

The 9-connected Excluded Minors for the Class of Quasi-graphic Matroids

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Abstract

The class of quasi-graphic matroids, recently introduced by Geelen, Gerards, and Whittle, is minor closed and contains both the class of lifted-graphic matroids and the class of frame matroids, each of which generalises the class of graphic matroids. In this paper, we prove that the matroids $U_{3,7}$ and $U_{4,7}$ are the only 9-connected excluded minors for the class of quasi-graphic matroids.

2010 Mathematics Subject Classification: 05B35

Keywords: matroids, quasi-graphic matroids, excluded minors

1 Introduction

Let H be a graph and let N be a matroid. For a vertex v of H we let $\text{loops}_H(v)$ denote the set of loops of H whose ends are v . We say that H is a *framework* for N if

(QG1) $E(H) = E(N)$,

(QG2) $r_N(E(H')) \leq |V(H')|$ for each component H' of H , and

(QG3) for each vertex v of H we have $\text{cl}_N(E(H - v)) \subseteq E(H - v) \cup \text{loops}_H(v)$, and

(QG4) for each circuit C of N , the graph $H[C]$ has at most two components.

A matroid is *quasi-graphic* if it has a framework. The class of quasi-graphic matroids, recently introduced by Geelen, Gerards, and Whittle [8], is minor closed and contains both lifted-graphic matroids and frame matroids. Recently, the author and Geelen [4] proved that there are infinitely many quasi-graphic excluded minors for the class of frame matroids and the class of lifted-graphic matroids, but we are confident that the class of quasi-graphic matroids admits a finite excluded-minor characterisation.

Conjecture 1.1. ([4], Conjecture 1.5.) *There are, up to isomorphism, only finitely many excluded-minors for the class of quasi-graphic matroids.*

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One of the difficulties to prove Conjecture 1.1 is that some graphic matroids have exponentially many different frameworks; for example, the rank- r wheel has at least 2^r “inequivalent” frameworks, see [3]. The same difficulty appears when considering problems on excluded minors for the class of frame matroids and for the class of lifted-graphic matroids. In fact, in the proof of Rota’s Conjecture, Geleen, Gerards, and Whittle encountered a similar difficulty. The interesting thing is: we have some kind of opposite versions in the proof of the two conjectures. For Rota’s Conjecture, the proof for the low branch-width case is not complicated, see [1, 9]; while the proof for the high branch-width case is very difficult. While, for Conjecture 1.1, the proof for low connectivity is thought to be difficult, while the proof for high connectivity is not complicated. In this paper, we prove

Theorem 1.2. *Other than $U_{3,7}$ and $U_{4,7}$, no excluded minor for the class of quasi-graphic matroids is 9-connected.*

Funk and Mayhew [7] recently proved that, for each positive integer r , the class of quasi-graphic matroids has only a finite number of excluded minors of rank r .

This paper is organized as follows. In Section 3, we prove that $U_{3,7}$ and $U_{4,7}$ are the only 9-connected excluded minors of rank less than nine for the class of quasi-graphic matroids. 9-connected excluded minors of rank at least nine are considered in Section 5. Some definitions and basic properties of quasi-graphic matroids are given in Section 2. Properties of frameworks for graphic matroids are presented in Section 4.

2 Preliminaries

We assume that the reader is familiar with matroid theory and we follow the terminology of Oxley [10].

For a graph G , let $\text{loops}(G)$ be the set of loops in G . An edge of G is a *link* if it is not a loop. For any $v \in V(G)$, let $\text{st}_G(v)$ denote the set of edges incident with v . For any $U \subseteq V(G)$ and $F \subseteq E(G)$, set $\text{st}_G(U) = \bigcup_{u \in U} \text{st}_G(u)$, and let $G[U]$ be the induced subgraph of G defined on U , and let $G[F]$ be the subgraph of G with F as its edge set and without isolated vertices. Let $c_G(F)$ be the number of components of $G[F]$, and let $V_G(F)$ denote $V(G[F])$. When $F = \{e\}$, we will let $V_G(e)$ denote $V_G(\{e\})$. When there is no confusion, all subscripts will be omitted. For a number k , we say that G is k -connected if $G - S$ has exactly one component for any $S \subset V(G)$ with $|S| < k$.

A *theta graph* is a graph that consists of a pair of distinct vertices joined by three internally disjoint paths. A *cycle* is a connected 2-regular graph. A collection \mathcal{B} of some cycles of G satisfies the *theta property* if no theta subgraph of G contains exactly two members of \mathcal{B} . A *biased graph* consists of a pair (G, \mathcal{B}) , where G is a graph and \mathcal{B} is a collection of some cycles of G that satisfies the theta property. A cycle C of G is *balanced* if $C \in \mathcal{B}$, otherwise, it is *unbalanced*.

Let H be a framework for a matroid N . For any cycle C of H , either $C \in \mathcal{C}(N)$ or $C \in \mathcal{I}(N)$ by ([8], Lemma 2.5.). Let \mathcal{B}_N be the set of cycles of H that are circuits of N . Since \mathcal{B}_N satisfies the theta property by ([8], Lemma 3.2.), (H, \mathcal{B}_N) is a biased graph. For convenience, we will also view H as the biased graph (H, \mathcal{B}_N) . A subgraph H' of H is *balanced* if each cycle in H' is balanced; otherwise, H' is *unbalanced*. If all cycles in H' are unbalanced, then H' is *contra-balanced*.

By ([8], Lemma 3.3) and (QG4), we have

Lemma 2.1. *Let H be a framework for a matroid N . When $C \in \mathcal{C}(N)$, either*

1. $H[C]$ is a balanced cycle,
2. $H[C]$ is a connected contra-balanced graph with minimum degree at least two with $|C| = |V(C)| + 1$, or
3. $H[C]$ is a union of two unbalanced cycles that meet in at most one single vertex.

Lemma 2.2. ([8], Lemma 2.6.) *Let H be a framework for a matroid N . If H' is a subgraph of H with $|E(H')| > |V(H')|$, then $E(H')$ is a dependent set of N .*

By Lemmas 2.1 and 2.2, we have

Lemma 2.3. *Let H be a framework for a matroid N . Let C_1, C_2 be unbalanced cycles of H with $|V(C_1) \cap V(C_2)| \leq 1$. Then the following hold.*

- $E(C_1 \cup C_2)$ is a circuit of N when $|V(C_1) \cap V(C_2)| = 1$.
- When $|V(C_1) \cap V(C_2)| = 0$, for each minimal path P in H linking C_1 and C_2 , we have $E(C_1 \cup C_2) \in \mathcal{C}(N)$ or $E(C_1 \cup C_2 \cup P) \in \mathcal{C}(N)$.

We say that H is a *frame representation* of a matroid N if a subset I of $E(H)$ is independent in N if and only if $H[I]$ has no balanced cycles and $|E(H')| \leq |V(H')|$ for each component H' of $H[I]$. We say that H is a *lifted-graphic representation* of N if a subset I of $E(H)$ is independent in N if and only if $H[I]$ has at most one cycle and when the cycle exists, it is unbalanced. Note that, when H is a lifted-graphic representation for a 3-connected matroid, H has at most one loop, and the loop is unbalanced.

Theorem 2.4. ([8], Theorems 7.1 and 7.2.) *Let H be a framework for a 3-connected matroid N . If H has an unbalanced loop, then H is a frame representation or a lifted-graphic representation for N .*

When H is a lifted-graphic representation for N with an unbalanced loop e , by the definition of lifted-graphic representation, all graphs obtained from $H \setminus e$ by attaching the loop e to any vertex of $H \setminus e$ or a new vertex not in $H \setminus e$ are also lifted-graphic representations of N . Under this condition, we view all graphs obtained in this way as equivalent. That is, when all frameworks for N can be obtained from H by this way, we view H as the unique framework for N .

Lemma 2.5. *Let H be a framework for a matroid N . If H is not connected but N is connected, then H is a lifted-graphic representation of N .*

Proof. Let H_1, H_2, \dots, H_n be the components of H . Since every pair of elements of $E(N)$ must be contained in a circuit of N , by Lemma 2.1 each edge of H is in an unbalanced cycle.

