

DISTRIBUTED ORDER ESTIMATION OF ARX MODEL UNDER COOPERATIVE EXCITATION CONDITION*

DIE GAN, ZHIXIN LIU[†]

Abstract. In this paper, we consider the distributed estimation problem of a linear stochastic system described by an autoregressive model with exogenous inputs (ARX) when both the system orders and parameters are unknown. We design distributed algorithms to estimate the unknown orders and parameters by combining the proposed local information criterion (LIC) with the distributed least squares method. The simultaneous estimation for both the system orders and parameters brings challenges for the theoretical analysis. Some analysis techniques, such as double array martingale limit theory, stochastic Lyapunov functions, and martingale convergence theorems are employed. For the case where the upper bounds of the true orders are available, we introduce a cooperative excitation condition, under which the strong consistency of the estimation for the orders and parameters is established. Moreover, for the case where the upper bounds of true orders are unknown, similar distributed algorithm is proposed to estimate both the orders and parameters, and the corresponding convergence analysis for the proposed algorithm is provided. We remark that our results are obtained without relying on the independency or stationarity assumptions of regression vectors, and the cooperative excitation conditions can show that all sensors can cooperate to fulfill the estimation task even though any individual sensor can not.

Key words. distributed order estimation, cooperative excitation condition, distributed least squares, convergence

AMS subject classifications. 68W15, 93B30, 93E24

1. Introduction. The statistical models are widely used in almost every field of engineering and science, and how to choose or identify an appropriate statistical models to fit observations is an important issue. The order estimation of statistical models is one of the key steps to construct the models. In fact, the investigation of the order estimation has many applications in engineering systems, such as radar [1], power systems [2], real seismic traces [3] and physiological systems [4].

In order to estimate the order of the statistical models, some criterions are proposed including AIC (Akaike's Information Criterion) [5], BIC (Bayesian Information Criterion) [6], CIC [7] (the first "C" emphasizes that the criterion is designed for feedback control systems) and their variants [8]. Based on these information criterions, considerable progresses have been made on the order estimation in time series analysis and adaptive estimation and control (e.g., [9]-[13]). Some theoretical results are also obtained for the order estimation problem. For example, Hannan and Kavalieris in [14] introduced an algorithm to estimate the model orders and system parameters, and the convergence of the algorithm were obtained with stationary input sequence. Chen and Guo in [8] introduced a modification of the BIC criterion to estimate the order of the multidimensional ARX system, where the true orders are assumed to belong to a known finite set. Furthermore, the relevant results for the estimation of the system orders were generalized in [15] to the case where the upper bounds of the true orders are unknown. After that, some development for the order estimation

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D. Gan and Z. X. Liu are with the Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P. R. China. (gandie@amss.ac.cn, lzx@amss.ac.cn)

[†] Corresponding author.

problem are provided (e.g., [16]-[19]). Recently, the genetic algorithm [20] and neural networks [21] were developed for model order estimation problem with good performance. However, the effectiveness of the proposed algorithms in [20] and [21] was verified by some simulation examples without rigorous theoretical analysis.

Over the past decade, with the development of communication technology and computer science, wireless sensor networks have attracted increasing attention of researchers due primarily to their practical applications in engineering systems, such as intelligent transportation and machine health monitoring [22]. We know that in a sensor network, each sensor can only measure partial information of the system due to its limited sensing capacity. In order to estimate the unknown states and system parameters by using data from sensor networks, the centralized and distributed methods are two common schemes, where the latter is gaining increasing popularity because of scalability, privacy and robustness against node and link failures. In distributed algorithms, each sensor only needs to communicate with its neighboring sensors in a certain domain. Some strategies including incremental strategies [23], consensus strategies [24], diffusion strategies [25], and combination of them [26] are proposed to construct the distributed algorithms. Based on these strategies, the performance analysis of the distributed estimation algorithms are investigated, for example, the consensus-based least mean squares (LMS) (e.g., [27][28]), the diffusion stochastic gradient descent algorithm [29], the diffusion Kalman filter (e.g., [30][31]), the diffusion least squares (LS) (e.g., [32]-[34]), the diffusion forgetting factor recursive least squares [35]. Most of the corresponding theoretical results are established by requiring the independency, stationarity or Gaussian assumptions for the regression vectors due to the mathematical difficulty in analyzing the product of random matrices. However, these requirements are hard to be satisfied since the regression signals may be correlated due to multi-path effect or feedback. In order to avoid using the independency and stationarity conditions of the regressors, some attempts are made for some distributed estimation algorithms. For the time-invariant unknown parameter, Xie and Guo studied the diffusion LS algorithm and established the convergence result in [36]. For time-varying unknown parameter, they investigated the consensus-based and diffusion LMS algorithm, and proposed the corresponding cooperative information condition to guarantee the stability of the algorithm (e.g., [24][37]). For the diffusion Kalman filter algorithm, we introduced the collective random observability condition and provided the stability analysis of distributed Kalman filter algorithm in [31]. We see that the analysis of all these results are established with known system orders. How to construct and analyze the distributed algorithms when the system orders are unknown brings challenges for us.

In this paper, we investigate the distributed estimation problem of linear stochastic systems described by an ARX model with unknown system orders and parameters. The estimates for the orders of each sensor are obtained by minimizing the proposed LIC, and the estimates for unknown system parameters are derived by the distributed least squares method where the system orders are replaced by the estimates of orders. The challenges in the theoretical analysis focus on the effect caused by the system noises and the coupled relationship between the estimates of system orders and parameters. We introduce some mathematical tools including the double array martingale limit theory, martingale convergence theorems, and the stochastic Lyapunov functions to study the convergence of the proposed distributed algorithms. The main contributions of this paper are summarized as follows.

- For the case where the true orders have known upper bounds, we design a distributed algorithm to simultaneously estimate both the system orders

and parameters by minimizing the proposed LIC and using the distributed LS method. A cooperative excitation condition is introduced to reflect the joint effect of multiple sensors: the estimation task can be still completed by the cooperation of the sensor networks even if any individual sensor can not. Under the cooperative excitation condition, the strong consistency of the estimates for both system orders and parameters is established.

- For the case where the upper bounds of true orders are unknown, a similar distributed algorithm is proposed where the growth rate for the upper bounds of the system orders are characterized by a nondecreasing positive function. We employ the double array martingale limit theory to deal with the difficulty arising in analyzing the cumulative effect of the system noises. The convergence analysis for system orders and parameters can also be provided.
- The theoretical results obtained in this paper do not require the assumptions of the independency and stationarity of the regression signals as used in almost all theoretical analysis of the distributed algorithms, which makes it possible for applications to the stochastic feedback systems.

The rest of this paper is organized as follows. We introduce some preliminaries including graph theory and the observation model in [section 2](#). In [section 3](#), we establish the convergence results when the upper bounds of the true orders are available. The case where the upper bounds of the true orders are unknown is investigated in [section 4](#). We present the conclusion of the paper in [section 5](#).

2. Problem Formulation.

2.1. Some Preliminaries. In this paper, we use $\mathbf{A} \in \mathbb{R}^{m \times n}$ to denote an $m \times n$ -dimensional real matrix. For a matrix \mathbf{A} , we use $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ to denote the largest and smallest eigenvalue of the matrix. $\|\mathbf{A}\|$ denotes the Euclidean norm, i.e., $\|\mathbf{A}\| = (\lambda_{\max}(\mathbf{A}\mathbf{A}^T))^{\frac{1}{2}}$, where the notation T denotes the transpose operator. We use $\det(\cdot)$ to denote the determinant of the corresponding matrix. For a symmetric matrix \mathbf{A} , if all eigenvalues of \mathbf{A} are positive (or nonnegative), then it is a positive definite (semipositive) matrix. Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ are two symmetric matrices, and \mathbf{C} is an $n \times m$ -dimensional matrix. Then by the Rayleigh quotient of the symmetric matrix, we can easily obtain the following inequality,

$$(2.1) \quad \lambda_{\min} \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \leq \lambda_{\min}(\mathbf{A}).$$

The matrix inversion formula is used in our analysis, and we list it here.

LEMMA 2.1. [38] *For any matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} with suitable dimensions, the following formula*

$$(\mathbf{A} + \mathbf{B}\mathbf{D}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}^{-1} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

holds, provided that the relevant matrices are invertible.

If all elements of a matrix $\mathbf{A} = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ are nonnegative, then it is a nonnegative matrix, and furthermore if $\sum_{j=1}^n a_{ij} = 1$ holds for all $i \in \{1, \dots, n\}$, then it is called a stochastic matrix.

Let $\{\mathbf{A}_k\}$ be a matrix sequence and $\{b_k\}$ be a positive scalar sequence. Then by $\mathbf{A}_k = O(b_k)$ we mean that there exists a constant $C > 0$ such that $\|\mathbf{A}_k\| \leq Cb_k$, $\forall k \geq 0$, and by $\mathbf{A}_k = o(b_k)$ we mean that $\lim_{k \rightarrow \infty} \|\mathbf{A}_k\|/b_k = 0$.

In this paper, our purpose is to estimate both system orders and parameters in a distributed way and establish the corresponding convergence results. We use an

undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to describe the relationship between sensors where \mathcal{V} is the set of sensors and \mathcal{E} is the edge set. The adjacency matrix $\mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ is introduced to reflect the weight of the corresponding edge. The elements of \mathcal{A} satisfy: $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The set of the neighbors of sensor i is denoted as $N_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. A path of length ℓ is a sequence of $\ell + 1$ sensors such that the subsequent sensors are connected. The graph \mathcal{G} is called connected if for any two sensors i and j , there is a path connecting them. The diameter $D_{\mathcal{G}}$ of the graph \mathcal{G} is defined as the maximum length of the path between any two sensors. For simplicity of analysis, the convergence of the estimates in this paper is considered under the condition that the weighted adjacency matrix \mathcal{A} is symmetric and stochastic. Thus, it is obvious that \mathcal{A} is doubly stochastic.

2.2. Observation model. We consider a network composed of n sensors. At each time instant t ($t = 0, 1, 2, \dots$), the input signal $u_{t,i} \in \mathbb{R}$ and the output signal $y_{t,i} \in \mathbb{R}$ of sensor $i \in \{1, \dots, n\}$ are assumed to obey the following linear stochastic ARX model,

$$(2.2) \quad \begin{aligned} y_{t+1,i} &= b_1 y_{t,i} + \dots + b_{p_0} y_{t+1-p_0,i} + c_1 u_{t,i} + \dots + c_{q_0} u_{t+1-q_0,i} + w_{t+1,i}, \\ y_{t,i} &= 0, \quad u_{t,i} = 0, \quad \text{for } t \leq 0, \end{aligned}$$

where $\{w_{t,i}\}$ is a noise process, p_0, q_0 are unknown true orders ($b_{p_0} \neq 0, c_{q_0} \neq 0$) and $b_1, \dots, b_{p_0}, c_1, \dots, c_{q_0}$ are unknown parameters.

