# EXPECTED NUMBER OF INDUCED SUBTREES SHARED BY TWO INDEPENDENT COPIES OF THE TERMINAL TREE IN A CRITICAL BRANCHING PROCESS 

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#### Abstract

Consider a rooted tree $T$ of minimum degree 2 at least, with leaf-set [ $n$ ]. A rooted tree $\mathcal{T}$ with leaf-set $S \subset[n]$ is induced by $S$ in $T$ if $\mathcal{T}$ is the lowest common ancestor subtree for $S$, with all its degree- 2 vertices suppressed. A "maximum agreement subtree" (MAST) for a pair of two trees $T^{\prime}$ and $T^{\prime \prime}$ is a tree $\mathcal{T}$ with a largest leaf-set $S \subset[n]$ such that $\mathcal{T}$ is induced by $S$ both in $T^{\prime}$ and $T^{\prime \prime}$. Bryant et al. 7] and Bernstein et al. [6] proved, among other results, that for $T^{\prime}$ and $T^{\prime \prime}$ being two independent copies of a random binary (uniform or Yule-Harding distributed) tree $T$, the likely magnitude order of $\operatorname{MAST}\left(T^{\prime}, T^{\prime \prime}\right)$ is $O\left(n^{1 / 2}\right)$. In this paper we prove this bound for a wide class of random rooted trees: $T$ is a terminal tree of a branching process with an offspring distribution of mean 1 , conditioned on "total number of leaves is $n$ ".


## 1. Introduction, Results

Consider a rooted binary tree $T$, with $n$ leaves (pendant vertices) labelled by elements from $[n]$. We visualize this tree with the root on top and the leaves at bottom. Given $S \subset[n]$, let $v(S) \in V(T)$ denote the lowest common ancestor of leaves in $S$, $(\operatorname{LCA}(S)$.) Introduce the subtree of $T$ formed by the paths from $v(S)$ to leaves in $S$. Ignoring (suppressing) degree-2 vertices of this subtree (except the root itself), we obtain a rooted binary tree with leaf-set $S$. This binary tree is called "a tree induced by $S$ in $T$ ".

Finden and Gordon [11] and Gordon [13] introduced a notion of a "maximum agreement subtree" (MAST) for a pair of such trees $T^{\prime}$ and $T^{\prime \prime}$ : it is a tree $\mathcal{T}$ with a largest leaf-set $S \subset[n]$ such that $\mathcal{T}$ is induced by $S$ both in $T^{\prime}$ and $T^{\prime \prime}$. In a pioneering paper [7], Bryant, McKenzie and Steel addressed the problem of a likely order of $\operatorname{MAST}\left(T_{n}^{\prime}, T_{n}^{\prime \prime}\right)$ when $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$ are two independent copies of a random binary tree $T_{n}$. To quote from [7], such a problem is "relevant when comparing evolutionary trees for the same set of species that have been constructed from two quite different types of data".

[^0]It was proved in [7] that $\operatorname{MAST}\left(T_{n}^{\prime}, T_{n}^{\prime \prime}\right) \leq(1+o(1)) e 2^{-1 / 2} n^{1 / 2}$ with probability $1-o(1)$ as $n \rightarrow \infty$. The proof was based on a remarkable property of the uniformly random rooted binary tree, and of few other tree models, known as "sampling consistency", see Aldous [1]. As observed by Aldous [3, 4], sampling consistency makes this model conceptually close to a uniformly random permutation of $[n]$. Combinatorially, it means that a rooted binary tree $\mathcal{T}$ with with a leaf-set $S \subset[n],|S|=s$, is induced by $S$ in exactly $\frac{(2 n-3)!!}{(2 a-3)!!}$ rooted binary trees with leaf-set $[n]$, regardless of choice of $\mathcal{T}$. Probabilistically, the rooted binary tree induced by $S$ in $T_{n}$ is distributed uniformly on the set of all $(2 s-3)!!$ such trees. Mike Steel [23] pointed out that the sampling consistency of the rooted binary tree follows directly from a recursive process for generating all the rooted trees in which $S$ induces $\mathcal{T}$.

Bernstein, Ho, Long, Steel, St. John, and Sullivant [6] established a qualitatively similar upper bound $O\left(n^{1 / 2}\right)$ for the likely size of a common induced subtree in a harder case of Yule-Harding tree, again relying on sampling consistency of this tree model. Recently Misra and Sullivant [19] were able to prove the estimate $\Theta\left(n^{1 / 2}\right)$ for the case when two independent binary trees with $n$ labelled leaves are obtained by selecting independently, and uniformly at random, two leaf-labelings of the same unlabelled tree. Using the classic results on the length of the longest increasing subsequence in the uniformly random permutation, the authors of [6] established a first power-law lower bound $\mathrm{cn}{ }^{1 / 8}$ for the likely size of the common induced subtree in the case of the uniform rooted binary tree, and a lower bound $c n^{a-o(1)}, a=0.344 \ldots$, for the Yule-Harding model. Very recently, Aldous [3] proved that a maximum agreement rooted subtree for two independent, uniform, unrooted trees is likely to have $n^{\frac{\sqrt{3}-1}{2}-o(1)} \approx n^{0.366}$ leaves, at least. It was mentioned in [3] that an upper bound $O\left(n^{1 / 2}\right)$ could be obtained by "the first moment method (calculating the expected number of large common subtrees)".

In this paper we show that the total number of unrooted trees with leafset $[n]$, which contains a rooted subtree induced by $S \subset[n], s<n$, is $\frac{(2 n-5)!!}{(2 s-3)!!}$. It follows that a rooted binary tree induced by $S$ in the uniformly random unrooted tree on $[n]$ is again distributed uniformly on the set of all $(2 s-3)!!$ rooted trees. Using the asymptotic estimate from [7], we have: a maximum agreement rooted subtree for two independent copies of the uniformly random unrooted tree is likely to have at most ( $1+o(1)) e 2^{-1 / 2} n^{1 / 2}$ leaves.

The proof of this $\frac{(2 n-5)!!}{(2 s-3)!!}$ result suggested that a bound $O\left(n^{1 / 2}\right)$ might, just might, be obtained for a broad class of random rooted trees that includes the rooted binary tree, by using a probabilistic counterpart of the two-phase counting procedure. Consider a Markov branching process with a given
offspring distribution $\mathbf{p}=\left\{p_{j}\right\}_{j \geq 0}$. If $p_{0}>0$ and $\sum_{j} j p_{j}=1$ (critical case), then the process is almost surely extinct.

Let $T_{t}$ be the random terminal tree, and let $T_{n}$ be $T_{t}$ conditioned on the event " $T_{t}$ has $n$ leaves", which we label, uniformly at random, by elements of $[n]$. The uniform binary rooted tree is a special case corresponding to $p_{0}=p_{2}=1 / 2$. In general, we assume that $p_{1}=0$, g.c.d. $\left(j: p_{j+1}>0\right)=1$, and that $P(s):=\sum_{j} p_{j} s^{j}$ has convergence radius $R>1$, all the conditions being met by the binary case. We will show that $\mathbb{P}_{n}:=\mathbb{P}\left(T_{t}\right.$ has $n$ leaves $)>$ 0 for all $n$, meaning that $T_{n}$ is well-defined for all $n$.

Finally, an out-degree of a vertex in $T_{n}$ may now exceed 2. So we add to the definition of a tree $\mathcal{T}$, induced by $S$ in $T_{n}$, a condition: the out-degree of every vertex from $V(\mathcal{T})$ in $T_{n}$ is the same as its out-degree in $\mathcal{T}$.

Under the conditions above, we prove that a rooted binary tree $\mathcal{T}$ with leaf-set $S \subset[n]$, and edge-set $E(\mathcal{T})$, is induced by $S$ in $T_{n}$ with probability

$$
\begin{gathered}
\frac{(n-s)!}{n!\mathbb{P}_{n}} \mathbb{P}(\mathcal{T})\left[y^{n-s}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y), \quad(e(\mathcal{T}):=|E(\mathcal{T})|), \\
\Phi(y)=P(\Phi(y))+p_{0}(y-1), \quad \Phi_{1}(y)=\left(1-p_{0}\right)^{-1} \sum_{j>1} p_{j} \Phi^{j-1}(y) ; \\
\mathbb{P}(\mathcal{T})=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})} ; \quad d(v, \mathcal{T}):=\text { out-degree of } v \text { in } \mathcal{T} .
\end{gathered}
$$

Here $\Phi(y)\left(\Phi_{1}(y)\right.$, respectively) is the probability generating function of the total number of leaves in the terminal tree (conditioned on the event "the progenitor has at least two children", respectively.). We will check that for $p_{0}=p_{2}=1 / 2$ this formula reduces to $\frac{1}{(2 s-3)!!}$. Note that in general, because of the factor $\mathbb{P}(\mathcal{T})$, and $e(\mathcal{T})$, the probability of $\mathcal{T}$ being induced by $S$ in $T_{n}$ depends not only on $|S|$, but also on the whole out-degree sequence of $\mathcal{T}$.

We use the above identity to prove the following claim. Let

$$
\begin{gathered}
c(\mathbf{p}):=\lambda e^{3 / 2}\left(1-\sum_{j=2}^{\infty} p_{j}^{2}\right)^{1 / 2} \\
\lambda:=\max \left(\chi^{-4}, \chi^{-2}\right), \quad \chi:=\left(2 p_{0}^{3}\right)^{-1 / 2}\left(1-p_{0}\right) \sigma, \quad \sigma^{2}:=\sum_{j=2}^{\infty} j(j-1) p_{j} .
\end{gathered}
$$

Then, for $\varepsilon \in(0,1 / 2]$, with probability $\geq 1-(1-\varepsilon)^{(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}}$, the largest number of leaves in an induced subtree shared by two independent copies of the conditioned terminal tree $T_{n}$ is at most $(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$.

For a wide ranging exposition of combinatorial/probabilistic problems and methods in theory of phylogeny, we refer the reader to a book [22] by Steel.

## 2. UNIFORM BINARY TREES

Consider a rooted binary tree $T$ with leaf-set $[n]$. For a given $S \subset[n]$, there exists a subtree with leaf-set $S$, which is rooted at the lowest vertex common to all $|S|$ paths leading away from $S$ toward the root of $T$. The vertex set of this lowest common ancestor (LCA) tree is the set of all vertices from the paths in question. Ignoring degree- 2 vertices of this subtree (except the root itself), we obtain a rooted binary tree $\mathcal{T}$. This LCA subtree has a name "a tree induced by $S$ in $T$ ", see [3].

Let $T_{n}^{\prime}, T_{n}^{\prime \prime}$ be two independent copies of the uniformly random rooted binary tree with leaf-set $[n]$. Let $X_{n, a}$ denote the random total number of leaf-sets $S \subset[n]$ of cardinality $a$ that induce the same rooted subtree in $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$. Bernstein et al. [6] proved that

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\frac{\binom{n}{a}}{(2 a-3)!!} \tag{2.1}
\end{equation*}
$$

The proof was based on sampling consistency of the random tree $T_{n}$, so that $N(\mathcal{T})$, the number of rooted trees on $[n]$ in which $S$ induces a given rooted tree $\mathcal{T}$ on $S$, is $\frac{(2 n-3)!!}{(2 a-3)!!}$, thus dependent only on the leaf-set size.

Following [3] (see Introduction), we consider the case when a binary tree with leaf-set $[n]$ is unrooted. Let now $T_{n}^{\prime}, T_{n}^{\prime \prime}$ be two independent copies of the uniformly random (unrooted) binary tree with leaf-set $[n]$. Let $X_{n, a}$ denote the random total number of leaf-sets $S \subset[n]$ of cardinality $a$ that induce the same rooted subtree in $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$.

Lemma 2.1. Let $a=|S|<n$. Then

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\frac{\binom{n}{a}}{(2 a-3)!!} . \tag{2.2}
\end{equation*}
$$

Equivalently $\mathcal{N}(\mathcal{T})$, the number of unrooted trees $T$ on $[n]$, in which $S$ induces a given rooted tree $\mathcal{T}$ with leaf set $S$, is $\frac{(2 n-5)!!}{(2 a-3)!!}$.

Note. So the expectation is the same as for the rooted trees $T$ on $[n]$.
Proof. For an unrooted tree $T$ with $n$ leaves, the notion of a rooted subtree induced by a leaf-set $S$ with $|S|=a>1$ makes sense only for $a<n$, and this subtree uniquely exists for any such $S$. Indeed a vertex $v$ adjacent to any fixed leaf $\ell^{*} \in[n] \backslash S$ is joined by a unique path to each leaf in $S$. Any other vertex $v^{\prime}$ common to some $a$ paths emanating from $a$ leaves must be common to all the paths from $S$ to $v$. It follows that there exists a unique vertex $v^{*}$ which is the LCA of the $a$ leaves. The paths from $v^{*}$ to $S$ form the subtree induced by $S$, and $\ell^{*}$ is connected by an external path to $v^{*}$, if $\ell^{*} \neq v^{*}$.

Let us evaluate $\mathcal{N}(\mathcal{T})$. Consider a generic rooted tree with $a$ leaves. For $\mathcal{T}$ to be induced by its leaves in $T$ with $n$ leaves, it has to be obtained by ignoring degree-2 (non-root) vertices in the LCA subtree for leaf-set $S$.

