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1 **STABILITY FOR FINITE ELEMENT DISCRETIZATION OF SOME**
2 **INVERSE PARAMETER PROBLEMS FROM INTERNAL DATA -**
3 **APPLICATION TO ELASTOGRAPHY**

4 ELIE BRETIN*, PIERRE MILLIEN†, AND LAURENT SEPPECHER‡

5 **Abstract.** In this article, we provide stability estimates for the finite element discretization of
6 a class of inverse parameter problems of the form $-\nabla \cdot (\mu S) = \mathbf{f}$ in a domain Ω of \mathbb{R}^d . Here μ is the
7 unknown parameter to recover, the matrix valued function S and the vector valued distribution \mathbf{f}
8 are known. As uniqueness is not guaranteed in general for this problem, we prove a Lipschitz-type
9 stability estimate in an hyperplane of $L^2(\Omega)$. This stability is obtained through an adaptation of
10 the so-called discrete *inf-sup* constant or LBB constant to a large class of first-order differential
11 operators. We then provide a simple and original discretization based on hexagonal finite element
12 that satisfies the discrete stability condition and shows corresponding numerical reconstructions.
13 The obtained algebraic inversion method is efficient as it does not require any iterative solving of the
14 forward problem and is very general as it only requires S and μ to be bounded and no additional
15 information at the boundary is needed.

16 **Key words.** Inverse problems, Reverse Weak Formulation, Inf-Sup constant, Linear Elastogra-
17 phy, Finite Element Method

18 **AMS subject classifications.** 65J22, 65N21, 35R30, 65M60

19 **1. Introduction.** This work deals with inverse problems of the form

20 (1.1)
$$-\nabla \cdot (\mu S) = \mathbf{f} \quad \text{in } \Omega,$$

21 where Ω is a smooth bounded domain of \mathbb{R}^d , $d \geq 2$ and where $\mu \in L^\infty(\Omega)$ is the
22 unknown parameter map. In this problem, $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ and $\mathbf{f} \in H^{-1}(\Omega, \mathbb{R}^d)$
23 are given from some measurements and may contain noise. If one defines the first
24 order differential operator

25 (1.2)
$$\begin{aligned} T : L^\infty(\Omega) \subset L^2(\Omega) &\rightarrow H^{-1}(\Omega, \mathbb{R}^d) \\ \mu &\mapsto -\nabla \cdot (\mu S), \end{aligned}$$

26 the inverse parameter problem that we aim to solve can be expressed as

27 (1.3)
$$\text{Find } \mu \in L^\infty(\Omega) \quad \text{s.t.} \quad T\mu = \mathbf{f}.$$

28 If S , \mathbf{f} and μ are assumed smooth enough, this problem reads as a first order
29 transport equation in μ that can be solved with the characteristics method knowing
30 μ in a part of the boundary (the incoming flow boundary). Here, as no additional
31 regularity is assumed and as the right-hand side \mathbf{f} belongs to $H^{-1}(\Omega, \mathbb{R}^d)$ this problem
32 shall be considered under its weak formulation :

33 (1.4)
$$\text{Find } \mu \in L^\infty(\Omega) \quad \text{s.t.} \quad \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d).$$

34 This weak form of the inverse problem (1.3) (introduced in [1]) will be called the
35 Reverse Weak Formulation (RWF). In this inverse problem, we do not assume the

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36 knowledge of any information on μ at the boundary. Note that the case $\mathbf{f} = \mathbf{0}$ can be
 37 considered and corresponds to the determination of the null space of the operator T .

38 In [1] the well-posedness of this problem has been established in an hyperplane of
 39 $L^2(\Omega)$ in a general setting and under light hypothesis of regularity and invertibility
 40 of the matrix S . See subsection 1.1 and [1]. Note that the hypotheses used in this
 41 reference will not be used to established the error estimates given in the present paper.

42 The goal of the present paper is to investigate the stability properties of the
 43 discretized version of the problem (1.4) and to provide error estimates based on the
 44 properties of the discretization spaces and on the discretized approximation of the
 45 operator T . More precisely, given a finite dimensional operator $T_h : M_h \rightarrow V_h'$ and
 46 $\mathbf{f}_h \in V_h'$ where M_h and V_h are finite dimensional subspaces that approach $M := L^2(\Omega)$
 47 and $V := H_0^1(\Omega, \mathbb{R}^d)$ respectively, we seek conditions on M_h , V_h and T_h for the L^2 -
 48 stability of the following discretized problem:

$$49 \quad (1.5) \quad \text{Find } \mu_h \in M_h \quad \text{s.t.} \quad T_h \mu_h = \mathbf{f}_h.$$

50 We also give conditions that guarantee the convergence of μ_h to μ for the L^2 -norm.
 51 In most cases, the stability only occurs in an hyperplane of $L^2(\Omega)$ which is the orthog-
 52 onal of the singular direction of the operator T_h with respect to its smallest singular
 53 value. This leads to a remaining scalar uncertainty that can be resolved using a single
 54 additional scalar information on μ .

55 The originality of this work lies here on the Reverse Weak Formulation (1.4)
 56 that exhibits the unknown parameter μ as the solution of a weak linear differential
 57 problem in the domain Ω without boundary condition. Hence the uniqueness is not
 58 guaranteed at first look and the stability has to be considered with respect to some
 59 possible errors on both \mathbf{f} and T . As we will see, the error term $T_h - T$ is not controlled
 60 in $\mathcal{L}(L^2(\Omega), H^{-1}(\Omega, \mathbb{R}^d))$ (definition in Section 2) in general but only for a weaker
 61 norm (see Subsection 2.3). This creates difficulties that are not covered by the classical
 62 literature on the theory of perturbations of linear operators.

63 **1.1. Scientific context and motivations.** Elastography is an imaging modal-
 64 ity that aims at reconstructing the mechanical properties of biological tissues. The
 65 local values of the elastic parameters can be used as a discriminatory criterion for
 66 differentiating healthy tissues from diseased tissues [20]. While numerous modalities
 67 of elastography exist (see for example [13, 18, 10, 7]), the most common procedure is
 68 to use an auxiliary imaging method (such as ultrasound imaging, magnetic resonance
 69 imaging, optical coherence tomography ...) to measure the displacement field \mathbf{u} in a
 70 medium when a mechanical perturbation is applied. See [21] and inside references for
 71 recent advances on this point. The inverse problem can be formulated as recovering
 72 the shear modulus μ in the linear elastic equation

$$73 \quad (1.6) \quad -\nabla \cdot (2\mu\mathcal{E}(\mathbf{u})) - \nabla(\lambda\nabla \cdot \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega,$$

74 where \mathbf{u} and \mathbf{f} are given in Ω and λ can be assumed known in Ω . The term $\mathcal{E}(\mathbf{u})$
 75 denotes the strain matrix which is the symmetric part of the gradient of \mathbf{u} . The
 76 stability of this inverse problem has been extensively studied under various regularity
 77 assumptions for the coefficients to be reconstructed [2, 3, 23, 16]. Recently, in [1] the
 78 authors introduced a new inversion method based on a finite element discretization
 79 of equation (1.1) where $S := 2\mathcal{E}(\mathbf{u})$. A study of the linear operator T defined by (1.2)
 80 or by the equivalent weak formulation

$$81 \quad (1.7) \quad \langle T\mu, \mathbf{v} \rangle_{H^{-1}, H_0^1} := \int_{\Omega} \mu S : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in H_0^1(\Omega, \mathbb{R}^{d \times d})$$

82 showed that, under a piecewise smoothness hypothesis on S and under an assumption
 83 of the form $|\det(S)| \geq c > 0$ *a.e.* in Ω , the operator T has a null space of dimension
 84 one at most and is a closed range operator. This ensures the theoretical stability of the
 85 reconstruction in the orthogonal complement of the null space. However, depending
 86 on the choice of discretization spaces, the discretized version of T may not satisfy the
 87 same properties and numerical instability may be observed. For instance, in [1] the
 88 authors approach (1.7) using the classical pair $(\mathbb{P}^0, \mathbb{P}^1)$ of finite element spaces. As
 89 it could have been expected, they faced a numerical instability that was successfully
 90 overcome by using a TV -penalization technique.

91 *Remark 1.1.* The classical elliptic theory says that the strain matrix belongs to
 92 $L^2(\Omega, \mathbb{R}^{d \times d})$. Here, we add the hypothesis $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ in order to control the
 93 error on μ in the Hilbert space $L^2(\Omega)$. This boundedness hypothesis is not restrictive
 94 as it is known that the strain is bounded as soon as the elastic parameters are piecewise
 95 smooth with smooth surfaces of discontinuity (see [17]).

96 Let us point out here that inverse problems of the form (1.1) may arise from
 97 various other physical situations. Note first that the reconstruction of the Young's
 98 modulus E when the Poisson's ratio ν is known is very similar to the problem defined
 99 in (1.6). In this case the governing linear elastic equation reads $-\nabla \cdot (E \Sigma) = \mathbf{f}$ where
 100 $\Sigma := a_\nu \mathcal{E}(\mathbf{u}) + b_\nu (\nabla \cdot \mathbf{u}) I$ and $a_\nu := 1/(1+\nu)$ and $b_\nu := \nu/((1+\nu)(1-2\nu))$ in dimension
 101 3. A second example is the electrical impedance imaging with internal data, where the
 102 goal is to recover the conductivity σ in the scalar elliptic equation $-\nabla \cdot (\sigma \nabla u) = 0$. If
 103 one can measure two potential fields u_1 and u_2 solutions of the previous equation and
 104 defines $S := [\nabla u_1 \ \nabla u_2]$, then the problem reads $-\nabla \cdot (\sigma S) = \mathbf{0}$. A third example is a
 105 classical problem corresponding to the particular case where S is the identity matrix
 106 everywhere. In this case, the problem reads $-\nabla \mu = \mathbf{f}$ which is the inverse gradient
 107 problem.

108 The properties of the gradient operator $\nabla : L^2(\Omega) \rightarrow H^{-1}(\Omega, \mathbb{R}^d)$ and its dis-
 109 cretization have been extensively studied in particular in the context of fluid dynam-
 110 ics and some tools developed in this framework are useful to treat our more general
 111 problem. For the reader convenience, let us recall the most important property which
 112 ensures the existence of a bounded left-inverse.

113 Hence, in the case where S is the identity matrix everywhere, *i.e.* $T := -\nabla$, the
 114 operator T is known to be a closed range operator from $L^2(\Omega)$ to $H^{-1}(\Omega, \mathbb{R}^d)$ if Ω is
 115 a Lipschitz domain (see [22, p.99] and references within). One can write

$$116 \quad \|q\|_{L^2(\Omega)} \leq C \|\nabla q\|_{H^{-1}(\Omega)} \quad \forall q \in L_0^2(\Omega),$$

118 where $C > 0$ and $L_0^2(\Omega)$ is the space of zero-mean, square-integrable functions. The
 119 norm of the pseudo-inverse of the gradient in $H^{-1}(\Omega, \mathbb{R}^d)$ is closely related with the
 120 *inf-sup* condition of the divergence:

$$121 \quad (1.8) \quad \beta := \inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in H_0^1(\Omega, \mathbb{R}^d)} \frac{\int_\Omega (\nabla \cdot \mathbf{v}) q}{\|\mathbf{v}\|_{H_0^1(\Omega)} \|q\|_{L^2(\Omega)}} > 0$$

123 Indeed, we have $C = 1/\beta$. Since the closed-range property of the gradient is equiv-
 124 alent to the surjectivity of the divergence in $L_0^2(\Omega)$, the study of behavior of β is an
 125 important step in establishing the well-posedness and stability of the Stokes problem
 126 [14, Chap. I, Theorem 4.1]. The constant β is also known as the *LBB* constant (for
 127 Ladyzhenskaya-Babuška-Brezzi). It is well known that in general, the constant β may

128 not behave well in finite element spaces, and may vanish when the mesh size goes to
 129 zero. More precisely, if one considers discrete spaces $M_h \subset L^2(\Omega)$ and $V_h \subset H_0^1(\Omega, \mathbb{R}^d)$
 130 with discretization parameter $h > 0$, the associated discrete *inf-sup* constant given by

$$131 \quad (1.9) \quad \beta_h := \inf_{q \in M_h} \sup_{\mathbf{v} \in V_h} \frac{\int_{\Omega} (\nabla \cdot \mathbf{v}) q}{\|\mathbf{v}\|_{H_0^1(\Omega)} \|q\|_{L^2(\Omega)}}$$

132
 133 may not satisfy the discrete *inf-sup* condition $\forall h > 0, \beta_h \geq \beta^* > 0$. Pairs of finite
 134 element spaces that satisfy the discrete *inf-sup* condition are known as *inf-sup* stable
 135 elements and play an important role in the stability of the Galerkin approximation
 136 for the Stokes problem. We refer to [5] for more details on the *inf-sup* constant of the
 137 gradient and its convergence.

138 **1.2. Main results.** Inspired by this approach, we introduce a generalization of
 139 the *inf-sup* constant and a corresponding definition of the discrete *inf-sup* constant
 140 that are suitable for operators of type (1.2) in particular. A major difference with the
 141 classical definition of the *inf-sup* constant of the gradient is that, here, the operator
 142 T may contain measurement noise and may have a trivial null space.

143 In a general framework, consider $T \in \mathcal{L}(M, V')$ where M and V are two Hilbert
 144 spaces. The problem $T\mu = \mathbf{f}$ is approached by a finite dimensional problem $T_h\mu_h =$
 145 \mathbf{f}_h where $T \in \mathcal{L}(M_h, V_h')$ and M_h, V_h approach M and V respectively.

146 The first main goal of this work is to provide a stability condition with respect to
 147 the M -norm for the discrete problem based on the associated discrete *inf-sup* constant.
 148 We consider the stability with respect to both the noise and the interpolation error
 149 on the right-hand side \mathbf{f} and on the operator T itself. The case $\mathbf{f} = \mathbf{0}$ corresponds
 150 to a null space identification problem and the condition $\|\mu\|_M = 1$ is added.