Denote the unknown parameter vector $\boldsymbol{\theta}(p, q)$ and the corresponding regression vector $\boldsymbol{\varphi}_{t,i}(p, q)$ as

$$(2.3) \quad \boldsymbol{\theta}(p, q) = [b_1, \dots, b_p, c_1, \dots, c_q]^T,$$

$$(2.4) \quad \boldsymbol{\varphi}_{t,i}(p, q) = [y_{t,i}, \dots, y_{t+1-p,i}, u_{t,i}, \dots, u_{t+1-q,i}]^T.$$

If $p > p_0$, then $b_j = 0$ for $p_0 < j \leq p$; and if $q > q_0$, then $c_m = 0$ for $q_0 < m \leq q$. The regression model (2.2) can be rewritten as

$$(2.5) \quad y_{t+1,i} = \boldsymbol{\theta}^T(p, q) \boldsymbol{\varphi}_{t,i}(p, q) + w_{t+1,i} \quad (\text{for } p \geq p_0 \text{ and } q \geq q_0)$$

$$(2.6) \quad = \boldsymbol{\theta}^T(p_0, q_0) \boldsymbol{\varphi}_{t,i}(p_0, q_0) + w_{t+1,i}.$$

The purpose of this paper is to design the distributed algorithm for each sensor by using the local information from its neighbors to estimate both the system orders p_0, q_0 and the parameter vector $\boldsymbol{\theta}(p_0, q_0)$. We know that for the case where the system orders p_0, q_0 are known, the distributed LS algorithm is one of the most basic algorithms to estimate the unknown parameters, and it has wide applications because of its fast convergence rate, e.g., in the area of cloud technologies (e.g., [39]). The details of the distributed LS algorithm can be found in the following [Algorithm 2.1](#) (see [36]).

In this section, for any given (p, q) , the estimation error between the true parameter and the estimate obtained by [Algorithm 2.1](#) is denoted as $\tilde{\boldsymbol{\theta}}_{t,i}(p, q)$,

$$(2.12) \quad \tilde{\boldsymbol{\theta}}_{t,i}(p, q) = [b_1 - b_{1,t}^i, \dots, b_p - b_{p,t}^i, c_1 - c_{1,t}^i, \dots, c_q - c_{q,t}^i]^T,$$

where $\{b_{j,t}^i\}_{j=1}^p$ and $\{c_{r,t}^i\}_{r=1}^q$ are denoted as the estimates of the corresponding components of $\boldsymbol{\theta}_{t,i}(p, q)$ obtained by [Algorithm 2.1](#).

We have the following result on the estimation error $\tilde{\boldsymbol{\theta}}_{t,i}(p, q)$, which will be helpful for the subsequent theoretical analysis.

Algorithm 2.1 Distributed LS Algorithm

For any given $i \in \{1, \dots, n\}$ and given system order (p, q) , begin with an initial estimate $\theta_{0,i}(p, q)$, and an initial positive definite matrix $\mathbf{P}_{0,i}(p, q)$. The distributed LS algorithm is recursively defined at time instant $t \geq 0$ as follows.

1: Adaptation.

$$(2.7) \quad \begin{aligned} \bar{\theta}_{t+1,i}(p, q) &= \theta_{t,i}(p, q) + d_{t,i}(p, q) \mathbf{P}_{t,i}(p, q) \varphi_{t,i}(p, q) \\ &\quad \cdot (y_{t+1,i} - \varphi_{t,i}^T(p, q) \theta_{t,i}(p, q)), \end{aligned}$$

$$(2.8) \quad \bar{\mathbf{P}}_{t+1,i}(p, q) = \mathbf{P}_{t,i}(p, q) - d_{t,i}(p, q) \mathbf{P}_{t,i}(p, q) \varphi_{t,i}(p, q) \varphi_{t,i}^T(p, q) \mathbf{P}_{t,i}(p, q),$$

$$(2.9) \quad d_{t,i}(p, q) = [1 + \varphi_{t,i}^T(p, q) \mathbf{P}_{t,i}(p, q) \varphi_{t,i}(p, q)]^{-1},$$

2: Diffusion.

$$(2.10) \quad \mathbf{P}_{t+1,i}^{-1}(p, q) = \sum_{j \in N_i} a_{ij} \bar{\mathbf{P}}_{t+1,j}^{-1}(p, q),$$

$$(2.11) \quad \theta_{t+1,i}(p, q) = \mathbf{P}_{t+1,i}(p, q) \sum_{j \in N_i} a_{ij} \bar{\mathbf{P}}_{t+1,j}^{-1}(p, q) \bar{\theta}_{t+1,j}(p, q).$$

LEMMA 2.2. For $p \geq p_0$ and $q \geq q_0$, the following equation holds,

$$(2.13) \quad \mathbf{P}_{t+1,i}^{-1}(p, q) \tilde{\theta}_{t+1,i}(p, q) = \sum_{j \in N_i} a_{ij} \mathbf{P}_{t,j}^{-1}(p, q) \tilde{\theta}_{t,j}(p, q) - \sum_{j \in N_i} a_{ij} \varphi_{t,j}(p, q) w_{t+1,j}.$$

Proof. For simplicity of expression, we use $d_{t,i}$, $\varphi_{t,i}$, $\mathbf{P}_{t,i}$, $\bar{\mathbf{P}}_{t+1,i}$, $\bar{\theta}_{t+1,i}$, $\tilde{\theta}_{t,i}$ and $\theta_{t+1,i}$ to denote $d_{t,i}(p, q)$, $\varphi_{t,i}(p, q)$, $\mathbf{P}_{t,i}(p, q)$, $\bar{\mathbf{P}}_{t+1,i}(p, q)$, $\bar{\theta}_{t+1,i}(p, q)$, $\tilde{\theta}_{t,i}(p, q)$ and $\theta_{t+1,i}(p, q)$. By (2.9), we have

$$(2.14) \quad d_{t,i} = 1 - d_{t,i} \varphi_{t,i}^T \mathbf{P}_{t,i} \varphi_{t,i}.$$

Combining this with (2.7) and (2.8), we have

$$\begin{aligned} \bar{\theta}_{t+1,i} &= (\mathbf{I} - d_{t,i} \mathbf{P}_{t,i} \varphi_{t,i} \varphi_{t,i}^T) \theta_{t,i} + d_{t,i} \mathbf{P}_{t,i} \varphi_{t,i} y_{t+1,i} \\ &= (\mathbf{I} - d_{t,i} \mathbf{P}_{t,i} \varphi_{t,i} \varphi_{t,i}^T) \theta_{t,i} + \mathbf{P}_{t,i} \varphi_{t,i} (1 - d_{t,i} \varphi_{t,i}^T \mathbf{P}_{t,i} \varphi_{t,i}) y_{t+1,i} \\ &= (\mathbf{P}_{t,i} - d_{t,i} \mathbf{P}_{t,i} \varphi_{t,i} \varphi_{t,i}^T \mathbf{P}_{t,i}) \mathbf{P}_{t,i}^{-1} \theta_{t,i} + (\mathbf{P}_{t,i} - d_{t,i} \mathbf{P}_{t,i} \varphi_{t,i} \varphi_{t,i}^T \mathbf{P}_{t,i}) \varphi_{t,i} y_{t+1,i} \\ &= \bar{\mathbf{P}}_{t+1,i} \mathbf{P}_{t,i}^{-1} \theta_{t,i} + \bar{\mathbf{P}}_{t+1,i} \varphi_{t,i} y_{t+1,i}. \end{aligned}$$

Hence we have

$$\bar{\mathbf{P}}_{t+1,i}^{-1} \bar{\theta}_{t+1,i} = \mathbf{P}_{t,i}^{-1} \theta_{t,i} + \varphi_{t,i} y_{t+1,i}.$$

Substituting this equation into (2.11) yields

$$(2.15) \quad \mathbf{P}_{t+1,i}^{-1} \theta_{t+1,i} = \sum_{j \in N_i} a_{ij} (\mathbf{P}_{t,j}^{-1} \theta_{t,j} + \varphi_{t,j} y_{t+1,j}).$$

By (2.8) and Lemma 2.1, we have

$$(2.16) \quad \bar{\mathbf{P}}_{t+1,i}^{-1} = \mathbf{P}_{t,i}^{-1} + \varphi_{t,i} \varphi_{t,i}^T.$$

Hence by (2.10), (2.15) and (2.16), we have

$$\begin{aligned}
& \mathbf{P}_{t+1,i}^{-1} \tilde{\boldsymbol{\theta}}_{t+1,i} = \mathbf{P}_{t+1,i}^{-1} \boldsymbol{\theta} - \mathbf{P}_{t+1,i}^{-1} \boldsymbol{\theta}_{t+1,i} \\
& = \sum_{j \in N_i} a_{ij} \bar{\mathbf{P}}_{t+1,j}^{-1} \boldsymbol{\theta} - \sum_{j \in N_i} a_{ij} (\mathbf{P}_{t,j}^{-1} \boldsymbol{\theta}_{t,j} + \boldsymbol{\varphi}_{t,j} y_{t+1,j}) \\
& = \sum_{j \in N_i} a_{ij} (\mathbf{P}_{t,j}^{-1} + \boldsymbol{\varphi}_{t,j} \boldsymbol{\varphi}_{t,j}^T) \boldsymbol{\theta} - \sum_{j \in N_i} a_{ij} (\mathbf{P}_{t,j}^{-1} \boldsymbol{\theta}_{t,j} + \boldsymbol{\varphi}_{t,j} \boldsymbol{\varphi}_{t,j}^T \boldsymbol{\theta} + \boldsymbol{\varphi}_{t,j} w_{t+1,j}) \\
& = \sum_{j \in N_i} a_{ij} \mathbf{P}_{t,j}^{-1} \tilde{\boldsymbol{\theta}}_{t,j} - \sum_{j \in N_i} a_{ij} \boldsymbol{\varphi}_{t,j} w_{t+1,j},
\end{aligned}$$

which completes the proof of the lemma. \square

For the case where the system orders p_0, q_0 are known, Xie and Guo in [36] proved that the distributed LS algorithm can converge to the true parameters almost surely (a.s.) under a cooperative excitation condition. However, when the system orders p_0, q_0 are unknown, the estimation for both the system orders and the parameters makes the design and analysis of the distributed algorithms quite complicated. We will deal with such a problem in the following two sections.

3. Case I: The upper bounds of true orders are known. In this section, we will first design the distributed algorithm to estimate both the system orders (p_0, q_0) and the parameter vector $\boldsymbol{\theta}(p_0, q_0)$ for the case where the system orders have known upper bounds, i.e.,

$$(p_0, q_0) \in M \triangleq \{(p, q), 0 \leq p \leq p^*, 0 \leq q \leq q^*\},$$

where p^* and q^* are known upper bounds of the system orders.