The outside (third) neighbors of the ignored vertices are the roots of subtrees with some $b$ leaves from the remaining $n-a$ leaves, selected in $\binom{n-a}{b}$ ways. The roots of possible trees, attached to internal points chosen from some of $2(a-1)$ edges of $\mathcal{T}$, can be easily ordered. Introduce $F(b, k)$, the total number of ordered forests of $k$ rooted trees with $b$ leaves altogether. By Lemma 4 of Carter et al. [9] (for the count of unordered trees), we have

$$
\begin{equation*}
F(b, k)=\frac{k(2 b-k-1)!}{(b-k)!2^{b-k}} \tag{2.3}
\end{equation*}
$$

It was indicated in [6] that (2.3) follows from

$$
\begin{equation*}
F(b, k)=b!\cdot\left[x^{b}\right] B(x)^{k}, \quad B(x):=1-\sqrt{1-2 x} \tag{2.4}
\end{equation*}
$$

(Semple and Steel [21]). For the reader's convenience here is a sketch proof of (2.4) and (2.3). We have

$$
\begin{aligned}
F(b, k) & =b!\sum_{t_{1}+\cdots+t_{k}=b} \prod_{j \in[k]} \frac{\left(2 t_{j}-3\right)!!}{t_{j}!}=b!\sum_{t_{1}+\cdots+t_{k}=b} \prod_{j \in[k]} \frac{1}{t_{j} 2^{t_{j}-1}}\binom{2\left(t_{j}-1\right)}{t_{j}-1} \\
& =b!\left[x^{b}\right]\left[\sum_{t \geq 1} \frac{x^{t}}{t 2^{t-1}}\binom{2(t-1)}{t-1}\right]^{k}=b!\left[x^{b}\right] B(x)^{k}=\frac{k(2 b-k-1)!}{(b-k)!2^{b-k}}
\end{aligned}
$$

for the last two steps we used Equations (2.5.10), (2.5.16) in Wilf [24].
Introduce $\mathcal{F}(b, k)$, the total number of the ordered forests of $k$ binary trees with roots attached to internal points of $\mathcal{T}$ 's edges, with $b$ leaves altogether. (These leaves have to be chosen from $[n] \backslash\left(S \cup\left\{\ell^{*}\right\}\right.$, so $b \leq n-a-1$. ) Since the total number of integer compositions (ordered partitions) of $k$ with $j \leq 2(a-1)$ positive parts is

$$
\binom{k-1}{j-1}\binom{2(a-1)}{j}=\binom{k-1}{j-1}\binom{2(a-1)}{2(a-1)-j}
$$

(2.3) implies

$$
\begin{align*}
\mathcal{F}(b, k)=F(b, k) & \sum_{j \leq 2(a-1)}\binom{k-1}{j-1}\binom{2(a-1)}{2(a-1)-j}  \tag{2.5}\\
= & F(b, k)\binom{k+2 a-3}{2 a-3}=b!\binom{k+2 a-3}{2 a-3}\left[x^{b}\right] B(x)^{k} .
\end{align*}
$$

Now, $\sum_{k \leq b} \mathcal{F}(b, k)$ is the total number of ways to expand the host subtree into a full binary subtree rooted at the lowest common ancestor of the $a$
leaves. To evaluate this sum, first denote $\alpha=2 a-3, \beta=B(x)$ and write

$$
\sum_{k \geq 0}\binom{k+\alpha}{\alpha} \beta^{k}=\sum_{k \geq 0}(-\beta)^{k}\binom{-\alpha-1}{k}=(1-\beta)^{-\alpha-1}
$$

Therefore

$$
\begin{aligned}
\sum_{k \leq b}\binom{k+\alpha}{\alpha}\left[x^{b}\right] B(x)^{k} & =\left[x^{b}\right] \sum_{k \geq 0}\binom{k+\alpha}{\alpha} B(x)^{k}=\left[x^{b}\right] \frac{1}{(1-B(x))^{\alpha+1}} \\
& =\left[x^{b}\right](1-2 x)^{-\frac{\alpha+1}{2}}
\end{aligned}
$$

We conclude that

$$
\begin{equation*}
\sum_{k \leq b} \mathcal{F}(b, k)=\left.b!\left[x^{b}\right](1-2 x)^{-\frac{\alpha+1}{2}}\right|_{\alpha=2 a-3}=b!\left[x^{b}\right](1-2 x)^{-(a-1)} . \tag{2.6}
\end{equation*}
$$

Recall that $b$ leaves were chosen from $[n] \backslash\left(S \cup\left\{\ell^{*}\right\}\right)$. If $b=n-a-1$, then attaching the single remaining leaf to the root $v^{*}$ we get a binary tree $T$ with leaf-set $[n]$. If $b \leq n-a-2$, we view the expanded subtree as a single leaf, and form an unrooted binary tree with $1+(n-a-b) \geq 3$ leaves, in $[2(n-a-b)-3]!$ ! ways. Therefore $\mathcal{N}(\mathcal{T})$ depends on $a$ only, and with $\nu:=n-a-1$, it is given by

$$
\begin{aligned}
\mathcal{N}(\mathcal{T})=\sum_{b \leq \nu} & \binom{\nu}{b} b!\left[x^{b}\right](1-2 x)^{-(a-1)}(2(\nu-b)-1)!! \\
& =\nu!\sum_{b \leq \nu}\left[x^{b}\right](1-2 x)^{-(a-1)} \cdot\left[x^{\nu-b}\right](1-2 x)^{-1 / 2} \\
& =\nu!\left[x^{\nu}\right](1-2 x)^{-a+1 / 2}=\prod_{j=0}^{n-a-2}(2 a-1+2 j)=\frac{(2 n-5)!!}{(2 a-3)!!}
\end{aligned}
$$

in the first line $(-1)!!:=1$, and for the second line we used

$$
\frac{(2 k-1)!!}{k!}=\frac{(2 k)!}{2^{k}(k!)^{2}}=2^{-k}\binom{2 k}{k}=\left[x^{k}\right](1-2 x)^{-1 / 2} .
$$

Consequently

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\binom{n}{a}(2 a-3)!!\left[\frac{\mathcal{N}(\mathcal{T})}{(2 n-5)!!}\right]^{2}=\frac{\binom{n}{a}}{(2 a-3)!!} . \tag{2.7}
\end{equation*}
$$

## 3. Branching Process Framework

Consider a branching process initiated by a single progenitor. This process is visualized as a growing rooted tree. The root is the progenitor, connected by edges to each of the vertices, that represent the root's "children", i.e. the root's immediate offspring. Each of the children becomes the root of the
corresponding (sub)tree, so that the ordered children of all these roots are the grandchildren of the progenitor. We obviously get a recursively defined process; it delivers a nested sequence of trees, which is either infinite, or terminates at a moment when none of the members of the last generation have children.

The classic Galton-Watson branching process is the case when the number of each member's children (a) is independent of those numbers for all members from the preceding and current generations and (b) has the same distribution $\left\{p_{j}\right\}_{j \geq 0},\left(\sum_{j} p_{j}=1\right)$. It is well-known that if $p_{0}>0$ and $\sum_{j \geq 0} j p_{j}=1$, then the process terminates with probability 1, Harris [15]. Let $T_{t}$ denote the terminal tree. Given a finite rooted tree, $T$, we have

$$
\mathbb{P}\left(T_{t}=T\right)=\prod_{v \in V(T)} p_{d(v, T)},
$$

where $d(v, T)$ is the out-degree of vertex $v \in V(T) . \quad X_{t}:=\left|V\left(T_{t}\right)\right|$, the total population size by the extinction time, has the probability generating function (p.g.f) $F(x):=\mathbb{E}\left[x^{X_{t}}\right],|x| \leq 1$, that satisfies

$$
\begin{equation*}
F(x)=x P(F(x)), \quad P(\xi):=\sum_{j \geq 0} p_{j} \xi^{j},(|\xi| \leq 1) \tag{3.1}
\end{equation*}
$$

Indeed, introducing $F_{\tau}(x)$ the p.g.f. of the total cardinality of the first $\tau$ generations, we have

$$
F_{\tau+1}(x)=x \sum_{j \geq 0} p_{j}\left[F_{\tau}(x)\right]^{j}=x P\left(F_{\tau}(x)\right) .
$$

So letting $\tau \rightarrow \infty$, we obtain (3.1). In the same vein, consider the pair $\left(X_{t}, Y_{t}\right)$, where $Y_{t}:\left|\left\{v \in V\left(T_{t}\right): d\left(v, T_{t}\right)=0\right\}\right|$ is the total number of leaves (zero out-degree vertices) of the terminal tree. Then denoting $G(x, y)=$ $\mathbb{E}\left[x^{X_{t}} y^{Y_{t}}\right],(|x|,|y| \leq 1)$, we have

$$
\begin{equation*}
G(x, y)=p_{0} x y+x \sum_{j \geq 1} p_{j}[G(x, y)]^{j}=x P(G(x, y))+p_{0} x(y-1) . \tag{3.2}
\end{equation*}
$$

So, with $\Phi(y):=\mathbb{E}\left[y^{Y}\right]=G(1, y)$, we get

$$
\begin{equation*}
\Phi(y)=\sum_{j \geq 1} p_{j} \Phi^{j}(y)+p_{0} y=P(\Phi(y))+p_{0}(y-1) . \tag{3.3}
\end{equation*}
$$

Importantly, this identity allows us to deal directly with the leaf set at the extinction moment: $\mathbb{P}_{k}:=\left[y^{k}\right] \Phi(y)$ is the probability that $T_{t}$ has $k$ leaves. In particular, $\mathbb{P}_{1}=\left[y^{1}\right] \Phi(y)=p_{0}>0$. More generally, $\mathbb{P}_{k}>0$ for all $k \geq 1$. meaning that $\mathbb{P}\left(T_{t}\right.$ has $k$ leaves $)>0$ for all $k \geq 1$. Indeed, for $k \geq 2$, we
have

$$
\mathbb{P}_{k}=\sum_{j \geq 1} p_{j} \sum_{\substack{k_{1}+\cdots+k_{j}=k \\ k_{1}, \ldots, k_{j} \geq 1}} \mathbb{P}_{k_{1}} \cdots \mathbb{P}_{k_{j}} ;
$$

so the claim follows by easy induction on $k$.
If $p_{0}=p_{2}=1 / 2$, then the branching process is a nested sequence of binary trees. The equation (3.3) yields

$$
\Phi(y)=1-(1-y)^{1 / 2}=\sum_{n \geq 1}\left(\frac{y}{2}\right)^{n} \frac{(2 n-3)!!}{n!}, \quad|y| \leq 1 .
$$

So the terminal tree $T_{t}$ has $n$ leaves with probability $\frac{(2 n-3)!!}{2^{n} n!}>0$. On this event, call it $A_{n}$, the total number of vertices is $2 n-1$, and each of rooted binary trees with $2 n-1$ vertices is a value of the terminal tree of the same probability $(1 / 2)^{2 n-1}$. Conditionally on the event $A_{n}$, we label, uniformly at random, the leaves of $T_{t}$ by elements of $[n]$ and use notation $T_{n}$ for the resulting uniformly random, rooted binary tree.

This is a promising sign that we can extend what we did for the uniformly random binary trees, i.e. for $p_{0}=p_{2}=1 / 2$, for a more general offspring distribution $\left\{p_{j}\right\}$.

We continue to assume that $p_{1}=0$. The notion of an induced subtree needs to be expanded, since an out-degree of a vertex now may exceed 2 . Let $\mathcal{T}$ be a tree with a leaf-set $S \subset[n]$, such that every non-leaf vertex of $\mathcal{T}$ has out-degree 2 , at least. We say that $S$ induces $\mathcal{T}$ in a tree $T_{n}$ provided that: (a) the LCA subtree for $S$ in $T_{n}$ is $\mathcal{T}$ if we ignore vertices of total degree 2 in this LCA subtree; (b) the out-degree of every other vertex in the LCA of $S$ in $T_{n}$ is the same as its out-degree in $\mathcal{T}$.

Let $\mathcal{T}$ be a tree with the leaf-set $S,|S|=a<n, b \leq n-a$. Let $A_{n}(\mathcal{T}, b) \subset A_{n}$ be the event: (i) some $b$ elements from $[n] \backslash S$ are chosen as the leaves of all the complementary subtrees rooted at degree -2 vertices sprinkled on the edges of $\mathcal{T}$, forming a composed tree with $a+b$ leaves; (ii) the tree with $n$ leaves is obtained by using the remaining $n-a-b$ leaves and an extra leaf which is the root of the tree composed in step (i). The event $A_{n}(\mathcal{T}, b)$ is partitioned into disjoint $\binom{n-a}{b}$ events corresponding to all choices to select $b$ elements of $[n] \backslash S$ in question.

Let $e(\mathcal{T})$ be the total number of edges in $\mathcal{T}$. For each of these choices, on the event $A_{n}(\mathcal{T}, b)$ we must have some $k \leq b$ trees with ordered roots on some of $e(\mathcal{T})$ edges, and with their respective, nonempty, leaf-set labels forming an ordered set partition of the set of $b$ leaves. The root of each of these trees has one child down the host "edge" of $\mathcal{T}$, and all the remaining children outside edges of $\mathcal{T}$. Since $p_{1}=0$, the number of other children of a root is $j$ with conditional probability $\left(1-p_{0}\right)^{-1} p_{j+1}$. So the number of
leaves of the subtrees rooted at the children is $i$ with probability $\left[y^{i}\right] \Phi_{1}(y)$, where

$$
\begin{equation*}
\Phi_{1}(y)=\left(1-p_{0}\right)^{-1} \sum_{j \geq 1} p_{j+1} \Phi^{j}(y) . \tag{3.4}
\end{equation*}
$$

Therefore, conditionally on " $S$ induces $\mathcal{T}$ ", a given set of $b$ elements of $[n]$ is the leaf-set of these complementary trees with probability

$$
\begin{aligned}
& b!\sum_{j \leq k \leq b}\binom{k-1}{j-1}\binom{e(\mathcal{T})}{j} \sum_{b_{1}+\cdots+b_{k}=b} \prod_{t=1}^{k}\left[y^{b_{t}}\right] \Phi_{1}(y) \\
& =b!\sum_{j \leq k \leq b}\binom{k-1}{j-1}\binom{e(\mathcal{T})}{e(\mathcal{T})-j}\left[y^{b}\right] \Phi_{1}^{k}(y)=b!\left[y^{b}\right] \sum_{k}\binom{k+e(\mathcal{T})-1}{k} \Phi_{1}^{k}(y) \\
& =b!\left[y^{b}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} .
\end{aligned}
$$

Explanation: $k$ is a generic total number of trees rooted at ordered internal points of some $j$ edges of $\mathcal{T} ; b_{t}$ is a generic number of leaves of a $t$-th tree; the product of two binomial coefficients in the top line is the number of ways to pick $j$ edges of $\mathcal{T}$ and to select an ordered, $j$-long, composition of $k$; the sum is the probability that the $k$ trees have $b$ leaves in total.

With these complementary trees attached, we obtain a tree with $a+b$ leaves. So for $A_{n}(\mathcal{T}, b)$ to hold, we view the root of this tree (i.e. the root of $\mathcal{T}$ ) as a leaf and complete determination of a tree with $n$ leaves by constructing an auxiliary tree with 1 plus $(n-a-b)$ remaining leaves. Therefore using the definition of $\Phi(y)$, we have

$$
\begin{aligned}
\mathbb{P}\left(A_{n}(\mathcal{T}, b)\right)= & \frac{1}{n!}\binom{n-a}{b} \times b!\left[y^{b}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \\
& \times \mathbb{P}(\mathcal{T}) \cdot(n-a-b+1)!\left[y^{n-a-b+1}\right] \Phi(y)
\end{aligned}
$$

Here $\mathbb{P}(\mathcal{T}):=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})}$, where $\{d(v, \mathcal{T})\}$ is the out-degree sequence of non-leaf vertices of $\mathcal{T}$.

