Omega_T\times\mathbb{T}))}\big)$, *with* $C_1>0$ *a constant depending only on the operator* $L(0,\partial_z)$ *and of* s, *but not on* $\mathbf{n}_0,\xi_0$. *Recall that the real number* $\mathcal{V}^*$, *bounding the group velocities* $\mathbf{v}_\alpha$, *has been defined in Lemma 2.14.*

First the $L^2(\omega_T\times\mathbb{T})$ estimate, analogous to estimate (6.44) for resonant modes, is investigated, and equation (6.62a) is rewritten. Write $S_{\mathbf{n}_0,\xi_0}$, $W_{\mathbf{n}_0,\xi_0}$ and $F_{\mathbf{n}_0,\xi_0}$ in $\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))$ as

$$S_{\mathbf{n}_0,\xi_0}(z,\Theta)=\sum_{\lambda\in\mathbb{Z}^*}\sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\lambda\Theta},\quad W_{\mathbf{n}_0,\xi_0}(z,\Theta)=\sum_{\lambda\in\mathbb{Z}^*}\omega_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\lambda\Theta},$$
$$F_{\mathbf{n}_0,\xi_0}(z,\Theta)=\sum_{\lambda\in\mathbb{Z}^*}f_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\lambda\Theta},$$

with $\sigma_{\lambda,\mathbf{n}_0,\xi_0}$, $\omega_{\lambda,\mathbf{n}_0,\xi_0}$ and $f_{\lambda,\mathbf{n}_0,\xi_0}$ scalar functions on Ω_T. Then equation (6.62a) writes

$$\sum_{\lambda\in\mathbb{Z}^*}f_{\lambda,\mathbf{n}_0,\xi_0}(z)\,e^{i\,\lambda\mathbf{n}_0\cdot\theta}\,e^{i\lambda\Theta}=\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\sum_{\lambda\in\mathbb{Z}^*}\sigma_{\lambda,\mathbf{n}_0,\xi_0}\,e^{i\lambda\Theta}, \tag{6.64a}$$

$$+\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}}i\,\lambda_2\,\omega_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,e^{i\lambda\Theta}. \tag{6.64b}$$

Recall that the vector field $\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}$, defined in Lemma 2.12, is given by

$$\widetilde{X}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}=\frac{-1}{\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\partial_t+\frac{1}{\partial_\xi\tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\nabla_\eta\tau_{k(\eta_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)\cdot\nabla_y+\partial_{x_d}.$$

By taking the double of the real part of the $L^2(\omega_T\times\mathbb{T})$ scalar product of equality (6.64) with the function $S_{\mathbf{n}_0,\xi_0}$, one obtains an equality, with on one side of it the term

$$2\,\mathrm{Re}\,\langle F_{\mathbf{n}_0,\xi_0}\,|\,S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}\,(x_d),$$

which is estimated in a similar manner than for the resonant incoming frequencies:

$$\Big|2\,\mathrm{Re}\,\langle F_{\mathbf{n}_0,\xi_0}\,|\,S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}\,(x_d)\Big|\leqslant C\,\|F_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}\,(x_d)+C\,\|S_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}\,(x_d). \tag{6.65}$$

One may note here the interest of having reduced the equations to scalar Burgers equations for the non-resonant modes, since the coefficients $|\widetilde{\pi}_{(\mathbf{n}_0\cdot\boldsymbol{\zeta},\xi_0)}\,E(\mathbf{n}_0,\xi_0)|^{-1}$ no longer appear, these ones being not uniformly bounded for the non resonant modes $(\mathbf{n}_0,\xi_0)$.

We now focus on the other side of the equality obtained by taking the double of the real part of the scalar product of equation (6.64) with the profile $S_{\mathbf{n}_0,\xi_0}$. The analysis of terms (6.64a) and (6.64b) is analogous to the one made for the outgoing non-resonant modes.

Concerning the transport term (6.64a), according to identity (3.15) and using an integration by parts, we obtain

$$2\,\mathrm{Re}\,\langle (6.64a) \,|\, S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}(x_d) = \frac{d}{dx_d}\,\|S_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}(x_d) - \frac{1}{\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)}\,\|S_{\mathbf{n}_0,\xi_0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{T})}(T). \tag{6.66}$$

Note that since $(\mathbf{n}_0,\xi_0)$ is an incoming mode, the quantity $-\partial_\xi \tau_{k(\mathbf{n}_0,\xi_0)}(\mathbf{n}_0\cdot\boldsymbol{\eta},\xi_0)$ is positive, allowing to omit the second term on the right hand side of the equality in the estimates.

For the self-interaction term (6.64b), with computations analogous to the ones used for the incoming resonant modes, we obtain

$$2\,\mathrm{Re}\,\langle (6.64b) \,|\, S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}(x_d) = - (2\pi)^m \sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} i\lambda_1\,\Gamma\big((\mathbf{n}_0,\xi_0),(\mathbf{n}_0,\xi_0)\big)\,\langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)}(x_d).$$

Therefore, using the upper bound (6.3), we get

$$\Big|2\,\mathrm{Re}\,\langle (6.64b) \,|\, S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}(x_d)\Big| \leqslant C \sum_{\lambda\in\mathbb{Z}^*}\sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}^*\\ \lambda_1+\lambda_2=\lambda}} |\lambda_1\mathbf{n}_0|^h \,\Big|\langle \omega_{\lambda_1,\mathbf{n}_0,\xi_0}\,\sigma_{\lambda_2,\mathbf{n}_0,\xi_0}\,|\,\sigma_{\lambda,\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T)}(x_d)\Big|.$$

Here the order of regularity must be taken down to h since we wish for an upper bound independent of $\mathbf{n}_0$, in the purpose of summing the inequality with respect to $\mathbf{n}_0$. An upper bound of the form $C(\mathbf{n}_0)|\lambda_1|$ instead of $C|\lambda_1\mathbf{n}_0|^h$ could be obtained, but where the constant $C(\mathbf{n}_0)$ depends on $\mathbf{n}_0$, and may be arbitrarily large since we consider modes $\mathbf{n}_0$ close to the glancing set. The right hand side of the equality being of the form $\langle f\,|\,g\rangle$, the following inequality holds

$$\Big|2\,\mathrm{Re}\,\langle (6.64b) \,|\, S_{\mathbf{n}_0,\xi_0}\rangle_{L^2(\omega_T\times\mathbb{T})}(x_d)\Big| \leqslant C\,\|W_{\mathbf{n}_0,\xi_0}\|_{\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))}\,\|S_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}(x_d), \tag{6.67}$$

using Sobolev inequality, s being such that $s>h+(d+m)/2$. Using equations (6.64) and (6.66) and estimates (6.65) and (6.67), it finally follows the differential inequality

$$\frac{d}{dx_d}\,\|S_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}(x_d) \leqslant C\,\|F_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}(x_d) + C\Big(1+\|W_{\mathbf{n}_0,\xi_0}\|_{\mathcal{C}(\mathbb{R}^+_{x_d},H^s(\omega_T\times\mathbb{T}))}\Big)\,\|S_{\mathbf{n}_0,\xi_0}\|_{L^2(\omega_T\times\mathbb{T})}(x_d). \tag{6.68}$$

To obtain the required $H^s(\omega_T\times\mathbb{T})$ estimate, we use commutators estimates analogous to the one for resonant incoming modes, which we do not detail here. Finally we obtain the sought estimate (6.63).

The a priori estimates (6.44) and (6.63) (for $s=0$) as well as the equivalence property of Proposition 6.22 ensure the uniqueness of the solution to (5.11).

6.6. **Construction of a solution.**

6.6.1. *Construction of an oscillating solution to the linearized system for the resonant incoming modes.* Thanks to the a priori estimate (6.39) of Proposition 6.24 on the linearized system (6.38), a solution to this system can be constructed, proving the following result.

Proposition 6.27. *Consider $s>s_0$ and $T>0$, and let $V^{\mathrm{osc}}_{\mathrm{res}}$ be a profile of $\mathcal{N}^{\mathrm{osc}}_{s,T}$ involving only resonant incoming modes, $F^{\mathrm{osc}}_{\mathrm{res}}$ be in $\mathcal{P}^{\mathrm{osc}}_{s,T}$ and G be in $H^s(\omega_T\times\mathbb{T}^m)$. Then there exists a solution $U^{\mathrm{osc}}_{\mathrm{res}}$ in $\mathcal{P}^{\mathrm{osc}}_{s,T}$ to system (6.38), involving only resonant incoming modes, that moreover satisfies the following estimate*

$$\|U^{\mathrm{osc}}_{\mathrm{res}}\|^2_{\mathcal{E}_{s,T}} \leqslant C_1\,e^{C(V)\,\mathcal{V}^*T}\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)} + \mathcal{V}^*T\,e^{C(V)\mathcal{V}^*T}\,\|F^{\mathrm{osc}}_{\mathrm{res}}\|^2_{\mathcal{E}_{s,T}},$$

where $C(V) := C_1(1 + \|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}})$, *with* C_1 *a positive constant depending only on* $L(0,\partial_z)$, *on the boundary frequencies* $\zeta_1,\dots,\zeta_m$, *and on* s.

The proof of such a result using an a priori estimate of the form (6.39) is detailed in [JMR95, Theorem 6.3.3]. Its main ideas are recalled here.

The uniqueness of the solution follows directly from the a priori estimate (6.39). Concerning the existence, a finite difference scheme is used. Since the operators ∂_{θ_j} for $j = 1,\dots,m$, are skew-symmetric, skew-symmetric finite difference operators must be considered. Denoting $e_1,\dots,e_m$ the canonical basis of $\mathbb{R}^m$, we define, for every function U of $\Omega_T \times \mathbb{T}^m \times \mathbb{R}_+$,

$$\delta^h_j U(z,\theta,\psi_d) := \big(U(z,\theta + he_j) - U(z,\theta - he_j)\big)/2h,$$

for $j = 1,\dots,m$ and $h > 0$. The proof then consists in showing that there exists, for $h > 0$, a unique solution $U^{\rm osc}_h$ to the regularized system

$$\mathbf{E}^{\rm in}_{\rm res}\, U^{\rm osc}_h = U^{\rm osc}_h \tag{6.69a}$$

$$\widetilde{\mathbf{E}^{i}}^{\rm in}_{\rm res}\Big[\widetilde{L}(0,\partial_z)\, U^{\rm osc}_h + \sum_{j=1}^m \widetilde{L}_1(V^{\rm osc}_{\rm res},\zeta_j)\,\delta^h_j U^{\rm osc}_h\Big] = \widetilde{\mathbf{E}^{i}}^{\rm in}_{\rm res}\, F^{\rm osc}_{\rm res} \tag{6.69b}$$

$$\big(U^{\rm osc}_h\big)_{|x_d=0,\psi_d=0} = H^{\rm osc}_{\rm res} \tag{6.69c}$$

$$\big(U^{\rm osc}_h\big)_{|t\leqslant 0} = 0, \tag{6.69d}$$

and that this solution satisfies the estimate uniform with respect to $h > 0$,

$$\|U^{\rm osc}_h\|^2_{\mathcal{E}_{s,T}} \leqslant C_1\, e^{C(V)\,\mathcal{V}^*T}\, \|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)} + \mathcal{V}^*T\, e^{C(V)\mathcal{V}^*T}\, \|F^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}}\,,$$

where $C(V) := C_1(1 + \|V^{\rm osc}_{\rm res}\|^2_{\mathcal{E}_{s,T}})$, with $C_1 > 0$ a constant depending only on the operator $L(0,\partial_z)$, on the boundary frequencies $\zeta_1,\dots,\zeta_m$, and on s. This uniform estimate allows to extract a sequence $(U^{\rm osc}_{h_n})_n$ weakly converging towards $U^{\rm osc}_{\rm res}$ in $\mathcal{E}_{s,T}$. Passing to the limit in system (6.69) leads to the result of Proposition 6.27.

6.6.2. *Construction of an oscillating solution to systems* (6.17) *and* (6.18). This part is devoted to the following result, constituting a part of the result of Theorem 4.1.

