# On the placement of an obstacle so as to optimize the Dirichlet heat content 

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#### Abstract

We prove that among all doubly connected domains of $\mathbb{R}^{n}(n \geq 2)$ bounded by two spheres of given radii, the Dirichlet heat content at any fixed time achieves its minimum when the spheres are concentric. This is shown to be a special case of a more general theorem concerning the optimal placement of a convex obstacle inside some larger domain so as to maximize or minimize the Dirichlet heat content.


Keywords: Heat kernel, heat content, Dirichlet boundary condition, principle of not feeling the boundary, maximum principles for parabolic equations

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## 1 Introduction

Let $\Omega$ be a bounded path-connected open subset of $\mathbb{R}^{n}(n \geq 2)$ with smooth boundary, on which we consider the (non-negative) Dirichlet Laplacian $\triangle_{\Omega}$ [19, 22] with eigenvalues

$$
\lambda_{1}(\Omega)<\lambda_{2}(\Omega) \leq \lambda_{3}(\Omega) \leq \cdots \leq \lambda_{k}:=\lambda_{k}(\Omega) \leq \cdots \rightarrow \infty
$$

and associated normalized eigenfunctions $\left\{\phi_{k}:=\phi_{k}(\Omega)\right\}_{k=1}^{\infty}$, that is, $-\triangle \phi_{k}=\lambda_{k} \phi_{k}$ in $\Omega$, $\left.\phi_{k}\right|_{\partial \Omega}=0$, and $\int_{\Omega} \phi_{k}^{2}=1$, where it is of no harm to assume that $\left\{\phi_{k}\right\}_{k=1}^{\infty}$ are real valued. It is well known that due to interior regularity [26, §6.3.1] and boundary regularity [26, §6.3.2], $\left\{\phi_{k}\right\}_{k=1}^{\infty}$ are elements of $C^{\infty}(\bar{\Omega})$ [26, §6.5.1]. Recall Weyl's celebrated asymptotic formula [14, 20]

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\lambda_{k}^{n / 2}}{k}=\frac{(4 \pi)^{n / 2} \Gamma\left(\frac{n+2}{2}\right)}{|\Omega|} \tag{1.1}
\end{equation*}
$$

where $|\Omega|$ means the volume of $\Omega$, and an optimal uniform bound of Grieser 35] asserting

$$
\begin{equation*}
\max _{x \in \Omega}\left|\phi_{k}(x)\right| \leq C_{\Omega} k^{\frac{n-1}{2 n}} \quad(k \in \mathbb{N}) \tag{1.2}
\end{equation*}
$$

where $C_{\Omega}$ is some positive constant depending only on $\Omega$.

The Dirichlet heat kernel $p_{\Omega}(x, y, t)$ for $\Omega$ on $\bar{\Omega} \times \bar{\Omega} \times(0, \infty)$ was originally introduced [20, 46] as the classical solution to the heat equation

$$
\triangle_{x} p_{\Omega}=\frac{\partial p_{\Omega}}{\partial t}
$$

in $\Omega \times(0, \infty)$ subject to $p(x, y, t)=0$ whenever $x \in \partial \Omega$ and

$$
\lim _{t \rightarrow 0} \int_{\Omega} p_{\Omega}(x, y, t) u(y) d y=u(x)
$$

uniformly for every function $u$ continuous on $\bar{\Omega}$ and vanishing on $\partial \Omega$. It is uniquely determined and traditionally written via Mercer's theorem in functional analysis 45] as

$$
\begin{equation*}
p_{\Omega}(x, y, t)=\sum_{k=1}^{\infty} e^{-\lambda_{k} t} \phi_{k}(x) \phi_{k}(y) \quad((x, y, t) \in \bar{\Omega} \times \bar{\Omega} \times(0, \infty)) \tag{1.3}
\end{equation*}
$$

According to (1.1) and (1.2), we see that the series in (1.3) converges uniformly on $\bar{\Omega} \times \bar{\Omega} \times[\epsilon, \infty)$ for every $\epsilon>0$. Actually, by considering (1.1), (1.2) as well as the elliptic regularity [26, §6.3], one easily gets that $p_{\Omega}$ is a smooth function satisfying

$$
\left(\partial^{\alpha} p_{\Omega}\right)(x, y, t)=\sum_{k=1}^{\infty} \partial^{\alpha}\left(e^{-\lambda_{k} t} \phi_{k}(x) \phi_{k}(y)\right)
$$

for an arbitrary multi-partial derivative $\partial^{\alpha}$ of $2 n+1$ variables [21]. A striking property of Dirichlet heat kernel is that $p_{\Omega}$ is positive in the interior region $\Omega \times \Omega \times(0, \infty)$ [21, 36]. One may also understand heat kernel from the viewpoint of probability theory [64, 65], or relate it to wave kernel by considering the functional calculus

$$
e^{-t \triangle_{\Omega}}=\frac{1}{2 \sqrt{\pi t}} \int_{\mathbb{R}} \cos \left(s \sqrt{\triangle_{\Omega}}\right) e^{-\frac{s^{2}}{4 t}} d s \quad(t>0)
$$