Using $j\left[y^{j}\right] \Phi(y)=\left[y^{j-1}\right] \Phi^{\prime}(y), j \geq 1$, and summing the last equation for $0 \leq b \leq n-a$, we obtain

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{(n-a)!}{n!} \mathbb{P}(\mathcal{T})\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \tag{3.5}
\end{equation*}
$$

For a partial check of (3.5), let us return to $p_{0}=p_{2}=1 / 2$. Here

$$
\mathbb{P}(\mathcal{T})=p_{2}^{a-1}=2^{-a+1}, \quad \Phi(y)=1-(1-y)^{1 / 2}, \quad \Phi_{1}(y)=\Phi(y)
$$

Therefore

$$
\begin{aligned}
& {\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y)=\left[y^{n-a}\right](1-\Phi(y))^{-2 a+2} \Phi^{\prime}(y) } \\
&=\left[y^{n-a}\right](1-y)^{-a+1} \cdot \frac{1}{2}(1-y)^{-1 / 2}=\frac{1}{2}\left[y^{n-a}\right](1-y)^{-a+1 / 2} \\
&=2^{-n+a-1} \frac{(2 n-3)!!}{(n-a)!(2 a-3)!!}
\end{aligned}
$$

so, by (3.5), we have

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{(2 n-3)!!}{2^{n} n!(2 a-3)!!} \tag{3.6}
\end{equation*}
$$

Since $\mathbb{P}\left(A_{n}\right)=\frac{(2 n-3)!!}{2^{n} n!}$, we conclude that

$$
\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right)=\frac{\mathbb{P}\left(A_{n}(\mathcal{T})\right)}{\mathbb{P}\left(A_{n}\right)}=\frac{1}{(2 a-3)!!},
$$

for every binary tree $\mathcal{T}$ with leaf-set $S \subset[n],|S|=a$. The LHS is the probability that $S$ induces $\mathcal{T}$ in the uniformly random binary tree $T_{n}$.

To summarize, we proved
Lemma 3.1. Consider the branching process with the immediate offspring distribution $\left\{p_{j}\right\}$, such that $p_{0}>0, p_{1}=0$, and $\sum_{j \geq 0} j p_{j}=1$. With probability 1 , the process eventually stops, so that a finite terminal tree $T_{t}$ is a.s. well-defined. (1) Setting $A_{n}=\left\{T_{t}\right.$ has $n$ leaves $\}$, we have

$$
\mathbb{P}\left(A_{n}\right)=\left[y^{n}\right] \Phi(y), \quad \Phi(y)=P(\Phi(y))+p_{0}(y-1),
$$

where $P(\eta):=\sum_{j \geq 0} \eta^{j} p_{j}$. (2) On the event $A_{n}$, we define $T_{n}$ as the tree $T_{t}$ with leaves labelled uniformly at random by elements from $[n]$. Given a rooted tree $\mathcal{T}$ with leaf-set $S \subset[n]$, set $A_{n}(\mathcal{T}):=$ " $A_{n}$ holds and $S$ induces $\mathcal{T}$ in $T_{n} "$. Then $\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right)=\frac{\mathbb{P}\left(A_{n}(\mathcal{T})\right)}{\mathbb{P}\left(A_{n}\right)}$, where

$$
\begin{gather*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{(n-a)!}{n!} \mathbb{P}(\mathcal{T})\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y), \\
\Phi(y)=P(\Phi(y))+p_{0}(y-1), \quad \Phi_{1}(y)=\left(1-p_{0}\right)^{-1} \sum_{j>1} p_{j} \Phi^{j-1}(y) ;  \tag{3.7}\\
\mathbb{P}(\mathcal{T})=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})} ;
\end{gather*}
$$

$V_{\text {int }}(\mathcal{T})$ and $e(\mathcal{T})$ are the set of non-leaves and the number of edges of $\mathcal{T}$.
Note. For $p_{0}=p_{2}=1 / 2, \mathbb{P}\left(A_{n}(\mathcal{T})\right)$ turned out to be dependent only on the number of leaves of $\mathcal{T}$. The formula (3.7) clearly shows that, in general, this probability depends on shape of $\mathcal{T}$. Fortunately, this dependence is confined to a single factor $\mathbb{P}(\mathcal{T})$, since the rest depends on two scalars, $a$
and $e(\mathcal{T})$. Importantly, these quantities are of the same order of magnitude. Indeed, if $\mathbb{P}(\mathcal{T})>0$ then $e(\mathcal{T}) \geq \max \left(a, 2\left|V_{\text {int }}(\mathcal{T})\right|\right)$, i.e.

$$
a \leq e(\mathcal{T})=|V(\mathcal{T})|-1=\left|V_{\mathrm{int}}(\mathcal{T})\right|+a-1 \leq \frac{e(\mathcal{T})}{2}+a-1
$$

so that

$$
\begin{equation*}
a \leq e(\mathcal{T}) \leq 2(a-1) \tag{3.8}
\end{equation*}
$$

3.1. Asymptotics. From now on we assume that the series $\sum_{j} p_{j} s^{j}$ has convergence radius $R>1$.

Lemma 3.2. Suppose that $d:=$ g.c.d. $\left\{j \geq 1: p_{j+1}>0\right\}=1$. Let $\sigma^{2}:=$ $\sum_{j \geq 0} j(j-1) p_{j}$, i.e. $\sigma^{2}$ is the variance of the immediate offspring, since $\sum_{j \geq 0} j p_{j}=1$. Then

$$
\mathbb{P}\left(A_{n}\right)=\frac{\left(2 p_{0}\right)^{1 / 2}}{\sigma} \frac{(2 n-3)!!}{2^{n} n!}+O\left(n^{-2}\right)=\left(\frac{p_{0}}{2 \pi \sigma^{2}}\right)^{1 / 2} n^{-3 / 2}+O\left(n^{-2}\right) .
$$

Proof. According to Lemma 3.1, we need to determine an asymptotic behavior of the coefficient in the power series $\Phi(z)=\sum_{n \geq 1} z^{n} \mathbb{P}\left(A_{n}\right)$, where $\Phi(z)$ is given implicitly by the functional equation $\Phi(z)=\sum_{j \geq 0} p_{j} \Phi^{j}(z)+$ $p_{0}(z-1),(|z| \leq 1)$.

In 1974 Bender [5] sketched a proof of the following general claim.
Theorem 3.3. Assume that the power series $w(z)=\sum_{n} a_{n} z^{n}$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose that there exist $r>0$ and $s>0$ such that: (i) for some $R>r$ and $S>s, F(z, w)$ is analytic for $|z|<R$ and $w<S$; (ii) $F(r, s)=F_{w}(r, s)=0$; (iii) $F_{z}(r, s) \neq 0$ and $F_{w w}(r, s) \neq 0$; (iv) if $|z| \leq r,|w| \leq s$, and $F(z, w)=F_{w}(z, w)=0$, then $z=r$ and $w=s$. Then $a_{n} \sim\left(\left(r F_{z}(r, s)\right) /\left(2 \pi F_{w w}(r, s)\right)\right)^{1 / 2} n^{-3 / 2} r^{-n}$.

The remainder term aside, that's exactly what we claim in Lemma 3.2 for our $\Phi(z)$. The proof in [5] relied on an appealing conjecture that, under the conditions (i)-(iv), $r$ is the radius of convergence for the power series for $w(z)$, and $z=r$ is the only singularity for $w(z)$ on the circle $|z|=r$. However, ten years later Canfield [8] found an example of $F(z, w)$ meeting the four conditions in which $r$ and the radius of convergence for $w(z)$ are not the same. Later Meir and Moon found some conditions sufficient for validity of the conjecture. Our equation $\Phi(z)=P(\Phi(z))+p_{0}(z-1)$ is a special case of $w=\phi(w)+h(z)$ considered in [17. For the conditions from [17 to work in our case, it would have been necessary to have $\left|P^{\prime}(w) / P(w)\right| \leq 1$ for complex $w$ with $|w| \leq 1$, a strong condition difficult to check. (An interesting discussion of these issues can be found in an encyclopedic book by Flajolet and Sedgewick [12] and an authoritative survey by Odlyzko [20].)

Let $r$ be the convergence radius for the powers series representing $\Phi(z)$; so that $r \geq 1$, since $\mathbb{P}\left(A_{n}\right) \leq 1$. By implicit differentiation, we have

$$
\lim _{x \uparrow 1} \Phi^{\prime}(x)=\lim _{x \uparrow 1} \frac{p_{0}}{1-\mathbb{P}_{w}(\Phi(x))}=\infty,
$$

since $\lim _{x \uparrow 1} P_{w}(\Phi(x))=\sum_{j} j p_{j}=1$. Therefore $r=1$. Turn to complex $z$. For $|z|<1$, we have $F(z, \Phi(z))=0$, where $F(z, w):=p_{0}(z-1)+P(w)-w$ is analytic as a function of $z$ and $w$ subject to $|w|<1$. Observe that $F_{w}(z, w)=P^{\prime}(w)-1=0$ is possible only if $|w| \geq 1$, since for $|w|<1$ we have $\left|P^{\prime}(w)\right| \leq P^{\prime}(|w|)<P^{\prime}(1)=1$. If $|w|=1$ then $P^{\prime}(w)=\sum_{j \geq 2} j p_{j} w^{j-1}=1$ if and only if $w=w_{k}:=\exp \left(i \frac{2 \pi k}{d}\right)$, and $k=1, \ldots, d$. Notice also that

$$
\begin{equation*}
P\left(w_{k}\right)=\sum_{j \geq 0} p_{j} w_{k}^{j}=p_{0}+w_{k} \sum_{j \geq 2} p_{j} w_{k}^{j-1}=p_{0}+w_{k}\left(1-p_{0}\right) . \tag{3.9}
\end{equation*}
$$

Now, $z$ is a singular point of $\Phi(z)$ if and only if $|z|=1$ and $\left(P^{\prime}(w)-\right.$ 1) $\left.\right|_{w=\Phi(z)}=0$, i.e. if and only if $\Phi(z)=w_{k}$ for some $k \in[d]$, which is equivalent to

$$
p_{0}(z-1)+P\left(w_{k}\right)-w_{k}=0 .
$$

Combination of this condition with (3.9) yields $z=w_{k}$. Therefore the set of all singular points of $\Phi(z)$ on the circle $|z|=1$ is the set of all $w_{k}$ such that $\Phi\left(w_{k}\right)=w_{k}$. Notice that

$$
P^{\prime \prime}\left(w_{k}\right)=w_{k}^{-1} \sum_{j \geq 2} j(j-1) p_{j} w_{k}^{j-1}=w_{k}^{-1} \sum_{j \geq 2} j(j-1) p_{j}=w_{k}^{-1} \sigma^{2} \neq 0 .
$$

So none of $w_{k}$ is an accumulation point of roots of $P^{\prime}(w)-1$ outside the circle $|w|=1$, i.e. $\left\{w_{k}\right\}_{k \in[d]}$ is the full root set of $P^{\prime}(w)-1$ inside the circle $|w|=1+\rho_{0}$, for some small $\rho_{0}>0$.

Consequently, if $d=1$, then $z=1$ is the only singular point of $\Phi(z)$ on the circle $|z|=1$. Define the argument $\arg (z)$ by the condition $\arg (z) \in[0,2 \pi)$. By the analytic implicit function theorem applied to $F(z, w)$, for every $z_{\alpha}=$ $e^{i \alpha}, \varepsilon \leq \alpha \leq 2 \pi-\varepsilon$, a small $\varepsilon>0$ being fixed, there exists an analytic function $\Psi_{\alpha}(z)$ defined on $D_{z_{\alpha}}(\rho)$-an open disc centered at $z_{\alpha}$, of a radius $\rho=\rho(\varepsilon)<\rho_{0}$ small enough to make $\varepsilon / 2 \leq \arg (z) \leq 2 \pi-\varepsilon / 2$ for all $z \in D_{z_{\alpha}-}$ such that $F\left(z, \Psi_{\alpha}(z)\right)=0$ for $z \in D_{z_{\alpha}}$ and $\Psi_{\alpha}(z)=\Phi(z)$ for $z \in D_{z_{\alpha}}$ with $|z| \leq 1$. Together, these local analytic continuations determine an analytic continuation of $\Phi(z)$ to a function $\hat{\Psi}(z)$ determined, and bounded, for $z$ with $|z|<1+\rho, \arg (z) \in[\varepsilon, 2 \pi-\varepsilon]$.

Since $z_{0}=1$ is the singular point of $\Phi(z)$, there is no analytic continuation of $\Phi(z)$ for $|z|>1$ and $|z-1|$ small. So instead we delete an interval $[1,1+\rho)$ and continue $\Phi(z)$ analytically into the remaining part of a disc centered at 1. Here is how. We have $F_{w w}(1, \Phi(1))=P^{\prime \prime}(1)=\sum_{j} j(j-1) p_{j}=\sigma^{2}>0$. By a "preparation" theorem due to Weierstrass, (Ebeling [10], Krantz [16]),
already used by Bender [5] for the same purpose in the general setting, there exist two open discs $D_{1}$ and $\mathcal{D}_{1}$ such that for $z \in D_{1}$ and $w \in \mathcal{D}_{1}$ we have

$$
F(z, w)=\left[(w-1)^{2}+c_{1}(z)(w-1)+c_{2}(z)\right] g(z, w),
$$

where $c_{j}(z)$ are analytic on $D_{1}, c_{j}(1)=0$, and $g(z, w)$ is analytic, nonvanishing, on $D_{1} \times \mathcal{D}_{1}$. (The degree 2 of the polynomial is exactly the order of the first non-vanishing derivative of $F(z, w)$ with respect to $w$ at $(1,1)$.) So for $z \in D_{1}$ and $w \in \mathcal{D}_{1}, F(z, w)=0$ is equivalent to

$$
(w-1)^{2}+c_{1}(z)(w-1)+c_{2}(z)=0 \Longrightarrow w=1+(z-1)^{1 / 2} h(z),
$$

where $h(z)$ is analytic at $z=1$. Plugging the power series $w=1+(z-$ $1)^{1 / 2} h(z)=1+(z-1)^{1 / 2} \sum_{j \geq 0} h_{j}(z-1)^{j}$ into equation $F(z, w)=0$, and expanding $P(w)$ in powers of $w-1$, we can compute the coefficients $h_{j}$. In particular,

$$
w(z)=1-\gamma(1-z)^{1 / 2}+O(|z-1|), \quad \gamma:=\left(2 p_{0}\right)^{1 / 2} \sigma^{-1} .
$$

For $z$ real and $z \in(0,1)$, we have $\Phi(z)=1-\gamma(1-z)^{1 / 2}+O(1-z)$. So to use $w(z)$ as an extension $\tilde{\Psi}(z)$ we need to choose $\sqrt{\xi}=|\xi|^{1 / 2} \exp (i \operatorname{Arg}(\xi) / 2)$, where $\operatorname{Arg}(\xi) \in(-\pi, \pi)$.