151 In Theorem 4.1 we provide an error estimate between the normalized solution of
 152 $\arg \min_{z \in M_h} \|T_h z_h\|_{V_h'}$ and the normalized solution of $Tz = \mathbf{0}$ of the form

$$153 \quad \|z_h - p_h(z)\|_M \leq \frac{C}{\beta(T_h)} (\|T - T_h\| + \|z - \pi_h z\|_M)$$

154 where $\pi_h z$ is the orthogonal projection of z on M_h , $p_h(z) := \pi_h z / \|\pi_h z\|_M$. The
 155 constant $\beta(T_h)$ is an adaptation of the *inf-sup* constant from (1.9) to general operators.
 156 (See Section 3). For the hypotheses and other details about the norms used, see
 157 directly Theorem 4.1.

158 In the case $\mathbf{f} \neq \mathbf{0}$, we consider two distinct situations. The first case is when T
 159 is invertible and $\alpha(T) := \inf_{z \in N} \|Tz\|_{V'} / \|z\|_M$ is not "too small". In Theorem 4.4
 160 we provide an error estimate between the solution of $T_h\mu_h = \mathbf{f}_h$ in the sense of least
 161 squares and the unique solution of $T\mu = \mathbf{f}$ of the form

$$162 \quad \frac{\|\mu_h - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{C}{\alpha(T_h)} (\|T - T_h\| + \|\mathbf{f} - \mathbf{f}_h\| + \|\pi_h \mu - \mu\|_M).$$

163 where $\alpha(T_h) := \min_{z_h \in M_h} \|T_h z_h\|_{V_h'} / \|z_h\|_M$. For the hypotheses and other details
 164 about the norms used, see directly Theorem 4.4.

165 The second case is when T has a non trivial null space (of dimension one) or
 166 remains invertible but with a constant $\alpha(T_h)$ too small to make the previous result
 167 applicable. In this case, the error estimate is only proved in an hyperplane of M (the
 168 orthogonal complement of the approximated null space). The approximation of μ is
 169 then obtained up to an unknown scalar constant.

170 In Theorem 4.7 we provide an error estimate between the solution of $T_h\mu_h = \mathbf{f}_h$
 171 in $\{z_h\}^\perp$ in the sense of least squares and the solution of $T\mu = \mathbf{f}$ up to an unknown
 172 translation in the direction z_h . This estimate is of the form: $\exists t \in \mathbb{R}$ such that

$$173 \quad \frac{\|\mu_h + tz_h - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{C}{\beta(T_h)} (\|T_h - T\| + \|\mathbf{f}_h - \mathbf{f}\| + \|\pi_h\mu - \mu\|_M + \alpha(T_h)).$$

174 For the hypotheses and other details about the norms used, see directly Theorem 4.7.

175 These error estimates are quantitative. They depend on the discrete *inf-sup* con-
 176 stant and can be explicitly computed in all practical situations dealing with experi-
 177 mental data. These estimates allow for a control of the quality of the reconstruction
 178 in the pair of approximation spaces (M_h, V_h) directly from the noisy interpolated
 179 data. The behavior of the discrete *inf-sup* constant with respect to the discretization
 180 parameter h gives a practical criterion for the convergence of μ_h towards μ .

181 The present paper is closely linked to the sensitivity analysis and discretization
 182 analysis for the Moore-Penrose generalized inverse of T when T is a closed range op-
 183 erator. There exist a vast literature on this subject (see [4, 9, 24, 15] and references
 184 herein) as well as on the finite dimensional interpolation of the generalized inverse
 185 [11]. However, there are fundamental differences between the present work and the
 186 existing literature. First, we do not know here whether the operator T has closed
 187 range. Second, we perform a sensitivity analysis of the left inverse of $T \in \mathcal{L}(M, V')$
 188 under perturbations that are controlled in a weaker norm. More precisely, pertur-
 189 bations are controlled here in $\mathcal{L}(E, V')$ where $E \subset M$ is a Banach space dense in
 190 M . This might seem a technical issue but it is mandatory if one wants to work with
 191 discontinuous parameters μ and S . This choice is motivated by the applications in
 192 bio-medical imaging where, in most cases, the biological tissues exhibit discontinuities
 193 in their physical properties. For instance, in the linear elasticity inverse problem (see
 194 equation (1.6)) the matrix $S = 2\mathcal{E}(\mathbf{u})$ has the same surfaces of discontinuities than the
 195 shear modulus of the medium and cannot be approached in $L^\infty(\Omega, \mathbb{R}^{d \times d})$ by smooth
 196 functions. This leads to perturbations of T in $\mathcal{L}(L^\infty(\Omega), H^{-1}(\Omega, \mathbb{R}^d))$ instead of
 197 $\mathcal{L}(L^2(\Omega), H^{-1}(\Omega, \mathbb{R}^d))$. More details and examples are given in Subsection 2.3.

198 **1.3. Outline of the paper.** The article is organized as follows: In Section 2, we
 199 describe the Galerkin approximation of the problem (1.3) and define all the approxi-
 200 mation errors involved. In Section 3, we generalize the notion of *inf-sup* constant to
 201 any operator $T \in \mathcal{L}(M, V')$ and we prove in Theorem 3.10 the upper semi-continuity
 202 of the discrete *inf-sup* constant. This is an asymptotic comparison between the dis-
 203 crete and the *continuous inf-sup* constants. In Section 4 we give and prove the main
 204 stability estimates (Theorems 4.1, 4.4 and 4.7) based on the discrete version of the
 205 *inf-sup* constant just defined. In Section 5 we present various numerical inversions,
 206 including stability tests and numerical computations of the *inf-sup* constant for var-
 207 ious pairs of finite element spaces. We also introduce in this section a pair of finite
 208 element spaces based on a hexagonal tiling of the domain Ω . It shows excellent numer-
 209 ical stability properties when compared to some more classical pair of discretization
 210 spaces.

211 **2. Discretization using the Galerkin approach.** We describe the Galerkin
 212 approximation of problem (1.3) and give the definitions of the various errors of ap-
 213 proximation.

214 **2.1. General notations.** In all this work, M and V are two Hilbert spaces with
 215 respective inner products denoted $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_V$. We denote $E \subset M$ a Banach

216 space dense in M . The space $V' := \mathcal{L}(V, \mathbb{R})$ is the space of the bounded linear forms
 217 on V endowed with the operator norm

$$218 \quad (2.1) \quad \|\varphi\|_{V'} := \sup_{\mathbf{v} \in V} \frac{\langle \varphi, \mathbf{v} \rangle_{V', V}}{\|\mathbf{v}\|_V},$$

219 where $\langle \cdot, \cdot \rangle_{V', V}$ is the duality pairing between V' and V . The space $\mathcal{L}(M, V')$ is the
 220 space of the bounded linear operator from M to V' endowed with the operator norm
 221 written $\|\cdot\|_{M, V'}$. For any $T \in \mathcal{L}(M, V')$, we denote its null space by $N(T)$.

222 *Example 2.1.* In the case of the inverse elastography problem using the operator
 223 T defined in (1.2), we take $M := L^2(\Omega)$, $V := H_0^1(\Omega, \mathbb{R}^d)$, $E := L^\infty(\Omega)$ and so
 224 $V' = H^{-1}(\Omega, \mathbb{R}^d)$. Here $H_0^1(\Omega, \mathbb{R}^d)$ is the space of all squared integrable vector-valued
 225 functions \mathbf{v} on Ω such that $\nabla \mathbf{v}$ is also square integrable and such that its trace on $\partial\Omega$
 226 vanishes. The space $H^{-1}(\Omega, \mathbb{R}^d)$ is the topological dual of $H_0^1(\Omega, \mathbb{R}^d)$.

227 **2.2. Spaces discretization and projection.** In order to approach the problem
 228 (1.3) by a finite dimensional problem, we first approach spaces M and V by finite
 229 dimensional spaces.

230 **DEFINITION 2.2.** For any Banach space X , we say that a sequence subspaces
 231 $(X_h)_{h>0}$ approaches X if this sequence is asymptotically dense in X . That means
 232 that for any $x \in X$, there exists a sequence $(x_h)_{h>0}$ such that $x_h \in X_h$ for all $h > 0$
 233 and $\|x_h - x\|_X$ converges to zero when h goes to zero. We naturally endow X_h with
 234 the restriction of the X -norm to make it a normed vector space.

235 Consider now two sequences of subspaces $(M_h)_{h>0}$ and $(V_h)_{h>0}$ that approach
 236 respectively the Hilbert spaces M and V . Naturally, M_h is endowed with the M -
 237 norm and V_h is endowed with the V -norm. In some cases we need to use the E -norm
 238 over M_h . To highlight the difference, we will denote $E_h := (M_h, \|\cdot\|_E)$ the space M_h
 239 endowed with the E -norm.

240 *Example 2.3.* In the case of Example 2.1, $M = L^2(\Omega)$ and one can choose M_h as
 241 the classical finite element space $\mathbb{P}^0(\Omega_h)$, i.e. the class of piecewise constant functions
 242 over a subdivision of Ω by elements of maximum diameter $h > 0$ [14].

243 **DEFINITION 2.4.** We denote $\pi_h : M \rightarrow M_h$ the orthogonal projection form M
 244 onto M_h . It naturally satisfies $\lim_{h \rightarrow 0} \|\pi_h m - m\|_M = 0$ and $\|\pi_h m\|_M \leq \|m\|_M$, for
 245 all $m \in M$. We also denote $p_h : M \setminus N(\pi_h) \rightarrow M_h$ the normalized projection form
 246 M onto M_h defined by $p_h(m) := \pi_h m / \|\pi_h m\|_M$, $\forall m \in M$, $\pi_h m \neq 0$. Note that if
 247 $\|m\|_M = 1$, $p_h(m)$ satisfies $\|p_h(m) - m\|_M \leq \sqrt{2} \|\pi_h m - m\|_M$.

248 In the following, we will assume that π_h is also a contraction for the E -norm.
 249 That means,

$$250 \quad (2.2) \quad \forall m \in E \subset M, \quad \|\pi_h m\|_E \leq \|m\|_E.$$

251 This hypothesis is true in the case $E := L^\infty(\Omega)$, $M := L^2(\Omega)$ and $M_h := \mathbb{P}^0(\Omega_h)$ as
 252 in Example 2.3.

253 **DEFINITION 2.5.** For any non zero $\mu \in M$, we define its relative error of inter-
 254 polation onto M_h by

$$255 \quad (2.3) \quad \varepsilon_h^{int}(\mu) := \frac{\|\pi_h \mu - \mu\|_M}{\|\mu\|_M}.$$

256 As the sequence of subspaces $V_h \subset V$ approaches V , we define V'_h the space of all
 257 linear forms over V_h endowed with the norm

$$258 \quad \|\varphi\|_{V'_h} := \sup_{\mathbf{v} \in V_h} \frac{\langle \varphi, \mathbf{v} \rangle_{V'_h, V_h}}{\|\mathbf{v}\|_V}.$$

259 Note that $\mathbf{f} \mapsto \mathbf{f}|_{V_h}$ defines a natural map from V' onto V'_h and then any $\mathbf{f} \in V$
 260 naturally defines a unique element $\mathbf{f}|_{V_h}$ of V'_h (and we continue to call it \mathbf{f}). Then
 261 any non zero right-hand side linear form $\mathbf{f} \in V'$ is approached by a finite dimensional
 262 linear form $\mathbf{f}_h \in V'_h$ and we define its relative error of interpolation as follows.

263 **DEFINITION 2.6.** *The relative error of interpolation ε_h^{rhs} between $\mathbf{f} \neq \mathbf{0}$ and \mathbf{f}_h*
 264 *is defined by*

$$265 \quad (2.4) \quad \varepsilon_h^{rhs} := \frac{\|\mathbf{f}_h - \mathbf{f}\|_{V'_h}}{\|\mathbf{f}\|_{V'}}.$$

266 **2.3. Interpolation of the operator.** We approach the operator $T \in \mathcal{L}(M, V')$
 267 by a finite dimensional operator $T_h \in \mathcal{L}(M_h, V'_h)$. The error of approximation is
 268 defined as $T - T_h$ for the $\mathcal{L}(E_h, V'_h)$ norm which is weaker than assuming that the
 269 distance between T and T_h is small in $\mathcal{L}(M_h, V'_h)$. We remind the reader that $E_h :=$
 270 $E \cap M_h$ endowed with the E -norm.

271 **DEFINITION 2.7.** *The interpolation error ε_h^{op} between T and T_h is defined by*

$$272 \quad (2.5) \quad \varepsilon_h^{op} := \|T_h - T\|_{E_h, V'_h} := \sup_{\mu \in E_h} \frac{\|(T_h - T)\mu\|_{V'_h}}{\|\mu\|_E}.$$

273 This error contains both the interpolation error over the approximation spaces and
 274 the possible noise in measurements used to build T_h .

275 *Remark 2.8.* The reason of the choice of norms comes from the main application
 276 where $M := L^2(\Omega)$, $E := L^\infty(\Omega)$, $V := H_0^1(\Omega, \mathbb{R}^d)$ and $T\mu := -\nabla \cdot (\mu S)$ with
 277 $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$. This operator is approached by $T_h\mu := -\nabla \cdot (\mu S_h)$ where S_h is a
 278 discrete and possibly noisy version of S . In this case, the interpolation error $S_h - S$ is
 279 expected to be small in $L^2(\Omega, \mathbb{R}^{d \times d})$ but not in $L^\infty(\Omega, \mathbb{R}^{d \times d})$. This conduces to small
 280 interpolation error ε_h^{op} thanks to the control

$$281 \quad (2.6) \quad \|(T_h - T)\mu\|_{H^{-1}(\Omega)} \leq \|S_h - S\|_{L^2(\Omega)} \|\mu\|_{L^\infty(\Omega)}, \quad \forall \mu \in M_h.$$

282 but $T_h - T$ has no reason to be small in $\mathcal{L}(M_h, V'_h)$ (See example 2.9). This defini-
 283 tion of ε_h^{op} matches well practical situations like medical imaging for instance where S
 284 might be a discontinuous map with *a priori* unknown surfaces of discontinuity. There-
 285 fore it makes sense to consider $S_h - S$ small in $L^2(\Omega, \mathbb{R}^{d \times d})$ but not in $L^\infty(\Omega, \mathbb{R}^{d \times d})$.
 286 The next example 2.9 below explains this situation in dimension one.