For convenience of analysis, we introduce some notations and assumptions,

$$\begin{aligned}
\mathbf{d}_t(p, q) &= \text{diag}\{d_{t,1}(p, q), \dots, d_{t,n}(p, q)\}, \\
\boldsymbol{\Phi}_t(p, q) &= \text{diag}\{\boldsymbol{\varphi}_{t,1}(p, q), \dots, \boldsymbol{\varphi}_{t,n}(p, q)\}, \\
\mathbf{W}_t &= [w_{t,1}, \dots, w_{t,n}]^T, \\
\mathbf{P}_t(p, q) &= \text{diag}\{\mathbf{P}_{t,1}(p, q), \dots, \mathbf{P}_{t,n}(p, q)\}, \\
\bar{\mathbf{P}}_t(p, q) &= \text{diag}\{\bar{\mathbf{P}}_{t,1}(p, q), \dots, \bar{\mathbf{P}}_{t,n}(p, q)\}, \\
\tilde{\boldsymbol{\Theta}}_t(p, q) &= \text{col}\{\tilde{\boldsymbol{\theta}}_{t,1}(p, q), \dots, \tilde{\boldsymbol{\theta}}_{t,n}(p, q)\},
\end{aligned}$$

where $\text{col}(\cdot, \dots, \cdot)$ denotes a vector stacked by the specified vectors, and $\text{diag}(\cdot, \dots, \cdot)$ denotes a block matrix formed in a diagonal manner of the corresponding vectors or matrices.

In order to propose and further analyze the distributed algorithm used to estimate both the system order and the parameters, we introduce some assumptions on the network topology, and the observation noise and the regression vectors.

ASSUMPTION 3.1. *The communication graph \mathcal{G} is connected.*

REMARK 3.1. *Denote $\mathcal{A}^l \triangleq [a_{ij}^{(l)}]$ with \mathcal{A} being the weighted adjacency matrix of graph \mathcal{G} , i.e., $a_{ij}^{(l)}$ is the (i, j) -th entry of the matrix \mathcal{A}^l , $l \geq 1$ and $a_{ij}^{(1)} = a_{ij}$. Under [Assumption 3.1](#), we can easily obtain that \mathcal{A}^l is a positive matrix for $l \geq D_{\mathcal{G}}$ by the theory of product of stochastic matrices, which means that $a_{ij}^{(l)} > 0$ for any i and j .*

ASSUMPTION 3.2. For any $i \in \{1, \dots, n\}$, the noise sequence $\{w_{t,i}, \mathcal{F}_t\}$ is a martingale difference sequence where \mathcal{F}_t is a sequence of nondecreasing σ -algebras generated by $\{y_{k,i}, u_{k,i}, k \leq t, i = 1, 2, \dots, n\}$, and there exists a constant $\beta > 2$ such that

$$\sup_{t \geq 0} E[|w_{t+1,i}|^\beta | \mathcal{F}_t] < \infty, \text{ a.s.}$$

where $E[\cdot | \cdot]$ denotes the conditional expectation operator.

ASSUMPTION 3.3. (Cooperative Excitation Condition I). There exists a sequence $\{a_t\}$ of positive real numbers satisfying $a_t \xrightarrow{t \rightarrow \infty} \infty$ and

$$(3.1) \quad \frac{\log r_t(p^*, q^*)}{a_t} \xrightarrow{t \rightarrow \infty} 0, \quad \frac{a_t}{\lambda_{\min}^{p,q}(t)} \xrightarrow{t \rightarrow \infty} 0, \quad \text{for } (p, q) \in M^* \quad \text{a.s.}$$

where $M^* = \{(p_0, q^*), (p^*, q_0)\}$, $r_t(p, q) = \lambda_{\max}\{\mathbf{P}_0^{-1}(p, q)\} + \sum_{i=1}^n \sum_{k=0}^t \|\varphi_{k,i}(p, q)\|^2$ and

$$\lambda_{\min}^{p,q}(t) = \lambda_{\min} \left\{ \sum_{j=1}^n \mathbf{P}_{0,j}^{-1}(p, q) + \sum_{j=1}^n \sum_{k=0}^{t-D_{\mathcal{G}+1}} \varphi_{k,j}(p, q) \varphi_{k,j}^T(p, q) \right\}.$$

REMARK 3.2. We give an explanation for the choice of $\{a_t\}$ in [Assumption 3.3](#) for two typical cases: (I) If the regression vectors $\varphi_{k,i}(p^*, q^*)$ are bounded for any $i \in \{1, \dots, n\}$, and satisfy the ergodicity property, i.e., there exists a matrix \mathbf{H}_i such that $\frac{1}{t} \sum_{k=1}^t \varphi_{k,i}(p^*, q^*) \varphi_{k,i}^T(p^*, q^*) \xrightarrow{t \rightarrow \infty} \mathbf{H}_i$ with $\sum_{i=1}^n \mathbf{H}_i$ being positive definite (see e.g., [40]), then a_t can be taken as $a_t = t^\rho$, $0 < \rho < 1$; (II) If there exist three positive constants s_1, s_2 and s_3 (they may depend on the sample ω) such that

$$\sum_{i=1}^n \sum_{k=0}^t (\|y_{k,i}\|^2 + \|u_{k,i}\|^2) = O(t^{s_1}), \quad \text{a.s.}$$

$$\lambda_{\min}^{p,q}(t) \geq s_2 (\log t)^{1+s_3}, \quad \text{for } (p, q) \in M^*, \quad \text{a.s.,}$$

then [Assumption 3.3](#) can be also satisfied by taking $a_t = (\log t) \log \log t$.

REMARK 3.3. For the case where there is only one sensor ($n = 1$), Guo et al. in [7] investigated the strong consistency of the order estimate under the following conditions,

$$(3.2) \quad \frac{\log(\sum_{k=0}^t \|\varphi_{k,1}(p^*, q^*)\|^2 + 1)}{a_t} \xrightarrow{t \rightarrow \infty} 0, \quad \text{a.s.}$$

$$\frac{a_t}{\lambda_{\min}(\sum_{k=0}^t \varphi_{k,1}(p, q) \varphi_{k,1}^T(p, q) + \gamma \mathbf{I})} \xrightarrow{t \rightarrow \infty} 0, \quad \text{for } (p, q) \in M^*, \quad \text{a.s.,}$$

where γ is a positive constant, and $\{a_t, t \geq 1\}$ is a sequence of positive numbers. [Assumption 3.3](#) can be considered as an extension of (3.2) to the case of multiple sensors.

REMARK 3.4. Cooperative Excitation Condition I (i.e., [Assumption 3.3](#)) can reflect the joint effect of multiple sensors to some extent: all sensors may cooperatively estimate the unknown orders and parameters under [Assumption 3.3](#) (see [Theorem 3.4](#) and [Theorem 3.5](#)), even though any individual sensor can not fulfill the estimation task since the single sensor may be lack of adequate excitation to satisfy the condition (3.2).

In the following, we propose an algorithm to estimate the system orders p_0 and q_0 in a distributed way. For this propose, we introduce a local information criterion $L_{t,i}(p, q)$ for the sensor i at the time instant $t \geq 0$,

$$(3.3) \quad L_{t,i}(p, q) = \sigma_{t,i}(p, q, \boldsymbol{\theta}_{t,i}(p, q)) + (p + q)a_t,$$

where $\sigma_{0,i}(p, q, \boldsymbol{\beta}(p, q)) = 0$, and $\sigma_{t,i}(p, q, \boldsymbol{\beta}(p, q))$ is recursively defined for $t > 0$ as follows,

$$(3.4) \quad \sigma_{t,i}(p, q, \boldsymbol{\beta}(p, q)) = \sum_{j \in N_i} a_{ij} (\sigma_{t-1,j}(p, q, \boldsymbol{\beta}(p, q)) + [y_{t,j} - \boldsymbol{\beta}^T(p, q)\boldsymbol{\varphi}_{t-1,j}(p, q)]^2).$$

By $\sigma_{0,i}(p, q, \boldsymbol{\beta}(p, q)) = 0$, (3.4) is equivalent to the following equation,

$$(3.5) \quad \sigma_{t,i}(p, q, \boldsymbol{\beta}(p, q)) = \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} [y_{k+1,j} - \boldsymbol{\beta}^T(p, q)\boldsymbol{\varphi}_{k,j}(p, q)]^2.$$

When the upper bounds of orders are known, the distributed algorithm to estimate the system orders and parameters is put forward by minimizing LIC (i.e., $L_{t,i}(p, q)$) and using [Algorithm 2.1](#). It is clear that in (3.3), the first term is used to minimize the error between the observation signals and the prediction, while the penalty term “ $(p + q)a_t$ ” is introduced to avoid overfitting. The details of the algorithm can be found in [Algorithm 3.1](#).

Algorithm 3.1

For any given $i \in \{1, \dots, n\}$, the distributed estimation of the system orders and parameters can be obtained at the time instant $t \geq 1$ as follows.

Step 1 : For any $(p, q) \in M$, based on $\{\boldsymbol{\varphi}_{k,j}(p, q), y_{k+1,j}\}_{k=1}^{t-1}$, $j \in N_i$, the estimate $\boldsymbol{\theta}_{t,i}(p, q)$ can be obtained by using [Algorithm 2.1](#).

Step 2 : (Order Estimation) With the estimates $\{\boldsymbol{\theta}_{t,i}(p, q)\}_{(p,q) \in M}$ obtained by Step 1, the estimates $(p_{t,i}, q_{t,i})$ of system orders are given by minimizing $L_{t,i}(p, q)$, i.e.,

$$(3.6) \quad (p_{t,i}, q_{t,i}) = \arg \min_{(p,q) \in M} L_{t,i}(p, q).$$

Step 3 : (Parameter Estimation) The estimate $\boldsymbol{\theta}_{t,i}(p_{t,i}, q_{t,i})$ for the unknown parameter $\boldsymbol{\theta}(p_0, q_0)$ can be obtained by using [Algorithm 2.1](#), where the orders (p, q) are replaced by the estimates $(p_{t,i}, q_{t,i})$ obtained in Step 2.

Repeating the above steps, we obtain the order estimates $p_{t,i}, q_{t,i}$ and parameter estimates $\boldsymbol{\theta}_{t,i}(p_{t,i}, q_{t,i})$ for $t \geq 0$ and $i = 1, 2, \dots, n$.

In the following, we will analyze the convergence of the estimation for system orders and parameters obtained in [Algorithm 3.1](#). To this end, we first introduce some preliminary lemmas.

LEMMA 3.1. [36] *In [Algorithm 2.1](#), for any fixed p, q and $t \geq 1$, we have*

$$\lambda_{\max}\{\mathbf{d}_t(p, q)\boldsymbol{\Phi}_t^T(p, q)\mathbf{P}_t(p, q)\boldsymbol{\Phi}_t(p, q)\} \leq \frac{\det(\mathbf{P}_{t+1}^{-1}(p, q)) - \det(\mathbf{P}_t^{-1}(p, q))}{\det(\mathbf{P}_{t+1}^{-1}(p, q))} \leq 1.$$

LEMMA 3.2. [36] *Under Assumptions 3.1 and 3.2, we have for $p \geq p_0$ and $q \geq q_0$,*

$$\sum_{i=1}^n \tilde{\boldsymbol{\theta}}_{t,i}^T(p, q)\mathbf{P}_{t,i}^{-1}(p, q)\tilde{\boldsymbol{\theta}}_{t,i}(p, q) = O(\log r_t(p, q)),$$

where $r_t(p, q)$ is defined in [Assumption 3.3](#).