The continuations $\hat{\Psi}(z)$ and $\tilde{\Psi}(z)$ together determine an analytic continuation of $\Phi(z)$ into a function $\Psi(z)$ which is analytic and bounded on a disc $D^{*}=D_{0}\left(1+\rho^{*}\right)$ minus a cut $\left[1,1+\rho^{*}\right), \rho^{*}>0$ being chosen sufficiently small, such that

$$
\Psi(z)_{\substack{z \in D^{*} \mid \overline{\left(1,1+1+\rho^{*}\right)} \\ z \rightarrow 1}} 1-\gamma(1-z)^{1 / 2}+O(|z-1|) .
$$

It follows that

$$
\begin{aligned}
& \mathbb{P}\left(A_{n}\right)=\frac{1}{2 \pi i} \oint_{|z|=1} \frac{\Phi(z)}{z^{n+1}} d z \\
&=\frac{1}{2 \pi} \oint_{|z|=1+\rho *} \frac{\Psi(z)}{z^{n+1}}+\frac{1}{2 \pi i} \int_{1}^{1+\rho *} \frac{2 i \gamma(x-1)^{1 / 2}+O(x-1)}{x^{n+1}} d x \\
&=O\left(\left(1+\rho^{*}\right)^{-n}+n^{-2}\right)+\frac{\gamma}{\pi} \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x .
\end{aligned}
$$

For the second line we integrated $\Psi(z) / z^{n+1}$ along the limit contour : it consists of the directed circular arc $z=\left(1+\rho^{*}\right) e^{i \alpha}, 0<\alpha<2 \pi$ and a detour part formed by two opposite-directed line segments, one from $z=(1+$ $\left.\rho^{*}\right) e^{i(2 \pi-0)}$ to $z=e^{i(2 \pi-0)}$ and another from $z=e^{i(+0)}$ to $z=\left(1+\rho^{*}\right) e^{i(+0)}$.

By the formula 3.191(2) from Gradshteyn and Ryzik [14], we have

$$
\begin{aligned}
& \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x=B(n-1 / 2,3 / 2)=\frac{\Gamma(n-1 / 2) \Gamma(3 / 2)}{\Gamma(n+1)} \\
& =\frac{\prod_{m=2}^{n}\left(n-\frac{2 m-1}{2}\right)}{n!} \cdot \frac{1}{2} \Gamma^{2}(1 / 2)=\pi \frac{(2 n-3)!!}{2^{n} n!} .
\end{aligned}
$$

Therefore

$$
\frac{\gamma}{\pi} \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x=\gamma \frac{(2 n-3)!!}{2^{n} n!}=\frac{\gamma}{2 \pi^{1 / 2} n^{3 / 2}}+O\left(n^{-2}\right)
$$

Recalling that $\gamma=\left(2 p_{0}\right)^{1 / 2} \sigma^{-1}$, we complete the proof of the Lemma.
We will use $\mathbf{p}$ to denote the offspring distribution $\left\{p_{j}\right\}$. Using Lemma 3.1 and Lemma 3.2 we prove
Theorem 3.4. Let $c(\mathbf{p}):=\lambda e^{3 / 2}\left(1-\sum_{j \geq 2} p_{j}^{2}\right)^{1 / 2}$, where $\lambda=\max \left(\chi^{-4}, \chi^{-2}\right)$ and $\chi=\left(2 p_{0}^{3}\right)^{-1 / 2}\left(1-p_{0}\right) \sigma$. Then, for $\varepsilon \in(0,1 / 2]$ and $a \geq(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$, we have $\mathbb{E}\left[X_{n, a}\right] \leq(1-\varepsilon)^{a}$. Consequently, with probability $\geq 1-(1-$ $\varepsilon)^{(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}}$, the largest number of leaves in a common induced subtree is at most $(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$.

Proof. By Lemma 3.1,

$$
\begin{gather*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{(n-a)!}{n!} \mathbb{P}(\mathcal{T})\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y), \\
\Phi(y)=P(\Phi(y))+p_{0}(y-1), \quad \Phi_{1}(y)=\left(1-p_{0}\right)^{-1} \sum_{j>1} p_{j} \Phi^{j-1}(y) ;  \tag{3.10}\\
\mathbb{P}(\mathcal{T})=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})} .
\end{gather*}
$$

Start with the $\left[y^{n-a}\right]$ factor. Observe that

$$
\begin{aligned}
\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} & =\sum_{j \geq 0}\left(\Phi_{1}(y)\right)^{j}\binom{e(\mathcal{T})+j-1}{j}, \\
\Phi^{\prime}(y) & =\frac{p_{0}}{1-P^{\prime}(\Phi(y))}=p_{0} \sum_{j \geq 0}\left(P^{\prime}(\Phi(y))\right)^{j} .
\end{aligned}
$$

The power series for both $\Phi(y)$ and $\Phi_{1}(y)$ around $y=0$, which start with $y^{1}$, have non-negative coefficients only, and so does the power series for $P^{\prime}(w)$ at $w=0$. Therefore the power series for $\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y)$ around $y=0$ has only non-negative coefficients. So we obtain a Chernoff-type bound: for all $r \in(0,1)$,

$$
\begin{equation*}
\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \leq \frac{\left(1-\Phi_{1}(r)\right)^{-e(\mathcal{T})} \Phi^{\prime}(r)}{r^{n-a}} . \tag{3.11}
\end{equation*}
$$

As $\Phi(r)=1-\gamma(1-r)^{1 / 2}+O(1-r)$, we have

$$
\begin{gathered}
1-\Phi_{1}(y)=1-\left(1-p_{0}\right)^{-1} \sum_{j>1} p_{j} \Phi^{j-1}(y)=\left(1-p_{0}\right)^{-1} \frac{P(\Phi(y))-p_{0}}{\Phi(y)} \\
=1-\frac{1-p_{0}+P^{\prime}(1)(\Phi(y)-1)+O(1-r)}{1-\gamma(1-r)^{1 / 2}+O(1-r)} \\
=1-\frac{1-\gamma(1-r)^{1 / 2} /\left(1-p_{0}\right)}{1-\gamma(1-r)^{1 / 2}}(1+O(1-r)) \\
=\frac{\gamma p_{0}}{1-p_{0}}(1-r)^{1 / 2}\left(1+O\left((1-r)^{1 / 2}\right)\right),
\end{gathered}
$$

and $\Phi^{\prime}(y)=O\left((1-r)^{-1 / 2}\right)$. So the RHS is essentially of order $f(r):=$ $(1-r)^{-e(\mathcal{T}) / 2} r^{-n+a}$, and $f(r)$ attains its maximum

$$
\frac{[n-a+e(\mathcal{T}) / 2]^{n-a+e(\mathcal{T}) / 2}}{(n-a)^{n-a}[e(\mathcal{T}) / 2]^{e(\mathcal{T}) / 2}} \leq c n^{1 / 2}\binom{n-a+e(\mathcal{T}) / 2}{n-a}
$$

at $r_{n}=\frac{n-a}{n-a+e(\mathcal{T}) / 2}$, which is $1-\Theta(a / n)$, since $e(\mathcal{T}) \in[a, 2 a]$ and $a=o(n)$, see (3.8). In addition, the binomial factor is at most $\binom{n}{n-a}=\binom{n}{a}$. So, denoting $\mathbf{p}=\left\{p_{j}\right\}$,

$$
\begin{aligned}
& {\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \leq c_{1} n\left(\frac{1-p_{0}}{\gamma p_{0}}-c_{2}(a / n)^{1 / 2}\right)^{-e(\mathcal{T})}\binom{n}{a}} \\
& \leq c_{1} n \lambda^{a}\binom{n}{a}, \quad \lambda>\lambda(\mathbf{p})=\left\{\begin{array}{l}
\left(\frac{1-p_{0}}{\gamma p_{0}}\right)^{-4}, \\
\frac{1-p_{0}}{\gamma p_{0}} \leq 1, \\
\left(\frac{1-p_{0}}{\gamma p_{0}}\right)^{-2}, \\
, \frac{1-p_{0}}{\gamma p_{0}}>1 .
\end{array}\right.
\end{aligned}
$$

$\left(\lambda(\mathbf{p})=1\right.$ for the benchmark case $p_{0}=p_{2}=1 / 2$. ) Hence, using the top line equation in (3.10), we obtain

$$
\mathbb{P}\left(A_{n}(\mathcal{T})\right) \leq \frac{c_{1} n}{a!} \lambda^{a} \mathbb{P}(\mathcal{T})
$$

Since $\mathbb{P}\left(A_{n}\right)=\Theta\left(n^{-3 / 2}\right)$, we conclude that

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right) \leq \frac{c_{1} n^{5 / 2}}{a!} \lambda^{a} \mathbb{P}(\mathcal{T}) \tag{3.12}
\end{equation*}
$$

Recall that $\mathbb{P}(\mathcal{T})=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})}$, where $\{d(v, \mathcal{T})\}$ is the out-degree sequence of non-leaf vertices of a generic $\mathcal{T}$ with $a$ leaves labelled by the elements of $S$. The RHS in (3.12) does not depend on how the $a$ labels are assigned to the leaves. Therefore we have the following upper bound for $\mathbb{E}\left[X_{n, a}\right]$ :

$$
\mathbb{E}\left[X_{n, a}\right]=\binom{n}{a} \sum_{\mathcal{T}}\left[\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right)\right]^{2} \leq a!\binom{n}{a}\left(\frac{c_{1} n^{5 / 2} \lambda^{a}}{a!}\right)^{2} \sum_{\mathcal{T}} \mathbb{P}^{2}(\mathcal{T})
$$

the last sum is over all (finite) rooted trees $\mathcal{T}$ with $a$ unlabelled leaves. Define the probability distribution $\mathbf{q}=\left\{q_{j}\right\}: q_{0}=1-\sum_{j \geq 2} p_{j}^{2}>0, q_{1}=0$, $q_{j}=p_{j}^{2}$ for $j \geq 2$. Then we have

$$
\mathbb{P}^{2}(\mathcal{T})=\prod_{v \in V_{\text {int }}(\mathcal{T})} p_{d(v, \mathcal{T})}^{2}=q_{0}^{-a} \prod_{v \in V(\mathcal{T})} q_{d(v, \mathcal{T})} .
$$

Observe that $\sum_{j \geq 2} j q_{j}<\sum_{j \geq 2} j p_{j}=1$. Therefore the process with the offspring distribution $\left\{q_{j}\right\}$ is almost surely extinct, implying that

$$
\sum_{\mathcal{T}} \mathbb{P}^{2}(\mathcal{T})=q_{0}^{-a} \sum_{\mathcal{T}} \prod_{v \in V(\mathcal{T})} q_{d(v, \mathcal{T})} \leq q_{0}^{-a}=\left(1-\sum_{j \geq 2} p_{j}^{2}\right)^{-a}
$$

A close look shows that, in fact,

$$
\sum_{\mathcal{T}} \mathbb{P}^{2}(\mathcal{T})=o\left(\rho^{-a}\right), \quad \rho:=\max _{\eta \geq 1}\left(\eta-\sum_{j \geq 2} p_{j}^{2} \eta^{j}\right)
$$

So using $a!\geq(a / e)^{a},\binom{n}{a} \leq(n e / a)^{a}$, we obtain then

$$
\mathbb{E}\left[X_{n, a}\right] \leq c \frac{n^{5}\binom{n}{a}}{a!}\left(\frac{\lambda^{2}}{q_{0}}\right)^{a} \leq c n^{5}\left(\frac{n}{a^{2}} \frac{(e \lambda)^{2}}{q_{0}}\right)^{a} \rightarrow 0
$$

if $a \geq(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}, c(\mathbf{p}):=e \lambda(\mathbf{p}) q_{0}^{-1 / 2}$.
Acknowledgment. I owe a debt of genuine gratitude to Ovidiu Costin and Jeff McNeal for guiding me to the Weierstrass separation theorem. I thank Daniel Bernstein, Mike Steel, and Seth Sullivant for an important feedback regarding the references [7, [6] and [22].

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# EXPECTED NUMBER OF INDUCED SUBTREES SHARED BY TWO INDEPENDENT COPIES OF A RANDOM TREE 

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#### Abstract

Consider a rooted tree $T$ with leaf-set [ $n$ ], and with all non-leaf vertices having out-degree 2 , at least. A rooted tree $\mathcal{T}$ with leaf-set $S \subset[n]$ is induced by $S$ in $T$ if $\mathcal{T}$ is the lowest common ancestor subtree for $S$, with all its degree- 2 vertices suppressed. A "maximum agreement subtree" (MAST) for a pair of two trees $T^{\prime}$ and $T^{\prime \prime}$ is a tree $\mathcal{T}$ with a largest leaf-set $S \subset[n]$ such that $\mathcal{T}$ is induced by $S$ both in $T^{\prime}$ and $T^{\prime \prime}$. Bryant et al. 8] and Bernstein et al. 6] proved, among other results, that for $T^{\prime}$ and $T^{\prime \prime}$ being two independent copies of a random binary (uniform or Yule-Harding distributed) tree $T$, the likely magnitude order of $\operatorname{MAST}\left(T^{\prime}, T^{\prime \prime}\right)$ is $O\left(n^{1 / 2}\right)$. We prove this bound for a wide class of random rooted trees : $T$ is a terminal tree of a branching, Galton-Watson, process with an ordered-offspring distribution of mean 1 , conditioned on "total number of leaves is $n$ ".


## 1. Introduction, results

Consider a rooted binary tree $T$, with $n$ leaves labelled by elements from $[n]$. We visualize this tree with the root on top and the leaves at bottom. Given $S \subset[n]$, let $v(S) \in V(T)$ denote the lowest common ancestor of leaves in $S$, (LCA $(S)$.) Introduce the subtree of $T$ formed by the paths from $v(S)$ to leaves in $S$. Ignoring (suppressing) degree-2 vertices of this subtree (except the root itself), we obtain a rooted binary tree with leaf-set $S$. This binary tree is called "a tree induced by $S$ in $T$ ".

Finden and Gordon [12] and Gordon [14] introduced a notion of a "maximum agreement subtree" (MAST) for a pair of such trees $T^{\prime}$ and $T^{\prime \prime}$ : it is a tree $\mathcal{T}$ with a largest leaf-set $S \subset[n]$ such that $\mathcal{T}$ is induced by $S$ both in $T^{\prime}$ and $T^{\prime \prime}$. In a pioneering paper [8], Bryant, McKenzie and Steel addressed the problem of a likely order of $\operatorname{MAST}\left(T_{n}^{\prime}, T_{n}^{\prime \prime}\right)$ when $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$ are two independent copies of a random binary tree $T_{n}$. To quote from [8], such a problem is "relevant when comparing evolutionary trees for the same set of species that have been constructed from two quite different types of data".

[^1]It was proved in 8 that $\operatorname{MAST}\left(T_{n}^{\prime}, T_{n}^{\prime \prime}\right) \leq(1+o(1)) e 2^{-1 / 2} n^{1 / 2}$ with probability $1-o(1)$ as $n \rightarrow \infty$. The proof was based on a remarkable property of the uniformly random rooted binary tree, and of few other tree models, known as "sampling consistency", see Aldous [1]. As observed by Aldous [3, 4], sampling consistency makes this model conceptually close to a uniformly random permutation of $[n]$. Combinatorially, it means that a rooted binary tree $\mathcal{T}$ with with a leaf-set $S \subset[n],|S|=s$, is induced by $S$ in exactly $\frac{(2 n-3)!!}{(2 a-3)!!}$ rooted binary trees with leaf-set $[n]$, regardless of choice of $\mathcal{T}$. Probabilistically, the rooted binary tree induced by $S$ in $T_{n}$ is distributed uniformly on the set of all $(2 a-3)!!$ such trees. Mike Steel [25] pointed out that the sampling consistency of the rooted binary tree follows directly from a recursive process for generating all the rooted trees in which $S$ induces $\mathcal{T}$.