2.5.1. *Let C_1 and C_2 be unbalanced cycles of H_1 and H_2 , respectively. If H_2 has an unbalanced cycle C'_2 with $E(C_2) \cap E(C'_2) \neq \emptyset$ and $E(C_1 \cup C'_2) \in \mathcal{C}(N)$, then $E(C_1 \cup C_2) \in \mathcal{C}(N)$.*

Subproof. Assume not. Without loss of generality we may assume that C'_2 is chosen with $E(C_2 \cup C'_2)$ as small as possible. When $C_2 \cup C'_2$ is a theta subgraph of H_2 , since $E(C_2 \cup C'_2)$ or the third cycle in $C_2 \cup C'_2$ that is neither C_2 nor C'_2 is a circuit of N by Lemmas 2.1 and 2.2, we have $E(C_1 \cup C_2) \in C(N)$ by the circuit elimination axiom and Lemma 2.1. So we may assume that $C_2 \cup C'_2$ is not a theta subgraph of H_2 . Since $E(C_2) \cap E(C'_2) \neq \emptyset$, there is a path $P \subsetneq C_2$ such that $C'_2 \cup P$ is a theta-graph. In a similar way we can show that $E(C_1 \cup C'_2)$ is a circuit of N for an unbalanced cycle C''_2 of H_2 with $P \subseteq C''_2 \subseteq C'_2 \cup P$, a contradiction to the choice of C'_2 as $|E(C_2 \cup C'_2)| < |E(C_2 \cup C''_2)|$. \square

2.5.2. *A union of each pair of unbalanced cycles coming from different components of H is a circuit of N .*

Subproof. Let C_1 and C_2 be unbalanced cycles of H_1 and H_2 , respectively. By symmetry, it suffices to show that $E(C_1 \cup C_2) \in C(N)$. Let $e_i \in C_i$ for each $1 \leq i \leq 2$. Since N is connected, N has a circuit C containing $\{e_1, e_2\}$. Since e_1 and e_2 are in different components of H , by Lemma 2.1, there is an unbalanced cycle C'_i of H_i containing e_i for each integer $1 \leq i \leq 2$ such that $C = E(C'_1 \cup C'_2)$. Since $e_2 \in E(C_2) \cap E(C'_2)$, we have $E(C'_1 \cup C_2) \in C(N)$ by 2.5.1. Moreover, since $e_1 \in E(C_1) \cap E(C'_1)$, using 2.5.1 again, $E(C_1 \cup C_2) \in C(N)$. \square

2.5.3. *For every $1 \leq i \leq n$, a union of every pair of vertex-disjoint unbalanced cycles of H_i is a circuit of N .*

Subproof. Assume that the claim does not hold for H_1 . Then there are vertex-disjoint unbalanced cycles C_1, C'_1 of H_1 and a path P minimal linking the two cycles such that $E(C_1 \cup C'_1 \cup P)$ is a circuit of N by Lemma 2.3. Let C be a union of C_1 and an unbalanced cycle of H_2 . By 2.5.2, C is a circuit of N . Let $f \in E(C_1)$ and $g \in E(P)$. By circuit elimination axiom, there is a circuit C' of N with $g \in C' \subseteq E(C_1 \cup C'_1 \cup P \cup C) - \{f\}$, a contradiction to Lemma 2.1 as $H[C']$ has degree-1 vertices. \square

By 2.5.2 and 2.5.3, a union of every pair of vertex-disjoint unbalanced cycles of H is a circuit of N , so the lemma holds. \square

After this paper was submitted to a journal in September 2017, one of the referees told the author in his/her referee report that Lemma 2.5 was also proved in ([2], Corollary 4.7) by Bowler, Funk, and Slilaty. The two proofs are totally different.

By ([8], Lemmas 3.6 and 4.2) or Lemma 2.5 and ([8], Lemmas 4.2) we have

Lemma 2.6. *Assume that H is a framework for a 3-connected matroid N with $|E(N)| \geq 4$ and H has no isolated vertices. Then*

1. H is connected, or
2. H is a lifted-graphic representation of N with exactly two components, one of which is a loop-component.

Moreover, N has a connected framework.

Lemma 2.7. *For any integer $k \geq 2$, if H is a connected framework for a k -connected matroid N , then H is $k - 1$ connected.*

Proof. Assume not. Let (X, Y) be a partition of $E(N)$ with $m = |V_H(X) \cap V_H(Y)| \leq k - 2$ and such that $H[X]$ and $H[Y]$ are connected graphs with at least $m + 1$ vertices. When $H[X]$ and $H[Y]$ are unbalanced, implying that $|X|, |Y| \geq m + 1$, we have that (X, Y) is an $m + 1$ -separation, a contradiction. When X is balanced, (X, Y) is an m -separation, a contradiction. \square

Theorem 2.8. ([8], Theorem 1.6.) *A 3-connected matroid N is quasi-graphic if and only if there exists a graph H such that*

1. $E(H) = E(N)$,
2. H is connected,
3. $r(N) \leq |V(H)|$, and
4. for each vertex v of H we have $\text{cl}_M(E(H - v)) \subseteq E(H - v) \cup \text{loops}_H(v)$.

Lemma 2.9. *Let H be a framework for a matroid N . For an edge e of H , if $H \setminus e$ is connected and unbalanced, then e is in a circuit of N .*

Proof. By considering a maximal independent set of $H \setminus e$, it follows from (QG2) that $r(N \setminus e) = |V(H)|$. Moreover, since $r(N) \leq |V(H)|$ by (QG2), we have $r(N) = r(N \setminus e)$. So the lemma holds. \square

Let H be a framework for a matroid N . Let $H' = H - \text{loops}(H)$ when H is a lifted-graphic representation of N , otherwise let $H' = H$. A vertex $v \in V(H)$ is a *blocking vertex* if H' is unbalanced and all unbalanced cycles of H' contain v . Set $\text{st}_H^*(v) = \text{st}_{H'}(v)$. Note that $\text{st}_H^*(v)$ is the same as $\text{st}_H(v)$ unless H is lifted-graphic of N and v is incident with a loop.

Lemma 2.10. *Let H be a connected framework for a 3-connected matroid N , and $v \in V(H)$.*

1. $\text{st}_H^*(v)$ is a union of cocircuits of N .
2. v is a blocking vertex of H if and only if $\text{st}_H^*(v) \notin C^*(N)$.

Proof. (1) follows from Lemma 2.1 and Theorem 2.4. Next, we prove that (2) is true.

Note that H and H' are 2-connected by Lemma 2.7. Assume that v is a blocking vertex of H . Since $H' - v$ is connected and balanced, $r(E(H' - v)) = |V(H')| - 2 = r(N) - 2$. So $\text{st}_H^*(v) \notin C^*(N)$.

Assume that $\text{st}_H^*(v) \notin C^*(N)$. Then $\text{st}_H^*(v)$ contains at least two cocircuits of N by (1), implying $r(E(H' - v)) \leq r(N) - 2 = |V(H')| - 2$. Moreover, since $H' - v$ is connected, $H' - v$ is balanced. So v is a blocking vertex of H . \square

Let H be a connected framework for a 3-connected matroid N . We say that a vertex v of H is *fixed* in H if $N \setminus \text{st}_H^*(v)$ is a 3-connected non-graphic matroid.

Lemma 2.11. *Let H be a connected framework for a 3-connected matroid N . For an edge f of H , if v is fixed in $H \setminus f$, then v is fixed in H .*

Proof. Evidently, it suffices to show that $N \setminus \text{st}_H^*(v)$ is 3-connected. Assume not. Then $f \notin \text{st}_H^*(v)$. Since $N \setminus (\text{st}_H^*(v) \cup \{f\})$ and N are 3-connected and non-graphic, f is a coloop of $N \setminus \text{st}_H^*(v)$ and $H' - \{v, f\}$ is connected and unbalanced by Lemma 2.7 and Theorem 2.4. Then $f \in \text{st}_H(v)$ by Lemma 2.9. Since $f \notin \text{st}_H^*(v)$, we have that $\{f\} = \text{loops}_H(v)$ and H is a lifted-graphic representation for N . Hence, $f \in \text{cl}(E(H - v))$ as $H - v$ is unbalanced, a contradiction to the fact that f is a coloop of $N \setminus \text{st}_H^*(v)$. \square

Lemma 2.12. *Let H and H' be 2-connected frameworks for a 3-connected matroid N . If v is a fixed vertex of H , then $\text{st}_H^*(v) \in C^*(N)$ and there is a fixed vertex v' of H' satisfying $\text{st}_{H'}^*(v') = \text{st}_H^*(v)$.*

Proof. Since $H' - v$ is unbalanced, $\text{st}_H^*(v) \in C^*(N)$ by Lemma 2.10. So $r(N \setminus \text{st}_H^*(v)) = r(N) - 1$. Since $N \setminus \text{st}_H^*(v)$ is a 3-connected non-graphic matroid, $|V(H')| = r(N) \geq 3$ and by Lemma 2.6 the graph $H' \setminus \text{st}_H^*(v)$ has exactly two components, one of which is an isolated vertex or a loop-component. Let $\{v'\}$ be the vertex set of the 1-vertex component of $H' \setminus \text{st}_H^*(v)$, and H'_1 be the other component. Since no circuit of N can intersect $\text{st}_H^*(v)$ with exactly one element and H'_1 is connected and unbalanced, by Lemma 2.9, each edge in $\text{st}_H^*(v)$ has at most one end in H'_1 . So $\text{st}_{H'}^*(v') = \text{st}_H^*(v)$ by Theorem 2.4, implying that v' is fixed in H' . \square

For convenience, we will say that the vertex v' in Lemma 2.12 is the *corresponding* vertex of v in H' and denote it by v too.

Lemma 2.13. *Let H be a 2-connected framework for a 3-connected matroid N . If at most one vertex is not fixed in H , then one of the following holds.*

1. H is the unique framework for N .
2. $H - \text{loops}(H)$ has a blocking vertex.

In particular, when all vertices are fixed in H , (1) holds.