Proposition 6.28. *Consider* $s > s_0$, *and* G *in* $H^\infty(\mathbb{R}^d \times \mathbb{T}^m)$, *zero for negative times* t. *There exists a time* $T > 0$, *depending only on the operator* $L(0,\partial_z)$, *on the boundary frequencies* $\zeta_1,\dots,\zeta_m$, *on the* $H^s(\mathbb{R}^d \times \mathbb{T}^m)$ *norm of* G *and on* s, *such that system* (6.17) *and, for every* $(\mathbf{n}_0,\xi_0)$ *in* $(\mathcal{B}_{\mathbb{Z}^m} \times \mathcal{C}_{\rm in}(\mathbf{n}_0)) \setminus \mathcal{F}^{\rm in}_{\rm res}$, *system* (6.18), *admit solutions* $U^{\rm osc}_{\rm res}$ *and* $S_{\mathbf{n}_0,\xi_0}$ *in* $\mathcal{P}^{\rm osc}_{s,T}$ *and* $\mathcal{C}(\mathbb{R}_+, H^s(\omega_T \times \mathbb{T}))$, *where the functions* $S_{\mathbf{n}_0,\xi_0}$ *are of zero mean. Furthermore, if we denote, for* $(\mathbf{n}_0,\xi_0)$ *in* $(\mathcal{B}_{\mathbb{Z}^m} \times \mathcal{C}_{\rm in}(\mathbf{n}_0)) \setminus \mathcal{F}^{\rm in}_{\rm res}$,

$$S_{\mathbf{n}_0,\xi_0}(z,\Theta) =: \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\lambda\,\Theta},$$

then the profile $U^{\rm osc}$ *defined, for* (z,θ,ψ_d) *in* $\Omega_T \times \mathbb{T}^m \times \mathbb{R}_+$, *by*

$$U^{\rm osc}(z,\theta,\psi_d) := U^{\rm osc}_{\rm res}(z,\theta,\psi_d) + \sum_{\substack{(\mathbf{n}_0,\xi_0)\in\\ (\mathcal{B}_{\mathbb{Z}^m}\times\mathcal{C}_{\rm in}(\mathbf{n}_0))\setminus\mathcal{F}^{\rm in}_{\rm res}}} \sum_{\lambda\in\mathbb{Z}^*} \sigma_{\lambda,\mathbf{n}_0,\xi_0}(z)\, e^{i\lambda\mathbf{n}_0\cdot\theta}\, e^{i\lambda\xi_0\psi_d}\, E(\mathbf{n}_0,\xi_0), \tag{6.70}$$

belongs to the space $\mathcal{P}^{\rm osc}_{s,T}$.

It is classical to deduce from an existence result of a solution to a linearized system with an estimate of the form (6.39), the existence of a solution to the original system. The main ideas of the method described in [BGS07, Théorème 10.1] are recalled here.

First system (6.17) is investigated, and the following iterative scheme is considered:

$$\mathbf{E}^{\mathrm{in}}_{\mathrm{res}}\, U^{\mathrm{osc}}_{\nu+1} = U^{\mathrm{osc}}_{\nu+1} \tag{6.71a}$$

$$\widetilde{\mathbf{E}^{i}}^{\mathrm{in}}_{\mathrm{res}}\Big[\widetilde{L}(0,\partial_z)\,\beta_T U^{\mathrm{osc}}_{\nu+1} + \sum_{j=1}^{m} \widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu},\zeta_j)\,\partial_{\theta_j}\beta_T U^{\mathrm{osc}}_{\nu+1}\Big] = 0 \tag{6.71b}$$

$$\big(U^{\mathrm{osc}}_{\nu+1}\big)_{|x_d=0,\psi_d=0} = H^{\mathrm{osc}}_{\mathrm{res}} \tag{6.71c}$$

$$\big(U^{\mathrm{osc}}_{\nu+1}\big)_{|t\leqslant 0} = 0, \tag{6.71d}$$

initialized with $U^{\mathrm{osc}}_0(.,x_d,.,\psi_d) := H^{\mathrm{osc}}_{\mathrm{res}}$, for all x_d,ψ_d in $\mathbb{R}_+$. Proposition 6.27 ensures that the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is well defined in $\mathcal{P}^{\mathrm{osc}}_{s,T}$. Then the proof consists in showing that the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is bounded in high norm, and contracting in low norm, in order to deduce its weak convergence in the Banach space $\mathcal{P}^{\mathrm{osc}}_{s,T}$.

Bound in high norm. According to estimate (6.39), we have, for $\nu \geqslant 0$,

$$\big\|U^{\mathrm{osc}}_{\nu+1}\big\|^2_{\mathcal{E}_{s,T}} \leqslant C_1 e^{C(U^{\mathrm{osc}}_{\nu})\,\mathcal{V}^*T}\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\,, \tag{6.72}$$

where $C(U^{\mathrm{osc}}_{\nu}) = C_1\big(1+\|U^{\mathrm{osc}}_{\nu}\|^2_{\mathcal{E}_{s,T}}\big)$. If the time $T>0$ is chosen sufficiently small so that

$$\exp\Big[C_1\big(1+2\,C_1\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\big)\,\mathcal{V}^*T\Big] \leqslant 2,$$

then an induction argument shows that $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is bounded in $\mathcal{E}_{s,T}$ by $\sqrt{2\,C_1}\,\|G\|_{H^s(\omega_T\times\mathbb{T}^m)}$. Indeed, the initial step is obvious, up to assuming $C_1 > 1/2$. On an other hand, assuming $\|U^{\mathrm{osc}}_{\nu}\|_{\mathcal{E}_{s,T}} \leqslant \sqrt{2\,C_1}\,\|G\|_{H^s(\omega_T\times\mathbb{T}^m)}$ for some $\nu\geqslant 0$, according to (6.72) and the assumption on T, we obtain

$$\begin{aligned}
\big\|U^{\mathrm{osc}}_{\nu+1}\big\|^2_{\mathcal{E}_{s,T}} &\leqslant C_1 \exp\Big[C_1\big(1+\|U^{\mathrm{osc}}_{\nu}\|^2_{\mathcal{E}_{s,T}}\big)\,\mathcal{V}^*T\Big]\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\\
&\leqslant C_1 \exp\Big[C_1\big(1+2\,C_1\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\big)\,\mathcal{V}^*T\Big]\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\\
&\leqslant 2\,C_1\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}\,,
\end{aligned}$$

which is the expected estimate.

Contraction in low norm. Denote, for $\nu\geqslant 1$, $W^{\mathrm{osc}}_{\nu} := U^{\mathrm{osc}}_{\nu} - U^{\mathrm{osc}}_{\nu-1}$, that satisfies the system

$$\mathbf{E}^{\mathrm{in}}_{\mathrm{res}}\, W^{\mathrm{osc}}_{\nu+1} = W^{\mathrm{osc}}_{\nu+1} \tag{6.73a}$$

$$\widetilde{\mathbf{E}^{i}}^{\mathrm{in}}_{\mathrm{res}}\Big[\widetilde{L}(0,\partial_z)\,\beta_T W^{\mathrm{osc}}_{\nu+1} + \sum_{j=1}^{m} \widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu},\zeta_j)\,\partial_{\theta_j}\beta_T W^{\mathrm{osc}}_{\nu+1}\Big] = \widetilde{\mathbf{E}^{i}}^{\mathrm{in}}_{\mathrm{res}}\, F_{\nu+1} \tag{6.73b}$$

$$\big(W^{\mathrm{osc}}_{\nu+1}\big)_{|x_d=0,\psi_d=0} = 0 \tag{6.73c}$$

$$\big(W^{\mathrm{osc}}_{\nu+1}\big)_{|t\leqslant 0} = 0, \tag{6.73d}$$

where $F_{\nu+1}$ is given by

$$F_{\nu+1} := \sum_{j=1}^{m}\big(\widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu-1},\zeta_j) - \widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu},\zeta_j)\big)\partial_{\theta_j}\beta_T U^{\mathrm{osc}}_{\nu}.$$

According to estimate (6.39) applied to system (6.73) for $s=0$, the following inequality holds

$$\big\|W^{\mathrm{osc}}_{\nu+1}\big\|^2_{\mathcal{E}_{0,T}} \leqslant \mathcal{V}^*T\, e^{C(U^{\mathrm{osc}}_{\nu})\mathcal{V}^*T}\,\|F_{\nu+1}\|^2_{\mathcal{E}_{0,T}}\,.$$

First note that by assumption on T, and since the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is bounded in $\mathcal{E}_{s,T}$ by $\sqrt{2}\,C_1\,\|G\|_{H^s(\omega_T\times\mathbb{T}^m)}$, we have, for $\nu\geqslant 0$,

$$e^{C(U^{\mathrm{osc}}_{\nu})\mathcal{V}^*T} \leqslant 2. \tag{6.74}$$

Now the $\mathcal{E}_{0,T}$ norm of $F_{\nu+1}$ is estimated. Thanks to the product estimate and the choice of the index s, we have

$$\|F_{\nu+1}\|^2_{\mathcal{E}_{0,T}} \leqslant C \sum_{j=1}^{m} \left\| \left(\widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu-1}, \zeta_j) - \widetilde{L}_1(\beta_T U^{\mathrm{osc}}_{\nu}, \zeta_j) \right) \right\|^2_{\mathcal{E}_{0,T}} \|U^{\mathrm{osc}}_{\nu}\|^2_{\mathcal{E}_{s,T}} .$$

According to the mean value inequality, and since the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is bounded in $\mathcal{E}_{s,T}$, one then obtains

$$\|F_{\nu+1}\|^2_{\mathcal{E}_{0,T}} \leqslant C \, \|W_{\nu}\|^2_{\mathcal{E}_{0,T}} \, \|G\|^2_{H^s(\omega_T \times \mathbb{T}^m)} . \tag{6.75}$$

Therefore, according to estimates (6.74) and (6.75), we get

$$\left\|W^{\mathrm{osc}}_{\nu+1}\right\|^2_{\mathcal{E}_{0,T}} \leqslant C\mathcal{V}^* T \, \|G\|^2_{H^s(\omega_T \times \mathbb{T}^m)} \, \|W^{\mathrm{osc}}_{\nu}\|^2_{\mathcal{E}_{0,T}} .$$

For $T > 0$ small enough, the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is therefore convergent in $\mathcal{E}_{0,T}$.

Thus the sequence $(U^{\mathrm{osc}}_{\nu})_{\nu}$ is a Cauchy sequence in the Banach space $\mathcal{P}^{\mathrm{osc}}_{0,T}$, and therefore converges to a function $U^{\mathrm{osc}}_{\mathrm{res}}$ of $\mathcal{P}^{\mathrm{osc}}_{0,T}$. It is possible to show, with arguments that will not be recalled here, that $U^{\mathrm{osc}}_{\mathrm{res}}$ is actually in $\mathcal{P}^{\mathrm{osc}}_{s,T}$ and satisfies system (6.17), see [BGS07, Theorem 10.1] for similar results.

The proof of the existence of a solution to (6.18) is identical, and is not detailed here. It relies on a result of existence of a solution to the linearized system (6.62), analogous to Proposition 6.27, that has not been spelled out. One may however note that the existence time T is indeed independent of $(\mathbf{n}_0, \xi_0)$, since the constants in estimate (6.63) are independent of $(\mathbf{n}_0, \xi_0)$, and since according to estimate (6.37), each boundary term $h_{\mathbf{n}_0,\xi_0}$ is controlled in $H^s(\omega_T \times \mathbb{T})$ by $C \, \|G\|_{H^s(\omega_T \times \mathbb{T}^m)}$, uniformly with respect to $(\mathbf{n}_0, \xi_0)$.