287 *Example 2.9.* In dimension one, take $\Omega := (-1, 1)$, $M = L^2(\Omega)$, $E = L^\infty(\Omega)$ and
 288 $V = H_0^1(\Omega)$. Take $S \in L^\infty(\Omega)$ and define $T\mu := -(\mu S)'$. Fix $h > 0$ and consider
 289 any uniform subdivision $\Omega_h \subset \Omega$ of size h containing the segment $I_h := (-h/2, h/2)$
 290 (hence 0 is not a node). Define the interpolation spaces $M_h := \mathbb{P}^0(\Omega_h)$, $V_h := \mathbb{P}_0^1(\Omega_h)$.
 291 Chose $S = 1 + \chi_{(0,1)}$ and $S_h = 1 + \chi_{(\frac{h}{2}, 1)} \in M_h$ and $T_h\mu := -(\mu S_h)'$. An explicit
 292 computation gives

$$293 \quad \|S_h - S\|_{L^2(\Omega)}^2 = \frac{h}{2} \quad \text{i.e.} \quad \|S_h - S\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{h}).$$

294 Thanks to (2.6), we also get that $\|T_h - T\|_{E_h, V'_h} = \mathcal{O}(\sqrt{h})$.

295 Consider now the sequence $\mu_h = h^{-1/2}\chi_{I_h}$ which satisfies $\|\mu_h\|_{L^2(\Omega)} = 1$ and a
 296 basis test function $v_h \in V_h$ supported in $[-h/2, 3h/2]$ and such that $v_h(h/2) = 1$. It
 297 satisfies $\|v_h\|_{H_0^1(-1,1)} = \sqrt{2/h}$. We can write

$$298 \quad \langle -(\mu_h(S_h - S))', v_h \rangle_{H^{-1}, H_0^1} = \int_{I_h} \mu_h(S_h - S)v'_h = h^{-1/2},$$

299 hence

$$300 \quad \sup_{v \in V_h} \frac{\langle -(\mu_h(S_h - S))', v \rangle_{H^{-1}, H_0^1}}{\|v\|_{H_0^1(-1,1)}} \geq \frac{\langle -(\mu_h(S_h - S))', v_h \rangle_{H^{-1}, H_0^1}}{\|v_h\|_{H_0^1(-1,1)}} = \frac{\sqrt{2}}{2},$$

301 and then $\|T_h - T\|_{M_h, V'_h} \geq \frac{\sqrt{2}}{2}$. As a consequence $T_h - T$ is not getting small for the
 302 $\mathcal{L}(M_h, V'_h)$ -norm.

303 **3. The generalized *inf-sup* constant.** In this section we generalize the notion
 304 of *inf-sup* constant to any operators T in $\mathcal{L}(M, V')$. Let us first define three useful
 305 constants for such operators.

306 DEFINITION 3.1. For any $T \in \mathcal{L}(M, V')$, we call

$$307 \quad \alpha(T) := \inf_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \quad \text{and} \quad \rho(T) := \sup_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

308 we also call $\delta(T) := \sqrt{\rho(T)^2 - \alpha(T)^2}$.

309 We now extend the notion of *inf-sup* constant of the gradient operator to any
 310 operators of $\mathcal{L}(M, V')$. As the existence of a null space of dimension one is not
 311 guaranteed¹, we first propose this very general definition of the generalized *inf-sup*
 312 constant called $\beta(T)$.

313 **3.1. Definition and properties.**

314 DEFINITION 3.2. The *inf-sup* constant of direction $e \in M$, $e \neq 0$ of the operator
 315 $T \in \mathcal{L}(M, V')$ is the non-negative number

$$316 \quad \beta_e(T) := \inf_{\substack{\mu \in M \\ \mu \perp e}} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

317 The generalized *inf-sup* constant of T is now defined by

$$318 \quad \beta(T) := \sup_{\substack{e \in M \\ \|e\|_M=1}} \beta_e(T).$$

319 It is mandatory here to show that this definition indeed extends the classical
 320 definition of the *inf-sup* constant known for ∇ -type operators (with a null space of
 321 dimension one).

¹Depending on $S(x)$, the operator $T : \mu \mapsto -\nabla \cdot (\mu S)$ may have various type of null spaces. In one hand, in [1] it has been shown that if S is smooth and everywhere invertible, then $N(T) = \{0\}$ if and only if $S^{-1}\nabla \cdot S$ is not a gradient. In the other hand, if S vanishes in a subset $\omega \subset \Omega$, then any function μ supported inside ω belongs to $N(T)$.

322 PROPOSITION 3.3. Let $T \in \mathcal{L}(M, V')$ and $z \in M$ such that $\|z\|_M = 1$ and
 323 $\|Tz\|_{V'}^2 \leq \alpha(T)^2 + \varepsilon^2$ for some $\varepsilon \geq 0$. We have

$$324 \quad \beta_z(T)^2 \leq \beta(T)^2 \leq \beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon).$$

325 In case where $\varepsilon = 0$, it implies that $\beta(T) = \beta_z(T)$.

326 The proof of this result uses the self-adjoint operator $S_T \in \mathcal{L}(M)$ canonically
 327 associated with T .

328 LEMMA 3.4. For any $T \in \mathcal{L}(M, V')$, there exists $S_T \in \mathcal{L}(M)$ self-adjoint posi-
 329 tive semi-definite such that for any $\mu \in M$, $\|T\mu\|_{V'}^2 = \langle S_T\mu, \mu \rangle_M$.

330 *Proof.* Call $\Phi : V' \rightarrow V$ the Riesz isometric identification defined by $\langle \Phi f, v \rangle_V =$
 331 $\langle f, v \rangle_{V', V}$ for any $f \in V'$, $v \in V$. Call also $T^* : V \rightarrow H$ the adjoint operator of T .
 332 We have for any $\mu \in M$,

$$333 \quad \|T\mu\|_{V'}^2 = \|\Phi T\mu\|_V^2 = \langle T\mu, \Phi T\mu \rangle_{V', V} = \langle \mu, T^* \Phi T\mu \rangle_M = \langle S_T\mu, \mu \rangle_M.$$

334 where $S_T := T^* \Phi T : M \rightarrow M$ is a self-adjoint positive semi-definite operator. \square

335 *Proof.* (of Proposition 3.3) The first inequality comes from the definition of $\beta(T)$.
 336 For the second, take $e \in M$ of norm one and consider $m \in E \cap \{z\}^\perp$ of norm
 337 one. If $e \perp z$ then $z \in \{e\}^\perp$ and immediately $\beta_e(T)^2 \leq \|Tz\|_{V'}^2 \leq \alpha(T)^2 + \varepsilon^2 \leq$
 338 $\beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon)$.

339 Suppose now that $\langle e, z \rangle_M \neq 0$. Consider $a = -\langle m, e \rangle_M / \langle z, e \rangle_M$ and $\mu := az + m$.
 340 It is clear that $\mu \in \{e\}^\perp$ and $\|\mu\|_M^2 = a^2 + 1$. Using Lemma 3.4, we write

$$\begin{aligned} 341 \quad \|T\mu\|_{V'}^2 &= \langle S_T\mu, \mu \rangle_M = a^2 \langle S_Tz, z \rangle_M + 2a \langle S_Tz, m \rangle_M + \langle S_Tm, m \rangle_M \\ &= a^2 \|Tz\|_{V'}^2 + 2a \langle S_Tz, m \rangle_M + \|Tm\|_{V'}^2 \\ &\leq (1 + a^2) \|Tm\|_{V'}^2 + a^2 \varepsilon^2 + 2|a| |\langle S_Tz, m \rangle_M|. \end{aligned}$$

342 Using Proposition A.1 we bound $|\langle S_Tz, m \rangle_M|$ by $\varepsilon\delta(T)$ and then

$$\begin{aligned} &\frac{\|T\mu\|_{V'}^2}{\|\mu\|_M^2} \leq \|Tm\|_{V'}^2 + \varepsilon^2 + \varepsilon\delta(T) \\ 343 \quad \inf_{\substack{\mu \in E \\ \mu \perp e}} \frac{\|T\mu\|_{V'}^2}{\|\mu\|_M^2} &\leq \|Tm\|_{V'}^2 + \varepsilon(\delta(T) + \varepsilon) \\ &\beta_e(T)^2 \leq \|Tm\|_{V'}^2 + \varepsilon(\delta(T) + \varepsilon). \end{aligned}$$

344 This last statement is true for any $m \in M \cap \{z\}^\perp$ of norm one so we can take the
 345 infimum over m to get $\beta_e(T)^2 \leq \beta_z(T)^2 + \varepsilon(\delta(T) + \varepsilon)$. We conclude now by taking
 346 the supremum over e . \square

347 As a consequence of Proposition 3.3, the generalized *inf-sup* constant has a simpler
 348 formula in the case of an operator with trivial null space.

349 COROLLARY 3.5. If $N(T) \neq \{0\}$, consider any $z \in N(T)$ such that $\|z\|_M = 1$.
 350 Then we have $\beta(T) = \beta_z(T)$.

351 If $T = \nabla$, the classical definition of $\beta(\nabla)$ given in (1.8) matches the definition
 352 3.2.

353 *Remark 3.6.* This corollary leads to an alternative definition of $\beta(T)$ which does
 354 not depend on the choice of z in $N(T)$ (even for a dimension greater than one).
 355 Moreover, we see that $\beta(T) > 0$ implies $\dim N(T) = 1$. Indeed, if $N(T) > 1$, then
 356 there exist $z_1, z_2 \in N(T)$ and $\beta(T) = \beta_{z_1}(T) > 0$ with $z_1 \perp z_2$ such that $T(z_1) =$
 357 $T(z_2) = 0$ and $z_1, z_2 \neq 0$. Moreover, as $\|Tz_2\| \geq \beta_{z_1}(T)\|z_2\|$, we have $z_2 = 0$ which is
 358 a contradiction.

359 It is possible to extend a little this corollary to a class of operators with trivial
 360 null space if the infimum value of the operator on the unit sphere is reached.

361 *Remark 3.7.* The case $\varepsilon = 0$ in Proposition 3.3 leads to an alternative definition
 362 of $\beta(T)$ if the infimum $\alpha(T)$ is reached. Note that this definition does not depend on
 363 the choice of z . Moreover the condition $\varepsilon = 0$ is fulfilled in particular if T is a finite
 364 rank or finite dimensional operator.

365 If the infimum value $\alpha(T)$ is not reached on the unit sphere, we keep the general
 366 definition 3.2.

367 **3.2. Discrete *inf-sup* constant.** The different constants related to the ap-
 368 proximated operator $T_h \in \mathcal{L}(M_h, V'_h)$ come from the same definition than for the
 369 operator $T \in \mathcal{L}(M, V')$. Simply remark that as T_h is a finite dimensional operator,
 370 the infimum in

$$371 \quad (3.1) \quad \alpha(T_h) := \inf_{\mu \in M_h} \frac{\|T_h \mu\|_{V'_h}}{\|\mu\|_M}$$

372 is reached by a direction $z_h \in M_h$ such that $\|z_h\|_M = 1$. This means that $\|T_h z_h\|_{V'_h} =$
 373 $\alpha(T_h)$. As a consequence, following Corollary ??, the *inf-sup* constant of T_h is given
 374 by

$$375 \quad (3.2) \quad \beta(T_h) := \inf_{\substack{\mu \in M_h \\ \mu \perp z_h}} \frac{\|T_h \mu\|_{V'_h}}{\|\mu\|_M}.$$

376 This discrete *inf-sup* constant is the key element to establish the stability of the
 377 discrete inverse problem and as we will see, its behaviors when $h \rightarrow 0$ will determine
 378 the convergence of the solution of the discrete problem to the exact solution. In
 379 a similar way than for the classical *inf-sup* constant, the behavior of the discrete
 380 inf – sup constant $\beta(T_h)$ can be catastrophic in the sense that it can vanish to zero if
 381 $h \rightarrow 0$. This strongly depends on the choice of interpolation pair of spaces (M_h, V_h) .
 382 For instance, if the discrete operator $T_h : M_h \rightarrow V'_h$ is under-determined, one may
 383 have $\beta(T_h) = 0$. In a same manner than in [8], we give a definition of the discrete
 384 *inf-sup* condition.

385 **DEFINITION 3.8.** *We say that the sequence of operators $(T_h)_{h>0}$ satisfies the dis-*
 386 *crete inf-sup condition if there exists $\beta^* > 0$ such that*

$$387 \quad (3.3) \quad \beta^* \leq \beta(T_h), \quad \forall h > 0.$$

388 *Remark 3.9.* In this work, we do not prove that the discrete *inf-sup* condition is
 389 satisfied by some specific choices of discretized operators $T_h : M_h \rightarrow V'_h$. We mention
 390 it here as a condition for uniform stability with respect to h , (see Theorems 4.1 4.7).
 391 We only aim at giving discrete stability estimates that involves $\beta(T_h)$ for a fixed $h > 0$.

392 **3.3. Upper semi-continuity of the *inf-sup* constant.** A legitimate question
393 about the discrete *inf-sup* constant is to know if it can be greater than the continuous
394 *inf-sup* constant if the discretization spaces are well chosen. Inspired by a classical
395 result on the discrete *inf-sup* of the divergence that can be found in [8] for instance,
396 we state and prove in this subsection that the discrete *inf-sup* constant is upper semi-
397 continuous when $h \rightarrow 0$. This concludes that the discrete *inf-sup* constant $\beta(T_h)$ is
398 always asymptotically worse than the continuous *inf-sup* constant $\beta(T)$.

399 **THEOREM 3.10** (Upper semi-continuity). *If the operator error ε^{op} defined in*
400 *(2.5) converges to 0 when $h \rightarrow 0$, then*

$$401 \quad \limsup_{h \rightarrow 0} \alpha(T_h) \leq \alpha(T).$$

402 *Moreover, if the problem $Tz = \mathbf{0}$ admits a solution $z \in E$ with $\|z\|_M = 1$ and if the*
403 *sequence $(T_h)_{h>0}$ satisfies the discrete *inf-sup* condition (see Definition 3.8), then*

$$404 \quad \limsup_{h \rightarrow 0} \beta(T_h) \leq \beta(T).$$

405 *Remark 3.11.* This result is useful to understand that no discretization can get
406 a better stability constant than $\beta(T)$. The question of the convergence of $\alpha(T_h)$ and
407 $\beta(T_h)$ toward respectively $\alpha(T)$ and $\beta(T)$ is not treated here; it is clearly not a simple
408 question. It is already known as a difficult issue concerning *inf-sup* constant of the
409 gradient operator. See [5] for more details about this question.