Proof. First, we prove

2.13.1. *When all vertices in H are fixed, (1) holds.*

Subproof. When $\text{loops}(H) = \emptyset$ or H has a loop but it is not a lifted-graphic representation for N , Lemma 2.12 implies that (1) holds. So we may assume that H is a lifted-graphic representation for N with a loop f by Theorem 2.4. Let H' be another connected framework for N . By Lemma 2.12, $\text{loops}(H) = \text{loops}(H') = \{f\}$ and $H \setminus f = H' \setminus f$. When H has no blocking vertex, since a union of f and each unbalanced cycle of H is a circuit of N by the structure of H , the graph H' must be a lifted-graphic representation for N , so H and H' are equivalent. When H has a blocking vertex, either f is incident with a blocking vertex of $H' \setminus f$ or H' must be a lifted-graphic representation for N . No matter which case happens, H' is a lifted-graphic representation for N , so H and H' are equivalent. That is, (1) holds. \square

By 2.13.1, we may therefore assume that H has a unique unfixed vertex v . Assume that (1) is not true. Let H' be a connected framework for N that is not equivalent to H . By Lemma 2.12, we may assume that $V(H) = V(H')$ and $\text{st}_H^*(u) = \text{st}_{H'}^*(u)$ for any $v \neq u \in V(H)$. Therefore,

2.13.2. *For any vertices $x, y \in V(H - v)$, we have that $xy \in E(H)$ if and only if $xy \in E(H')$.*

By symmetry, 2.13.1 and Lemma 2.12, we may assume that v is the unique unfixed vertex of H' .

2.13.3. *When H is a lifted-graphic representation of N with a loop f , both H and $H' - \text{loops}(H')$ have v as their blocking vertex.*

Subproof. Since N is 3-connected, $\text{loops}(H) = \{f\}$. Since $\text{st}_H(u) - \{f\} = \text{st}_H^*(u) = \text{st}_{H'}^*(u)$ for any $v \neq u \in V(H)$, we have $f \in \text{loops}(H')$. When H' is a lifted-graphic representation for N , since $\{f\} = \text{loops}(H')$, we have $H \setminus f = H' \setminus f$, so H and H' are equivalent, a contradiction. Hence, H' is a frame representation for N by Theorem 2.4. Then $\text{st}_H(u) - \{f\} = \text{st}_{H'}(u)$ for any $v \neq u \in V(H)$, implying that $f \in \text{loops}_{H'}(v)$ and $H - \{v, f\} = H' - \{v, \text{loops}(H')\}$.

Assume that $H - \{v, f\}$ has an unbalanced cycle C . Then $E(C) \cup \{f\} \in \mathcal{C}(N)$ as H is a lifted-graphic representation for N . On the other hand, since C is a cycle of $H - \{v, f\}$ of length at least 2, C is also an unbalanced cycle of $H' - \{v, f\}$ by 2.13.2. Since H' is a frame representation for N and $f \in \text{loops}_{H'}(v)$, we have $E(C) \cup \{f\} \in \mathcal{I}(N)$, a contradiction. Hence, $H - \{v, f\}$ is balanced. That is, the claim holds. \square

By 2.13.3 and symmetry, we may therefore assume that neither H nor H' is a lifted-graphic representation of N with a loop. Then $\text{st}_H(u) = \text{st}_{H'}(u)$ for any $v \neq u \in V(H)$. Since $H \neq H'$, there is a link $e = vu$ of H (or H'), which is a loop of H' (or H) incident with u . Assume that $H - \{v, \text{loops}(H)\}$ has an unbalanced cycle C . Since $|E(C)| \geq 2$, it follows from 2.13.2 that C is also an unbalanced cycle of $H' - v$. Let P be a minimal path in $H - v$ joining u and C . Note that $P = \{u\}$ when $u \in V_H(C)$. Comparing $H[E(C \cup P) \cup \{e\}]$ and $H'[E(C \cup P) \cup \{e\}]$, we will get a contradiction. Hence, $H - \{v, \text{loops}(H)\}$ is balanced. That is, (2) holds. \square

In Section 5 we will need a number of simple conditions which prevent a matroid from being an excluded minor for the class of quasi-graphic matroids. In the following Lemmas we gather a few such conditions. Lemmas 2.13-2.15 will be only used in the proof of Theorem 5.16.

Lemma 2.14. *Let e, f be elements of a 3-connected matroid N such that $N \setminus e$, $N \setminus f$, and $N \setminus e, f$ are 3-connected. Let H be a 2-connected unbalanced framework for $N \setminus e, f$ that has no blocking vertices. If H can be extended to frameworks for $N \setminus e$ and $N \setminus f$, then N is quasi-graphic.*

Proof. Let G be a graph with $H = G \setminus e, f$ such that $G \setminus e$ and $G \setminus f$ are frameworks for $N \setminus e$ and $N \setminus f$, respectively. Since H is connected, by Lemmas 2.6 and 2.7 we may assume that G is 2-connected. We claim that G is a framework for N . Evidently, (QG1) and (QG2) hold. Since N is 3-connected, by Theorem 2.8, it suffices to show that (QG3) holds. Let v be a vertex of G . When $e, f \in \text{st}_G(v)$, (QG3) obviously holds for v . So by symmetry we may assume that $e \notin \text{st}_G(v)$. Since H is 2-connected and has no blocking vertices, $H - v$ is connected and unbalanced. Then it follows from Lemma 2.9 that $e \in \text{cl}_N(E(H - v))$ as $G \setminus f$ is a framework for $N \setminus f$. When $f \in \text{st}_G(v)$, since $\text{cl}_N(E(G - v)) = \text{cl}_N(E(H - v))$ and $G \setminus e$ is a framework for $N \setminus e$, (QG3) holds for v . When $f \notin \text{st}_G(v)$, by the symmetry between e and f , we have $f \in \text{cl}_N(E(H - v))$. So $\text{cl}_N(E(G - v)) = \text{cl}_N(E(H - v))$, implying that (QG3) holds for v as $\text{st}_G(v) = \text{st}_H(v)$. \square

Lemma 2.15. *Let e, f be elements of a 3-connected matroid N such that $N \setminus e$, $N \setminus f$, and $N \setminus e, f$ are 3-connected. Let H be a 2-connected framework for $N \setminus e$. Assume that there is a balanced cycle C of H with $f \in E(C)$ such that all vertices in $V_H(C)$ are fixed in $H \setminus f$. If $N \setminus f$ is quasi-graphic, so is N .*

Proof. Let G'' be a framework for $N \setminus e, f$ that can be extended to a framework G' for $N \setminus f$. By Lemmas 2.6 and 2.7 we may further assume that G'' and G' are 2-connected. Since G'' and $H \setminus f$ are frameworks for $N \setminus e, f$ and all vertices in $V_H(C)$ are fixed in $H \setminus f$, by Lemma 2.12, we may assume that corresponding vertices in $G''[E(C) - \{f\}]$ and $H[E(C) - \{f\}]$ are labelled by same symbols and

$$\text{st}_{G''}^*(v) = \text{st}_{H \setminus f}^*(v) = \text{st}_H^*(v) - \{f\} \in C^*(N \setminus e, f),$$

for any $v \in V_H(C)$. Hence, $G'[E(C) - \{f\}]$ and $H[E(C) - \{f\}]$ are isomorphic paths. Let G be the graph obtained from G' by adding f to G' such that $G[E(C)]$ is a cycle. That is, $G[E(C)]$ and $H[E(C)]$ are isomorphic.

We claim that G is a framework for N . (QG1) obviously holds. Since G is 2-connected and $r(N) = r(N \setminus f) = |V(G')|$, (QG2) holds for G . Since N is 3-connected, by Theorem 2.8 it suffices to show that (QG3) holds. Since $E(C)$ is a circuit of N and G' is a framework for $N \setminus f$, (QG3) holds for each vertex in $V(G) - V_G(E(C)) + V_G(f)$. For any $v \in V_G(E(C)) - V_G(f)$, since v is fixed in both H and $H \setminus f$ by Lemma 2.11, we have

$$\text{st}_{G'}^*(v) - \{e\} = \text{st}_{G''}^*(v) = \text{st}_H^*(v) \in C^*(N \setminus e, f) \cap C^*(N \setminus e) \quad (2.1)$$

by Lemma 2.12. Since $N \setminus (\text{st}_{G''}^*(v) \cup \{e, f\})$ is 3-connected and non-graphic, $G' - v$ is unbalanced, so $\text{st}_{G'}^*(v) \in C^*(N \setminus f)$. Combined with (2.1), $\text{st}_{G'}^*(v) \in C^*(N)$ or $\{e, f\} \in C^*(N)$. Since N is 3-connected, $\text{st}_{G'}^*(v) \in C^*(N)$. Hence, (QG3) holds for v . \square

3 9-connected excluded minors with rank less than nine.

In this section, we prove that, if M is a 9-connected excluded minor for the class of quasi-graphic matroids with $r(M) \leq 8$, then M is isomorphic to $U_{3,7}$ or $U_{4,7}$. To prove this, we need one more definition.

Let G be a simple graph. For a positive integer k , let kG denote the graph obtained from G by replacing each edge of G by a parallel class with exactly k edges.