[16, 48, 63].
There are many derived concepts from Dirichlet heat kernel including the Dirichlet heat trace of $\Omega$ defined by

$$
\begin{equation*}
Z_{\Omega}(t)=\int_{\Omega} p_{\Omega}(x, x, t) d x=\sum_{k=1}^{\infty} e^{-\lambda_{k} t} \quad(t>0) \tag{1.4}
\end{equation*}
$$

and the Dirichlet heat content of $\Omega$ given as

$$
\begin{equation*}
H_{\Omega}(t)=\int_{\Omega} \int_{\Omega} p_{\Omega}(x, y, t) d x d y=\sum_{k=1}^{\infty} e^{-\lambda_{k} t}\left(\int_{\Omega} \phi_{k}(x) d x\right)^{2} \quad(t>0) \tag{1.5}
\end{equation*}
$$

Both functions of positive time are known to have full short-time asymptotic expansions:

$$
\begin{equation*}
Z_{\Omega}(t) \sim \frac{|\Omega|}{(4 \pi t)^{n / 2}}+\sum_{k=1}^{\infty} \alpha_{k}(\Omega) t^{\frac{k-n}{2}} \quad(t \rightarrow 0) \tag{1.6}
\end{equation*}
$$

$$
\begin{equation*}
H_{\Omega}(t) \sim|\Omega|+\sum_{k=1}^{\infty} \beta_{k}(\Omega) t^{\frac{k}{2}} \quad(t \rightarrow 0) \tag{1.7}
\end{equation*}
$$

A tremendous amount of effort has been put to establish the existence of both formulae and compute the coefficients in terms of the geometry of $\partial \Omega$ in the current [7, 8, 9, 52, 53, 61] and more specific (such as smooth planar regions [66], polygonal domains [10, 11]) or general (such as vector-valued elliptic operators [33, 34]) settings. The Dirichlet spectral zeta function for $\Omega$ is defined as the meromorphic extension of

$$
\begin{equation*}
\zeta_{\Omega}: z \mapsto \sum_{k=1}^{\infty} \frac{1}{\lambda_{k}^{z}}=\frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-1} Z_{\Omega}(t) d t \quad\left(\operatorname{Re}(z)>\frac{n}{2}\right) \tag{1.8}
\end{equation*}
$$

to the complex plane $\mathbb{C}$ whose singularities can be deduced from (1.6) and basic properties of the Mellin transform [27] to be simple poles at $\frac{n}{2}, \frac{n-1}{2}, \ldots, \frac{1}{2}$, and negative half-integers. The regularized Dirichlet determinant of $\Omega$, denoted by $\operatorname{det}(\Omega)$, is then defined as $\exp \left(-\frac{d \zeta_{\Omega}}{d z}(0)\right)$. In much the same way,

$$
\begin{equation*}
\mathcal{T}_{\Omega}: z \mapsto \sum_{k=1}^{\infty} \frac{\left(\int_{\Omega} \phi_{k}\right)^{2}}{\lambda_{k}^{z}}=\frac{1}{\Gamma(z)} \int_{0}^{\infty} t^{z-1} H_{\Omega}(t) d t \quad\left(\operatorname{Re}(z)>\frac{n}{2}\right) \tag{1.9}
\end{equation*}
$$

admits a meromorphic extension to $\mathbb{C}$ whose singularities are simple poles at negative halfintegers. We also note that

$$
\begin{equation*}
k!\cdot \mathcal{T}_{\Omega}(k)=k \int_{0}^{\infty} t^{k-1} H_{\Omega}(t) d t \quad(k \in \mathbb{N}) \tag{1.10}
\end{equation*}
$$

is called $k$-th exit time moment of $\Omega[18,62]$.
In this paper we are particularly interested in $H_{\Omega}(t)$, which represents the total amount of heat at time $t$ of the (weak) solution to the heat equation $\triangle_{x} \psi=\frac{\partial \psi}{\partial t}$ in $\Omega \times(0, \infty)$ with initial temperature 1 and zero boundary conditions on $\partial \Omega \times(0, \infty)$. Since $\phi_{1}$ can be assumed to be positive in $\Omega$, we see that as $t$ goes to infinity, the dominating term of the series in (1.5) is the first one. In contrast to the heat trace series in (1.4), some higher terms in (1.5) could vanish: if $\Omega$ is a ball [30, 39] or an annulus [47], then

$$
\int_{\Omega} \phi_{2}=\cdots=\int_{\Omega} \phi_{n+1}=0
$$

which in turn yields

$$
H_{\Omega}(t)=e^{-\lambda_{1} t}\left(\int_{\Omega} \phi_{1}\right)^{2}+\sum_{k=n+2}^{\infty} e^{-\lambda_{k} t}\left(\int_{\Omega} \phi_{k}\right)^{2} \quad(t>0) .
$$

A beautiful result of Burchard and Schmuckenschläger [13] claims that $H_{\Omega}(t) \leq H_{B}(t)$ for all $t>0$ provided that $B$ is an open ball in $\mathbb{R}^{n}$ having the same volume of $\Omega$. To compare, Luttinger [50] established $Z_{\Omega}(t) \leq Z_{B}(t)$ still for all $t>0$, which, to the best of the author's knowledge, may be the first all-time comparison result in spectral theory. An advantage of such kind of theorems is that they may deliver more information than isoperimetric inequalities for a single quantity such as the Faber-Krahn inequality [14, 15] for the lowest eigenvalue $\lambda_{1}(\Omega)$.