How to deal with the effect of the noises is a crucial step for the convergence analysis of [Algorithm 3.1](#), and the following lemma provides an upper bound for the cumulative summation of the noises.

LEMMA 3.3. *Under Assumptions 3.1 and 3.2, we have for any fixed p, q ,*

$$\mathbf{S}_{t+1,i}^T(p, q) \mathbf{P}_{t+1,i}(p, q) \mathbf{S}_{t+1,i}(p, q) = O(\log r_t(p, q)),$$

where $\mathbf{S}_{t+1,i}(p, q) = \sum_{j=1}^n \sum_{k=0}^t a_{ij}^{(t+1-k)} \boldsymbol{\varphi}_{k,j}(p, q) w_{k+1,j}$, and $a_{ij}^{(t+1-k)}$ is the i -th row, j -th column entry of the weight matrix \mathcal{A}^{t+1-k} .

Proof. For the convenience of expression, we use $\mathbf{S}_{t,i}$, \mathbf{P}_k , $\boldsymbol{\Phi}_k$ and \mathbf{d}_k to denote $\mathbf{S}_{t,i}(p, q)$, $\mathbf{P}_k(p, q)$, $\boldsymbol{\Phi}_k(p, q)$ and $\mathbf{d}_k(p, q)$.

Set $\mathbf{S}_0 = 0$ and $\mathbf{S}_t = \text{col}\{\mathbf{S}_{t,1}, \dots, \mathbf{S}_{t,n}\}$. Then we have

$$\mathbf{S}_{k+1} = \sum_{l=0}^k \mathcal{A}^{k+1-l} \boldsymbol{\Phi}_l \mathbf{W}_{l+1} = \mathcal{A}(\mathbf{S}_k + \boldsymbol{\Phi}_k \mathbf{W}_{k+1}).$$

By (2.8) and Lemma 4.2 in [36], we have

$$\begin{aligned} \mathbf{S}_{k+1}^T \mathbf{P}_{k+1} \mathbf{S}_{k+1} &= (\mathbf{S}_k^T + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T) \mathcal{A}^T \mathbf{P}_{k+1} \mathcal{A} (\mathbf{S}_k + \boldsymbol{\Phi}_k \mathbf{W}_{k+1}) \\ &\leq (\mathbf{S}_k^T + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T) \bar{\mathbf{P}}_{k+1} (\mathbf{S}_k + \boldsymbol{\Phi}_k \mathbf{W}_{k+1}) \\ &= (\mathbf{S}_k^T + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T) (\mathbf{P}_k - \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k) (\mathbf{S}_k + \boldsymbol{\Phi}_k \mathbf{W}_{k+1}) \\ &= \mathbf{S}_k^T \mathbf{P}_k \mathbf{S}_k + 2 \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \mathbf{S}_k + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{W}_{k+1} - \mathbf{S}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \mathbf{S}_k \\ (3.7) \quad &- 2 \mathbf{S}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{W}_{k+1} - \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{W}_{k+1}. \end{aligned}$$

Moreover, by the definition of \mathbf{d}_k and (2.14), we have

$$(3.8) \quad \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k = \mathbf{I} - \mathbf{d}_k.$$

By (2.8) and (3.8), we derive that

$$(3.9) \quad \begin{aligned} \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k &= \mathbf{P}_k \boldsymbol{\Phi}_k - \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \\ &= \mathbf{P}_k \boldsymbol{\Phi}_k - \mathbf{P}_k \boldsymbol{\Phi}_k (\mathbf{I} - \mathbf{d}_k) = \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k. \end{aligned}$$

Substituting (3.8) into (3.7), we have by (3.9)

$$\begin{aligned} \mathbf{S}_{k+1}^T \mathbf{P}_{k+1} \mathbf{S}_{k+1} &\leq \mathbf{S}_k^T \mathbf{P}_k \mathbf{S}_k + 2 \mathbf{S}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \mathbf{W}_{k+1} - \mathbf{S}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \boldsymbol{\Phi}_k^T \mathbf{P}_k \mathbf{S}_k \\ &\quad + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \mathbf{W}_{k+1} \\ &= \mathbf{S}_k^T \mathbf{P}_k \mathbf{S}_k + 2 \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k \mathbf{W}_{k+1} - \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k \mathbf{d}_k^{-1} \boldsymbol{\Phi}_k^T \bar{\mathbf{P}}_{k+1} \mathbf{S}_k \\ &\quad + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \mathbf{W}_{k+1} \\ &\leq \mathbf{S}_k^T \mathbf{P}_k \mathbf{S}_k + 2 \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k \mathbf{W}_{k+1} - \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k \boldsymbol{\Phi}_k^T \bar{\mathbf{P}}_{k+1} \mathbf{S}_k \\ &\quad + \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \mathbf{W}_{k+1}. \end{aligned}$$

By the summation of both sides of the above inequality, we have

$$(3.10) \quad \begin{aligned} &\mathbf{S}_{t+1} \mathbf{P}_{t+1} \mathbf{S}_{t+1} + \sum_{k=0}^t \|\mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k\|^2 \\ &\leq 2 \sum_{k=0}^t \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \boldsymbol{\Phi}_k \mathbf{W}_{k+1} + \sum_{k=0}^t \mathbf{W}_{k+1}^T \boldsymbol{\Phi}_k^T \mathbf{P}_k \boldsymbol{\Phi}_k \mathbf{d}_k \mathbf{W}_{k+1}. \end{aligned}$$

Next, we estimate the two terms on the right hand side of (3.10) separately. By [Assumption 3.2](#) and the martingale estimation theorem (see, e.g., [41]), we can get the following inequality,

$$(3.11) \quad \sum_{k=0}^t \mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \Phi_k \mathbf{W}_{k+1} = O(1) + o\left(\sum_{k=0}^t \|\mathbf{S}_k^T \bar{\mathbf{P}}_{k+1} \Phi_k\|^2\right).$$

Then by the proof of Lemma 4.4 in [36], we obtain

$$(3.12) \quad \sum_{k=0}^t \mathbf{W}_{k+1}^T \Phi_k^T \mathbf{P}_k \Phi_k \mathbf{d}_k \mathbf{W}_{k+1} = \sum_{k=0}^t \mathbf{W}_{k+1}^T \mathbf{d}_k \Phi_k^T \mathbf{P}_k \Phi_k \mathbf{W}_{k+1} = O(\log r_t).$$

Substituting (3.11) and (3.12) into (3.10) yields

$$\mathbf{S}_{t+1} \mathbf{P}_{t+1} \mathbf{S}_{t+1} = O(\log r_t),$$

which completes the proof of the lemma. \square

REMARK 3.5. If [Assumption 3.2](#) is relaxed to the following weaker noise condition

$$(3.13) \quad \sup_{t \geq 0} E[|w_{t+1,i}|^2 | \mathcal{F}_t] < \infty, \text{ a.s.},$$

then under [Assumption 3.1](#) similar results as those of [Lemma 3.2](#) and [Lemma 3.3](#) can also be obtained, save that the term “ $\log r_t(p, q)$ ” in [Lemma 3.2](#) and [Lemma 3.3](#) is replaced by “ $\log r_t(p, q) (\log \log r_t(p, q))^\tau$ (for some $\tau > 1$)”.

Now, we present the main results concerning the convergence of the order estimates obtained by [Algorithm 3.1](#).

THEOREM 3.4. Under [Assumptions 3.1-3.3](#), the order estimate sequence $(p_{t,i}, q_{t,i})$ given by (3.6) converges to the true order (p_0, q_0) almost surely, i.e.,

$$(p_{t,i}, q_{t,i}) \xrightarrow[t \rightarrow \infty]{} (p_0, q_0), \quad \text{a.s. for } i \in \{1, \dots, n\}.$$

Proof. For $i \in \{1, \dots, n\}$, we need to show that the sequence $(p_{t,i}, q_{t,i})$ has only one limit point (p_0, q_0) . Let $(p'_i, q'_i) \in M$ be a limit point of $(p_{t,i}, q_{t,i})$, i.e., there is a subsequence $\{t_k\}$ such that

$$(3.14) \quad (p_{t_k,i}, q_{t_k,i}) \xrightarrow[k \rightarrow \infty]{} (p'_i, q'_i).$$

In order to prove $(p_{t,i}, q_{t,i}) \xrightarrow[t \rightarrow \infty]{} (p_0, q_0)$, we just need to show the impossibility of the following two situations,

- (i) $p'_i \geq p_0, q'_i \geq q_0$ and $p'_i + q'_i > p_0 + q_0$,
- (ii) $p'_i < p_0$ or $q'_i < q_0$.

Note that both $p_{t_k,i}$ and $q_{t_k,i}$ are integers, by (3.14) we have $(p_{t_k,i}, q_{t_k,i}) \equiv (p'_i, q'_i)$ for sufficiently large k . We first show that the situation (i) will not happen by reduction to absurdity.