Theorem 3.1. *$U_{3,7}$ is an excluded minor for the class of quasi-graphic matroids.*

Proof. First we show that $2K_3$ is the unique framework for $U_{3,6}$. Let G be a framework for $U_{3,6}$. By Lemma 2.6 we may assume that G is connected. Then $|V(G)| = 3$. Since $|E(G)| = 6$, either each vertex in G is incident with exactly four edges or some vertex v is incident with at most three edges. When the former case happens, G is isomorphic to $2K_3$. When the latter case happens, since $G - v$ has no balanced cycles with at most two edges, Lemma 2.1 implies that $U_{3,6}$ has a triangle, a contradiction.

Since $2K_3$ is the unique framework for $U_{3,6}$, it is easy to verify that $U_{3,7}$ is not quasi-graphic. Moreover, since $6K_2$ is a framework for $U_{2,6}$, the theorem holds. \square

Theorem 3.2. *$U_{4,7}$ is an excluded minor for the class of quasi-graphic matroids.*

Proof. Let C_4 be a 4-edge cycle, let K be the graph obtained from $2C_4$ by deleting a pair of non-adjacent edges. Evidently, K_4 and K are frame representations for $U_{4,6}$. Note that, neither K_4 nor K can be extended to a framework for $U_{4,7}$. Since $2K_3$ is a framework for $U_{3,6}$, to prove the theorem, it suffices to show that, besides K_4 and K , $U_{4,6}$ has no other frameworks.

Let G be a framework for $U_{4,6}$. By Lemma 2.6 we may assume that G is connected. Then $|V(G)| = 4$, G is 2-connected and each vertex of G is incident with at least three edges. Assume that G has a blocking vertex u . Since each circuit in $U_{4,6}$ has five elements and $G - u$ is balanced, $G - u$ is a forest, so $|\text{st}_G(u)| \geq 4$. Since G is 2-connected, $G - u$ is a 2-edge path; that is, $|\text{st}_G(u)| = 4$. Let v_1, v_2 be the degree-1 vertices of $G - u$. Since $E(G) - \{f\}$ is a circuit of $U_{4,6}$ for each edge $f \in \text{st}_G(u)$, there are exactly two edges joining u and v_i for each $1 \leq i \leq 2$. So $\text{st}_G(u)$ is dependent in $U_{4,6}$, a contradiction. So G has no blocking vertices. For each vertex v of G , since $G - v$ is connected and unbalanced, $|E(G - v)| = 3$ as $|\text{st}_G(v)| \geq 3$. So $|\text{st}_G(v)| = 3$. Since

$|E(G)| = 6$, by the arbitrary choice of v , the graph G has no loops and G is isomorphic to K_4 or K . \square

Theorem 3.3. *Let M be an excluded minor for the class of quasi-graphic matroids. If M is 9-connected with rank at most eight, then M is isomorphic to $U_{3,7}$ or $U_{4,7}$.*

Proof. We claim that M is isomorphic to $U_{r,2r-1}$, $U_{r,2r}$, $U_{r,2r+1}$, or $U_{8,n}$ for a number r , where $n \geq 15$. Assume that M has a circuit C with $|C| \leq r(M)$. Without loss of generality we may further assume that C is chosen as small as possible. When $|E(M) - C| \geq |C|$, the partition $(C, E(M) - C)$ is a $|C|$ -separation, a contradiction to the fact that M is 9-connected. When $|E(M) - C| < |C|$, since $|C| \leq r(M)$ and $E(M) - C$ is independent by the choice of C , the partition $(C, E(M) - C)$ is an $|E(M) - C|$ -separation, a contradiction. So M is uniform. Then it follows from ([10], Corollary 8.6.3) that the claim holds.

Since kK_2 is a framework for $U_{2,k}$, we have $r(M) \geq 3$. Since $U_{4,7}$ is a minor of $U_{r,2r-1}$, $U_{r,2r}$, $U_{r,2r+1}$, and $U_{8,n}$ when $r \geq 4$ and $n \geq 15$, by Theorem 3.2 either $r(M) = 3$ or M is isomorphic to $U_{4,7}$. Moreover, since $U_{3,6}$ is quasi-graphic, the theorem holds from Theorem 3.1. \square

4 Frameworks for graphic matroids

Let G be a graph, and $M(G)$ its cycle matroid. A *signed graph* is a pair (G, Σ) with $\Sigma \subseteq E(G)$, each edge in Σ is labelled by -1 and other edges are labelled by 1 . A cycle C of G is Σ -even if $|E(C) \cap \Sigma|$ is even, otherwise it is Σ -odd. A set $\Sigma' \subseteq E(G)$ is a *signature* of (G, Σ) if (G, Σ) and (G, Σ') have the same Σ -even cycles and the same Σ -odd cycles. Evidently, for any cut C^* of G , the set $\Sigma \Delta E(C^*)$ is a signature of (G, Σ) . For a framework H for a matroid N , we say that H is a *signed graph* if there is a set $\Sigma \subseteq E(H)$ such that a cycle C of H is balanced if and only if C is a Σ -even cycle. We also say that Σ is a *signature* of H .

All definitions in the following five paragraphs were first given by Chen, DeVos, Funk and Pivotto [3].

Fat thetas. Let G_1, G_2, G_3 be non-empty graphs with distinct vertices $x_i, y_i \in V(G_i)$. Let G be obtained from G_1, G_2, G_3 by identifying y_i and x_{i+1} to a vertex w_i for every $1 \leq i \leq 3$ (where the indices are modulo 3). Let H be obtained from G_1, G_2, G_3 by identifying x_1, x_2, x_3 to a vertex x and identifying y_1, y_2, y_3 to a vertex y . A cycle of H is balanced if and only if $E(C)$ is completely contained in one of G_1, G_2 or G_3 . Then we say that H is a *fat theta* obtained from G .

Simple curlings. Let G be a graph and $v \in V(G)$. Let H be the signed graph obtained from G by first labelling all edges incident with v by -1 , and then changing any such edge $e = vu$ to a loop incident with u while keeping all other edges not incident with v unchanged and labelled by 1 . Then we say that H is a *simple curling* of G .

Pinches. If H is obtained from a graph G by identifying two vertices v_1 and v_2 to a new vertex v and labeling all edges originally incident with v_1 by -1 and all other edges by 1 , then we say H is a *pinch*. An edge with ends v_1, v_2 becomes an unbalanced loop incident to v .

4-twistings. Let G_1, G_2, G_3, G_4 be graphs (not necessarily all non-empty) with distinct vertices $x_i, y_i, z_i \in V(G_i)$. Let G be obtained from G_1, G_2, G_3, G_4 by identifying x_i, y_{3-i}, z_{i+2} to a vertex w_i for every $1 \leq i \leq 4$ (where the indices are modulo 4). Let H be a signed graph obtained from G_1, G_2, G_3, G_4 by identifying x_1, x_2, x_3, x_4 to a vertex x , identifying y_1, y_2, y_3, y_4 to a vertex y and identifying z_1, z_2, z_3, z_4 to a vertex z , and with all edges originally incident with

x_1, y_2 or z_3 labelled by -1 and all other edges labelled by 1 . Then we say that H is a *4-twisting* of G .

Consecutive twistings. Let G_1, \dots, G_k (for $k \geq 3$), be graphs with distinct vertices $x_i, y_i, z_i \in V(G_i)$ for $1 \leq i \leq k$. Let G be a graph obtained from G_1, \dots, G_k by identifying z_1, z_2, \dots, z_k to a vertex z and for each $1 \leq i \leq k$ identifying y_{i-1} and x_i to a vertex w_i (where the indices are modulo k). Let H be the signed graph obtained from G_1, \dots, G_k by identifying y_{i-1}, z_i, x_{i+1} to a vertex u_i for every $1 \leq i \leq k$ (where the indices are modulo k), and with all edges originally incident with y_1 or x_2 labelled by -1 and all other edges labelled by 1 . Then we say that H is a *consecutive twisting* or a *consecutive k -twisting* of G . If k is odd then H is a *consecutive odd-twisting* of G .

Theorem 4.1. ([3], Corollary 1.3.) *Let G be a 3-connected graph with $|V(G)| \geq 5$. Let H be a frame representation of $M(G)$. Then either H is balanced, or H is obtained from G as a simple curling, a pinch, a 4-twisting, or a consecutive odd-twisting.*

Recall that $c(H)$ is the number of components of H .

Theorem 4.2. ([8], Theorem 2.7.) *Let H be a framework for a matroid N . If $r(N) \leq |V(H)| - c(H)$, then $N = M(H)$.*

Theorem 4.3. ([15], Theorem 2.) *Let H be a lifted-graphic representation of a matroid N . Then N is binary if and only if H is a signed graph or H has a unique unbalanced component which is a fat theta.*

Let G be a graph, and let (X_1, X_2) be a partition of $E(G)$ such that $V(X_1) \cap V(X_2) = \{u_1, u_2\}$. We say that G' is obtained by a *Whitney flip* of G on $\{u_1, u_2\}$ if G' is a graph obtained by identifying vertices u_1, u_2 of $G[X_1]$ with vertices u_2, u_1 of $G[X_2]$, respectively. A graph G' is *2-isomorphic to G* if G' is obtained from G by a sequence of the operations: Whitney flips, identifying two vertices from distinct components of a graph, or partitioning a graph into components each of which is a block of the original graph.