410 *Remark 3.12.* An interesting consequence of this result is that, in case of an
411 operator T with non-trivial null space, the fact that $(T_h)_{h>0}$ satisfies the discrete
412 *inf-sup* condition implies that $\beta(T) > 0$ which means that T has closed range (see [6,
413 p. 47]). It could be used to prove the closed range property for some operators. For
414 instance, to our knowledge, the minimal conditions on $S \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ that make
415 $T : \mu \mapsto -\nabla \cdot (\mu S)$ a closed range operator are not known.

416 *Proof.* (of Theorem 3.10) First define the sequence of set

$$417 \quad C_h := \left\{ \mu \in M_h \mid (\varepsilon_h^{op})^{1/2} \|\mu\|_E \leq \|\mu\|_M \right\}.$$

418 For any $h > 0$ and $\mu \in C_h$ we get

$$419 \quad (3.4) \quad \begin{aligned} \|T_h \mu\|_{V'_h} &\leq \|T \mu\|_{V'_h} + \|(T_h - T) \mu\|_{V'_h} \leq \|T \mu\|_{V'} + \varepsilon_h^{op} \|\mu\|_E \\ &\leq \|T \mu\|_{V'} + (\varepsilon_h^{op})^{1/2} \|\mu\|_M. \end{aligned}$$

420 Hence

$$421 \quad \begin{aligned} \alpha(T_h) &\leq \frac{\|T \mu\|_{V'}}{\|\mu\|_M} + (\varepsilon_h^{op})^{1/2}, \quad \forall \mu \in C_h \\ \alpha(T_h) &\leq \inf_{\mu \in C_h} \frac{\|T_h \mu\|_{V'_h}}{\|\mu\|_M} + (\varepsilon_h^{op})^{1/2}. \end{aligned}$$

422 This is true for any $h > 0$ so $\limsup_{h \rightarrow 0} \alpha(T_h) \leq \limsup_{h \rightarrow 0} \inf_{\mu \in C_h} \frac{\|T \mu\|_{V'}}{\|\mu\|_M}$.

423 As proposition B.4 shows that $\lim_{h \rightarrow 0} C_h = M$ in the sense of Definition B.1, using
424 that T is continuous over the sphere $\{\mu \in M \mid \|\mu\|_M = 1\}$ we can use Proposition

425 **B.3** that says

$$426 \quad \limsup_{h \rightarrow 0} \inf_{\mu \in C_h} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \leq \inf_{\mu \in M} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} = \alpha(T)$$

427 which gives the first result.

428 For the second result, consider the sequence $(z_h)_{h>0}$ that satisfies $\|z_h\|_M = 1$ and
 429 $\|T_h z_h\|_{V'_h} = \alpha(T_h)$. Then $\beta(T_h) = \beta_{z_h}(T_h)$. For any $h > 0$ and $\mu \in C_h \cap \{z_h\}^\perp$,
 430 similarly to (3.4), we get

$$431 \quad \|T_h \mu\|_{V'_h} \leq \|T\mu\|_{V'} + (\varepsilon_h^{\text{op}})^{1/2} \|\mu\|_M,$$

432 and then by definition of $\beta(T_h)$,

$$433 \quad \beta(T_h) \leq \frac{\|T\mu\|_{V'}}{\|\mu\|_M} + (\varepsilon_h^{\text{op}})^{1/2}, \quad \forall \mu \in C_h \cap \{z_h\}^\perp$$

$$433 \quad \beta(T_h) \leq \inf_{\mu \in C_h \cap \{z_h\}^\perp} \frac{\|T_h \mu\|_{V'_h}}{\|\mu\|_M}.$$

434 This is true for any $h > 0$ so we deduce

$$435 \quad \limsup_{h \rightarrow 0} \beta(T_h) \leq \limsup_{h \rightarrow 0} \inf_{\mu \in C_h \cap \{z_h\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M}.$$

436 Now as Theorem 4.1 says that the sequence z_h converges to z in M and Proposition
 437 **B.5** gives that $\lim_{h \rightarrow 0} C_h \cap \{z_h\}^\perp = M \cap \{z\}^\perp$, we can use Proposition **B.3** that says

$$438 \quad \limsup_{h \rightarrow 0} \inf_{\mu \in C_h \cap \{z_h\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} \leq \inf_{\mu \in M \cap \{z\}^\perp} \frac{\|T\mu\|_{V'}}{\|\mu\|_M} = \beta_z(T) = \beta(T)$$

439 which gives the second result. □

440 **4. Error estimates.** In this section, we state and prove the error estimates that
 441 are stability estimates for the approximated problem $T_h \mu_h = \mathbf{f}_h$.

442 **4.1. Error estimate in the case $\mathbf{f} = \mathbf{0}$.**

443 **THEOREM 4.1** (Error estimate in the case $\mathbf{f} = \mathbf{0}$). *Consider $T \in \mathcal{L}(M, V')$ and*
 444 *let $z \in E$ be a solution of $Tz = \mathbf{0}$ with $\|z\|_M = 1$ and assume that h is small enough*
 445 *to have $\varepsilon_h^{\text{int}}(z) \leq 1/2$. Consider $z_h \in M_h$ a solution of*

$$446 \quad (4.1) \quad \|T_h z_h\|_{V'_h} = \alpha(T_h) \quad \text{with} \quad \|z_h\|_M = 1 \quad \text{and} \quad \langle z_h, z \rangle_M \geq 0.$$

447 *If $\beta(T_h) > 0$ we have*

$$448 \quad \|z_h - p_h(z)\|_M \leq \frac{4}{\beta(T_h)} (\sqrt{2} \|z\|_E \varepsilon_h^{\text{op}} + 2\rho(T) \varepsilon_h^{\text{int}}(z)).$$

449 *Where $\varepsilon_h^{\text{op}}$ and $\varepsilon_h^{\text{int}}$ are defined in (2.5) and (2.3). Moreover, if $\varepsilon_h^{\text{op}} \rightarrow 0$ and (T_h)*
 450 *satisfies the discrete inf-sup condition (3.3), then $\|z_h - z\|_M \rightarrow 0$.*

451 **Remark 4.2.**

452 1. Note that if $\varepsilon_h^{\text{op}} \rightarrow 0$, since $\alpha(T) = 0$, we have, from Theorem 3.10, that
 453 $\alpha(T_h) \rightarrow 0$. Moreover, if the discrete inf sup condition (equation (3.3)) is
 454 satisfied, then z_h is defined uniquely.

- 455 2. It is necessary to assume $z \in E$ to overcome the fact that $T_h - T$ is controlled
456 in $\mathcal{L}(E_h, V'_h)$ but not in $\mathcal{L}(M_h, V'_h)$. See section 2.3 for more details. In
457 the framework of the inverse elastography problem, the hypothesis $z \in E :=$
458 $L^\infty(\Omega)$ is not restrictive as physical parameters of biological tissues have
459 bounded values with some known *a priori* bounds.
460 3. The normalized projection $p_h(z)$ of z is the best possible approximation of z
461 in M_h with the constraint of norm one.
462 4. Problem (4.1) admits a solution z_h as T_h is a finite dimensional operator.
463 The condition $\langle z_h, z \rangle_M \geq 0$ is only here to choose between the two solutions
464 z_h and $-z_h$ and is not of crucial importance.
465 5. This result provides a quantitative error estimate as $\beta(T_h)$ can be computed
466 from T_h as the second smallest singular value (see Subsection 5.1) and all the
467 error terms on the right-hand side can be estimated (at least an upper bound
468 can be given).

469 Before giving the proof of Theorem 4.1, we first establish and prove a more general
470 result.

471 PROPOSITION 4.3. Consider $T_1 \in \mathcal{L}(M, V')$ let $z_1 \in E$ be a solution of

$$472 \quad \|T_1 z_1\|_{V'} \leq \alpha(T_1) + \varepsilon_1 \quad \text{with} \quad \|z_1\|_M = 1$$

473 where $\varepsilon_1 \geq 0$. Fix $r \geq \|z_1\|_E$. For any $T_2 \in \mathcal{L}(M, V')$, consider a solution $z_2 \in E$ of

$$474 \quad \|T_2 z_2\|_{V'} \leq \alpha(T_2) + \varepsilon_2 \quad \text{with} \quad \|z_2\|_M = 1 \quad \text{and} \quad \langle z_1, z_2 \rangle_M \geq 0.$$

475 If $\beta_{z_2}(T_2) > 0$ we have $\|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta_{z_2}(T_2)} \left(2r \|T_2 - T_1\|_{E, V'} + 2\alpha(T_1) + 2\varepsilon_1 + \varepsilon_2 \right)$

476 and if $\varepsilon_2 = 0$ this reads $\|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta(T_2)} \left(2r \|T_2 - T_1\|_{E, V'} + 2\alpha(T_1) + 2\varepsilon_1 \right)$.

477 *Proof.* Write $z_1 = tz_2 + m$ where $t \in [0, 1]$ and $m \perp z_2$. We have that $1 = t^2 +$
478 $\|m\|_M^2$. Then $z_1 - z_2 = (t-1)z_2 + m$ and so $\|z_2 - z_1\|_M^2 = 2(1-t) \leq 2(1-t^2) \leq 2\|m\|_M^2$.
479 Then $\|z_2 - z_1\|_M \leq \sqrt{2}\|m\|_M$. Now use the definition of $\beta_{z_2}(T_2)$ to write

$$480 \quad \begin{aligned} \beta_{z_2}(T_2) \|m\|_M &\leq \|T_2 m\|_{V'} \leq \|T_2 z_1\|_{V'} + \|T_2 z_2\|_{V'} \leq \|T_2 z_1\|_{V'} + \alpha(T_2) + \varepsilon_2 \\ &\leq 2 \|T_2 z_1\|_{V'} + \varepsilon_2 \end{aligned}$$

481 and remark that $\|T_2 z_1\|_{V'} \leq \|(T_2 - T_1)z_1\|_{V'} + \|T_1 z_1\|_{V'} \leq r \|T_2 - T_1\|_{E, V'} + \|T_1 z_1\|_{V'}$
482 which implies that

$$483 \quad \|T_2 z_1\|_{V'} \leq r \|T_2 - T_1\|_{E, V'} + \alpha(T_1) + \varepsilon_1.$$

484 We deduce that $\beta_{z_2}(T_2) \|m\|_M \leq 2r \|T_2 - T_1\|_{E, V'} + 2\alpha(T_1) + 2\varepsilon_1 + \varepsilon_2$ and then

$$485 \quad \|z_2 - z_1\|_M \leq \frac{\sqrt{2}}{\beta_{z_2}(T_2)} \left(2r \|T_2 - T_1\|_{E, V'} + 2\alpha(T_1) + 2\varepsilon_1 + \varepsilon_2 \right). \quad \square$$

486 We now give the proof of Theorem 4.1:

487 *Proof.* First remark that the infimum in (4.1) is reached here because T_h is a
488 finite dimensional operator. Consider $T|_{M_h} : M_h \rightarrow V'_h$ and call $g_h := Tp_h(z)$. This
489 quantity is small in V'_h as

$$490 \quad \begin{aligned} \|g_h\|_{V'_h} &= \|Tp_h(z)\|_{V'_h} = \|T(p_h(z) - z)\|_{V'_h} \leq \|T\|_{M, V'} \|p_h(z) - z\|_M \\ &\leq \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z). \end{aligned}$$

491 From this, we deduce that $\alpha(T|_{M_h}) \leq \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z)$ and that $p_h(z)$ is solution of

$$492 \quad \|T|_{M_h} p_h(z)\|_{V'_h} \leq \alpha(T|_{M_h}) + \varepsilon \quad \text{with} \quad \|p_h(z)\|_M = 1,$$

493 with $\varepsilon = \sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z)$. Due to Hypothesis (2.2) and $\varepsilon_h^{\text{int}}(z) \leq 1/2$ we have

$$494 \quad \|p_h(z)\|_E = \frac{\|\pi_h z\|_E}{\|\pi_h z\|_M} \leq 2 \frac{\|z\|_E}{\|z\|_M} \leq 2r.$$

495 Applying now Proposition (4.3) on operators $T_1 = T|_{M_h}$ and $T_2 = T_h$ both in
496 $\mathcal{L}(M_h, V'_h)$ with $z_1 = p_h(z)$, $z_2 = z_h$, $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 0$. We get

$$\begin{aligned} \|z_h - p_h(z)\|_M &\leq \frac{\sqrt{2}}{\beta(T_h)} (4r \varepsilon_h^{\text{op}} + 2\alpha(T|_{M_h}) + 2\varepsilon) \\ 497 \quad &\leq \frac{\sqrt{2}}{\beta(T_h)} \left(4r \varepsilon_h^{\text{op}} + 4\sqrt{2}\rho(T)\varepsilon_h^{\text{int}}(z) \right) \\ &\leq \frac{4}{\beta(T_h)} \left(\sqrt{2}r \varepsilon_h^{\text{op}} + 2\rho(T)\varepsilon_h^{\text{int}}(z) \right). \end{aligned}$$

498 For the convergence, the additional hypothesis give the convergence of the right-hand
499 side. We use that $p_h(z) \rightarrow z$ to conclude. \square

500 **4.2. Error estimates in the case $\mathbf{f} \neq \mathbf{0}$.** We give and prove a first stability
501 result based on the constant $\alpha(T_h)$.