The main purpose of this paper is to establish an all-time comparison theorem for the Dirichlet heat content of domains with "holes". Our result was motivated by the paper [24] by El Soufi and Harrell concerning a similar result for heat trace. Let $B$ be an open ball of radius $r_{1}$ in $\mathbb{R}^{n}$ such that its closure $\bar{B}$ is contained in another larger concentric open ball $\mathscr{B}$ of radius $r_{2}$ in $\mathbb{R}^{n}$. Consider the Dirichlet eigenvalue problem on $\mathscr{A}_{s}:=\mathscr{B} \backslash(\bar{B}+s V)$, where $V$ is a fixed unit vector in $\mathbb{R}^{n}$ and $s \in\left[0, r_{2}-r_{1}\right)$ is a displacement parameter.


It is known that

- (D1) $\lambda_{1}\left(\mathscr{A}_{s}\right)$ is a strictly decreasing function of $s$ [38, 43, 60] (see also 40]),
- (D2) $\lambda_{2}\left(\mathscr{A}_{s}\right)$ attains its maximal value uniquely at $s=0$ [25],
- (D3) $Z_{\mathscr{A}_{s}}(t)$ is a non-decreasing function of $s$ for every $t>0$ [24],
- (D4) $Z_{\mathscr{A}_{s_{1}}}(t)<Z_{\mathscr{A}_{s_{2}}}(t)$ as long as $s_{1}<s_{2}$ are fixed and $t>0$ is sufficiently small [6],
- (D5) $\operatorname{det}\left(\mathscr{A}_{s}\right)$ is a strictly decreasing function of $s$ [24].

We mention that (D5) also follows from (D3) and (D4) because due to Kac's principle of not feeling the boundary [4, 5, 48], the short-time asymptotic coefficients $\left\{\alpha_{k}\left(\mathscr{A}_{s}\right)\right\}_{k=1}^{\infty}$ in (1.6) (with $\Omega$ replaced by $\mathscr{A}_{s}$ ) are independent of $s$, from which it is routine [24, 67] to deduce

$$
\frac{d \zeta_{\mathscr{A}_{s_{2}}}}{d z}(0)-\frac{d \zeta_{\mathscr{A}_{s_{1}}}}{d z}(0)=\int_{0}^{\infty} \frac{Z_{\mathscr{A}_{s_{2}}}(t)-Z_{\mathscr{A}_{s_{1}}}(t)}{t} d t
$$

Apart from the Dirichlet eigenvalue problem on $\mathscr{A}_{s}$, one may also consider Neumann boundary conditions [2, 68], or mixed boundary conditions [40, 58], or the Steklov eigenvalue problem [31, 55], or the mixed Steklov-Dirichlet problem [41, 68], or the $p$-Laplacian [1, 17], and so on. Our main result reads as follows.

Theorem 1.1. For any $t>0, H_{\mathscr{A}_{s}}(t)$ is a strictly increasing function of $s$.
As an immediate corollary, we see that any order exit time moment of $\mathscr{A}_{s}$ is also a strictly increasing function of the displacement parameter $s$.

Our proof relies on (weak, strong, and Friedman's) maximum principles for parabolic equations [26, 28, 29, 49, 59], Kac's principle of not feeling the boundary, and Savo's variational formula for Dirichlet heat content 62].

## 2 Heat kernel comparison



We assume that $\Omega$, a bounded connected open subset of $\mathbb{R}^{n}$ with smooth boundary, can be written as the union of pairwise disjoint non-empty sets

$$
\begin{equation*}
\Omega=\Omega_{--} \cup \Omega_{-} \cup\left(H \cap \Omega \cap \partial \Omega_{+}\right) \cup \Omega_{+}, \tag{2.1}
\end{equation*}
$$

where $H$ (in black) is a hyperplane, $\Omega_{+}$(in purple) and $\Omega_{-}$(in blue) are contained in distinct connected components of $\mathbb{R}^{n} \backslash H$ and they are symmetric with respect to the hyperplane $H$, and $\Omega_{--}$(in red) adheres somewhere to $\Omega_{-}$but nowhere to $\Omega_{+}$. In other words, $\Omega \backslash\left(H \cap \Omega \cap \partial \Omega_{+}\right)$ consists of two parts, one is $\Omega_{+}$on one side of $\mathbb{R}^{n} \backslash H$, the other is $\Omega_{--} \cup \Omega_{-}$, and the reflection image of $\Omega_{+}$with respect to $H$ is a proper subset of $\Omega_{--} \cup \Omega_{-}$. We remark that $\Omega$ may have several different decompositions of the form (2.1), and each partition is uniquely determined by identifying $\Omega_{+}$. For any $x \in \overline{\Omega_{+}}$, we let $x^{*}$ stand for the reflection point of $x$ with respect to $H$. As continuous curves in $\Omega$ connecting two points of $\Omega_{+}$may have to pass through $\Omega_{--}$, we see that $\Omega_{+}$is not necessarily connected. The reflection image of a continuous curve in $\Omega_{+}$is a continuous curve in $\Omega_{-}$, so $\Omega_{--} \cup \Omega_{-}$is easily seen to be connected. To freely apply maximum principles for parabolic equations in the space-time $\overline{\Omega_{+}} \times[0, \infty)$, we further assume that $\Omega_{+}$is connected, unless otherwise stated. Later on we will see that this condition can be dropped in many situations (see Remark 2.8).
M. van den Berg [4] showed that

$$
\begin{equation*}
\left|p_{\Omega}(x, y, t)-\frac{1}{(4 \pi t)^{n / 2}} \exp \left(-\frac{|x-y|^{2}}{4 t}\right)\right| \leq \frac{2 n}{(4 \pi t)^{n / 2}} \exp \left(-\frac{3-2 \sqrt{2}}{n t} d_{x}^{2}\right) \tag{2.2}
\end{equation*}
$$

for all $x, y \in \Omega$ and $t>0$, where $d_{x}:=d(x, \partial \Omega)$ is the distance of $x$ from $\partial \Omega$. Although this quantified version of Kac's principle of not feeling the boundary is not optimal for small times [5, 48, 51], it well serves the purpose of the current paper.