Bernstein, Ho, Long, Steel, St. John, and Sullivant [6] established a qualitatively similar upper bound $O\left(n^{1 / 2}\right)$ for the likely size of a common induced subtree in a harder case of Yule-Harding tree, again relying on sampling consistency of this tree model. Recently Misra and Sullivant [21] were able to prove the two-sided estimate $\Theta\left(n^{1 / 2}\right)$ for the case when two independent binary trees with $n$ labelled leaves are obtained by selecting independently, and uniformly at random, two leaf-labelings of the same unlabelled tree. Using the classic results on the length of the longest increasing subsequence in the uniformly random permutation, Bernstein et al. [6] established a first power-law lower bound $c n^{1 / 8}$ for the likely size of the common induced subtree in the case of the uniform rooted binary tree, and a lower bound $c n^{a-o(1)}, a=0.344 \ldots$, for the Yule-Harding model. Very recently, Aldous [3] proved that a maximum agreement rooted subtree for two independent, uniform, unrooted trees is likely to have $n^{\frac{\sqrt{3}-1}{2}-o(1)} \approx n^{0.366}$ leaves, at least. It was mentioned in [3] that an upper bound $O\left(n^{1 / 2}\right)$ could be obtained by "the first moment method (calculating the expected number of large common subtrees)".

In this paper we show that the total number of unrooted trees with leafset [n], which contains a rooted subtree induced by $S \subset[n], a=|S|<n$, is $\frac{(2 n-5)!!}{(2 a-3)!!}$. The proof is based on a two-phase counting procedure, indirectly inspired by the well-known process of generating a uniformly random unrooted, leaf-labelled, tree. It follows that a rooted binary tree induced by $S$ in the uniformly random unrooted tree on $[n]$ is again distributed uniformly on the set of all $(2 a-3)!!$ rooted trees, so that the expected number of agreement trees with $a$ leaves is $\binom{n}{a} /(2 a-3)!!$. Mike Steel [25] informed us recently that this $\frac{(2 n-5)!!}{(2 a-3)!!}$ formula can also be obtained by observing that the number of unrooted binary trees on $[n]$ in which a leaf-set $\mathcal{S} \subseteq[n]$ induces a given unrooted tree is $\frac{(2 n-5)!!}{(2|\mathcal{S}|-5)!}$. Using the asymptotic estimate
from [8], we have: a maximum agreement rooted subtree for two independent copies of the uniformly random unrooted tree is likely to have at most $(1+o(1)) e 2^{-1 / 2} n^{1 / 2}$ leaves.

Our proof of this $\frac{(2 n-5)!!}{(2 a-3)!!}$ result suggested, strongly, that a bound $O\left(n^{1 / 2}\right)$ might, just might, be obtained for a broad class of random rooted trees by using a probabilistic two-phase counting procedure, where the random tree grows from the root, rather than from leaves.

Consider a Markov branching process initiated by a single progenitor, with a given offspring distribution $\mathbf{p}=\left\{p_{j}\right\}_{j \geq 0}$. If $p_{0}>0$ and $\sum_{j} j p_{j}=1$ (critical case), then the process is almost surely extinct. This process is visualized as a growing rooted tree such that children of each father are ordered, by "seniority" say.

Let $T_{t}$ be the random terminal tree, and let $T_{n}$ be $T_{t}$ conditioned on the event " $T_{t}$ has $n$ leaves", that we label, uniformly at random, by elements of $[n]$. For $p_{0}=p_{2}=1 / 2, T_{n}$ is doubly-random, obtained by picking uniformly at random a binary tree with $2 n-1$ vertices, such that two children of each father are ordered, and labeling the tree's $n$ leaves uniformly at random by elements of $[n]$. This scheme certainly resembles a process studied by Harding [16] (Section 3.2). A key difference is that Harding considered the case when the children of a parent are indistinguishable.

In general, we assume that $p_{1}=0$, g.c.d. $\left(j: p_{j+1}>0\right)=1$, and that $P(s):=\sum_{j} p_{j} s^{j}$ has convergence radius $R>1$. We will show that $\mathbb{P}_{n}:=$ $\mathbb{P}\left(T_{t}\right.$ has $n$ leaves $)>0$ for all $n$, meaning that $T_{n}$ is well-defined for all $n$.

Finally, an out-degree of a vertex in $T_{n}$ may now exceed 2 . So we add to the definition of a tree $\mathcal{T}$, induced by $S$ in $T_{n}$, the condition: the out-degree of every vertex from $V(\mathcal{T})$ in $T_{n}$ is the same as its out-degree in $\mathcal{T}$.

Under the conditions above, we prove that a rooted binary tree $\mathcal{T}$ with leaf-set $S \subset[n]$, the vertex set $V(\mathcal{T})$ and the edge set $E(\mathcal{T})$, is induced by $S$ in $T_{n}$ with probability

$$
\begin{gather*}
\frac{(n-a)!}{n!p_{0} \mathbb{P}_{n}} \mathbb{P}(\mathcal{T})\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y), \quad(e(\mathcal{T}):=|E(\mathcal{T})|), \\
\Phi(y)=P(\Phi(y))+p_{0}(y-1), \quad \Phi_{1}(y)=\sum_{j>1} j p_{j} \Phi^{j-1}(y) ;  \tag{1.1}\\
\mathbb{P}(\mathcal{T})=\prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}, \quad d(v, \mathcal{T}):=\text { out-degree of } v \text { in } \mathcal{T} .
\end{gather*}
$$

Here $\Phi(y)$ is the probability generating function of the total number of leaves in the terminal tree. We will use (1.1) to show that for $p_{0}=p_{2}=1 / 2$ the expected number of agreement trees with $a$ leaves is $\binom{n}{a} /\left[2^{a-1}(2 a-3)!!\right]$. Consequently a maximum-agreement rooted subtree for two independent
copies of the terminal Galton-Watson tree with $n$ leaves, labelled uniformly at random, is likely to have at most $(1+o(1))(e / 2) n^{1 / 2}$ leaves.

Note that in general, because of the factor $\mathbb{P}(\mathcal{T})$, and $e(\mathcal{T})$, the probability of $\mathcal{T}$ being induced by $S$ in $T_{n}$ depends not only on $|S|$, but also on the whole out-degree sequence of $\mathcal{T}$.

We use the above identity to prove the following claim. Let

$$
\begin{gathered}
c(\mathbf{p}):=e p_{0} \lambda\left[\max _{r}\left(r-\sum_{j=2}^{\infty} p_{j}^{2} r^{j}\right)\right]^{-1 / 2}, \\
\lambda:=\max \left(\chi^{-2}, \chi^{-1}\right), \quad \chi:=\left(2 p_{0} \sigma^{2}\right)^{1 / 2}, \quad \sigma^{2}:=\sum_{j=2}^{\infty} j(j-1) p_{j} ;
\end{gathered}
$$

$\left(c(\mathbf{p})=e / 2\right.$ for $\left.p_{0}=p_{2}=1 / 2\right)$. Then, for $\varepsilon \in(0,1 / 2$ ], with probability $\geq 1-(1-\varepsilon)^{(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}}$, the largest number of leaves in an induced subtree shared by two independent copies of the conditioned terminal tree $T_{n}$ is at most $(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$.

For a wide ranging exposition of combinatorial/probabilistic problems and methods in theory of phylogeny, we refer the reader to the books [23] by Semple and Steel, and [24] by Steel. The reader may wish to consult Bóna and Flajolet [7] for a thought-provoking study of algebraic-analytic properties of binary trees from the references above.

## 2. Uniform binary trees

Consider a rooted binary tree $T$ with leaf-set $[n]$. For a given $S \subset[n]$, there exists a subtree with leaf-set $S$, which is rooted at the lowest vertex common to all $|S|$ paths leading away from $S$ toward the root of $T$. The vertex set of this lowest common ancestor (LCA) tree is the set of all vertices from the paths in question. Ignoring degree-2 vertices of this subtree (except the root itself), we obtain a rooted binary tree $\mathcal{T}$. This LCA subtree has a name "a tree induced by $S$ in $T$ ", see 3].

Let $T_{n}^{\prime}, T_{n}^{\prime \prime}$ be two independent copies of the uniformly random rooted binary tree with leaf-set $[n]$. Let $X_{n, a}$ denote the random total number of leaf-sets $S \subset[n]$ of cardinality $a$ that induce the same rooted subtree in $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$. Bryant et al. [8] proved that

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\frac{\binom{n}{a}}{(2 a-3)!!} . \tag{2.1}
\end{equation*}
$$

The proof was based on sampling consistency of the random tree $T_{n}$, so that $N(\mathcal{T})$, the number of rooted trees on $[n]$ in which $S$ induces a given rooted tree $\mathcal{T}$ on $S$, is $\frac{(2 n-3)!!}{(2 a-3)!!}$, thus dependent only on the leaf-set size.

Following Aldous [3] (see Introduction), we consider the case when a binary tree $T$ with leaf-set $[n]$ is unrooted. Here the definition of a rooted subtree induced by a leaf-set $S$ with $|S|=a>1$ remains the same, except
that it makes sense only for $a<n$. An induced subtree uniquely exists for any such $S$. Indeed, a vertex $v$ adjacent to any fixed leaf $\ell^{*} \in[n] \backslash S$ is joined by a unique path to each leaf in $S$. By tracing these $a$ paths toward $v$, we determine their first common vertex $v^{*}$. The subtree formed by the paths from $v^{*}$ to $S$ is induced by $S$ in $T$. Since $T$ is a tree, a subtree induced by $S$ is unique.

Let now $T_{n}^{\prime}, T_{n}^{\prime \prime}$ be two independent copies of the uniformly random (unrooted) binary tree with leaf-set $[n]$. Let $X_{n, a}$ denote the random total number of leaf-sets $S \subset[n]$ of cardinality $a$ that induce the same rooted subtree in $T_{n}^{\prime}$ and $T_{n}^{\prime \prime}$.

Lemma 2.1. Let $a=|S|<n$. Then

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\frac{\binom{n}{a}}{(2 a-3)!!} . \tag{2.2}
\end{equation*}
$$

Equivalently $\mathcal{N}(\mathcal{T})$, the number of unrooted trees $T$ on $[n]$, in which $S$ induces a given rooted tree $\mathcal{T}$ with leaf-set $S$, is $\frac{(2 n-5)!!}{(2 a-3)!!}$.

Note. So the expectation is the same as for the rooted trees $T$ on $[n]$. Compared to a recent combinatorial argument by Steele [24], our longer proof is based on machinery of generating functions. An advantage of our argument is its being a precursor of an avoidably more complicated argument in Section 3. There, a random tree grows from root to leaves, rather than from leaves to root, as it happens for the classic algorithm, used by Steele: the uniformly random binary tree is generated by attaching labelled leaves to a current tree, a leaf at a time.

Proof. Let us evaluate $\mathcal{N}(\mathcal{T})$. Consider a generic rooted tree with $a$ leaves. For $\mathcal{T}$ to be induced by its leaves in $T$ with $n$ leaves, it has to be obtained by ignoring degree- 2 (non-root) vertices in the LCA subtree for leaf-set $S$.

The outside (third) neighbors of the ignored vertices are the roots of subtrees with some $b$ leaves from the remaining $n-a$ leaves, selected in $\binom{n-a}{b}$ ways. The roots of possible trees, attached to internal points chosen from some of $2(a-1)$ edges of $\mathcal{T}$, can be easily ordered. Introduce $F(b, k)$, the total number of ordered forests of $k$ rooted trees with $b$ leaves altogether. By Lemma 4 of Carter et al. [10] (for the count of unordered trees), we have

$$
\begin{equation*}
F(b, k)=\frac{k(2 b-k-1)!}{(b-k)!2^{b-k}} \tag{2.3}
\end{equation*}
$$

It was indicated in [6] that (2.3) follows from

$$
\begin{equation*}
F(b, k)=b!\cdot\left[x^{b}\right] B(x)^{k}, \quad B(x):=1-\sqrt{1-2 x}, \tag{2.4}
\end{equation*}
$$

(Semple and Steel [23]). For the reader's convenience here is a sketch proof of (2.4) and (2.3). We have

$$
\begin{aligned}
F(b, k) & =b!\sum_{t_{1}+\cdots+t_{k}=b} \prod_{j \in[k]} \frac{\left(2 t_{j}-3\right)!!}{t_{j}!}=b!\sum_{t_{1}+\cdots+t_{k}=b} \prod_{j \in[k]} \frac{1}{t_{j} 2^{t_{j}-1}}\binom{2\left(t_{j}-1\right)}{t_{j}-1} \\
& =b!\left[x^{b}\right]\left[\sum_{t \geq 1} \frac{x^{t}}{t 2^{t-1}}\binom{2(t-1)}{t-1}\right]^{k}=b!\left[x^{b}\right] B(x)^{k}=\frac{k(2 b-k-1)!}{(b-k)!2^{b-k}}
\end{aligned}
$$

for the last two steps we used Equations (2.5.10), (2.5.16) in Wilf [26].
Introduce $\mathcal{F}(b, k)$, the total number of the ordered forests of $k$ binary trees with roots attached to internal points of $\mathcal{T}$ 's edges, with $b$ leaves altogether. ( $b$ leaves have to be chosen from $[n] \backslash\left(S \cup\left\{\ell^{*}\right\}\right)$, so $b \leq n-a-1$.) Since the total number of integer compositions (ordered partitions) of $k$ with $j \leq$ $2(a-1)$ positive parts is

$$
\binom{k-1}{j-1}\binom{2(a-1)}{j}=\binom{k-1}{j-1}\binom{2(a-1)}{2(a-1)-j},
$$

(2.3) implies

$$
\begin{align*}
\mathcal{F}(b, k)=F(b, k) & \sum_{j \leq 2(a-1)}\binom{k-1}{j-1}\binom{2(a-1)}{2(a-1)-j}  \tag{2.5}\\
= & F(b, k)\binom{k+2 a-3}{2 a-3}=b!\binom{k+2 a-3}{2 a-3}\left[x^{b}\right] B(x)^{k} .
\end{align*}
$$

Now, $\sum_{k \leq b} \mathcal{F}(b, k)$ is the total number of ways to expand the host subtree into a full binary subtree rooted at the lowest common ancestor of the $a$ leaves. To evaluate this sum, first denote $\alpha=2 a-3, \beta=B(x)$ and write

$$
\sum_{k \geq 0}\binom{k+\alpha}{\alpha} \beta^{k}=\sum_{k \geq 0}(-\beta)^{k}\binom{-\alpha-1}{k}=(1-\beta)^{-\alpha-1}
$$

Therefore

$$
\begin{aligned}
\sum_{k \leq b}\binom{k+\alpha}{\alpha}\left[x^{b}\right] B(x)^{k} & =\left[x^{b}\right] \sum_{k \geq 0}\binom{k+\alpha}{\alpha} B(x)^{k}=\left[x^{b}\right] \frac{1}{(1-B(x))^{\alpha+1}} \\
& =\left[x^{b}\right](1-2 x)^{-\frac{\alpha+1}{2}} .
\end{aligned}
$$