Suppose that (i) holds. By (2.5) and (3.5), we see that $\sigma_{t_k,i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k,i}(p'_i, q'_i))$

can be calculated by the following equation,

$$\begin{aligned}
& \sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) \\
&= \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} [y_{l+1, j} - \boldsymbol{\theta}_{t_k, i}^T(p'_i, q'_i) \boldsymbol{\varphi}_{l, j}(p'_i, q'_i)]^2 \\
&= \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} [\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) + w_{l+1, j}]^2 \\
&= \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) \boldsymbol{\varphi}_{l, j}^T(p'_i, q'_i) \right) \tilde{\boldsymbol{\theta}}_{t_k, i}(p'_i, q'_i) \\
(3.15) \quad &+ 2\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) w_{l+1, j} \right) + \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1, j}^2.
\end{aligned}$$

By [Lemma 3.2](#) and [Lemma 3.3](#), we have the following relationship,

$$\begin{aligned}
& \left| \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) w_{l+1, j} \right) \right| \\
& \leq \left\| \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \mathbf{P}_{t_k, i}^{-\frac{1}{2}}(p'_i, q'_i) \right\| \left\| \mathbf{P}_{t_k, i}^{\frac{1}{2}}(p'_i, q'_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) w_{l+1, j} \right) \right\| \\
(3.16) \quad &= O(\log(r_{t_k}(p'_i, q'_i))) = O(\log(r_{t_k}(p^*, q^*))).
\end{aligned}$$

By [\(2.10\)](#) and [\(2.16\)](#), we have for any p and q

$$(3.17) \quad \mathbf{P}_{t_k, i}^{-1}(p, q) = \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p, q) \boldsymbol{\varphi}_{l, j}^T(p, q) + \sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p, q).$$

By this equation and [Lemma 3.2](#), we can easily obtain that

$$\begin{aligned}
& \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p'_i, q'_i) \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p, q) \boldsymbol{\varphi}_{l, j}^T(p'_i, q'_i) \tilde{\boldsymbol{\theta}}_{t_k, i}(p'_i, q'_i) \\
(3.18) \quad &= O(\log r_{t_k}(p'_i, q'_i)) = O(\log(r_{t_k}(p^*, q^*))).
\end{aligned}$$

Substituting [\(3.16\)](#) and [\(3.18\)](#) into [\(3.15\)](#), we see that there exists a positive constant C_1 satisfying

$$(3.19) \quad \sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1, j}^2 \geq -C_1 \log(r_{t_k}(p^*, q^*)).$$

Now, we will consider $\sigma_{t_k, i}(p_0, q_0, \boldsymbol{\theta}_{t_k, i}(p_0, q_0))$. By [Lemma 2.2](#), we have for $p \geq p_0$ and $q \geq q_0$,

$$\begin{aligned}
& \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p, q) w_{l+1, j} \\
(3.20) \quad &= \sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p, q) \tilde{\boldsymbol{\theta}}_{0, j}(p, q) - \mathbf{P}_{t_k, i}^{-1}(p, q) \tilde{\boldsymbol{\theta}}_{t_k, i}(p, q).
\end{aligned}$$

By a similar way as that used in (3.15), we obtain

$$\begin{aligned}
& \sigma_{t_k, i}(p_0, q_0, \boldsymbol{\theta}_{t_k, i}(p_0, q_0)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1, j}^2 \\
&= \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(p_0, q_0) \boldsymbol{\varphi}_{l, j}^T(p_0, q_0) \right) \tilde{\boldsymbol{\theta}}_{t_k, i}(p_0, q_0) \\
& \quad + 2\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}((p_0, q_0)) w_{l+1, j} \right).
\end{aligned}$$

Combining this with (3.17) and (3.20) yields

$$\begin{aligned}
& \sigma_{t_k, i}(p_0, q_0, \boldsymbol{\theta}_{t_k, i}(p_0, q_0)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1, j}^2 \\
&= \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \mathbf{P}_{t_k, i}^{-1}(p_0, q_0) \tilde{\boldsymbol{\theta}}_{t_k, i}(p_0, q_0) \\
& \quad - \tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p_0, q_0) \right) \tilde{\boldsymbol{\theta}}_{t_k, i}(p_0, q_0) \\
& \quad + 2\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p_0, q_0) \tilde{\boldsymbol{\theta}}_{0, j}(p_0, q_0) - \mathbf{P}_{t_k, i}^{-1}(p_0, q_0) \tilde{\boldsymbol{\theta}}_{t_k, i}(p_0, q_0) \right) \\
& \leq -\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p_0, q_0) \right) \tilde{\boldsymbol{\theta}}_{t_k, i}(p_0, q_0) \\
& \quad + 2\tilde{\boldsymbol{\theta}}_{t_k, i}^T(p_0, q_0) \left(\sum_{j=1}^n a_{ij}^{(t_k)} \mathbf{P}_{0, j}^{-1}(p_0, q_0) \tilde{\boldsymbol{\theta}}_{0, j}(p_0, q_0) \right) \\
(3.21) \quad & \leq \left(\sum_{j=1}^n a_{ij}^{(t_k)} \tilde{\boldsymbol{\theta}}_{0, j}^T(p_0, q_0) \mathbf{P}_{0, j}^{-1}(p_0, q_0) \tilde{\boldsymbol{\theta}}_{0, j}(p_0, q_0) \right) = O(1),
\end{aligned}$$

where the last inequality is obtained by

$$(3.22) \quad 2\mathbf{x}^T \mathbf{A} \mathbf{y} \leq \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{y}^T \mathbf{A} \mathbf{y} \quad \text{for } \mathbf{A} \geq 0.$$

From (3.19) and (3.21), we have,

$$\sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) - \sigma_{t_k, i}(p_0, q_0, \boldsymbol{\theta}_{t_k, i}(p_0, q_0)) \geq -C_1 \log r_{t_k}(p^*, q^*) - C_2,$$

where C_2 is a positive constant. Note that $(p_{t_k, i}, q_{t_k, i}) = \arg \min_{p, q \in M} L_{t_k, i}(p, q)$. By [Assumption 3.3](#), we have

$$\begin{aligned}
0 & \geq L_{t_k, i}(p_{t_k, i}, q_{t_k, i}) - L_{t_k, i}(p_0, q_0) = L_{t_k, i}(p'_i, q'_i) - L_{t_k, i}(p_0, q_0) \\
& = \sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) - \sigma_{t_k, i}(p_0, q_0, \boldsymbol{\theta}_{t_k, i}(p_0, q_0)) + (p'_i + q'_i - p_0 - q_0) a_{t_k} \\
& \geq -C_1 \log r_{t_k}(p^*, q^*) - C_2 + (p'_i + q'_i - p_0 - q_0) a_{t_k} \\
& = a_{t_k} \left(\frac{-C_1 \log r_{t_k}(p^*, q^*)}{a_{t_k}} + (p'_i + q'_i - p_0 - q_0) \right) - C_2 \rightarrow \infty, \text{ as } k \rightarrow \infty,
\end{aligned}$$

which leads to the contradiction. Thus, the situation (i) will not happen.

In the following, we will show the impossibility of situation (ii) by reduction to absurdity. Suppose that (ii) holds, i.e., $p'_i < p_0$ or $q'_i < q_0$. In order to analyze the properties of the estimate error, we introduce the following $(s_i + v_i)$ -dimensional vector with $s_i = \max\{p_0, p'_i\}$, $v_i = \max\{q_0, q'_i\}$,

$$\boldsymbol{\theta}_{t_k, i}(s_i, v_i) = [b_{1, t_k}^i, \dots, b_{s_i, t_k}^i, c_{1, t_k}^i, \dots, c_{v_i, t_k}^i]^T.$$

If $p'_i < p_0$, then $b_{m, t_k}^i \triangleq 0$ for $p'_i < m \leq p_0$; and if $q'_i < q_0$, then $c_{m, t_k}^i \triangleq 0$ for $q'_i < m \leq q_0$.

Denote $\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i) = \boldsymbol{\theta}(s_i, v_i) - \boldsymbol{\theta}_{t_k, i}(s_i, v_i)$. It is clear that

$$(3.23) \quad \|\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i)\|^2 \geq \min\{|b_{p_0}|^2, |c_{q_0}|^2\} \triangleq \alpha_0 > 0.$$

Then by (2.6), we have

$$\begin{aligned} & \sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) \\ &= \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} [\boldsymbol{\theta}^T(p_0, q_0) \boldsymbol{\varphi}_{l, j}(p_0, q_0) - \boldsymbol{\theta}_{t_k, i}^T(p'_i, q'_i) \boldsymbol{\varphi}_{l, j}(p'_i, q'_i) + w_{l+1, j}]^2. \end{aligned}$$

Hence combining this equation and the definition $\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i)$, we obtain

$$\begin{aligned} & \sigma_{t_k, i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k, i}(p'_i, q'_i)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1, j}^2 \\ &= \tilde{\boldsymbol{\theta}}_{t_k, i}^T(s_i, v_i) \mathbf{P}_{t_k, i}^{-1}(s_i, v_i) \tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i) - \tilde{\boldsymbol{\theta}}_{t_k, i}^T(s_i, v_i) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0, j}^{-1}(s_i, v_i) \right) \tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i) \\ & \quad + 2\tilde{\boldsymbol{\theta}}_{t_k, i}^T(s_i, v_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l, j}(s_i, v_i) w_{l+1, j} \right) \\ (3.24) \quad & \triangleq \tilde{\boldsymbol{\theta}}_{t_k, i}^T(s_i, v_i) \mathbf{P}_{t_k, i}^{-1}(s_i, v_i) \tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i) - M_1 + M_2. \end{aligned}$$

By (3.17) and Remark 3.1, we have for any p and q

$$(3.25) \quad \lambda_{\min}(\mathbf{P}_{t_k, i}^{-1}(p, q)) \geq a_{\min} \lambda_{\min}^{p, q}(t_k),$$

where $a_{\min} = \min_{i, j \in \mathcal{V}} a_{ij}^{(D_G)} > 0$. Hence, by (3.25) and Lemma 3.2, we have for $p \geq p_0$ and $q \geq q_0$,

$$(3.26) \quad \sum_{i=1}^n \|\tilde{\boldsymbol{\theta}}_{t+1, i}(p, q)\|^2 = O\left(\frac{\log r_t(p, q)}{\lambda_{\min}^{p, q}(t)}\right).$$

When $p'_i < p_0$ (so does the case $q'_i < q_0$), we can use (3.26) in the first p'_i components of $\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i)$. Then by (2.1) and Assumption 3.3, we obtain $\|\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i)\| = O(1)$, hence we have

$$(3.27) \quad M_1 \leq \lambda_{\max} \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0, j}^{-1}(s_i, v_i) \right) \|\tilde{\boldsymbol{\theta}}_{t_k, i}(s_i, v_i)\|^2 = O(1).$$

In the following, we will analyze M_2 . Similar to the analysis of (3.16), by Lemma 3.3, we have

$$\begin{aligned}
|M_2| &\leq \left\| \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-\frac{1}{2}}(s_i, v_i) \right\| \cdot \left\| \mathbf{P}_{t_k,i}^{\frac{1}{2}}(s_i, v_i) \left(\sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} \boldsymbol{\varphi}_{l,j}(s_i, v_i) w_{l+1,j} \right) \right\| \\
&= O \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-\frac{1}{2}}(s_i, v_i) \right\| \cdot \log^{\frac{1}{2}}(r_{t_k}(s_i, v_i)) \right\} \\
(3.28) &= O \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-\frac{1}{2}}(s_i, v_i) \right\| \cdot \log^{\frac{1}{2}}(r_{t_k}(p^*, q^*)) \right\}.
\end{aligned}$$

Therefore, by (3.24)-(3.28), we see that there exist two positive constants C_3 and C_4 such that

$$\begin{aligned}
&\sigma_{t_k,i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k,i}(p'_i, q'_i)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1,j}^2 \\
&\geq \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-1}(s_i, v_i) \tilde{\boldsymbol{\theta}}_{t_k,i}(s_i, v_i) - C_3 \\
(3.29) \quad &- C_4 \left\| \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-\frac{1}{2}}(s_i, v_i) \right\| \cdot \log^{\frac{1}{2}}(r_{t_k}(p^*, q^*)).
\end{aligned}$$

By Assumption 3.3, (3.23) and (3.25), we have

$$\left\| \tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-\frac{1}{2}}(s_i, v_i) \right\| \cdot \log^{\frac{1}{2}}(r_{t_k}(p^*, q^*)) = o(\tilde{\boldsymbol{\theta}}_{t_k,i}^T(s_i, v_i) \mathbf{P}_{t_k,i}^{-1}(s_i, v_i) \tilde{\boldsymbol{\theta}}_{t_k,i}(s_i, v_i)).$$

Furthermore, by (3.23) and (3.25), we have

$$\begin{aligned}
&\sigma_{t_k,i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k,i}(p'_i, q'_i)) - \sum_{j=1}^n \sum_{l=0}^{t_k-1} a_{ij}^{(t_k-l)} w_{l+1,j}^2 \\
&= a_{\min} \alpha_0 \lambda_{\min}^{s_i, v_i}(t_k) (1 + o(1)) - C_3 \\
(3.30) \quad &\geq \frac{a_{\min} \alpha_0 \min\{\lambda_{\min}^{p_0, q^*}(t_k), \lambda_{\min}^{p^*, q_0}(t_k)\}}{2} - C_3,
\end{aligned}$$

where (2.1) is used in the last inequality.