We will consider smooth ( $C^{2}$ in $x$ and $C^{1}$ in $t$ are actually enough) solutions to the heat equation

$$
\begin{equation*}
\triangle_{x} \psi=\frac{\partial \psi}{\partial t} \tag{2.3}
\end{equation*}
$$

in $\Omega_{+} \times(0, T)$, where $T$ is usually set to be arbitrarily large. Suppose some solution $\psi$ admits a unique continuous extension to $\overline{\Omega_{+}} \times[0, T]$, then the weak maximum principle for parabolic equations [26, p. 368] ensures that both the maximal and minimal values of $\psi$ over the compact cylinder $\overline{\Omega_{+}} \times[0, T]$ are attained on the parabolic boundary $\left(\partial \Omega_{+} \times(0, T]\right) \cup\left(\Omega_{+} \times\{0\}\right)$ of the cylinder. This principle does not exclude the possibility of attaining extremal values inside $\Omega_{+} \times(0, T]$, and if that situation indeed occurs, say for example at $\left(x_{0}, t_{0}\right) \in \Omega_{+} \times(0, T]$, then the strong maximum principle for parabolic equations [26, p. 375] guarantees that $\psi$ is a constant on $\overline{\Omega_{+}} \times\left[0, t_{0}\right]$. We will also apply Friedman's strong maximum principle [28], which generalizes Hopf's maximum principle [32, 37, 42] from elliptic equations to parabolic ones.

Lemma 2.1. For any $x \in H \cap \Omega \cap \partial \Omega_{+}, y \in \overline{\Omega_{+}}$and $t>0$, one has $p_{\Omega}(x, y, t) \leq p_{\Omega}\left(x, y^{*}, t\right)$.


Proof. Let $x \in H \cap \Omega \cap \partial \Omega_{+}$be fixed, and consider

$$
\psi(y, t):=p_{\Omega}(x, y, t)-p_{\Omega}\left(x, y^{*}, t\right)
$$

on $\overline{\Omega_{+}} \times(0, \infty)$. It is well known that $\psi$ is a continuous function (see the Introduction). Since $x$ is an interior point of $\Omega$, one gets $d_{x}>0$. Given an arbitrary $y \in \overline{\Omega_{+}}$, we now have two cases to consider.

Case 1: Suppose $|x-y|<d_{x}$. Obviously, $|x-y|=\left|x-y^{*}\right|$. It then follows from (2.2) that

$$
|\psi(y, t)| \leq \frac{4 n}{(4 \pi t)^{n / 2}} \exp \left(-\frac{3-2 \sqrt{2}}{n t} d_{x}^{2}\right)
$$

for all $t>0$.
Case 2: Suppose $|x-y| \geq d_{x}$. Since the Dirichlet heat kernel of an arbitrary open domain is bounded above by the full space counterpart (see e.g. [20, (3.3)]), one gets

$$
|\psi(y, t)| \leq \frac{2}{(4 \pi t)^{n / 2}} \exp \left(-\frac{|x-y|^{2}}{4 t}\right) \leq \frac{2}{(4 \pi t)^{n / 2}} \exp \left(-\frac{d_{x}^{2}}{4 t}\right)
$$

for all $t>0$.
Considering both cases, we see that $\psi(y, t)$ converges uniformly to the zero function on $\overline{\Omega_{+}}$as $t$ goes to 0 . Consequently, $\psi$ admits a unique continuous zero extension to $\overline{\Omega_{+}} \times[0, \infty)$. Next, let $y \in \partial \Omega_{+}$and $t>0$ be arbitrary. If $y \in H \cap \Omega \cap \partial \Omega_{+}$, then $y=y^{*}$, hence $\psi(y, t)=0$; else suppose $y \in \partial \Omega_{+} \backslash\left(H \cap \Omega \cap \partial \Omega_{+}\right)$, then $p_{\Omega}(x, y, t)=0$, which implies that $\psi(y, t) \leq 0$. To summarize, we see that $\psi$ is non-positive on the parabolic boundary of $\overline{\Omega_{+}} \times[0, \infty)$. Obviously, $\psi$ is a smooth solution (see the Introduction) to the equation (2.3) in $\Omega_{+} \times(0, T)$ (the variable $x$ in (2.3) is accordingly changed to $y$ ) for arbitrarily large $T$, hence it follows the weak maximum principle for parabolic equations by letting $T \rightarrow \infty$ that $\psi$ is non-positive on $\overline{\Omega_{+}} \times[0, \infty)$. This proves the lemma.

Corollary 2.2. For any $x, y \in \overline{\Omega_{+}}$and $t>0$, one has $p_{\Omega}(x, y, t) \leq p_{\Omega}\left(x^{*}, y^{*}, t\right)$.


Proof. Let $\varphi$ be an arbitrary non-negative function in $C_{c}^{\infty}\left(\Omega_{+}\right)$, and let $d_{\varphi}$ denote the distance between the support of $\varphi$, denoted as usual as $\operatorname{supp}(\varphi)$, and $\partial \Omega_{+}$. Consider

$$
\psi(x, t):=\int_{\Omega_{+}}\left(p_{\Omega}(x, y, t)-p_{\Omega}\left(x^{*}, y^{*}, t\right)\right) \varphi(y) d y
$$

on $\overline{\Omega_{+}} \times(0, \infty)$, which is easily seen to be a continuous function. Given an arbitrary $x \in \overline{\Omega_{+}}$, we now have two cases to consider.

