

EFFECTIVE MEDIUM THEORY FOR EMBEDDED OBSTACLES IN ELASTICITY WITH APPLICATIONS TO INVERSE PROBLEMS

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ABSTRACT. We consider the time-harmonic elastic wave scattering from a general (possibly anisotropic) inhomogeneous medium with an embedded impenetrable obstacle. We show that the impenetrable obstacle can be effectively approximated by an isotropic elastic medium with a particular choice of material parameters. We derive sharp estimates to rigorously verify such an effective approximation. Our study is strongly motivated by the related studies of two challenging inverse elastic problems including the inverse boundary problem with partial data and the inverse scattering problem of recovering mediums with buried obstacles. The proposed effective medium theory readily yields some interesting applications of practical significance to these inverse problems.

Keywords: elastic scattering, embedded obstacle, effective medium theory, asymptotic analysis, variational analysis, inverse elastic problem, partial data

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1. INTRODUCTION

1.1. Motivations and background. Our study is strongly motivated by the related studies of two challenging inverse elastic problems, which we shall discuss in what follows. To that end, we first introduce the Lamé system that governs the elastic wave propagation in \mathbb{R}^n , $n = 2, 3$. Throughout, we let \mathcal{C} and ρ signify the constitutive material parameters of an elastic medium. Here, $\mathcal{C}(\mathbf{x}) = (\mathcal{C}_{ijkl}(\mathbf{x}))_{i,j,k,l=1}^n$ is a four-rank real-valued tensor satisfying the following symmetry property:

$$\mathcal{C}_{ijkl} = \mathcal{C}_{klij} \quad \text{and} \quad \mathcal{C}_{ijkl} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk}, \quad i, j, k, l = 1, 2, \dots, n. \quad (1.1)$$

$\rho(\mathbf{x})$ is a bounded measurable complex-valued function with $\Re\rho > 0$ and $\Im\rho \geq 0$. Physically, \mathcal{C} signifies the stiffness tensor, and $\Re\rho$ and $\Im\rho$ characterize the density and damping of an elastic medium, respectively. Let $\mathbf{u}(\mathbf{x}) = (u_j(\mathbf{x}))_{j=1}^n \in \mathbb{C}^n$ denote the displacement field in the elastic medium. In linear elasticity, one has the following Lamé system:

$$\mathcal{L}_{\mathcal{C}}\mathbf{u} + \omega^2\rho\mathbf{u} = \mathbf{0}, \quad \mathcal{L}_{\mathcal{C}}\mathbf{u} := \nabla \cdot (\mathcal{C} : \nabla\mathbf{u}) = \left(\sum_{j,k,l=1}^n \partial_j(\mathcal{C}_{ijkl}\partial_l u_k) \right)_{i=1}^n, \quad (1.2)$$

where $\omega \in \mathbb{R}_+$ signifies the angular frequency and $\mathcal{L}_{\mathcal{C}}$ is referred to as the Lamé operator associated with \mathcal{C} . In (1.2), the symbol “ $:$ ” indicates an action of double contraction, which is defined for two matrices $\mathbf{A} = (a_{ij})_{i,j=1}^n$ and $\mathbf{B} = (b_{ij})_{i,j=1}^n$:

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^n a_{ij}b_{ij} \quad \text{and} \quad \mathcal{C} : \mathbf{A} = (\mathcal{C} : \mathbf{A})_{ij} = \left(\sum_{k,l=1}^n \mathcal{C}_{ijkl}a_{kl} \right).$$

Throughout we assume that the elastic tensor \mathcal{C} satisfies the uniform Legendre ellipticity condition:

$$c_{\min}\|\boldsymbol{\xi}\|_2^2 \leq \boldsymbol{\xi} : \mathcal{C} : \boldsymbol{\xi}^* \leq c_{\max}\|\boldsymbol{\xi}\|_2^2, \quad \forall \boldsymbol{\xi} \in \mathbb{C}^{n \times n} \text{ being a symmetric matrix,} \quad (1.3)$$

where c_{\min} and c_{\max} are two positive constants. If there exist scalar real functions $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ such that

$$\mathcal{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (1.4)$$

where δ is the Kronecker delta function, then the elastic medium is said to be isotropic, otherwise it is said anisotropic.

Let $\Sigma \Subset \mathbb{R}^n$ be a bounded Lipschitz domain. Consider the following boundary value problem associated with the Lamé system:

$$\mathbf{u} \in H^1(\Sigma)^n, \quad \mathcal{L}_{\mathcal{C}}\mathbf{u} + \omega^2\rho\mathbf{u} = \mathbf{0} \text{ in } \Sigma, \quad \mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u}) = \boldsymbol{\psi} \in H^{-1/2}(\partial\Sigma)^n \text{ on } \partial\Sigma, \quad (1.5)$$

where $\mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u}) := \boldsymbol{\nu} \cdot (\mathcal{C} : \nabla\mathbf{u})$ with $\boldsymbol{\nu} \in \mathbb{S}^{n-1}$ signifying the exterior unit normal vector to $\partial\Sigma$. It is known that there exists a unique solution to (1.5), provided that ω does not belong to a discrete set (known as the eigenvalues) [26]. Assuming that ω is not an eigenvalue, the following boundary Neumann-to-Dirichlet (NtD) map is well-defined:

$$\Lambda_{\Sigma; \mathcal{C}, \rho} : H^{-1/2}(\partial\Sigma)^n \mapsto H^{1/2}(\partial\Sigma)^n, \quad \Lambda_{\Sigma; \mathcal{C}, \rho}(\boldsymbol{\psi}) = \mathbf{u}|_{\partial\Sigma}, \quad (1.6)$$

where \mathbf{u} is the solution to (1.5). The NtD map $\Lambda_{\mathcal{C}, \rho}$ encodes all the possible Cauchy data $(\mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u})|_{\partial\Sigma}, \mathbf{u}|_{\partial\Sigma})$ associated with the Lamé system (1.5). An inverse problem of industrial importance arising in the elastic probing is to recover the elastic body $(\Sigma; \mathcal{C}, \rho)$ by the boundary observations, namely:

$$\Lambda_{\Sigma; \mathcal{C}, \rho} \rightarrow (\Sigma; \mathcal{C}, \rho). \quad (1.7)$$

In practice, it means that one exerts the traction force on the boundary of the elastic body (i.e. $\mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u})|_{\partial\Sigma} = \boldsymbol{\psi}$) to induce the elastic field \mathbf{u} inside the body, and then measures the response on the boundary (i.e. \mathbf{u}), and in such a non-destructive way to infer knowledge of the interior of the elastic body. The inverse problem (1.7) is nonlinear and ill-conditioned and has been extensively and intensively investigated in the literature, see e.g. [5, 13–15, 18, 19, 27, 28] and the references cited therein. In many practical scenarios, one cannot achieve the measurements of the elastic field on the full boundary $\partial\Omega$, and instead, one can only measure on part of the boundary, say $(\mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u})|_{\Gamma}, \mathbf{u}|_{\Gamma})$, where $\Gamma \Subset \partial\Omega$. This is particular the case that Σ is not a solid body and possesses a hole, say $\Sigma = \Omega \setminus \overline{D}$, where $D \Subset \Omega$, and Ω and D are both solid bodies.¹ In such a case, $\partial\Sigma = \partial\Sigma_{\text{interior}} \cup \partial\Sigma_{\text{exterior}}$, where the interior boundary $\partial\Sigma_{\text{interior}} = \partial D$ and the exterior boundary $\partial\Sigma_{\text{exterior}} = \partial\Omega$. From a practical point of view, the interior boundary is inaccessible in the elastic probing, and hence in the inverse problem (1.7), one can only exert the input and measure the output on the exterior boundary, namely $\Gamma = \partial\Omega$. That is, one needs to require in (1.5) that $\text{supp}(\boldsymbol{\psi}) \subset \partial\Sigma_{\text{exterior}} = \partial\Omega$, which leads to the following system:

$$\mathcal{L}_{\mathcal{C}}\mathbf{u} + \omega^2\rho\mathbf{u} = \mathbf{0} \text{ in } \Omega \setminus \overline{D}, \quad \mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u}) = \mathbf{0} \text{ on } \partial D, \quad \mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u}) = \boldsymbol{\psi} \in H^{-1/2}(\partial\Omega)^n \text{ on } \partial\Omega. \quad (1.8)$$

Then the inverse problem (1.7) becomes:

$$(\mathcal{T}_{\boldsymbol{\nu}}(\mathbf{u})|_{\partial\Omega}, \mathbf{u}|_{\partial\Omega}) \rightarrow (\Omega \setminus \overline{D}; \mathcal{C}, \rho), \quad (1.9)$$

where $\mathbf{u} \in H^1(\Omega \setminus \overline{D})^n$ is the solution to (1.8). The partial-data inverse problem constitutes a class of highly challenging open problems in the literature, and it even remains

¹One can think that in two dimensions, both Ω and D are simply connected.

largely open for the case associated with the differential equation $\nabla \cdot (\sigma \nabla u) = 0$ where σ is a scalar function [21, 24] (the so-called Calderón's inverse conductivity problem), a fortiori the one associated with the Lamé system (1.8). We are aware that the partial-data inverse elastic problem was recently studied in [13] following the spirit of the related studies of the partial-data Calderón problem within a certain restricted and special setup.

In this paper, we propose a different perspective to tackle the partial-data inverse elastic problem that can work in an extremely general scenario. To that end, we note that physically, D represents a traction-free impenetrable obstacle embedded in the elastic medium $(\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. In what follows, we set $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ to signify such an elastic object as described above. Let $\Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^p : H^{-1/2}(\partial\Omega)^n \mapsto H^{1/2}(\partial\Omega)^n$ denote the partial NtD map associated with the Lamé system (1.8). That is,

$$\Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^p(\boldsymbol{\psi}) = \mathbf{u}|_{\partial\Omega}, \quad (1.10)$$

where $\mathbf{u} \in H^1(\Omega \setminus \overline{D})^n$ is the solution to (1.8). For comparison, we also write $\Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^f = \Lambda_{\Sigma; \mathcal{C}, \rho}$, where $\Lambda_{\Sigma; \mathcal{C}, \rho}$ is defined in (1.6), to signify that it encodes the full boundary measurements.

Definition 1.1. Consider $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ as described above. If there exist an elastic medium $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ with $(\tilde{\mathcal{C}}, \tilde{\rho})|_{\Omega \setminus \overline{D}} = (\mathcal{C}, \rho)|_{\Omega \setminus \overline{D}}$, and $\varepsilon \in \mathbb{R}_+$ with $\varepsilon \ll 1$ such that

$$\left\| \Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^p - \Lambda_{\Omega; \tilde{\mathcal{C}}, \tilde{\rho}}^f \right\|_{\mathcal{L}(H^{-1/2}(\partial\Omega)^n, H^{1/2}(\partial\Omega)^n)} \leq C\varepsilon, \quad (1.11)$$

where C is a generic positive constant depending on Ω, D and $\mathcal{C}, \rho, \omega$, then $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is said to be an effective ε -realization of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. If $\varepsilon \equiv 0$, then $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is said to be an effective realization of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$.

It is conjectured that the unique identifiability holds generically for the aforementioned partial-data inverse problem, namely the correspondence between $\Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^p$ and $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ is one-to-one. It means that the (perfect) effective realization of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ should not exist in generic scenarios. However, we shall show in this paper that there always exist effective ε -realizations of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ for any given $\varepsilon \ll 1$. If so, the partial-data inverse problem of recovering $(\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ by knowledge of $\Lambda_{\Omega \setminus \overline{D}; \mathcal{C}, \rho}^p$ can be (at least approximately) reduced to the full-data inverse problem of recovering $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ by knowledge of $\Lambda_{\Omega; \tilde{\mathcal{C}}, \tilde{\rho}}^f$, whereby one has rich results for the unique identifiability and reconstruction methods; see the references cited earlier as well as the references therein. We shall present more discussions in what follows on the interesting implications of our study to the inverse elastic problem.

So far, we have mainly considered the inverse boundary problem of making use of the traction field $\mathcal{T}_\nu(\mathbf{u})$ as the boundary input and the displacement field \mathbf{u} on the boundary as the measured output. An alternative way is to make use of the displacement field as the boundary input and the boundary traction field as the output. By following a similar discussion, one can show that the homogeneous condition $\mathcal{T}_\nu(\mathbf{u})|_{\partial D} = \mathbf{0}$ should be replaced by $\mathbf{u}|_D = \mathbf{0}$. In such a case, D is referred to as a rigid impenetrable obstacle in the literature. Clearly, Definition 1.1 also applies for the (perfect or approximate) effective realization of an embedded rigid obstacle.