502 **THEOREM 4.4** (Error estimate using $\alpha(T_h)$). *Consider $\mu \in E$ a solution of*
503 *$T\mu = \mathbf{f}$ with $\mathbf{f} \neq 0$ and assume that h is small enough to have $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$. Fix*
504 *$r := \|\mu\|_E / \|\mu\|_M$. Consider now $\mu_h \in M_h$ a solution of $\mu_h = \arg \min_{m \in M_h} \|T_h m - \mathbf{f}_h\|_{V'_h}$.*

505 *If $\alpha(T_h) > 0$, we have*

$$506 \quad \frac{\|\mu_h - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{4}{\alpha(T_h)} [r \varepsilon_h^{\text{op}} + \rho(T) (\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu))].$$

507 *Where $\varepsilon_h^{\text{op}}$, $\varepsilon_h^{\text{rhs}}$ and $\varepsilon_h^{\text{int}}$ are defined in (2.5), (2.4) and (2.3). Moreover, if there exists*
508 *$\alpha^* > 0$ such that $\alpha(T_h) \geq \alpha^*$ for all $h > 0$ and if $\varepsilon_h^{\text{op}} \rightarrow 0$ and $\varepsilon_h^{\text{rhs}} \rightarrow 0$ when $h \rightarrow 0$,*
509 *then $\|\mu_h - \mu\|_M \rightarrow 0$ when $h \rightarrow 0$.*

510 **Remark 4.5.** Note that if $\alpha(T_h) > 0$ for all $h > 0$, then μ_h is uniquely defined and
511 moreover $\varepsilon_h^{\text{op}} \rightarrow 0$ and if $\alpha(T_h) \geq \alpha_* > 0$, Theorem 3.10 assures that $\alpha(T) \geq \alpha_* > 0$
512 which guarantee the uniqueness of μ .

513 **Remark 4.6.** This result makes sense in practice even if $\alpha(T_h)$ goes to zero. In-
514 deed, at a fixed $h > 0$, $\alpha(T_h)$ can be computed from T_h as the first singular value and
515 all the error terms on the right-hand side can be estimated (at least an upper bound
516 can be given). It then gives a quantitative error bound on the reconstruction that can
517 be useful no matter with the asymptotic behavior of $\alpha(T_h)$.

518 **Proof.** First note that from the hypothesis $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$ we have that $\|\mu\|_M \leq$
519 $2\|\pi_h \mu\|_M$ and $\|\pi_h \mu\|_E \leq \|\mu\|_E \leq r\|\mu\|_M \leq 2r\|\pi_h \mu\|_M$ and $\|\mathbf{f}\|_{V'} \leq \rho(T)\|\mu\|_M$.

520 From the definition of $\alpha(T_h)$ we write

$$\begin{aligned}
\alpha(T_h) \|\mu_h - \pi_h \mu\|_M &\leq \|T_h \mu_h - T_h \pi_h \mu\|_{V'_h} \leq \|T_h \mu_h - \mathbf{f}_h\|_{V'_h} + \|T_h \pi_h \mu - \mathbf{f}_h\|_{V'_h} \\
&\leq 2 \|T_h \pi_h \mu - \mathbf{f}_h\|_{V'_h} \\
&\leq 2 \|T \mu - \mathbf{f}_h\|_{V'_h} + 2 \|T \pi_h \mu - T \mu\|_{V'_h} + 2 \|(T_h - T) \pi_h \mu\|_{V'_h} \\
521 &\leq 2 \|\mathbf{f} - \mathbf{f}_h\|_{V'_h} + 2 \rho(T) \|\pi_h \mu - \mu\|_M + 2 \varepsilon_h^{\text{op}} \|\pi_h \mu\|_E \\
&\leq 2 \varepsilon_h^{\text{rhs}} \|\mathbf{f}\|_{V'} + 2 \rho(T) \varepsilon_h^{\text{int}}(\mu) \|\mu\|_M + 4r \varepsilon_h^{\text{op}} \|\pi_h \mu\|_M \\
&\leq 2 \rho(T) (\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu)) \|\mu\|_M + 4r \varepsilon_h^{\text{op}} \|\pi_h \mu\|_M \\
&\leq 4 [\rho(T) (\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu)) + r \varepsilon_h^{\text{op}}] \|\pi_h \mu\|_M.
\end{aligned}$$

522

□

523 We now state and prove the main stability estimate concerning the general prob-
524 lem $T\mu = \mathbf{f}$ with a non zero right-hand side. This result uses $\beta(T_h)$ which is always
525 better than $\alpha(T_h)$. The price of this change is that the stability estimates only holds
526 in the hyperplane $\{z_h\}^\perp$, where z_h is the vector that minimizes $\|T_h z_h\|_{V'_h}$ on the unit
527 sphere.

528 **THEOREM 4.7** (Error estimate using $\beta(T_h)$). *Consider $\mu \in E$ a solution of*
529 *$T\mu = \mathbf{f}$ with $\mathbf{f} \neq 0$ and assume that h is small enough to have $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$. Fix*
530 *$r := \|\mu\|_E / \|\mu\|_M$. Consider $z_h \in M_h$ a solution of*

$$531 \quad \|T_h z_h\|_{V'_h} = \alpha(T_h) \quad \text{with} \quad \|z_h\|_M = 1.$$

532 Consider now $\mu_h \in M_h$ a solution of

$$533 \quad (4.2) \quad \mu_h = \arg \min_{\substack{m \in M_h \\ m \perp z_h}} \|T_h m - \mathbf{f}_h\|_{V'_h}, \quad \text{with } \mu_h \perp z_h.$$

534 If $\beta(T_h) > 0$, there exists $t \in \mathbb{R}$ such that $\mu_{h,t} := tz_h + \mu_h$ satisfies

$$535 \quad \frac{\|\mu_{h,t} - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{4}{\beta(T_h)} \left[r \varepsilon_h^{\text{op}} + \rho(T) (\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu)) + \frac{\alpha(T_h)}{2} \right].$$

536 Where $\varepsilon_h^{\text{op}}$, $\varepsilon_h^{\text{rhs}}$ and $\varepsilon_h^{\text{int}}$ are defined in (2.5), (2.4) and (2.3).

537 **Remark 4.8.** This result has to be used as soon as Theorem 4.4 is irrelevant
538 because $\alpha(T_h)$ is too small. It somehow kills the degenerated direction z_h and gives a
539 possibly better estimate for the computed solution up to an unknown component in
540 the direction z_h .

541 **Remark 4.9.** This result gives also the algorithmic procedure to approach the
542 exact solution μ :

- 543 1. Identify z_h with stability thanks to Theorem 4.1.
- 544 2. Solve the problem (4.2) to identify μ_h .
- 545 3. Find the best approximation $tz_h + \mu_h$ by choosing a correct coefficient $t \in \mathbb{R}$
546 using any additional scalar information on the exact solution such as its mean,
547 its background value, a punctual value, etc. . .

548 **Remark 4.10.** This result provides a quantitative error estimate as $\alpha(T_h)$ and
549 $\beta(T_h)$ can be computed from T_h as the two first singular values and all the error terms
550 on the right-hand side can be estimated (at least an upper bound can be given).

551 Before giving the proof of this Theorem, let us state and prove an intermediate
552 result.

553 PROPOSITION 4.11. Consider $T_1 \in \mathcal{L}(M, V')$ $\mathbf{f}_1 \in V'$, $\mathbf{f}_1 \neq 0$ and let $\mu_1 \in E$
554 be a solution of $T_1 \mu_1 = \mathbf{f}_1$. Fix $r := \|\mu_1\|_E / \|\mu_1\|_M$ and for any $T_2 \in \mathcal{L}(M, V')$,
555 consider a solution $z_2 \in E$ of

$$556 \quad \|T_2 z_2\|_{V'} \leq \alpha(T_2) + \varepsilon_2 \quad \text{and} \quad \|z_2\|_M = 1$$

557 and consider a solution $\mu_2 \in E$ of

$$558 \quad T_2 \mu_2 = \mathbf{f}_2 \quad \text{and} \quad \mu_2 \perp z_2.$$

559 If $\beta_{z_2}(T_2) > 0$, there exists $t \in \mathbb{R}$ such that $\mu_{2,t} := tz_2 + \mu_2$ satisfies

$$560 \quad \frac{\|\mu_{2,t} - \mu_1\|_M}{\|\mu_1\|_M} \leq \frac{1}{\beta_{z_2}(T_2)} \left(\frac{\|\mathbf{f}_2 - \mathbf{f}_1\|_{V'}}{\|\mu_1\|_M} + r \|T_2 - T_1\|_{E, V'} + \alpha(T_2) + \varepsilon_2 \right).$$

561 Moreover, if $\varepsilon_2 = 0$ it reads

$$562 \quad \frac{\|\mu_{2,t} - \mu_1\|_M}{\|\mu_1\|_M} \leq \frac{1}{\beta(T_2)} \left(\frac{\|\mathbf{f}_2 - \mathbf{f}_1\|_{V'}}{\|\mu_1\|_M} + r \|T_2 - T_1\|_{E, V'} + \alpha(T_2) \right).$$

563 *Proof.* Denote $\mu_{2,t} := tz_2 + \mu_2$ with $t := \langle \mu_1, z_2 \rangle_M$. With this choice, we have
564 that $(\mu_{2,t} - \mu_1) \perp z_2$. From the definition of $\beta_{z_2}(T_2)$, we write

$$\begin{aligned} \beta_{z_2}(T_2) \|\mu_{2,t} - \mu_1\|_M &\leq \|T_2 \mu_{2,t} - T_2 \mu_1\|_{V'} \\ &\leq \|T_2 \mu_2 - T_1 \mu_1\|_{V'} + |t| \|T_2 z_2\|_{V'} + \|(T_2 - T_1) \mu_1\|_{V'} \\ 565 \quad &\leq \|\mathbf{f}_2 - \mathbf{f}_1\|_{V'} + \|\mu_1\|_M (\alpha(T_2) + \varepsilon_2) + \|T_2 - T_1\|_{E, V'} \|\mu_1\|_E \cdot \\ &\leq \|\mathbf{f}_2 - \mathbf{f}_1\|_{V'} + \|\mu_1\|_M \left(\alpha(T_2) + \varepsilon_2 + r \|T_2 - T_1\|_{E, V'} \right). \end{aligned}$$

566

□

567 We can now give the proof of Theorem 4.7.

568 *Proof.* (of Theorem 4.7) Consider $T|_{M_h} : E_h \rightarrow V'_h$ and call $\mathbf{g}_h := T\pi_h\mu$. Remark
569 that $\|\pi_h\mu\|_E \leq \|\mu\|_E \leq r \|\mu\|_M \leq 2r \|\pi_h\mu\|_M$. Applying Proposition 4.11 to the
570 operators $T_1 := T|_{M_h}$, $T_2 := T_h$ both in $\mathcal{L}(M_h, V'_h)$, with $\mathbf{f}_1 := \mathbf{g}_h$, $\mathbf{f}_2 := T_h\mu_h$ both
571 in V'_h and with $\mu_1 := \pi_h\mu$, $\mu_2 := \mu_h$. We get the existence of $t \in \mathbb{R}$ such that

$$572 \quad \frac{\|\mu_{h,t} - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{1}{\beta(T_h)} \left(\frac{\|T_h\mu_h - \mathbf{g}_h\|_{V'_h}}{\|\pi_h\mu\|_M} + 2r \varepsilon_h^{\text{op}} + \alpha(T_h) \right).$$

Now we bound $\|T_h\mu_h - \mathbf{g}_h\|_{V'_h}$ as follows:

$$\|T_h\mu_h - \mathbf{g}_h\|_{V'_h} \leq \|T_h\mu_h - \mathbf{f}_h\|_{V'_h} + \|\mathbf{g}_h - \mathbf{f}_h\|_{V'_h}.$$

573 To deal with the first term, we define $p := \pi_h\mu - \langle \pi_h\mu, z_h \rangle_M z_h$ orthogonal to z_h . We
574 have

$$\begin{aligned} \|T_h\mu_h - \mathbf{f}_h\|_{V'_h} &\leq \|T_h p - \mathbf{f}_h\|_{V'_h} \leq \|T_h \pi_h\mu - \mathbf{f}_h\|_{V'_h} + \|T_h z_h\|_{V'_h} \|\pi_h\mu\|_M \\ &\leq \|T \pi_h\mu - \mathbf{f}_h\|_{V'_h} + \|(T_h - T) \pi_h\mu\|_{V'_h} + \alpha(T_h) \|\pi_h\mu\|_M \\ 575 \quad &\leq \|\mathbf{g}_h - \mathbf{f}_h\|_{V'_h} + \varepsilon_h^{\text{op}} \|\pi_h\mu\|_E + \alpha(T_h) \|\pi_h\mu\|_M \\ &\leq \|\mathbf{g}_h - \mathbf{f}_h\|_{V'_h} + (2r \varepsilon_h^{\text{op}} + \alpha(T_h)) \|\pi_h\mu\|_M. \end{aligned}$$

576 Now the second term is bounded as follows:

$$\begin{aligned}
\|\mathbf{g}_h - \mathbf{f}_h\|_{V'_h} &\leq \|\mathbf{g}_h - \mathbf{f}\|_{V'_h} + \|\mathbf{f} - \mathbf{f}_h\|_{V'_h} \leq \|T\pi_h\mu - T\mu\|_{V'_h} + \varepsilon_h^{\text{rhs}} \|\mathbf{f}\|_V, \\
577 \quad &\leq \rho(T)\varepsilon_h^{\text{int}}(\mu) \|\mu\|_M + \rho(T)\varepsilon_h^{\text{rhs}} \|\mu\|_M \leq \rho(T) \|\mu\|_M (\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}) \\
&\leq 2\rho(T) \|\pi_h\mu\|_M (\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}).
\end{aligned}$$

578 This last line is true because the hypothesis $\varepsilon_h^{\text{int}}(\mu) \leq 1/2$ implies that $\|\mu\|_M \leq$
579 $2 \|\pi_h\mu\|_M$. Putting things together, it come that

$$580 \quad \frac{\|T_h\mu_h - \mathbf{g}_h\|_{V'_h}}{\|\pi_h\mu\|_M} \leq 4\rho(T) (\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}) + 2r \varepsilon_h^{\text{op}} + \alpha(T_h)$$

581 and then

$$582 \quad \frac{\|\mu_{h,t} - \pi_h\mu\|_M}{\|\pi_h\mu\|_M} \leq \frac{2}{\beta(T_h)} [2\rho(T) (\varepsilon_h^{\text{int}}(\mu) + \varepsilon_h^{\text{rhs}}) + 2r \varepsilon_h^{\text{op}} + \alpha(T_h)].$$

583

□

584 **5. Numerical results.** In this section we provide numerical applications of The-
585 orems 4.1 and 4.7 and we present the general methodology to numerically approach
586 the solution of the equation (1.1) in various contexts. In the whole section, we stay
587 in the framework where $M := L^2(\Omega)$, $E := L^\infty(\Omega)$ and $V := H_0^1(\Omega, \mathbb{R}^d)$.