In his Ph.D. thesis, Shih [12] proved the following characterization of graphic lifted-graphic matroids (see also [11], Theorem 4.1.).

Theorem 4.4 (Theorem 1, Chapter 2 in [12]). *Let G be a graph and let H be a lifted-graphic representation of $M(G)$. Assume that H is an unbalanced signed graph. Then there exists a graph G' 2-isomorphic to G such that one of the following holds.*

- (1) H is obtained from G' by a pinch.
- (2) H is obtained from G' by a 4-twisting.
- (3) H is obtained from G' by a consecutive twisting.

Following a similar way as the proof of ([8], Theorem 1.4.), we prove

Theorem 4.5. *Let H be a 2-connected framework for a 3-connected matroid N . If N is representable, then H is a frame representation or a lifted-graphic representation of N .*

Proof. Without loss of generality we may assume that H is unbalanced. Then $|V(H)| = r(N)$. By Theorem 2.4 we may assume that H has no loops. Assume that there is a vertex v of H such that $r_N(E(H-v)) \leq r(N) - 2$. Since $H-v$ is connected, it follows from Theorem 4.2 that $H-v$ is balanced. Then v is a blocking vertex of H , so H is a frame representation and a lifted-graphic representation of N . So we may assume that $r_N(E(H-v)) = r(N) - 1$ for each vertex v of H . Moreover, since H has no loops, $\text{st}_H(v)$ is a cocircuit of N by (QG3).

Let A be a matrix over a field \mathbb{F} with linearly independent rows satisfying $N = M(A)$, where $M(A)$ is the matroid represented by A . Since $\text{st}_H(v)$ is a cocircuit of N for each vertex v of H , there is a matrix $B \in \mathbb{F}^{V(H) \times E(H)}$ such that

1. the row-space of B is contained in the row-space of A , and
2. for each $v \in V(H)$ and $e \in E(H)$, the element of B in the row labelled by v and the column labelled by e is non-zero if and only if v is incident with e in H .

Note that $M(B)$ is a frame matroid and H is a framework for $M(B)$. Since H is connected, we have that

$$|V(H)| = r(M(A)) \geq r(M(B)) \geq |V(H)| - 1, \quad (4.1)$$

and if $r(M(A)) = r(M(B))$ then $M(A) = M(B)$ by (1) and (2). So we may assume that $r(M(A)) > r(M(B))$. Then $M(B) = M(H)$ by Theorem 4.2, up to row-operations we may assume that A is obtained from B by appending a single row by (4.1). Hence, H is a lifted-graphic representation of N . \square

By Lemma 2.6 (or Lemma 2.5) and Theorem 4.5, we have

Corollary 4.6. *Let H be a framework for a 3-connected representable matroid N . Then H is a frame representation or a lifted-graphic representation of N .*

The following result is an immediate consequence of Theorems 4.1, 4.3, 4.4, and Corollary 4.6.

Theorem 4.7. *Let G be a 3-connected graph with $|V(G)| \geq 5$, and H a connected framework for $M(G)$. Then H is isomorphic to G , or H is obtained from G by a simple curling, a pinch, a 4-twisting, or a consecutive twisting.*

By Theorem 4.7 we have

Corollary 4.8. *Let G be a 4-connected graph with $|V(G)| \geq 5$, and H a connected framework for $M(G)$. Then H is isomorphic to G or H is obtained from G by a simple curling or a pinch.*

Lemma 4.9. *Let G be a 3-connected simple graph, and H a 4-connected unbalanced framework for $M(G)$ with $|V(H)| \geq 4$. Then*

1. H is obtained from G by a simple curling or a pinch, or
2. H is a signed graph with a signature X such that $H[X]$ is a triangle.

Proof. Assume that (1) is not true. Since H is a 4-connected unbalanced graph with $|V(H)| \geq 4$, by Theorem 4.7, the graph H is obtained from G by a 4-twisting or a consecutive 3-twisting. Without loss of generality that it is a 4-twisting, since the consecutive 3-twisting is (up to relabelling of vertices) the special case of this in which one of the G_i has no edges. By symmetry we

may assume without loss of generality that none of G_1, G_2, G_3 has more than 3 vertices, where G_i and symbols that will be used but not defined in the proof, say w_i, x_i, y_i, x, y, z , are defined as in the definition of 4-twistings. By 3-connectivity and simplicity of G , there is precisely one edge e_i from w_1 to each w_i with $i \in \{2, 3, 4\}$, and there are no other edges incident with w_1 . By the definition of 4-twistings, the signature of H is $\{e_2, e_3, e_4\}$. The edge e_2 can only arise from an edge x_1y_1 in G_1 or an edge x_2y_2 in G_2 : in either case it joins x to y in H . Similarly e_3 joins x to z in H and e_4 joins y to z in H . Thus the signature of H is the set of edges of a triangle. \square

5 Proof of Theorem 1.2.

Recall that $c(H)$ is the number of components of a graph H . Lemmas 5.1-5.5 will be frequently used in this section.

Lemma 5.1. *Let H be a framework for a matroid. For an edge $f \in E(H)$, if $H \setminus f$ is balanced and H has a balanced cycle containing f , then H is balanced.*

Proof. Since $r(E(H)) = r(E(H \setminus f)) = |V(H)| - c(H)$, the graph H is balanced. \square

Note that Lemma 5.1 also follows immediately from the theta property.

For any subset X of $E(H) \cup V(H)$, if $H \setminus X$ is balanced, we say that X is a *balancing set* of H . Note that, when H is balanced, each subset of $E(H) \cup V(H)$ is balancing. We say a balancing set X is *minimal* if no proper subset of X is a balancing set of H . Note that, when H has a non-empty minimal balancing set $V \cup E$ with $V \subseteq V(H)$ and $E \subseteq E(H)$, the graph H is unbalanced and $E \cap \text{st}(V) = \emptyset$ by the definition of minimal balancing sets.

Lemma 5.2. *Let H be a connected unbalanced framework for a matroid N . If X is a minimal balancing edge set of H with $X \subseteq E(H)$, then $X \in C^*(N)$.*

Proof. Since $H \setminus X$ is connected and balanced, $r(E(H \setminus X)) = |V(H)| - 1 = r(N) - 1$. On the other hand, since each cycle in $H \setminus (X \setminus \{f\})$ containing f is unbalanced for each $f \in X$ by Lemma 5.1, $r(E(H \setminus X) \cup \{f\}) = r(N)$. Hence, X is a cocircuit of N . \square

Lemma 5.3. *Let H be a connected framework for an n -connected matroid N with $|V(H)| \geq n$. When H is unbalanced, each balancing set of H that contains only edges has rank at least n .*

Proof. Assume not. Let X be a minimal balancing set of H with $X \subseteq E(H)$ and $r(X) \leq n - 1$. Then $r(E(N) - X) = r(N) - 1$ by Lemma 5.2. Since $H \setminus X$ contains a spanning tree of H , $|E(N) - X| \geq n - 1$, so $(X, E(N) - X)$ is an $r(X)$ -separation of N , a contradiction. \square

For any subset X of $E(H) \cup V(H)$, if $c(H \setminus X) > c(H)$, then we say that X is a *cut* of H .

Lemma 5.4. *Let H be a framework for a matroid. Let $X_i = V_i \cup E_i$ be a balancing set of H with $V_i \subseteq V(H)$ and $E_i \subseteq E(H)$ for each $1 \leq i \leq 2$.*

1. *If X_1 is minimal and contains a link f satisfying $f \notin E_2 \cup \text{st}(V_2)$, then $X_1 \cup X_2$ contains a cut of H .*
2. *If $H - (X_1 \cup X_2)$ is connected and $V_1 \cap V_2 = \emptyset$, then $E_1 \cup E_2 \cup E(H[V_1 \cup V_2])$ is a balancing set of H .*

Proof. First we prove that (1) is true. Since X_1 is minimal, each cycle in $H \setminus (X_1 \setminus \{f\})$ containing f is unbalanced by Lemma 5.1. Moreover, since $H - X_2$ is balanced and $f \in E(H - X_2)$, the graph $H \setminus (X_1 \setminus \{f\})$ has a cut contained in $X_2 \cup \{f\}$, so (1) holds.

Assume that (2) is not true. Let C be an unbalanced cycle of $H \setminus (E_1 \cup E_2 \cup E(H[V_1 \cup V_2]))$ with $|V(C) \cap (V_1 \cup V_2)|$ as small as possible. Since $H - X_i$ is balanced for each $1 \leq i \leq 2$, we have $V(C) \cap V_i \neq \emptyset$. Then $|V(C) \cap (V_1 \cup V_2)| \geq 2$ as $V_1 \cap V_2 = \emptyset$. Since C does not contain edges in $H[V_1 \cup V_2]$, the subgraph $C - (V_1 \cup V_2)$ is disconnected. Moreover, since $H - (X_1 \cup X_2)$ is connected, there is a path P of $H - (X_1 \cup X_2)$ connecting two components of $C - (V_1 \cup V_2)$ such that $C \cup P$ is a theta subgraph. For any cycle C' of H with $P \subseteq C' \subseteq C \cup P$, since $|V(C') \cap (V_1 \cup V_2)| \leq |V(C) \cap (V_1 \cup V_2)| - 1$, the cycle C' is balanced by the choice of C . Therefore C is balanced by the theta property, a contradiction. So (2) holds. \square

Note that, the set E_i in Lemma 5.4 may be empty.