Another inverse problem of close interest is the simultaneous recovery of buried obstacles and surrounding mediums in the elastic scattering theory. Let $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ be

described as earlier, where D can either a traction-free or a rigid obstacle. Let λ_e, μ_e and ρ_e be real constants satisfying the strong convexity condition (induced by the ellipticity condition (1.3)):

$$\mu_e > 0, \quad n\lambda_e + 2\mu_e > 0 \quad \text{and} \quad \rho_e > 0. \quad (1.12)$$

Let \mathcal{C}^e be an isotropic elastic tensor as defined in (1.4) with $\lambda = \lambda_e$ and $\mu = \mu_e$. Let (\mathcal{C}, ρ) be extended into $\mathbb{R}^n \setminus \overline{\Omega}$ such that $(\mathcal{C}, \rho) = (\mathcal{C}^e, \rho_e)$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Let \mathbf{u}^{in} be an entire solution to the following Lamé system:

$$\mu_e \Delta \mathbf{u}^{in} + (\lambda_e + \mu_e) \nabla (\nabla \cdot \mathbf{u}^{in}) + \omega^2 \rho_e \mathbf{u}^{in} = \mathbf{0}. \quad (1.13)$$

Consider the following elastic scattering system:

$$\begin{cases} \mathcal{L}_{\mathcal{C}} \mathbf{u} + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ \mathbf{u} = \mathbf{u}^{in} + \mathbf{u}^s & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathcal{B}(\mathbf{u}) = \mathbf{0} & \text{on } \partial D, \\ \mathbf{u}|_{\partial \Omega} = \mathbf{u}^s|_{\partial \Omega} + \mathbf{u}^{in}, \quad \mathcal{T}_{\nu}(\mathbf{u}) = \mathcal{T}_{\nu}(\mathbf{u}^s) + \mathcal{T}_{\nu}(\mathbf{u}^{in}) & \text{on } \partial \Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla (\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(n-1)/2} \left(\frac{\partial \mathbf{u}^{t,s}}{\partial |\mathbf{x}|} - i\kappa_t \mathbf{u}^{t,s} \right) = \mathbf{0}, & t = p, s, \end{cases} \quad (1.14)$$

where $\mathbf{f}(\mathbf{x})$ indicates a source and is compactly supported outside Ω , namely $\text{supp}(\mathbf{f}) \subset B_{r_0} \setminus \overline{\Omega}$ for some ball B_{r_0} with center at the origin and a radius of r_0 . $\iota := \sqrt{-1}$, $\kappa_s := \omega \sqrt{1/\mu_e}$ and $\kappa_p := \omega \sqrt{1/(\lambda_e + 2\mu_e)}$, and $\mathcal{B}(\mathbf{u}) = \mathbf{u}$ or $\mathcal{B}(\mathbf{u}) = \mathcal{T}_{\nu}(\mathbf{u})$ correspond, respectively, to the cases that D is rigid or traction-free. The system (1.14) describes the time-harmonic scattering due to an incident field \mathbf{u}^{in} and the scatter $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. \mathbf{u}^s is referred to as the scattered field, which characterizes the perturbation of the propagation of the incident field due to the presence of the inhomogeneous scatterer. $\mathbf{u}^{p,s}$ and $\mathbf{u}^{s,s}$ are the compressional and shear parts of \mathbf{u}^s , respectively. The last limit in (1.14) is known as the Kupradze radiation condition, which holds uniformly in the angular variable $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{n-1}$. We couldn't find a convenient reference for the well-posedness of the scattering problem (1.14) in such a general scenario and shall provide a proof in Subsection 2.4. The solution $\mathbf{u}^s \in H_{loc}^1(\mathbb{R}^n \setminus \overline{D})^n$ admits the following asymptotic expansion (cf. [15]):

$$\mathbf{u}^s(\mathbf{x}) = \frac{\exp(i\kappa_p |\mathbf{x}|)}{|\mathbf{x}|^{(n-1)/2}} \mathbf{u}^{p,\infty}(\hat{\mathbf{x}}) + \frac{\exp(i\kappa_s |\mathbf{x}|)}{|\mathbf{x}|^{(n-1)/2}} \mathbf{u}^{s,\infty}(\hat{\mathbf{x}}) + \mathcal{O}(|\mathbf{x}|^{-\frac{n+1}{2}}) \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad (1.15)$$

uniformly in all directions $\hat{\mathbf{x}} \in \mathbb{S}^{n-1}$. $\mathbf{u}^{p,\infty}$ and $\mathbf{u}^{s,\infty}$ are defined on the unit sphere \mathbb{S}^{n-1} , and are known as the longitudinal and transversal far field patterns corresponding to $\mathbf{u}^{p,s}$ and $\mathbf{u}^{s,s}$, respectively. The far-field pattern \mathbf{u}^{∞} of the scattered field \mathbf{u}^s is defined as the sum of $\mathbf{u}^{p,\infty}$ and $\mathbf{u}^{s,\infty}$, i.e.,

$$\mathbf{u}^{\infty} := \mathbf{u}^{p,\infty} + \mathbf{u}^{s,\infty}.$$

It is known that $\mathbf{u}^{p,\infty}$ is normal to \mathbb{S}^{n-1} and $\mathbf{u}^{s,\infty}$ is tangential to \mathbb{S}^{n-1} . Thus we have $\mathbf{u}^{p,\infty} = (\mathbf{u}^{\infty} \cdot \hat{\mathbf{x}}) \hat{\mathbf{x}}$ and $\mathbf{u}^{s,\infty} = \hat{\mathbf{x}} \times \mathbf{u}^{\infty} \times \hat{\mathbf{x}}$.

An inverse scattering problem arising in practical applications including seismology and elastography is to recover the inhomogeneous scatterer $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ by knowledge of the associated far-field pattern \mathbf{u}^{∞} , namely,

$$\mathbf{u}^{\infty}(\hat{\mathbf{x}}; \mathbf{u}^{in}, \mathbf{f}, D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)) \rightarrow D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho). \quad (1.16)$$

Here, it is noted that there are two different kinds of sources, f and \mathbf{u}^{in} , in (1.13), which correspond to the so-called passive and active measurements in the context of the inverse problem (1.16). In order to give a general study, we include both of them into our study, and either one of them can be taken to be zero, corresponding to different scenarios in the context of the inverse problem (1.16) in the literature. In (1.16), the presence of the impenetrable obstacle D make the study of the inverse problem radically more challenging compared to the case without the obstacle, i.e. $D = \emptyset$. In fact, to our best knowledge, there is no result available in the literature for the inverse problem (1.16) in the case when $D \neq \emptyset$, whereas there are rich results in the case $D = \emptyset$; see e.g. [15, 16] and the references cited therein. Nevertheless, we would like to mention some related studies for the inverse acoustic and electromagnetic scattering problems in simultaneously recovering a buried obstacle and its surrounding medium [12, 22, 23] where one needs to make use of multiple-frequency measurements, namely severely over-determined data were used. Similar to the treatment for the inverse boundary problem (1.7) with partial measurements, we intend reduce the inverse scattering problem (1.16) to a simpler case with no buried obstacles in an effective way. To that end, we introduce the following definition.

Definition 1.2. Consider $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ as described above. If there exist an elastic medium $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ with $(\tilde{\mathcal{C}}, \tilde{\rho})|_{\Omega \setminus \overline{D}} = (\mathcal{C}, \rho)|_{\Omega \setminus \overline{D}}$, and $\varepsilon \in \mathbb{R}_+$ with $\varepsilon \ll 1$ such that

$$\begin{aligned} & \left\| \mathbf{u}^\infty(\hat{\mathbf{x}}; \mathbf{u}^{in}, \mathbf{f}, (\Omega; \tilde{\mathcal{C}}, \tilde{\rho})) - \mathbf{u}^\infty(\hat{\mathbf{x}}; \mathbf{u}^{in}, \mathbf{f}, D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)) \right\|_{C(\mathbb{S}^{n-1})^n} \\ & \leq C\varepsilon \left(\|\mathbf{u}^{in}\|_{H^1(B_{r_0})^n} + \|\mathbf{f}\|_{L^2(B_{r_0})^n} \right), \end{aligned} \quad (1.17)$$

where B_{r_0} is any given central ball containing Ω , and C is a generic positive constant depending on the a-priori parameters, then $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is said to be an effective ε -realization of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. If $\varepsilon \equiv 0$, then $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is said to be an effective realization of $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. For simpler terminologies, we also call $(D; \tilde{\mathcal{C}}, \tilde{\rho})$ an effective realization of the obstacle D .

Hence, if one can find an effective realization of the embedded obstacle D , the inverse problem (1.16) can then be effectively reduced to the recovery of $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$, which possesses a much simpler topological structure.

1.2. Summary of the main results. Motivated by the studies of the inverse problems discussed earlier, we establish in this paper that there are always approximate effective realizations of the embedded obstacles. It is clear that the two problems (1.8) and (1.14) are closely related. Indeed, they are equivalent if an appropriate truncation is introduced for truncating the unbounded domain $\mathbb{R}^n \setminus \overline{D}$ in (1.14) into a bounded one. In the rest of our paper, we shall present our study mainly for the scattering system (1.14). On the one hand, the scattering model (1.14) is physically more relevant in the context of the inverse elastic problem study, and on the other hand, the corresponding mathematical argument for the effective medium theory associated with (1.14) is technically more involved than associated with (1.8). Nevertheless, it is emphasized that the results established in our study hold equally for the corresponding problem associated with (1.8).

Our main result can be summarized in the following theorem.

Theorem 1.1. *Consider the scattering problem (1.14) associated with the scatterer $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$. Let λ_0, μ_0 be real constants satisfying the strong convexity condition in (1.12), and η_0, τ_0 be positive constants. Let $\varepsilon \in \mathbb{R}_+$ and $\varepsilon \ll 1$.*

- (I) *Case 1: If D is a traction-free obstacle, then $(D; \mathcal{C}^0, \rho_0)$ with \mathcal{C}^0 given in the form (1.4):*

$$\lambda = \varepsilon \lambda_0, \quad \mu = \varepsilon \mu_0, \quad \rho_0 = \eta_0 + \nu \tau_0, \quad (1.18)$$

is an $\varepsilon^{1/2}$ -realization of D in the sense of Definition 1.2;

- (II) *Case 2: If D is a rigid obstacle, then $(D; \mathcal{C}^0, \rho_0)$ with \mathcal{C}^0 given in the form (1.4):*

$$\lambda = \varepsilon^{-2} \lambda_0, \quad \mu = \varepsilon^{-2} \mu_0, \quad \rho_0 = (\eta_0 + \nu \varepsilon^{-1} \tau_0), \quad (1.19)$$

is an $\varepsilon^{1/2}$ -realization of D in the sense of Definition 1.2.

Remark 1.1. In (1.18) and (1.19), we assume that λ_0, μ_0 and η_0, τ_0 are all constants. Indeed, they can be replaced to be variable functions satisfying the strong convexity condition and this can easily be seen from our subsequent argument in proving Theorem 1.1. However, we stick to the simpler case with constants in order to ease the exposition. The main point is that if D is a traction-free obstacle, as long as the effective medium is lossy with asymptotically small bulk moduli, one can have the approximate effective realization effect. A similar remark can be made for the case if D is a rigid obstacle.

1.3. Discussion. We present more discussion on the implications of Theorem 1.1 to the inverse problem (1.9) or (1.16). As remarked earlier, we focus our discussion on (1.16). A standard approach for solving the inverse problem (1.16) is the following optimization formulation:

$$\min_{\hat{D} \oplus (\Omega \setminus \overline{\hat{D}}; \hat{\mathcal{C}}, \hat{\rho}) \in \mathcal{C}} \left\| \mathbf{u}^\infty(\hat{\mathbf{x}}; \mathbf{u}^{in}, \mathbf{f}, \hat{D} \oplus (\Omega \setminus \overline{\hat{D}}; \hat{\mathcal{C}}, \hat{\rho})) - \mathcal{M}(D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)) \right\|_{C(\mathbb{S}^{n-1})^n}, \quad (1.20)$$

where \mathcal{C} and \mathcal{M} signify the a-priori class of admissible scatterers and the measured far-field data, respectively. Clearly, $D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)$ is a global minimizer to (1.20). By Theorem 1.1, we replace (1.20) by the following optimization problem for the reconstruction:

$$\min_{(\Omega; \hat{\mathcal{C}}, \hat{\rho}) \in \mathcal{C}} \left\| \mathbf{u}^\infty(\hat{\mathbf{x}}; \mathbf{u}^{in}, \mathbf{f}, (\Omega; \hat{\mathcal{C}}, \hat{\rho})) - \mathcal{M}(D \oplus (\Omega \setminus \overline{D}; \mathcal{C}, \rho)) \right\|_{C(\mathbb{S}^{n-1})^n}. \quad (1.21)$$

By Definition 1.2 and Theorem 1.1, $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is an asymptotically global minimizer to (1.21). Three remarks are in order. First, it is expected that in generic scenarios the reconstruction result from (1.21) can (approximately) locate the topological defect of the underlying scatterer, namely the buried obstacle. In fact, considering the case that D is a traction-free obstacle, one can see that for the reconstructed medium, it should possess asymptotically small bulk moduli in the region where the obstacle is located. Second, for illustration, we only considered a simpler case with a single ‘‘hole’’ above. It is clear that the same idea works for the case that there are multiple ‘‘holes’’ within the scatterer. That is, one can start the reconstruction with the optimization formulation (1.21) without any requirement of the a-priori knowledge of the topological structure of the underlying scatterer. Using the reconstruction result, one should be able to (approximately) profile the topological structure of the scatterer, namely to (approximately) identify the buried obstacles, by locating the regions where the reconstructed medium show a certain asymptotically peculiar behaviour. One can then use such a reconstruction result as an initial guess for the optimization formulation (1.20) to further refine the reconstruction. Third, it is clear that the above described reconstruction procedure is rather heuristic. One would need to establish the uniqueness and stability results in order to guarantee the qualitative and quantitative properties of the minimizers to (1.20) and (1.21) required in the reconstruction procedure described above. The ill-posedness of the inverse problem

shall add extra complexities to the desired theoretical justification. Hence, in this paper, in order to have a focusing theme of our study, we mainly consider the effective realization of embedded obstacles and postpone the more comprehensive inverse problem study in a forthcoming paper.

The rest of the paper is organized as follows. In Section 2, we mainly recall some preliminary results and give one important auxiliary lemma and give the proof of the well-posedness of (1.14). The proof of Theorem 1.1 for Case 1 and Case 2 will be provided in Section 3 and 4, respectively.

2. AUXILIARY RESULTS

2.1. Preliminary. In this subsection, we present some preliminary results for our subsequent use. We first recall the following lemma on the conormal derivative of the vector field in the linear elasticity, which is a special case of Lemma 4.3 in [26].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipschitz boundary. Suppose $\mathbf{u} \in H^1(\Omega)^n$ and $\mathbf{h} \in H^{-1}(\Omega)^n$ satisfying*

$$\mathcal{L}\mathbf{u} = \mathbf{h} \quad \text{in } \Omega, \quad (2.1)$$

where $\mathcal{C}(\mathbf{x})$ is an elastic tensor satisfying the uniform Legendre ellipticity condition (1.3). Then there exists $\mathbf{g} \in H^{-1/2}(\partial\Omega)^n$ such that

$$\Psi(\mathbf{u}, \mathbf{v}) = -(\mathbf{h}, \mathbf{v})_\Omega + (\mathbf{g}, \gamma\mathbf{v})_{\partial\Omega} \quad \forall \mathbf{v}(\mathbf{x}) \in H^1(\Omega)^n \quad (2.2)$$

with

$$\Psi(\mathbf{u}, \mathbf{v}) = \int_{\Omega} [\mathcal{C}(\mathbf{x}) : \overline{\nabla\mathbf{u}}] : \nabla\mathbf{v} d\mathbf{x}, \quad (\mathbf{h}, \mathbf{v})_\Omega = \int_{\Omega} \mathbf{h}(\mathbf{x}) \cdot \mathbf{v} d\mathbf{x}, \quad (\mathbf{g}, \gamma\mathbf{v})_{\partial\Omega} = \int_{\partial\Omega} \mathbf{g} \cdot \gamma\mathbf{v} ds(\mathbf{x}),$$

where γ is the trace operator from $H^1(\Omega)^n$ to $H^{1/2}(\Omega)^n$.