588 In subsection 5.2, we exhibit a simple and efficient pair of approximation spaces
589 (M_h, V_h) called the honeycomb discretization pair, that numerically satisfies the dis-
590 crete *inf-sup* condition.

591 For all the numerical experiments, we use the Matlab environment with some
592 elements of the PDE toolbox. We first determine the matrix \mathcal{M} and then, the de-
593 termination of the constants α and β and the determination of the solution of the
594 homogeneous problem is done using the singular values decomposition method (svds
595 in Matlab). The determination of the solution for the heterogeneous problem is sim-
596 ply done using the classical linear system solver (mldivide in Matlab). For the high
597 degree finite element spaces $(\mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^4)$, we use the getfem (see [19]) environnement
598 on Matlab to generate the matrix \mathcal{M} .

599 **5.1. Matrix formulation of the discretized problem.** In this section, we
600 describe the matrix formulation of the discrete problem (1.5) which gives a way to
601 use the stability theorems in practice. Let us fix a discretization size $h > 0$ and pick
602 a pair of finite dimensional subspaces $M_h \subset M$ and $V_h \subset V$. Let $(\varepsilon_1, \dots, \varepsilon_n)$ be a
603 basis of M_h and let $(\mathbf{e}_1, \dots, \mathbf{e}_p)$ be a basis of V_h . We define $\mathcal{T} \in \mathbb{R}^{p \times n}$ and $\mathbf{b} \in \mathbb{R}^p$ the
604 matrix versions of the discrete operator T_h and the right-hand side \mathbf{f}_h as the matrices

$$605 \quad (5.1) \quad \mathcal{T}_{ij} := \langle T_h \varepsilon_j, \mathbf{e}_i \rangle_{V'_h, V_h}, \quad \text{and} \quad \mathbf{b}_i := \langle \mathbf{f}_h, \mathbf{e}_i \rangle_{V'_h, V_h}.$$

606 As no ambiguity can occur, we adopt the notation for $\mu := \sum_j \mu_j \varepsilon_j \in M_h$
607 and $\boldsymbol{\mu} := (\mu_1, \dots, \mu_n)^T$ and the same notation for $\mathbf{v} := \sum_i v_i \mathbf{e}_i \in V_h$ and $\mathbf{v} =$
608 $(v_1, \dots, v_p)^T \in \mathbb{R}^p$. We have the correspondence

$$609 \quad \langle T_h \mu, \mathbf{v} \rangle_{V'_h, V_h} = \mathbf{v}^T \mathcal{T} \boldsymbol{\mu}.$$

610 We now call $(\mathcal{S}_M)_{ij} := \langle \varepsilon_i, \varepsilon_j \rangle_M$ and $(\mathcal{S}_V)_{ij} := \langle \mathbf{e}_i, \mathbf{e}_j \rangle_V$. They enable to compute
611 the norm in M and V through the formulas $\|\mu\|_M^2 = \sum_{i,j} \mu_i \mu_j \langle \varepsilon_i, \varepsilon_j \rangle_M = \boldsymbol{\mu}^T \mathcal{S}_M \boldsymbol{\mu}$,

612 and $\|\mathbf{v}\|_V^2 = \sum_{i,j} v_i v_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle_V = \mathbf{v}^T \mathcal{S}_V \mathbf{v}$. If we denote \mathcal{B}_M and \mathcal{B}_V the square root
613 matrices of \mathcal{S}_M and \mathcal{S}_V (i.e. such that $\mathcal{B}_M^2 = \mathcal{S}_M$), we have that $\|\boldsymbol{\mu}\|_M = \|\mathcal{B}_M \boldsymbol{\mu}\|_2$
614 and $\|\mathbf{v}\|_V = \|\mathcal{B}_V \mathbf{v}\|_2$. Hence the constant $\alpha(T_h)$ is given by

$$615 \quad (5.2) \quad \begin{aligned} \alpha(T_h) &= \inf_{\boldsymbol{\mu} \in \mathbb{R}^n} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T \mathcal{T} \boldsymbol{\mu}}{\|\mathcal{B}_M \boldsymbol{\mu}\|_2 \|\mathcal{B}_V \mathbf{v}\|_2} \\ &= \inf_{\boldsymbol{\mu} \in \mathbb{R}^n} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2 \|\mathbf{v}\|_2} = \inf_{\boldsymbol{\mu} \in \mathbb{R}^n} \frac{\|\mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}\|_2}{\|\boldsymbol{\mu}\|_2}. \end{aligned}$$

616 which is the smallest singular value of the matrix

$$617 \quad \mathcal{M} := \mathcal{B}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$$

618 or also the square root of the smallest eigenvalue of $\mathcal{M}^T \mathcal{M} = \mathcal{B}_M^{-1} \mathcal{T}^T \mathcal{S}_V^{-1} \mathcal{T} \mathcal{B}_M^{-1}$.

619 Call now $\mathbf{z} \in \mathbb{R}^n$ the first singular vector of \mathcal{M} (hence associated with $\alpha(T_h)$) or
620 the first eigenvector of $\mathcal{M}^T \mathcal{M}$. It is equal to the solution $z_h := \sum_j z_j \varepsilon_j \in M_h$ of (4.1)
621 up to a change of sign.

622 *Remark 5.1.* The basis matrices \mathcal{B}_M and \mathcal{B}_V are mandatory to get the exact
623 solution $\alpha(T_h)$ and z_h as defined in (4.1). As $\alpha(T_h)$ is expected to be small, it is
624 possible to consider directly the first singular vector of the matrix \mathcal{T} itself. The
625 numerical computation gets a bit simpler but creates an additional error which is not
626 controlled by the theory described herein.

627 We can now compute the discrete *inf-sup* constant of T_h :

$$628 \quad (5.3) \quad \begin{aligned} \beta(T_h) &= \inf_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathcal{S}_M \mathbf{z}}} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T \mathcal{T} \boldsymbol{\mu}}{\|\mathcal{B}_M \boldsymbol{\mu}\|_2 \|\mathcal{B}_V \mathbf{v}\|_2} \\ &= \inf_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathbf{z}}} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T (\mathcal{B}_V^{-1})^T \mathcal{T} \mathcal{B}_M^{-1} \boldsymbol{\mu}}{\|\boldsymbol{\mu}\|_2 \|\mathbf{v}\|_2} = \inf_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathbf{z}}} \frac{\|\mathcal{M} \boldsymbol{\mu}\|_2}{\|\boldsymbol{\mu}\|_2} \end{aligned}$$

629 which is the second smallest singular value of the matrix \mathcal{M} or also the square root
630 of the second smallest eigenvalue of $\mathcal{M}^T \mathcal{M}$. Finally, in order to give the solution of
631 (4.2) in Theorem 4.7, we rewrite the problem under a matrix formulation:

$$632 \quad \begin{aligned} \min_{\substack{m \in M_h \\ m \perp z_h}} \|T_h m - \mathbf{f}_h\|_{V'_h} &= \min_{\substack{m \in M_h \\ m \perp z_h}} \sup_{\mathbf{v} \in V_h} \frac{\langle T_h m - \mathbf{f}_h, \mathbf{v} \rangle_{V'_h, V_h}}{\|\mathbf{v}\|_V} = \min_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathbf{z}}} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T (\mathcal{T} \boldsymbol{\mu} - \mathbf{b})}{\|\mathcal{B}_V \mathbf{v}\|_2} \\ &= \min_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathbf{z}}} \sup_{\mathbf{v} \in \mathbb{R}^p} \frac{\mathbf{v}^T \mathcal{B}_V^{-1} (\mathcal{T} \boldsymbol{\mu} - \mathbf{b})}{\|\mathbf{v}\|_2} = \min_{\substack{\boldsymbol{\mu} \in \mathbb{R}^n \\ \boldsymbol{\mu} \perp \mathbf{z}}} \|\mathcal{B}_V^{-1} (\mathcal{T} \boldsymbol{\mu} - \mathbf{b})\|_2. \end{aligned}$$

633 Call now $\tilde{\mathcal{T}} := \begin{bmatrix} \mathcal{T} \\ \mathbf{z}^T \end{bmatrix}$, $\tilde{\mathbf{b}} := \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$ and $\tilde{\mathcal{B}}_V := \begin{bmatrix} \mathcal{B}_V & 0 \\ 0 & 1 \end{bmatrix}$ we aim at solving

$$634 \quad \tilde{\mathcal{B}}_V^{-1} \tilde{\mathcal{T}} \boldsymbol{\mu} = \tilde{\mathcal{B}}_V^{-1} \tilde{\mathbf{b}}$$

635 in the sense of least squares which is equivalent to define $\boldsymbol{\mu} := (\tilde{\mathcal{T}}^T \tilde{\mathcal{S}}_V^{-1} \tilde{\mathcal{T}})^{-1} \tilde{\mathcal{T}}^T \tilde{\mathcal{S}}_V^{-1} \tilde{\mathbf{b}}$.

636 **5.2. The honeycomb pair of finite element spaces.** After numerous tests
637 with various finite element pair of spaces, it appears that a specific pair of spaces
638 gather a large amount of advantages for the specific use in our inverse parameter

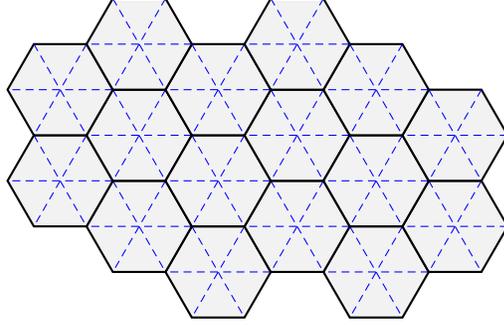


FIG. 1. Honeycomb space discretization. In plain black, the hexagonal subdivision and in dashed blue, the triangular subdivision.

639 problem. This pair (M_h, V_h) is the so called honeycomb discretization pair. Like in
 640 Figure 1, define a regular hexagonal subdivision of Ω denoted $\{\Omega_{h,j}^{\text{hex}}\}_{j=1,\dots,N_h^{\text{hex}}}$ where
 641 $h > 0$ is the edge length of the hexagons and N_h^{hex} is the number of hexagons used.
 642 We then call $\Omega_h \subset \Omega$ the subdomain defined by this subdivision. That means

$$643 \quad \overline{\Omega}_h = \bigcup_{j=1}^{N_h^{\text{hex}}} \overline{\Omega_{h,j}^{\text{hex}}}.$$

644 Now we consider the uniform triangular sub-mesh defined by subdividing each hexagon
 645 in six equilateral triangles of size h . This subdivision is denoted $\{\Omega_{h,k}^{\text{tri}}\}_{k=1,\dots,N_h^{\text{tri}}}$
 646 where $N_h^{\text{tri}} := 6N_h^{\text{hex}}$. It is represented in dashed blue in figure 1.

647 We now define the finite dimensional discretization space M_h of M as the collec-
 648 tion of functions $\mu \in L^2(\Omega_h)$ that are constant in each hexagon. In other terms,

$$649 \quad M_h := \mathbb{P}^0(\Omega_h^{\text{hex}}) = \left\{ \mu \in L^2(\Omega_h) \mid \forall j \mu|_{\Omega_{h,j}^{\text{hex}}} \text{ is constant} \right\}.$$

650 Functions in M_h can be extended by 0 out of Ω_h to get $M_h \subset M$. For the discretization
 651 space of V , we chose the classical finite element class \mathbb{P}_0^1 over the triangulation. It is
 652 made of all the functions of $H_0^1(\Omega_h)$ that are linear over all the triangles. In other
 653 terms,

$$654 \quad V_h := \mathbb{P}_0^1(\Omega_h^{\text{tri}}, \mathbb{R}^2) = \left\{ \mathbf{v} \in H_0^1(\Omega_h, \mathbb{R}^2) \mid \forall k \mathbf{v}|_{\Omega_{h,k}^{\text{tri}}} \text{ is linear} \right\}.$$

655 Functions in V_h can be extended by $\mathbf{0}$ out of Ω_h to get $V_h \subset V$.

656 *Remark 5.2.* This particular choice of finite element spaces gathers several ad-
 657 vantages to compare to other more classical pairs:

- 658 1. The space $\mathbb{P}^0(\Omega_h^{\text{hex}})$ is suitable for discontinuous functions interpolation. This
 659 is important as we aim at recovering discontinuous mechanical parameters of
 660 biological tissues for instance.
- 661 2. The hexagonal discretization of Ω is optimal among the other regular plane
 662 tilings (triangle and square) in the sense that it minimizes the ratio of the
 663 number of elements N_h^{hex} over the size h . As we see in all the numerical tests,
 664 it also provides the smallest error on the reconstruction among all the other
 665 pairs of spaces that we have tried. As a conjecture, we believe that this pair

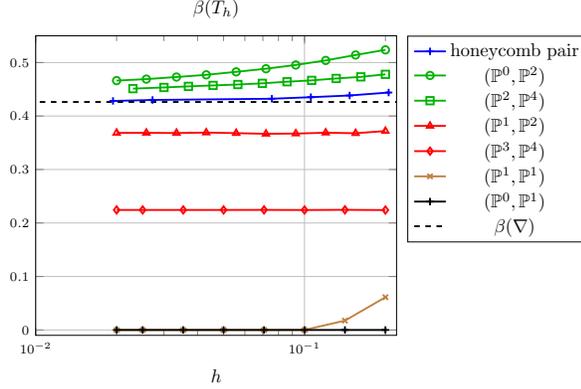


FIG. 2. Behavior of the discrete inf-sup constant $\beta(T_h)$ for the inverse gradient problem in the unit square $\Omega := (0, 1)^2$, for various choices of pair of discretization spaces. The dashed line represents the conjectured value of the inf-sup constant $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ of the gradient operator in Ω .