Finally, it is shown that the profile U^{osc} defined by (6.70) actually belongs to $\mathcal{P}^{\mathrm{osc}}_{s,T}$. Indeed, according to Lemma 6.21, we have

$$\begin{aligned}
\|U^{\mathrm{osc}}\|^2_{\mathcal{E}_{s,T}} &\leqslant C \sup_{x_d>0} \sum_{|\alpha| \leqslant s} \left\langle \partial^{\alpha}_{z',\theta} U^{\mathrm{osc}} \,\middle|\, \partial^{\alpha}_{z',\theta} U^{\mathrm{osc}} \right\rangle_{\mathrm{in}} (x_d) \\
&\leqslant C \sup_{x_d>0} \sum_{|\alpha| \leqslant s} \left\langle \partial^{\alpha}_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \,\middle|\, \partial^{\alpha}_{z',\theta} U^{\mathrm{osc}}_{\mathrm{res}} \right\rangle_{\mathrm{in}} (x_d) \\
&\quad + C \sup_{x_d>0} \sum_{\substack{(\mathbf{n}_0,\xi_0) \in \\ (\mathcal{B}_{\mathbb{Z}^m} \times \mathcal{C}_{\mathrm{in}}(\mathbf{n}_0)) \setminus \mathcal{F}^{\mathrm{in}}_{\mathrm{res}}}} \sum_{\lambda \in \mathbb{Z}^*} \|\sigma_{\lambda,\mathbf{n}_0,\xi_0}\|^2_{H^s(\omega_T)} (x_d) \\
&\leqslant C \, \|U^{\mathrm{osc}}_{\mathrm{res}}\|^2_{\mathcal{E}_{s,T}} + C \sum_{\substack{(\mathbf{n}_0,\xi_0) \in \\ (\mathcal{B}_{\mathbb{Z}^m} \times \mathcal{C}_{\mathrm{in}}(\mathbf{n}_0)) \setminus \mathcal{F}^{\mathrm{in}}_{\mathrm{res}}}} \|S_{\mathbf{n}_0,\xi_0}\|^2_{\mathcal{C}(\mathbb{R}_+, H^s(\omega_T \times \mathbb{T}))} ,
\end{aligned}$$

so that, using the a priori estimates (6.39) and (6.63) as well as the boundary term estimates (6.37), one gets

$$\|U^{\mathrm{osc}}\|^2_{\mathcal{E}_{s,T}} \leqslant C \, \|G\|^2_{H^s(\omega_T \times \mathbb{T}^m)} + C \sum_{\substack{(\mathbf{n}_0,\xi_0) \in \\ (\mathcal{B}_{\mathbb{Z}^m} \times \mathcal{C}_{\mathrm{in}}(\mathbf{n}_0)) \setminus \mathcal{F}^{\mathrm{in}}_{\mathrm{res}}}} \|h_{\mathbf{n}_0,\xi_0}\|^2_{H^s(\omega_T \times \mathbb{T})} \leqslant C \, \|G\|^2_{H^s(\omega_T \times \mathbb{T}^m)} . \tag{6.76}$$

6.6.3. *Determination of the evanescent part and conclusion.* To conclude as to the proof of Theorem 4.1, it must be proved that there exists a solution U^{ev} in $\mathcal{P}^{\mathrm{ev}}_{s,T}$ to system (6.19), where the parameters s and T are those given in Proposition 6.28.

The polarization condition (6.19a) results, according to Remark 6.20, to

$$U^{\mathrm{ev}}(z, \theta, \psi_d) = \sum_{\mathbf{n} \in \mathbb{Z}^m \setminus \{0\}} e^{\psi_d \, \mathcal{A}(\mathbf{n} \cdot \boldsymbol{\zeta})} \, \Pi^{e}_{\mathbb{C}^N}(\mathbf{n} \cdot \boldsymbol{\zeta}) \, U^{\mathrm{ev}}_{\mathbf{n}}(z, 0) \, e^{i\mathbf{n} \cdot \theta}.$$

The traces $\left(\Pi^{e}_{\mathbb{C}^N}(\mathbf{n} \cdot \boldsymbol{\zeta}) \, U^{\mathrm{ev}}_{\mathbf{n}}\right)_{|\psi_d = 0}$ for $\mathbf{n}$ in $\mathbb{Z}^m \setminus \{0\}$ must therefore be determined to find the profile U^{ev}. The boundary condition (6.19b) gives the double trace on the boundary, for $\mathbf{n}$ in

$\mathbb{Z}^m \setminus \{0\}$,

$$U^{\mathrm{ev}}_{\mathbf{n}}(z',0,0) = \Pi^e_{\mathbb{C}^N}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{\mathrm{ev}}_{\mathbf{n}}(z',0,0) = \Pi^e_-(\mathbf{n}\cdot\boldsymbol{\zeta})\left(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\right)^{-1} G_{\mathbf{n}}(z').$$

Then this trace is lifted with respect to x_d using a function χ of $\mathcal{C}^\infty_0(\mathbb{R}_+)$, equaling 1 in 0. Namely we set

$$U^{\mathrm{ev}}(z,\theta,\psi_d) := \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \chi(x_d)\, e^{\psi_d\,\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_-(\mathbf{n}\cdot\boldsymbol{\zeta})\left(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\right)^{-1} G_{\mathbf{n}}(z')\, e^{i\mathbf{n}\cdot\theta}.$$

Note that, by construction, the profile U^{ev} satisfies the polarization condition (6.19a) as well as the boundary condition (6.19b). It must be now verified that it belongs to the space of evanescent profiles $\mathcal{P}^{\mathrm{ev}}_{s,T}$. First we note that the profile U^{ev} belongs to $L^\infty\big(\mathbb{R}^+_{x_d}\times\mathbb{R}^+_{\psi_d}, H^s_+(\omega_T\times\mathbb{T}^m)\big)$. Indeed, on one hand, the functions $G_{\mathbf{n}}$ being zero for negative times t, the profile U^{ev} is zero for negative times t. On the other hand, since the function χ is bounded, the inverse map $\left(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\right)^{-1}$ is uniformly bounded according to remark 2.8, and the terms $e^{\psi_d\,\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^e_-(\mathbf{n}\cdot\boldsymbol{\zeta})$ are also uniformly bounded according to estimate (6.12a) of Proposition 6.16, for $x_d,\psi_d \geqslant 0$, the following estimate holds:

$$\|U^{\mathrm{ev}}\|^2_{H^s(\omega_T\times\mathbb{T}^m)}(x_d,\psi_d) \leqslant C \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} \|G_{\mathbf{n}}\|^2_{H^s(\omega_T)} = C\,\|G\|^2_{H^s(\omega_T\times\mathbb{T}^m)}. \tag{6.77}$$

From now on we denote, for $\mathbf{n}$ in $\mathbb{Z}^m\setminus\{0\}$,

$$U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d) := \chi(x_d)\, e^{\psi_d\,\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e_-(\mathbf{n}\cdot\boldsymbol{\zeta})\left(B_{|E_-(\mathbf{n}\cdot\boldsymbol{\zeta})}\right)^{-1} G_{\mathbf{n}}(z'),$$

so that $U^{\mathrm{ev}}(z,\theta,\psi_d) = \sum_{\mathbf{n}\in\mathbb{Z}^m\setminus\{0\}} U^{\mathrm{ev}}_{\mathbf{n}}(z,\psi_d)\, e^{i\mathbf{n}\cdot\theta}$.

Then it is proven that the profile U^{ev} is continuous with respect to (x_d,ψ_d) in $\mathbb{R}_+\times\mathbb{R}_+$ with values in $H^s(\omega_T\times\mathbb{T}^m)$. Consider (x^0_d,ψ^0_d) in $\mathbb{R}_+\times\mathbb{R}_+$, and $\varepsilon>0$. There holds, for $x_d,\psi_d\geqslant 0$,

$$\begin{aligned}\left\|U^{\mathrm{ev}}(x_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi^0_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)} &\leqslant \left\|U^{\mathrm{ev}}(x_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)}\\ &\quad + \left\|U^{\mathrm{ev}}(x^0_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi^0_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)},\end{aligned}$$

and we seek to estimate the two terms on the right hand side of the inequality. For the first one, according to estimate (6.12a) of Proposition (6.16) and Remark 2.8, for $\psi_d\geqslant 0$, we get

$$\left\|U^{\mathrm{ev}}(x_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)} \leqslant C\left|\chi(x_d)-\chi(x^0_d)\right|\|G\|_{H^s(\omega_T\times\mathbb{T}^m)}.$$

By continuity of χ, there exists therefore $\delta_1>0$, depending only on ε, such that for all x_d such that $|x_d - x^0_d|<\delta_1$ and for all $\psi_d\geqslant 0$, we have

$$\left\|U^{\mathrm{ev}}(x_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)} < \varepsilon.$$

For the second one, we denote by M an integer such that

$$\left\|\sum_{|\mathbf{n}|>M} G_{\mathbf{n}}\, e^{i\mathbf{n}\cdot\theta}\right\|_{H^s(\omega_T\times\mathbb{T}^m)} < \varepsilon.$$

Thus, for $\psi_d\geqslant 0$,

$$\left\|U^{\mathrm{ev}}(x^0_d,\psi_d) - U^{\mathrm{ev}}(x^0_d,\psi^0_d)\right\|_{H^s(\omega_T\times\mathbb{T}^m)}$$

$$\leqslant \left\|\sum_{0<|\mathbf{n}|\leqslant M}\left[U^{\mathrm{ev}}_{\mathbf{n}}(y,x^0_d,\psi_d) - U^{\mathrm{ev}}_{\mathbf{n}}(y,x^0_d,\psi^0_d)\right] e^{i\mathbf{n}\cdot\theta}\right\|_{H^s(\omega_T\times\mathbb{T}^m)} \tag{6.78a}$$

$$+ \left\|\sum_{|\mathbf{n}|>M}\left[U^{\mathrm{ev}}_{\mathbf{n}}(y,x^0_d,\psi_d) - U^{\mathrm{ev}}_{\mathbf{n}}(y,x^0_d,\psi^0_d)\right] e^{i\mathbf{n}\cdot\theta}\right\|_{H^s(\omega_T\times\mathbb{T}^m)}. \tag{6.78b}$$

The sum in term (6.78a) being finite and the functions $U^{\mathrm{ev}}_{\mathbf{n}}$ being continuous with respect to ψ_d , there exists $\delta_2>0$ such that for all ψ_d such that $|\psi_d-\psi^0_d|<\delta_2$, we have (6.78a) $<\varepsilon$. On an other hand, according to estimate (6.12a), Remark 2.8 and since χ is bounded, we have, by

construction of M, for all $\psi_d \geqslant 0$, (6.78b) $< C\varepsilon$ where $C > 0$ does not depend on ε. It is then possible to conclude: for all (x_d, ψ_d) such that $|(x_d, \psi_d) - (x_d^0, \psi_d^0)| < \min(\delta_1, \delta_2)$, we have

$$\left\| U^{\mathrm{ev}}(x_d, \psi_d) - U^{\mathrm{ev}}(x_d^0, \psi_d^0) \right\|_{H^s(\omega_T \times \mathbb{T}^m)} < (2 + C)\varepsilon,$$

showing the required continuity.

Finally, with similar arguments as above for the continuity property, it is possible to show that the profile U^{ev} converges towards zero in the space $\mathcal{C}\big(\mathbb{R}^+_{x_d}, H^s(\omega_T \times \mathbb{T}^m)\big)$ when ψ_d goes to infinity (so in particular in $H^s(\omega_T \times \mathbb{T}^m)$ for every fixed x_d).

All points of Definition 3.4 of evanescent profiles have therefore been verified, so it has been proven that the profile U^{ev} belongs to the space $\mathcal{P}^{\mathrm{ev}}_{s,T}$ of evanescent profiles.

Remark 6.29. It has been shown in the previous paragraph, in estimate (6.76), that the oscillating part U^{osc} is controlled in $\mathcal{E}_{s,T}$ by the $H^s(\omega_T \times \mathbb{T}^m)$ norm of the boundary term G. On an other hand, according to estimate (6.77), the evanescent part U^{ev} is also controlled by the $H^s(\omega_T \times \mathbb{T}^m)$ norm of G. Thus the leading profile U satisfies

$$\|U\|_{\mathcal{P}_{s,T}} \leqslant C \, \|G\|_{H^s(\omega_T \times \mathbb{T}^m)} \, .$$

6.7. **Conclusion and perspectives.** It has therefore been proven that for $s > h + (d+m)/2$, there exists a time $T > 0$ small enough such that systems (6.17), (6.18) and (6.19) admit solutions $U^{\mathrm{osc}}_{\mathrm{res}}$, $S_{\mathbf{n}_0,\xi_0}$ and U^{ev} in $\mathcal{P}^{\mathrm{osc}}_{s,T}$, $\mathcal{C}(\mathbb{R}_+, H^s(\omega_T \times \mathbb{T}))$ and $\mathcal{P}^{\mathrm{ev}}_{s,T}$. According to Proposition 6.22, the profile $U = U^{\mathrm{osc}} + U^{\mathrm{ev}}$ (where U^{osc} is defined from $U^{\mathrm{osc}}_{\mathrm{res}}$ and $S_{\mathbf{n}_0,\xi_0}$ by equation (6.70)) is therefore a solution in $\mathcal{P}_{s,T}$ to system (5.11). It concludes the proof of Theorem 4.1.

Estimate (6.39) is not tame since the norm of V^{osc} in the estimate depends on the regularity index s. Therefore, it is a priori not possible to obtain the existence of a solution U^{osc} of infinite regularity considering a boundary term G infinitely regular, since without a tame estimate, the existence time T a priori depends on the considered index s. It has been chosen not to attempt to keep the estimates tame until the end for the sake of simplicity, for example in estimate (6.44), but it is however conceivable to achieve this more precise statement in further work.