Furthermore, \mathbf{g} is uniquely determined by \mathbf{u} and \mathbf{h} in the sense that the following estimate holds for some constant $\eta > 0$:

$$\|\mathbf{g}\|_{H^{-1/2}(\partial\Omega)^n} \leq \eta(\|\mathbf{u}\|_{H^1(\Omega)^n} + \|\mathbf{h}\|_{H^{-1}(\Omega)^n}). \quad (2.3)$$

In general, we write $\mathbf{g} = \boldsymbol{\nu} \cdot [\mathcal{C}(\mathbf{x}) : \nabla\mathbf{u}]$ in the distribution sense, which is called the conormal derivative of \mathbf{u} .

Corollary 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain with a Lipschitz boundary. Suppose $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^n$ and $\mathbf{h} = -\omega^2\rho\mathbf{u} \in L^2(\Omega)^n \Subset H^{-1}(\Omega)^n$. Then we have*

$$\Psi(\mathbf{u}, \mathbf{v}) = (\omega^2\rho\mathbf{u}, \mathbf{v})_\Omega + (\mathcal{T}_\nu(\mathbf{u}), \gamma\mathbf{v})_{\partial\Omega} \quad (2.4)$$

and

$$\|\mathcal{T}_\nu(\mathbf{u})\|_{H^{-1/2}(\partial\Omega)^n} \leq \eta\|\mathbf{u}\|_{H^1(\Omega)^n}, \quad (2.5)$$

where η is a positive constant.

Proof. We can easily obtain (2.4) by using (2.2). Using the definitions of dual norms $\|\cdot\|_{H^{-1}(\Omega)^n}$ and $\|\cdot\|_{L^2(\Omega)^n}$ we have

$$\begin{aligned} \|\mathbf{h}\|_{H^{-1}(\Omega)^n} &= \|\omega^2\rho\mathbf{u}\|_{H^{-1}(\Omega)^n} = \sup_{\mathbf{0} \neq \mathbf{w} \in H^1(\Omega)^n} \frac{|(\omega^2\rho\mathbf{u}, \mathbf{w})_\Omega|}{\|\mathbf{w}\|_{H^1(\Omega)^n}} \leq \sup_{\mathbf{0} \neq \mathbf{w} \in L^2(\Omega)^n} \frac{|(\omega^2\rho\mathbf{u}, \mathbf{w})_\Omega|}{\|\mathbf{w}\|_{L^2(\Omega)^n}} \\ &= \|\omega^2\rho\mathbf{u}\|_{L^2(\Omega)^n} \leq \eta\|\mathbf{u}\|_{H^1(\Omega)^n}, \end{aligned}$$

where $\eta = \eta(\rho, \omega, \Omega)$ is a positive constant. Therefore, (2.5) follows from (2.3). The proof is complete. \square

The next lemma, which can be named as Rellich's lemma in the linear elasticity, can be proved by generalizing the arguments in [15].

Lemma 2.2. *Let B_r be an appropriate ball centered at origin with a radius $r \in \mathbb{R}_+$, and assume that \mathbf{u}^s is a radiating solution to*

$$\mu \Delta \mathbf{u}^s + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}^s) + \omega^2 \rho \mathbf{u}^s = \mathbf{0}, \quad \mu > 0, \quad n\lambda + 2\mu > 0, \quad \rho > 0$$

in $|\mathbf{x}| \geq r$. If

$$\Im \left(\int_{\partial B_r} \mathcal{T}_\nu \mathbf{u}^s \cdot \overline{\mathbf{u}^s} ds \right) \leq 0, \quad (2.6)$$

then $\mathbf{u}^s = \mathbf{0}$ in $|\mathbf{x}| \geq r$.

By Lemma 2.2, we can easily derive the following result.

Theorem 2.2. *Suppose $\mathbf{u} \in H^1(\mathbb{R}^n)^n$ solves*

$$\mathcal{L}\mathcal{C}\mathbf{u} + \omega^2 \rho(\mathbf{x})\mathbf{u} = \mathbf{0}, \quad (2.7)$$

where $\mathcal{C}(\mathbf{x})$, ω , and ρ are defined as in (1.2). If \mathbf{u} satisfies the Kupradze radiation conditions, then \mathbf{u} vanishes in \mathbb{R}^n .

Proof. Let B_r be an appropriate large ball with center at origin and radius r . Multiplying $\overline{\mathbf{u}}$ on both sides of (2.7) and integrating over B_r , we have

$$\int_{B_r} \mathcal{L}\mathcal{C}\mathbf{u} \cdot \overline{\mathbf{u}} dx + \int_{B_r} \omega^2 \rho(\mathbf{x})\mathbf{u} \cdot \overline{\mathbf{u}} dx = 0.$$

By applying Betti's first formula (cf. [2]), we get

$$\int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \mathbf{u}) \cdot \overline{\mathbf{u}} ds(\mathbf{x}) - \int_{B_r} (\mathcal{C}(\mathbf{x}) : \nabla \overline{\mathbf{u}}) : \nabla \mathbf{u} dx + \int_{B_r} \omega^2 \rho(\mathbf{x}) |\mathbf{u}|^2 dx = 0.$$

Taking the imaginary part of the above equation, we obtain

$$\Im \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \mathbf{u}) \cdot \overline{\mathbf{u}} ds(\mathbf{x}) = - \int_{B_r} \omega^2 \Im \rho |\mathbf{u}|^2 dx \leq 0.$$

By Lemma 2.2, we have $\mathbf{u} = \mathbf{0}$ outside B_r . Then it follows from the unique continuation that $\mathbf{u} = \mathbf{0}$ in \mathbb{R}^n . \square

2.2. Auxiliary lemmas for Case 1. We derive several technical auxiliary lemmas for proving Theorem 1.1 in Sections 3 and 4. We first consider Case 1 in Theorem 1.1, where D is a traction-free obstacle. In what follows, we let B_r signify a central ball of radius r containing Ω , and consider the following two scattering problems: Given $\mathbf{p} \in H^{-1/2}(\partial D)^n$, $\mathbf{h}_1 \in H^{1/2}(\partial \Omega)^n$, $\mathbf{h}_2 \in H^{-1/2}(\partial \Omega)^n$ and \mathbf{f} with $\text{supp}(\mathbf{f}) \subset B_{r_0} \setminus \overline{\Omega} \subset B_r \setminus \overline{\Omega}$,

find $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(\mathbb{R}^n \setminus \overline{\Omega})^n$ such that

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{v} + \omega^2 \rho(\mathbf{x}) \mathbf{v} = \mathbf{0} & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}_C \mathbf{e} \mathbf{u}^s + \omega^2 \rho_e \mathbf{u}^s = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}^s = \mathbf{u}^{p,s} + \mathbf{u}^{s,s} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{u}) = \mathbf{p} & \text{on } \partial D, \\ \mathbf{v} = \mathbf{u}^s + \mathbf{h}_1, \quad \mathcal{T}_\nu(\mathbf{v}) = \mathcal{T}_\nu(\mathbf{u}^s) + \mathbf{h}_2 & \text{on } \partial \Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(n-1)/2} \left(\frac{\partial \mathbf{u}^{t,s}}{\partial |\mathbf{x}|} - i \kappa_t \mathbf{u}^{t,s} \right) = 0, & t = p, s, \end{array} \right. \quad (2.8)$$

and find $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(B_r \setminus \overline{\Omega})^n$ satisfying the following truncated system:

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{v} + \omega^2 \rho(\mathbf{x}) \mathbf{v} = \mathbf{0} & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}_C \mathbf{e} \mathbf{u}^s + \omega^2 \rho_e \mathbf{u}^s = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}^s = \mathbf{u}^{p,s} + \mathbf{u}^{s,s} & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{u}) = \mathbf{p} & \text{on } \partial D, \\ \mathbf{u}|_{\partial \Omega} = \mathbf{u}^s|_{\partial \Omega} + \mathbf{h}_1, \quad \mathcal{T}_\nu(\mathbf{u}) = \mathcal{T}_\nu(\mathbf{u}^s) + \mathbf{h}_2 & \text{on } \partial \Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{u}^s) = \Lambda \mathbf{u}^s & \text{on } \partial B_r, \end{array} \right. \quad (2.9)$$

where Λ is the Dirichlet-to-Neumann (DtN) map introduced in [6, 20] such that

$$\begin{aligned} \Lambda : H^{1/2}(\partial B_r)^n &\longrightarrow H^{-1/2}(\partial B_r)^n, \\ \tilde{\mathbf{g}} &\longmapsto \mathcal{T}_\nu(\tilde{\mathbf{q}}) \end{aligned} \quad (2.10)$$

with a radiating solution $\tilde{\mathbf{q}}$ for Navier equation

$$\left\{ \begin{array}{ll} \mu_e \Delta \tilde{\mathbf{q}} + (\lambda_e + \mu_e) \nabla(\nabla \cdot \tilde{\mathbf{q}}) + \omega^2 \rho_e \tilde{\mathbf{q}} = \mathbf{0} & \text{in } \mathbb{R}^n \setminus \overline{B_r}, \\ \tilde{\mathbf{q}} = \tilde{\mathbf{g}} & \text{on } \partial B_r, \end{array} \right.$$

where λ_e, μ_e and ρ_e are real constants satisfying the strong convexity condition (1.12).

In the following, we establish the equivalence of problem (2.8) and problem (2.9) in Lemma 2.3. Therefore we can prove that (2.8) admits a unique solution and satisfies certain a priori estimates.

Lemma 2.3. *The scattering problems (2.8) and (2.9) are equivalent.*

Proof. By applying the definition of Λ , it is easy to see that if $(\mathbf{v}, \mathbf{u}^s)$ is a solution to the scattering problem (2.8), then $(\mathbf{v}, \mathbf{u}^s)|_{B_r \setminus \overline{D}}$ solves the scattering problem (2.9).

On the other hand, suppose $(\mathbf{v}, \mathbf{u}^s)$ is a solution to the truncated system (2.9). By applying the integral representation and $\mathcal{T}_\nu \mathbf{u}^s = \Lambda \mathbf{u}^s$ on ∂B_r , we can derive that

$$\begin{aligned} \mathbf{u}^s(\mathbf{x}) &= \int_{\partial B_r} \left\{ \left\{ \mathcal{T}_\nu^y \Phi(\mathbf{x}, \mathbf{y}) \right\}^\top \cdot \mathbf{u}^s(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \cdot \Lambda \mathbf{u}^s(\mathbf{y}) \right\} ds(\mathbf{y}) + \int_{B_r \setminus \overline{\Omega}} \Phi(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \\ &\quad - \int_{\partial \Omega} \left\{ \left\{ \tilde{\mathcal{T}}_\nu^y \Phi(\mathbf{x}, \mathbf{y}) \right\}^\top \cdot \mathbf{u}^s(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \cdot \tilde{\mathcal{T}}_\nu^y \mathbf{u}^s(\mathbf{y}) \right\} ds(\mathbf{y}), \end{aligned} \quad (2.11)$$

where $\Phi(\mathbf{x}, \mathbf{y})$ is the fundamental solution to the Lamé system (1.13) with the form

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{\kappa_s^2}{4\pi\omega^2} \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} I + \frac{1}{4\pi\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|} - e^{i\kappa_p|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \quad (2.12)$$

and

$$\mathcal{T}_{\nu}^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}) = \left[\mathcal{T}_{\nu}^{\mathbf{y}} \left(\Phi(\mathbf{x}, \mathbf{y})(:, 1) \right), \mathcal{T}_{\nu}^{\mathbf{y}} \left(\Phi(\mathbf{x}, \mathbf{y})(:, 2) \right), \mathcal{T}_{\nu}^{\mathbf{y}} \left(\Phi(\mathbf{x}, \mathbf{y})(:, 3) \right) \right]. \quad (2.13)$$

Here, I is the identity matrix, $\Phi(\mathbf{x}, \mathbf{y})(:, j)$ denotes the j -th column of $\Phi(\mathbf{x}, \mathbf{y})$, $j = 1, 2, 3$. $\mathcal{T}_{\nu}^{\mathbf{y}}$ is the exterior unit normal vector to the boundaries with respect to \mathbf{y} . Notice that $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y})^{\top}$. Then, by combining the definition of Λ with the fact that each column of $\Phi(\mathbf{x}, \mathbf{y})$ satisfies the Kupradze radiation condition, we can obtain that

$$\int_{\partial B_r} \left\{ \left\{ \mathcal{T}_{\nu}^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}) \right\}^{\top} \cdot \mathbf{u}^s(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \cdot \Lambda \mathbf{u}^s(\mathbf{y}) \right\} ds(\mathbf{y}) = 0. \quad (2.14)$$

Substituting (2.14) into (2.11) yields

$$\begin{aligned} \mathbf{u}^s(\mathbf{x}) &= - \int_{\partial \Omega} \left\{ \left\{ \mathcal{T}_{\nu}^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}) \right\}^{\top} \cdot \mathbf{u}^s(\mathbf{y}) - \Phi(\mathbf{x}, \mathbf{y}) \cdot \mathcal{T}_{\nu}^{\mathbf{y}} \mathbf{u}^s(\mathbf{y}) \right\} ds(\mathbf{y}) \\ &\quad + \int_{B_r \setminus \overline{\Omega}} \Phi(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Clearly, \mathbf{u}^s can be extended to a function belong to $H_{loc}^1(\mathbb{R}^n \setminus \overline{\Omega})^n$ (still denoted by \mathbf{u}^s). Since each column of $\Phi(\mathbf{x}, \mathbf{y})$ or $\mathcal{T}_{\nu}^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$ satisfies the Kupradze radiation condition, the new function $\mathbf{u}^s \in H_{loc}^1(\mathbb{R}^n \setminus \overline{\Omega})^n$ also satisfies the Kupradze radiation condition. Hence, $(\mathbf{v}, \mathbf{u}^s)$ solves problem (2.8). \square

In next lemma, we prove that there exists a unique solution to the system (2.8), which is determined by the inputs \mathbf{p} , \mathbf{h}_1 , \mathbf{h}_2 and \mathbf{f} .

Lemma 2.4. *Given $\mathbf{p} \in H^{-1/2}(\partial D)^n$, $\mathbf{h}_1 \in H^{1/2}(\partial \Omega)^n$, $\mathbf{h}_2 \in H^{-1/2}(\partial \Omega)^n$ and \mathbf{f} with $\text{supp}(\mathbf{f}) \subset B_{r_0} \setminus \overline{\Omega}$, there exists a unique solution $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(\mathbb{R}^3 \setminus \overline{\Omega})^n$ to the system (2.8) such that the following estimate holds*

$$\begin{aligned} \|\mathbf{v}\|_{H^1(\Omega \setminus \overline{D})^n} + \|\mathbf{u}^s\|_{H^1(\mathbb{R}^n \setminus \overline{\Omega})^n} &\leq C \left(\|\mathbf{p}\|_{H^{-1/2}(\partial D)^n} + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} \right. \\ &\quad \left. + \|\mathbf{h}_2\|_{H^{-1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right) \end{aligned} \quad (2.15)$$

for some constant $C > 0$ depending only on $\mathcal{C}(\mathbf{x})$, κ_p , κ_s , \mathcal{C}^e , $\rho(\mathbf{x})$, Ω , D , B_r and ω .