By (3.3), (3.21), (3.30) and Assumption 3.3, for large k and some positive constant C_5 , we have the following inequality for $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
0 &\geq L_{t_k,i}(p_{t_k,i}, q_{t_k,i}) - L_{t_k,i}(p_0, q_0) = L_{t_k,i}(p'_i, q'_i) - L_{t_k,i}(p_0, q_0) \\
&= \sigma_{t_k,i}(p'_i, q'_i, \boldsymbol{\theta}_{t_k,i}(p'_i, q'_i)) - \sigma_{t_k,i}(p_0, q_0, \boldsymbol{\theta}_{t_k,i}(p_0, q_0)) + (p'_i + q'_i - p_0 - q_0) a_{t_k} \\
&\geq \frac{a_{\min} \alpha_0 \min\{\lambda_{\min}^{p_0, q^*}(t_k), \lambda_{\min}^{p^*, q_0}(t_k)\}}{2} - C_5 + (p'_i + q'_i - p_0 - q_0) a_{t_k} \\
&\geq \frac{a_{\min} \alpha_0 \min\{\lambda_{\min}^{p_0, q^*}(t_k), \lambda_{\min}^{p^*, q_0}(t_k)\}}{2} \left(1 + \frac{2(p'_i + q'_i - p_0 - q_0) a_{t_k}}{a_{\min} \alpha_0 \min\{\lambda_{\min}^{p_0, q^*}(t_k), \lambda_{\min}^{p^*, q_0}(t_k)\}} \right) - C_5 \\
&\geq \frac{a_{\min} \alpha_0 \min\{\lambda_{\min}^{p_0, q^*}(t_k), \lambda_{\min}^{p^*, q_0}(t_k)\}}{4} - C_5 \rightarrow \infty,
\end{aligned}$$

which leads to contradiction. The proof of the theorem is complete. \square

REMARK 3.6. Under Assumption 3.1 and the weaker noise condition (3.13), by Remark 3.5, we can verify that the result of Theorem 3.4 still holds by taking the

sequence $\{a_t\}$ in [Assumption 3.3](#) to satisfy the following conditions,

$$\frac{\log r_t(p^*, q^*) (\log \log r_t(p^*, q^*))^\tau}{a_t} \xrightarrow{t \rightarrow \infty} 0, \quad \frac{a_t}{\lambda_{\min}^{p,q}(t)} \xrightarrow{t \rightarrow \infty} 0, \quad \text{a.s.},$$

where $(p, q) \in M^*$.

REMARK 3.7. In [Algorithm 3.1](#), if the estimates $(p_{t,i}, q_{t,i})$ for the system order (p_0, q_0) are obtained by the following step,

$$(3.31) \quad p_{t,i} = \arg \min_{1 \leq p \leq p^*} L_{t,i}(p, q^*), \quad q_{t,i} = \arg \min_{1 \leq q \leq q^*} L_{t,i}(p^*, q),$$

then the sequence $(p_{t,i}, q_{t,i})$ can also converge to the true order (p_0, q_0) by a similar argument as that used in [Theorem 3.4](#). At each time instant t , we need to search $p^* \times q^*$ points to find the minimum of the function $L_{t,i}(p, q)$ in [\(3.6\)](#), while in [\(3.31\)](#) we just need to search at most $p^* + q^*$ points.

Note that both the estimates $(p_{t,i}, q_{t,i})$ and the true orders (p_0, q_0) are integers, from [Theorem 3.4](#), we see that there exists a large enough T such that $p_{t,i} = p_0$ and $q_{t,i} = q_0$ for $t \geq T$. Thus, from [\(3.26\)](#) and [Assumption 3.3](#), we have the following consistent estimation of the parameter vector $\boldsymbol{\theta}(p_0, q_0)$.

THEOREM 3.5. Under the conditions of [Theorem 3.4](#), for any $i \in \{1, \dots, n\}$, we have

$$\boldsymbol{\theta}_{t,i}(p_{t,i}, q_{t,i}) \xrightarrow{t \rightarrow \infty} \boldsymbol{\theta}(p_0, q_0), \quad \text{a.s.}$$

where $\boldsymbol{\theta}_{t,i}(p_{t,i}, q_{t,i})$ is obtained by [Algorithm 3.1](#).

4. Case II: The upper bounds of true orders are unknown. In this section, we consider a general case where the upper bounds of the system orders are unknown. We first give some assumptions on the system signals and the noise.

ASSUMPTION 4.1. For $i \in \{1, \dots, n\}$, the noise sequence $\{w_{t,i}, \mathcal{F}_t\}$ is a martingale difference sequence satisfying

$$\sup_{t \geq 0} E[|w_{t+1,i}|^2 | \mathcal{F}_t] < \infty, \quad \|w_{t,i}\| = O(\eta_i(t)) \quad \text{a.s.}$$

where \mathcal{F}_t is defined in [Assumption 3.2](#) and $\eta_i(t)$ is a positive, deterministic, nondecreasing function satisfying

$$\sup_t \eta_i(e^{t+1}) / \eta_i(e^t) < \infty.$$

In order to simplify the analysis of the estimation error, we need to introduce an assumption on the input and output signals which implies that the system is not explosive. This assumption is commonly used in the stability analysis of the closed-loop feedback control systems for a single sensor case (see e.g., [\[7\]](#), [\[8\]](#) and [\[15\]](#)).

ASSUMPTION 4.2. There exists a positive constant b such that the input and output signals satisfy

$$(4.1) \quad \sum_{i=1}^n \sum_{k=0}^{t-1} (\|y_{k,i}\|^2 + \|u_{k,i}\|^2) = O(t^b) \quad \text{a.s.}$$

Similar to [Assumption 3.3](#) in [section 3](#), we introduce the following cooperative excitation condition which can be considered as an extension of the excitation condition used in [\[15\]](#) for a single sensor to the distributed order estimation algorithm when the upper bounds of true orders are unknown. This condition can also reflect the joint effect of multiple sensors as illustrated in [Remark 3.4](#).

ASSUMPTION 4.3. (Cooperative Excitation Condition II). *A sequence $\{\bar{a}_t\}$ of positive real numbers can be found such that $\bar{a}_t \xrightarrow[t \rightarrow \infty]{} \infty$ and*

$$(4.2) \quad \frac{h_t \log t + [\eta(t) \log \log t]^2}{\bar{a}_t} \xrightarrow[t \rightarrow \infty]{} 0, \quad \frac{\bar{a}_t}{\lambda_{\min}^0(t)} \xrightarrow[t \rightarrow \infty]{} 0,$$

hold almost surely, where

$$\lambda_{\min}^0(t) = \lambda_{\min} \left\{ \sum_{j=1}^n P_{0,j}^{-1}(m_0, m_0) + \sum_{i=1}^n \sum_{k=0}^{t-D_G} \varphi_{k,i}^0 (\varphi_{k,i}^0)^T \right\},$$

with $\eta(t) \triangleq (\sum_{i=1}^n \eta_i^2(t))^{\frac{1}{2}}$, $\varphi_{t,i}^0 = [y_{t,i}, \dots, y_{t-m_0+1,i}, u_{t,i}, \dots, u_{t-m_0+1,i}]^T$, $m_0 \triangleq \max\{p_0, q_0\}$ and the regression lag h_t is chosen as $h_t = O((\log t)^\alpha)$ ($\alpha > 1$), and $\log t = o(h_t)$.

We are now to construct the algorithm to estimate both the system orders and parameters in a distributed way when the upper bounds of orders are unknown. For estimating the unknown orders (p_0, q_0) , we introduce the following local information criterion $\bar{L}_{t,i}(p, q)$ for the sensor i ,

$$(4.3) \quad \bar{L}_{t,i}(p, q) = \sigma_{t,i}(p, q, \boldsymbol{\theta}_{t,i}(p, q)) + (p + q)\bar{a}_t,$$

where $\sigma_{t,i}(p, q, \boldsymbol{\beta}(p, q))$ is recursively defined in [\(3.4\)](#).

By minimizing the local information criterion [\(4.3\)](#) and using [Algorithm 2.1](#), we obtain the following distributed algorithm.

Algorithm 4.1

For any given sensor $i \in \{1, \dots, n\}$, the distributed algorithm for the estimation of the system orders and the parameters is defined at the time instant $t \geq 1$ as follows.

Step 1 : For any $0 \leq s \leq \lfloor \log t \rfloor$, based on $\{\varphi_{k,j}(s, s), y_{k+1,j}\}_{k=1}^{t-1}$, $j \in N_i$, the estimate $\boldsymbol{\theta}_{t,i}(s, s)$ can be obtained by using [Algorithm 2.1](#), where the orders (p, q) are replaced by (s, s) ($0 \leq s \leq \lfloor \log t \rfloor$).

Step 2 : (Order estimation) With the estimates $\{\boldsymbol{\theta}_{t,i}(s, s)\}_{s=0}^{\lfloor \log t \rfloor}$ obtained by Step 1,
 take $\hat{m}_{t,i}$ by minimizing $\bar{L}_{t,i}(s, s)$ for $0 \leq s \leq \lfloor \log t \rfloor$;
 take $\hat{p}_{t,i}$ by minimizing $\bar{L}_{t,i}(p, \hat{m}_{t,i})$ for $0 \leq p \leq \hat{m}_{t,i}$;
 take $\hat{q}_{t,i}$ by minimizing $\bar{L}_{t,i}(\hat{p}_{t,i}, q)$ for $0 \leq q \leq \hat{m}_{t,i}$.

Step 3 : (Parameter estimation) The estimate $\boldsymbol{\theta}_{t,i}(\hat{p}_{t,i}, \hat{q}_{t,i})$ for the unknown parameter vector $\boldsymbol{\theta}(p_0, q_0)$ is obtained by using [Algorithm 2.1](#), where the orders (p, q) are replaced by the estimates $(\hat{p}_{t,i}, \hat{q}_{t,i})$ obtained by Step 2.