- 666 of spaces should be optimal, in term of error estimate, for a large class of
667 operators T .
- 668 3. From a given hexagonal mesh and triangular sub-mesh, spaces $\mathbb{P}^0(\Omega_h^{\text{hex}})$ and
669 $\mathbb{P}_0^1(\Omega_h^{\text{tri}}, \mathbb{R}^2)$ are easy to build from the most classical pair of finite element
670 spaces $(\mathbb{P}^0(\Omega_h^{\text{tri}}), \mathbb{P}^1(\Omega_h^{\text{tri}}))$.
 - 671 4. The system of equations $T_h \mu_h = \mathbf{f}_h$ is (most of the time) over-determinate
672 as it involves around $2N_h^{\text{hex}}$ equations for N_h^{hex} unknown. Note that as we
673 solve the problem in the sense of least squares, over-determination is not a
674 problem while under-determination is.
 - 675 5. This pair gives an excellent evaluation of the discrete *inf-sup* constant $\beta(T_h)$
676 in our numerical exemples that is a key element for discrete stability.

677 **5.3. Inverse gradient problem.** Let Ω be the unit square $(0, 1)^2$. We approach
678 here the solution $\mu \in L^\infty(\Omega)$ of the problem $-\nabla \mu = \mathbf{f}$ where \mathbf{f} is given vectorial
679 function. This case correspond to (1.1) where $S = I$ everywhere. In this case, many
680 simplification occur as $T_h := -\nabla|_{M_h}$ and then $\varepsilon_h^{\text{op}} = 0$. Moreover $\rho(T) \leq 1$. In the
681 absence of noise, the result of Theorem 4.7 reads : $\frac{\|\mu_h - \pi_h \mu\|_M}{\|\pi_h \mu\|_M} \leq \frac{4}{\beta(T_h)} (\varepsilon_h^{\text{rhs}} + \varepsilon_h^{\text{int}}(\mu))$
682 where μ_h is the solution of $\min_{\mu \in M_h} \|T_h \mu - \mathbf{f}\|_{V_h'}$ under the condition $\mu_h \in L_0^2(\Omega_h)$
683 i.e. $\int_{\Omega_h} \mu_h = 0$.

684 Let first compute $\beta(T_h)$ using (5.3) at check its behavior when h go to 0. In
685 figure 2 we see that it seem to converge to some $\beta_0 > 0$ lower than the conjectured
686 *inf-sup* constant $\beta(\nabla) = \sqrt{1/2 - 1/\pi}$ in the unit square (see [8, Theorem 3.3] for
687 details about this conjectured value).

688 Consider now a smooth map $\mu_1(x) := \cos(10x_1) + \cos(10x_2)$ for $x \in \Omega$, for such
689 a smooth function we expect an error of interpolation in M_h of order $\varepsilon_h^{\text{int}}(\mu_1) = \mathcal{O}(h)$
690 and an error of interpolation of its gradient on V_h' of order $\varepsilon_h^{\text{rhs}} = \mathcal{O}(h^2)$. Hence the
691 relative error $E_1(h) := \|\mu_{1,h} - \pi_h \mu_1\|_M / \|\pi_h \mu_1\|_M$ is expected to be at least of order
692 $\mathcal{O}(h)$. In figure 4 we observe a convergence of order 2 in absence of noise. We retry
693 the same test with piecewise constant μ_2 . Its derivative is approached first in $\mathbb{P}^0(\Omega_h^{\text{tri}})$
694 to deduce its vectorial form in V_h' . We observe a convergence of order 1/2 in absence
695 of noise.

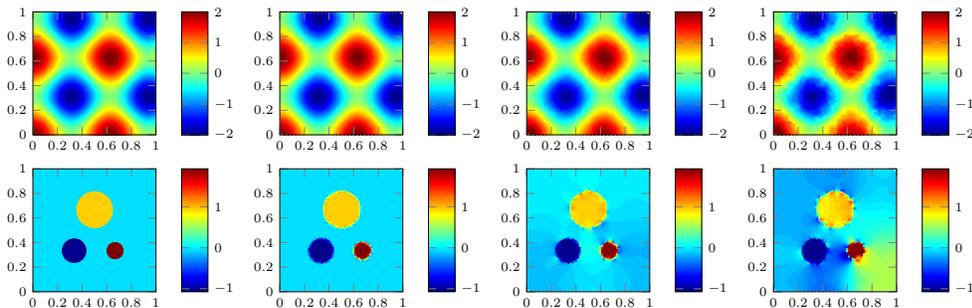


FIG. 3. Numerical stability of the reconstruction of maps μ_1 and μ_2 using method given by Theorem 4.7 with resolution $h = 0.01$. From left to right: column 1: exact map to recover, 2. reconstruction with no noise, column 3: reconstruction with noise level $\sigma = 1$, column 4: reconstruction with noise level $\sigma = 2$.

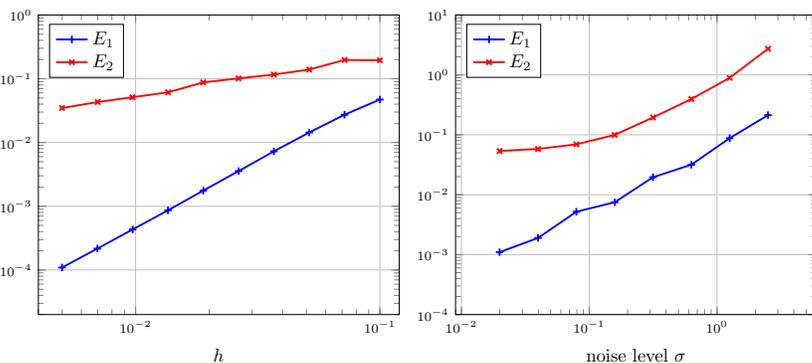


FIG. 4. Left : relative L^2 -error on the reconstruction with respect to h in the absence of noise. Right : relative L^2 -error on the reconstruction with respect to the noise level σ with $h = 0.01$.

696 To illustrate the stability with respect to noise on the right-hand side, we corrupt
 697 the data $-\nabla\mu$ with the multiplication term-by-term by $1 + \sigma\mathcal{N}$ where $\sigma > 0$ is the
 698 noise level and \mathcal{N} is a Gaussian random variable of variance one.

699 5.4. Quasi-static elastography.

700 *Forward problem.* To illustrate the ability of solving a quasi-static elastography
 701 problem in the case $\lambda = 0$ from a single measurement, we compute a virtual data field
 702 by solving the linear elastic forward problem

$$703 \quad (5.4) \quad \begin{cases} -\nabla \cdot (2\mu_{\text{exact}} \mathcal{E}(\mathbf{u})) = \mathbf{0} & \text{in } (0, 1)^2, \\ 2\mu_{\text{exact}} \mathcal{E}(\mathbf{u}) \cdot \boldsymbol{\nu} = \mathbf{g} & \text{on } (0, 1) \times \{1\}, \\ \mathcal{E}(\mathbf{u}) \cdot \boldsymbol{\nu} = \mathbf{0} & \text{on } (0, 1) \times \{0\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \{0, 1\} \times (0, 1). \end{cases}$$

704 where μ_{exact} is described in Figure 5. We chose here a constant boundary force
 705 $\mathbf{g} := (1, -1)^T$. This problem is solved using classical \mathbb{P}^1 finite element method over
 706 an unstructured triangular mesh. The computed data field \mathbf{u} is then stored in a
 707 cartesian grid to avoid any numerical inverse crime. It is represented in Figure 5.

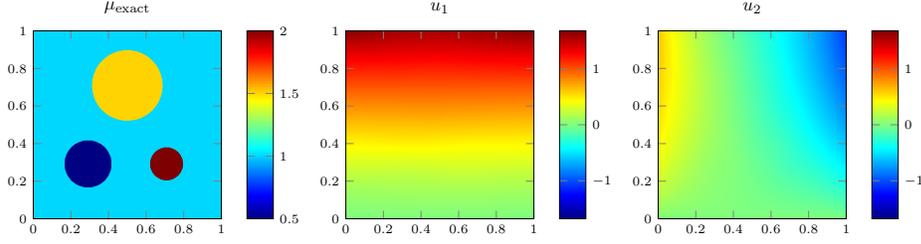


FIG. 5. First line, from left to right: The exact map μ_{exact} , the two components of the data field $\mathbf{u} = (u_1, u_2)$ computed via (5.4), the only data used to invert the problem.

708 Note that μ_{exact} is chosen bounded and piecewise constant, and thanks to the clas-
 709 sical elliptic regularity theory the exact strain tensor $S := \mathcal{E}(\mathbf{u})$ is piecewise smooth
 710 and bounded in Ω (see [17]). Hence, both μ_{exact} and S satisfy the hypotheses used in
 711 the error estimates.

712 *Inverse problem.* We first choose a pair of approximation spaces (M_h, V_h) and we
 713 interpolate the displacement field \mathbf{u} on the corresponding mesh nodes as a continuous
 714 and piecewise linear map. From this interpolated data, we compute its approached
 715 derivative $\nabla \mathbf{u}$ by computing the exact piecewise constant derivative of the interpo-
 716 lated displacement field. We deduce the approximation of the strain tensor $S := 2\mathcal{E}(\mathbf{u})$
 717 as a piecewise constant map. We then construct the matrix form of the approached
 718 operator T_h by formula (5.1). Before applying Theorem 4.1 we compute the discrete
 719 values of $\alpha(T_h)$ and $\beta(T_h)$ for few pairs of spaces (see Figure 6). We here control that
 720 $\beta(T_h)$ does not vanish and that the ratio $\alpha(T_h)/\beta(T_h)$ is small enough. We recall that
 721 this is needed for good error estimates using Theorem 4.1. Note that the honeycomb
 722 pair shows a much better behavior than the other consider pairs of spaces. In the
 723 results, we denote by “honeycomb pair” the pair of spaces defined in subsection 5.2
 724 and we denote \mathbb{P}^k the classical space of Lagrangian finite element space over an un-
 725 structured triangulation (see [12] for precise definitions).

726
 727 We plot now solutions μ_h of the numerical inversion with various choices of pair
 728 of spaces in Figure 7. Then in Figure 8 we present tables of comparisons of different
 729 pair of spaces in terms of relative error and complexity through the number of degrees
 730 of freedom and number of equations. In particular,

- 731 • As expected and for all choice of pair of spaces satisfying inf-sup condition,
 732 the numerical approximation \mathbf{u} gives some nice reconstruction of the elastic
 733 coefficient $2\mu_{\text{exact}}$. Moreover, in each case, we also clearly observe a conver-
 734 gence as $h \rightarrow 0$.
- 735 • The numerical solutions obtained with the honeycomb approach give some
 736 better reconstruction than using other pair of spaces. It can be explained by
 737 a better ratio $\alpha(T_h)/\beta(T_h)$.
- 738 • The use of high degree as with the pair of spaces $(\mathbb{P}^4, \mathbb{P}^2)$ raises some numerical
 739 memory issues in the computation the matrix \mathcal{B}_M^{-1} and \mathcal{S}_V^{-1} . In particular, we
 740 don’t succeed to reach time steps h smaller than $h = 0.025$ with a standard
 741 laptop.
- 742 • From a computation cost point of view, the honeycomb approach has also
 743 many advantages. The matrix \mathcal{S}_M and \mathcal{S}_V are respectively diagonal and tri-
 744 diagonal which greatly facilitate the computation of $\mathcal{B}_M^{-1} = \sqrt{\mathcal{S}_M^{-1}}$ and \mathcal{S}_V^{-1} .

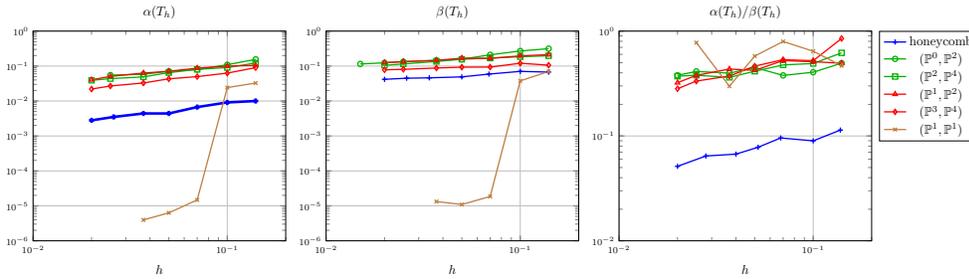


FIG. 6. Behavior of the constants $\alpha(T_h)$, $\beta(T_h)$ and the ratio $\alpha(T_h)/\beta(T_h)$ for the inverse static elastography problem in the unit square $\Omega := (0, 1)^2$, for various choices of pair of discretization spaces.

745 Finally, we can reach much finer resolutions than using other finite element
 746 space proposed in this paper.
 747 • For all the tested pairs, the matrix is over-determined and the measured al-
 748 gebraic rank is equal to n . However as the the first singular value is very small
 749 to compare to the others, the matrix rank should be considered "numerically
 750 speaking equal to $n - 1$ ".

751 **6. Concluding remarks.** In this article we have proved the numerical stability
 752 of the Galerkin approximation of the inverse parameter problem arising from the elas-
 753 tography in medical imaging. It has been done through a direct discretization of the
 754 Reverse Weak Formulation without boundary conditions. The obtained stability esti-
 755 mates arise from a generalization of the *inf-sup* constant (continuous and discrete) to
 756 a large class of first order differential operator. These results shed light on the impor-
 757 tance of the choice of finite element spaces to assure uniqueness and stability. Various
 758 numerical applications have been presented which illustrate the stability theorems. A
 759 new pair of finite element spaces based on a hexagonal tiling has been introduced. It
 760 showed excellent stability behavior for the specific purpose of this inverse problem.