Proof. Firstly, let $\mathbf{p} = \mathbf{0}$, $\mathbf{h}_1 = \mathbf{0}$, $\mathbf{h}_2 = \mathbf{0}$, $\mathbf{f} = \mathbf{0}$. It is sufficient to show that there exists only a trivial solution to (2.8). Post-multiplying the first equation of (2.8), respectively, by $\overline{\mathbf{v}}$ and $\overline{\mathbf{u}^s}$ and using the Betti's first formula (cf. [1, 7, 8]) over $\Omega \setminus \overline{D}$ and $B_r \setminus \overline{\Omega}$ and the boundary conditions on ∂D and $\partial \Omega$, we have

$$\begin{aligned} \int_{\Omega \setminus \overline{D}} [\mathcal{C}(\mathbf{x}) : \nabla \overline{\mathbf{v}}] : \nabla \mathbf{v} d\mathbf{x} &= \int_{\Omega \setminus \overline{D}} \omega^2 \rho |\mathbf{v}|^2 d\mathbf{x} - \int_{B_r \setminus \overline{\Omega}} [\mathcal{C}^e : \nabla \overline{\mathbf{u}^s}] : \nabla \mathbf{u}^s ds(\mathbf{x}) \\ &\quad + \int_{\partial B_r} \boldsymbol{\nu} \cdot [\mathcal{C}^e : \nabla \mathbf{u}^s] \cdot \overline{\mathbf{u}^s} ds(\mathbf{x}) + \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e |\mathbf{u}^s|^2 d\mathbf{x}. \end{aligned} \quad (2.16)$$

Taking the imaginary part of the equation above, we obtain

$$\Im \int_{\partial B_r} \boldsymbol{\nu} \cdot [\mathcal{C}^e : \nabla \mathbf{u}^s] \overline{\mathbf{u}^s} ds(\mathbf{x}) = - \int_{\Omega \setminus \overline{D}} \omega^2 \Im \rho |\mathbf{u}|^2 d\mathbf{x} \leq 0.$$

From Lemma 2.2 and the unique continuation principle, we know $\mathbf{u}^s = \mathbf{0}$ in $\Omega \setminus \overline{D}$ and $\mathbf{v} = \mathbf{0}$ in D . Therefore, the uniqueness of the solution to (2.8) is established.

By Lemma 2.3, problems (2.8) and (2.9) are equivalent. Thus, we only need to verify the existence of solution to (2.9) by the variational technique. Without loss of generality, we assume $\omega^2 \rho_e$ is not a Dirichlet eigenvalue in $B_r \setminus \overline{\Omega}$. It is easy to check that the vector field \mathbf{w} , which is defined by $\mathbf{w}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$ in $\Omega \setminus \overline{D}$ and $\mathbf{w}(\mathbf{x}) = \mathbf{u}^s(\mathbf{x}) + \tilde{\mathbf{v}}(\mathbf{x})$ in $B_r \setminus \overline{\Omega}$, satisfies

$$\begin{cases} \mathcal{L}_C \mathbf{w} + \omega^2 \rho(\mathbf{x}) \mathbf{w} = \mathbf{f} & \text{in } B_r \setminus \overline{D}, \\ \mathbf{w}^s = \mathbf{w}^{p,s} + \mathbf{w}^{s,s} & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{w}) = \mathbf{p} & \text{on } \partial D, \\ \mathbf{w}^- = \mathbf{w}^+ & \text{on } \partial \Omega, \\ \mathcal{T}_\nu(\mathbf{w}^-) = \mathcal{T}_\nu(\mathbf{w}^+) + \mathcal{T}_\nu(\mathbf{u}^{in}) - \mathcal{T}_\nu(\tilde{\mathbf{v}}) & \text{on } \partial \Omega, \\ \mathcal{T}_\nu(\mathbf{w}^-) = \Lambda \mathbf{w}^+ + \mathcal{T}_\nu(\tilde{\mathbf{v}}) & \text{on } \partial B_r, \end{cases} \quad (2.17)$$

where \mathbf{w}^- and \mathbf{w}^+ stand for the limits from outside and inside $\partial \Omega$, respectively, Λ is the DtN operator given in (2.10), $\tilde{\mathbf{v}}$ is a solution to the following equation:

$$\begin{cases} \mu_e \Delta \tilde{\mathbf{v}} + (\lambda_e + \mu_e) \nabla(\nabla \cdot \tilde{\mathbf{v}}) + \omega^2 \rho_e \tilde{\mathbf{v}} = \mathbf{0} & \text{in } B_r \setminus \overline{\Omega} \\ \tilde{\mathbf{v}} = \mathbf{u}^{in} & \text{on } \partial \Omega, \\ \tilde{\mathbf{v}} = \mathbf{0} & \text{on } \partial B_r. \end{cases} \quad (2.18)$$

By [26, Theorem 4.10], we know that $\tilde{\mathbf{v}}$ is unique and $\|\tilde{\mathbf{v}}\|_{H^1(B_r \setminus \overline{\Omega})^n} = O(\|\mathbf{u}^{in}\|_{H^{1/2}(\partial \Omega)^n})$.

Next, we introduce a bounded operator

$$\Lambda_0 : H^{1/2}(\partial B_r)^n \longrightarrow H^{-1/2}(\partial B_r)^n$$

which maps Φ to $\mathcal{T}_\nu(\tilde{\mathbf{w}}) \Big|_{\partial B_r}$ where $\tilde{\mathbf{w}} \in H_{loc}^1(\mathbb{R}^n \setminus \overline{B_r})^n$ is the unique solution of the following system:

$$\begin{cases} \mu_e \Delta \tilde{\mathbf{w}} + (\lambda_e + \mu_e) \nabla(\nabla \cdot \tilde{\mathbf{w}}) + \omega^2 \rho_e \tilde{\mathbf{w}} = \mathbf{0} & \text{in } \mathbb{R}^n \setminus \overline{B_r}, \\ \tilde{\mathbf{w}} = \Phi \in H^{1/2}(\partial B_r)^n & \text{on } \partial B_r. \end{cases} \quad (2.19)$$

The operator Λ_0 has the following properties

$$- \int_{\partial B_r} \overline{\Phi} \Lambda_0 \Phi ds(x) \geq 0, \quad \Phi \in H^{1/2}(\partial B_r)^n, \quad (2.20)$$

and the difference $\Lambda - \Lambda_0$ is a compact operator from $H^{1/2}(\partial B_r)^n \rightarrow H^{-1/2}(\partial B_r)^n$. It is proved in [6] that these properties still hold for dyadic field by the similar analysis for the Laplace operator [4, 17]. Hence, for any $\varphi \in H^1(B_r \setminus \overline{D})^n$, using the test function $\overline{\varphi}$ we can easily derive the variational formulation of (2.17): find $\mathbf{w} \in H^1(B_r \setminus \overline{D})^n$ such that

$$a_1(\mathbf{w}, \varphi) + a_2(\mathbf{w}, \varphi) = \mathcal{F}(\varphi), \quad (2.21)$$

where the bilinear forms a_1 , a_2 and the linear functional $\mathcal{F}(\cdot)$ are defined by

$$\begin{aligned} a_1(\mathbf{w}, \varphi) &:= \int_{\Omega \setminus \overline{D}} (\mathcal{C}(\mathbf{x}) : \nabla \overline{\varphi}) : \nabla \mathbf{w} \, dx + \int_{\Omega \setminus \overline{D}} \rho \omega^2 \mathbf{w} \cdot \overline{\varphi} \, dx + \int_{B_r \setminus \overline{\Omega}} (\mathcal{C}^e : \nabla \overline{\varphi}) : \nabla \mathbf{w} \, dx \\ &\quad + \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e \mathbf{w} \cdot \overline{\varphi} \, dx - \int_{\partial B_r} \Lambda_0 \mathbf{w} \cdot \overline{\varphi} \, ds(\mathbf{x}), \\ a_2(\mathbf{w}, \varphi) &:= -2 \int_{\Omega \setminus \overline{D}} \rho \omega^2 \mathbf{w} \cdot \overline{\varphi} \, dx - 2 \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e \mathbf{w} \cdot \overline{\varphi} \, dx \\ &\quad - \int_{\partial B_r} (\Lambda - \Lambda_0) \mathbf{w} \cdot \overline{\varphi} \, ds(\mathbf{x}), \\ \mathcal{F}(\varphi) &:= - \int_{\partial D} \mathbf{p} \cdot \overline{\varphi} \, ds(\mathbf{x}) + \int_{\partial \Omega} (\mathbf{h}_2 - \mathcal{T}_\nu(\tilde{\mathbf{v}})) \cdot \overline{\varphi} \, ds(\mathbf{x}) + \int_{\partial B_r} \mathcal{T}_\nu(\tilde{\mathbf{v}}) \cdot \overline{\varphi} \, ds(\mathbf{x}) \\ &\quad - \int_{B_r \setminus \overline{D}} \mathbf{f} \cdot \overline{\varphi} \, dx. \end{aligned}$$

By using the assumptions about $\rho(\mathbf{x})$ and $\mathcal{C}(\mathbf{x})$ given in Subsection 1.1, Cauchy-Schwarz inequality and the definition of operator Λ_0 , one can show the boundedness of the bilinear form a_1 : for any $\phi, \varphi \in H^1(B_r \setminus \overline{D})^n$,

$$|a_1(\phi, \varphi)| \leq C_1 \|\phi\|_{H^1(B_r \setminus \overline{D})^n} \|\varphi\|_{H^1(B_r \setminus \overline{D})^n}$$

for some constant C_1 . Furthermore, by virtue of Poincaré's inequality and (2.20), we have the coercivity property of the bilinear form a_1 : for any $\varphi \in H^1(B_r \setminus \overline{D})^n$,

$$a_1(\varphi, \varphi) \geq C_2 \|\varphi\|_{H^1(B_r \setminus \overline{D})^n}^2$$

for some constant C_2 . According to Lax-Milgram lemma, there exists a bounded inverse operator $\mathcal{L} : H^1(B_r \setminus \overline{D})^n \rightarrow H^1(B_r \setminus \overline{D})^n$ such that

$$a_1(\mathbf{w}, \varphi) = \langle \mathcal{L} \mathbf{w}, \varphi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H^1(B_r \setminus \overline{D})^n$, and the inverse of \mathcal{L} is also bounded. In view of the expression of the bilinear form a_2 , we introduce two bounded operators \mathcal{K}_1 and \mathcal{K}_2 given by

$$\begin{aligned} \langle \mathcal{K}_1 \mathbf{w}, \varphi \rangle &:= 2 \int_{\Omega \setminus \overline{D}} \rho \omega^2 \mathbf{w} \cdot \overline{\varphi} \, dx + 2 \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e \mathbf{w} \cdot \overline{\varphi} \, dx, \\ \langle \mathcal{K}_2 \mathbf{w}, \varphi \rangle &:= \int_{\partial B_r} (\Lambda - \Lambda_0) \mathbf{w} \cdot \overline{\varphi} \, ds(\mathbf{x}). \end{aligned} \tag{2.22}$$

We claim that the operators \mathcal{K}_1 and \mathcal{K}_2 are both compact. In fact, let $\{\mathbf{w}_n\}_{n=1}^\infty$ be a bounded sequence in $H^1(B_r \setminus \overline{D})^n$ and weakly converge to \mathbf{w}_* in the sense of $\|\cdot\|_{H^1(B_r \setminus \overline{D})^n}$ (denoted by $\mathbf{w}_n \rightharpoonup \mathbf{w}_*$). Since $\mathcal{I} : H^1(B_r \setminus \overline{D})^n \rightarrow L^2(B_r \setminus \overline{D})^n$ is a compact embedding operator, we get

$$\langle \mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*), \varphi \rangle = 2 \int_{\Omega \setminus \overline{D}} \rho \omega^2 (\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\varphi} \, dx + 2 \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e (\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\varphi} \, dx$$

and thus

$$\left\| \mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*) \right\|_{H^1(B_r \setminus \overline{D})^n}^2 = \langle \mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*), \mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*) \rangle$$

$$\begin{aligned}
&= 2 \int_{\Omega \setminus \overline{D}} \rho \omega^2 (\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*)} \, dx \\
&\quad + 2 \int_{B_r \setminus \overline{\Omega}} \rho_e \omega^2 (\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\mathcal{K}_1(\mathbf{w}_n - \mathbf{w}_*)} \, dx \\
&\leq 2C \omega^2 \max \{ \|\rho(\mathbf{x})\|_{\mathbf{L}^\infty(\Omega \setminus \overline{D})}, \rho_e \} \|\mathbf{w}_n - \mathbf{w}_*\|_{L^2(B_r \setminus \overline{D})}^2,
\end{aligned}$$

which implies that \mathcal{K}_1 is compact. Similarly, we can verify the compactness of \mathcal{K}_2 . Since $\mathbf{w}_n \rightharpoonup \mathbf{w}_*$ in $H^1(B_r \setminus \overline{D})^n$, we have $\mathbf{w}_n|_{\partial B_r} \rightharpoonup \mathbf{w}_*|_{\partial B_r}$ in $H^{1/2}(\partial B_r)^n$ by the trace operator. Together with the compactness of $\Lambda - \Lambda_0$, it is easy to obtain that

$$(\Lambda - \Lambda_0)\mathbf{w}_n|_{\partial B_r} \longrightarrow (\Lambda - \Lambda_0)\mathbf{w}_*|_{\partial B_r}$$

in $H^{-1/2}(\partial B_r)^n$. For any $\varphi \in H^1(B_r \setminus \overline{D})^n$, it holds that

$$\langle \mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*), \varphi \rangle = \int_{\partial B_r} (\Lambda - \Lambda_0)(\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\varphi} \, ds(x).$$

Therefore we have

$$\begin{aligned}
\left\| \mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*) \right\|_{H^1(B_r \setminus \overline{D})^n}^2 &= \langle \mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*), \mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*) \rangle \\
&= \int_{\partial B_r} (\Lambda - \Lambda_0)(\mathbf{w}_n - \mathbf{w}_*) \cdot \overline{\mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*)} \, ds(x) \\
&\leq \|(\Lambda - \Lambda_0)(\mathbf{w}_n - \mathbf{w}_*)\|_{H^{-1/2}(\partial B_r)^n} \|\mathcal{K}_2(\mathbf{w}_n - \mathbf{w}_*)\|_{H^{1/2}(\partial B_r)^n} \\
&\leq C \|(\Lambda - \Lambda_0)(\mathbf{w}_n - \mathbf{w}_*)\|_{H^{-1/2}(\partial B_r)^n} \|\mathbf{w}_n - \mathbf{w}_*\|_{L^2(B_r \setminus \overline{D})^n},
\end{aligned}$$

which implies that \mathcal{K}_2 is compact.