We conclude that

$$
\begin{equation*}
\sum_{k \leq b} \mathcal{F}(b, k)=\left.b!\left[x^{b}\right](1-2 x)^{-\frac{\alpha+1}{2}}\right|_{\alpha=2 a-3}=b!\left[x^{b}\right](1-2 x)^{-(a-1)} . \tag{2.6}
\end{equation*}
$$

Recall that $b$ leaves were chosen from $[n] \backslash\left(S \cup\left\{\ell^{*}\right\}\right)$. If $b=n-a-1$, then attaching the single remaining leaf to the root $v^{*}$ we get a binary tree $T$ with leaf-set $[n]$. If $b \leq n-a-2$, we view the expanded subtree as a single leaf, and form an unrooted binary tree with $1+(n-a-b) \geq 3$ leaves,
in $[2(n-a-b)-3]!$ ! ways. Therefore $\mathcal{N}(\mathcal{T})$ depends on $a$ only, and with $\nu:=n-a-1$, it is given by

$$
\begin{aligned}
\mathcal{N}(\mathcal{T})=\sum_{b \leq \nu} & \binom{\nu}{b} b!\left[x^{b}\right](1-2 x)^{-(a-1)}(2(\nu-b)-1)!! \\
& =\nu!\sum_{b \leq \nu}\left[x^{b}\right](1-2 x)^{-(a-1)} \cdot\left[x^{\nu-b}\right](1-2 x)^{-1 / 2} \\
& =\nu!\left[x^{\nu}\right](1-2 x)^{-a+1 / 2}=\prod_{j=0}^{n-a-2}(2 a-1+2 j)=\frac{(2 n-5)!!}{(2 a-3)!!}
\end{aligned}
$$

in the first line $(-1)!!:=1$, and for the second line we used

$$
\frac{(2 k-1)!!}{k!}=\frac{(2 k)!}{2^{k}(k!)^{2}}=2^{-k}\binom{2 k}{k}=\left[x^{k}\right](1-2 x)^{-1 / 2} .
$$

Consequently

$$
\begin{equation*}
\mathbb{E}\left[X_{n, a}\right]=\binom{n}{a}(2 a-3)!!\left[\frac{\mathcal{N}(\mathcal{T})}{(2 n-5)!!}\right]^{2}=\frac{\binom{n}{a}}{(2 a-3)!!} . \tag{2.7}
\end{equation*}
$$

## 3. Branching Process Framework

Consider a branching process initiated by a single progenitor. This process is visualized as a growing rooted tree. The root is the progenitor, connected by edges to each of its immediate descendants (children), that are ordered, say by seniority. Each of the children becomes the root of the corresponding subtree, so that the children of all these roots are the grandchildren of the progenitor. We obviously get a recursively defined process. It delivers a nested sequence of trees, which is either infinite, or terminates at a moment when none of the current leaves have children.

The classic Galton-Watson branching process is the case when the number of each member's children (a) is independent of those numbers for all members from the preceding and current generations and (b) has the same distribution $\left\{p_{j}\right\}_{j \geq 0},\left(\sum_{j} p_{j}=1\right)$. It is well-known that if $p_{0}>0$ and $\sum_{j \geq 0} j p_{j}=1$, then the process terminates with probability 1 , Harris [17]. Let $T_{t}$ denote the terminal tree. Given a finite rooted tree $T$, we have

$$
\mathbb{P}\left(T_{t}=T\right)=\prod_{v \in V(T)} p_{d(v, T)},
$$

where $d(v, T)$ is the out-degree of vertex $v \in V(T) . \quad X_{t}:=\left|V\left(T_{t}\right)\right|$, the total population size by the extinction time, has the probability generating
function (p.g.f) $F(x):=\mathbb{E}\left[x^{X_{t}}\right],|x| \leq 1$, that satisfies

$$
\begin{equation*}
F(x)=x P(F(x)), \quad P(\xi):=\sum_{j \geq 0} p_{j} \xi^{j},(|\xi| \leq 1) . \tag{3.1}
\end{equation*}
$$

Indeed, introducing $F_{\tau}(x)$ the p.g.f. of the total cardinality of the first $\tau$ generations, we have

$$
F_{\tau+1}(x)=x \sum_{j \geq 0} p_{j}\left[F_{\tau}(x)\right]^{j}=x P\left(F_{\tau}(x)\right) .
$$

So letting $\tau \rightarrow \infty$, we obtain (3.1). In the same vein, consider the pair $\left(X_{t}, Y_{t}\right)$, where $Y_{t}:\left|\left\{v \in V\left(T_{t}\right): d\left(v, T_{t}\right)=0\right\}\right|$ is the total number of leaves (zero out-degree vertices) of the terminal tree. Then denoting $G(x, y)=$ $\mathbb{E}\left[x^{X_{t}} y^{Y_{t}}\right],(|x|,|y| \leq 1)$, we have

$$
\begin{equation*}
G(x, y)=p_{0} x y+x \sum_{j \geq 1} p_{j}[G(x, y)]^{j}=x P(G(x, y))+p_{0} x(y-1) . \tag{3.2}
\end{equation*}
$$

So, with $\Phi(y):=\mathbb{E}\left[y^{Y}\right]=G(1, y)$, we get

$$
\begin{equation*}
\Phi(y)=\sum_{j \geq 1} p_{j} \Phi^{j}(y)+p_{0} y=P(\Phi(y))+p_{0}(y-1) . \tag{3.3}
\end{equation*}
$$

Importantly, this identity allows us to deal directly with the leaf set at the extinction moment: $\mathbb{P}_{k}:=\left[y^{k}\right] \Phi(y)$ is the probability that $T_{t}$ has $k$ leaves. In particular, $\mathbb{P}_{1}=\left[y^{1}\right] \Phi(y)=p_{0}>0$. More generally, $\mathbb{P}_{k}>0$ for all $k \geq 1$. meaning that $\mathbb{P}\left(T_{t}\right.$ has $k$ leaves $)>0$ for all $k \geq 1$. Indeed, for $k \geq 2$, we have

$$
\mathbb{P}_{k}=\sum_{j \geq 1} p_{j} \sum_{\substack{k_{1}+\cdots+k_{j} \geq k \\ k_{1}, \ldots, k_{j} \geq 1}} \mathbb{P}_{k_{1}} \cdots \mathbb{P}_{k_{j}} ;
$$

so the claim follows by easy induction on $k$. Introducing the event $A_{k}:=$ $\left\{T_{t}\right.$ has $k$ leaves $\}$, we have $\mathbb{P}\left(A_{k}\right)=\left[y^{k}\right] \Phi(y)$.

If $p_{0}=p_{2}=1 / 2$, then the branching process is a nested sequence of binary trees. The equation (3.3) yields

$$
\Phi(y)=1-(1-y)^{1 / 2}=\sum_{n \geq 1}\left(\frac{y}{2}\right)^{n} \frac{(2 n-3)!!}{n!}, \quad|y| \leq 1 ;
$$

so $\mathbb{P}\left(A_{n}\right)=\frac{(2 n-3)!!}{2^{n} n!}>0$. On the event $A_{n}$, the total number of vertices is $2 n-1$, and each of rooted binary trees with ordered pairs of children is a value of the terminal tree with the same probability $(1 / 2)^{2 n-1}$. Now,

$$
\frac{\frac{(2 n-3)!!}{2^{n} n!}}{(1 / 2)^{2 n-1}}=\frac{1}{n}\binom{2(n-1)}{n-1}
$$

is the Catalan number $C(n-1)$, which is the total number of rooted binary trees with $n$ leaves, and $n-1$ non-leaves, each having 2 ordered children.

Thus, conditionally on $A_{n}$, the terminal tree is distributed uniformly on the set of these $C(n-1)$ trees. We do not have such uniformity for a general $\left\{p_{j}\right\}$, of course.

For a general $\left\{p_{j}\right\}$, on the event $A_{n}$ we label, uniformly at random, the leaves of $T_{t}$ by elements of $[n]$. We take liberty to use the same notation $T_{n}$, as in Section 2, for the resulting doubly random, leaf-labelled tree.

That $T_{n}$ is again associated with a recursive process is a hopeful sign that we can get a counterpart of what we proved for the uniformly random, leaf-labelled binary tree, and also extend an analysis to a more general distribution $\left\{p_{j}\right\}$.

We continue to assume that $p_{1}=0$. The notion of an induced subtree needs to be expanded, since an out-degree of a vertex now may exceed 2 . Let $\mathcal{T}$ be a tree with a leaf-set $S \subset[n]$, such that every non-leaf vertex of $\mathcal{T}$ has at least two (ordered) children. We say that $S$ induces $\mathcal{T}$ in a tree $T_{n}$ provided that: (a) the LCA subtree for $S$ in $T_{n}$ is $\mathcal{T}$ if we ignore vertices of total degree 2 in this LCA subtree; (b) the out-degree of every other vertex in the LCA of $S$ in $T_{n}$ is the same as its out-degree in $\mathcal{T}$. We call this event $A_{n}(\mathcal{T})$. We evaluate $\mathbb{P}\left(A_{n}(\mathcal{T})\right)$ in steps. Let $|S|=a<n$. Given $b \leq n-a$, let $A_{n}(\mathcal{T}, b)$ be the event:
(i) $A_{n}$ holds, i.e. the terminal tree $T_{t}$ has $n$ leaves; the uniformly random labelling of the leaves of $T_{t}$, that results in the random tree $T_{n}$ is such that: (ii) some $b$ elements from $[n] \backslash S$ are chosen as leaf labels for all the complementary extinction subtrees rooted at degree-2 vertices sprinkled on the edges of $\mathcal{T}$, forming-together with $\mathcal{T}$ on leaf-set $S$-an expanded terminal tree on leaf-set $S \cup\{b$ leaves $\}$, of cardinality $a+b$;
(iii) the terminal tree with $n$ labelled leaves is obtained as follows: we build up a terminal tree with leaf-set $[n] \backslash(S \cup\{b$ leaves $\})$ plus an extra super-leaf, which is the root of the tree built up in (ii), and replace the super-leaf with this tree.

In summary, a terminal tree $T_{n}$, compatible with the event $A_{n}(\mathcal{T}, b)$, is built up of terminal subtrees with a certain number of leaves, each subtree being delivered by a branching process that starts at the subtree's root.

Clearly $A_{n}(\mathcal{T})$ is the disjoint union of the events $A_{n}(\mathcal{T}, b)$. By the very definition, on the event $A_{n}(\mathcal{T}, b)$ the leaf-set $S$ certainly induces $\mathcal{T}$ in $T_{n}$.

To evaluate $\mathbb{P}\left(A_{n}(\mathcal{T}, b)\right)$ we partition $A_{n}(\mathcal{T}, b)$ into disjoint $\binom{n-a}{b}$ events corresponding to all choices to select $b$ elements of $[n] \backslash S$. Let $e(\mathcal{T})$ be the total number of edges in $\mathcal{T}$. For each of these choices, on the event $A_{n}(\mathcal{T}, b)$ we must have some $k \leq b$ terminal subtrees whose roots are some degree- 2 vertices, selected from some of $e(\mathcal{T})$ edges, with their respective, nonempty, leaf-sets forming an ordered set partition of the set of $b$ leaves. The root of each of these trees has one child down the host edge of $\mathcal{T}$,
and all the remaining children are outside of edges of $\mathcal{T}$. The number of those children is $j$ with probability $(j+1) p_{j+1}$, and $\left\{(j+1) p_{j+1}\right\}_{j \geq 1}$ is a probability distribution, as

$$
\sum_{j \geq 1}(j+1) p_{j+1}=\sum_{j \geq 0} j p_{j}=1 .
$$

So the total number of leaves of terminal subtrees rooted at those outside children is $i$ with probability $\left[y^{i}\right] \Phi_{1}(y)$, where

$$
\begin{equation*}
\Phi_{1}(y)=\sum_{j \geq 1}(j+1) p_{j+1} \Phi^{j}(y) . \tag{3.4}
\end{equation*}
$$

The probability $\left[y^{i}\right] \Phi_{1}(y)$ is positive for each $i \geq 1$, since $\left[y^{i}\right] \Phi^{j}(y)>0$ for each $j \geq 1$. Therefore a given set of $b$ elements of $[n]$ is the leaf-set of these terminal subtrees with probability

$$
\begin{aligned}
& b!\sum_{j \leq k \leq b}\binom{k-1}{j-1}\binom{e(\mathcal{T})}{j} \sum_{b_{1}+\cdots+b_{k}=b} \prod_{t=1}^{k}\left[y^{b_{t}}\right] \Phi_{1}(y) \\
& =b!\sum_{j \leq k \leq b}\binom{k-1}{j-1}\binom{e(\mathcal{T})}{e(\mathcal{T})-j}\left[y^{b}\right] \Phi_{1}^{k}(y)=b!\left[y^{b}\right] \sum_{k}\binom{k+e(\mathcal{T})-1}{k} \Phi_{1}^{k}(y) \\
& =b!\left[y^{b}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})}
\end{aligned}
$$

Explanation: $k$ is a generic total number of trees rooted at ordered internal points of some $j$ edges of $\mathcal{T} ; b_{t}$ is a generic number of leaves of a $t$-th tree; the product of two binomial coefficients in the top line is the number of ways to pick $j$ edges of $\mathcal{T}$ and to select an ordered, $j$-long, composition of $k$; the sum is the probability that the $k$ trees have $b$ leaves in total. $b$ ! accounts for the number of ways to assign the chosen $b$ elements as labels of $b$ leaves.

With these complementary trees attached, we obtain a terminal tree with $a+b$ leaves, rooted at the root of $\mathcal{T}$. We denote this expanded tree $\mathcal{E}(\mathcal{T})$. For $b=n-a$, this is our terminal tree $T$ with $n$ labelled leaves. If $b<n-a$, then $\mathcal{E}(\mathcal{T})$ is a subtree of $T$. Specifically, a branching process tree, grown from a progenitor root, terminates when there is a single active leave, meaning that this leaf is about to produce children in accordance with offspring distribution $\mathbf{p}$, and that all other leaves are childless. This single leaf becomes the root of $\mathcal{E}(\mathcal{T})$, which completes construction of $T$. Now, $p_{0}$ times the probability that at this moment the number of childless leaves is $n-a-b$ equals $(n-a-b+1)!\left[y^{n-a-b+1}\right] \Phi(y)$, which is the probability of a terminal tree with $n-a-b+1$ leaves labelled by remaining $n-a-b$ elements of $[n]$ plus 1 , accounting for the root of $\mathcal{E}(\mathcal{T})$.