Possible extensions of the result of this article to less restrictive assumptions are now discussed. It seems reasonable to consider a similar result under the assumption that the system under study is hyperbolic with constant multiplicity, and not strictly hyperbolic (Assumption 2). Similarly, Assumption 7 could be removed to allow outgoing frequencies to exist within the domain. It is a situation of this type which is considered in [CGW11]. But in this case it is no longer possible to determine beforehand the traces of incoming modes, as done in Proposition 6.22. This may also open the way to an infinite number of resonances with outgoing phases, which complicates the functional framework. The weakening of the uniform Kreiss-Lopatinskii condition Assumption 3 shall be discussed in a future work. Concerning the glancing frequencies, Assumption 4 stating that all glancing frequencies are of order 2 seems to be crucial, see [Wil00]. Likewise, it seems difficult to do without Assumption 5 ensuring that no glancing frequencies are created on the boundary.

Finally, this work raises the question of the justification of the geometric optic expansion that has been constructed, namely to prove that the function

$$z \mapsto \varepsilon \, U_1(z, z' \cdot \zeta_1/\varepsilon, \dots, z' \cdot \zeta_m/\varepsilon, x_d/\varepsilon)$$

is indeed a good approximation on a fixed time interval of the exact solution to (2.1) as ε goes to zero. To do so, two main methods are practicable. As conducted in [Wil96], if there exists a solution on a time interval independent on the parameter ε, it is conceivable to show that this exact solution and the function defined above draw near each other when ε goes to 0, see [JMR95] and [CGW11]. The problem is that in this work we do not have an exact solution on a fixed time interval. An other strategy relies on using a large number of corrector profiles, which we do not dispose either here (constructing correctors relies on small divisor accurate controls for noncharacteristic modes, which goes even further beyond Assumption 6). Both of these points (getting an existence time of the exact solution independent of epsilon and building a large number of correctors) do not seem to be within our reach for the moment, but will be the topics of future studies.

Appendix A. Additional proofs

A.1. **Proof of Proposition 2.21.** We detail here the proof of Proposition 2.21, omitted at first because of its length.

Recall that, for ζ in Ξ_0, the projectors $\Pi_j(\zeta)$, for j in $\mathcal{G}(\zeta)\cup\mathcal{I}(\zeta)$, are defined as the projectors from $E_-(\zeta)$ on $E_-^j(\zeta)$ according to decomposition (2.23), and that $\Pi_-^e(\zeta)$ is defined as the projector from $E_-(\zeta)$ on the elliptic stable component $E_-^e(\zeta)=\oplus_{j\in\mathcal{P}(\zeta)}E_-^j(\zeta)$ according to the same decomposition. Proposition 2.21 then reads as follows.

Proposition A.1 ([Wil96])**.** *Under assumption 2 and 4, for $\zeta\in\Xi_0$ the projectors $\Pi_-^j(\zeta)$ for j in $\mathcal{G}(\zeta)\cup\mathcal{I}(\zeta)$, and the projectors $\Pi_-^e(\zeta)$ are uniformly bounded with respect to ζ in Ξ_0.*

Proof. In all the proof we indistinctly denote by e every analytic function which, evaluated in a particular point $\underline{\alpha}$ precised below, is nonzero, and which is therefore nonzero in a neighborhood of the point $\underline{\alpha}$. Since the projectors $\Pi_-^j(\zeta)$, $j\in\mathcal{G}(\zeta)\cup\mathcal{I}(\zeta)$ and $\Pi_-^e(\zeta)$ are homogeneous of degree 0 with respect to ζ, the claim is proved locally in Σ_0, and the result follows from the compactness of the sphere Σ_0. The study is therefore reduced locally in a neighborhood of every point of Σ_0.

Consider $\underline{\zeta}=(\underline{\tau},\underline{\eta})\in\Sigma_0$. We are interested in the behavior, on a neighborhood of $\underline{\zeta}$ in Σ_0, of the projectors $\Pi_-^j(\zeta)$, $j\in\mathcal{G}(\zeta)\cup\mathcal{I}(\zeta)$ and $\Pi_-^e(\zeta)$, and therefore in the behavior, in a neighborhood of $\underline{\zeta}$, of the eigenvalues of $\mathcal{A}(\zeta)$. According to Proposition 2.9 there exists a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Σ_0, an integer $L\geqslant 0$, and a regular basis $\mathbb{C}^N$ in which the matrix $\mathcal{A}(\zeta)$ is a block diagonal matrix of the form

$$\operatorname{diag}\big(\mathcal{A}_-(\zeta),\mathcal{A}_+(\zeta),\mathcal{A}_1(\zeta),\dots,\mathcal{A}_L(\zeta)\big),\tag{A.1}$$

where the block $\mathcal{A}_-(\zeta)$ (resp. $\mathcal{A}_+(\zeta)$), eventually of size zero, is of negative definite (resp. positive definite) real part, and where the blocks $\mathcal{A}_j(\zeta)$ are of type *iii)* or *iv)* with the notations of Proposition 2.9. According to this proposition, the eigenvalues associated with the blocks of type *iii)* remain imaginary for $\zeta\in\Sigma_0$ in a neighborhood of $\underline{\zeta}$ and therefore do not contribute to the elliptic parts of the stable and unstable subspaces. However, the eigenvalues of the blocks of type *iv)* may have a nonzero real part in a neighborhood of $\underline{\zeta}$ and thus contribute to the elliptic parts. Thus, in a neighborhood of $\underline{\zeta}$ in Σ_0, the elliptic part $\oplus_{j\in\mathcal{P}(\zeta)}E_-^j(\zeta)$ writes as the direct sum of the stable subspace for $\mathcal{A}(\zeta)$ associated with the block $\mathcal{A}_-(\zeta)$ and of the generalized eigenspaces associated with the potential eigenvalues of negative real part of the blocks $\mathcal{A}_j(\zeta)$ of type *iv)*. The detailed description of these eigenspaces constitutes the central point of the analysis below.

In the basis adapted to decomposition (A.1), which is analytic with respect to $\zeta\in\Sigma_0$, we consider the first vectors associated with the block $\mathcal{A}_-(\zeta)$ and the aim is to complete this set of vectors into an analytic basis of the stable subspace $E_-(\zeta)$. The purpose is to construct, in a neighborhood of $\underline{\zeta}$, a determination, continuous with respect to ζ, of the stable eigenvectors of $\mathcal{A}(\zeta)$ associated with the blocks $\mathcal{A}_j(\zeta)$ of type *iii)* and *iv)* (which are therefore imaginary in $\underline{\zeta}$) and to deduce from that the existence of a linearly independent set of generalized eigenvectors continuously depending on ζ. To this end, the analyis of [Mét00] is followed.

Let $i\underline{\xi}_j$ be an imaginary eigenvalue of $\mathcal{A}(\underline{\zeta})$ of algebraic multiplicity n_j. By definition of $\mathcal{A}(\underline{\zeta})$ and with the notations of Assumption 2, there exists a unique index k_j between 1 and N such that

$$\underline{\tau}=\tau_{k_j}(\underline{\eta},\underline{\xi}_j).$$

Two cases may occur, depending on the cancellation of the quantity $\frac{\partial\tau_{k_j}}{\partial\xi}(\underline{\eta},\underline{\xi}_j)$. In the first case we shall see that there exists a continuous extension of the eigenvalue $i\underline{\xi}_j$ which remains imaginary for $\zeta\in\Sigma_0$ in a neighborhood of $\underline{\zeta}$, and that there exists a regular projector on the associated subspace. In the second case, the eigenvalue $i\underline{\xi}_j$ is degenerate (i.e. is not semisimple) and extends to a continuous eigenvalue $i\xi_j$, which, depending on the position of ζ in the

neighborhood of $\underline{\zeta}$, may become of nonzero real part, or imaginary and simple, or even remains imaginary and degenerate.

First suppose that

$$\frac{\partial \tau_{k_j}}{\partial \xi}(\underline{\eta}, \underline{\xi}_j) \neq 0,$$

that is $(\underline{\tau}, \underline{\eta}, \underline{\xi}_j)$ is incoming or outgoing. According to Assumption 2, for (ζ, ξ) in $\mathbb{R}^{d+1} \setminus \{0\}$, we have

$$\det\big(\mathcal{A}(\zeta) - i\xi I\big) = \det(A_d(0))^{-1} i^N \det L\big(0, (\zeta, \xi)\big) = \big(\tau - \tau_{k_j}(\eta, \xi)\big)\, e(\zeta, \xi), \tag{A.2}$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$. Since $\frac{\partial \tau_{k_j}}{\partial \xi}(\underline{\eta}, \underline{\xi}_j) \neq 0$, according to the Weierstrass preparation theorem [Hör90], there exists a unique real analytic function ξ_j defined in a neighborhood of $\underline{\zeta}$ in Σ_0 satisfying $\xi_j(\underline{\zeta}) = \underline{\xi}_j$ and such that in a neighborhood of $(\underline{\zeta}, \underline{\xi}_j)$ in $\Sigma_0 \times \mathbb{R}$ we have

$$\tau - \tau_{k_j}(\eta, \xi) = \big(\xi - \xi_j(\zeta)\big)\, e(\zeta, \xi), \tag{A.3}$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$. Thus, in a neighborhood of $(\underline{\zeta}, \underline{\xi}_j)$ in $\Sigma_0 \times \mathbb{R}$ we have

$$\det\big(\mathcal{A}(\zeta) - i\xi I\big) = \big(\xi - \xi_j(\zeta)\big)\, e(\zeta, \xi),$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$, so in a neighborhood of $\underline{\zeta}$ in Σ_0, $i\,\xi_j(\zeta)$ is an eigenvalue (analytic with respect to ζ) of $\mathcal{A}(\zeta)$ of algebraic multiplicity 1. On an other hand, according to identity (A.3), we have $\tau = \tau_{k_j}\big(\eta, \xi_j(\zeta)\big)$, thus

$$\mathcal{A}(\zeta)\, \pi_{k_j}(\eta, \xi_j(\zeta)) = i\,\xi_j(\zeta)\, \pi_{k_j}(\eta, \xi_j(\zeta)).$$

In a neighborhood of $\underline{\zeta}$ in Σ_0, $i\,\xi_j(\zeta)$ is therefore an eigenvalue of $\mathcal{A}(\zeta)$ of geometric multiplicity 1, thus simple. Furthermore the projector $\pi_{k_j}\big(\eta, \xi_j(\zeta)\big)$ is analytic with respect to ζ and is a projector on the eigenspace of $\mathcal{A}(\zeta)$ associated with $i\,\xi_j(\zeta)$. Thus, in the block decomposition (A.1), there is a unique scalar block among the blocks $\mathcal{A}_l(\zeta)$ corresponding to the eigenvalue $i\,\xi_j(\zeta)$. In the incoming case, we then obtain associated eigenvectors depending analyticly on $\zeta \in \Sigma_0$ in a neighborhood of $\underline{\zeta}$, contributing to the stable subspace $E_-(\zeta)$.

If now $\frac{\partial \tau_{k_j}}{\partial \xi}(\underline{\eta}, \underline{\xi}_j) = 0$, then, according to Assumption 4, we have $\frac{\partial^2 \tau_{k_j}}{\partial \xi^2}(\underline{\eta}, \underline{\xi}_j) \neq 0$ and in that case we say that $\underline{\xi}_j$ is *glancing*. Thus there exists a function e defined in a neighborhood of $\underline{\xi}_j$ with $e(\underline{\xi}_j) \neq 0$ such that for ξ close to $\underline{\xi}_j$, we have

$$\underline{\tau} - \tau_{k_j}(\underline{\eta}, \xi) = (\xi - \underline{\xi}_j)^2\, e(\xi).$$

We deduce, according to (A.2) that for ξ close to $\underline{\xi}_j$,

$$\det\big(\mathcal{A}(\underline{\zeta}) - i\xi I\big) = \big(\xi - \underline{\xi}_j\big)^2 e(\xi),$$

where $e(\underline{\xi}_j) \neq 0$. The algebraic multiplicity n_j of the eigenvalue $i\underline{\xi}_j$ is therefore equal to 2 whereas its geometric multiplicity equals 1 since

$$\ker\big(\mathcal{A}(\underline{\zeta}) - i\,\underline{\xi}_j\, I\big) = \ker L\big(0, (\underline{\zeta}, \underline{\xi}_j)\big) = \operatorname{Im} \pi_{k_j}(\underline{\eta}, \underline{\xi}_j),$$

and since the projector $\pi_{k_j}(\underline{\eta}, \underline{\xi}_j)$ is of rank 1. The aim is therefore to find a basis of the generalized eigenspace associated with $i\,\underline{\xi}_j$, which is of dimension 2. By definition of the analytic function τ_{k_j} and of the projector π_{k_j}, we have, for ξ close to $\underline{\xi}_j$,

$$L\big(0, \tau_{k_j}(\underline{\eta}, \xi), \underline{\eta}, \xi\big)\, \pi_{k_j}(\underline{\eta}, \xi) = 0.$$