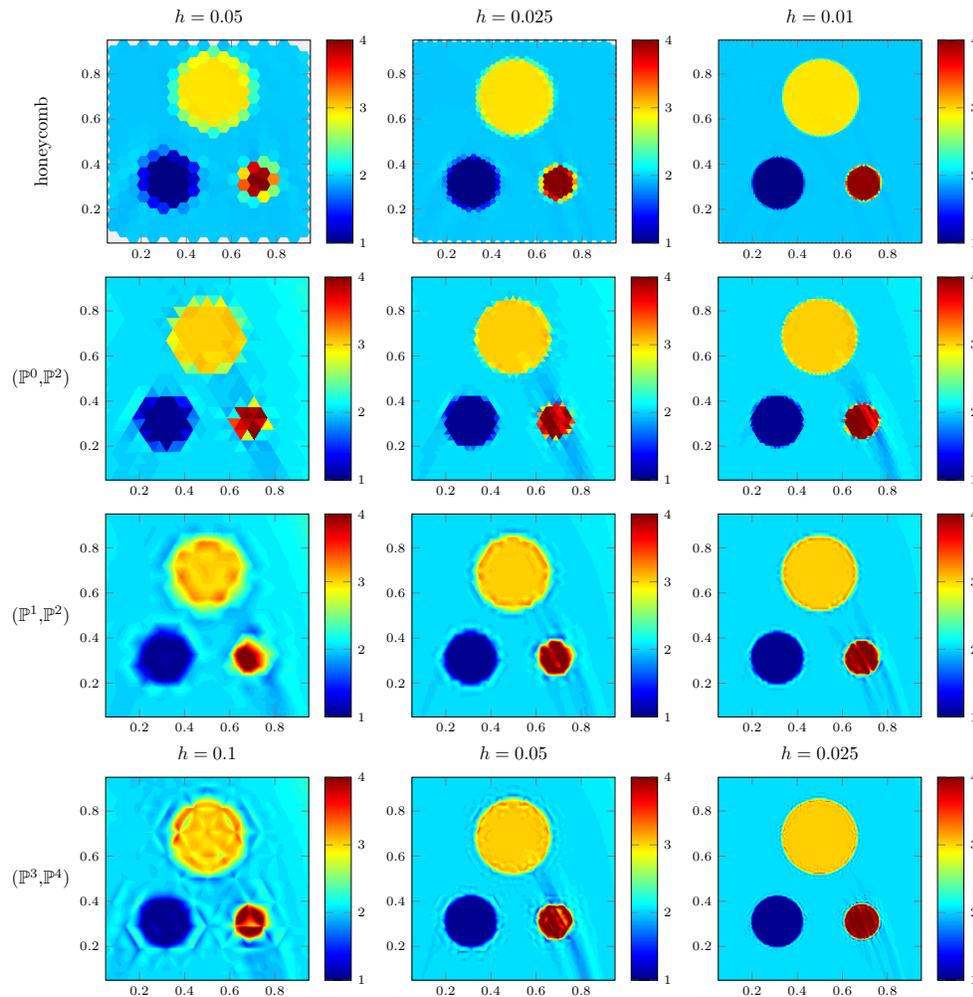


FIG. 7. Reconstruction of the shear modulus map μ using various pairs of finite element spaces in the subdomain of interest $(0.1, 0.9)^2$.

$h = 0.05$	E	n	p	$h = 0.025$	E	n	p
honeycomb	9.2%	338	1888	honeycomb	6.3%	1510	8765
$(\mathbb{P}^0, \mathbb{P}^2)$	9.1%	735	2788	$(\mathbb{P}^0, \mathbb{P}^2)$	6.7%	2982	12k
$(\mathbb{P}^1, \mathbb{P}^2)$	8.5%	407	2800	$(\mathbb{P}^1, \mathbb{P}^2)$	5.7%	1570	12k
$(\mathbb{P}^3, \mathbb{P}^4)$	5.4%	3424	11k	$(\mathbb{P}^3, \mathbb{P}^4)$	3.4%	13654	47k

FIG. 8. Comparison of four pairs of finite element spaces in term of relative error E of the reconstruction, degrees of freedom n , and number of equations p . The product np is an indication of the algorithmic complexity.

761 **Appendix A. A result on self-adjoint operators.**

762 LEMMA A.1. Let H be an Hilbert space and $S : H \rightarrow H$ be a self-adjoint positive
 763 semi-definite linear operator. Call $\alpha^2 := \inf\{\langle Sx, x \rangle_H \mid \|x\|_H = 1\}$ and $z \in H$ such
 764 that $\|z\|_H = 1$ and take $\langle Sz, z \rangle_H \leq \alpha^2 + \varepsilon^2$ with $\varepsilon > 0$. For any $p \perp z$ with $\|p\|_H = 1$

765 we have

$$766 \quad |\langle Sz, p \rangle_H| \leq \varepsilon \sqrt{\rho^2 - \alpha^2}$$

767 x where $\rho^2 := \sup\{\langle Sx, x \rangle_H \mid \|x\|_H = 1\}$.

768 *Proof.* Consider $t \in (0, 1)$, $u_t := -\text{sign} \langle Sz, p \rangle_H \sqrt{1 - t^2}$ and $z_t := tz + u_t p$ of
769 norm one. By definition of α we have

$$770 \quad \begin{aligned} \alpha^2 &\leq \langle Sz_t, z_t \rangle_H = t^2 \langle Sz, z \rangle_H + 2t u_t \langle Sz, p \rangle_H + u_t^2 \langle Sp, p \rangle_H \\ &\leq t^2(\alpha^2 + \varepsilon^2) + 2t u_t \langle Sz, p \rangle_H + u_t^2 \rho^2. \end{aligned}$$

771 Then

$$772 \quad \begin{aligned} -2t u_t \langle Sz, p \rangle_H &\leq (t^2 - 1)\alpha^2 + t^2 \varepsilon^2 + u_t^2 \rho^2 \\ 2t |u_t| |\langle Sz, p \rangle_H| &\leq t^2 \varepsilon^2 + u_t^2 (\rho^2 - \alpha^2) \\ 2 |\langle Sz, p \rangle_H| &\leq \frac{t}{|u_t|} \varepsilon^2 + \frac{|u_t|}{t} (\rho^2 - \alpha^2). \end{aligned}$$

773 This statement is true for any $t \in (0, 1)$ so for any $\tau \in (0, 1)$ we have

$$774 \quad 2 |\langle Sz, p \rangle_H| \leq \tau \varepsilon^2 + \frac{1}{\tau} (\rho^2 - \alpha^2).$$

775 The minimum of the right-hand side is reached for $\tau = \sqrt{(\rho^2 - \alpha^2)/\varepsilon^2}$ which implies
776 that $2 |\langle Sz, p \rangle_H| \leq 2\sqrt{\varepsilon^2(\rho^2 - \alpha^2)}$. \square

777 **Appendix B. Limit of subsets and infimum.** Let M be a Hilbert space and
778 let $E \subset M$ be Banach space dense in M . Let $(M_h)_{h>0}$ be a sequence of subspace of E
779 endowed with the M -norm. We assume that the orthogonal projection $\pi_h : M \rightarrow M_h$
780 satisfies

$$781 \quad \forall x \in E, \quad \|\pi_h x\|_E \leq \|x\|_E.$$

782 **DEFINITION B.1.** For any sequence $(A_h)_{h>0}$ of subsets of M , we define its limit
783 as

$$784 \quad \lim_{h \rightarrow 0} A_h := \left\{ x \in M \mid \exists (x_h)_{h>0} \subset M, \lim_{h \rightarrow 0} \|x_h - x\|_M = 0, \forall h > 0 \ x_h \in A_h \right\}.$$

785 **PROPOSITION B.2.** $\lim_{h \rightarrow 0} A_h$ is a closed subset of M and, if $A_h \subset X \subset M$ for
786 all $h > 0$, then $\lim_{h \rightarrow 0} A_h \subset \overline{X}$.

787 *Proof.* Call $A := \lim_{h \rightarrow 0} A_h$ and take $x \in \overline{A}$. There exists a sequence $(x_n)_{n \in \mathbb{N}}$
788 of A such that $\|x - x_n\|_M \leq 1/(2n)$ for all $n \in \mathbb{N}^*$. For all $n \in \mathbb{N}^*$, there exists
789 a sequence $(x_n^h)_{h>0}$ such that $\lim_{h \rightarrow 0} \|x_n^h - x_n\|_M = 0$ and $x_n^h \in A_h$ for all $h > 0$.
790 Hence there exists $h_n > 0$ such that for all $h \leq h_n$ we have $\|x_n^h - x_n\|_M \leq 1/(2n)$.
791 We can decrease h_n to satisfy $h_n < h_{n-1}$ for all $n \geq 2$. Now define the sequence
792 $(y_h)_{h>0}$ as follows: If $h > h_1$, y_h is any element of A_h . If $h \in [h_{n+1}, h_n)$, we take
793 $y_h = x_n^h$. It is clear that $y_h \in A_h$ for all $h > 0$. Moreover, for any $h \leq h_n$,
794 $\|y_h - x_n\|_M \leq 1/(2n)$ and $\|x - x_n\|_M \leq 1/(2n)$ which give $\|y_h - x\|_M \leq 1/n$. This
795 shows that $\lim_{h \rightarrow 0} \|y_h - x\|_M = 0$ and therefore $x \in A$. The second part of the
796 statement is trivial.

797 PROPOSITION B.3. Assume that $A := \lim_{h \rightarrow 0} A_h$ is not empty and consider a
 798 fonction $f : M \rightarrow \mathbb{R}$. If there exists a subset $B \subset A$ such that f is continuous in B
 799 and $\inf_A f = \inf_B f$ then we have

$$800 \quad \limsup_{h \rightarrow 0} \inf_{A_h} f \leq \inf_A f.$$

801 *Proof.* Take $x \in B$. As $x \in A$, there exists $(x_h)_{h>0}$ such that $x_h \in A_h$ for all
 802 $h > 0$ and $\lim_{h \rightarrow 0} x_h = x$. For any $h > 0$, $f(x_h) \leq f(x) + |f(x_h) - f(x)|$ and
 803 $\inf_{A_h} f \leq f(x) + |f(x_h) - f(x)|$. Taking the superior limit when $h \rightarrow 0$ it comes from
 804 the continuity of f at x , $\limsup_{h \rightarrow 0} \inf_{A_h} f \leq f(x)$ which if true for any $x \in B$ so
 805 $\limsup_{h \rightarrow 0} \inf_{A_h} f \leq \inf_B f = \inf_A f$. \square

806 We assume now that the sequence (M_h) satisfies $\lim_{h \rightarrow 0} M_h = M$. We consider a
 807 sequence of positive real number α_h that converges zero and a corresponding sequence
 808 of subsets $C_h := \{x \in M_h \mid \alpha_h \|x\|_E \leq \|x\|_M\}$.

809 PROPOSITION B.4. The following limit holds: $\lim_{h \rightarrow 0} C_h = M$.

810 *Proof.* We prove that $E \subset C := \lim_{h \rightarrow 0} C_h$. Take $x \in E \setminus \{0\}$, for h small enough
 811 it satisfies $2\alpha_h \|x\|_E \leq \|x\|_M$. Consider now its orthogonal projection $\pi_h x$ of x onto
 812 M_h . It satisfies $\lim_{h \rightarrow 0} \pi_h x = x$. For h small enough $\|x\|_M \leq 2\|\pi_h x\|_M$ and then

$$813 \quad \alpha_h \|\pi_h x\|_E \leq \alpha_h \|x\|_E \leq \frac{1}{2} \|x\|_M \leq \|\pi_h x\|_M$$

814 which means that $\pi_h x \in C_h$. As a consequence, $x \in \lim_{h \rightarrow 0} C_h$. \square

815 PROPOSITION B.5. Let $(z_h)_{h>0}$ be sequence of M such that $\|z_h\|_M = 1$ and which
 816 converges weakly to $z \neq 0$. Then

$$817 \quad \lim_{h \rightarrow 0} (C_h \cap \{z_h\}^\perp) = M \cap \{z\}^\perp.$$

818 *Proof.* Take $x \in \lim_{h \rightarrow 0} (C_h \cap \{z_h\}^\perp)$. There exists (x_h) such that $x_h \in C_h$ and
 819 $x_h \perp z_h$ and $x_h \rightarrow x$. We have $\langle x, z \rangle_M = \lim_{h \rightarrow 0} \langle x, z_h \rangle_M = \lim_{h \rightarrow 0} \langle x - x_h, z_h \rangle_M =$
 820 0 . Then $x \in M \cap \{z\}^\perp$.

821 Reversely, take $x \in M \cap \{z\}^\perp$, and fix $\varepsilon > 0$. There exists $x_\varepsilon \in E \setminus \{0\}$ such that
 822 $\|x_\varepsilon - x\|_M \leq \varepsilon$ and $x_\varepsilon \perp z$ and Consider now the orthogonal projection $\pi_h x_\varepsilon$ of x_ε
 823 onto M_h . It satisfies $\lim_{h \rightarrow 0} \pi_h x_\varepsilon = x_\varepsilon$. For h small enough $\|x_\varepsilon\|_M \leq 2\|\pi_h x_\varepsilon\|_M$.
 824 Consider now $\tilde{z} \in E$ such that $\langle z, \tilde{z} \rangle_M \geq 1/2$ and $\|\tilde{z}\|_M = 1$. We define now

$$825 \quad x_\varepsilon^h = \pi_h x_\varepsilon + \beta_h \pi_h \tilde{z} \in M_h,$$

826 with $\beta_h = -\langle \pi_h x_\varepsilon, z_h \rangle_M / \langle \pi_h \tilde{z}, z_h \rangle_M$ in order to have $x_\varepsilon^h \perp z_h$ for all h . Re-
 827 mark that β_h is wel defined for h small enough as $\langle \pi_h \tilde{z}, z_h \rangle_M$ converges to $\langle z, \tilde{z} \rangle_M$
 828 and converges to zero as $\langle \pi_h x_\varepsilon, z_h \rangle_M = \langle x_\varepsilon, z_h \rangle_M + \langle \pi_h x_\varepsilon - x_\varepsilon, z_h \rangle_M$ converges to
 829 $\langle x_\varepsilon, z \rangle_M = 0$. Then $x_\varepsilon^h \rightarrow x_\varepsilon$. Now we write

$$830 \quad \|x_\varepsilon^h\|_E \leq \|\pi_h x_\varepsilon\|_E + \beta_h \|\pi_h \tilde{z}\|_E \leq \|x_\varepsilon\|_E + \beta_h \|\tilde{z}\|_E,$$

831 and $\|x_\varepsilon^h\|_M \rightarrow \|x_\varepsilon\|_M \neq 0$. As a consequence, for h small enough, $\alpha_h \|x_\varepsilon^h\|_E \leq$
 832 $\|x_\varepsilon^h\|_M$ which means that $x_\varepsilon^h \in C_h \cap \{z_h\}^\perp$ for h small enough. This shows that
 833 $x_\varepsilon \in \lim_{h \rightarrow 0} (C_h \cap \{z_h\}^\perp)$. This is true for any $\varepsilon > 0$ and as the limit set is closed,
 834 $x \in \lim_{h \rightarrow 0} (C_h \cap \{z_h\}^\perp)$. \square

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