Case 1: Suppose $d(x, \operatorname{supp}(\varphi))<\frac{d_{\varphi}}{2}$. This condition implies $d\left(x, \partial \Omega_{+}\right) \geq \frac{d_{\varphi}}{2}$. Since any continuous curve with starting point $x$ has to leave $\Omega_{+}$first if it wants to escape from $\Omega$, one gets $d(x, \partial \Omega) \geq d\left(x, \partial \Omega_{+}\right)$. Thus $d_{x} \geq \frac{d_{\varphi}}{2}$. Similarly, we have $d\left(x^{*}, \partial \Omega\right) \geq d\left(x^{*}, \partial \Omega_{-}\right)$, which combined with $d\left(x^{*}, \partial \Omega_{-}\right)=d\left(x, \partial \Omega_{+}\right)$, yields $d_{x^{*}} \geq \frac{d_{\varphi}}{2}$. It then follows from (2.2) by considering $|x-y|=\left|x^{*}-y^{*}\right|$ for all $y \in \Omega_{+}$that

$$
|\psi(x, t)| \leq \frac{4 n}{(4 \pi t)^{n / 2}} \exp \left(-\frac{3-2 \sqrt{2}}{4 n t} d_{\varphi}^{2}\right) \int_{\Omega_{+}} \varphi(y) d y
$$

for all $t>0$.
Case 2: Suppose $d(x, \operatorname{supp}(\varphi)) \geq \frac{d_{\varphi}}{2}$. Then $\left|x^{*}-y^{*}\right|=|x-y| \geq \frac{d_{\varphi}}{2}$ for all $y \in \operatorname{supp}(\varphi)$. Since the Dirichlet heat kernel of an arbitrary open domain is bounded above by the full space counterpart, one gets

$$
|\psi(x, t)| \leq \frac{2}{(4 \pi t)^{n / 2}} \exp \left(-\frac{d_{\varphi}^{2}}{16 t}\right) \int_{\Omega_{+}} \varphi(y) d y
$$

for all $t>0$.
Considering both cases, we see that $\psi(x, t)$ converges uniformly to the zero function on $\overline{\Omega_{+}}$as $t$ goes to 0 . Consequently, $\psi$ admits a unique continuous zero extension to $\overline{\Omega_{+}} \times[0, \infty)$. Next, let $x \in \partial \Omega_{+}$and $t>0$ be arbitrary. If $x \in H \cap \Omega \cap \partial \Omega_{+}$, then it follows from Lemma 2.1 that $\psi(x, t) \leq 0$; else suppose $x \in \partial \Omega_{+} \backslash\left(H \cap \Omega \cap \partial \Omega_{+}\right)$, then $p_{\Omega}(x, y, t)=0$ for all $y \in \operatorname{supp}(\varphi)$, which implies that $\psi(x, t) \leq 0$. To summarize, we see that $\psi$ is non-positive on the parabolic boundary of $\overline{\Omega_{+}} \times[0, \infty)$. Obviously, $\psi$ is a smooth solution to the equation (2.3) in $\Omega_{+} \times(0, T)$ for arbitrarily large $T$, hence it follows the weak maximum principle for parabolic equations by letting $T \rightarrow \infty$ that $\psi$ is non-positive on $\overline{\Omega_{+}} \times[0, \infty)$.

Finally, letting $y \in \Omega_{+}$be fixed and $\left\{\varphi_{i} \in C_{c}^{\infty}\left(\Omega_{+}\right)\right\}_{i=1}^{\infty}, \varphi_{i} \geq 0$, be an approximation of the Dirac measure at $y$, we get $p_{\Omega}(x, y, t) \leq p_{\Omega}\left(x^{*}, y^{*}, t\right)$ for all $x \in \overline{\Omega_{+}}$and $t>0$. This suffices to prove the corollary by continuity.

Remark 2.3. We should remark that Corollary 2.2 was first given by El Soufi and Harrell in [24, line 8, p. 890]. They claimed that for any fixed $x \in \partial \Omega_{+}$, one can apply the weak maximum principle for parabolic equations [26, §7.1] to the solution

$$
\psi(y, t) \mapsto p_{\Omega}(x, y, t)-p_{\Omega}\left(x^{*}, y^{*}, t\right)
$$

of the heat equation $\triangle_{y} \psi=\frac{\partial \psi}{\partial t}$ in $\Omega_{+} \times(0, \infty)$. But $\psi$ does not admit any continuous extension to $\overline{\Omega_{+}} \times[0, \infty)$ if $x$ is an element of $\partial \Omega_{+} \backslash\left(H \cap \Omega \cap \partial \Omega_{+}\right)$such that its reflection $x^{*}$ with respect to $H$ lies inside $\Omega$ (see the figure at the end of this remark), because due to Kac's principle of not feeling the boundary, we should have

$$
\psi(x, t)=p_{\Omega}(x, x, t)-p_{\Omega}\left(x^{*}, x^{*}, t\right)=-p_{\Omega}\left(x^{*}, x^{*}, t\right) \rightarrow-\infty \quad(t \rightarrow 0)
$$