Let X and Y be subsets of the ground set of a matroid N . Set

$$\square_N(X, Y) = r_N(X) + r_N(Y) - r_N(X \cup Y).$$

When (X, Y) is a partition of $E(N)$, we often denote $\square_N(X, Y)$ by $\lambda_N(X)$. When there is no confusion, subscripts will be omitted.

Lemma 5.5. *Let H be a 4-connected framework for a simple and non-3-connected matroid N with $|V(H)| \geq 4$. Then H is unbalanced and has a balancing set X with $r(X) \leq 2$. In particular, when N has no triangles, $|X| \leq 2$.*

Proof. Since N is not 3-connected and H is 4-connected, H is unbalanced. Let (X, Y) be an exact k -separation of N for an integer $1 \leq k \leq 2$. We may assume that (X, Y) is chosen with $\lambda(X) + c(H[X]) + c(H[Y])$ as small as possible.

Case 1. $H[X]$ and $H[Y]$ are connected.

Set $m = |V_H(X) \cap V_H(Y)|$. Then $m \in \{k-1, k, k+1\}$ as $\lambda(X) = k-1$. Since H is 4-connected, by symmetry we may assume that $V_H(Y) = V(H)$ and $m = |V_H(X)|$. When $m = k-1$, we have that $k = 2$ and $H[X]$ consists of loops, so N has a circuit contained in X of size at most 2, which is not possible as N is simple. When $m = k+1 \leq 3$, both $H[X]$ and $H[Y]$ are balanced, that is, X is a balancing set of H with $r(X) = k$. When $m = k \leq 2$, one of $H[X]$ and $H[Y]$ is balanced and the other is unbalanced. If $H[X]$ is balanced, since $|X| \geq k$ and $k = |V_H(X)|$, the set X contains a circuit of N whose size is at most 2, a contradiction. So $H[Y]$ is balanced. That is, X is a balancing set of H with $r(X) = k-1$.

Case 2. $H[X]$ is disconnected, implying $|X| \geq 2$.

Let X_1 be the edge set of a component of $H[X]$.

5.5.1. *Either $|X - X_1| < k$ or $\square(X_1, X - X_1) = 1$ and $\square(X_1, Y) = 0$.*

Subproof. Assume that $|X - X_1| \geq k$. Since $c(H[X]) + c(H[Y]) > c(H[X - X_1]) + c(H[Y \cup X_1])$, we have $\lambda(X - X_1) > \lambda(X)$ by the choice of (X, Y) , so $\square(X_1, X - X_1) > \square(X_1, Y) \geq 0$. Since $\square(X_1, X - X_1) \leq 1$, the claim holds. \square

Assume that $\cap(X_1, X - X_1) = 0$. By 5.5.1, we have $1 \leq |X - X_1| < k$. So $k = 2 = c(H[X])$. Using 5.5.1 again, we have $|X_1| = 1$, so $|X| = 2$, implying that $H[Y]$ is a connected spanning subgraph of H . Since N is simple, $r(X) = 2$. Then $H[Y]$ is balanced as $\lambda(X) = 1$, so the lemma holds. Hence, we may assume that $\cap(X', X - X') = 1$ for the edge set X' of each component of $H[X]$, implying that $H[X']$ is unbalanced by Lemma 2.1. By symmetry we may further assume that (a) either $H[Y]$ is connected or each component of $H[Y]$ is unbalanced.

When $|X| = 2$, since $\cap(X_1, X - X_1) = 1$, we have $X \in C(N)$, which is not possible as N is simple. So $|X| \geq 3$. By symmetry assume that $|X - X_1| \geq 2$. By 5.5.1, we have $\cap(X_1, Y) = 0$. Since X_1 is unbalanced, each component of $H[Y]$ that shares vertices with $H[X_1]$ is balanced by Lemma 2.1. Then $H[Y]$ is connected and balanced by (a), implying that $r(Y) = |V_H(Y)| - 1$ and X is a balancing set of H .

Let $X_1, \dots, X_{c(H[X])}$ be the components of $H[X]$. Since an unbalanced spanning unicyclic subgraph of $H[X]$ is an independent set in N , we have $r(X) \geq \sum_1^{c(H[X])} |V_H(X_i)| - c(H[X]) + 1$. Then

$$1 \geq k - 1 = \lambda(X) \geq |V_H(X) \cap V_H(Y)| - c(H[X])$$

as $r(Y) = |V_H(Y)| - 1$. Hence, $|V_H(X_i) \cap V_H(Y)| \leq 2$ for each $1 \leq i \leq c(H[X])$ and at most one $|V_H(X_i) \cap V_H(Y)|$ is not equal to 1. Since H is 4-connected and $H[X]$ is disconnected, $V_H(Y) = V(H)$. Since $r(Y) = r(N) - 1$ and $\lambda(X) \leq 1$, we have $r(X) \leq 2$. Hence, the lemma holds as X is a balancing set of H . \square

Recall that we define $H' = H - \text{loops}(H)$ when H is a lifted-graphic representation of N , otherwise let $H' = H$. For $v_1, v_2 \in V(H)$, we say $\{v_1, v_2\}$ is a *blocking pair* of H if v_i is a blocking vertex of $H' - v_{3-i}$ for each $1 \leq i \leq 2$. Note that, by our definition, balanced frameworks have no blocking vertices, and no vertex in a blocking pair is a blocking vertex.

Lemma 5.6. *Let H be a 7-connected unbalanced framework for a matroid with $|V(H)| \geq 8$. Assume that H has no blocking pairs and $H - \text{loops}(H)$ has no blocking vertices. Then there is an edge f of H such that $H \setminus f$ has no blocking pairs and $H - (\text{loops}(H) \cup \{f\})$ has no blocking vertices.*

Proof. Let e be an edge of H and S_e be a minimal subset of $V(H)$ such that $H - (\text{loops}(H) \cup \{e\} \cup S_e)$ is balanced. We can further assume that $|S_e| \leq 2$ otherwise the lemma holds. Let f be a link of $H - (S_e \cup \{e\})$. Assume that $H \setminus f$ has a blocking pair S_f . Since $S_f \cup \{f\}$ is a minimal balancing set of H , it follows from Lemma 5.4 (1) that $S_e \cup S_f \cup \{e, f\}$ contains a cut of H , a contradiction to the fact that H is 7-connected, a contradiction. Following a similar way, we show that $H \setminus (\text{loops}(H) \cup \{f\})$ has no blocking vertices. Hence, the lemma holds for f . \square

Lemma 5.7. *Let H be a 6-connected framework for a 7-connected matroid N with $|V(H)| \geq 7$. Assume that H has no blocking pairs and $H - \text{loops}(H)$ has no blocking vertices. Then at most one vertex of H is not fixed.*

Proof. Assume not. Let v_1, v_2 be unfixed vertices of H . Then $N \setminus \text{st}_H^*(v_i)$ is graphic or non-3-connected for each $1 \leq i \leq 2$. Since H has no blocking pairs and $H - \text{loops}(H)$ has no blocking vertices, by Lemma 4.9 or Lemma 5.5, for each $1 \leq i \leq 2$, there is a minimal balancing set X_i of $H \setminus \text{st}_H^*(v_i)$ such that the following (a) or (b) happens. (a) $|X_i| = 2$, the two edges in X_i have no common vertex and if X_i contains a loop then H is not a lifted-graphic representation of N , for otherwise H has a blocking pair by the definition of blocking pairs. (b) $H[X_i]$ is a triangle.

Hence, $v_i \notin V(X_i)$ for each $1 \leq i \leq 2$ no matter which case happens. Since N is 7-connected, $X_i \cup \{v_i\}$ is a minimal balancing set of H for each $1 \leq i \leq 2$.

We claim that $X_1 = X_2$ when $v_2 \notin V(X_1)$. When $X_1 \subseteq X_2$, it follows from (a) and (b) that $X_1 = X_2$. Hence, it suffices to show that $X_1 \subseteq X_2$. Assume to the contrary that there is an edge $x \in X_1 - X_2$. Then x is not a loop of H , otherwise $x \in X_2$ as $v_2 \notin V(X_1)$. Since $X_1 \cup \{v_1\}$ is a minimal balancing set of H , by Lemma 5.4 (1), $X_1 \cup X_2 \cup \{v_1, v_2\}$ contains a cut of H . Since H is 6-connected, $X_1 \cup X_2$ is a matching in H of size 4 and $X_1 \cup X_2 \cup \{v_1, v_2\}$ is cut of H by (a) and (b). Let H_1 be a component of $H - \{v_1, v_2, X_1, X_2\}$. Set $H_1^+ = H[V(H_1) \cup \{v_1, v_2\}] \setminus E(H[\{v_1, v_2\}])$. Since each balancing set of edges in H has size at least 7 by Lemma 5.3, $H \setminus (X_1 \cup X_2)$ is unbalanced with v_1, v_2 as its blocking vertices. Then H_1^+ is balanced, otherwise H_1^+ has an unbalanced cycle containing exactly one vertex of $\{v_1, v_2\}$ by the theta property. So $\lambda(E(H_1^+)) \leq 5$, a contradiction to the fact that N is 7-connected.