Differentiating this equation with respect to ξ and evaluating in $\xi = \underline{\xi}_j$, one gets, since $\tau_{k_j}(\underline{\eta}, \underline{\xi}_j) = \underline{\tau}$,

$$\partial_\xi \tau_{k_j}(\underline{\eta}, \underline{\xi}_j)\, \partial_\tau L\big(0, \underline{\tau}, \underline{\eta}, \underline{\xi}_j\big)\, \pi_{k_j}(\underline{\eta}, \underline{\xi}_j) + \partial_\xi L\big(0, \underline{\tau}, \underline{\eta}, \underline{\xi}_j\big)\, \pi_{k_j}(\underline{\eta}, \underline{\xi}_j) + L\big(0, \underline{\tau}, \underline{\eta}, \underline{\xi}_j\big)\, \partial_\xi \pi_{k_j}(\underline{\eta}, \underline{\xi}_j) = 0,$$

that is to say, according to the expression of $L\big(0,(\tau,\eta,\xi)\big)$ and using $\partial_\xi \tau_{k_j}(\underline{\eta},\underline{\xi}_j)=0$,

$$A_d(0)\,\pi_{k_j}(\underline{\eta},\underline{\xi}_j)+iA_d(0)\big(\mathcal{A}(\underline{\zeta})-i\underline{\xi}_j\big)\frac{\partial \pi_{k_j}}{\partial \xi}(\underline{\eta},\underline{\xi}_j)=0.$$

Denoting $\underline{P}_0:=\pi_{k_j}(\underline{\eta},\underline{\xi}_j)$ and $\underline{P}_1:=\frac{\partial \pi_{k_j}}{\partial \xi}(\underline{\eta},\underline{\xi}_j)$ we obtain

$$\big(\mathcal{A}(\underline{\zeta})-i\underline{\xi}_j\big)\underline{P}_1=i\underline{P}_0. \tag{A.4}$$

We then denote by $\underline{E}_j$ a nonzero vector of the linear line $\operatorname{Im}\pi_{k_j}(\underline{\eta},\underline{\xi}_j)$. Equation (A.4) thus leads to

$$\big(\mathcal{A}(\underline{\zeta})-i\underline{\xi}_j\big)\underline{P}_1\underline{E}_j=i\underline{E}_j. \tag{A.5}$$

One can then verify that $(\underline{E}_j,\underline{P}_1\underline{E}_j)$ is a family of linearly independent vectors and that it therefore forms a basis of the generalized eigenspace associated with $i\,\underline{\xi}_j$. In this basis, according to (A.5), the operator $\mathcal{A}(\underline{\zeta})$ restricted to the generalized eigenspace associated with $i\,\underline{\xi}_j$ is given by the following matrix:

$$Q(\underline{\zeta})=\begin{pmatrix} i\underline{\xi}_j & i\\ 0 & i\underline{\xi}_j\end{pmatrix}. \tag{A.6}$$

We have therefore obtained a triangularization of the matrix $\mathcal{A}(\underline{\zeta})$ restricted to the generalized eigenspace associated with $i\,\underline{\xi}_j$, and we seek to extend this structure in a neighborhood of $\underline{\zeta}$ and to study the behavior of the stable eigenvalues of the matrix $\mathcal{A}(\underline{\zeta})$ restricted to the generalized eigenspace associated with $i\,\underline{\xi}_j$ in a neighborhood of $\underline{\zeta}$. In [Mét00] and using a result of [Ral71], it is proved that there exists a linearly independent set of vectors $E_j^0(\zeta),E_j^1(\zeta)$, analytic with respect to $\zeta\in\Sigma$ in a neighborhood of $\underline{\zeta}$, generating a subspace $F_j(\zeta)$ which is stable under $\mathcal{A}(\zeta)$, such that $E_j^0(\underline{\zeta})=\underline{E}_j$ and $E_j^1(\underline{\zeta})=\underline{P}_1\underline{E}_j$ and such that the restriction of $\mathcal{A}(\zeta)$ to the subspace $F_j(\zeta)$ is given by

$$Q(\zeta)=i\begin{pmatrix} \underline{\xi}_j+q_1(\zeta) & 1\\ q_2(\zeta) & \underline{\xi}_j\end{pmatrix}, \tag{A.7}$$

where $q_1(\underline{\zeta})=q_2(\underline{\zeta})=0$ and where $\frac{\partial q_2}{\partial \tau}(\underline{\zeta})\neq 0$. Among the blocks $\mathcal{A}_l(\zeta)$ of the block diagonalization (A.1) of the matrix $\mathcal{A}(\zeta)$ in a neighborhood of $\underline{\zeta}$, there is therefore a 2×2 block given by $Q(\zeta)$.

The aim is now to study the eigenvalues of the 2×2 block $Q(\zeta)$ above and to find a continuous determination of the stable eigenvalue in a neighborhood of $\underline{\zeta}$ in Σ (and not only in Σ_0), namely the Laplace parameter γ is allowed to be positive. First the expression of the characteristic polynomial of $Q(\zeta)$ is investigated. It is of degree 2, allowing to obtain an explicit formula for the eigenvalues of $Q(\zeta)$. According to (A.2), in a neighborhood of $(\underline{\zeta},\underline{\xi}_j)$, we have

$$\det\big(\mathcal{A}(\zeta)-i\xi I\big)=\big(\tau-\tau_{k_j}(\eta,\xi)\big)\,e(\zeta,\xi).$$

On an other hand, according to the Weierstrass preparation theorem and since $\partial_\xi\tau_{k_j}(\underline{\eta},\underline{\xi}_j)=0$ and $\partial_\xi^2\tau_{k_j}(\underline{\eta},\underline{\xi}_j)\neq 0$, there exists a couple of functions (f_0,f_1), analytic with respect to $\underline{\zeta}$, satisfying $f_0(\underline{\zeta})=f_1(\underline{\zeta})=0$ and such that for (ζ,ξ) close to $(\underline{\zeta},\underline{\xi}_j)$,

$$\tau-\tau_{k_j}(\eta,\xi)=\Big(\xi^2-\big(2\underline{\xi}_j+f_1(\zeta)\big)\,\xi+\underline{\xi}_j^2+f_0(\zeta)\Big)\,e(\zeta,\xi), \tag{A.8}$$

where $e(\underline{\zeta},\underline{\xi}_j)\neq 0$. Thus

$$\det\big(\mathcal{A}(\zeta)-i\xi I\big)=\Big(\xi^2-\big(2\underline{\xi}_j+f_1(\zeta)\big)\,\xi+\underline{\xi}_j^2+f_0(\zeta)\Big)\,e(\zeta,\xi),$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$. But according to the block decomposition of $\mathcal{A}(\zeta)$ we have

$$\det\big(\mathcal{A}(\zeta) - i\xi I\big) = \det\big(Q(\zeta) - i\xi I\big)\, e(\zeta, \xi),$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$ so that

$$\xi^2 - \big(2\underline{\xi}_j + f_1(\zeta)\big)\,\xi + \underline{\xi}_j^2 + f_0(\zeta) = \det\big(Q(\zeta) - i\xi I\big)\, e(\zeta, \xi),$$

where $e(\underline{\zeta}, \underline{\xi}_j) \neq 0$. Since according to (A.7) the ξ polynomial given by $\det\big(Q(\zeta) - i\xi I\big)$ is of degree 2 and of leading coefficient -1, we obtain

$$\det\big(Q(\zeta) - i\xi I\big) = -\Big(\xi^2 - \big(2\underline{\xi}_j + f_1(\zeta)\big)\,\xi + \underline{\xi}_j^2 + f_0(\zeta)\Big).$$

By identification, according to (A.7), we get $f_1 = q_1$ and $f_0 = \underline{\xi}_j\, q_1 - q_2$.

The interest is now made on the behavior of the eigenvalues of $Q(\zeta)$, and therefore on the roots of the polynomial $\xi^2 - \big(2\underline{\xi}_j + f_1(\zeta)\big)\,\xi + \underline{\xi}_j^2 + f_0(\zeta)$, for ζ in a neighborhood of $\underline{\zeta}$ in Σ. The Puiseux expansion theory ensure that for $\gamma > 0$ small, the eigenvalues of $Q(\zeta)$ with $\zeta = (\underline{\tau} - i\gamma, \underline{\eta})$ admit an expansion of the form

$$\xi(\zeta) = \underline{\xi}_j + \alpha_{1,2}\,\gamma^{1/2} + O(\gamma),$$

where the coefficients $\alpha_{1,2}$ are obtained resolving

$$\alpha_{1,2}^2 = i\big(\partial_\tau f_0(\underline{\zeta}) - \partial_\tau f_1(\underline{\zeta})\,\underline{\xi}_j\big).$$

But since $f_1 = q_1$ and $f_0 = \underline{\xi}_j\, q_1 - q_2$, we have

$$\partial_\tau f_0(\underline{\zeta}) - \partial_\tau f_1(\underline{\zeta})\,\underline{\xi}_j = -\partial_\tau q_2(\underline{\zeta}) \neq 0,$$

so that $\operatorname{Im}\alpha_{1,2} = \pm c$ where $c > 0$. Thus for $\gamma > 0$, $Q(\underline{\tau} - i\gamma, \underline{\eta})$ admits a unique stable eigenvalue $\xi_j^-(\zeta)$ (namely such that $\operatorname{Im}\xi_j^-(\zeta) > 0$) and a unique unstable eigenvalue $\xi_j^+(\zeta)$ (such that $\operatorname{Im}\xi_j^+(\zeta) < 0$). It is deduced that for ζ in a neighborhood of $\underline{\zeta}$ in $\Sigma \setminus \Sigma_0$, $Q(\zeta)$ admits a unique stable eigenvalue denoted by $\xi_j(\zeta)$. We then seek to continuously extend the eigenvalue ξ_j for $\gamma = 0$, that is to say we are interested in the root $\xi^2 - \big(2\underline{\xi}_j + f_1(\zeta)\big)\,\xi + \underline{\xi}_j^2 + f_0(\zeta)$ that extends $\xi_j(\zeta)$ to a neighborhood of $\underline{\zeta}$ in Σ. The behavior of this extension ξ_j shall then depends on the sign of the discriminant (real when ζ is real) $\Delta_j(\zeta) := 4\,\underline{\xi}_j\, f_1(\zeta) + f_1(\zeta)^2 - 4f_0(\zeta)$ which has been represented in Figure 7.

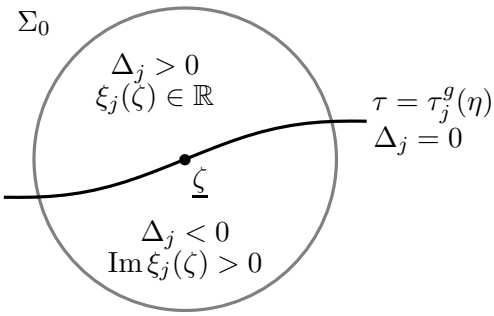

FIGURE 7. Sign of the discriminant $\Delta_j(\zeta)$ in a neighborhood of $\underline{\zeta}$.