In the rest part of the section we will prepare some strict inequalities for later use.
Theorem 2.4. For any $x, y \in \Omega_{+}$and $t>0$, one has $p_{\Omega}(x, y, t)<p_{\Omega}\left(x^{*}, y^{*}, t\right)$.
Proof. We argue by contradiction and suppose, by considering Corollary 2.2, that there exists $\left(x_{0}, y_{0}, t_{0}\right) \in \Omega_{+} \times \Omega_{+} \times(0, \infty)$ such that $p_{\Omega}\left(x_{0}, y_{0}, t_{0}\right)=p_{\Omega}\left(x_{0}^{*}, y_{0}^{*}, t_{0}\right)$. Define

$$
\psi(x, t):=p_{\Omega}\left(x, y_{0}, t\right)-p_{\Omega}\left(x^{*}, y_{0}^{*}, t\right)
$$

on $\overline{\Omega_{+}} \times(0, \infty)$. It is straightforward to check that $\psi$ is a smooth solution to the equation (2.3) in $\Omega_{+} \times(0, \infty)$, and due to Corollary[2.2, $\psi$ is everywhere non-positive. Note also $\psi\left(x_{0}, t_{0}\right)=0$. Thus we can apply, the strong maximum principle for parabolic equations to the restriction of $\psi$ to $\overline{\Omega_{+}} \times\left[\frac{t_{0}}{2}, t_{0}\right]$, to see that $\psi$ is identical to zero on $\overline{\Omega_{+}} \times\left[\frac{t_{0}}{2}, t_{0}\right]$. In particular, $\psi\left(\cdot, t_{0}\right) \equiv 0$ is equivalent to the fact that

$$
p_{\Omega}\left(x, y_{0}, t_{0}\right)=p_{\Omega}\left(x^{*}, y_{0}^{*}, t_{0}\right)
$$

for all $x \in \overline{\Omega_{+}}$. Recall that the reflection image of $\Omega_{+}$with respect to $H$ is a proper subset of $\Omega_{--} \cup \Omega_{-}$, so there exists an element $\widetilde{x}$ of $\partial \Omega_{+} \backslash\left(\Omega \cap H \cap \partial \Omega_{+}\right)$such that its reflection $\widetilde{x}^{*}$ with respect to $H$ lies inside $\Omega$. At this point we get

$$
p_{\Omega}\left(\widetilde{x}, y_{0}, t_{0}\right)=p_{\Omega}\left(\widetilde{x}^{*}, y_{0}^{*}, t_{0}\right)
$$

which is an absurd fact because the left hand side is zero while the right hand side is positive. This finishes the proof of the theorem.

Remark 2.5. Based on the "log-concavity" results of Brascamp and Lieb [12], Bañuelos et al. [3, Prop. 5.2] showed that for any $t>0, p_{\mathbb{U}}(x, x, t)$ is a strictly decreasing function of $|x|$, where $\mathbb{U}$ is the open ball $\left\{x \in \mathbb{R}^{n}:|x|<1\right\}$. We remark that Theorem 2.4 provides a second proof of this result by suitably identifying $\Omega$ and $\Omega_{+}$. To compare, Pascu and Gageonea [56, 57] confirmed a conjecture of Laugesen and Morpurgo [44], which asserts that the diagonal of the Neumann heat kernel for $\mathbb{U}$ is a strictly increasing radial function for any fixed time.
Theorem 2.6. For any $x, y \in \Omega_{+}$and $t>0$, one has

$$
p_{\Omega}(x, y, t)+p_{\Omega}\left(x, y^{*}, t\right)<p_{\Omega}\left(x^{*}, y, t\right)+p_{\Omega}\left(x^{*}, y^{*}, t\right)
$$



As the proof of Theorem 2.6 is so similar to those of Corollary 2.2 and Theorem 2.4 we only present a sketch.

Step 1: Consider the continuous function

$$
\psi(x, t):=\int_{\Omega_{+}}\left(p_{\Omega}(x, y, t)+p_{\Omega}\left(x, y^{*}, t\right)-p_{\Omega}\left(x^{*}, y, t\right)-p_{\Omega}\left(x^{*}, y^{*}, t\right)\right) \varphi(y) d y
$$

on $\overline{\Omega_{+}} \times(0, \infty)$, where $\varphi \in C_{c}^{\infty}\left(\Omega_{+}\right)$is non-negative and fixed.
Step 2: Show, by applying (2.2) suitably, that $\psi$ admits a unique continuous zero extension to $\overline{\Omega_{+}} \times[0, \infty)$ that is everywhere non-positive on the parabolic boundary of $\overline{\Omega_{+}} \times[0, \infty)$. The weak maximum principle for parabolic equations then ensures that $\psi$ is non-positive on $\overline{\Omega_{+}} \times[0, \infty)$.