When $X_1 = X_2$, since $H - \{v_1, v_2, X_1\}$ is connected, by Lemma 5.4 (2), $X_1 \cup E(H[\{v_1, v_2\}])$ is a balancing set of rank at most 5, a contradiction to Lemma 5.3. So $X_1 \neq X_2$. By symmetry and the claim proved in the last paragraph, $v_i \in V(X_{3-i})$ for each $1 \leq i \leq 2$. Let x be the edge in X_1 that is not incident with v_2 . Since each cycle in the 4-connected graph $H - \{v_1, v_2\}$ containing x is unbalanced by Lemma 5.1, $x \in X_1 \cap X_2$. Combined with (a) and (b), we have $r(X_1 \cup X_2 \cup E(H[\{v_1, v_2\}])) \leq 6$. On the other hand, since $v_i \in V(X_{3-i})$, the graph $H - \{v_1, v_2, X_1, X_2\}$ is 2-connected. By Lemma 5.4 (2) again, $X_1 \cup X_2 \cup E(H[\{v_1, v_2\}])$ is a balancing set, a contradiction to Lemma 5.3. \square

To prevent a matroid from being an excluded minor for the class of quasi-graphic matroids, we can use Lemmas 5.6 and 5.7 to show that H and $H \setminus f$ have enough fixed vertices for some edge f , as long as $H - \text{loops}(H)$ has no blocking vertex and H has no blocking pair. In the rest of this section, we will show that when H has a blocking vertex or a blocking pair, there is a balanced cycle C of H and $f \in E(C)$ such that all vertices in $V(C)$ are fixed in both $H \setminus f$ and H (namely in Lemmas 5.11 and 5.14). The case that H has a blocking vertex will be dealt with first.

A biased graph H is *contra-balanced* if each cycle of H is unbalanced.

Lemma 5.8. ([14], Theorem 6.) *A biased graph is a signed graph if and only if it has no contra-balanced theta-subgraphs.*

Lemma 5.9. *Let v be a blocking vertex of a biased graph H . Let $x \in V(H - v) \cup E(H - v)$ that is not adjacent with v when $x \in V(H - v)$. If $H - \{x, v\}$ is connected, then H is a signed graph if and only if $H - x$ is a signed graph.*

Proof. Evidently, it suffices to show that “if” part. Assume that $H - x$ is a signed graph but H is not. Then H has a contra-balanced theta subgraph T containing x by Lemma 5.8. Since v is a blocking vertex of H , v is a degree-3 vertex of T . Since v and x are not adjacent when $x \in V(H - v)$, the graph $T - \{x, v\}$ has exactly two or three components. Since $H - \{x, v\}$ is connected, $H - \{x, v\}$ has a minimal forest P that joins different components of $T - \{x, v\}$ such that $(T \cup P) - \{x, v\}$ is connected. Then $(T \cup P) - x$ consists of a theta subgraph T' and some vertex-disjoint paths that are not in any cycle. Since T is contra-balanced and $H - v$ is balanced, T' is also a contra-balanced theta-subgraph by theta property. Hence, $H - x$ is not a signed graph by Lemma 5.8, a contradiction. \square

Suppose that v is a blocking vertex of H and $H - v$ is connected. In this case we define a relation \sim_v on the edges in $\text{st}_H(v) - \text{loops}_H(v)$ by declaring $e \sim_v f$ if either $e = f$ or all

cycles containing e and f are balanced. This is an equivalence relation, as we show next. Let e_1, e_2, e_3 be distinct edges in $\text{st}_H(v)$ with $e_1 \sim_v e_2$ and $e_2 \sim_v e_3$. Let T be a theta subgraph of H containing all of e_1, e_2 and e_3 ; such a theta subgraph exists because $H - v$ is connected. The cycle in T containing both e_1 and e_2 is balanced, and so is the cycle containing both e_2 and e_3 . Therefore the cycle C in T containing e_1 and e_3 is balanced. Any other cycle containing e_1 and e_3 may be obtained from C by rerouting along balanced cycles (contained in $H - v$), hence all the cycles containing e_1 and e_3 are balanced and $e_1 \sim_v e_3$, showing that \sim_v is an equivalence relation. The same argument shows that a cycle of H (that is not a loop) is unbalanced if and only if it contains two edges in $\text{st}_H(v)$ which are not equivalent. We call the partition given by the equivalence classes of \sim_v the *standard partition* of $\text{st}_H(v) - \text{loops}_H(v)$. For more details, the reader can refer to ([5], Section 2) or ([6], Section 1). Definitions and results introduced in this paragraph will only be used in the proof of Lemma 5.10.

When H is a signed graph with a blocking vertex v , since H has no contra-balanced theta subgraph by Lemma 5.8, it is easy to show that there is a partition (X_1, X_2) of $\text{st}_H(v) - \text{loops}_H(v)$ such that $H - (\text{loops}(H) \cup X_i)$ is balanced for each $1 \leq i \leq 2$. Split v into v_1, v_2 such that X_1, X_2 are incident with v_1 and v_2 , respectively, and such that each unbalanced loop in H joins v_1 and v_2 and each balanced loop in $\text{loops}_H(v)$ is a loop incident with any v_i . Let G denote the new graph. Then H is a lifted-graphic representation of $M(G)$. Hence, if a framework for a matroid N is a signed graph with a blocking vertex, then N is graphic. This fact will be frequently used in the rest of this section without reference.

Let $X, Y \subseteq E(H)$ and $\mathcal{P} = (P_1, \dots, P_n)$ be a partition of X . We will let $\mathcal{P} - Y$ denote the partition $(P_1 - Y, \dots, P_n - Y)$ of $X - Y$.

Lemma 5.10. *Let H be a 5-connected framework for a 5-connected matroid N with $|V(H)| \geq 5$. Assume that N is non-graphic and H has a blocking vertex v . Then the following hold.*

1. *For each vertex $v \neq u \in V(H)$, the graph $H - u$ is unbalanced and $N \setminus \text{st}_H(u)$ is 3-connected.*
2. *A vertex u with $u \neq v$ is not fixed in H if and only if $H \setminus E(H[\{v, u\}])$ is an unbalanced signed graph.*
3. *At most one vertex in $V(H) - \{v\}$ is not fixed in H .*

Proof. First we prove that (1) is true. If $H - u$ is balanced, then it follows from Lemma 5.4 (2) that $E(H[\{u, v\}])$ is a balancing set of H with rank at most 2, which is not possible by Lemmas 5.2 and 5.3. Hence, $H - u$ is unbalanced.

Assume that $N \setminus \text{st}_H(u)$ is not 3-connected for some vertex $v \neq u \in V(H)$. Since N has no triangles, by Lemma 5.5, $H - u$ has a minimal balancing set X with $|X| \leq 2$. Since each cycle of $H - u$ containing exactly one edge of X is unbalanced by Lemma 5.1, $X \subseteq \text{st}_H(v) \cup \text{loops}(H)$. Moreover, since $H - \{u, v\}$ is connected, it follows from Lemma 5.4 (2) that $X \cup E(H[\{u, v\}])$ is a balancing set of H with rank at most 4, which is not possible by Lemmas 5.2 and 5.3.

Secondly, we prove that (2) is true. When $H \setminus E(H[\{v, u\}])$ is an unbalanced signed graph with v as its blocking vertex, $N \setminus E(H[\{v, u\}])$ is graphic, and thus so is $N \setminus \text{st}_H(u)$. Hence, u is not fixed in H . Next, we prove the ‘‘only if’’ part of (2) is true. Since u is not fixed, $N \setminus \text{st}_H(u)$ is graphic by (1). Then $H - u$ is an unbalanced signed graph with v as its blocking vertex by (1), hence so is $H \setminus E(H[\{v, u\}])$ by repeatedly using Lemma 5.9.

Thirdly, we prove that (3) is true. Assume not. There are vertices $u, u' \in V(H) - \{v\}$ such that $H \setminus E(H[\{v, u\}])$ and $H \setminus E(H[\{v, u'\}])$ are signed graphs with v as their blocking vertex by (2). Let

\mathcal{P} be the standard partition of $\text{st}_H(v)$ - $\text{loops}_H(v)$ in H . Then \mathcal{P} - $\text{st}_H(u)$ and \mathcal{P} - $\text{st}_H(u')$ have exactly two non-empty members. When $\mathcal{P} - (\text{st}_H(u) \cup \text{st}_H(u'))$ has exactly two non-empty members, \mathcal{P} has exactly two non-empty members, implying that H is a signed-graph, a contradiction to the fact that N is non-graphic. When $\mathcal{P} - (\text{st}_H(u) \cup \text{st}_H(u'))$ has exactly one non-empty member, since N is simple, H has a minimal balancing set $\{e, e'\}$ with $e \in E(H[\{v, u\}])$ and $e' \in E(H[\{v, u'\}])$. Then $\{e, e'\} \in C^*(N)$ by Lemma 5.2, a contradiction. \square

Lemma 5.11. *Let H be a 6-connected framework for a 6-connected matroid N with $|V(H)| \geq 6$. Assume that N is non-graphic and H has a blocking vertex v . Then $H - v$ has a balanced cycle C such that all vertices in $V(C)$ are fixed in both $H \setminus f$ and H for each edge f of C .*

Proof. We claim that a vertex $u \in V(H - v)$ is not fixed in H if and only if it is not fixed in $H \setminus f$ for an arbitrary $f \in E(H - v)$. Evidently, it suffices to show that the “if” part is true. Let u be a vertex in $V(H - v)$ that is not fixed in $H \setminus f$. By Lemma 5.10 (2), $H \setminus (E(H[\{v, u\}]) \cup \{f\})$ is an unbalanced signed graph with v as its blocking vertex. Since $H \setminus (E(H[\{v, u\}]) \cup \{f, v\})$ is connected, $H \setminus E(H[\{v, u\}])$ is a signed graph by Lemma 5.9. So u is not fixed in H by Lemma 5.10 (2) again.