Therefore

$$
\begin{align*}
\mathbb{P}\left(A_{n}(\mathcal{T}, b)\right)= & \frac{\mathbb{P}(\mathcal{T})}{p_{0} n!}\binom{n-a}{b} \times b!\left[y^{b}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})}  \tag{3.5}\\
& \times(n-a-b+1)!\left[y^{n-a-b+1}\right] \Phi(y) .
\end{align*}
$$

Here $\mathbb{P}(\mathcal{T}):=\prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}$, where $\{d(v, \mathcal{T})\}$ is the out-degree sequence of vertices in $\mathcal{T}$, that includes the actual out-degree of $\mathcal{T}$ 's root. Using $(j+1)\left[y^{j+1}\right] \Phi(y)=\left[y^{j}\right] \Phi^{\prime}(y)$, we simplify (3.5):

$$
\mathbb{P}\left(A_{n}(\mathcal{T}, b)\right)=\frac{(n-a)!\mathbb{P}(\mathcal{T})}{p_{0} n!}\left[y^{b}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})}\left[y^{n-a-b}\right] \Phi^{\prime}(y)
$$

Summing the last equation for $0 \leq b \leq n-a$, we obtain

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{\mathbb{P}(\mathcal{T})(n-a)!}{p_{0} n!}\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \tag{3.6}
\end{equation*}
$$

For $p_{0}, p_{2}=1 / 2$, we have $\mathbb{P}(\mathcal{T})=(1 / 2)^{2 a-1}, \Phi_{1}(y)=\Phi(y)$, see (3.4). So

$$
\begin{aligned}
& {\left[y^{n-a}\right](1-\Phi(y))^{-e(\mathcal{T})} \Phi(y)=\left[y^{n-a}\right](1-\Phi(y))^{-2 a+2} \Phi^{\prime}(y) } \\
&=\left[y^{n-a}\right](1-y)^{-a+1} \cdot \frac{1}{2}(1-y)^{-1 / 2}=\frac{1}{2}\left[y^{n-a}\right](1-y)^{-a+1 / 2} \\
&=2^{-n+a-1} \frac{(2 n-3)!!}{(n-a)!(2 a-3)!!}
\end{aligned}
$$

and, by (3.6), we have

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{(2 n-3)!!}{2^{n+a-1} n!(2 a-3)!!} \tag{3.7}
\end{equation*}
$$

Since $\mathbb{P}\left(A_{n}\right)=\frac{(2 n-3)!!}{2^{n} n!}$, we conclude that

$$
\begin{equation*}
\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right)=\frac{\mathbb{P}\left(A_{n}(\mathcal{T})\right)}{\mathbb{P}\left(A_{n}\right)}=\frac{1}{2^{a-1}(2 a-3)!!} \tag{3.8}
\end{equation*}
$$

for every binary tree $\mathcal{T}$ with leaf-set $S \subset[n],|S|=a$. The LHS is the probability that $S$ induces $\mathcal{T}$ in the uniformly random binary tree $T_{n}$.

Theorem 3.1. Let $X_{n, a}$ denote the total number of leaf-sets $S \subset[n]$ that induce the same binary tree in two independent copies of the random binary tree $T_{n}$. We have

$$
\mathbb{E}\left[X_{n, a}\right]=\binom{n}{a}\left[2^{a-1}(2 a-3)!!\right]^{-1}
$$

in particular,

$$
\mathbb{E}\left[X_{n, 1}\right]=n, \quad \mathbb{E}\left[X_{n, 2}\right]=n(n-1) / 4, \quad \mathbb{E}\left[X_{n, 3}\right]=n(n-1)(n-2) / 72 .
$$

Consequently a maximum-agreement rooted subtree for two independent copies of the terminal Galton-Watson tree with n leaves, labelled uniformly at random, is likely to have at most $(1+o(1))(e / 2) n^{1 / 2}$ leaves.

Proof. The total number of the binary trees with $a$ leaves in question is $a!C(a-1)=2^{a-1}(2 a-3)!!$. Therefore, by (3.8),

$$
\mathbb{E}\left[X_{n, a}\right]=\binom{n}{a}\left[2^{a-1}(2 a-3)!!\right]^{-2} \cdot a!C(a-1)=\binom{n}{a}\left[2^{a-1}(2 a-3)!!\right]^{-1} .
$$

For a general distribution $\left\{p_{j}\right\}$, we have proved
Lemma 3.2. Consider the branching process with the immediate offspring distribution $\left\{p_{j}\right\}$, such that $p_{0}>0, p_{1}=0$, and $\sum_{j \geq 0} j p_{j}=1$. With probability 1 , the process eventually stops, so that a finite terminal tree $T_{t}$ is a.s. well-defined. (1) Setting $A_{n}=\left\{T_{t}\right.$ has $n$ leaves $\}$, we have

$$
\mathbb{P}\left(A_{n}\right)=\left[y^{n}\right] \Phi(y), \quad \Phi(y)=P(\Phi(y))+p_{0}(y-1),
$$

where $P(\eta):=\sum_{j \geq 0} \eta^{j} p_{j}$. (2) On the event $A_{n}$, we define $T_{n}$ as the tree $T_{t}$ with leaves labelled uniformly at random by elements from $[n]$. Given a rooted tree $\mathcal{T}$ with leaf-set $S \subset[n]$, set $A_{n}(\mathcal{T}):=$ " $A_{n}$ holds; $S$ is a subset of $T_{t}^{\prime} s$ leaf-set ; $S$ induces $\mathcal{T}$ in $T_{n}$ ". Then $\mathbb{P}\left(A_{n}(\mathcal{T}) \mid A_{n}\right)=\frac{\mathbb{P}\left(A_{n}(\mathcal{T})\right)}{\mathbb{P}\left(A_{n}\right)}$, where

$$
\begin{align*}
\mathbb{P}\left(A_{n}(\mathcal{T})\right)=\frac{\mathbb{P}(\mathcal{T})(n-a)!}{p_{0} n!}\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y), \\
\Phi(y)=P(\Phi(y))+p_{0}(y-1), \quad \Phi_{1}(y)=\sum_{j>1} j p_{j} \Phi^{j-1}(y),  \tag{3.9}\\
\mathbb{P}(\mathcal{T})=\prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})} ;
\end{align*}
$$

$V(\mathcal{T})$ and $e(\mathcal{T})$ are the set of vertices and the number of edges of $\mathcal{T}$.
Note. For $p_{0}=p_{2}=1 / 2, \mathbb{P}\left(A_{n}(\mathcal{T})\right)$ turned out to be dependent only on the number of leaves of $\mathcal{T}$. The formula (3.9) clearly shows that, in general, this probability depends on shape of $\mathcal{T}$. Fortunately, this dependence is confined to a single factor $\mathbb{P}(\mathcal{T})$, since the rest depends on two scalars, $a$ and $e(\mathcal{T})$. Importantly, these parameters are of the same order of magnitude. Indeed, if if $a>1$ and $\mathbb{P}(\mathcal{T})>0$ then $e(\mathcal{T}) \geq \max \left(a, 2\left|V_{\text {int }}(\mathcal{T})\right|\right)$, where $V_{\text {int }}(\mathcal{T})$ is the set of non-leaf vertices of $\mathcal{T}$. Hence

$$
a \leq e(\mathcal{T})=|V(\mathcal{T})|-1=\left|V_{\mathrm{int}}(\mathcal{T})\right|+a-1 \leq \frac{e(\mathcal{T})}{2}+a-1
$$

so that

$$
\begin{equation*}
a \leq e(\mathcal{T}) \leq 2(a-1) \tag{3.10}
\end{equation*}
$$

3.1. Asymptotics. From now on we assume that the series $\sum_{j} p_{j} s^{j}$ has convergence radius $R>1$.

Lemma 3.3. Suppose that $d:=$ g.c.d. $\left\{j \geq 1: p_{j+1}>0\right\}=1$. Let $\sigma^{2}:=$ $\sum_{j \geq 0} j(j-1) p_{j}$, i.e. $\sigma^{2}$ is the variance of the immediate offspring, since $\sum_{j \geq 0} j p_{j}=1$. Then

$$
\mathbb{P}\left(A_{n}\right)=\frac{\left(2 p_{0}\right)^{1 / 2}}{\sigma} \frac{(2 n-3)!!}{2^{n} n!}+O\left(n^{-2}\right)=\left(\frac{p_{0}}{2 \pi \sigma^{2}}\right)^{1 / 2} n^{-3 / 2}+O\left(n^{-2}\right)
$$

Proof. According to Lemma 3.2, we need to determine an asymptotic behavior of the coefficient in the power series $\Phi(z)=\sum_{n \geq 1} z^{n} \mathbb{P}\left(A_{n}\right)$, where $\Phi(z)$ is given implicitly by the functional equation $\Phi(z)=\sum_{j \geq 0} p_{j} \Phi^{j}(z)+$ $p_{0}(z-1),(|z| \leq 1)$.

In 1974 Bender [5] sketched a proof of the following general claim.
Theorem 3.4. Assume that the power series $w(z)=\sum_{n} a_{n} z^{n}$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose that there exist $r>0$ and $s>0$ such that: (i) for some $R>r$ and $S>s, F(z, w)$ is analytic for $|z|<R$ and $w<S$; (ii) $F(r, s)=F_{w}(r, s)=0$; (iii) $F_{z}(r, s) \neq 0$ and $F_{w w}(r, s) \neq 0$; (iv) if $|z| \leq r,|w| \leq s$, and $F(z, w)=F_{w}(z, w)=0$, then $z=r$ and $w=s$. Then $a_{n} \sim\left(\left(r F_{z}(r, s)\right) /\left(2 \pi F_{w w}(r, s)\right)\right)^{1 / 2} n^{-3 / 2} r^{-n}$.

The remainder term aside, that's exactly what we claim in Lemma 3.3 for our $\Phi(z)$. The proof in [5] relied on an appealing conjecture that, under the conditions (i)-(iv), $r$ is the radius of convergence for the power series for $w(z)$, and $z=r$ is the only singularity for $w(z)$ on the circle $|z|=r$. However, ten years later Canfield [9] found an example of $F(z, w)$ meeting the four conditions in which $r$ and the radius of convergence for $w(z)$ are not the same. Later Meir and Moon found some conditions sufficient for validity of the conjecture. Our equation $\Phi(z)=P(\Phi(z))+p_{0}(z-1)$ is a special case of $w=\phi(w)+h(z)$ considered in [19]. For the conditions from [19] to work in our case, it would have been necessary to have $\left|P^{\prime}(w) / P(w)\right| \leq 1$ for complex $w$ with $|w| \leq 1$, a strong condition difficult to check. (An interesting discussion of these issues can be found in an encyclopedic book by Flajolet and Sedgewick [13] and an authoritative survey by Odlyzko [22].)

Let $r$ be the convergence radius for the powers series representing $\Phi(z)$; so that $r \geq 1$, since $\mathbb{P}\left(A_{n}\right) \leq 1$. By implicit differentiation, we have

$$
\lim _{x \uparrow 1} \Phi^{\prime}(x)=\lim _{x \uparrow 1} \frac{p_{0}}{1-\mathbb{P}_{w}(\Phi(x))}=\infty
$$

since $\lim _{x \uparrow 1} P_{w}(\Phi(x))=\sum_{j} j p_{j}=1$. Therefore $r=1$. Turn to complex $z$. For $|z|<1$, we have $F(z, \Phi(z))=0$, where $F(z, w):=p_{0}(z-1)+P(w)-w$ is analytic as a function of $z$ and $w$ subject to $|w|<1$. Observe that $F_{w}(z, w)=P^{\prime}(w)-1=0$ is possible only if $|w| \geq 1$, since for $|w|<1$ we have $\left|P^{\prime}(w)\right| \leq P^{\prime}(|w|)<P^{\prime}(1)=1$. If $|w|=1$ then $P^{\prime}(w)=\sum_{j \geq 2} j p_{j} w^{j-1}=1$ if and only if $w=w_{k}:=\exp \left(i \frac{2 \pi k}{d}\right)$, and $k=1, \ldots, d$. Notice also that

$$
\begin{equation*}
P\left(w_{k}\right)=\sum_{j \geq 0} p_{j} w_{k}^{j}=p_{0}+w_{k} \sum_{j \geq 2} p_{j} w_{k}^{j-1}=p_{0}+w_{k}\left(1-p_{0}\right) . \tag{3.11}
\end{equation*}
$$

Now, $z$ is a singular point of $\Phi(z)$ if and only if $|z|=1$ and $\left(P^{\prime}(w)-\right.$ 1) $\left.\right|_{w=\Phi(z)}=0$, i.e. if and only if $\Phi(z)=w_{k}$ for some $k \in[d]$, which is equivalent to

$$
p_{0}(z-1)+P\left(w_{k}\right)-w_{k}=0 .
$$

Combination of this condition with (3.11) yields $z=w_{k}$. Therefore the set of all singular points of $\Phi(z)$ on the circle $|z|=1$ is the set of all $w_{k}$ such that $\Phi\left(w_{k}\right)=w_{k}$. Notice that

$$
P^{\prime \prime}\left(w_{k}\right)=w_{k}^{-1} \sum_{j \geq 2} j(j-1) p_{j} w_{k}^{j-1}=w_{k}^{-1} \sum_{j \geq 2} j(j-1) p_{j}=w_{k}^{-1} \sigma^{2} \neq 0 .
$$

So none of $w_{k}$ is an accumulation point of roots of $P^{\prime}(w)-1$ outside the circle $|w|=1$, i.e. $\left\{w_{k}\right\}_{k \in[d]}$ is the full root set of $P^{\prime}(w)-1$ inside the circle $|w|=1+\rho_{0}$, for some small $\rho_{0}>0$.

Consequently, if $d=1$, then $z=1$ is the only singular point of $\Phi(z)$ on the circle $|z|=1$. Define the $\operatorname{argument} \arg (z)$ by the condition $\arg (z) \in[0,2 \pi)$. By the analytic implicit function theorem applied to $F(z, w)$, for every $z_{\alpha}=$ $e^{i \alpha}, \varepsilon \leq \alpha \leq 2 \pi-\varepsilon$, a small $\varepsilon>0$ being fixed, there exists an analytic function $\Psi_{\alpha}(z)$ defined on $D_{z_{\alpha}}(\rho)$-an open disc centered at $z_{\alpha}$, of a radius $\rho=\rho(\varepsilon)<\rho_{0}$ small enough to make $\varepsilon / 2 \leq \arg (z) \leq 2 \pi-\varepsilon / 2$ for all $z \in D_{z_{\alpha}-}$ such that $F\left(z, \Psi_{\alpha}(z)\right)=0$ for $z \in D_{z_{\alpha}}$ and $\Psi_{\alpha}(z)=\Phi(z)$ for $z \in D_{z_{\alpha}}$ with $|z| \leq 1$. Together, these local analytic continuations determine an analytic continuation of $\Phi(z)$ to a function $\hat{\Psi}(z)$ determined, and bounded, for $z$ with $|z|<1+\rho, \arg (z) \in[\varepsilon, 2 \pi-\varepsilon]$.