Step 3: Taking a non-negative approximation of the Dirac measure at an arbitrary $y \in \Omega_{+}$ can yield

$$
p_{\Omega}(x, y, t)+p_{\Omega}\left(x, y^{*}, t\right) \leq p_{\Omega}\left(x^{*}, y, t\right)+p_{\Omega}\left(x^{*}, y^{*}, t\right)
$$

for all $x \in \overline{\Omega_{+}}$and $t>0$.
Step 4: Suppose there was $\left(x_{0}, y_{0}, t_{0}\right) \in \Omega_{+} \times \Omega_{+} \times(0, \infty)$ such that

$$
p_{\Omega}\left(x_{0}, y_{0}, t_{0}\right)+p_{\Omega}\left(x_{0}, y_{0}^{*}, t_{0}\right)=p_{\Omega}\left(x_{0}^{*}, y_{0}, t_{0}\right)+p_{\Omega}\left(x_{0}^{*}, y_{0}^{*}, t_{0}\right) .
$$

Then apply the strong maximum principle for parabolic equations to the smooth solution

$$
\psi(x, t):=p_{\Omega}\left(x, y_{0}, t\right)+p_{\Omega}\left(x, y_{0}^{*}, t\right)-p_{\Omega}\left(x^{*}, y_{0}, t\right)-p_{\Omega}\left(x^{*}, y_{0}^{*}, t\right)
$$

of the equation (2.3) in $\Omega_{+} \times\left(\frac{t_{0}}{2}, t_{0}\right)$ to get $\psi\left(x, t_{0}\right)=0$ for all $x \in \overline{\Omega_{+}}$. But by picking an element $\widetilde{x}$ of $\partial \Omega_{+} \backslash\left(\Omega \cap H \cap \partial \Omega_{+}\right)$such that its reflection $\widetilde{x}^{*}$ with respect to $H$ lies inside $\Omega$, we get $\psi\left(\widetilde{x}, t_{0}\right)<0$, a contradiction.

Theorem 2.7. For any $x \in \Omega_{+}, y \in \Omega_{--} \cup \Omega_{-}$and $t>0$, one has $p_{\Omega}(x, y, t)<p_{\Omega}\left(x^{*}, y, t\right)$.


Proof. Let $y \in \Omega_{--} \cup \Omega_{-}$be fixed, and consider the continuous function

$$
\psi(x, t):=p_{\Omega}(x, y, t)-p_{\Omega}\left(x^{*}, y, t\right)
$$

on $\overline{\Omega_{+}} \times(0, \infty)$. It is straightforward to check that $\psi$ solves the equation (2.3) in $\Omega_{+} \times(0, \infty)$ and is non-positive on $\partial \Omega_{+} \times(0, \infty)$. Hence for any $0<\epsilon<T<\infty$, the weak maximum principle for parabolic equations ensures that

$$
\max _{(x, t) \in \bar{\Omega}_{+} \times[\epsilon, T]} \psi(x, t) \leq \max \left\{0, \sup _{x \in \Omega_{+}} \psi(x, \epsilon)\right\} .
$$

We then note

$$
\begin{aligned}
\sup _{x \in \Omega_{+}} \psi(x, \epsilon) & \leq \sup _{x \in \Omega_{+}} p_{\Omega}(x, y, \epsilon) \\
& \leq \sup _{x \in \Omega_{+}} \frac{1}{(4 \pi \epsilon)^{n / 2}} \exp \left(-\frac{|x-y|^{2}}{4 \epsilon}\right) \\
& =\frac{1}{(4 \pi \epsilon)^{n / 2}} \exp \left(-\frac{d\left(y, \Omega_{+}\right)^{2}}{4 \epsilon}\right)
\end{aligned}
$$

where it is crucial to mention that

$$
\begin{equation*}
d\left(y, \Omega_{+}\right)>0 \tag{2.4}
\end{equation*}
$$

Consequently, by combining the above inequalities and letting first $\epsilon \rightarrow 0$ then $T \rightarrow \infty$, we see that $\psi$ is non-positive on $\overline{\Omega_{+}} \times(0, \infty)$. The remaining issue of proving $\psi$ being strictly negative in $\Omega_{+} \times(0, \infty)$ is similar to the corresponding part of Theorem [2.4, thus omitted.

Remark 2.8. According to the proof of Theorem 2.7, we point out that the connectedness assumption on $\Omega_{+}$in Lemma 2.1, Corollary 2.2, Theorems 2.4, 2.6 and 2.7 all can be dropped as it suffices to consider the heat equation (2.3) in the Cartesian product of each individual connected component of $\Omega_{+}$with $(0, \infty)$. Take Corollary 2.2 for example: in case $x$ and $y$ stay in the same connected component of $\Omega_{+}$, then apply Corollary 2.2 itself by redefining $\Omega_{+}$ as this component (see the top two figures at the end of this remark); otherwise, apply the approximation technique introduced in the proof of Theorem 2.7 and note $d\left(y, \Omega_{x}\right)>0$, where $\Omega_{x}$ denotes the connected component of $\Omega_{+}$that contains $x$ (see the bottom two figures). The other theorems can be dealt with in much the same way.


Remark 2.9. We point out that the smoothness assumption on $\Omega$ in Theorems 2.4, 2.6 and 2.7 can be dropped in two steps. Step 1: Given a bounded connected open subset of $\mathbb{R}^{n}$ of the form (2.1), we can approximate it by an increasing sequence of connected compactly supported smooth subdomains to get analogues of these theorems, obtaining inequalities rather than strict inequalities at the moment. Step 2: These inequalities can then be improved to strict ones by suitably appealing to the strong maximum principle for parabolic equations.

Corollary 2.10. For any $x \in \Omega_{+}$and $t>0$, one has

$$
\int_{\Omega} p_{\Omega}(x, y, t) d y<\int_{\Omega} p_{\Omega}\left(x^{*}, y, t\right) d y
$$

To be clear, by considering Remarks 2.8 and $2.9, \Omega \subset \mathbb{R}^{n}$ is assumed to be a bounded connected open set of the form (2.1).