Output : $\hat{p}_{t,i}$, $\hat{q}_{t,i}$ and $\boldsymbol{\theta}_{t,i}(\hat{p}_{t,i}, \hat{q}_{t,i})$.

REMARK 4.1. *In Step 2 of the above [Algorithm 4.1](#), we first estimate the maximum value m_0 of true orders whose upper bound is characterized by the function $\log t$. Then the true orders p_0, q_0 are obtained by searching among at most $2\hat{m}_{t,i}$ points at each time instant t .*

In the following, we will provide the consistency analysis of [Algorithm 4.1](#) when the upper bounds of orders are unknown, in which a crucial step is to prove that for any i , $\hat{m}_{t,i} \rightarrow m_0$ as $t \rightarrow \infty$. Then by the order estimation procedure in [Algorithm 4.1](#), the convergence of the estimates for the system orders and parameters can be obtained by a similar analysis as those in [section 3](#). To this end, we need to introduce the following double array martingale estimation lemma to deal with the noise effect in the form of $\max_{1 \leq m \leq h_t} \left\| \sum_{k=1}^t f_k(m) w_{k+1} \right\|$.

LEMMA 4.1. [15] *Let $\{v_t, \mathcal{F}_t\}$ be an s' -dimensional martingale difference sequence satisfying $\|v_t\| = o(\rho(t))$ a.s., $\sup_t E(\|v_{t+1}\|^2 | \mathcal{F}_t) < \infty$ a.s., where the properties of $\rho(t)$ is described as same as $\eta_i(t)$ in [Assumption 4.1](#). Assume that $f_t(m), t, m = 1, 2, \dots$, is an \mathcal{F}_t -measurable, $r' \times s'$ -dimensional random matrix satisfying $\|f_t(m)\| \leq C < \infty$ a.s. for all t, m and some deterministic constant C . Then for $h_t = O([\log t]^\alpha)$ ($\alpha > 1$), the following property holds as $t \rightarrow \infty$,*

$$\max_{1 \leq m \leq h_t} \max_{1 \leq j \leq t} \left\| \sum_{k=1}^j f_k(m) v_{k+1} \right\| = O \left(\max_{1 \leq m \leq h_t} \sum_{k=1}^t \|f_k(m)\|^2 \right) + o(\rho(t) \log \log t), \quad \text{a.s.}$$

In order to simplify the expression of the following lemmas and theorems, we write (l) for (l, l) in $\theta_{t,i}, \varphi_{t,i}$ and $P_{t,i}$ when $p = q = l$.

LEMMA 4.2. *Let $V_t(l) = \tilde{\Theta}_t^T(l) P_t^{-1}(l) \tilde{\Theta}_t(l)$. Then under [Assumption 3.1](#) and [Assumptions 4.1-4.3](#), we have*

$$\max_{m_0 \leq l \leq h_t} V_{t+1}(l) = O(h_t \log t) + o(\eta^2(t) \log \log t),$$

where h_t and $\eta(t)$ are defined in [Assumption 4.3](#).

Proof. By the proof of Lemma 4.4 in [36], we have for $l \geq m_0$

$$(4.4) \quad V_{t+1}(l) = O(1) + \sum_{k=0}^t \mathbf{W}_{k+1}^T \mathbf{d}_k(l) \Phi_k^T(l) P_k(l) \Phi_k(l) \mathbf{W}_{k+1}.$$

By (3.17) and [Lemma 3.1](#), we have

$$(4.5) \quad \begin{aligned} & \max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) P_k(l) \Phi_k(l) \} \\ & \leq \max_{1 \leq l \leq h_t} [\log \det(P_{t+1}^{-1}(l)) - \log \det(P_0^{-1}(l))] \\ & = O \left\{ \max_{1 \leq l \leq h_t} \left(l \cdot \log \left(\lambda_{\max} P_0^{-1}(l) + \sum_{j=1}^n \sum_{k=0}^t \|\varphi_{k,j}(l)\|^2 \right) \right) \right\} = O(h_t \log t), \end{aligned}$$

where [Assumption 4.2](#) is used in the last equation. By [Assumption 4.1](#), [Lemma 3.1](#) and [Lemma 4.1](#), we have

$$\begin{aligned} & \max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) P_k(l) \Phi_k(l) \} (\|\mathbf{W}_{k+1}\|^2 - E(\|\mathbf{W}_{k+1}\|^2 | \mathcal{F}_k)) \\ & = o(\eta^2(t) \log \log t) + O \left(\max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) P_k(l) \Phi_k(l) \} \right). \end{aligned}$$

Hence by (4.5) and Assumption 4.1, we have

$$\begin{aligned}
& \max_{1 \leq l \leq h_t} \sum_{k=0}^t \mathbf{W}_{k+1}^T \mathbf{d}_k(l) \Phi_k^T(l) \mathbf{P}_k(l) \Phi_k(l) \mathbf{W}_{k+1} \\
& \leq \max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) \mathbf{P}_k(l) \Phi_k(l) \} \| \mathbf{W}_{k+1} \|^2 \\
& \leq \max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) \mathbf{P}_k(l) \Phi_k(l) \} (\| \mathbf{W}_{k+1} \|^2 - E(\| \mathbf{W}_{k+1} \|^2 | \mathcal{F}_k)) \\
& \quad + \max_{1 \leq l \leq h_t} \sum_{k=0}^t \lambda_{\max} \{ \mathbf{d}_k(l) \Phi_k^T(l) \mathbf{P}_k(l) \Phi_k(l) \} E(\| \mathbf{W}_{k+1} \|^2 | \mathcal{F}_k) \\
(4.6) \quad & = o(\eta^2(t) \log \log t) + O(h_t \log t).
\end{aligned}$$

Substituting (4.6) into (4.4) yields the result of the lemma. \square

LEMMA 4.3. Under Assumption 3.1, Assumptions 4.1-4.3, for any $i \in \{1, \dots, n\}$, we have

$$\max_{1 \leq l \leq h_t} \{ \mathbf{S}_{t,i}^T(l) \mathbf{P}_{t,i}(l) \mathbf{S}_{t,i}(l) \} = O(h_t \log t) + o(\{\eta(t) \log \log t\}^2),$$

where $\mathbf{S}_{t,i}(l) = \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \varphi_{k,j}(l) w_{k+1,j} \right)$, and h_t and $\eta(t)$ are defined in Assumption 4.3.

The above lemma can be derived by following the proof of Lemma 3.3, and we omit it here.

The following theorem will establish the convergence of the estimates $\hat{m}_{t,i}$, $\hat{p}_{t,i}$, $\hat{q}_{t,i}$ and $\boldsymbol{\theta}_{t,i}(\hat{p}_{t,i}, \hat{q}_{t,i})$ given by Algorithm 4.1 to the true values.

THEOREM 4.4. Under Assumption 3.1, Assumptions 4.1-4.3, we have for any $i \in \{1, \dots, n\}$,

$$(4.7) \quad \hat{m}_{t,i} \xrightarrow[t \rightarrow \infty]{} m_0 \quad \text{a.s.}$$

$$(4.8) \quad (\hat{p}_{t,i}, \hat{q}_{t,i}) \xrightarrow[t \rightarrow \infty]{} (p_0, q_0) \quad \text{a.s.}$$

$$(4.9) \quad \boldsymbol{\theta}_{t,i}(\hat{p}_{t,i}, \hat{q}_{t,i}) \xrightarrow[t \rightarrow \infty]{} \boldsymbol{\theta}(p_0, q_0) \quad \text{a.s.}$$

Proof. We first show that $\limsup_{t \rightarrow \infty} \hat{m}_i(t) \leq m_0$ a.s.
For $p > p_0, q > q_0$, set

$$\begin{aligned}
\boldsymbol{\theta}(p, q) &= [b_1, \dots, b_p, c_1, \dots, c_q]^T, \\
\tilde{\boldsymbol{\theta}}_{t,i}(p, q) &= \boldsymbol{\theta}(p, q) - \boldsymbol{\theta}_{t,i}(p, q),
\end{aligned}$$

where $b_p = 0, p > p_0, c_q = 0, q > q_0, \boldsymbol{\theta}_{t,i}(p, q)$ is obtained by Algorithm 2.1.

Then by (3.5), for $l \geq m_0$, we have

$$\begin{aligned}
\sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) &= \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} [\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \boldsymbol{\varphi}_{k,j}(l) + w_{k+1,j}]^2 \\
&= \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(l) \boldsymbol{\varphi}_{k,j}^T(l) \right) \tilde{\boldsymbol{\theta}}_{t,i}(l) \\
&\quad + 2\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(l) w_{k+1,j} \right) + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 \\
(4.10) \quad &\triangleq I_1 + I_2 + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2.
\end{aligned}$$

In the following, we estimate I_1, I_2 separately.

On the one hand, by (3.17) (3.20) and (3.22), we have

$$\begin{aligned}
I_1 + I_2 &= \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\mathbf{P}_{t,i}^{-1}(l) - \sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(l) \right) \tilde{\boldsymbol{\theta}}_{t,i}(l) \\
&\quad + 2\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(l) \tilde{\boldsymbol{\theta}}_{0,j}(l) - \mathbf{P}_{t,i}^{-1}(l) \tilde{\boldsymbol{\theta}}_{t,i}(l) \right) \\
&= -\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-1}(l) \tilde{\boldsymbol{\theta}}_{t,i}(l) + 2\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(l) \tilde{\boldsymbol{\theta}}_{0,j}(l) \right) \\
&\quad - \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(l) \right) \tilde{\boldsymbol{\theta}}_{t,i}(l) \\
(4.11) \quad &\leq -\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-1}(l) \tilde{\boldsymbol{\theta}}_{t,i}(l) + \sum_{j=1}^n \left(a_{ij}^{(t)} \tilde{\boldsymbol{\theta}}_{0,j}^T(l) \mathbf{P}_{0,j}^{-1}(l) \tilde{\boldsymbol{\theta}}_{0,j}(l) \right).
\end{aligned}$$

Hence by (4.10) and (4.11), we have for $l \geq m_0$,

$$(4.12) \quad \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) - \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 \leq -\tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-1}(l) \tilde{\boldsymbol{\theta}}_{t,i}(l) + O(1).$$

On the other hand, by Lemma 4.3, we have

$$\begin{aligned}
|I_2| &\leq 2 \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-\frac{1}{2}}(l) \right\| \cdot \left\| \mathbf{P}_{t,i}^{\frac{1}{2}}(l) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(l) w_{k+1,j} \right) \right\| \\
&\leq O \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-\frac{1}{2}}(l) \right\| \cdot \left\{ o([\eta(t) \log \log t]^2) + O(h_t \log t) \right\}^{\frac{1}{2}} \right\}.
\end{aligned}$$

Then for $l \geq m_0$ and sufficiently large t , by (3.17), (4.10), and Assumption 4.3, we

have for some positive constant C_6 ,

$$\begin{aligned} & \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) - \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 \\ & \geq \frac{1}{2} \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-1}(l) \tilde{\boldsymbol{\theta}}_{t,i}(l) \\ & \quad - C_6 \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-\frac{1}{2}}(l) \right\| \cdot \{o([\eta(t) \log \log t]^2) + h_t \log t\}^{\frac{1}{2}} \right\}. \end{aligned}$$

Hence by (4.3) and Lemma 4.2, we have for sufficiently large t ,

$$\begin{aligned} & \max_{m_0 < l \leq \log t} \{ \bar{L}_{t,i}(m_0, m_0) - \bar{L}_{t,i}(l, l) \} \\ & = \max_{m_0 < l \leq \log t} \left\{ \sigma_{t,i}(m_0, m_0, \boldsymbol{\theta}_{t,i}(m_0, m_0)) - \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 \right. \\ & \quad \left. - \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 - 2(l - m_0) \bar{a}_t \right\} \\ & \leq \max_{m_0 < l \leq \log t} O \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(l) \mathbf{P}_{t,i}^{-\frac{1}{2}}(l) \right\| \{o([\eta(t) \log \log t]^2) + O(h_t \log t)\}^{\frac{1}{2}} \right\} + O(1) - 2\bar{a}_t \\ & = o([\eta(t) \log \log t]^2) + O(h_t \log t) + O(1) - 2\bar{a}_t < 0. \end{aligned}$$

By the above equation, we have

$$\bar{L}_{t,i}(m_0, m_0) < \min_{m_0 < l \leq \log t} \bar{L}_{t,i}(l, l),$$

which implies that $\limsup_{t \rightarrow \infty} \hat{m}_{t,i} \leq m_0$.