Since $z_{0}=1$ is the singular point of $\Phi(z)$, there is no analytic continuation of $\Phi(z)$ for $|z|>1$ and $|z-1|$ small. So instead we delete an interval $[1,1+\rho)$ and continue $\Phi(z)$ analytically into the remaining part of a disc centered at 1. Here is how. We have $F_{w w}(1, \Phi(1))=P^{\prime \prime}(1)=\sum_{j} j(j-1) p_{j}=\sigma^{2}>0$. By a "preparation" theorem due to Weierstrass, (Ebeling [11], Krantz [18]), already used by Bender [5] for the same purpose in the general setting, there exist two open discs $D_{1}$ and $\mathcal{D}_{1}$ such that for $z \in D_{1}$ and $w \in \mathcal{D}_{1}$ we have

$$
F(z, w)=\left[(w-1)^{2}+c_{1}(z)(w-1)+c_{2}(z)\right] g(z, w),
$$

where $c_{j}(z)$ are analytic on $D_{1}, c_{j}(1)=0$, and $g(z, w)$ is analytic, nonvanishing, on $D_{1} \times \mathcal{D}_{1}$. (The degree 2 of the polynomial is exactly the order of the first non-vanishing derivative of $F(z, w)$ with respect to $w$ at $(1,1)$.) So for $z \in D_{1}$ and $w \in \mathcal{D}_{1}, F(z, w)=0$ is equivalent to

$$
(w-1)^{2}+c_{1}(z)(w-1)+c_{2}(z)=0 \Longrightarrow w=1+(z-1)^{1 / 2} h(z),
$$

where $h(z)$ is analytic at $z=1$. Plugging the power series $w=1+(z-$ $1)^{1 / 2} h(z)=1+(z-1)^{1 / 2} \sum_{j \geq 0} h_{j}(z-1)^{j}$ into equation $F(z, w)=0$, and expanding $P(w)$ in powers of $w-1$, we can compute the coefficients $h_{j}$. In particular,

$$
w(z)=1-\gamma(1-z)^{1 / 2}+O(|z-1|), \quad \gamma:=\left(2 p_{0}\right)^{1 / 2} \sigma^{-1} .
$$

For $z$ real and $z \in(0,1)$, we have $\Phi(z)=1-\gamma(1-z)^{1 / 2}+O(1-z)$. So to use $w(z)$ as an extension $\tilde{\Psi}(z)$ we need to choose $\sqrt{\xi}=|\xi|^{1 / 2} \exp (i \operatorname{Arg}(\xi) / 2)$, where $\operatorname{Arg}(\xi) \in(-\pi, \pi)$.

The continuations $\hat{\Psi}(z)$ and $\tilde{\Psi}(z)$ together determine an analytic continuation of $\Phi(z)$ into a function $\Psi(z)$ which is analytic and bounded on a disc $D^{*}=D_{0}\left(1+\rho^{*}\right)$ minus a cut $\left[1,1+\rho^{*}\right), \rho^{*}>0$ being chosen sufficiently small, such that

It follows that

$$
\begin{aligned}
& \mathbb{P}\left(A_{n}\right)=\frac{1}{2 \pi i} \oint_{|z|=1} \frac{\Phi(z)}{z^{n+1}} d z \\
& =\frac{1}{2 \pi} \oint_{|z|=1+\rho *} \frac{\Psi(z)}{z^{n+1}}+\frac{1}{2 \pi i} \int_{1}^{1+\rho *} \frac{2 i \gamma(x-1)^{1 / 2}+O(x-1)}{x^{n+1}} d x \\
& \quad=O\left(\left(1+\rho^{*}\right)^{-n}+n^{-2}\right)+\frac{\gamma}{\pi} \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x .
\end{aligned}
$$

For the second line we integrated $\Psi(z) / z^{n+1}$ along the limit contour : it consists of the directed circular arc $z=\left(1+\rho^{*}\right) e^{i \alpha}, 0<\alpha<2 \pi$ and a detour part formed by two opposite-directed line segments, one from $z=(1+$ $\left.\rho^{*}\right) e^{i(2 \pi-0)}$ to $z=e^{i(2 \pi-0)}$ and another from $z=e^{i(+0)}$ to $z=\left(1+\rho^{*}\right) e^{i(+0)}$. By the formula 3.191(2) from Gradshteyn and Ryzik [15], we have

$$
\begin{aligned}
& \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x=B(n-1 / 2,3 / 2)=\frac{\Gamma(n-1 / 2) \Gamma(3 / 2)}{\Gamma(n+1)} \\
& =\frac{\prod_{m=2}^{n}\left(n-\frac{2 m-1}{2}\right)}{n!} \cdot \frac{1}{2} \Gamma^{2}(1 / 2)=\pi \frac{(2 n-3)!!}{2^{n} n!} .
\end{aligned}
$$

Therefore

$$
\frac{\gamma}{\pi} \int_{1}^{\infty} \frac{(x-1)^{1 / 2}}{x^{n+1}} d x=\gamma \frac{(2 n-3)!!}{2^{n} n!}=\frac{\gamma}{2 \pi^{1 / 2} n^{3 / 2}}+O\left(n^{-2}\right)
$$

Recalling that $\gamma=\left(2 p_{0}\right)^{1 / 2} \sigma^{-1}$, we complete the proof of the Lemma.
Using Lemma 3.2 and Lemma 3.3 we prove
Theorem 3.5. Suppose that $\mathbf{p}=\left\{p_{j}\right\}$ is such that $p_{0}>0, p_{1}=0$ and g.c.d. $\left(j \geq 1: p_{j+1}>0\right)=1$. (i) Then $T_{n}$, the random finite terminal tree of of Galton-Watson process with offspring distribution $\mathbf{p}=\left\{p_{j}\right\}$, and $n$ leaves, labelled uniformly at random by elements of $[n]$, is well defined for every $n$. (ii) Let $X_{n, a}$ be the total number of subsets $S \subset[n],|S|=a$, such that $S$ induces the same subtree in two independent copies of $T_{n}$. Then there is an explicit constant $c(\mathbf{p})$, such that: for $\varepsilon \in(0,1 / 2]$ and $a \geq(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$, we have $\mathbb{E}\left[X_{n, a}\right] \leq(1-\varepsilon)^{a}$. So, with probability $\geq 1-(1-\varepsilon)^{(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}}$, the largest number of leaves in a common induced subtree is $(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}$, at most. ( $c(\mathbf{p})=e / 2$ if $p_{0}=p_{2}=1 / 2$.)

Proof. By Lemma 3.2,

$$
\begin{gather*}
=\frac{(n-a)!}{\mathbb{P}\left(A_{n}\right) p_{0} n!}\left(\prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}\right)\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y),  \tag{3.12}\\
\Phi(y)=\sum_{j>1} p_{j} \Phi^{j}(y)+p_{0} y, \quad \Phi_{1}(y)=\sum_{j>1} j p_{j} \Phi^{j-1}(y) .
\end{gather*}
$$

$\Phi(y)$ and $\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})}$ are generating functions with positive coefficients; then so is their product. So we obtain a Chernoff-type bound: for $r \in(0,1)$,

$$
\begin{equation*}
\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \leq \frac{\left(1-\Phi_{1}(r)\right)^{-e(\mathcal{T})} \Phi^{\prime}(r)}{r^{n-a}} . \tag{3.13}
\end{equation*}
$$

As $\Phi(r)=1-\gamma(1-r)^{1 / 2}+O(1-r)$, we have

$$
\begin{aligned}
1-\Phi_{1}(y) & =1-\sum_{j>1} j p_{j} \Phi^{j-1}(y)=1-P^{\prime}(\Phi) \\
& =1-P^{\prime}(1)-P^{\prime \prime}(1)(\Phi-1)+O\left((\Phi-1)^{2}\right) \\
& =\chi(1-r)^{1 / 2}\left(1+O\left((1-r)^{1 / 2}\right)\right), \quad \chi:=\gamma \sigma^{2}=\left(2 p_{0} \sigma^{2}\right)^{1 / 2},
\end{aligned}
$$

and $\Phi^{\prime}(y)=O\left((1-r)^{-1 / 2}\right)$. So the RHS of (3.13) is of order $\chi^{-e(\mathcal{T})} f(r)$, with $f(r):=(1-r)^{-e(\mathcal{T}) / 2} r^{-n+a} . f(r)$ attains its maximum

$$
\frac{[n-a+e(\mathcal{T}) / 2]^{n-a+e(\mathcal{T}) / 2}}{(n-a)^{n-a}[e(\mathcal{T}) / 2]^{e(\mathcal{T}) / 2}} \leq c n^{1 / 2}\binom{n-a+e(\mathcal{T}) / 2}{n-a}
$$

at $r_{n}=\frac{n-a}{n-a+e(\mathcal{T}) / 2}$, which is $1-\Theta(a / n)$, since $e(\mathcal{T}) \in[a, 2 a]$ and $a=o(n)$, see (3.10). In addition, the binomial factor is at most $\binom{n}{n-a}=\binom{n}{a}$. Hence

$$
\begin{aligned}
& {\left[y^{n-a}\right]\left(1-\Phi_{1}(y)\right)^{-e(\mathcal{T})} \Phi^{\prime}(y) \leq c_{1} n(\chi+O(\sqrt{a / n}))^{-e(\mathcal{T})}\binom{n}{a}} \\
& \leq c_{1} n \lambda^{a}\binom{n}{a}, \quad \lambda>\lambda(\mathbf{p})= \begin{cases}\chi^{-2}, & \chi \leq 1, \\
\chi^{-1}, & \chi>1 .\end{cases}
\end{aligned}
$$

Hence, using the second line equation in (3.12), and $\mathbb{P}\left(A_{n}\right)=\Theta\left(n^{-3 / 2}\right)$, we obtain

$$
\begin{equation*}
\mathbb{P}\left(S \text { induces } \mathcal{T} \text { in } T_{n}\right) \leq \frac{c_{1} n}{a!} \lambda^{a} \prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})} \tag{3.14}
\end{equation*}
$$

Therefore we have:

$$
\mathbb{E}\left[X_{n, a}\right] \leq(n)_{a}\left(\frac{c_{1} n^{5 / 2} \lambda^{a}}{a!}\right)^{2} \sum_{\mathcal{T}} \prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}^{2}
$$

where the last sum is over all (finite) rooted trees $\mathcal{T}$, with ordered children and $a$ unlabelled leaves. Define the probability distribution $\mathbf{q}=\left\{q_{j}\right\}: q_{0}=$ $1-\sum_{j \geq 2} p_{j}^{2}>0, q_{1}=0, q_{j}=p_{j}^{2}$ for $j \geq 2$. Then we have

$$
\prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}^{2}=\left(\frac{p_{0}^{2}}{q_{0}}\right)^{a} \prod_{v \in V(\mathcal{T})} q_{d(v, \mathcal{T})} .
$$

Observe that $\sum_{j \geq 2} j q_{j}<\sum_{j \geq 2} j p_{j}=1$. Therefore the process with the offspring distribution $\mathbf{q}$ is almost surely extinct, implying that

$$
\begin{aligned}
\sum_{\mathcal{T}} \prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}^{2} & =\left(\frac{p_{0}^{2}}{q_{0}}\right)^{a} \sum_{\mathcal{T}} \prod_{v \in V(\mathcal{T})} q_{d(v, \mathcal{T})} \leq\left(\frac{p_{0}^{2}}{q_{0}}\right)^{a}\left[y^{a}\right] \Psi(y), \\
\Psi(y) & =\sum_{j \geq 0} q_{j} \Psi^{j}(y)+q_{0}(y-1),
\end{aligned}
$$

i.e. $\Psi(y)$ is the p.g.f. of the number of leaves in the terminal tree for the distribution $\mathbf{q}$. Let $C$ be a contour around $y=0$ within the circle $|y|<1$ and let $\mathcal{C}$ be a circle of radius $\rho$ which is the maximum point of $r-\sum_{j \geq 2} q_{j} r^{j}$. Then, using $y=q_{0}^{-1}\left(\Psi(y)-\sum_{j \geq 2} q_{j} \Psi^{j}(y)\right)$, we have

$$
\begin{aligned}
& {\left[y^{a}\right] \Psi(y)=a^{-1}\left[y^{a-1}\right] \Psi^{\prime}(y)=\frac{1}{2 \pi i a} \oint_{C} \frac{\Psi^{\prime}(y)}{y^{a}} d y} \\
& \quad=\frac{q_{0}^{a}}{2 \pi i a} \oint_{\mathcal{C}} \frac{d \eta}{\left(\eta-\sum_{j \geq 2} q_{j} \eta^{j}\right)^{a}} \leq \frac{q_{0}^{a} \rho}{a\left(\rho-\sum_{j \geq 2} q_{j} \rho^{j}\right)^{a}}
\end{aligned}
$$

Therefore

$$
\sum_{\mathcal{T}} \prod_{v \in V(\mathcal{T})} p_{d(v, \mathcal{T})}^{2} \leq \frac{\rho p_{0}^{2 a}}{a\left(\rho-\sum_{j \geq 2} q_{j} \rho^{j}\right)^{a}}
$$

So using $a!\geq(a / e)^{a},\binom{n}{a} \leq(n e / a)^{a}$, we obtain: for all $\lambda>\lambda(\mathbf{p})$,

$$
\mathbb{E}\left[X_{n, a}\right] \leq c_{2} \frac{n^{5}\binom{n}{a}}{a!}\left(\frac{\left(p_{0} \lambda\right)^{2}}{\rho-\sum_{j \geq 2} q_{j} \rho^{j}}\right)^{a} \leq c_{3} n^{5}\left(\frac{n}{a^{2}} \frac{\left(e p_{0} \lambda\right)^{2}}{\rho-\sum_{j \geq 2} q_{j} \rho^{j}}\right)^{a} \rightarrow 0,
$$

if $a \geq(1+\varepsilon) c(\mathbf{p}) n^{1 / 2}, c(\mathbf{p}):=e p_{0} \lambda(\mathbf{p})\left(\rho-\sum_{j \geq 2} q_{j} \rho^{j}\right)^{-1 / 2}$. Here $\lambda(\mathbf{p})=$ $\max \left(\chi(\mathbf{p})^{-2}, \chi(\mathbf{p})^{-1}\right)$, and $\chi(\mathbf{p})=\left(2 p_{0} \sigma^{2}\right)^{1 / 2}$.

Acknowledgment. I owe a debt of genuine gratitude to Ovidiu Costin and Jeff McNeal for guiding me to the Weierstrass separation theorem. I thank Daniel Bernstein, Mike Steel, and Seth Sullivant for an important feedback regarding the references [8], [6] and [24]. Mike steered me away from pursuing a false lead regarding the shapes of binary trees.

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[^0]:    Date: August 12, 2021.
    2010 Mathematics Subject Classification. 05C30, 05C80, 05C05, 34E05, 60C05.
    Key words and phrases. Random, binary tree, asymptotics.

[^1]:    Date: August 12, 2021.
    2010 Mathematics Subject Classification. 05C30, 05C80, 05C05, 34E05, 60C05.
    Key words and phrases. Random, terminal tree, asymptotics.