When the discriminant $\Delta_j(\zeta)$ of this polynomial is negative, the eigenvalue $\xi_j(\zeta)$ is necessarily given by

$$\xi_j(\zeta) = \frac{2\,\underline{\xi}_j + f_1(\zeta) + i\sqrt{4f_0(\zeta) - 4\,\underline{\xi}_j\, f_1(\zeta) - f_1(\zeta)^2}}{2},$$

since it must be of non-negative imaginary part. When the discriminant $\Delta_j(\zeta)$ is zero, $\xi_j(\zeta)$ is given by

$$\xi_j(\zeta) = \frac{2\,\underline{\xi}_j + f_1(\zeta)}{2}.$$

The location of the discriminant roots may even be made precise, since it satisfies $\partial_\tau \Delta_j(\underline{\zeta}) = 4\big(\partial_\tau f_1(\underline{\zeta})\,\underline{\xi}_j - \partial_\tau f_0(\underline{\zeta})\big) \neq 0$, so according to the implicit functions theorem, there exists an analytic function τ_j^g defined in a neighborhood of $\underline{\eta}$ which parameterizes in a neighborhood of $\underline{\zeta}$ in Σ_0 the set of the discriminant's roots, see Figure 7. Finally, when the discriminant $\Delta_j(\zeta)$ is positive, we must determine which one of the real roots

$$\frac{2\,\underline{\xi}_j + f_1(\zeta) \pm \sqrt{4\,\underline{\xi}_j\, f_1(\zeta) + f_1(\zeta)^2 - 4 f_0(\zeta)}}{2}, \tag{A.9}$$

continuously extends the stable eigenvalue $\xi_j(\zeta)$ when $\gamma = 0$. If $\xi_j(\zeta)$ refers to the sought eigenvalue until $\gamma = 0$, and if we denote $\zeta = (\sigma, \eta) := (\tau - i\gamma, \eta)$, since $\xi_j(\zeta)$ is real when $\gamma = 0$ and $\operatorname{Im}\xi_j(\zeta) > 0$ when $\gamma > 0$, we have necessarily

$$\frac{\partial \operatorname{Im}\xi_j}{\partial \operatorname{Im}\sigma}\Big|_{\gamma=0} \leqslant 0,$$

so that according to the Cauchy-Riemann equations, we must have

$$\partial_\tau\big(\operatorname{Re}\xi_j\big)|_{\gamma=0} \leqslant 0.$$

Thus, if $\partial_\tau f_0(\underline{\zeta}) - \partial_\tau f_1(\underline{\zeta})\,\underline{\xi}_j > 0$, the real root

$$\frac{2\,\underline{\xi}_j + f_1(\zeta) + \sqrt{4\,\underline{\xi}_j\, f_1(\zeta) + f_1(\zeta)^2 - 4 f_0(\zeta)}}{2}$$

is the one that continuously extends the stable eigenvalue $\xi_j(\zeta)$ when $\gamma = 0$, and in the other case, the other root must be chosen. We have therefore obtained a continuous determination of the stable eigenvalue $\xi_j(\zeta)$ of the matrix $Q(\zeta)$ in a neighborhood of $\underline{\zeta}$. Note now that an eigenvector of the matrix

$$\begin{pmatrix} \underline{\xi}_j + q_1(\zeta) & 1 \\ q_2(\zeta) & \underline{\xi}_j \end{pmatrix}$$

associated with the eigenvalue $\xi_j(\zeta)$ writes $\Big(1, \frac{q_2(\zeta)}{\xi_j(\zeta) - \underline{\xi}_j}\Big)$. One thus gets, using the linearly independent vectors $E_j^0(\zeta), E_j^1(\zeta)$, an eigenvector $\mathcal{A}(\zeta)$ associated with the stable eigenvalue $i\,\xi_j(\zeta)$ continuous with respect to ζ[4].

In a nutshell, in a neighborhood of $\underline{\zeta}$ in Σ_0, the degenerate imaginary eigenvalue $i\,\underline{\xi}_j$ continuously extends in a stable eigenvalue $i\,\xi_j(\zeta)$ of which the behavior depends on the sign of the discriminant $\Delta_j(\zeta)$ in the neighborhood of $\underline{\zeta}$, which has been figured in Figure 7. If $\Delta_j(\zeta)$ is negative, then $i\,\xi_j(\zeta)$ is of negative real part so the eigenvalue $i\,\xi_j(\zeta)$ contributes to the elliptic part of the stable subspace $E_-(\zeta)$. If $\Delta_j(\zeta)$ is positive, then $i\,\xi_j(\zeta)$ is a simple imaginary eigenvalue of $\mathcal{A}(\zeta)$ so it contributes to a subspace $E_-^l(\zeta)$ with l in $\mathcal{I}(\zeta)$. Finally if $\Delta_j(\zeta)$ is zero, the eigenvalue $i\,\xi_j(\zeta)$ remains imaginary and degenerate so it contributes to a subspace $E_-^l(\zeta)$ with l in $\mathcal{G}(\zeta)$.

We denote now by $i\,\underline{\xi}_l$, $l = 1, \dots, r$ the real incoming eigenvalues and by $i\,\underline{\xi}_l$, $l = r+1, \dots, r+g$ the glancing eigenvalues of the matrix $\mathcal{A}(\underline{\zeta})$. Using the notations of the beginning of the proof, we have found continuous extensions $i\,\xi_l$, $l = 1, \dots, r+g$, of these eigenvalues in a neighborhood of $\underline{\zeta}$. Therefore, a continuous determination of the stable eigenvalues of the blocks $\mathcal{A}_1(\zeta), \dots, \mathcal{A}_L(\zeta)$ has been determined, as well as a continuous basis of the stable subspace $E_-(\zeta)$ constituted of generalized eigenvectors of the matrix $\mathcal{A}_-(\zeta)$ and of eigenvectors of the matrix $\mathcal{A}(\zeta)$ associated with the eigenvalues $i\,\xi_l(\zeta)$ for $l = 1, \dots, r+g$. Then we denote by $\widetilde{\Pi}_-^e(\zeta)$ the analytic projector from $E_-(\zeta)$ to the stable subspace associated with the elliptic block $\mathcal{A}_-(\zeta)$, and, for $l = 1, \dots, r+g$, $\widetilde{\Pi}_-^l(\zeta)$ the continuous projector from $E_-(\zeta)$ to the eigenspace associated

[4]Since $\xi_j(\zeta)$ is a root of the polynomial $\xi^2 - \big(2\underline{\xi}_j + q_1(\zeta)\big)\,\xi + \underline{\xi}_j^2 + \underline{\xi}_j q_1(\zeta) - q_2(\zeta)$, according to $f_1 = q_1$ and $f_0 = \underline{\xi}_j q_1 - q_2$, we have $\frac{q_2(\zeta)}{\xi_j(\zeta) - \underline{\xi}_j} = \xi_j(\zeta) - \underline{\xi}_j - q_1(\zeta)$. Thus the following limit holds $\frac{q_2(\zeta)}{\xi_j(\zeta) - \underline{\xi}_j} \underset{\zeta \to \underline{\zeta}}{\longrightarrow} 0$, and the considered eigenvector continuously depends on ζ in a neighborhood of $\underline{\zeta}$

with $i\,\xi_l(\zeta)$. Since these projectors are continuous with respect to ζ in a neighborhood of $\underline{\zeta}$, they can be assumed to be bounded on this neighborhood.

If $l = 1, \dots, r$, the eigenvalue $i\,\xi_l(\zeta)$ is imaginary and simple in a neighborhood of $\underline{\zeta}$, so, for all ζ, the projector $\widetilde{\Pi}^l_-(\zeta)$ contributes to a projector $\Pi^j_-(\zeta)$ for some j (depending on ζ) in $\mathcal{I}(\zeta)$. If $l = r+1, \dots, r+g$, then, depending on the sign of Δ_l, the eigenvalue $i\,\xi_l(\zeta)$ may be imaginary and simple, or imaginary and degenerate, or even of nonzero real part, so depending on where ζ is in a neighborhood of $\underline{\zeta}$, the projector $\widetilde{\Pi}^l_-(\zeta)$ contributes to $\Pi^e_-(\zeta)$ (when $\Delta_l < 0$), to $\Pi^j_-(\zeta)$ for some j in $\mathcal{G}(\zeta)$ (when $\Delta_l = 0$) or to $\Pi^j_-(\zeta)$ for some j in $\mathcal{I}(\zeta)$ (when $\Delta_l > 0$). As for it, the projector $\widetilde{\Pi}^e_-(\zeta)$ always contributes to $\Pi^e_-(\zeta)$.

We seek now to explicitly describe the projectors $\Pi^e_-(\zeta)$ and $\Pi^j_-(\zeta)$ for $j \in \mathcal{I}(\zeta) \cup \mathcal{G}(\zeta)$. To simplify the notations, we assume that among the imaginary eigenvalues $i\,\underline{\xi}_l$, there is only one of them which is glancing, namely that $g = 1$. The expressions of the sought projectors depend on whether the eigenvalue $i\,\xi_{r+1}(\zeta)$ is incoming, glancing, or of negative real part, and therefore on where is ζ in the neighborhood of $\underline{\zeta}$, see Figure 7. If ζ belongs to the area of the neighborhood of $\underline{\zeta}$ where $\Delta_{r+1}(\zeta) > 0$, then the eigenvalue $i\,\xi_{r+1}(\zeta)$ is incoming, so $\mathcal{I}(\zeta)$ is of cardinality $r+1$ and $\mathcal{G}(\zeta)$ is empty. In this case, for all index j in $\mathcal{I}(\zeta)$, we have

$$\Pi^j_-(\zeta) = \widetilde{\Pi}^l_-(\zeta),$$

for some l between 1 and $r+1$, and

$$\Pi^e_-(\zeta) = \widetilde{\Pi}^e_-(\zeta).$$

If ζ is, in the neighborhood of $\underline{\zeta}$, on the hypersurface defined by $\Delta_{r+1} = 0$, then $i\,\xi_{r+1}(\zeta)$ is glancing and in that case $\mathcal{I}(\zeta)$ is of cardinality r and $\mathcal{G}(\zeta)$ is if cardinality 1. We have therefore, for j in $\mathcal{I}(\zeta)$,

$$\Pi^j_-(\zeta) = \widetilde{\Pi}^l_-(\zeta),$$

for some l between 1 and r, for the index j of $\mathcal{G}(\zeta)$,

$$\Pi^j_-(\zeta) = \widetilde{\Pi}^{r+1}_-(\zeta),$$

and

$$\Pi^e_-(\zeta) = \widetilde{\Pi}^e_-(\zeta).$$

Finally, if ζ belongs to the area of the neighborhood of $\underline{\zeta}$ where $\Delta_j(\zeta) < 0$, then $i\,\xi_{r+1}(\zeta)$ is of negative real part so it contributes to the elliptic part $E_-(\zeta)$. Thus $\mathcal{I}(\zeta)$ is of cardinality r, $\mathcal{G}(\zeta)$ is empty and the extension of the degenerate eigenvalue $i\xi_{r+1}(\zeta)$ contributes to the elliptic part. In this case, for all index j in $\mathcal{I}(\zeta)$, we have

$$\Pi^j_-(\zeta) = \widetilde{\Pi}^l_-(\zeta),$$

for some l between 1 and r, and

$$\Pi^e_-(\zeta) = \widetilde{\Pi}^e_-(\zeta) + \widetilde{\Pi}^{r+1}_-(\zeta).$$

Therefore, since the projectors $\widetilde{\Pi}^l_-(\zeta)$, $l = 1, \dots, r+1$ and $\widetilde{\Pi}^e_-(\zeta)$ are bounded uniformly with respect to ζ in a neighborhood of $\underline{\zeta}$, we deduce that the projectors $\Pi^e_-(\zeta)$ and $\Pi^j_-(\zeta)$ for $j \in \mathcal{I}(\zeta) \cup \mathcal{G}(\zeta)$ are bounded uniformly with respect to ζ in a neighborhood of $\underline{\zeta}$, which concludes the proof of Proposition 2.21.

In the general case where there are multiple glancing eigenvalues $i\,\underline{\xi}_l$ (namely when $g \geqslant 1$), the projectors $\Pi^e_-(\zeta)$ and $\Pi^j_-(\zeta)$ for $j \in \mathcal{I}(\zeta) \cup \mathcal{G}(\zeta)$ can still be expressed using the projectors $\widetilde{\Pi}^l_-(\zeta)$, $l = 1, \dots, r+g$ and $\widetilde{\Pi}^e_-(\zeta)$. Since the expression of $\Pi^e_-(\zeta)$ will be needed in the proof of Proposition 6.16 below, it is given here. For ζ in a neighborhood of $\underline{\zeta}$ in Σ_0, we have

$$\Pi^e_-(\zeta) = \widetilde{\Pi}^e_-(\zeta) + \sum_{j=r+1}^{r+g} \mathbb{1}_{\Delta_j(\zeta)<0}\, \widetilde{\Pi}^j_-(\zeta). \tag{A.10}$$

□

A.2. **Proof of Lemma 2.22.** The following proof of Lemma 2.22 uses results and notations from the previous one, and is therefore given now. First we recall the statement of Lemma 2.22.