By Lemma 5.10 (3), $H - v$ has a balanced cycle C such that all vertices in $V(C)$ are fixed in H . By the claim proved in the last paragraph, for each $f \in E(C)$, all vertices in $V(C)$ are also fixed in $H \setminus f$. \square

Next, the case that H has a blocking pair S but $H - \text{loops}(H)$ has no blocking vertices will be dealt with. To deal with this case, we need to introduce a characterization of the structure of biased graphs that have at least two blocking vertices. Lemma 5.12 will be only used in the proof of Lemma 5.13.

Lemma 5.12. ([16], Corollary 2.) *Let $V^* = \{v_1, \dots, v_n\}$ be the set of blocking vertices of a biased graph H . Assume that $n \geq 2$. Then one of the following holds.*

1. H is obtained from mK_2 by replacing each edge e_i with a balanced graph H_i such that all cycles of H not contained in some H_i are unbalanced, where $m \geq 2$.
2. H is obtained from a cycle $v_1v_2 \dots v_nv_1$ by replacing each edge v_iv_{i+1} with a graph H_i and a cycle in H is unbalanced if and only if it contains $\{v_1, \dots, v_n\}$, where no vertex in H_i separates v_i and v_{i+1} and all subscripts are modulo n .

Lemma 5.13. *Let H be a 5-connected unbalanced framework for a 6-connected matroid N with $|V(H)| \geq 6$. Assume that H has a blocking pair S and $H - \text{loops}(H)$ has no blocking vertices. Then we have*

1. *If some $v \in V(H) - S$ is not fixed in H , then there is a vertex $u \in S$ such that $\{u, v\}$ is a blocking pair of H .*
2. *H has at most two blocking pairs, and they have a common vertex.*

Note that at most one vertex in $V(H) - S$ can be contained in a blocking pair of H and at most one vertex in $V(H) - S$ is not fixed in H by Lemma 5.13.

Proof of Lemma 5.13. When H is a lifted-graphic representation for N with a loop, we may assume that the loop is in $\text{st}(S)$.

5.13.1. *If S_1, S_2 are blocking pairs of H , then $S_1 \cap S_2 \neq \emptyset$.*

Subproof. Assume otherwise. By Lemma 5.4 (2), $E(H[S_1 \cup S_2])$ is a balancing set of H of rank at most 4, a contradiction to Lemma 5.3. \square

First, we prove that (1) is true. Since $N \setminus \text{st}_H(v)$ is graphic or non-3-connected, by Lemma 4.9 or Lemma 5.5, either (a) $H - v$ has a blocking vertex u or (b) $H \setminus \text{st}_H(v)$ has a minimal balancing set X such that $|X| \leq 2$ or $H[X]$ is a triangle. When (a) happens, $u \in S$ by 5.13.1, so (1) holds. Assume that (b) happens. Since each cycle C in $H - v$ with $|C \cap X| = 1$ is unbalanced by Lemma 5.1, we have $X \subseteq \text{st}(S)$, so $H - (S \cup X \cup \{v\})$ is 2-connected. Then $X \cup E(H[S \cup \{v\}])$ is a balancing set of H with rank at most 5 by Lemma 5.4 (2), a contradiction to Lemmas 5.2 and 5.3.

Now, we prove that (2) holds. Assume that, besides S , the graph H has two other blocking pairs S_1, S_2 . By 5.13.1, we may assume that $S_1 = \{u, v\}$ and $S_2 = \{u', v'\}$, where $u, u' \in S$ and $v, v' \in V(H) - S$. When $u = u'$, let w be the unique vertex in $S - \{u\}$. Since v, v', w are distinct blocking vertices of $H - u$, at least one pair of vertices in $\{v, v', w\}$ is a cut of $H - u$ by Lemma 5.12 (2), so H is not 4-connected, a contradiction. Hence, $u \neq u'$, implying $v = v'$ using 5.13.1 again. Let E_1 be the set of edges from u to u' . Since u, u' are blocking vertices of $G - v$, by Lemma 5.12 (1) or Lemma 5.12 (2), either $\{u, u'\}$ is a cut of $G - v$ or $\{v\} \cup E_1$ is a balancing set. Since H is 5-connected, $\{v\} \cup E_1$ is a balancing set. Since u, v are blocking vertices of $G - u'$, by symmetry we have that $\{u'\} \cup E_2$ is a balancing set, where E_2 is the set of edges between u and v . Hence, by Lemma 5.4 (2), the set of all edges between u, u' and v is a balancing set, a contradiction to Lemmas 5.2 and 5.3. \square

Lemma 5.14. *Let H be a 6-connected unbalanced framework for a 7-connected matroid N with $|V(H)| \geq 7$. Assume that H has a blocking pair S and $H - \text{loops}(H)$ has no blocking vertices. Then $H - S$ has a balanced cycle C such that all vertices in $V(C)$ are fixed in $H \setminus f$ and H for every edge f of C .*

Proof. By Lemma 5.13 (2), $H - S$ has a balanced cycle C such that each vertex in $V(C)$ is not contained in a blocking pair of H . Lemma 5.13 (1) implies that all vertices in $V(C)$ are fixed in H . Let f be an arbitrary edge in C . Assume that the lemma does not hold for f . Then there is some vertex $v \in V(C)$ that is not fixed in $H \setminus f$. By Lemma 5.13 (1), $\{u, v\}$ is a blocking pair of $H \setminus f$ for some $u \in S$. Since $\{u, v\}$ is not a blocking pair of H , there is a minimal balancing set X of H with $f \in X \subseteq \{u, v, f\}$. By Lemma 5.4 (1), $X \cup S$ contains a cut of H , a contradiction to the fact that H is 6-connected. \square

To prove Theorem 1.2, we need one more result. Tutte [13] proved

Theorem 5.15. ([10], Theorem 10.3.1.) *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}, F_7, F_7^*, M^*(K_5)$ and $M^*(K_{3,3})$.*

Now, we prove Theorem 1.2, which is restated here in a slightly different way.

Theorem 5.16. *Let M be an excluded minor for the class of quasi-graphic matroids. Then M is isomorphic to $U_{3,7}$ or $U_{4,7}$, or M is not 9-connected.*

Proof. Assume that M is 9-connected. When $r(M) \leq 8$, it follows from Theorem 3.3 that M is isomorphic to $U_{3,7}$ or $U_{4,7}$. So we may assume that $r(M) \geq 9$. Since M is non-graphic and the matroids in Theorem 5.15 each have a cocircuit of size less than 9, there is an element e of

M such that $M \setminus e$ is non-graphic by Theorem 5.15. Let G be a 7-connected framework for $M \setminus e$ with $|V(G)| \geq 9$.

First, consider the case that $G - \text{loops}(G)$ has no blocking vertices and G has no blocking pairs. By Lemma 5.6, there is an edge f of G such that $G \setminus f$ has no blocking pairs and $G - (\text{loops}(G) \cup \{f\})$ has no blocking vertices. It follows from Lemmas 5.7 and 2.13 that $G \setminus f$ is a unique framework for $M \setminus e, f$. Then $G \setminus f$ can be extended to a framework for $M \setminus f$. Moreover, since G has no blocking pairs, $G \setminus f$ has no blocking vertices, so M is quasi-graphic by Lemma 2.14, a contradiction.

Secondly, consider the case that G has a blocking pair or $G - \text{loops}(G)$ has a blocking vertex. When $G - \text{loops}(G)$ has a blocking vertex v and G is not a lifted-graphic representation for N , let G' be obtained from G by changing each loop in $\text{loops}(G) - \text{loops}_G(v)$ to a link joining its original end and v ; otherwise, set $G' = G$. By Theorem 2.4, G' is also a 7-connected framework for $M \setminus e$ that has a blocking pair or a blocking vertex. By Lemma 5.11 or Lemma 5.14, there is a balanced cycle C of G' such that all vertices in $V(C)$ are fixed in $G' \setminus f$ for each edge f in C . Since $M \setminus f$ is quasi-graphic, so is M by Lemma 2.15, a contradiction. \square

Acknowledgements

The author thanks the referees for their careful reading of this manuscript and thanks one of them for pointing out an error in the original proof of Lemma 4.9.

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