Since \mathcal{L} is bounded and $\mathcal{K}_1 + \mathcal{K}_2$ is compact, we know that $\mathcal{L} - (\mathcal{K}_1 + \mathcal{K}_2)$ is a Fredholm operator of index zero. According to the Fredholm alternative theorem, Riesz representation theory and the uniqueness of (2.8), we know there must exist a solution to (2.8). Since the inverse of $\mathcal{L} - (\mathcal{K}_1 + \mathcal{K}_2)$ is bounded, by applying the Lax-Milgram lemma to

$$\langle (\mathcal{T} - \mathcal{K}_1 - \mathcal{K}_2) \mathbf{w}, \varphi \rangle = \mathcal{F}(\varphi),$$

we get

$$\|\mathbf{w}\|_{H^1(B_R \setminus \overline{D})^n} \leq C \|\mathcal{F}\|.$$

On the other hand, it is straightforward to verify that

$$|\mathcal{F}(\varphi)| \leq C \left(\|\mathbf{p}\|_{H^{-1/2}(\partial D)^n} + \|\mathbf{h}_2\|_{H^{-1/2}(\partial D)^n} \right) + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{H^{-1/2}(B_r \setminus \overline{D})^n} \|\varphi\|_{H(B_r \setminus \overline{D})^n},$$

which can directly imply the inequality (2.25). \square

2.3. Auxiliary lemmas for Case 2. In this subsection, we shall establish several key lemmas for Case 2 in Theorem 1.1. Considering that D is a rigid obstacle, the unbounded and truncated scattering systems associated with Case 2 are given as follows:

find $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(\mathbb{R}^n \setminus \overline{\Omega})^n$ satisfying

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{v} + \omega^2 \rho(\mathbf{x}) \mathbf{v} = \mathbf{0} & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}_{C^e} \mathbf{u}^s + \omega^2 \rho_e \mathbf{u}^s = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}^s = \mathbf{u}^{p,s} + \mathbf{u}^{s,s} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}|_{\partial D} = \mathbf{p} & \text{on } \partial D, \\ \mathbf{v} = \mathbf{u}^s + \mathbf{h}_1, \quad \mathcal{T}_\nu(\mathbf{v}) = \mathcal{T}_\nu(\mathbf{u}^s) + \mathbf{h}_2 & \text{on } \partial \Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(n-1)/2} \left(\frac{\partial \mathbf{u}^{t,s}}{\partial |\mathbf{x}|} - i \kappa_t \mathbf{u}^{t,s} \right) = 0, & t = t, s \end{array} \right. \quad (2.23)$$

and find $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(B_r \setminus \overline{\Omega})^n$ satisfying

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{v} + \omega^2 \rho(\mathbf{x}) \mathbf{v} = \mathbf{0} & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}_{C^e} \mathbf{u}^s + \omega^2 \rho_e \mathbf{u}^s = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}|_{\partial D} = \mathbf{p} & \text{on } \partial D, \\ \mathbf{v}|_{\partial \Omega} = \mathbf{u}^s|_{\partial \Omega} + \mathbf{h}_1, \quad \mathcal{T}_\nu(\mathbf{v}) = \mathcal{T}_\nu(\mathbf{u}^s) + \mathbf{h}_2 & \text{on } \partial \Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{u}^s) = \Lambda \mathbf{u}^s & \text{on } \partial B_r, \end{array} \right. \quad (2.24)$$

where $\mathbf{p} \in H^{1/2}(\partial D)^n$, $\mathbf{h}_1 \in H^{1/2}(\partial \Omega)^n$, $\mathbf{h}_2 \in H^{-1/2}(\partial \Omega)^n$ and \mathbf{f} with $\text{supp}(\mathbf{f}) \subset B_{r_0} \setminus \overline{\Omega} \subset B_r \setminus \overline{\Omega}$. In fact, we can easily obtain the equivalence of (2.23) and (2.24) by the similar argument of Lemma 2.3. In addition, similar to Lemma 2.4, we have the following result.

Lemma 2.5. *Given $\mathbf{p} \in H^{1/2}(\partial D)^n$, $\mathbf{h}_1 \in H^{1/2}(\partial \Omega)^n$, $\mathbf{h}_2 \in H^{-1/2}(\partial \Omega)^n$ and \mathbf{f} with $\text{supp}(\mathbf{f}) \subset B_{r_0} \setminus \overline{\Omega}$, there exists a unique solution $(\mathbf{v}, \mathbf{u}^s) \in H^1(\Omega \setminus \overline{D})^n \times H^1(\mathbb{R}^3 \setminus \overline{\Omega})^n$ to the system (2.8) such that the following estimate holds*

$$\begin{aligned} \|\mathbf{v}\|_{H^1(\Omega \setminus \overline{D})^n} + \|\mathbf{u}^s\|_{H^1(\mathbb{R}^n \setminus \overline{\Omega})^n} \leq C & \left(\|\mathbf{p}\|_{H^{1/2}(\partial D)^n} + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} \right. \\ & \left. + \|\mathbf{h}_2\|_{H^{-1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right), \end{aligned} \quad (2.25)$$

where C is a positive constant.

Proof. The uniqueness can be easily proved. Thus, we only need to verify that there exists a solution to (2.24) such that the estimate (2.25) holds. Without loss of generality, we assume $\omega^2 \rho_e$ is not a Dirichlet eigenvalue in $B_r \setminus \overline{\Omega}$. The PDE system (2.24) can be

converted into the following system:

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{w} + \omega^2 \rho(\mathbf{x}) \mathbf{w} = \mathbf{f} & \text{in } B_r \setminus \overline{D}, \\ \mathbf{w}^s = \mathbf{w}^{p,s} + \mathbf{w}^{s,s} & \text{in } B_r \setminus \overline{\Omega}, \\ \mathbf{w}|_{\partial D} = \mathbf{p} & \text{on } \partial D, \\ \mathbf{w}^-|_{\partial \Omega} = \mathbf{w}^+|_{\partial \Omega} & \text{on } \partial \Omega, \\ \mathcal{T}_\nu(\mathbf{w}^-) = \mathcal{T}_\nu(\mathbf{w}^+) + \mathcal{T}_\nu(\mathbf{u}^{in}) - \mathcal{T}_\nu(\tilde{\mathbf{v}}) & \text{on } \partial \Omega, \\ \mathbf{w}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{w}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{w}) = \Lambda \mathbf{w} + \mathcal{T}_\nu(\tilde{\mathbf{v}}) & \text{on } \partial B_r, \end{array} \right. \quad (2.26)$$

where $\mathbf{w}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$ in $\Omega \setminus \overline{D}$ and $\mathbf{w}(\mathbf{x}) = \mathbf{u}^s(\mathbf{x}) + \tilde{\mathbf{v}}(\mathbf{x})$ in $B_r \setminus \overline{\Omega}$, $\tilde{\mathbf{v}}$ is a solution to

$$\left\{ \begin{array}{ll} \mathcal{L}_{C^e} \mathbf{w} + \omega^2 \rho_e \mathbf{w} = \mathbf{f} & \text{in } B_r \setminus \overline{\Omega}, \\ \tilde{\mathbf{v}} = \mathbf{u}^{in} & \text{on } \partial \Omega, \\ \tilde{\mathbf{v}} = \mathbf{0} & \text{on } \partial B_r. \end{array} \right.$$

By [26, Theorem 4.10], it is obvious to see that $\tilde{\mathbf{v}}$ is unique and

$$\|\tilde{\mathbf{v}}\|_{H^1(B_r \setminus \overline{\Omega})^n} = O(\|\mathbf{u}^{in}\|_{H^{1/2}(\partial \Omega)^n}).$$

Similar to the proof of Lemma 2.5, we also use the bounded operator Λ_0 and its corresponding properties. Here, we introduce a new Sobolve space

$$X := \{\mathbf{w} \in H^1(B_r \setminus \overline{D})^n; \mathbf{w} = \mathbf{0} \text{ on } \partial D\}$$

and let $\mathbf{w}_0 \in H^1(B_r \setminus \overline{D})^n$ be such that $\mathbf{w}_0 = \mathbf{p}$ on ∂D and $\|\mathbf{w}_0\|_{H^1(B_r \setminus \overline{D})^n} \leq C\|\mathbf{q}\|_{H^{1/2}(\partial D)^n}$. Then for any $\varphi \in X$, using the test function $\overline{\varphi}$ we can easily derive the variational formulation of (2.26): find $\mathbf{w} \in H^1(B_r \setminus \overline{D})^n$ such that

$$a_1(\mathbf{w} - \mathbf{w}_0, \varphi) + a_2(\mathbf{w} - \mathbf{w}_0, \varphi) = \mathcal{F}(\varphi), \quad (2.27)$$

where the bilinear forms a_1, a_2 and the linear functional $\mathcal{F}(\cdot)$ are defined by

$$\begin{aligned} a_1(\mathbf{w} - \mathbf{w}_0, \varphi) &:= \int_{\Omega \setminus \overline{D}} (\mathcal{C}(\mathbf{x}) : \nabla \overline{\varphi}) : \nabla(\mathbf{w} - \mathbf{w}_0) dx + \int_{\Omega \setminus \overline{D}} \rho \omega^2 (\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} dx \\ &+ \int_{B_r \setminus \overline{\Omega}} (\mathcal{C}^e : \nabla \overline{\varphi}) : \nabla(\mathbf{w} - \mathbf{w}_0) dx + \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e (\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} dx \\ &- \int_{\partial B_r} \Lambda_0(\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} ds(x), \\ a_2(\mathbf{w} - \mathbf{w}_0, \varphi) &:= -2 \int_{\Omega \setminus \overline{D}} \rho \omega^2 (\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} dx - 2 \int_{B_r \setminus \overline{\Omega}} \omega^2 \rho_e (\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} dx \\ &- \int_{\partial B_r} (\Lambda - \Lambda_0)(\mathbf{w} - \mathbf{w}_0) \cdot \overline{\varphi} ds(x), \\ \mathcal{F}(\varphi) &:= \int_{\partial \Omega} (\mathbf{h}_2 - \mathcal{T}_\nu(\tilde{\mathbf{v}})) \cdot \overline{\varphi} ds(x) + \int_{\partial B_r} \mathcal{T}_\nu(\tilde{\mathbf{v}}) \cdot \overline{\varphi} ds(x) - \int_{B_r \setminus \overline{D}} \mathbf{f} \cdot \overline{\varphi} dx. \end{aligned}$$

Since Λ is a bounded operator from $H^{1/2}(\partial B_r)$ to $H^{1/2}(\partial B_r)$, \mathcal{F} is a bounded conjugate linear functional on X and both $a_1(\cdot, \cdot)$ and $a_2(\cdot, \cdot)$ are continuous on $X \times X$: for any

$\phi, \varphi \in X$,

$$|a_1(\phi, \varphi)| \leq C \|\phi\|_{H^1(B_r \setminus \bar{D})^n} \|\varphi\|_{H^1(B_r \setminus \bar{D})^n}$$

for some constant C .

From the properties of Λ_0 and (1.3), we see that for any $\varphi \in X$,

$$a_1(\varphi, \varphi) \geq C \|\varphi\|_{H^1(B_r \setminus \bar{D})^n}^2$$

with some constant C . Therefore, by Lax-Milgram lemma, there exists a bounded inverse operator $\mathcal{L} : X \rightarrow X$ such that

$$a_1(\mathbf{w} - \mathbf{w}_0, \varphi) = \langle \mathcal{L}(\mathbf{w} - \mathbf{w}_0), \varphi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $H^1(B_r \setminus \bar{D})^n$, and the inverse of \mathcal{L} is also bounded. Note that including a L^2 -inner product term in $a_1(\cdot, \cdot)$ is important since the Poincaré inequality does not hold in X any longer. From the expression of the bilinear form a_2 , we introduce two bounded operators \mathcal{K}_1 and \mathcal{K}_2 given by

$$\langle \mathcal{K}_1(\mathbf{w} - \mathbf{w}_0), \varphi \rangle := 2 \int_{\Omega \setminus \bar{D}} \rho \omega^2 (\mathbf{w} - \mathbf{w}_0) \cdot \bar{\varphi} \, dx + 2 \int_{B_r \setminus \bar{\Omega}} \omega^2 \rho_e (\mathbf{w} - \mathbf{w}_0) \cdot \bar{\varphi} \, dx, \quad (2.28)$$

$$\langle \mathcal{K}_2(\mathbf{w} - \mathbf{w}_0), \varphi \rangle := \int_{\partial B_r} (\Lambda - \Lambda_0)(\mathbf{w} - \mathbf{w}_0) \cdot \bar{\varphi} \, ds(x). \quad (2.29)$$

By the similar argument as in Lemma 2.4, we can verify that the operators \mathcal{K}_1 and \mathcal{K}_2 defined by (2.28) and (2.29) are also compact. Similarly, we also have

$$\langle (\mathcal{T} - \mathcal{K}_1 - \mathcal{K}_2)(\mathbf{w} - \mathbf{w}_0), \varphi \rangle = \mathcal{F}(\varphi),$$

By Lax-Milgram lemma, we see that

$$\|(\mathbf{w} - \mathbf{w}_0)\|_{H^1(B_r \setminus \bar{D})^n} \leq C \|\mathcal{F}\|.$$

On the other hand,

$$|\mathcal{F}(\varphi)| \leq C \left(\|\mathbf{p}\|_{H^{1/2}(\partial D)^n} + \|\mathbf{h}_2\|_{H^{-1/2}(\partial D)^n} \right) + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{H^{-1/2}(B_r \setminus \bar{D})^n} \|\varphi\|_{H(B_r \setminus \bar{D})^n},$$

which can directly imply the inequality

$$\begin{aligned} \|\mathbf{v}\|_{H^1(\Omega \setminus \bar{D})^n} + \|\mathbf{u}^s\|_{H^1(\mathbb{R}^n \setminus \bar{\Omega})^n} &\leq C \left\{ \|\mathbf{w}_0\|_{H^1(\mathbb{R}^n \setminus \bar{D})^n} + \|\mathbf{p}\|_{H^{1/2}(\partial D)^n} + \|\mathbf{h}_2\|_{H^{-1/2}(\partial D)^n} \right. \\ &\quad \left. + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{H^{-1/2}(B_r \setminus \bar{D})^n} \right\} \\ &\leq \tilde{C} \left(\|\mathbf{p}\|_{H^{1/2}(\partial D)^n} + \|\mathbf{h}_1\|_{H^{1/2}(\partial \Omega)^n} + \|\mathbf{h}_2\|_{H^{-1/2}(\partial \Omega)^n} \right. \\ &\quad \left. + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \end{aligned}$$

The proof is complete. \square

2.4. The well-posedness of the scattering problem (1.14). In this subsection, we can adopt a similar variational technique used in Subsections 2.2 and 2.3 to verify the well-posedness of the scattering problem (1.14).