We now show that $\liminf_{t \rightarrow \infty} \hat{m}_{t,i} \geq m_0$ holds almost surely. For any $l \leq m_0$, let us write $\boldsymbol{\theta}_{t,i}(l)$ given by Algorithm 2.1 in its component form

$$\boldsymbol{\theta}_{t,i}(l) = [b_{1,t}^i, \dots, b_{l,t}^i, c_{1,t}^i, \dots, c_{l,t}^i]^T \in \mathbb{R}^{2l}.$$

In order to avoid confusion, for any $l \leq m_0$, we denote the following m_0 -dimensional vector,

$$\boldsymbol{\theta}_{t,i}(m_0) = [b_{1,t}^i, \dots, b_{m_0,t}^i, c_{1,t}^i, \dots, c_{m_0,t}^i]^T \in \mathbb{R}^{2m_0},$$

where $b_{j,t}^i = 0, c_{j,t}^i = 0$ for $m_0 > j > l$.

For any $l \leq m_0$, we have

$$\begin{aligned} & y_{k+1,j} - \boldsymbol{\theta}_{t,i}^T(l) \boldsymbol{\varphi}_{k,j}(l) = y_{k+1,j} - \boldsymbol{\theta}_{t,i}^T(m_0) \boldsymbol{\varphi}_{k,j}(m_0) \\ & = y_{k+1,j} - \boldsymbol{\theta}^T(m_0) \boldsymbol{\varphi}_{k,j}(m_0) + [\boldsymbol{\theta}^T(m_0) - \boldsymbol{\theta}_{t,i}^T(m_0)] \boldsymbol{\varphi}_{k,j}(m_0) \\ & = w_{k+1,j} + \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \boldsymbol{\varphi}_{k,j}(m_0), \end{aligned}$$

where $\tilde{\boldsymbol{\theta}}_{t,i}(m_0) = \boldsymbol{\theta}(m_0) - \boldsymbol{\theta}_{t,i}(m_0) \in \mathbb{R}^{2m_0}$.

Hence by (3.5), we have for any $l \leq m_0$

$$\sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) = \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} (w_{k+1,j} + \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \boldsymbol{\varphi}_{k,j}(m_0))^2.$$

Thus, we have for any $l \leq m_0$

$$\begin{aligned}
& \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) - \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 \\
&= \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(m_0) \boldsymbol{\varphi}_{k,j}^T(m_0) \right) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) \\
(4.13) \quad & + 2\tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(m_0) w_{k+1,j} \right) \triangleq J_1 + J_2.
\end{aligned}$$

In the following, we estimate J_1, J_2 .

For $l < m_0$, by the definition of $\tilde{\boldsymbol{\theta}}_{t,i}(m_0)$, we have

$$(4.14) \quad \|\tilde{\boldsymbol{\theta}}_{t,i}(m_0)\|^2 \geq \min\{|b_{p_0}|^2, |c_{q_0}|^2\} = \alpha_0 > 0.$$

Then by (3.17), we have

$$(4.15) \quad \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-1}(m_0) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) \geq a_{\min} \lambda_{\min}^0(t) \alpha_0.$$

Moreover, by (2.1), Assumption 4.3 and Lemma 4.2, we have

$$\begin{aligned}
& \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(m_0) \right) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) \\
& \leq \lambda_{\max} \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(m_0) \right) \|\tilde{\boldsymbol{\theta}}_{t,i}(m_0)\|^2 = O(1).
\end{aligned}$$

Then for $l < m_0$, by (3.17), we obtain for some positive constant C_7 ,

$$\begin{aligned}
J_1 &= \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-1}(m_0) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) - \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \left(\sum_{j=1}^n a_{ij}^{(t)} \mathbf{P}_{0,j}^{-1}(m_0) \right) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) \\
(4.16) \quad & \geq \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-1}(m_0) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) - C_7.
\end{aligned}$$

By Lemma 4.3, we have

$$\begin{aligned}
|J_2| &\leq 2 \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-\frac{1}{2}}(m_0) \right\| \cdot \left\| \mathbf{P}_{t,i}^{\frac{1}{2}}(m_0) \left(\sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} \boldsymbol{\varphi}_{k,j}(m_0) w_{k+1,j} \right) \right\| \\
(4.17) \quad & \leq O \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-\frac{1}{2}}(m_0) \right\| \cdot \{o([\eta(t) \log \log t]^2) + O(h_t \log t)\}^{\frac{1}{2}} \right\}.
\end{aligned}$$

Then by (4.15)-(4.17) and Assumption 4.3, we have for large t

$$\begin{aligned}
J_1 + J_2 &\geq \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-1}(m_0) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) - C_8 \left\{ \left\| \tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-\frac{1}{2}}(m_0) \right\| \right. \\
&\quad \cdot \left. \{o([\eta(t) \log \log t]^2) + O(h_t \log t)\}^{\frac{1}{2}} \right\} - C_7 \\
&\geq a_{\min} \alpha_0 \lambda_{\min}^0(t) (1 + o(1)) \text{ a.s.},
\end{aligned}$$

where C_8 is a positive constant.

Hence by (4.13), we have for any $l < m_0$,

$$(4.18) \quad \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) \geq a_{\min} \alpha_0 \lambda_{\min}^0(t) (1 + o(1)) + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2.$$

Note that when $l = m_0$, by (4.12) and Lemma 4.2, we have

$$(4.19) \quad \begin{aligned} & \sigma_{t,i}(m_0, m_0, \boldsymbol{\theta}_{t,i}(m_0, m_0)) \\ & \leq -\tilde{\boldsymbol{\theta}}_{t,i}^T(m_0) \mathbf{P}_{t,i}^{-1}(m_0) \tilde{\boldsymbol{\theta}}_{t,i}(m_0) + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 + O(1) \\ & \leq O(h_t \log t) + o([\eta(t) \log \log t]^2) + \sum_{j=1}^n \sum_{k=0}^{t-1} a_{ij}^{(t-k)} w_{k+1,j}^2 + O(1). \end{aligned}$$

For any $l < m_0$, by (4.18)-(4.19) and Assumption 4.3, we have

$$\begin{aligned} & \bar{L}_{t,i}(l, l) - \bar{L}_{t,i}(m_0, m_0) \\ & = \sigma_{t,i}(l, l, \boldsymbol{\theta}_{t,i}(l, l)) - \sigma_{t,i}(m_0, m_0, \boldsymbol{\theta}_{t,i}(m_0, m_0)) + 2(l - m_0) \bar{a}_t \\ & \geq a_{\min} \alpha_0 \lambda_{\min}^0(t) (1 + o(1)) - C_9 (h_t \log t + o([\eta(t) \log \log t]^2)) + C_{10} - C_{11} \bar{a}_t \\ & = a_{\min} \lambda_{\min}^0(t) (\alpha_0 + o(1)) > 0 \text{ as } t \rightarrow \infty, \end{aligned}$$

where C_9, C_{10}, C_{11} are positive constants. Hence we have

$$\bar{L}_{t,i}(m_0, m_0) < \min_{1 \leq l < m_0} \bar{L}_{t,i}(l, l),$$

which implies that $\liminf_{t \rightarrow \infty} \hat{m}_{t,i} \geq m_0$ a.s. Thus the first assertion (4.7) has been proved.

By $\hat{m}_{t,i} \xrightarrow[t \rightarrow \infty]{} m_0$, the proof of (4.8) can be carried out by a similar argument as that used in section 3.

Note that both the estimates $(p_{t,i}, q_{t,i})$ and the true orders (p_0, q_0) are integers, we see that there exists a large enough T such that $p_{t,i} = p_0$ and $q_{t,i} = q_0$ for $t \geq T$. By the proof of Lemma 4.2, we have

$$V_t(p_0, q_0) = O(h_t \log t) + o(\eta^2(t) \log \log t).$$

Therefore,

$$\|\boldsymbol{\theta}_{t,i}(p_0, q_0) - \boldsymbol{\theta}(p_0, q_0)\|^2 = \frac{O(h_t \log t) + o(\eta^2(t) \log \log t)}{\lambda_{\min}^0(t)}.$$

The convergence of the parameters can be obtained by Assumption 4.3. This completes the proof of the theorem. \square

REMARK 4.2. From Theorem 4.4 (also Theorem 3.4 and Theorem 3.5), we see that the convergence of the estimates for both the system orders and parameters are derived without using the independency or stationarity assumptions on the regression vectors, which makes it possible to apply our distributed algorithms to practical feedback systems.

5. Conclusion. In this paper, we proposed distributed algorithms to simultaneously estimate both the unknown system orders and parameters by minimizing the LIC and using the distributed LS algorithm. For the case where the upper bounds of true orders are known, we show that the estimates of the parameters and the orders can converge to the true values under the cooperative excitation condition introduced in this paper. We note that the convergence results are obtained without using the independency and stationarity assumptions of regression vectors as commonly used in most existing literatures. Moreover, for the case where the upper bounds of true orders are unknown, we constructed similar distributed algorithm to estimate both the parameters and the orders by introducing a time-varying regression lag, and obtained the strong consistency of the distributed algorithm. The cooperative excitation condition can reveal the joint effect of multiple sensors. Many interesting problems deserve to be further investigated, for example, the distributed order estimation problem of the autoregressive moving average model with exogenous inputs (ARMAX), the recursive distributed algorithm for the order estimation problem.

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