Lemma A.2. *There exists a positive constant $C > 0$ such that, if the real frequency $\alpha = (\tau, \eta, \xi)$ in $\mathbb{R}^{1+d} \setminus \{0\}$ is characteristic, and if k between 1 and N is such that $\tau = \tau_k(\eta, \xi)$, then we have*

$$|\partial_\xi \tau_k(\eta, \xi)| \geqslant C \frac{\operatorname{dist}\big((\tau, \eta), \mathcal{G}\big)^{1/2}}{|(\tau, \eta)|^{1/2}}.$$

Using Lemma 2.15, we therefore obtain the following estimate

$$|\widetilde{\pi}_\alpha E_k(\eta, \xi)| \geqslant C \frac{\operatorname{dist}\big((\tau, \eta), \mathcal{G}\big)^{1/2}}{|(\tau, \eta)|^{1/2}}. \tag{A.11}$$

Proof. The interest is made at first in the first equality, which is proved using the homogeneity of degree zero of $\partial_\xi \tau_k(\eta, \xi)$ and of degree one of the distance. The analysis is therefore made on the sphere Σ_0, and we denote, for ζ in Σ_0,

$$m_g(\zeta) = \begin{cases} 1 & \text{if } \operatorname{sp}(\mathcal{A}(\zeta)) \cap i\mathbb{R} = \emptyset, \\ \displaystyle\min_{\substack{j \in \mathcal{G}(\zeta) \cup \\ \mathcal{I}(\zeta) \cup \mathcal{O}(\zeta)}} \big|\partial_\xi \tau_{k_j}\big(\eta, \xi_j(\zeta)\big)\big| & \text{otherwise,} \end{cases}$$

where $\operatorname{sp}(\mathcal{A}(\zeta))$ refers to the spectrum of the matrix $\mathcal{A}(\zeta)$, and where the notations k_j and $\xi_j(\zeta)$ has been introduced in Proposition 2.16. Using the compactness of the sphere Σ_0, it will be proved that m_g satisfies

$$m_g(\zeta) \geqslant C \operatorname{dist}(\zeta, \mathcal{G})^{1/2}, \tag{A.12}$$

for all ζ in Σ_0, where $C > 0$ is a suitable fixed constant. We thus consider $\underline{\zeta} := (\underline{\tau}, \underline{\eta})$ in Σ_0, and we show that there exists a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in which the previous equality (A.12) is satisfied.

We recall the results obtained in the proof of Proposition 2.21, in which a continuous determination of the eigenvalues of $\mathcal{A}(\zeta)$ for ζ in a neighborhood of $\underline{\zeta}$ has been determined. The previous proof focused on describing the stable eigenvalues, but it can be immediately extended to all eigenvalues of $\mathcal{A}(\zeta)$. Denote by $i\,\underline{\xi}_j$ the imaginary eigenvalues of $\mathcal{A}(\underline{\zeta})$. If the imaginary eigenvalue $i\,\underline{\xi}_j$ is not glancing, the proof of Proposition 2.21 yields to a continuous extension $i\,\xi_j(\zeta)$ in a neighborhood of $\underline{\zeta}$, which is an eigenvalue of $\mathcal{A}(\zeta)$. If $i\,\underline{\xi}_j$ is glancing, then we obtain, in a neighborhood of $\underline{\zeta}$, two continuous eigenvalues $i\,\xi_j^-(\zeta)$ and $i\,\xi_j^+(\zeta)$ extending $i\,\underline{\xi}_j$, which are possibly equal (when they are glancing). Finally, the block structure (see Proposition 2.9) and Proposition 2.21 provide a basis of $\mathbb{C}^N$ in which the matrix $\mathcal{A}(\zeta)$ is, in a neighborhood of $\underline{\zeta}$, block diagonal, with a block $\mathcal{A}_\pm(\zeta)$ with eigenvalues of nonzero real part, and scalar blocks corresponding to the eigenvalues $i\xi_j(\zeta)$. Three cases are then to be investigated.

- All eigenvalues of $\mathcal{A}_\pm(\zeta)$ are of nonzero real part in a neighborhood of $\underline{\zeta}$, so they don't contribute to $m_g(\zeta)$.
- If $i\,\underline{\xi}_j$ is imaginary and $\partial_\xi \tau_{k_j}\big(\underline{\eta}, \xi_j(\underline{\tau}, \underline{\eta})\big) \neq 0$, namely if the real characteristic frequency $\alpha_j(\underline{\zeta})$ is incoming or outgoing, then it has been proven that the eigenvalue $i\,\xi_j(\zeta)$ is still incoming or outgoing in a neighborhood of $\underline{\zeta}$. Furthermore, according to equation (A.3) differentiated with respect to ξ and evaluated in $\xi = \xi_j(\zeta)$, for ζ in a neighborhood of $\underline{\zeta}$, we have
$$\partial_\xi \tau_{k_j}\big(\eta, \xi_j(\zeta)\big) = -e\big(\eta, \xi_j(\zeta)\big),$$
where e is an analytic function nonzero in $\underline{\zeta}$, which is therefore lower bounded in a neighborhood of $\big(\underline{\zeta}, \xi_j(\underline{\zeta})\big)$. Thus, for ζ in a neighborhood of $\underline{\zeta}$, we have
$$\big|\partial_\xi \tau_{k_j}\big(\eta, \xi_j(\zeta)\big)\big| \geqslant C,$$
with $C > 0$.

- Finally, if $i\,\underline{\xi}_j$ is glancing, namely if $i\,\underline{\xi}_j$ is imaginary and $\partial_\xi \tau_{k_j}(\underline{\eta}, \underline{\xi}_j) = 0$, then $i\,\underline{\xi}_j$ is extended by two eigenvalues $i\,\xi_j^{\pm}(\zeta)$, of which the behavior depends on where ζ is in the neighborhood of $\underline{\zeta}$, see Figure 7. Denote by $\Delta_j(\zeta)$ the discriminant of the characteristic polynomial of the 2×2 block associated with the glancing eigenvalue. If ζ is such that $\Delta_j(\zeta) < 0$, then the two eigenvalues $i\,\xi_j^{\pm}(\zeta)$ are of nonzero real part, so they do not contribute to $m_g(\zeta)$. If $\Delta_j(\zeta) = 0$, then $\xi_j^-(\zeta) = \xi_j^+(\zeta)$ and the characteristic frequency $\big(\zeta, \xi_j^{\pm}(\zeta)\big)$ is glancing, so equality (A.12) is immediately satisfied. Finally, if $\Delta_j(\zeta) > 0$, then the two distinct eigenvalues $i\,\xi_j^{\pm}(\zeta)$ are imaginary, and contributes to $m_g(\zeta)$. According to the relation (A.8), differentiate with respect to ξ and evaluated in $\xi = \xi_j^+(\zeta)$, we have, for $\zeta = (\tau, \eta)$ in a neighborhood of $\underline{\zeta}$,
$$\partial_\xi \tau_{k_j}\big(\eta, \xi_j^+(\zeta)\big) = 2\big(f_1(\zeta)/2 + \underline{\xi}_j - \xi_j^+(\zeta)\big)\, e\big(\zeta, \xi_j^+(\zeta)\big),$$
where e is an analytic function, nonzero in $(\underline{\zeta}, \underline{\xi}_j)$. According to expression (A.9) of the roots $\xi_j^{\pm}(\zeta)$, we obtain
$$\partial_\xi \tau_{k_j}\big(\eta, \xi_j^+(\zeta)\big) = \pm\sqrt{\Delta_j(\zeta)}\, e\big(\zeta, \xi_j^+(\zeta)\big).$$
But, according to the proof of Proposition 2.21 above, one may write, for $\zeta = (\tau, \eta)$ in a neighborhood of $\underline{\zeta}$,
$$\Delta_j(\zeta) = \big(\tau - \tau_j^g(\eta)\big)\, e(\zeta),$$
where e is an analytic function, nonzero in $\underline{\zeta}$, and where the function τ_j^g parameterizes the surface of the zeros of Δ_j. We finally infer
$$\Big|\partial_\xi \tau_{k_j}\big(\eta, \xi_j^+(\zeta)\big)\Big| \geqslant C\,\Big|\tau - \tau_j^g(\eta)\Big|^{1/2} = C\,\Big|(\tau, \eta) - \big(\tau_j^g(\eta), \eta\big)\Big|^{1/2} \geqslant C\,\mathrm{dist}\,\big(\zeta, \mathcal{G}\big)^{1/2},$$
since the frequency $\big(\tau_j^g(\eta), \eta\big)$ is glancing by construction of τ_j^g. The same arguments apply to $i\,\xi_j^-(\zeta)$.

Up to reducing the constant C, we have therefore proved the existence of a neighborhood $\mathcal{V}$ in which equality (A.12) is satisfied. The result follows from the compactness of Σ_0 and by homogeneity.

The second inequality of Lemma 2.22 is obtained immediately using the result of Lemma 2.15. □

A.3. **Proof of Proposition 6.16.** The following proof also comes after the one of Proposition 2.21.

We recall that $\Pi_-^e(\zeta)$ is the projector from $E_-(\zeta)$ on the elliptic stable component $E_-^e(\zeta) = \oplus_{j\in\mathcal{P}(\zeta)} E_-^j(\zeta)$ according to decomposition (2.23) and that, when ζ is not glancing, $\Pi_{\mathbb{C}^N}^e(\zeta)$ is the projection from $\mathbb{C}^N$ on the stable elliptic component $E_-^e(\zeta)$ according decomposition (2.25). The statement of Proposition 6.16 reads as follows.

Proposition A.3. *Under Assumption 6, there exists a constant $c_1 > 0$ and a real number b_1 such that, for all ζ in $\mathcal{F}_b \setminus \{0\}$, the following estimates hold*

$$\Big|e^{t\mathcal{A}(\zeta)}\,\Pi_-^e(\zeta)\Big| \leqslant c_1\, e^{-c_1\, t\,|\zeta|^{-b_1}} \leqslant c_1, \qquad \forall t \geqslant 0, \tag{A.13a}$$

$$\Big|e^{t\mathcal{A}(\zeta)}\,\Pi_{\mathbb{C}^N}^e(\zeta)\Big| \leqslant c_1\,|\zeta|^{b_1}\, e^{-c_1\, t\,|\zeta|^{-b_1}}, \qquad \forall t \geqslant 0, \tag{A.13b}$$

$$\Big|e^{t\mathcal{A}(\zeta)}\big(I - \Pi_{\mathbb{C}^N}^e(\zeta)\big)\Big| \leqslant c_1\,|\zeta|^{b_1}, \qquad \forall t \leqslant 0. \tag{A.13c}$$

Proof. The homogeneity of degree 1 of the matrix $\mathcal{A}(\zeta)$ and of degree zero of the projectors $\Pi_-^e(\zeta)$ and $\Pi_{\mathbb{C}^N}^e(\zeta)$, and the compactness of the unit ball Σ_0 are used, and we therefore work in a neighborhood of every point $\underline{\zeta}$ of Σ_0. The result is then extended to a finite conic covering of Ξ_0. Since the projector $\Pi_{\mathbb{C}^N}^e(\zeta)$ is defined only for ζ non glancing, for inequalities (A.13b) and (A.13c) where it occurs, we are only interested in the points of the neighborhood of $\underline{\zeta}$ which are

not glancing. Thus we consider a point $\underline{\zeta}$ of Σ_0 and we come back to the notations of the proof of Proposition 2.21.

The interest is first made on the first estimate (A.13a). In the proof of Proposition 2.21, we have constructed, in a neighborhood of $\underline{\zeta}$, a continuous basis of $E_-(\zeta)$ associated with a regular change-of-basis matrix $T(\zeta)$ in which the matrix $\mathcal{A}(\zeta)$ restricted to $E_-(\zeta)$ is the following block diagonal matrix of size $p \times p$

$$\begin{pmatrix} \mathcal{A}_-(\zeta) & & & 0 \\ & i\xi_1(\zeta) I_{\omega_1} & & \\ & & \ddots & \\ 0 & & & i\xi_{r+g}(\zeta) I_{\omega_{r+g}} \end{pmatrix},$$

constituted of a block $\mathcal{A}_-(\zeta)$ of negative definite real part, of diagonal blocks $i\xi_j(\zeta)\, I_{\omega_j}$, $j = 1, \dots, r$ associated with the incoming eigenvalues $\underline{\xi}_j$ and of diagonal blocks $i\xi_j(\zeta) I_{\omega_j}$, $j = r+1, \dots, r+g$ associated with the eigenvalues $\underline{\xi}_j$ which are glancing in $\underline{\zeta}$. In that case, according to expression (A.10) of the projector $\Pi^e_-(\zeta)$ in a neighborhood of $\underline{\zeta}$ in Σ_0, the linear map $e^{t\mathcal{A}(\zeta)}\, \Pi^e_-(\zeta)$ from $E_-(\zeta)$ to itself is given, in the basis associated with the matrix $T(\zeta)$, by the following $p \times p$ block diagonal matrix

$$\begin{pmatrix} e^{t\mathcal{A}_-(\zeta)} & & & & \\ & 0 & & & \\ & & e^{i\,t\,\xi_{r+1}(\zeta)}\, \mathbb{1}_{\Delta_{r+1}(\zeta)<0}\, I_{\omega_{r+1}} & & \\ & & & \ddots & \\ & & & & e^{i\,t\,\xi_{r+g}(\zeta)}\, \mathbb{1}_{\Delta_{r+g}(\zeta)<0}\, I_{\omega_{r+g}} \end{pmatrix}.$$