Proposition 2.1. *There exists a unique solution $\mathbf{u} \in H^1(\mathbb{R}^n \setminus \bar{D})^n$ to the scattering problem (1.14). Furthermore, it holds that*

$$\|\mathbf{u}\|_{H^1(\mathbb{R}^n \setminus \bar{D})^n} \leq C \left(\|\mathbf{u}^{in}\|_{H^{1/2}(\partial \Omega)^n} + \|\mathcal{T}_\nu(\mathbf{u}^{in})\|_{H^{-1/2}(\partial \Omega)^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right), \quad (2.30)$$

where C is a positive constant, $\Omega \Subset B_{r_0}$ and B_{r_0} is a ball centered at the origin with the radius $r_0 \in \mathbb{R}_+$.

Proof. As discussed in Subsection 1.2, by using an appropriate truncation we can truncate the unbounded domain $\mathbb{R}^n \setminus \overline{D}$ in (1.14) into a bounded one. Indeed, (1.14) can be transformed to the following PDE system: Find $\mathbf{u} \in H^1(B_r \setminus \overline{D})^n$ such that

$$\left\{ \begin{array}{ll} \mathcal{L}_C \mathbf{u} + \omega^2 \rho(\mathbf{x}) \mathbf{u} = \mathbf{0} & \text{in } \Omega \setminus \overline{D}, \\ \mathcal{L}_{C^e} \mathbf{u}^s + \omega^2 \rho_e \mathbf{u}^s = \mathbf{f} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \mathbf{u}^s = \mathbf{u}^{p,s} + \mathbf{u}^{s,s} & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{B}(\mathbf{u}) = \mathbf{0} & \text{on } \partial D, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{u}^s|_{\partial\Omega} + \mathbf{u}^{in}, \quad \mathcal{T}_\nu(\mathbf{u}) = \mathcal{T}_\nu(\mathbf{u}^s) + \mathcal{T}_{\mathbf{u}^{in}} & \text{on } \partial\Omega, \\ \mathbf{u}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \mathbf{u}^s), \quad \mathbf{u}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \mathbf{u}^s) & \text{in } B_r \setminus \overline{\Omega}, \\ \mathcal{T}_\nu(\mathbf{u}^s) = \Lambda \mathbf{u}^s & \text{on } \partial B_r. \end{array} \right. \quad (2.31)$$

In fact, we can use a completely similar argument of Lemma 2.3 to verify the equivalence of (2.31) and (1.14), so we only need to illustrate that there exists a unique solution to (1.14) and it is relied on the input data \mathbf{u}^{in} and \mathbf{f} . Here we replace the product space $H^1(\Omega \setminus \overline{D})^n \times H^1(\mathbb{R}^n \setminus \overline{\Omega})^n$ in Lemma 2.4 (or $X \times X$ in Lemma 2.5) with $\mathbb{R}^n \setminus \overline{D}$ and take $\mathcal{B}(\mathbf{u}) = \mathbf{0}$ on ∂D , $\mathbf{h}_1 = \mathbf{u}^{in} \in H^{1/2}(\partial\Omega)^n$, $\mathbf{h}_2 = \mathcal{T}_\nu(\mathbf{u}^{in}) \in H^{-1/2}(\partial\Omega)^n$. By using a similar proof of Lemma 2.4 (or Lemma 2.5), we can easily obtain the uniqueness of solution to (1.14) and derive (2.30). The proof is complete. \square

3. PROOF OF THEOREM 1.1 FOR CASE 1

In this section, we mainly consider that the traction-free obstacle D has an $\varepsilon^{1/2}$ -realization $(D; \mathcal{C}^0, \rho_0)$ in the sense of Definition 1.2, where \mathcal{C}^0 is given in the form (1.4) and λ, μ, ρ_0 satisfy the conditions (1.18). Considering an elastic medium $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ with $(\tilde{\mathcal{C}}, \tilde{\rho})|_{\Omega \setminus \overline{D}} = (\mathcal{C}, \rho)|_{\Omega \setminus \overline{D}}$ and $(\tilde{\mathcal{C}}, \tilde{\rho})|_D = (\mathcal{C}^0, \rho_0)|_D$, let $(\tilde{\mathcal{C}}, \tilde{\rho})$ be extended into $\mathbb{R}^n \setminus \overline{\Omega}$ such that $(\tilde{\mathcal{C}}, \tilde{\rho}) = (\mathcal{C}^e, \rho_e)$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Then the medium scattering system described above is given as follows:

$$\left\{ \begin{array}{ll} \mathcal{L}_{\tilde{\mathcal{C}}} \tilde{\mathbf{u}} + \omega^2 \tilde{\rho} \tilde{\mathbf{u}} = \mathbf{f} & \text{in } \mathbb{R}^n, \\ \tilde{\mathbf{u}} = \mathbf{u}^{in} + \tilde{\mathbf{u}}^s & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \tilde{\mathbf{u}}^-|_{\partial D} = \tilde{\mathbf{u}}^+|_{\partial D}, \quad \mathcal{T}_\nu(\tilde{\mathbf{u}}^-) = \mathcal{T}_\nu(\tilde{\mathbf{u}}^+) & \text{on } \partial D, \\ \tilde{\mathbf{u}}|_{\partial\Omega} = \tilde{\mathbf{u}}^s|_{\partial\Omega} + \mathbf{u}^{in}, \quad \mathcal{T}_\nu(\tilde{\mathbf{u}}) = \mathcal{T}_\nu(\tilde{\mathbf{u}}^s) + \mathcal{T}_\nu(\mathbf{u}^{in}) & \text{on } \partial\Omega, \\ \tilde{\mathbf{u}}^{p,s} = -\frac{1}{k_p^2} \nabla(\nabla \cdot \tilde{\mathbf{u}}^s), \quad \tilde{\mathbf{u}}^{s,s} = \frac{1}{k_s^2} \nabla \times (\nabla \times \tilde{\mathbf{u}}^s) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{|\mathbf{x}| \rightarrow \infty} |\mathbf{x}|^{(n-1)/2} \left(\frac{\partial \tilde{\mathbf{u}}^{t,s}}{\partial |\mathbf{x}|} - \imath \kappa_t \tilde{\mathbf{u}}^{t,s} \right) = \mathbf{0}, & t = p, s, \end{array} \right. \quad (3.1)$$

where $\tilde{\mathbf{u}}^-$ and $\tilde{\mathbf{u}}^+$ stand for the limits from outside and inside ∂D , respectively. In the following lemma, we first derive the unique solution $\tilde{\mathbf{u}}$ of (3.1) in regions $B_r \setminus \overline{D}$ and D can be estimated well by \mathbf{u}^{in} and \mathbf{f} , which plays an important role in the subsequent proof.

Lemma 3.1. *Let $\tilde{\mathbf{u}}$ be the unique solution of (3.1). Then there exist positive constants r_0, C_1 and C_2 such that the following estimates hold for all $\varepsilon \ll 1$ and $r \geq r_0$:*

$$\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \bar{D})^n} \leq C_1 \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right), \quad (3.2)$$

$$\sqrt{\varepsilon} \|\tilde{\mathbf{u}}\|_{H^1(D)^n} \leq C_2 \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \quad (3.3)$$

Proof. From (3.1), we know that $\mathcal{L}_{\tilde{c}} \tilde{\mathbf{u}} + \omega^2 \tilde{\rho} \tilde{\mathbf{u}} = \mathbf{f}$ in D can be described as

$$\nabla \cdot (\mathcal{C}^0(\mathbf{x}) : \nabla \tilde{\mathbf{u}}) + \omega^2 \rho_0 \tilde{\mathbf{u}} = \mathbf{0}.$$

Multiplying it by $\bar{\tilde{\mathbf{u}}}$ and integrating over D , and then utilizing the Betti's first formula, we get

$$- \int_D (\mathcal{C}^0 : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} + \int_{\partial D} \boldsymbol{\nu} \cdot (\mathcal{C}^0 : \nabla \tilde{\mathbf{u}}) \cdot \bar{\tilde{\mathbf{u}}} \, ds(\mathbf{x}) + \omega^2 \int_D \rho_0 |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} = 0. \quad (3.4)$$

Repeating the similar deduction for $\mathcal{L}_{\tilde{c}} \tilde{\mathbf{u}} + \omega^2 \tilde{\rho} \tilde{\mathbf{u}} = \mathbf{f}$ in $\Omega \setminus \bar{D}$, we have

$$\begin{aligned} \int_{\Omega \setminus \bar{D}} (\mathcal{C}(\mathbf{x}) : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} &= \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{u}}) \cdot \bar{\tilde{\mathbf{u}}} \, ds(\mathbf{x}) - \int_{\partial D} \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{u}}) \cdot \bar{\tilde{\mathbf{u}}} \, ds(\mathbf{x}) \\ &\quad + \omega^2 \int_{\Omega \setminus \bar{D}} \rho(\mathbf{x}) |\tilde{\mathbf{u}}|^2 \, d\mathbf{x}. \end{aligned} \quad (3.5)$$

Similarly, we obtain the following integral equation over $B_r \setminus \bar{\Omega}$

$$\begin{aligned} \int_{B_r \setminus \bar{\Omega}} (\mathcal{C}^e : \nabla \bar{\tilde{\mathbf{u}}^s}) : \nabla \tilde{\mathbf{u}}^s \, d\mathbf{x} &= \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) - \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ &\quad + \omega^2 \rho_e \int_{B_r \setminus \bar{\Omega}} |\tilde{\mathbf{u}}^s|^2 \, d\mathbf{x} - \int_{B_r \setminus \bar{\Omega}} \mathbf{f}(x) \cdot \bar{\tilde{\mathbf{u}}^s} \, d\mathbf{x}. \end{aligned} \quad (3.6)$$

Adding up the integrals (3.4), (3.5) and (3.7), using the transmission conditions given in (3.1), we derive that

$$\begin{aligned} &- \int_D (\mathcal{C}^0 : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} + \int_D (\eta_0 \omega^2 |\tilde{\mathbf{u}}|^2 + \nu \tau_0 \omega^2 |\tilde{\mathbf{u}}|^2) \, d\mathbf{x} - \int_{\Omega \setminus \bar{D}} (\mathcal{C}(x) : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} \\ &+ \omega^2 \int_{\Omega \setminus \bar{D}} \rho(\mathbf{x}) |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} - \int_{B_r \setminus \bar{\Omega}} (\mathcal{C}^e : \nabla \bar{\tilde{\mathbf{u}}^s}) : \nabla \tilde{\mathbf{u}}^s \, d\mathbf{x} + \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ &+ \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\mathbf{u}}^{in} \, ds(\mathbf{x}) + \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) + \omega^2 \int_{B_r \setminus \bar{\Omega}} \rho_e |\tilde{\mathbf{u}}^s|^2 \, d\mathbf{x} \\ &+ \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \bar{\mathbf{u}}^{in} \, ds(\mathbf{x}) = \int_{B_r \setminus \bar{\Omega}} \mathbf{f}(x) \cdot \bar{\tilde{\mathbf{u}}^s} \, d\mathbf{x}. \end{aligned} \quad (3.7)$$

Taking the real and imaginary parts of (3.7), it is easy to obtain that

$$\begin{aligned} \int_D (\mathcal{C}^0 : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} &= \int_D \eta_0 \omega^2 |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} - \int_{\Omega \setminus \bar{D}} (\mathcal{C}(\mathbf{x}) : \nabla \bar{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, d\mathbf{x} + \int_{\Omega \setminus \bar{D}} \omega^2 \Re \rho |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} \\ &\quad - \int_{B_r \setminus \bar{\Omega}} (\mathcal{C}^e : \nabla \bar{\tilde{\mathbf{u}}^s}) : \nabla \tilde{\mathbf{u}}^s \, d\mathbf{x} + \Re \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ &\quad + \Re \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \bar{\mathbf{u}}^{in} \, ds(\mathbf{x}) + \Re \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \bar{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \end{aligned}$$

$$\begin{aligned}
& + \Re \int_{\partial\Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\mathbf{u}^{in}} \, d\mathbf{x} + \omega^2 \rho_e \int_{B_r \setminus \overline{\Omega}} |\tilde{\mathbf{u}}^s|^2 \, d\mathbf{x} \\
& - \Re \int_{B_r \setminus \overline{\Omega}} \mathbf{f}(\mathbf{x}) \cdot \overline{\tilde{\mathbf{u}}^s} \, d\mathbf{x}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\omega^2 \tau_0 \int_D |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} & = -\omega^2 \int_{\Omega \setminus \overline{D}} \Im \rho |\tilde{\mathbf{u}}|^2 \, d\mathbf{x} - \Im \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\
& - \Im \int_{\partial\Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) - \Im \int_{\partial\Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\
& - \Im \int_{\partial\Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) + \Im \int_{B_r \setminus \overline{\Omega}} \mathbf{f}(\mathbf{x}) \cdot \overline{\tilde{\mathbf{u}}^s} \, d\mathbf{x}.
\end{aligned} \tag{3.9}$$

Using the Kron's inequality (cf. [25, 26]), the definition of the norm of conormal derivatives, Hölder inequality, and Corollary 2.1, we can directly obtain the following inequalities

$$\varepsilon \|\nabla \tilde{\mathbf{u}}\|_{L^2(D)^n}^2 \leq C_1 \left(\|\tilde{\mathbf{u}}\|_{L^2(D)^n}^2 + \|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \overline{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \overline{\Omega})^n}^2 \right), \tag{3.10}$$

$$\|\tilde{\mathbf{u}}\|_{L^2(D)^n}^2 \leq C_2 \left(\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \overline{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \overline{\Omega})^n}^2 \right), \tag{3.11}$$

where C_1, C_2 are positive constants only related to $\lambda_0, \mu_0, \eta_0, \tau_0, \omega, \Omega, B_r, \mathcal{C}(\mathbf{x}), \mathcal{C}^e$ and ρ . Thus, we easily prove the important estimate by adding up (3.11) and (3.10),

$$\sqrt{\varepsilon} \|\tilde{\mathbf{u}}\|_{H^1(D)^n} \leq \tilde{C} \left(\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \overline{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \overline{\Omega})^n}^2 \right)^{1/2}, \tag{3.12}$$

where $\tilde{C} = \max\{\sqrt{C_1}, \sqrt{C_3}\}$.