Proof. It follows from Theorem 2.6 (see also Remarks 2.8 and 2.9) that

$$
\int_{\Omega_{-} \cup \Omega_{+}} p_{\Omega}(x, y, t) d y<\int_{\Omega_{-} \cup \Omega_{+}} p_{\Omega}\left(x^{*}, y, t\right) d y
$$

and from Theorem 2.7 (see also Remarks 2.8 and 2.9) that

$$
\int_{\Omega_{--}} p_{\Omega}(x, y, t) d y<\int_{\Omega_{--}} p_{\Omega}\left(x^{*}, y, t\right) d y
$$

Combining both strict inequalities proves the corollary.

## 3 Heat content optimization

In the previous section we have assumed that $\Omega$, a bounded connected open subset of $\mathbb{R}^{n}$ with smooth boundary, can be written as (2.1) with $\Omega_{+}$being connected. Now we further require that $\Omega$ is of the form $\Omega=\Theta \backslash \bar{B}$, where $\Theta$ is some open subset of $\mathbb{R}^{n}, B$ is a relatively compact open convex subset of $\Theta$ such that it is symmetric with respect to the hyperplane $H$ given in (2.1), and $\Omega_{+}$shares part of its boundary with $B$. Obviously, both $\Theta$ and $B$ are of smooth boundary, and $\Omega_{+}$is uniquely determined by $\Omega, H$ and $B$. Let $V$ denote the unit normal vector of $H$ pointing toward $\Omega_{+}$, and consider

$$
\Omega_{\epsilon}:=\Theta \backslash(\bar{B}+\epsilon V)
$$

for $\epsilon \in \mathbb{R}$ with small enough modulus.


Theorem 3.1. For any $t>0$, one has

$$
\left.\frac{d H_{\Omega_{\epsilon}}(t)}{d \epsilon}\right|_{\epsilon=0}>0
$$

Proof. It follows from Savo's variational formula [62, Thm. 10] that for any $t>0$,

$$
\left.\frac{d H_{\Omega_{\epsilon}}(t)}{d \epsilon}\right|_{\epsilon=0}=-\int_{0}^{t}\left[\int_{\partial B}\langle V, N\rangle \frac{\partial u}{\partial N}(x, \tau) \frac{\partial u}{\partial N}(x, t-\tau) d S(x)\right] d \tau
$$

where $N$ is the unit inner normal to the boundary of $\Omega$, and

$$
u(x, \tau):=\int_{\Omega} p_{\Omega}(x, y, \tau) d y \quad(x \in \bar{\Omega}, \tau>0) .
$$

Note that $(\partial B) \backslash H$ consists of two symmetric connected components, denoted by $(\partial B)_{+}$and $(\partial B)_{-}$respectively, and suppose $(\partial B)_{+}$is contained in $\partial \Omega_{+}$. Thus by symmetry, one gets

$$
\left.\frac{d H_{\Omega_{\epsilon}}(t)}{d \epsilon}\right|_{\epsilon=0}=-\int_{0}^{t}\left[\int_{(\partial B)_{+}}\langle V, N\rangle\left[\frac{\partial u}{\partial N}(x, \tau) \frac{\partial u}{\partial N}(x, t-\tau)-\frac{\partial u}{\partial N}\left(x^{*}, \tau\right) \frac{\partial u}{\partial N}\left(x^{*}, t-\tau\right)\right] d S(x)\right] d \tau .
$$

Since $u(x, \tau)$ vanishes on $(\partial B)_{+} \times(0, \infty)$ and is positive in $\Omega_{+} \times(0, \infty)$, we see that $\frac{\partial u}{\partial N}(x, \tau) \geq 0$ for all $x \in(\partial B)_{+}$and $\tau>0$. Actually, this trivial inequality can be improved to

$$
\begin{equation*}
\frac{\partial u}{\partial N}(x, \tau)>0 \tag{3.1}
\end{equation*}
$$

if we appeal to Friedman's strong maximum principle concerning extremal values attained at non-bottom part of the parabolic boundary [28, Thm. 2] (see also the books [29, 49, 59]), which is an extension of Hopf's maximum principle from elliptic equations to parabolic ones. To employ this theorem, it remains to check that $\Omega_{+}$has the interior ball property at every element of $(\partial B)_{+}$, which obviously stands because of the convexity of $B$. Applying the same maximum principle to $u\left(x^{*}, \tau\right)-u(x, \tau)$, which vanishes on $(\partial B)_{+} \times(0, \infty)$ and is positive in $\Omega_{+} \times(0, \infty)$ because of Corollary 2.10, we see that

$$
\begin{equation*}
\frac{\partial u}{\partial N}\left(x^{*}, \tau\right)>\frac{\partial u}{\partial N}(x, \tau) \tag{3.2}
\end{equation*}
$$

for all $x \in(\partial B)_{+}$and $\tau>0$. Observe from the convexity of $B$ that the inner product between $V$ and $N$, as a function of $x \in(\partial B)_{+}$, is everywhere non-negative and assumes maximal value 1 at points with longest distance to $H$. Combining this observation with (3.2), (3.1), and the variation formula displayed earlier proves the theorem.

Theorem 1.1 is an immediate consequence of our main result Theorem 3.1.
We mention that many illustrating examples about fundamental eigenvalue optimization given in [38 can be transformed into the current heat content context with suitable modifications, notably sign-changing (maximum $\rightleftharpoons$ minimum).

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[^0]:    ${ }^{1}$ We refer the interested reader to [23] [54] for the first variation formula for the Dirichlet heat trace.