On one hand, the block $\mathcal{A}_-(\zeta)$ is of negative definite real part, uniformly with respect to ζ. On the other hand, one can check that

$$\operatorname{Im} \xi_{r+l}(\zeta) = |\Delta_{r+l}(\zeta)|^{1/2}/2,$$

for $l = 1, \dots, g$, where $\Delta_{r+l}(\zeta)$ refers to the discriminant of the characteristic polynomial associated with the glancing eigenvalue $\underline{\xi}_{r+l}$ defined in the proof of Proposition 2.21 and is depicted in Figure 7. But one can write, in a neighborhood of $\underline{\zeta}$ in Σ_0,

$$\Delta_j(\zeta) = \big[\tau - \tau_j^g(\eta)\big] e(\zeta),$$

with $e(\underline{\zeta}) \neq 0$, where we recall that τ_j^g parameterizes in Σ_0 the surface of cancellation of Δ_j. It yields to the following estimate on Δ_j,

$$|\Delta_j(\zeta)| \geqslant C \left|\tau - \tau_j^g(\eta)\right| = C \left|\zeta - (\tau_j^g(\eta), \eta)\right| \geqslant C \operatorname{dist}(\zeta, \mathcal{G}). \tag{A.14}$$

Since the matrix T is regular, and therefore uniformly bounded with respect to ζ in a neighborhood of $\underline{\zeta}$ in Σ_0, according to (A.14), we get

$$\left| e^{t\mathcal{A}(\zeta)}\, \Pi^e_-(\zeta) \right| \leqslant C e^{-C \operatorname{dist}(\zeta, \mathcal{G})^{1/2}\, t}. \tag{A.15}$$

We consider now ζ in a conic neighborhood of $\underline{\zeta}$ in Ξ_0 with $\zeta = \lambda \zeta^*$ where $\lambda = |\zeta| \in \mathbb{R}_+^*$ and ζ^* is in a neighborhood of $\underline{\zeta}$ in Σ_0. Then, by homogeneity and using (A.15) and Assumption 6, we obtain

$$\begin{aligned} \left| e^{t\mathcal{A}(\zeta)}\, \Pi^e_-(\zeta) \right| = \left| e^{\lambda t \mathcal{A}(\zeta^*)}\, \Pi^e_-(\zeta^*) \right| &\leqslant C e^{-C \operatorname{dist}(\zeta^*, \mathcal{G})^{1/2}\, \lambda t} \\ &= C e^{-C \operatorname{dist}(\lambda\zeta^*, \mathcal{G})^{1/2}\, \lambda^{1/2} t} \leqslant C e^{-C|\zeta|^{(a_1+1)/2}\, t}. \end{aligned}$$

Finally the inequality extends to the whole space Ξ_0 by compactness of Σ_0, yielding to the required inequality (A.13a) for all ζ in $\mathcal{F}_b \setminus \{0\}$.

Concerning estimate (A.13b), note that, for $t \geqslant 0$ and for ζ non glancing,

$$e^{t\mathcal{A}(\zeta)}\,\Pi^e_{\mathbb{C}^N}(\zeta) = e^{t\mathcal{A}(\zeta)}\,\Pi^e_-(\zeta)\,\Pi^-_{\mathbb{C}^N}(\zeta),$$

where $\Pi^-_{\mathbb{C}^N}(\zeta)$ is the projector from $\mathbb{C}^N$ to the stable subspace $E_-(\zeta)$ according to decomposition (2.25), defined for ζ non glancing. The aim is therefore to control the projector $\Pi^-_{\mathbb{C}^N}(\zeta)$, and then use inequality (A.13a) to conclude. We still work in a neighborhood of $\underline{\zeta}$ in Σ_0 and, to simplify the notations, we assume that $\mathcal{A}(\underline{\zeta})$ admits a unique glancing eigenvalue $\underline{\xi}_g$ of algebraic multiplicity 2.

Applying the arguments of the proof of Proposition 2.21 to the unstable part $E^+(\zeta)$, one obtain an analytic basis

$$E_1(\zeta), \dots, E_N(\zeta)$$

of $\mathbb{C}^N$ associated with a change-of-basis matrix $\widetilde{T}(\zeta)$, analytic in a neighborhood of $\underline{\zeta}$ in Σ_0, such that in this basis, the linear map $\mathcal{A}(\zeta)$ writes

$$\widetilde{T}(\zeta)^{-1}\,\mathcal{A}(\zeta)\,\widetilde{T}(\zeta) = \operatorname{diag}\big(\mathcal{A}_-(\zeta), \mathcal{A}_1(\zeta), Q(\zeta), \mathcal{A}_+(\zeta), \mathcal{A}_2(\zeta)\big),$$

where $\mathcal{A}_-(\zeta)$ is of negative definite real part, $\mathcal{A}_1(\zeta)$ is the diagonal block associated with the incoming eigenvalues, $\mathcal{A}_+(\zeta)$ is of positive definite real part, $\mathcal{A}_2(\zeta)$ is the diagonal block associated with the outgoing eigenvalues, and the unique 2×2 block $Q(\zeta)$, associated with the glancing eigenvalue $\underline{\xi}_g$, writes

$$Q(\zeta) = i \begin{pmatrix} \underline{\xi}_g + q_1(\zeta) & 1 \\ q_2(\zeta) & \underline{\xi}_g \end{pmatrix},$$

where $q_1(\underline{\zeta}) = q_2(\underline{\zeta}) = 0$ and $\frac{\partial q_2}{\partial \tau}(\underline{\zeta}) \neq 0$.

We want now to construct, using the basis $E_1, \dots, E_N$, a new basis $F_1, \dots, F_N$ adapted to the decomposition

$$\mathbb{C}^N = E_-(\zeta) \oplus E_+(\zeta)$$

for ζ non glancing. If $E_1(\zeta), \dots, E_{p-1}(\zeta)$ are the $p-1$ first vectors corresponding to the blocks $\mathcal{A}_-(\zeta)$ and $\mathcal{A}_1(\zeta)$ of the basis of $\mathbb{C}^N$ associated with $T(\zeta)$, we set, for $j = 1, \dots, p-1$, $F_j(\zeta) := E_j(\zeta)$. Note that $\big(F_1(\zeta), \dots, F_{p-1}(\zeta)\big)$ is therefore a set of linearly independent vectors of $E_-(\zeta)$. We set as well $F_j(\zeta) := E_j(\zeta)$ for $j = p+2, \dots, N$, where $E_{p+2}(\zeta), \dots, E_N(\zeta)$ are the vectors of the basis of $\mathbb{C}^N$ defined by $T(\zeta)$ associated with the blocks $\mathcal{A}_+(\zeta)$ and $\mathcal{A}_2(\zeta)$, constituting a set of linearly independent vectors of $E_+(\zeta)$.

The two vectors $F_p(\zeta)$ and $F_{p+1}(\zeta)$ are now to be determined, which are the stable and unstable eigenvectors of $\mathcal{A}(\zeta)$ associated with the block $Q(\zeta)$. If $\xi_-(\zeta), \xi_+(\zeta)$ are the two stable and unstable eigenvalues (equal for ζ glancing) associated with the glancing eigenvalue $\underline{\xi}_g$, then the stable and unstable eigenvectors of $\mathcal{A}(\zeta)$ associated with $\xi_-(\zeta)$ and $\xi_+(\zeta)$ are given by

$$F_p(\zeta) := E_p(\zeta) + \frac{q_2(\zeta)}{\xi_-(\zeta) - \underline{\xi}_g}\,E_{p+1}(\zeta), \qquad F_{p+1}(\zeta) := E_p(\zeta) + \frac{q_2(\zeta)}{\xi_+(\zeta) - \underline{\xi}_g}\,E_{p+1}(\zeta).$$

Indeed, an eigenvector of the matrix $\widetilde{T}(\zeta)^{-1}\,\mathcal{A}(\zeta)\,\widetilde{T}(\zeta)$ associated with the eigenvalue $\xi_\pm(\zeta)$ is given by ${}^t(0, \dots, 0, 1, \frac{q_2(\zeta)}{\xi_\pm(\zeta) - \underline{\xi}_g}, 0, \dots, 0)$. Note that when ζ is glancing, namely when $\xi_-(\zeta) = \xi_+(\zeta)$ and $q_2(\zeta) = 0$, we have $F_p(\zeta) = F_{p+1}(\zeta) = E_p(\zeta)$.

The change-of-basis matrix from the canonical basis of $\mathbb{C}^N$ to the basis $F_1, \dots, F_N$ is therefore given by the product of the matrix $T(\zeta)$ and the block diagonal matrix

$$P(\zeta) := \operatorname{diag}\left(I_{p-1}, \begin{pmatrix} 1 & 1 \\ \dfrac{q_2(\zeta)}{\xi_-(\zeta) - \underline{\xi}_g} & \dfrac{q_2(\zeta)}{\xi_+(\zeta) - \underline{\xi}_g} \end{pmatrix}, I_{N-p-1}\right).$$

Thus the projector $\Pi^-_{\mathbb{C}^N}$ writes

$$\Pi^-_{\mathbb{C}^N} = T(\zeta)\, P(\zeta) \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} P(\zeta)^{-1}\, T(\zeta)^{-1}.$$

The matrix $T(\zeta)$ is analytic and therefore bounded as well as its inverse in a neighborhood of $\underline{\zeta}$. Since it has already been proven that $\frac{q_2(\zeta)}{\xi_-(\zeta)-\underline{\xi}_g}$ and $\frac{q_2(\zeta)}{\xi_+(\zeta)-\underline{\xi}_g}$ are bounded in a neighborhood of $\underline{\zeta}$, the matrix $P(\zeta)$ and ${}^t\mathrm{com}\, P(\zeta)$ are bounded. The determinant $\det P(\zeta)$ shall now be estimated. Since $\xi_-(\zeta)$ and $\xi_+(\zeta)$ are the two (possibly equal) roots of the polynomial $\xi^2 - \big(2\underline{\xi}_g + q_1(\zeta)\big)\,\xi + \underline{\xi}_g^2 + \underline{\xi}_g q_1(\zeta) - q_2(\zeta)$, we obtain

$$\big(\xi_-(\zeta) - \underline{\xi}_g\big)\big(\xi_+(\zeta) - \underline{\xi}_g\big) = -q_2(\zeta),$$

so that

$$\det P(\zeta) = \frac{q_2(\zeta)\big(\xi_-(\zeta) - \xi_+(\zeta)\big)}{\big(\xi_-(\zeta) - \underline{\xi}_g\big)\big(\xi_+(\zeta) - \underline{\xi}_g\big)} = -\big(\xi_-(\zeta) - \xi_+(\zeta)\big).$$

It yields to

$$|\det P(\zeta)| = |\Delta_g(\zeta)|^{1/2} \geqslant C\, \mathrm{dist}(\zeta, \mathcal{G})^{1/2},$$

according to estimate (A.14). The control (A.13b) follows in the same way as the one of (A.13a) using estimate (A.13a) and Assumption 6.

Finally, for estimate (A.13c), taking back the notations and the results of the previous point, the matrix $e^{t\mathcal{A}(\zeta)}\big(I - \Pi^e_{\mathbb{C}^N}(\zeta)\big)$ can be written as

$$e^{t\mathcal{A}(\zeta)}\big(I - \Pi^e_{\mathbb{C}^N}(\zeta)\big) = \widetilde{T}(\zeta)^{-1}\, P(\zeta)^{-1} \begin{pmatrix} 0 & & & & \\ & e^{t\mathcal{A}_1(\zeta)} & & 0 & \\ & & e^{tQ(\zeta)}\mathbb{1}_{\Delta_g(\zeta)>0} & & \\ & 0 & & e^{t\mathcal{A}_+(\zeta)} & \\ & & & & e^{t\mathcal{A}_2(\zeta)} \end{pmatrix} P(\zeta)\, \widetilde{T}(\zeta).$$

Note that the eigenvalues of the matrices $\mathcal{A}_1(\zeta)$, $\mathcal{A}_2(\zeta)$ and of the matrix $Q(\zeta)$ when $\Delta_g(\zeta) > 0$, are imaginary, and that the matrix $\mathcal{A}_+(\zeta)$ is of positive definite real part uniformly with respect to ζ. Thus, using the estimate on the change-of-basis matrix $P(\zeta)^{-1}$ proved above, one may conclude as for estimate (A.13c). □

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Institut de Mathématiques de Toulouse ; UMR5219, Université de Toulouse ; CNRS, UPS, F-31062 Toulouse Cedex 9, France

Email address: `corentin.kilque@math.univ-toulouse.fr`