In what follows, we shall prove (3.2) by contradiction. Suppose (3.2) is not true. Without loss of generality, we assume that for any nonnegative integer n , there exists a set of data $(\mathbf{f}^n, \mathbf{u}_n^{in}, \tilde{\mathbf{u}}^n)$, where $\tilde{\mathbf{u}}^n$ is the unique solution of (3.1) with \mathbf{f}^n and \mathbf{u}_n^{in} as inputs, satisfy the restriction

$$\begin{cases} \|\mathbf{f}^n\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} + \|\mathbf{u}_n^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} = 1, \\ \tilde{\mathbf{u}}^n \rightarrow \infty, \quad \text{as } \varepsilon \rightarrow 0. \end{cases} \tag{3.13}$$

We can construct another set of data $(\tilde{\mathbf{f}}^n, \tilde{\mathbf{g}}^{in}, \tilde{\mathbf{g}})$ as follows:

$$\begin{cases} \tilde{\mathbf{f}}^n = \frac{\mathbf{f}^n}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \overline{D})^n}}, & \tilde{\mathbf{g}} = \frac{\tilde{\mathbf{u}}^n}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \overline{D})^n}}, \\ \tilde{\mathbf{g}}^{in} = \frac{\mathbf{u}_n^{in}}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \overline{D})^n}}, & \tilde{\mathbf{g}}^s = \frac{\tilde{\mathbf{u}}^{n,s}}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \overline{D})^n}}, \\ \tilde{\mathbf{u}}^{n,s} = \tilde{\mathbf{u}}^n - \mathbf{u}_n^{in}, & \tilde{\mathbf{g}}^s = \tilde{\mathbf{g}} - \tilde{\mathbf{g}}^{in}, \end{cases} \tag{3.14}$$

where $\tilde{\mathbf{g}}$ is the unique solution of (3.1) with $\tilde{\mathbf{f}}^n$ and $\tilde{\mathbf{g}}^{in}$ as inputs. Obviously,

$$\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \overline{D})^n} = 1, \quad \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \rightarrow 0, \quad \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{3.15}$$

In fact, we can verify that $(\tilde{\mathbf{g}}|_{\Omega \setminus \bar{D}}, \tilde{\mathbf{g}}^s|_{\mathbb{R}^n \setminus \bar{\Omega}})$ is the unique solution to problem (2.8) with $\mathbf{p} = \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{g}})|_{\partial D}$, $\mathbf{h}_1 = \tilde{\mathbf{g}}^{in}|_{\partial \Omega}$, and $\mathbf{h}_2 = \mathcal{T}_{\boldsymbol{\nu}}(\tilde{\mathbf{g}}^{in})|_{\partial \Omega}$. According to Lemma 2.4 and Corollary 2.1, we have

$$\begin{cases} \|\tilde{\mathbf{g}}\|_{H^1(B_R \setminus \bar{D})^n} \leq C(\|\boldsymbol{\nu} \cdot (\mathcal{C}(x) : \nabla \tilde{\mathbf{g}})\|_{H^{-1/2}(\partial D)^n} + \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}), \\ \|\boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{g}})\|_{H^{-1/2}(\partial D)^n} \leq \tilde{C} \varepsilon \|\tilde{\mathbf{g}}\|_{H^1(D)^n}, \end{cases}$$

where C and \tilde{C} are positive constants not relying on ε . Similar to (3.12), we can adopt a completely similar argument for (3.1) with the set of data $(\tilde{\mathbf{f}}^n, \tilde{\mathbf{g}}^{in}, \tilde{\mathbf{g}})$ to derive that

$$\sqrt{\varepsilon} \|\tilde{\mathbf{g}}\|_{H^1(D)^n} \leq (\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} + \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}).$$

Hence,

$$\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

which contradicts with the equality $\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} = 1$. Therefore, the inequality (3.2) holds.

Substituting (3.2) into (3.12), using the inequality of arithmetic and geometric means, one can easily get (3.3). This completes the proof. \square

Next, we shall derive some sharp estimations of the systems (1.14) and (3.1), which can indicate that $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is an effective $\varepsilon^{1/2}$ -realization of $D \oplus (\Omega \setminus \bar{D}; \mathcal{C}, \rho)$ with traction-free obstacle. Firstly, we would like to show that the conormal derivation of $\mathcal{C}(\mathbf{x})$ on the boundary ∂D can be estimated by the input data.

Proposition 3.1. *Let $\tilde{\mathbf{u}} \in H_{loc}^1(\mathbb{R}^n)^n$ be the solution to the system (3.1). Then there exists a constant C such that the following estimate holds for $\varepsilon \ll 1$ and $r > r_0$:*

$$\|\boldsymbol{\nu} \cdot [\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{u}}]\|_{H^{-1/2}(\partial D)^n} \leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \quad (3.16)$$

Proof. By using the transmission on ∂D in the system (3.1), we have

$$\begin{aligned} \|\boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} &= \|\mathcal{T}_{\boldsymbol{\nu}}(\tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} = \|\boldsymbol{\nu} \cdot (\mathcal{C}^e(\mathbf{x}) : \tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} \\ &= \varepsilon \|\boldsymbol{\nu} \cdot (\mathcal{C}_0(\mathbf{x}) : \tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} \leq C_0 \varepsilon \|\tilde{\mathbf{u}}\|_{H^1(D)^n}, \end{aligned}$$

where $\mathcal{C}_0(\mathbf{x})$ is a fourth-rank tensor with Lamé constants both equaling to one. From Lemma 3.1, it can be verified by straightforward calculations that

$$\|\boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} \leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right).$$

The proof is complete. \square

The following proposition show that the solution $\tilde{\mathbf{u}} \in H_{loc}^1(\mathbb{R}^n)^n$ to (3.1) can approximate the solution $\mathbf{u} \in H_{loc}^1(\mathbb{R}^n \setminus \bar{D})^n$ to (1.14) with the respect to the parameter ε .

Proposition 3.2. *Suppose $\tilde{\mathbf{u}} \in H_{loc}^1(\mathbb{R}^n)^n$ is the solution to system (3.1) and $\mathbf{u} \in H_{loc}^1(\mathbb{R}^n \setminus \bar{D})^n$ is the solution to system (1.14). Then there exist two constants ε_0 and C such that the following estimate holds for $\varepsilon < \varepsilon_0$ and $r > r_0$:*

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{H^1(B_r \setminus \bar{D})^n} \leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \quad (3.17)$$

Proof. Let $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}$, where $\tilde{\mathbf{u}}$ and \mathbf{u} are the total fields of system (3.1) and system (1.14), respectively. We can easily verify that $(\mathbf{v}|_{\Omega \setminus \bar{D}}, \mathbf{v}^s|_{\mathbb{R}^n \setminus \bar{\Omega}})$ is the unique solution of system (2.8) with the boundary conditions: $\mathbf{f} = \mathbf{h}_1 = \mathbf{h}_2 = 0$, $\mathbf{p} = \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \mathbf{v}) = \boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{u}})$. Combining $\mathbf{v}^s = \tilde{\mathbf{u}}^s - \mathbf{u}^s = (\tilde{\mathbf{u}} - \mathbf{u}^{in}) - (\mathbf{u} - \mathbf{u}^{in}) = \mathbf{v}$ and Lemma 2.4 with Lemma 3.1, we obtain

$$\begin{aligned} \|\mathbf{v}\|_{H^1(B_r \setminus \bar{D})^n} &\leq C \|\boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \mathbf{v})\|_{H^{-1/2}(\partial D)^n} = C \|\boldsymbol{\nu} \cdot (\mathcal{C}(\mathbf{x}) : \nabla \tilde{\mathbf{u}})\|_{H^{-1/2}(\partial D)^n} \\ &\leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{\mathbf{H}^1(B_r \setminus \bar{\Omega})} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \end{aligned}$$

The completes the proof. \square

We are in the position to give the proof of Theorem 1.1 for Case 1.

Proof of Theorem 1.1 for Case 1. Let $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}$ and $\mathbf{v}^s = \tilde{\mathbf{u}}^s - \mathbf{u}^s$. We note that $\mathbf{v}^s = \mathbf{v}$. We use the following explicit expressions of the scattering amplitude of $\tilde{\mathbf{u}}^s$ and \mathbf{u}^s given by [1, 3] (see more details in [2, 9–11]):

$$\begin{aligned} \tilde{\mathbf{u}}^{p,\infty} &= \frac{\kappa_p^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left(\mathcal{T}_\nu^y (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \right)^\top \cdot \tilde{\mathbf{u}}^s(\mathbf{y}) \right. \\ &\quad \left. - (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \cdot \mathcal{T}_\nu \tilde{\mathbf{u}}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} \mathbf{u}^{p,\infty} &= \frac{\kappa_p^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left(\mathcal{T}_\nu^y (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \right)^\top \cdot \mathbf{u}^s(\mathbf{y}) \right. \\ &\quad \left. - (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \cdot \mathcal{T}_\nu \mathbf{u}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \tilde{\mathbf{u}}^{s,\infty} &= \frac{\kappa_s^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left\{ \mathcal{T}_\nu^y \left((I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \right) \right\}^\top \cdot \tilde{\mathbf{u}}^s(\mathbf{y}) \right. \\ &\quad \left. - (I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \cdot \mathcal{T}_\nu \tilde{\mathbf{u}}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \mathbf{u}^{s,\infty} &= \frac{\kappa_s^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left\{ \mathcal{T}_\nu^y \left((I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \right) \right\}^\top \cdot \mathbf{u}^s(\mathbf{y}) \right. \\ &\quad \left. - (I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \cdot \mathcal{T}_\nu \mathbf{u}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}, \end{aligned} \quad (3.21)$$

where $\tilde{\mathbf{u}}^{p,\infty}$ and $\mathbf{u}^{p,\infty}$ respectively are the longitudinal far-field patterns of $\tilde{\mathbf{u}}^s$ and \mathbf{u}^s , which are normal to \mathbb{S}^{n-1} , whereas $\tilde{\mathbf{u}}^{s,\infty}$ and $\mathbf{u}^{s,\infty}$ respectively are the transversal far-field patterns of $\tilde{\mathbf{u}}^s$ and \mathbf{u}^s , which are tangential to \mathbb{S}^{n-1} . Then

$$\begin{aligned} \tilde{\mathbf{u}}^{p,\infty} - \mathbf{u}^{p,\infty} &= \frac{\kappa_p^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left(\mathcal{T}_\nu^y (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \right)^\top \cdot \mathbf{v}^s(\mathbf{y}) \right. \\ &\quad \left. - (\hat{\mathbf{x}} \hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}}) \cdot \mathcal{T}_\nu \mathbf{v}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}, \end{aligned} \quad (3.22)$$

$$\begin{aligned} \tilde{\mathbf{u}}^{s,\infty} - \mathbf{u}^{s,\infty} &= \frac{\kappa_s^2}{4\pi\omega^2} \int_{\partial B_r} \left\{ \left\{ \mathcal{T}_\nu^y \left((I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \right) \right\}^\top \cdot \mathbf{v}^s(\mathbf{y}) \right. \\ &\quad \left. - (I - \hat{\mathbf{x}} \hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} \cdot \mathcal{T}_\nu \mathbf{v}^s(\mathbf{y}) \right\} ds(y), \quad \hat{\mathbf{x}} \in \mathbb{S}^{n-1}. \end{aligned} \quad (3.23)$$

Let $A = \hat{\mathbf{x}}\hat{\mathbf{x}}^\top e^{-i\kappa_p \hat{\mathbf{x}} \cdot \mathbf{y}} = [A_1, A_2, A_3]$, $B = (I - \hat{\mathbf{x}}\hat{\mathbf{x}}^\top) e^{-i\kappa_s \hat{\mathbf{x}} \cdot \mathbf{y}} = [B_1, B_2, B_3]$, where A_j is the j th column of A and B_j is the j th column of B , $j = 1, 2, 3$. Using the following fact that, for any vector field $\boldsymbol{\psi}$,

$$\begin{cases} |\boldsymbol{\nu} \cdot \nabla \boldsymbol{\psi}| \leq C_1 |\nabla \boldsymbol{\psi}|, \\ |\boldsymbol{\nu} \nabla \cdot \boldsymbol{\psi}| \leq C_2 |\nabla \boldsymbol{\psi}|, \\ |\boldsymbol{\nu} \times \nabla \times \boldsymbol{\psi}| \leq C_3 |\nabla \boldsymbol{\psi}|, \end{cases}$$

where $|\cdot|$ denote Frobenius norm for a matrix or Euclidean norm for a vector, C_1, C_2, C_3 are positive constants, $\boldsymbol{\nu}$ denotes the unit outward normal to the boundary, we have

$$|\mathcal{T}_{\boldsymbol{\nu}}^{\mathbf{y}} A_j| \leq C_4, \quad |\nabla A_j| \leq C_5 \kappa_p, \quad |A_j| \leq C_6 r$$

for $j = 1, 2, 3$, where C_4, C_5, C_6 are positive constants. One can derive the following estimate by using trace theorem, Theorem 2.2 and Proposition 3.2,

$$\begin{aligned} \left\| \tilde{\mathbf{u}}^{\mathbf{p}, \infty} - \mathbf{u}^{\mathbf{p}, \infty} \right\|_{C(\mathbb{S}^{n-1})^n} &\leq C_7 \left(\|\mathbf{v}\|_{H^{1/2}(B_r \setminus \overline{D})^n} + \|\mathcal{T}_{\boldsymbol{\nu}}^{\mathbf{y}} \mathbf{v}\|_{H^{-1/2}(\partial B_r)^n} \right) \\ &\leq C_8 \left(\|\mathbf{v}\|_{H^1(B_r \setminus \overline{D})^n} + \|\mathcal{T}_{\boldsymbol{\nu}}^{\mathbf{y}} \mathbf{v}\|_{H^{-1/2}(\partial B_r)^n} \right) \\ &\leq C_9 \left(\|\mathbf{v}\|_{H^1(B_r \setminus \overline{D})^n} + C_{10} \|\mathbf{v}\|_{H^1(B_r \setminus \overline{D})^n} \right) \\ &\leq C_{10} \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right), \end{aligned}$$

where C_7, C_8, C_9, C_{10} are positive constants depending only on $\omega, \kappa_p, \mathcal{C}(\mathbf{x}), \mathcal{C}^e, B_{r_0} \setminus \overline{\Omega}$ and $B_r \setminus \overline{D}$. Similarly, we obtain that

$$\left\| \tilde{\mathbf{u}}^{\mathbf{p}, \infty} - \mathbf{u}^{\mathbf{p}, \infty} \right\|_{C(\mathbb{S}^{n-1})^n} \leq C_{11} \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right),$$

where $C_{11} = C_{11}(\omega, \kappa_s, B_{r_0} \setminus \overline{\Omega}, B_r \setminus \overline{D}, \mathcal{C}(\mathbf{x}), \mathcal{C}^e)$. Hence,

$$\begin{aligned} \left\| \tilde{\mathbf{u}}^\infty - \mathbf{u}^\infty \right\|_{C(\mathbb{S}^2)^n} &\leq \left\| \tilde{\mathbf{u}}^{\mathbf{p}, \infty} - \mathbf{u}^{\mathbf{p}, \infty} \right\|_{C(\mathbb{S}^{n-1})^n} + \left\| \tilde{\mathbf{u}}^{\mathbf{s}, \infty} - \mathbf{u}^{\mathbf{s}, \infty} \right\|_{C(\mathbb{S}^{n-1})^n} \\ &\leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right), \end{aligned}$$

where $C = C(\omega, \kappa_p, \kappa_s, B_r \setminus \overline{D}, B_{r_0} \setminus \overline{\Omega}, \mathcal{C}(\mathbf{x}), \mathcal{C}^e) \in \mathbb{R}_+$. The proof is complete. \square

4. PROOF OF THEOREM 1.1 FOR CASE 2

In this section, we are committed to proving that $(D; \mathcal{C}^0, \rho_0)$ is an $\varepsilon^{1/2}$ -realization of the rigid obstacle D in the sense of Definition 1.2, where \mathcal{C}^0 is given in the form (1.4) and λ, μ, ρ_0 satisfy the conditions (1.19). An elastic medium $(\Omega; \tilde{\mathcal{C}}, \tilde{\rho})$ is considered, which satisfies that $(\tilde{\mathcal{C}}, \tilde{\rho})|_{\Omega \setminus \overline{D}} = (\mathcal{C}, \rho)|_{\Omega \setminus \overline{D}}$, $(\tilde{\mathcal{C}}, \tilde{\rho})|_D = (\mathcal{C}^0, \rho_0)|_D$ and $(\tilde{\mathcal{C}}, \tilde{\rho})|_{\mathbb{R}^n \setminus \overline{\Omega}} = (\mathcal{C}^e, \rho_e)|_{\mathbb{R}^n \setminus \overline{\Omega}}$. Consider the medium scattering system (3.1) except that (\mathcal{C}^0, ρ_0) satisfying (1.18) is replaced by (\mathcal{C}^0, ρ_0) with the parameters in (1.19).

In the following lemma, we derive that the unique solution $\tilde{\mathbf{u}}$ of (3.1) in regions $B_r \setminus \overline{D}$ and D can be estimated well by \mathbf{u}^{in} and \mathbf{f} , which play an important role in the subsequent proof.

Lemma 4.1. *Let $\tilde{\mathbf{u}}$ be the unique solution of (3.1) with (\mathcal{C}^0, ρ_0) satisfying (1.19). Then there exist positive constants r_0, C_1, C_2 such that the following estimate holds for all $\varepsilon \ll 1$ and $r \geq r_0$:*

$$\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \overline{D})^n} \leq C_1 \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right), \quad (4.1)$$

$$\|\tilde{\mathbf{u}}\|_{H^1(D)^n} \leq C_2 \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \overline{\Omega})^n} \right). \quad (4.2)$$

Proof. Multiplying $\mathcal{L}_{\tilde{\mathcal{C}}} \tilde{\mathbf{u}} + \omega^2 \tilde{\rho} \tilde{\mathbf{u}} = \mathbf{f}$ by $\overline{\tilde{\mathbf{u}}}$ and integrating it over $D, \Omega \setminus \overline{D}, B_r \setminus \overline{\Omega}$, respectively. By adding up them and the transmissions on ∂D and $\partial \Omega$, we have

$$\begin{aligned} & - \int_D (\mathcal{C}^0 : \nabla \overline{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, dx + \int_D (\eta_0 \omega^2 |\tilde{\mathbf{u}}|^2 + \varepsilon^{-1} \tau_0 \omega^2 |\tilde{\mathbf{u}}|^2) \, dx - \int_{\Omega \setminus \overline{D}} (\mathcal{C}(x) : \nabla \overline{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, dx \\ & + \omega^2 \int_{\Omega \setminus \overline{D}} \rho(\mathbf{x}) |\tilde{\mathbf{u}}|^2 \, dx - \int_{B_r \setminus \overline{\Omega}} (\mathcal{C}^e : \nabla \overline{\tilde{\mathbf{u}}^s}) : \nabla \tilde{\mathbf{u}}^s \, dx + \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ & + \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) + \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) + \omega^2 \int_{B_r \setminus \overline{\Omega}} \rho_e |\tilde{\mathbf{u}}^s|^2 \, dx \\ & + \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) = \int_{B_r \setminus \overline{\Omega}} \mathbf{f}(x) \cdot \overline{\tilde{\mathbf{u}}^s} \, dx. \end{aligned} \quad (4.3)$$

Taking the real and imaginary parts of (4.3), it is easy to obtain that

$$\begin{aligned} \int_D (\mathcal{C}^0 : \nabla \overline{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, dx &= \int_D \eta_0 \omega^2 |\tilde{\mathbf{u}}|^2 \, dx - \int_{\Omega \setminus \overline{D}} (\mathcal{C}(\mathbf{x}) : \nabla \overline{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, dx + \int_{\Omega \setminus \overline{D}} \omega^2 \Re \rho |\tilde{\mathbf{u}}|^2 \, dx \\ & - \int_{B_r \setminus \overline{\Omega}} (\mathcal{C}^e : \nabla \overline{\tilde{\mathbf{u}}^s}) : \nabla \tilde{\mathbf{u}}^s \, dx + \Re \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ & + \Re \int_{\partial \Omega} \boldsymbol{\nu} \cdot [\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s] \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) + \Re \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ & + \Re \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) + \omega^2 \rho_e \int_{B_r \setminus \overline{\Omega}} |\tilde{\mathbf{u}}^s|^2 \, dx \\ & - \Re \int_{B_r \setminus \overline{\Omega}} \mathbf{f}(\mathbf{x}) \cdot \overline{\tilde{\mathbf{u}}^s} \, dx \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \omega^2 \varepsilon^{-1} \tau_0 \int_D |\tilde{\mathbf{u}}|^2 \, dx &= -\omega^2 \int_{\Omega \setminus \overline{D}} \Im \rho |\tilde{\mathbf{u}}|^2 \, dx - \Im \int_{\partial B_r} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ & - \Im \int_{\partial \Omega} \boldsymbol{\nu} \cdot [\mathcal{C}^e : \nabla \tilde{\mathbf{u}}^s] \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) - \Im \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\tilde{\mathbf{u}}^s} \, ds(\mathbf{x}) \\ & - \Im \int_{\partial \Omega} \boldsymbol{\nu} \cdot (\mathcal{C}^e : \nabla \mathbf{u}^{in}) \cdot \overline{\mathbf{u}^{in}} \, ds(\mathbf{x}) + \Im \int_{B_r \setminus \overline{\Omega}} \mathbf{f}(\mathbf{x}) \cdot \overline{\tilde{\mathbf{u}}^s} \, dx. \end{aligned} \quad (4.5)$$

Since \mathcal{C}^0 satisfies the uniform Legendre ellipticity condition (1.3) and $\lambda = \varepsilon^{-1} \lambda_0, \mu = \varepsilon^{-1} \mu_0$,

$$\int_D (\mathcal{C}^0 : \nabla \overline{\tilde{\mathbf{u}}}) : \nabla \tilde{\mathbf{u}} \, dx \geq C_0 \varepsilon^{-1} \|\nabla \tilde{\mathbf{u}}\|_{L^2(D)^n}^2,$$

where C_0 only depends on λ_0 and μ_0 . And then we can directly obtain

$$\|\nabla \tilde{\mathbf{u}}\|_{L^2(D)^n}^2 \leq C_1 \varepsilon \left(\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \overline{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \overline{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \overline{D})^n}^2 \right), \quad (4.6)$$

$$\|\tilde{\mathbf{u}}\|_{L^2(D)^n}^2 \leq C_2 \varepsilon \left(\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \bar{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \bar{\Omega})^n}^2 \right), \quad (4.7)$$

where C_1, C_2 are positive constants not related to ε . Thus,

$$\|\tilde{\mathbf{u}}\|_{H^1(D)^n} \leq C \varepsilon^{1/2} \left(\|\tilde{\mathbf{u}}\|_{H^1(B_r \setminus \bar{D})^n}^2 + \|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}^2 + \|\mathbf{f}\|_{L^2(B_r \setminus \bar{\Omega})^n}^2 \right)^{1/2}. \quad (4.8)$$

As we did in proving (3.2), we can construct two sets of data $(\mathbf{f}^n, \mathbf{u}_n^{in}, \tilde{\mathbf{u}}^n)$ and $(\tilde{\mathbf{f}}^n, \tilde{\mathbf{g}}^{in}, \tilde{\mathbf{g}})$ as follows, where $\tilde{\mathbf{u}}^n$ is the unique solution of (3.1) with \mathbf{f}^n and \mathbf{u}_n^{in} as inputs and $\tilde{\mathbf{g}}$ is the unique solution of (3.1) with $\tilde{\mathbf{f}}^n$ and $\tilde{\mathbf{g}}^{in}$ as inputs,

$$\begin{cases} \tilde{\mathbf{f}}^n = \frac{\mathbf{f}^n}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \bar{D})^n}}, & \tilde{\mathbf{g}} = \frac{\tilde{\mathbf{u}}^n}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \bar{D})^n}}, \\ \tilde{\mathbf{g}}^{in} = \frac{\mathbf{u}_n^{in}}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \bar{D})^n}}, & \tilde{\mathbf{g}}^s = \frac{\tilde{\mathbf{u}}^{n,s}}{\|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \bar{D})^n}}, \\ \tilde{\mathbf{u}}^{n,s} = \tilde{\mathbf{u}}^n - \mathbf{u}_n^{in}, & \tilde{\mathbf{g}}^s = \tilde{\mathbf{g}} - \tilde{\mathbf{g}}^{in}, \\ \|\mathbf{f}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\mathbf{u}_n^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} = 1, \\ \|\tilde{\mathbf{u}}^n\|_{H^1(B_r \setminus \bar{D})^n} \rightarrow \infty, & \text{as } \varepsilon \rightarrow 0. \end{cases} \quad (4.9)$$

And then we have

$$\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} = 1, \quad \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \rightarrow 0, \quad \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+, \quad (4.10)$$

$$\|\tilde{\mathbf{g}}\|_{H^1(D)^n} \leq C \varepsilon^{1/2} (\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} + \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}). \quad (4.11)$$

From Lemma 2.5, $(\tilde{\mathbf{g}}|_{\Omega \setminus \bar{D}}, \tilde{\mathbf{g}}^s|_{\mathbb{R}^n \setminus \bar{\Omega}})$ is the unique solution of (2.8) with $\mathbf{p} = \tilde{\mathbf{g}}|_{\partial D}$, $\mathbf{h}_1 = \tilde{\mathbf{g}}^{in}|_{\partial \Omega}$, and $\mathbf{h}_2 = \mathcal{T}_\nu(\tilde{\mathbf{g}}^{in})|_{\partial \Omega}$ such that

$$\begin{aligned} \|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} &\leq d(\|\tilde{\mathbf{g}}\|_{H^{1/2}(\partial D)^n} + \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}), \\ &\leq \tilde{d}(\|\tilde{\mathbf{g}}\|_{H^1(D)^n} + \|\tilde{\mathbf{f}}^n\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} + \|\tilde{\mathbf{g}}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n}), \end{aligned}$$

where d and \tilde{d} are positive constants not relying on ε . Hence, we can see that

$$\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

which contradicts with the equality $\|\tilde{\mathbf{g}}\|_{H^1(B_r \setminus \bar{D})^n} = 1$. Hence, the inequality (4.1) holds.

Next, we prove (4.2). From (4.1) and (4.8), it is easy to obtain that (4.2) holds. This completes the proof. \square

Proposition 4.1. *Suppose $\tilde{\mathbf{u}} \in H_{loc}^1(\mathbb{R}^n)^n$ is the solution to system (3.1) and $\mathbf{u} \in H_{loc}^1(\mathbb{R}^n \setminus \bar{D})^n$ is the solution to system (1.14). Then there exist a constant C such that the following estimate holds for $\varepsilon \ll 1$ and $r > r_0$:*

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{H^1(B_r \setminus \bar{D})^n} \leq C \varepsilon^{1/2} \left(\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n} \right). \quad (4.12)$$

Proof. Let $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}$, where $\tilde{\mathbf{u}}$ and \mathbf{u} are the total fields of system (3.1) and system (1.14), respectively. We can easily verify that $(\mathbf{v}|_{\Omega \setminus \bar{D}}, \mathbf{v}^s|_{\mathbb{R}^n \setminus \bar{\Omega}})$ is the unique solution of system (2.8) with the boundary conditions: $\mathbf{f} = \mathbf{h}_1 = \mathbf{h}_2 = 0$, $\mathbf{p} = \mathbf{v}|_{\partial D} = \tilde{\mathbf{u}}|_{\partial D}$. By using Lemma 2.5, Trace Theorem and Lemma 4.1, we obtain

$$\|\mathbf{v}\|_{H^1(B_r \setminus \bar{D})^n} \leq C_1 \|\tilde{\mathbf{u}}\|_{H^{1/2}(\partial D)^n} \leq C_2 \|\tilde{\mathbf{u}}\|_{H^1(D)^n} \leq C \varepsilon^{1/2} (\|\mathbf{u}^{in}\|_{H^1(B_r \setminus \bar{\Omega})^n} + \|\mathbf{f}\|_{L^2(B_{r_0} \setminus \bar{\Omega})^n}).$$

The proof is complete. \square

Using Lemma 2.5 and Proposition 4.1 we can establish Theorem 1.1 for Case 2 by using similar arguments of Theorem 1.1 for Case 1. The detailed proof is skipped.

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